Problem1

(a) we have :

$(-1)\,2 - (0)\,4 = -2$

$(0)\,2 - (0)\,4 = 0$

$(-1)\,2 - (-1)\,4 = -2$

$(-1)\,2 - (0)\,4 = -2$

$(1)\,2 - (2)\,4 = -6$

$(1)\,2 - (3)\,4 = -10$

So, $S_{2,-4} = \{\ldots, -10, -6, -2, 0, 2, \ldots\} = Z$

(b) we have:

$(-1)\,12 + (-1)\,18 = -30$

$(-1)\,12 + (0)\,18 = -12$

$(0)\,12 + (-1)\,18 = -18$

$(0)\,12 + (0)\,18 = 0$

$(0)\,12 + (1)\,18 = 18$

$(1)\,12 + (-1)\,18 = -6$

So, $S_{12,18} = \{\ldots,-30, -12, -18, 0, 18, \ldots\} = 3Z$

(c) (i) for d|n, d|x and d|y, so d|(mx+my) for any integers m, n.

So, if $a \in S_{x,y}$, d|a. Therefore, $S_{x,y} \subseteq \{n : n \in Z \text{ and } d|n\}$.

The proof is complete.

(ii) because $z \in S_{x,y,}$ and from (i), we have that $S_{x,y} \subseteq \{n : n \in Z$ and $d|n\}$, so $d|z$, so that $z = kd$, $k \geq 1$. z is positive number in $S_{x,y}$, d is also positive, and $k \geq 1$, so $z = kd \geq d$.

The proof is complete.

(d) (i) because $z \in S_{x,y}$, so $z = mx+ny$ for some $m,n \in Z$.

Suppose $z|w$, and $w \in Z$, so $w = kz$.

Then we have $w = (km)x + (kn)y \in S_{x,y}$.

Let $w1 = (1)x + (0)y$, and $w2 = (0)x + (1)y$, so w1, w2 both elements of $S_{x,y}$. Then we have $z|w1$ and $z|w1$.

Therefore $z|x$ and $z|y$.

(ii) from (i), we have that $z|x$ and $z|y$, so z is a common divisor of x and y. $d = \gcd(x, y)$, so d is the greatest common divisor of x and y.

Therefore, $z \leq d$.

The proof is complete.

Problem2

(a) Since $\gcd(x,y) = 1$,

From Bézout' s Identity, we have that $mx + ny = \gcd(x,y) = 1$, for some integer m, n. So there exists $w \in [0,y) \cap N$, make $wx + ny = 1$.

So we have : $wx-1 = -ny$, therefore, $y|(wx-1)$, so that there is at least one $w \in [0,y) \cap N$ such that $wx \xrightarrow{mod\, y} 1$.

The proof is complete.

(b) If gcd(x,y) = 1, mx+ny = 1.

Multiply this by k on both side: Mkx + nky = k.

Since y|kx and y|nky, so that y|(mkx + nky)

Therefore, y|k.

The proof is complete.

(c) From (a) we have that :

there exists a w∈[0,y)∩N such that wx $\xrightarrow{mod\, y}$ 1.

wx $\xrightarrow{mod\, y}$ 1 if and only if wx%y = 1%y =1

if w∈[0,y), then wx ∈[0,yx), so there is at most one w such that wx%y= 1.

Therefore, there is at most one w∈[0,y)∩N such that wx $\xrightarrow{mod\, y}$ 1.

The proof is complete.

Problem3

We have that: for all m, n $\in N_{>0}$ with n ≤ m, so (m%n) ∈ [0,m-n].

When (m%n) = 0:

$\frac{3}{2}$ (n + (m%n)) = $\frac{3}{2}$ n

To prove $\frac{3}{2}$ (n + (m%n)) < m+n, is to prove: $\frac{3}{2}$ n <m+n.

we adjust the inequality, is to prove: $\frac{1}{2}$ n < m. And always n ≤ m, so the inequality holds.

When (m%n) = m-n:

$\frac{3}{2}$ (n + (m%n)) = $\frac{3}{2}$ m

To prove $\frac{3}{2}$ (n + (m%n)) < m+n, is to prove: $\frac{3}{2}$ m <m+n.

we adjust the inequality, is to prove: m<2n

while (m%n) ϵ [0,m-n], it also have: (m%n) ϵ [0,n]

so (m-n) < n, m<2n. so the inequality holds.

The proof is complete.

