Problem1

(a) From the definition, $R_1 \cap R_2 = \{(s1, s2) : (s1, s2) \in R1 \text{ and } (s1, s2) \in R2\}$.

- Reflective: Since R_1 and R_2 are both reflexive, So for each $s \in S$, $(s, s) \in R$, then $(s, s) \in R_1 \cap R_2$. Therefore, $R_1 \cap R_2$ is reflexive.
- Symmetric: Because R_1 and $R2$ are both symmetric, so if $(s_1, s_2) \in R_1 \cap R_2$, we have $(s_1, s_2) \in R_1$ and $(s_1, s_2) \in R_2$, then $(s_2, s_1) \in R_1$, $(s_2, s_1) \in R_2$. So $(s_2, s_1) \in R_1 \cap R_2$. Therefore, $R_1 \cap R_2$ is symmetric.
- Transitive: Let $s_1, s_2, s_3 \in S$. Since R_1 and R_2 are both transitive, if $(s_1, s_2) \in R1$, $(s_2, s_3) \in R_1$, then $(s_1, s_3) \in R_1$. Similarly, $(s_1, s_3) \in R_2$. So $(s_1, s_3) \in R_1 \cap R_2$. Therefore, $R_1 \cap R_2$ is transitive.

Therefore, $R_1 \cap R_2$ is an equivalence relation.

(b) $[x]_1 = [s: s \in S \text{ and } sR_1x]$

$[x]_2 = [s: s \in S \text{ and } sR_2x]$

$[x] = [s: s \in S \text{ and } s(R_1 \cap R_2)x] = [s: s \in S, x \in [x]_1 \cap [x]_2]$

(c) $R_1 \cup R_2$ is not an equivalence relation.

- Reflective: Since R_1 and R_2 are both reflexive, So for each $s \in S$, $(s, s) \in R$, then $(s, s) \in R_1 \cup R_2$. Therefore, $R_1 \cup R_2$ is reflexive.
- Symmetric: Let $(s_1, s_2) \in R_1 \cup R_2$, if $(s_1, s_2) \in R_1$, Because R_1 is Symmetric, so $(s_2, s_1) \in R_1$, then $(s_2, s_1) \in R_1 \cup R_2$. If $(s_1, s_2) \in R_2$, Because R_2 is Symmetric, so $(s_2, s_1) \in R_2$, then $(s2, s1) \in R_1 \cup R_2$. Therefore, $R_1 \cup R_2$ is symmetric.
- Transitive: Let $S = \{1, 2, 3\}$, $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$, $R2 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$. $R_1 \cup R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$. For element (1, 2) and (2, 3) in $R_1 \cup R_2$, if $R_1 \cup R_2$ is transitive, (1, 3) will in $R_1 \cup R_2$. But (1, 3) is not in $R1 \cup R_2$. There $R_1 \cup R_2$ is not transitive.

Therefore, $R_1 \cup R_2$ is not an equivalence relation.

Problem2

(a) $R_1; R_2 = \{(a, c) : \text{there is a } b \text{ with } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$. If R_1 and R_2 are both reflexive, for $s \in S$, let $(s, s) \in R_1$, if $R_1; R_2$, then $(s, s) \in S$. So that $(s, s) \in R_1; R_2$. Therefore, if R_1 and R_2 are reflexive then $R_1; R_2$ is reflexive.

(b) Use a counterexample to disapprove. R_1 and R_2 are both symmetric, so let S = {1, 2, 3}, $R_1 = \{(1, 1), (1, 2), (1, 3)\}$, $R_2 = \{(2,1), (2, 2), (2, 3)\}$. So $R1; R2 = \{(1, 1), (1, 2), (1, 3)\}$. $R_1; R_2$ is not symmetric.

(c) Use a counterexample from (b) to disapprove. $R_1; R_2$ is not transitive.

Problem3

(a) Use induction

Base case j = i: Clearly $R^i = R^i$. So $R^j = R^i$ holds.

Inductive case: Assume that $R^j = R^i$ holds. We will show that $R^{j+1} = R^i$ holds.

$$\begin{aligned} R^{j+1} &= R^j \cup (R; R^j) \\ &= R^i \cup (R; R^j) \\ &= R^{i+1} \\ &= R^i \end{aligned}$$

Therefore, by induction, $R^j = R^i$ holds for all $j \geq i$.

