Problem1

(a) $(x \vee 0') \wedge (x' \vee 0)$

$= (x \vee 0') \wedge x'$	identity
$= (x \vee 1) \wedge x'$	definition
$= x' \wedge (x \vee 1)$	commutative
$= (x' \wedge x) \vee (x' \wedge 1)$	distributive
$= 0 \vee (x' \wedge 1)$	complementation
$= 0 \vee x'$	identity
$= x'$	identity

(b) $(x \vee y) \wedge x$

$= (x \vee y) \wedge (x \vee 0)$	identity
$= x \vee (y \wedge 0)$	distributive
$= x \vee (y \wedge (y \wedge y'))$	identity
$= x \vee ((y \wedge y) \wedge y'))$	commutative
$= x \vee (y \wedge y')$	definition
$= x \vee 0$	complementation
$= x$	identity

(c) $y' \vee ((x \wedge y) \vee x')$

$= y' \vee (x' \vee (x \wedge y))$	commutative
$= y' \vee ((x' \vee x) \wedge (x' \vee y))$	distributive
$= y' \vee (1 \wedge (x' \vee y))$	complementation
$= y' \vee (x' \vee y)$	identity
$= (y' \vee y) \vee x'$	associative
$= 1 \vee x'$	complementation
$= (x \vee x') \vee x'$	complementation
$= x \vee (x' \vee x')$	associative
$= x \vee x'$	definition
$= 1$	complementation

Problem2

(a) $(p \wedge q) \rightarrow r$

$\equiv \neg(p \wedge q) \vee r$	Implication
$\equiv (\neg p \vee \neg q) \vee r$	De Morgan's
$\equiv \neg p \vee (\neg q \vee r)$	associative
$\equiv \neg p \vee (q \rightarrow r)$	Implication
$\equiv p \rightarrow (q \rightarrow r)$	Implication

(b) $(p \to q) \to r \not\equiv p \to (q \to r)$

Counterexample:

p	q	r	$(p \to q) \to r$	$p \to (q \to r)$
F	T	F	F	T

(c) we use truth tables to prove it:

p	q	r	$q \vee r$	$p \vee (q \vee r)$	$r \vee p$	$q \vee r$	$p \wedge q$	$(p \vee (q \vee r)) \wedge (r \vee p)$	$(p \wedge q) \vee (r \vee p)$
T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T	T
T	F	T	T	T	T	T	F	T	T
T	F	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	F	T	T
F	T	F	T	T	F	T	F	F	F
F	F	T	T	T	T	T	F	T	T
F	F	F	F	F	F	F	F	F	F

Therefore, $((p \vee (q \vee r)) \wedge (r \vee p)) \equiv ((p \wedge q) \vee (r \vee p))$

Problem3

(a)
- dual($\perp$) = $\top$
- dual($\top$) = $\perp$
- dual(p) = p for all p $\in$ P_{ROP}
- dual($\neg$ p) = $\neg$ p for all p $\in$ P_{ROP}
- For all formulars φ and ψ:
  - dual(($\varphi \wedge \psi$)) = dual (φ) $\vee$ dual(ψ)
  - dual(($\varphi \vee \psi$)) = dual(φ) $\wedge$ dual(ψ)

(b) proof by structural induction

Base case: let ψ be the smallest set such that ψ only contains the atomic proposition.

(1) If a single variable $\rho \in \psi$, a is in DNF, flip $\circ$ dual(ρ) = $\neg\rho$ clearly ρ is in CNF

(2) If atom proposition $\rho \vee \mu \in \psi$, flip $\circ$ dual(ψ) = $\neg \rho \wedge \neg \mu$, which is

in CNF.

(3) If atom proposition $(\rho \land \mu) \in \psi$, flip ∘ dual($\psi$) = $(\neg \rho \lor \neg \mu)$ is also an atom proposition, clearly an atom proposition is in CNF.

Therefore, for the smallest set ψ that only contains the atomic proposition, flip ∘ dual(ψ) is in CNF.

Inductive case: Assume that $\psi(n_1, n_1, n_3, \ldots\ldots n_k)$ is in DNF, $n_i \in$Prop, then flip ∘ dual(ψ) is in CNF.

When i = k+1, $\psi(n_1, n_1, n_3, \ldots\ldots, n_k, n_{k+1})$ is in DNF.

So ψ: $n_1 \land n_1 \land n_3 \land \ldots\ldots \land n_k \land n_{k+1}$.

dual(ψ) = $n_1 \lor n_1 \lor n_3 \lor \ldots\ldots \lor n_k \lor n_{k+1}$

flip ∘ dual(ψ) = $n_1 \lor n_1 \lor n_3 \lor \ldots\ldots \lor n_k \lor n_{k+1}$

(1) If n_{k+1} is an atom proposition, $n_1 \lor n_1 \lor n_3 \lor \ldots\ldots \lor n_k \lor n_{k+1}$ satisfy the CNF.

(2) n_{k+1} contains k formulas, because ψ is in DNF, so n_{k+1} is minterms.