Problem4

(a) $A \oplus A = (A \setminus A) \cup (A \setminus A)$ (Definition)

$= (A \cap A^C) \cup (A \cap A^C)$ (Definition)

$= \Phi \cup \Phi$ (Complementation)

$= \Phi$

(b) $A \cup U = A \cup (A^C \cap U)$ (Identity)

$= (A \cap U) \cup (A^C \cap U)$ (Identity)

$= U \cap (A \cup A^C)$ (Distribution)

$= U \cap U$

$= U$

(c) $A \oplus B = (A - B) \cup (B - A)$ (Definition)

$= (A \cap B^C) \cup (B \cap A^C)$ (Definition)

$= ((A \cap B^C) \cup B) \cap ((A \cap B^C) \cup A^C)$ (Distribution)

$= ((A \cup B) \cap (B^C \cup B)) \cap ((A \cup A^C) \cap (A^C \cup B^C))$ (Distribution)

$= ((A \cup B) \cap U) \cap (U \cap (A^C \cup B^C))$ (Complementation)

$= (A \cup B) \cap (A^C \cup B^C)$ (Identity)

(d) $(A \cup B)^C$

Let $x \in (A \cup B)^C$:

$\Rightarrow$ $x \in A^C$ and $x \in B^C$

$\Rightarrow$ $x \in (A^C \cap B^C)$

$\Rightarrow$ $(A \cup B)^C = A^C \cap B^C$

Problem5

$\sum^* = \{\lambda, 0, 1, 00, 11, 01, 10, 000, 111, \ldots\}$

(a) False

Let $X = \{0\}$, $Y = \{1\}$,

$(X \cup Y)^* = \{\lambda, 0, 1, 00, 11, 01, 10, 000, 111, \ldots\}$

$X^* = \{\lambda, 0, 00, 000, \ldots\}$

$Y^* = \{\lambda, 1, 11, 111, \ldots\}$

$X^* \cup Y = \{\lambda, 0, 1, 00, 11, 000, 111, 0000, 1111, \ldots.\}^*$

There are no elements that combine 0 and 1 in $X^* \cup Y$, therefore, false.

(b) True

(c) False

Let $X = \{0\}$

$X^* = \{\lambda, 0, 00, 000, \ldots\}$

$X(X^*) = \{0, 00, 000, 0000, \ldots.\}$

There is no λ in $X(X^*)$. Therefore false.

Problem6

(a) List all possible functions $f : \{a, b, c\} \rightarrow \{0, 1\}$:

1. $f(a) = 0, f(b) = 0, f(c) = 0$
2. $f(a) = 0, f(b) = 0, f(c) = 1$
3. $f(a) = 0, f(b) = 1, f(c) = 0$
4. $f(a) = 0, f(b) = 1, f(c) = 1$
5. $f(a) = 1, f(b) = 0, f(c) = 0$
6. $f(a) = 1, f(b) = 0, f(c) = 1$
7. $f(a) = 1, f(b) = 1, f(c) = 0$
8. $f(a) = 1, f(b) = 1, f(c) = 1$

(b) pow({a, b, c}) = {Φ, {a}, {b}, {c}, {a.b}, {a,c}, {b,c}, {a,b,c}}

|pow({a, b, c})| = 8

The cardinality of pow({a, b, c}) is equal to the number of function a.

(c) For each $\{w \in \{0,1\}^*: \text{length}(w) = 3\}$, there exists a function a to associate an element with w.

Problem7

Let $f \in (A^B)^C$, $g \in A^{BXC}$

Define $\varphi : (A^B)^C \rightarrow A^{BXC}$,

$\varphi(f)(b)(c) = f(b)(c)$

Define ω : $A^{BXC} \rightarrow (A^B)^C$

$\omega(g)(b)(c) = g(b,c)$

Therefore, there is a bijection between $(A^B)^C$ and A^{BXC}

The proof is complete.

Problem8

(a) (R1; R2); R3 = R1;(R2; R3) True

Let R1 = {(a, b)}, R2 = {(b, c)}, R3 = {(c, d)}

(R1; R2) = {(a, c)}, (R1; R2); R3 = {(c, d)}

(R2; R3) = {(b, d)}, R1;(R2; R3) = {(c, d)}

There is always a same element that both in the two binary relation R1, R2, and in R2, R3, so (R1; R2); R3 = R1;(R2; R3).

(b) I; R1 = R1; I = R1 where I = {(x, x) : x $\in$ S} True

Let I ={(x, x)}, R1 = {(a, b)}

If I; R1, then a = x, and I; R1 = {(x, b)}

If R1; I, then b = x, and R1; I = {(a, x)} = {(x, x)}

So I; R1 = {(x, b)} = {(x, x)}

Therefore true.

(c) (R1 $\cup$ R2); R3 = (R1; R3) $\cup$ (R2; R3) True

If (R1 $\cup$ R2); R3, then there must have an element b with a binary relation (a, b) $\in$ R1 $\cup$ R2,

And there must have a same element b with another binary relation (b, c)∊ R3.

However, if (a, b)∊ R1 but ∉ R2,

(R1 ∪ R2); R3 = R1; R3, R2; R3 = {Φ},

So (R1 ∪ R2); R3 = (R1; R3) ∪ (R2; R3) always holds

(d) R1;(R2 ∩ R3) = (R1; R2) ∩ (R1; R3) False

Let R1 = {(0, 0), (0, 1)}, R2 = {(0, 0), (1, 1)}, R3 = {(1, 0), (1, 1)}

R2 ∩ R3 ={(1, 1)}, then:

R1;(R2 ∩ R3) = {(0,1)}

(R1; R2) ∩ (R1; R3) = {(0, 0), (0, 1)} ∩ {(0, 0), (0, 1)} = {(0, 0), (0, 1)}

So that R1;(R2 ∩ R3) ≠ (R1; R2) ∩ (R1; R3)

Therefore false.