Assignment 4

1) a) $a_n = 4a_{n-1}$ $\quad a_0 = 1$ $\quad \forall n > 1$

$$a_n = 4a_{n-1}$$

(substitute) $= 4[4a_{n-2}] = 4^2 a_{n-2}$

(substitute) $= 4[4^2 a_{n-3}] = 4^3 a_{n-3}$

$\vdots$

$= 4^k a_{n-k}$

where k is some integer, & we must choose an appropriate value for k

Let $k = n$

$\Rightarrow a_{n-n} = a_0 = 1$ [Given]

$\Rightarrow 4^n a_{n-n} = 4^n a_0 = \boxed{4^n \ \forall\ n \geq 0}$

b) $a_n = ca_{n-1}$ $\quad a_0 = 1$ $\quad \forall n > 1$.

Given that, $a_n = ca_{n-1}$

(substitute) $= c(ca_{n-2}) = c^2 a_{n-2}$

(substitute) $= c[c^2 a_{n-3}] = c^3 a_{3n-3}$

$\vdots$

for some value of $k = c^k a_{n-k}$

where k is some integer, and we must choose an appropriate value of k

Suppose that, $k = n$

$$\Rightarrow c^n a_{n-n}$$

$$= c^n a_0$$

$$\Rightarrow c^n \quad \forall n \geq 0 \text{ and } c > 1$$

c) $b_n = b_{n-1} + \frac{n}{2}$

$b_0 = 0 \qquad \forall n > 1$

Given that, $b_n = b_{n-1} + \frac{n}{2}$

$$= \left[b_{n-2} + \left(\frac{n}{2} - 1\right) + \frac{n}{2} \right]$$

$$= \left[\underset{k+1}{b_{n-3}} + \underset{k}{\left(\frac{n}{2} - 2\right)} \right] + \left(\frac{n}{2} - 1\right) + \frac{n}{2}$$

$$= \left[b_{n-(k+1)} + \left(\frac{n}{2} - k\right) \right] + \cdots + \left(\frac{n}{2} - 2\right) + \left(\frac{n}{2} - 1\right) + \frac{n}{2}$$

$$= b_{n-(k+1)} + \left(\frac{n}{2} - k\right) + \cdots\cdots + \left(\frac{n}{2} - 2\right) + \left(\frac{n}{2} - 1\right) + \frac{n}{2}$$

We want to stop at b_0, we know that $b_0 = 0$

So, let $\frac{n}{2} - (k+1) = 0$

$$\Rightarrow \frac{n}{2} - k - 1 = 0$$

$$\Rightarrow \frac{n}{2} = k+1$$

$$\Rightarrow n = 2(k+1) = 2k+2$$

$\therefore$ $\frac{n}{2} - (k+1) = 0$

$$\frac{n}{2} - k = 1$$

$$\frac{n}{2} - (k-1) = 2 \text{ etc.}$$

So, we get

$$\overset{=0}{b_0} + 1 + 2 + 3 + \cdots\cdots + \left(\frac{n}{2} - 1\right) + \frac{n}{2}$$

$$\Rightarrow 1 + 2 + 3 + \cdots\cdots + \left(\frac{n}{2} - 1\right) + \frac{n}{2}$$

$$= \frac{n/2}{2}\left[1 + \frac{n}{2}\right] = \frac{n}{4}\left(\frac{2+n}{2}\right)$$

$$= \boxed{\frac{n(n+2)}{8}}$$

d) $c_n = 4c_{n-1} + n$

$c_0 = 0$

$\forall n > 1$

Given that, $c_n = 4c_{n-1} + n$

$$= 4[4c_{n-2} + (n-1)] + n$$

$$= 4^2 c_{n-2} + 4(n-1) + n$$

$$= 4^2[4c_{n-3} + (n-2)] + 4(n-1) + n$$

$$= 4^3 c_{n-3} + 4^2(n-2) + 4(n-1) + n$$

$$= 4^k c_{n-k} + 4^{k-1}(n-(k-1)) + \text{----} + 4(n-1) + 4^0 n$$

When $n = k$, we get

$$= 4^n c_0 + 4^{n-1} . 1 + 4^{n-2} . 2 + 4^{n-3} . 3 + \text{----} + 4^1(n-1) + n$$

