Back-Substitution Analysis

1. T(n) = 2T(n–1) + 1, T(0) = 1

T(n) = 2T(n - 1) + 1  
T(0) = 1  
  
# K = 1  
T(n) = 2T(n - 1) + 1  
T(n - 1) = 2T((n - 1) - 1) + 1  
T(n - 1) = 2T(n - 2) + 1  
  
# Plug in  
T(n) = 2(2T(n - 2) + 1) + 1

T(n) = 4T(n - 2) + 2 + 1  
  
# K = 2  
T(n - 2) = 2T((n - 2) - 1) + 1

T(n) = 2T(n - 3) + 1  
T(n) = 4(2T(n - 3) + 1) + 2 + 1

T(n) = 8T(n - 3) + 4 + 2 + 1  
  
# K = 3  
T(n - 3) = 2T(n - 4) + 1  
T(n) = 8(2T(n - 4) + 1) + 4 + 2 + 1

T(n) = 16T(n - 4) + 8 + 4 + 2 + 1  
  
# pattern  
T(n) = 2^k \* T(n - k) + (2^k - 1)  
  
  
T(n) = 2^n \* T(0) + (2^n - 1)  
T(0) = 1  
T(n) = 2^n \* 1 + 2^n - 1 = 2^(n+1) - 1  
  
O(2^n) class exponential

2. T(n) = T(n–2) + n^2, T(0) = 1

T(n) = T(n - 2) + n^2  
T(n - 2) = T(n - 4) + (n - 2)^2  
T(n - 4) = T(n - 6) + (n - 4)^2  
  
Continue back-substituting:  
T(n) = n^2 + (n - 2)^2 + (n - 4)^2 + ... + 4^2+ 2^2 + T(0)  
T(0) = 1  
  
Sum of squares of even numbers:  
Let n = 2k ⇒ all terms are (2i)² = 4i² for i = 1 to k  
  
So,  
T(n) = 4(1² + 2² + 3² + ... + k²) + 1  
Sum of squares formula:  
∑ i² = k(k + 1)(2k + 1)/6  
  
T(n) = 4[k(k + 1)(2k + 1)/6] + 1  
  
Asymptotic behavior: O(k³) ⇒ since k = n/2 ⇒ O(n³)  
  
➡ O(n³), class polynomial

3. T(n) = T(n–1) + 1/n, T(1) = 1

T(n) = T(n - 1) + 1/n  
T(n - 1) = T(n - 2) + 1/(n - 1)  
T(n - 2) = T(n - 3) + 1/(n - 2)  
...  
  
T(n) = 1 + 1/2 + 1/3 + ... + 1/n  
  
This is the harmonic series:  
H(n) = ∑ (1/i) from i = 1 to n ≈ ln(n) + γ  
(γ is the Euler-Mascheroni constant)  
  
➡ O(log n), class logarithmic