(b) Use induction

When $j \leq i$, clearly $R^j \subseteq R^i$

Base case j = i: $R^j = R^i$, So $R^j = R^i$ holds.

Inductive case: Assume that $R^j \subseteq R^i$ holds. We will show that $R^{j+1} \subseteq R^i$ holds.

From (a) we have that $R^j = R^i$ holds for all $j \geq i$. so $R^{j+1} = R^j \subseteq R^i$

Therefore, by induction, $R^j \subseteq R^i$ for all $j \geq 0$

(c) Use contradiction

Assume that $R^{k^2} \neq R^{k^2+1}$ ($|S| = k$)

So that $\exists(a, b) \in R^{k^2+1}$, but $(a, b) \notin R^{k^2}$

$R^{k^2+1} = R^{k^2} \cup (R; R^{k^2})$, so $(a, b) \in R^{k^2} \cup (R; R^{k^2})$

So that $(a, c_k) \in R$ and $(c_k, b) \in R^{k^2}$ ($c \neq a$).

$R^{k^2} = R^{k^2-1} \cup (R; R^{k^2-1})$, so $(c_k, b) \in R^{k^2-1} \cup (R; R^{k^2-1})$

Because $(a, c_k) \in R$, if $(c_k, b) \in R^{k^2-1}$, then $(a, b) \in (R; R^{k^2-1}) \in R^{k^2}$, it is contrary to the assumption.

So $(c_k, b) \notin R^{k^2-1}$, $(c_k, b) \in (R; R^{k^2-1})$

……

$(c_1, c_0) \in R$ and $(c_0, b) \in R^{k^2-1}$ $(c_1 \neq c_0)$

Because $a \in S$, $|S| = k$, and count($c_0, c_1, c_2, \ldots\ldots c_k$) = k+1, so there exists $c_i = a$, $i \in [0, k]$

This is contrary to the assumption. So the assumption is not true.

Therefore, if $|S| = k$, then $R^{k^2} = R^{k^2+1}$

(d) Use induction

Base case n = 0: For all $m \in N$, $R^{0+m} = R^m$

$$\begin{aligned} &= I ; R^m \text{ (From assignment1 problem 8(b))} \\ &= R^0 ; R^m \end{aligned}$$

So P(0) : $R^0;R^m = R^{0+m}$ holds

Inductive case: Assume for all m, P(n) holds, that is $R^n ; R^m = R^{n+m}$, we will show that

P(n+1) : $R^{n+1} ; R^m = R^{m+n+1}$ holds, that is :

$R^{n+1} ; R^m = (R^n \cup (R ; R^n)) ; R^m$

$= (R^n ; R^m) \cup ((R ; R^n) ; R^m)$ (From assignment1 problem8 (c))
$= (R^n ; R^m) \cup (R ; (R^n ; R^m))$ (From assignment1 problem8 (a))
$= R^{n+m} \cup (R ; R^{n+m})$
$= R^{n+m+1}$
$= R^{(n+1)+m}$

So P(n+1) holds.

Therefore, by induction, $R^n ; R^m = R^{m+n}$ holds for all $n \in N$

(e) From (c), we have that if |S| = k, then $R^{k^2} = R^{k^{2+1}}$
Assume that $(a, b) \in R^{k^2}$ and $(b, c) \in R^{k^2}$,
then $(a, c) \in R^{k^2} ; \in R^{k^2}$
$= R^{2k^2}$ (from (d))
$= R^{k^2}$ (from (a))
Therefore, R^{k^2} is transitive.

(f) We need to prove the reflective, symmetric and transitive.