$$= \sum_{j=0}^{n-1} 4^{n-j} . j$$

$$c_n = \sum_{j=0}^{n-1} 4^{n-j} . j$$

$$= 0 . 4^n + 1 . 4^{n-1} + 2 . 4^{n-2} + \text{---} + (n-1)4 + n$$

$$= \sum_{j=0}^{n-1} (n-j) 4^j = n \sum_{j=0}^{n-1} 4^j - \sum_{j=0}^{n-1} j . 4^j$$

$$\Rightarrow C_n = \frac{n.(1-4^n)}{1-4} - \sum_{j=0}^{n-1} j.4^j$$

$$= \frac{n}{2}(4^n-1) - \sum_{j=0}^{n-1} j.4^j$$

$$S_n = \sum_{j=0}^{n-1} j.4^j = ?$$

Now, $4^{k+1} - 4^k = 4^k(4-1) = 4^k.3 = 4^k$

$$S_n = \sum_{j=0}^{n-1} j.\frac{4^{j+1}-4^j}{2} = \frac{1}{2}\sum_{j=0}^{n-1} j(4^{j+1}-4^j)$$

e) $t_n = 10t_{n-1} + 21t_{n-2}$ $\quad t_0 = 0 \quad \forall n \geq 2$

$t_1 = 1$

$t_2 = 10t_0 + 21t_1$

$= 10(0) + 21(1) = 21$

$t_3 = 10t_1 + 21t_2 = 10 + 21^2$

$t_4 = 10t_2 + 21t_3 = 10(10 + 21^2) + 21^3$

Also, here $A = 10$, $B = -21$

Characterstics equation is $x^2 - 10x + 21 = 0$

$\Rightarrow x^2 - 10x + 21 = 0$

$x^2 - 3x - 7x + 21 = 0$

$x(x-3) - 7(x-3) = 0$

$(x-7)(x-3) = 0$

$x = 3, 7$

$\therefore$ Solution for $t_n = ax_1^n + bx_2^n = a(3)^n + b(7)^n$

where a and b are constants

Given conditions are $t_0 = 0$ and $t_1 = 1$

$t_0 = a(3)^0 + b(7)^0 \Rightarrow a + b = 0$ --- ①

$t_1 = a(3)^1 + b(7)^1$

$\Rightarrow 3a + 7b = 1$ --- ②

from eq^n 1 and 2

$$\cancel{3}a + 3b = 0$$
$$\cancel{3}a + 7b = 1$$
$$- \quad - \quad -$$
$$-4b = -1$$

$$\boxed{b = \frac{1}{4}}$$

$$\therefore \boxed{a = -\frac{1}{4}}$$

$\therefore$ The solutions of recurrence is:

$$\boxed{-\frac{1}{4}(3)^n + \frac{1}{4}(7)^n = t_n}$$

f) $t_n = -5t_{n-1} + 14t_{n-2} + 3^n \qquad t_0 = 0, \ t_1 = 1 \ \forall n \geq 2$

$$t_n + 5t_{n-1} - 14t_{n-2} = \underbrace{3^n}_{b^n \cdot 1}$$

$b = 3 \qquad P(n) = 1$

$\therefore$ degree $d = 0$

Characteristic equation :

$$(x-2)(x+7) = \text{not } 0$$
$$(x-2)(x+7)(x-3) = 0$$

$$a_n = c_1 2^n - c_2 7^n + c_3 3^n$$

From recurrence → boundary values

$t_0 = 0$ — 1

$t_1 = 1$ — 2

$t_2 = -5t_1 + 14t_0 + 3^2$

$= 4$ — 3

So, we get 3 equations:

$\underline{n=0}$ $\quad c_1 - c_2 + c_3 = 0$ — A

$\underline{n=1}$ $\quad 2c_1 - 7c_2 + 3c_3 = 1$ — B

$\underline{n=2}$ $\quad 4c_1 - 49c_2 + 9c_3 = 4$ — C

$\boxed{c_1 = -\frac{6}{5}}$, $\boxed{c_2 = \frac{1}{20}}$, $\boxed{c_3 = \frac{5}{4}}$

