COMP9020 Week 2
Sets and Formal Languages

- [RW] - Ch. 1, Ch. 3
- [LLM] - Sections 4.1, 4.2

Applications of Sets and Formal Languages

- Sets are the building blocks of nearly all mathematical structures
- Data structures based around sets can be a space-efficient storage system
- Formal languages are essential for compilers and programming language design
- Set theory is a good introduction to formal reasoning (logic)
- Formal languages provide a good introduction to recursive structures (recursion and induction)

Outline

- Introduction to Sets
- Formal Languages
- Set Equality
- Laws of Set Operations

Sets

Definition

A **set** is a collection of objects (**elements**). If x is an element of A we write $x \in A$.

NB

- *Elements are taken from a* **universe**, $\mathcal{U}$, *– but this can be quite complex. e.g. numbers, and sets of numbers, and sets of sets of numbers, etc.*
- *Not all "well-defined" universes are possible. e.g.*
  - *No "set of all sets" (Cantor's paradox)*
  - *No "sets which do not contain themselves" (Russell's paradox)*

Sets

- A set is defined by the collection of its elements. Order and multiplicity of elements is not considered.
- We distinguish between an element and the set comprising this single element. Thus always $a \neq \{a\}$.
- Set $\emptyset = \{\}$ is empty (no elements);
- Set $\{\{\}\}$ is nonempty — it has one element.
- There is only one empty set; only one set consisting of a single a; only one set of all natural numbers.

Subsets

Definition

For sets S and T, we say S is a **subset** of T, written $S \subseteq T$, if every element of S is an element of T.

NB

- $S \subseteq T$ *includes the case of* $S = T$
- $S \subset T$ *— a* **proper** *subset:* $S \subseteq T$ *and* $S \neq T$
- $\emptyset \subseteq S$ *for all sets* S
- $S \subseteq \mathcal{U}$ *for all sets* S
- $\mathbb{N}_{>0} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- *An element of a set; and a subset of that set are two different concepts*

$$a \in \{a, b\}, \quad a \not\subseteq \{a, b\}; \qquad \{a\} \subseteq \{a, b\}, \quad \{a\} \notin \{a, b\}$$

Defining sets

Sets are typically described by:
(a) Explicit enumeration of their elements

$$
\begin{aligned}
S_1 &= \{a, b, c\} = \{a, a, b, b, b, c\} \\
&= \{b, c, a\} = \ldots \quad \text{three elements} \\
S_2 &= \{a, \{a\}\} \quad \text{two elements} \\
S_3 &= \{a, b, \{a, b\}\} \quad \text{three elements} \\
S_4 &= \{\} \quad \text{zero elements} \\
S_5 &= \{\{\{\}\}\} \quad \text{one element} \\
S_6 &= \{\, \{\}, \{\{\}\} \,\} \quad \text{two elements}
\end{aligned}
$$

Defining sets

(b) Defining a subset of the universal set $\mathcal{U}$. Including:

- Specifying the properties their elements must satisfy. A typical description involves a **logical** property $P(x)$. For example, with $\mathcal{U} = \mathbb{N}$ and $P(x) =$ "x is even":

$$\{x : x \in \mathbb{N} \text{ and } x \text{ is even}\} = \{0, 2, 4, \ldots\}$$

- Derived sets of integers

$$2\mathbb{Z} = \{\ 2x : x \in \mathbb{Z}\ \} \qquad \text{the even numbers}$$
$$3\mathbb{Z} + 1 = \{\ 3x + 1 : x \in \mathbb{Z}\ \}$$

- Using interval notation.

Intervals

Intervals of numbers (applies to any type)

$$[a, b] = \{x : a \le x \le b\}; \qquad (a, b) = \{x : a < x < b\}$$
$$[a, b) = \{x : a \le x < b\}; \qquad (a, b] = \{x : a < x \le b\}$$

$$(-\infty, b] = \{x : x \le b\}; \qquad (-\infty, b) = \{x : x < b\}$$
$$[a, \infty) = \{x : a \le x\}; \qquad (a, \infty) = \{x : a < x\}$$

NB

$(a, a) = (a, a] = [a, a) = \emptyset$; *however* $[a, a] = \{a\}$.