Let $n_{k+1} = c_1 \land c_2 \land \ldots\ldots \land c_i$, c_i are literals.

flip ∘ dual(n_{k+1}) = $\neg c_1 \lor \neg c_2 \lor \ldots\ldots \lor \neg c_i$

so $n_1 \lor n_1 \lor n_3 \lor \ldots\ldots \lor n_k \lor n_{k+1}$ is CNF,

that is flip ∘ dual(ψ) is in CNF

Therefore, if ψ is in DNF, flip ∘ dual(ψ) is in CNF.

(c) For all $\varphi \in$ F, if φ is single variable:

$$\text{flip} \circ \text{dual}(\neg\varphi) \equiv \text{flip}(\neg\varphi) \equiv \varphi$$

If φ is literal:

$$\text{flip} \circ \text{dual}(\neg\varphi) \equiv \text{flip}(\neg\varphi) \equiv \varphi$$

Therefore, for all $\varphi \in$ F, flip ∘ dual($\neg\varphi$) is logically equivalent to φ.

Problem4

Define a Boolean algebra as having at least two elements, so it has a minimal element, we let it as 0, and a maximal element, we let it as 1.

Each element has a complement. 0’ = 1, 1’ = 0.

Assume that we can add the third element x, which is distinct from 0 and 1. Then x must have a complement x’. Because x $\neq$ 1 and x $\neq$ 0, so x’ $\neq$ 1 and x’ $\neq$ 0. Clearly x $\neq$ x’, therefore, the Boolean algebra has 4 elements, which is contrary to the assumption.

Therefore, there are no three element Boolean algebra.

Problem5

(a)

(i) R_1: House 1 is red.

R_2: House 2 is red.

R_3: House 3 is red.

R_4: House 4 is red.

R_5: House 5 is red.

B_1: House 1 is bule.

B_2: House 2 is bule.

B_3: House 3 is bule.

B_4: House 4 is bule.

B_5: House 5 is bule.

(ii) $R_1 \lor B_1$: House 1 is red or blue.

$\neg R_1 \land B_1$: House 1 is not both red and blue.

$R_1 \rightarrow R_2$: If house 1 is red, then house 2 is red, house 1 and house 2 are neighbour.

$B_1 \rightarrow B_2$: If house 1 is blue, then house 2 is blue, house 1 and house 2 are neighbour.

$(R_1 \rightarrow R_2) \lor (B_1 \rightarrow B_2)$: House 1 and house 2 are neighbour.

$(R_1 \land R_2) \lor (B_1 \land B_2)$: House 1 and house 2 are the same colour.

$\neg (R_1 \land R_2) \land \neg (B_1 \land B_2)$: House 1 and house 2 are not the same colour.

(iii) To prove the problem cannot be done, we just need to choose three consecutive houses, they have the following properties:

(a) house 1 and house 2 are neighbour, house 2 and house 3 are neighbour,

(b) house1 and house 2 are the same colour, house 2 and house 3 are the same colour,

(c) house 1 and house 3 can not be the same colour.

It can be expressed by the following formulas.

$$((R_1 \rightarrow R_2) \lor (B_1 \rightarrow B_2)) \land ((R_2 \rightarrow R_3) \lor (B_2 \rightarrow B_3)) \qquad ①$$

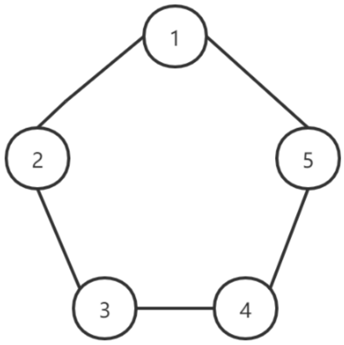
$$((R_1 \land R_2) \lor (B_1 \land B_2)) \land ((R_1 \land R_2) \lor (B_1 \land B_2)) \qquad ②$$

$$(\neg (R_1 \land R_3) \land \neg (B_1 \land B_3)) \qquad ③$$

So, we just need to prove for every three consecutive houses:

(① ∧ ② ∧ ③) is a contradiction.

(b)



(i)V = {1, 2, 3, 4, 5}

E = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)}

V represent house, such as 1 represents *house 1*, 2 represents *house 2*.

E represent neighbour, such as (1, 2) represents house 1 and house 2 is neighbour.

(ii)Colouring problem: we need to paint blue or red on each vertex, and the vertices connected by an edge have the same colour, the vertices that are not connected have the different colour.

That is: a mapping c: V $\rightarrow$ {red, blue} such that for every e = (v, w) $\in$ E: c(v) = c(w)

(iii) According to (ii), we have c(1) = c(2), c(2) = c(3), so we can get c(1) = c(3), which is contrary to the (ii).

Therefore, it cannot be done.