$\Rightarrow \boxed{t_n = -\frac{6}{5} 2^n - \frac{1}{20} 7^n + \frac{5}{4} 3^n}$

g) $t_n = 3t_{n-1} + n + 3^n$ $\quad t_0 = 0$

$\forall n \geq 1$

$$\underbrace{t_n - 3t_{n-1}}_{(x-3)} = n + 3^n$$

$b_1 = 1$ $\quad b_2 = 3$

$p_1(n) = n$ $\quad p_2(n) = 1$

$d_1 = 1$ $\quad d_2 = 0$

Characterstic equation:

$$(x-3)(x-1)^2(x-3) = 0$$

$$\Rightarrow (x-3)^2(x-1)^2 = 0$$

$$x = \underbrace{1, 3}_{\text{multiplicity} = 2}$$

General solution; $t_n = c_1 1^n + c_2 1^n . n + c_3 . 3^n + c_4 n . 3^n$

From recurrence $\longrightarrow$ boundary values

$t_0 = 0$ — (1)

$t_1 = 3t_0 + 1 + 3^1 = 4$ — (2)

$t_2 = 3t_1 + 2 + 3^2 = 23$ — (3)

$t_3 = 3t_2 + 3 + 3^3 = 98$ — (4)

So, we get

$\underline{n=0}$ $\quad c_1 + c_3 = 0$

$\underline{n=1}$ $\quad c_1 + c_2 + 3c_3 + 3c_4 = 4$

$\underline{n=2}$ $\quad c_1 + 2c_2 + 9c_3 + 18c_4 = 23$

$\underline{n=3}$ $\quad c_1 + 3c_2 + 27c_3 + 81c_4 = 98$

$\therefore c_1 = -1, \quad c_2 = \frac{-3}{4}; \quad c_3 = 1, \quad c_4 = \frac{11}{12}$

$$\Rightarrow \boxed{t_n = -1 - \frac{3}{4} n + 3^n + \frac{11}{12} n \cdot 3^n}$$

2) a) $T(n) = 4T\left(\frac{n}{2}\right) + n \quad \forall n > 1$

<u>By Induction:</u>

Let a = first term
r = Common ratio
m = Total number of n terms

$\Rightarrow$ we can simplify it, as the sum of indeterminate size is a G.P

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots ar^{m-1}$$

$$= \frac{a(r^m - 1)}{r-1}$$

for $n = 2^k$

$$\therefore S_n = 2^n + 2^{n+1} + 2^{n+2} + \cdots + 2^{2n-1}$$

$$= \frac{2^n(2^n - 1)}{2-1} = 4^n - 2^n$$

$\therefore$ The general formula for a is:

$$a(n) = 4^n - 2^n + 4^n \cdot m$$
$$= 4^n(m+1) - 2^n$$

Hence, $T(k) = a(2^k)$

$$\Rightarrow 4^{2^k}(m+1) - 2^{2^k}$$

By Big-oh of $T(n)$

(say we don't know the boundary conditions)

Let $n = 2^k$ for $k \in I_+$

$\Rightarrow$ $\underbrace{k \log_2 2}_{=1} = \log_2 n$ $\Rightarrow k = \log_2 n = \lg n$ ($\lg \equiv \log_2$)

rec: $T(n) = 4T\left(\frac{n}{2}\right) + n$

$T(2^k) = 4T(2^{k-1}) + 2^k$

$\Rightarrow t_k = 4t_{k-1} + 2^k$

where $t_k \triangleq T(2^k) = T(n)$

$\Rightarrow \underbrace{t_k - 4t_{k-1}}_{} = 2^k$

$(x-4)(x-2)^1 = 0$

roots $= 4, 2$

$\Rightarrow t^k = c_1 4^k + c_2 2^k$

$\Rightarrow T(\underbrace{2^k}_{n=2^k}) = c_1 4^{\underbrace{k}_{k=\lg n}} + c_2 \underbrace{2^k}_{n=2^k}$

$\Rightarrow T(n) = c_1 4^{\lg n} + c_2 n$

$\Rightarrow T(n) \in O(4^{\lg n} \mid n \text{ is power of } 2)$

b) $T(n) = 5\,T\left(\frac{n}{2}\right) + n^2 \quad \forall n > 1$

Let $n = 2^k$ for some $k \in I_+$

$\Rightarrow T(2^k) = 5T(2^{k-1}) + \underbrace{(2^k)^2}_{(2^k)^2 = 2^{2k} = 4^k}$

