PS 0 Stanford CS 229

Q1 ⓐ $2Ax + b$ $\left[\frac{\partial(x^T \cdot Ax)}{\partial x} = \frac{\partial x^T Ax}{\partial x} + (Ax)^T \frac{\partial x}{\partial x} \right]$

ⓑ $\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_i} & \cdots \end{bmatrix}^T$

$\underbrace{\begin{bmatrix} \frac{\partial g(h(x))}{\partial x_i} & \cdots \end{bmatrix}^T}$

$h \Rightarrow [a \; b \; c \ldots]_{n \times 1}$

$h(x) \Rightarrow ax_1 + bx_2 + \ldots$

$$\frac{\partial g(h(x))}{\partial x_i} = \frac{\partial g(h(x))}{\partial (h(x))} \cdot \frac{\partial (h(x))}{\partial x_i}$$

$$= g'(h(x)) \cdot \frac{\partial h(x)}{\partial x_i}$$

$$\nabla f(x) = \begin{bmatrix} g'(h(x)) \cdot \frac{\partial h(x)}{\partial x_1} \\ \vdots \\ g'(h(x)) \cdot \frac{\partial (h(x))}{\partial x_n} \end{bmatrix}$$

$$\therefore \nabla f(x) = g'(h(x)) \cdot \begin{bmatrix} \frac{\partial h(x)}{\partial x_1} \\ \vdots \end{bmatrix}$$

$$\nabla f(x) = g'(h(x)) \nabla h(x)$$

ⓒ $\nabla f(x) = \frac{\partial f(x)}{\partial x} = Ax + b$

$$\nabla^2 f(x) = \frac{\partial (Ax + b)}{\partial x} = A$$

$$Ax \Rightarrow \begin{bmatrix} A_{11} & A_{12} & \cdots \\ \vdots & & \vdots \\ A_{N1} & & A_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_{11} \cdot x_1 + A_{12} \cdot x_2 \cdots \\ \vdots \\ A_{N1} \cdot x_1 + \ldots \end{bmatrix}$$

vector

$$\frac{\partial vec}{\partial x} \Rightarrow \begin{bmatrix} \frac{\partial A_{11} x_1 + \ldots}{\partial x_1} & \frac{\partial \ldots}{\partial x_2} & \cdots \\ \vdots & & \\ \frac{\partial A_{N1} x_1 + \ldots}{\partial x_1} & \cdots & \end{bmatrix}$$

(d) $f(x) = g(a^T x)$ $\quad g: \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}^n$

$\nabla f(x) = \frac{\partial f(x)}{\partial x}$

$\Rightarrow \frac{\partial g(a^T x)}{\partial (a^T x)} \cdot \frac{\partial (a^T x)}{\partial x}$

$\Rightarrow g'(a^T x) \cdot a$

$\nabla^2 f(x) = \frac{\partial \nabla f(x)}{\partial x}$

$\Rightarrow a^T \frac{\partial g'(a^T x)}{\partial x}$

$\Rightarrow a^T \cdot \frac{\partial g'(a^T x)}{\partial a^T x} \cdot \frac{\partial a^T x}{\partial x}$

$a^T \cdot g''(a^T x) \cdot a$ $\quad \times$

correct

$\nabla^2 f(x) = \frac{\partial g'(\bar{a}^T \bar{x}) \cdot \bar{a}}{\partial \bar{x}} = \frac{\partial g'(a^T x)}{\partial a^T x} \cdot \frac{\partial a^T x}{\partial x} \cdot a^T$

$\nabla \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \end{bmatrix} x \to \nabla \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_1} \cdots \right]$ $\quad \Rightarrow g''(a^T x)\, a a^T$

$\to \frac{\partial^2 f}{\partial x_1 \partial x_1} \cdots \frac{\partial^2 f}{\partial x_2 \partial x_1}$

$\frac{\partial^2 f}{\partial x_1 \partial x_2}$

$\vdots$

Q2 (a) ZZ^T $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} z_1 z_1 & z_1 z_2 & z_1 z_3 \\ z_2 z_1 & z_2 z_2 & z_2 z_3 \\ z_3 z_1 & z_3 z_2 & z_3 z_3 \end{bmatrix}$ $\quad \therefore (ZZ^T) = (ZZ^T)^T$ $\quad ZZ^T$

scalar x scal

$\therefore z_i z_j = z_j z_i$

$x^T (ZZ^T) x \geq 0$

$\therefore (x^T Z)(Z^T x)$

$\Rightarrow (Z^T x)^T (Z^T x)$

$\Rightarrow \|Z^T x\|_2 \geq 0$

$\therefore$ Proved

(b) Null $A \to N(A) \Rightarrow \quad Ax = 0$

$$\therefore \underbrace{ZZ^Tx}_{\text{scalar}} = 0$$

$$\therefore C \cdot Z = 0 \quad C: \text{const}$$

$$\therefore N(A) = \{ x \in R^n \mid Z^T x = 0 \}$$

Rank = # indep col

$$\therefore Ax = C \cdot Z$$

$$\therefore \text{rank}(A) = 1$$

(c) $\underset{\text{iso}}{B}A\underset{\text{rand}}{B^T}$

let $B = C^T$ → C^TAB

$\therefore B^T = C$

$$\cancel{C_1 C_1} \begin{bmatrix} - C_1 - \\ - C_1 - \\ \vdots \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | & | \\ C_1 & C_2 \\ | & | \end{bmatrix}$$

$C_1^T A C_1 \geq 0$ similarly others

$\therefore BAB^T$ is PSD

Q3 (a) $A = T \Lambda T^{-1}$

$$AT = T\Lambda$$

$$A[t^1 \dots] = [t^1 \dots] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots \end{bmatrix}$$

$$[\;]\underset{t_1}{[\;]} = [t_1]\lambda_1$$

(b) $AU = U\Lambda$

$$[A][U^1 \; U^2 \dots] = [U^1 U^2 \dots] \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \\ & & \ddots \end{bmatrix}$$

$$[AU^1 \; AU^2 \dots] = [U^1\lambda_1 \; U^2\lambda_2 \dots]$$

(c) $x^T U \Lambda U^T x \geq 0$

$$(U^Tx)^T \Lambda (U^Tx) \geq 0$$

$$\therefore U^{1T} A U^1 = U^{1T} \lambda U^1 \quad \therefore \lambda \underbrace{U^{1T}U^1}_{\geq 0} = \lambda \|U^1\|_2$$

LHS $U^{1T} A U^1 \geq 0$

$\therefore \lambda \geq 0$