SOLUTION TO THE "CHRISTMAS EVE" RIDDLE

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Problem. Let n be a positive integer such that $n + 1$ is divisible by 24. Show that then also the sum of the divisors of n is divisible by 24.

Solution. First, more generally, recall that if a positive integer n has unique prime factorisation

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

then the number of divisors of n is given by

$$(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1).$$

To see this, observe that any divisor of n must have the form

$$p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k},$$

where $0 \leqslant \beta_j \leqslant \alpha_j$ for each j. There are therefore $\alpha_j + 1$ choices for the value of β_j, with each different tuple $(\beta_1, \beta_2, \ldots, \beta_k)$ mapping to a different divisor of n.

It follows immediately that

Fact I *The number of divisors of a positive integer n is odd if and only if n is a perfect square.*

Note also the following, both of which are trivially checked.

Fact II *The square of an integer can only be congruent to* 0 *or* 1 *modulo* 3.[1]

Fact III *The square of an integer can only be congruent to* 0, 1, *or* 4 *modulo* 8.

Now suppose that $n + 1$ is divisible by 24. One has $n \equiv 2 \pmod 3$, and so n cannot be a perfect square by **Fact** II, and must have an even number of divisors by **Fact** I.

Let $1 = d_1 < d_2 < \cdots < d_m$ be the divisors of n which do not exceed $\sqrt{n}$. The divisors of n can be paired-off and listed as

$$d_1, \frac{n}{d_1}, d_2, \frac{n}{d_2}, \ldots, d_m, \frac{n}{d_m},$$

and their sum may be written as

$$\sum_{j=1}^{m} \left(d_j + \frac{n}{d_j} \right).$$

[1] An integer which appears as a remainder of some square integer modulo a positive integer x is called a *quadratic residue* of x.

It is sufficient to prove that each term in the sum above is divisible by 24.

Let d be an arbitrary divisor of n and consider the expression

$$d + \frac{n}{d} = \frac{d^2 + n}{d}.$$

Since $n + 1$ is divisible by 24, one must have that n is co-prime to 24, whence so are any of its divisors.

Thus, the problem is reduced to proving that the numerator of the expression above is divisible by 24.

By **Fact** II, d^2 must be congruent to 0 or 1 modulo 3. The former implies that d is divisible by 3, which is impossible since d is co-prime to 24. Hence, $d^2 \equiv 1 \pmod 3$.

By **Fact** III, d^2 must be congruent to 0, 1, or 4 modulo 8. Either of $d^2 \equiv 0 \pmod 8$ or $d^2 \equiv 4 \pmod 8$ implies that d is even, which is impossible since d is co-prime to 24. Hence, $d^2 \equiv 1 \pmod 8$.

This shows that $d^2 \equiv 1 \pmod{24}$, from which it follows easily that

$$d^2 + n = (d^2 - 1) + (n + 1)$$

is divisible by 24, as required. $\square$