$\Rightarrow T(2^k) = 5T(2^{k-1}) + 4^k$

Let $t_k \triangleq T(2^k)$. Then:

$$t_k = 5t_{k-1} + 4^k$$

$$\Rightarrow \underbrace{t_k - 5t_{k-1}}_{} = 4^k$$

$$(x-5)(x-4)' = 0$$

$$\Rightarrow t_k = c_1 5^k + c_2 k \cdot 4^k$$

Putting n back

$T(n) = c_1 \underbrace{n^2}_{n = 2^k} + c_2 n^2 \underbrace{\lg n}_{2^k = n}$

So, $n^2 = 4^k \Rightarrow k = \lg n$

$\Rightarrow T(n) = O(n^2 \lg n \mid n$ is a power of 2)

3)

Tower of Hanoi

$t_n \triangleq$ No. of steps required to move n discs from 1 to 3

$$t_n = 2t_{n-1} + 1 \quad \forall n \geq 2$$

$$t_1 = 1$$ ← Hanoi recurrence

pattern: $t_n = 2^n - 1 \quad \forall n \geq 1$
(FORMULA)

PROOF : <u>By induction</u>

CLAIM: Solution to $t_n = 2t_n - 1 \quad \forall n \geq 2$
with $t_1 = 1$ is the formula

$$t_n = 2^n - 1 \quad \forall n \geq 1$$

Value given by recurrence = value given by formula?

check base case(s), & then use induction hypothesis.

<u>Base case</u>	<u>recursion</u>	<u>formula</u>
n = 2	3	3
n = 3	7	7

<u>Induction hypothesis</u> : Assume the formula solves the recurrence for n = 1, 2, 3 .... k

We need to prove that n = k+1

Recursion: $t_{k+1} = \underbrace{2t_k}_{\downarrow} + 1$

We have to show that the formula for n=k

$$= 2\underbrace{(2^k - 1)}_{t_k \text{ from formula}} + 1$$

$$= 2^{k+1} - 2 + 1$$

$$= 2^{k+1} - 1$$

which is exactly the same formula, but k is replaced by k+1

Solving by characterstic eqn method:

$$t_n - 3t_{n-1} - 4t_{n-2} = 0 \quad \text{etc.}$$

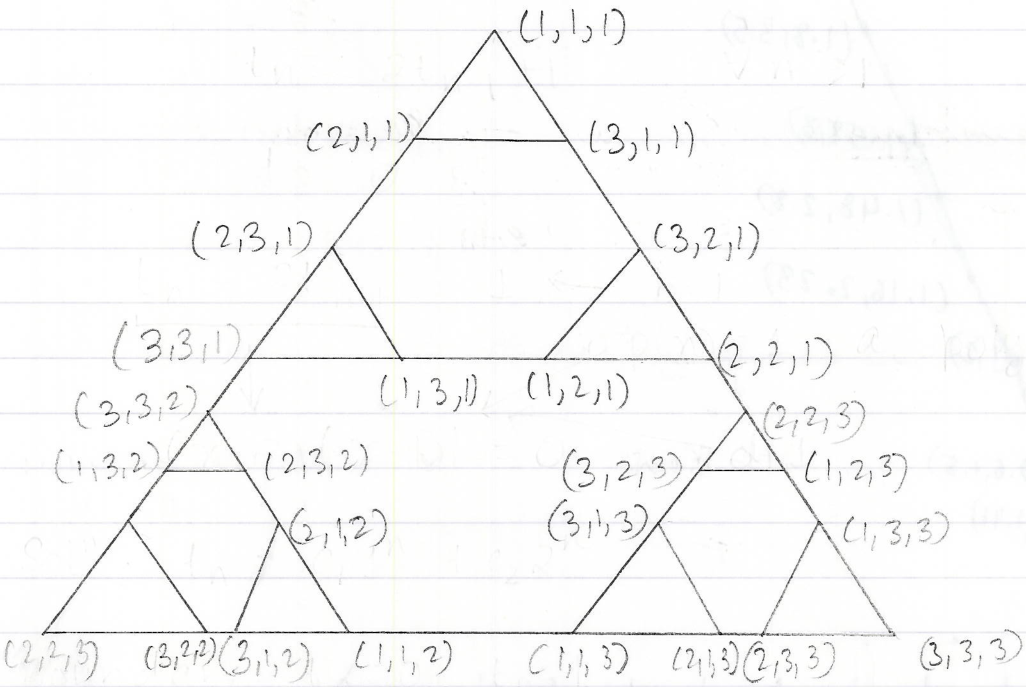
$$x^2 - 3x - 4 = 0$$

3 or more t_n values

Degree

3 ⇒ quadratic eqn
4 ⇒ cubic eqn
5 ⇒ quartic eqn $(x^4 + \dots)$

Now, look at the Hanoi recursion:

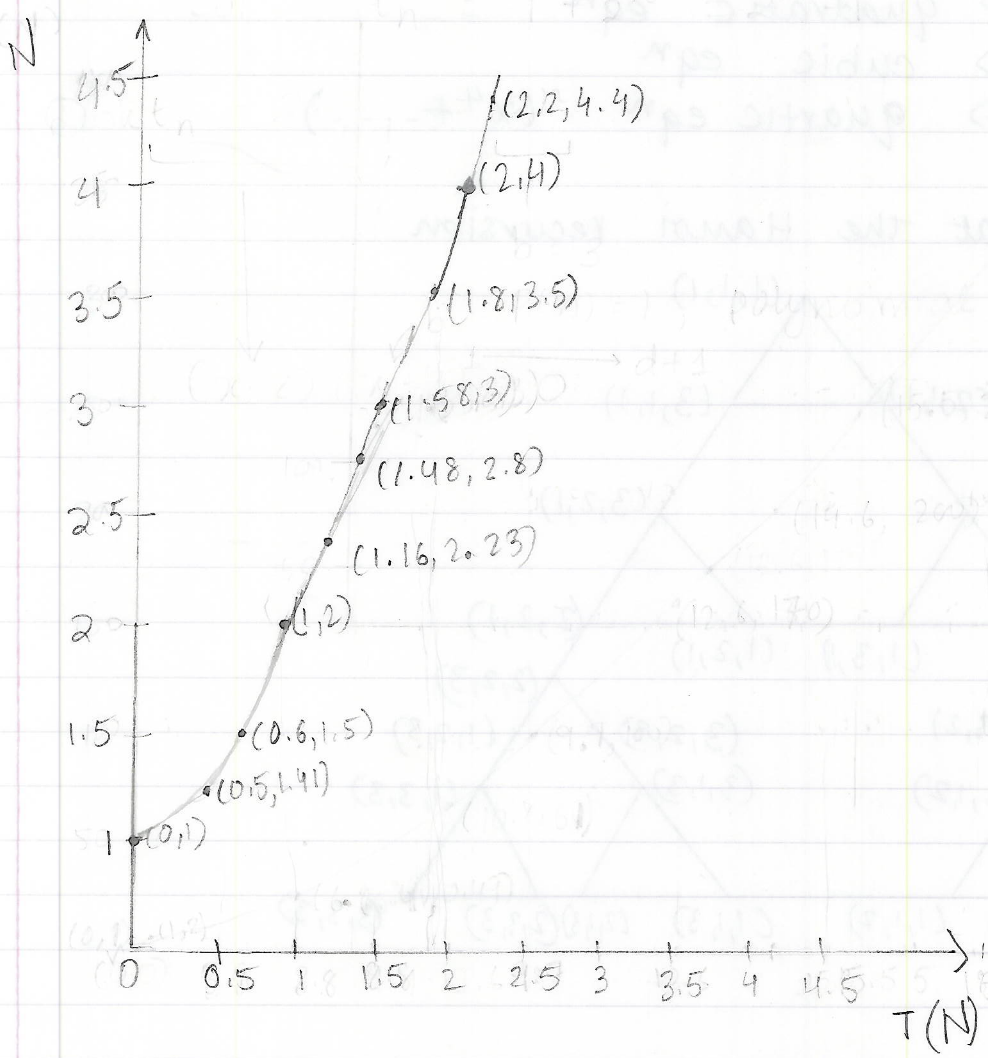


$$T(N) = \sum_{i=0}^{N-1} 2^i = \sum_{i=0}^{N-1} 2^i = 2^N - 1$$

$$S_n = a\left(\frac{r^N - 1}{r-1}\right)$$

The complexity of the algorithm is exponential
It grows very fast, as a power of 2

$$\therefore\ T(N) = 2^N$$



4) <u>Binary-search</u>

Initial condition: $T(1) = O(1) = 1$

Recurrence relation: The time to search in an array of N elements = time to search in an array of N/2 elements + 1 comparison

$$T(N) = T(N/2) + 1$$

Next, we perform telescoping :

$$T(N) = T(N/2) + 1$$
$$T(N/2) = T(N/4) + 1$$
$$T(N/4) = T(N/8) + 1$$
$$\text{———}$$
$$T(4) = T(2) + 1$$
$$T(2) = T(1) + 1$$