- Reflective: Since $(R \cup R^{\leftarrow})^0 = I$, and from (b) we have that $I \subseteq (R \cup R^{\leftarrow})^{k^2}$,
  so for any $a \in S$, $(a, a) \in I \subseteq (R \cup R^{\leftarrow})^{k^2}$.
  Therefore, $(R \cup R^{\leftarrow})^{k^2}$ is transitive.
- Symmetric: Let $p = (R \cup R^{\leftarrow})^{k^2}$ be a binary relation.
  Assume that $(a, b) \in (R \cup R^{\leftarrow})^{k^2} \cup (R \cup R^{\leftarrow} ; (R \cup R^{\leftarrow})^{k^2})$
  From (c) we know that :
  $(a, c_0) \in R \cup R^{\leftarrow}$ and $(c_0, b) \in (R \cup R^{\leftarrow})^{k^2-1}$
  $(c_0, c_1) \in R \cup R^{\leftarrow}$ and $(c_1, b) \in (R \cup R^{\leftarrow})^{k^2-2}$
  ……
  $(c_i, c_{i+1}) \in R \cup R^{\leftarrow}$ and $(c_{i+1}, b) \in (R \cup R^{\leftarrow})^{k^2}$
  So for all $i \in [0, k-1]$, $(c_i, c_{i+1}) \in R \cup R^{\leftarrow}$ ($c_0 = a$, $c_k = b$)
  If $(c_i, c_{i+1}) \in R \cup R^{\leftarrow}$, then $(c_{i+1}, c_i) \in R \cup R^{\leftarrow}$. So $(b, a) \in (R \cup R^{\leftarrow})^{k^2}$
  Therefore, $(R \cup R^{\leftarrow})^{k^2}$ is symmetric.
- Transitive: From (e) we have that, $(R \cup R^{\leftarrow})^{k^2}$ is transitive.

Therefore, if |S| = k, $(R \cup R^{\leftarrow})^{k^2}$ is an equivalence relation.

(g)

Problem4

If n = 1: $f(1) \in O(1)$

If $n \geq 1$:

Because $\lfloor \frac{n}{5} \rfloor \leq \lfloor \frac{n}{3} \rfloor$,

so $f(n) \leq f(\lfloor \frac{n}{3} \rfloor) + 3f(\lfloor \frac{n}{3} \rfloor) + n = 4f(\lfloor \frac{n}{3} \rfloor) + n = 4f(\lfloor \frac{n}{3} \rfloor) + O(1) \in O(n)$

Therefore, $f(n) \in O(n)$

Problem5

(a) count(T):

```
        if T = τ:
            return 0
        else:
            return count( T_left ) + count( T_right ) + 1
```

(b) leaves(T):

```
        if T = τ:
            return 0
        else if T =((τ, τ)):
            return 1
        else:
            return leaves( T_left ) + leaves( T_rignt )
```

(c) half-leaves(T):

```
        if T = τ:
            return 0
        else if T = (τ, T_right) or T = (T_letf, τ):
            return 1
        else:
            return half-leaves( T_left ) + half-leaves( T_right )
```

(d) count(T) = 2*leaves(T) + half-leaves(T) – 1

We prove P(T) by induction on T

Base case T = (τ, τ): count(T) = 1, leaves(T) = 1, half-leaves(T) = 0, so P((τ, τ)) holds.

Inductive case:T = (T_{left} , T_{right})

Assume P(T_{left}) holds and P(T_{right}) holds, and T_{left} , T_{right} are both not empty.

That is, count(T_{left}) = 2 * leaves(T_{left}) + half-leaves(T_{left}) – 1,

count(T_{right}) = 2 * leaves(T_{right}) + half-leaves(T_{right}) – 1,

we will show that P((T_{left} , T_{right})) holds.

count(T) = count((T_{left} , T_{right}))

= count(T_{left}) + count(T_{right}) + 1 + half-leaves(T_{right}) + half-leaves(T_{left}) – 2

= 2*leaves((T_{left} , T_{right})) + half-leaves((T_{left} , T_{right})) -1

= 2leaves(T) + half-leaves(T) – 1

So P(T) holds when T_{left} and T_{right} are not empty.

Therefore, by the induction, P(T) holds for all binary trees T.

Problem6

(a) $O(n^2)$

(b) $O(n^3)$

(c) recursive-product(A, B):

```
    if n = 1:
        AB = S*W
    if n = 2:
```

$$AB = \begin{pmatrix} SW + TY & SX + TZ \\ VW + VY & VX + VZ \end{pmatrix}$$

```
    else:
        recursive-product (SW)
        recursive-product (TY)
        recursive-product (SX)
        recursive-product (TZ)
        recursive-product (VW)
        recursive-product (VY)
        recursive-product (VX)
        recursive-product (VZ)
```

We have:

T(1) = O(1)

T(n) = O(1) + T(logn) $\in$ O(logn)

(d) T(n) $\in$ O(logn)