Problem6

(a) $p_1(n+1) = \frac{1}{2}\ p_1(n) + \frac{1}{3}\ p_2(n) + \frac{1}{3}\ p_5(n)$

$p_2(n+1) = \frac{1}{4}\ p_1(n) + \frac{1}{3}\ p_2(n) + \frac{1}{3}\ p_3(n)$

$p_3(n+1) = \frac{1}{3}\ p_2(n) + \frac{1}{3}\ p_3(n) + \frac{1}{3}\ p_4(n)$

$p_4(n+1) = \frac{1}{3}\ p_3(n) + \frac{1}{3}\ p_4(n) + \frac{1}{3}\ p_5(n)$

$p_5(n+1) = \frac{1}{4}\ p_1(n) + \frac{1}{3}\ p_4(n) + \frac{1}{3}\ p_5(n)$

(b) The transition matrix is $\begin{bmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \end{bmatrix}$

When $n \rightarrow \infty$, $p_i(n+1) = p_i(n)$, we let the stable probabilities for five houses p_1, p_2, p_3, p_4, p_5, so we have:

$$[p_1, p_2, p_3, p_4, p_5] \begin{bmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \end{bmatrix} = [p_1, p_2, p_3, p_4, p_5]$$

Therefore, we get:

$$\frac{1}{2}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_5 = p_1$$

$$\frac{1}{4}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3 = p_2$$

$$\frac{1}{3}p_2 + \frac{1}{3}p_3 + \frac{1}{3}p_4 = p_3$$

$$\frac{1}{3}p_3 + \frac{1}{3}p_4 + \frac{1}{3}p_5 = p_4$$

$$\frac{1}{4}p_1 + \frac{1}{3}p_4 + \frac{1}{3}p_5 = p_5$$

And we have $p_1 + p_2 + p_3 + p_4 + p_5 = 1$

Therefore, we get the steady state probabilities are:

$p_1 = \frac{1}{4}$, $p_2 = \frac{3}{16}$, $p_3 = \frac{3}{16}$, $p_4 = \frac{3}{16}$, $p_5 = \frac{3}{16}$

(c) Distance between house 1 and house 3 is 0.
Distance between house 1 and house 2 is 1.
Distance between house 1 and house 3 is 2.
Distance between house 1 and house 4 is 3.
Distance between house 1 and house 5 is 1.
When in the steady state, the expected distance is:

$$\frac{1}{4} * 0 + \frac{3}{16} * 1 + \frac{3}{16} * 2 + \frac{3}{16} * 3 + \frac{3}{16} * 1 = \frac{21}{16}$$

Problem7

(a) When n = 0, the binary tree is empty or contains a single root node. Therefore T(0) = 1.
When $n \geq 1$, T(n) = T(0).T(n) +T(1).T(n - 1) + …… + T(left-subtree).T(right-subtree) +…,
When there are totally n nodes, there is 1 root node, we assume that there are m nodes in the left subtree, $m \in [0, n-1]$, so there are $(n - m - 1)$ nodes in the right subtree. Therefore, we can conclude:

$$T(n) = \sum_{m=0}^{n-1} T(m).T(n - m - 1)$$

(b) We know that a full binary tree is combined with fully – internal nodes and leaves.
Let n be the number of nodes of a full binary tree, from Assignment 2, leaves(T) = 1 + internal(T).
(leaves(T) represents the number of leaves, internal(T) represents the number of fully-internal nodes)
Thus, n = leaves(T) + internal(T) = 1 + 2*internal(T), as internal(T) is an integer, n is odd.
Therefore, a full binary tree must have an odd number of nodes.

(c) From (b) we know that a full binary tree must have an odd number of nodes.
If we have a full binary tree T and delete all the leaves, we will get a binary tree with internal(T) nodes.
If we have a binary tree with n nodes, replace all empty subtrees with a new node, then these new nodes are new leaf, and all original nodes are fully-internal nodes, so the new binary tree is a full binary tree with n fully-internal nodes.
Thus, a full-binary tree is bijection to a binary tree. From (b), if a full binary tree has n nodes and n is odd, it has $\frac{n-1}{2}$ fully-internal nodes; if n is even, there are no such full binary tree. Therefore,

$$\mathrm{B(n)} = \begin{cases} 0, & \text{n is even} \\ \mathrm{T}(\frac{\mathrm{n-1}}{2}), & \text{n is odd} \end{cases}$$

$$= \begin{cases} 0, & \text{n is even} \\ \sum_{m=0}^{\frac{n-3}{2}} T(m)\,.\,\mathrm{T}(\frac{\mathrm{n-3}}{2} - \mathrm{m}), & \text{n is odd} \end{cases}$$

(d) F can be designed as a full binary tree, *n* represents the number of leaves, $\wedge$ and $\vee$ represents fully-internal nodes, literals represents leaves.
• From (b), a full binary tree with n leaves has (2n – 1) nodes,
so that B(2n - 1) = T(n - 1).
• From Assignment 2, a full binary tree with n leaves has (n-1) fully-internal nodes, these can be represented by $\wedge$ or $\vee$, so that there are 2^{n-1} $\wedge$ or $\vee$ symbols.
• As literals represent leaves, and they have the ¬literal, so that there are 2^n of (¬literal).
• As we have n leaves, so there are (n!) kinds of arrangement.
Therefore, we can get the formula of F(n):

$$F(n) = T(2n - 1) .\, 2^{n-1} .\, 2^n .\, n!$$
$$= 2^{2n-1} .\, n! .\, T(2n - 1)$$