Summing up the left & right sides

$$T(N) + T(N/2) + T(N/4) + T(N/8) + \dots + T(2)$$

$$= T(N/2) + T(N/4) + T(N/8) + \dots + T(2) + T(1) + (1 + 1 + \dots + 1)$$

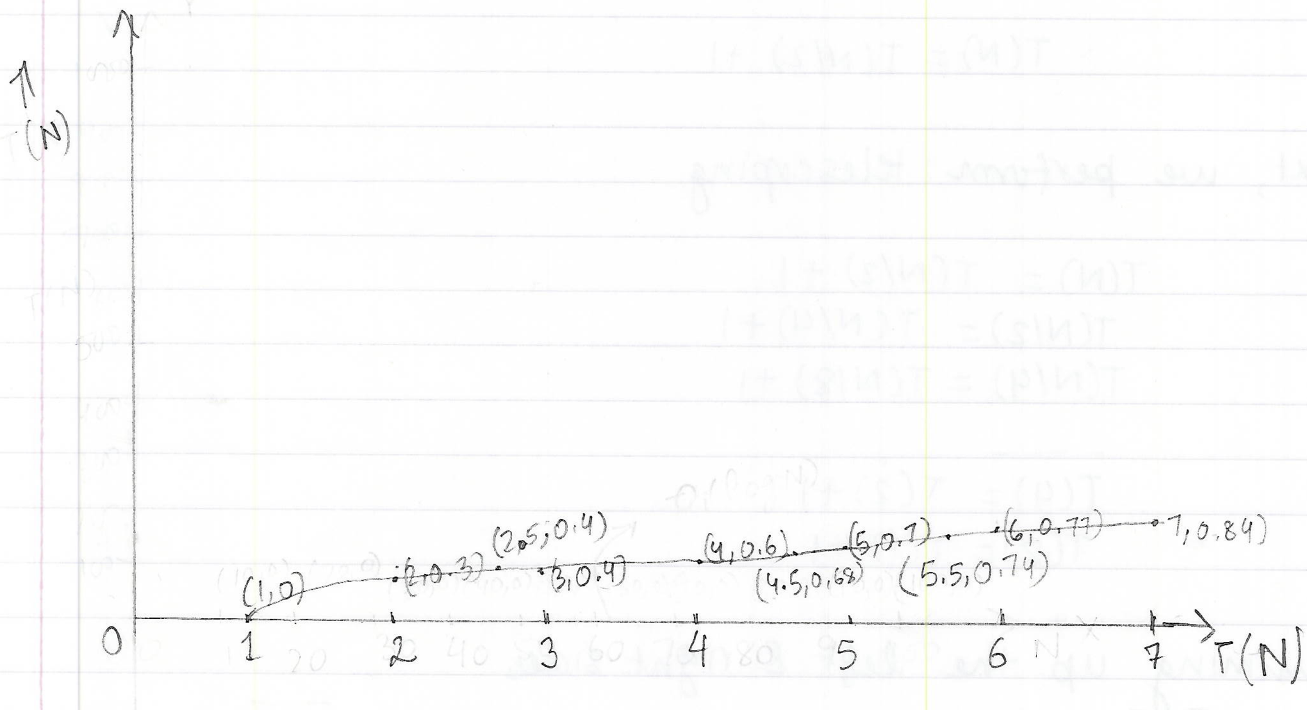
The no. of 1's on the right side is $\log N$

$\Rightarrow T(N) = T(1) + \log N$

$\therefore T(N) = 1 + \log N$

$\therefore$ The runtime of binary search is

$$T(N) = O(\log N)$$



The binary search algorithm reduces the search for an element in a search sequence of n to size of n/2, when n = even. Hence, if f(n) is the no. of comparisons required to search

Then, $f(n) = f(n/2) + 2$ when $n =$ even