Intervals of $\mathbb{N}, \mathbb{Z}$ are finite: if $m \le n$

$$[m, n] = \{m, m + 1, \ldots, n\}$$

Examples

Examples

- $[1, 5] = \{1, 2, 3, 4, 5\}$ (when $\mathcal{U} = \mathbb{Z}$)
- $[1, 5] = \{1, 1.1, 1.01, 1.001, \ldots, 2, \ldots, \pi, e, \ldots\}$ (when $\mathcal{U} = \mathbb{R}$)
- Number of multiples of k between n and m (inclusive):

$$\left\lfloor \frac{m}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor$$

- $0 \leq (m \,\%\, n) < n$

Examples

Examples

- $[1, 5] = \{1, 2, 3, 4, 5\}$ (when $\mathcal{U} = \mathbb{Z}$)
- $[1, 5] = \{1, 1.1, 1.01, 1.001, \ldots, 2, \ldots, \pi, e, \ldots\}$ (when $\mathcal{U} = \mathbb{R}$)
- Number of multiples of k in $[n, m]$:

$$\left\lfloor \frac{m}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor$$

- $0 \leq (m \,\%\, n) < n$

Examples

Examples

- $[1,5] = \{1,2,3,4,5\}$ (when $\mathcal{U} = \mathbb{Z}$)
- $[1,5] = \{1, 1.1, 1.01, 1.001, \ldots, 2, \ldots, \pi, e, \ldots\}$ (when $\mathcal{U} = \mathbb{R}$)
- Number of multiples of k in $[n, m]$:

$$\left\lfloor \frac{m}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor$$

- $(m \,\%\, n) \in [0, n)$

Defining sets

(c) Constructions from other, already defined, sets

- Union ($\cup$), intersection ($\cap$), complement ($\cdot^c$), set difference ($\setminus$), symmetric difference ($\oplus$)
- Power set $\mathsf{Pow}(X) = \{ A : A \subseteq X \}$
- Cartesian product ($\times$)

Set Operations

Definition

$A \cup B$ – **union** (a or b):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

$A \cap B$ – **intersection** (a and b):

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A^c – **complement** (with respect to a universal set $\mathcal{U}$):

$$A^c = \{x : x \in \mathcal{U} \text{ and } x \notin A\}.$$

We say that A, B are **disjoint** if $A \cap B = \emptyset$

Set Operations

Other set operations

Definition

$A \setminus B$ – **set difference**, relative complement (a but not b):

$$A \setminus B = A \cap B^c$$

$A \oplus B$ – **symmetric difference** (a and not b or b and not a; also known as a or b exclusively; a xor b):

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

Set Operations

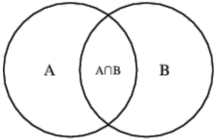
Fact

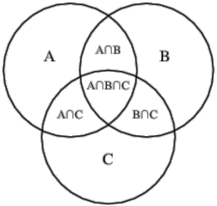
$$A \cup B = B \quad \textit{iff} \quad A \cap B = A \quad \textit{iff} \quad A \subseteq B$$

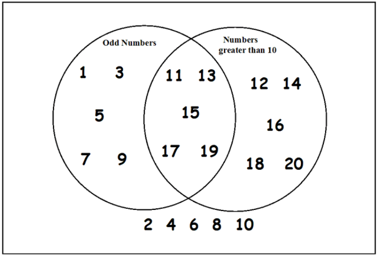
There is a correspondence between set operations and logical operators (to be discussed in Week 7).

Venn Diagrams

A simple graphical approach to reason about the algebraic properties of set operations.







Exercises

Exercises

RW: 1.4.4 (d)	All subsets of $\{a, b\}$:
RW: 1.4.7 (a)	$A \oplus A =$
RW: 1.4.7 (b)	$A \oplus \emptyset =$

Exercises

Exercises

RW: 1.4.4 (d)	All subsets of $\{a, b\} : \emptyset, \{a\}, \{b\}, \{a, b\}$
RW: 1.4.7 (a)	$A \oplus A =$
RW: 1.4.7 (b)	$A \oplus \emptyset =$

Exercises

Exercises

$\boxed{\text{RW: 1.4.4 (d)}}$	All subsets of $\{a, b\}$: $\emptyset, \{a\}, \{b\}, \{a, b\}$
$\boxed{\text{RW: 1.4.7 (a)}}$	$A \oplus A = \emptyset$
$\boxed{\text{RW: 1.4.7 (b)}}$	$A \oplus \emptyset =$

Exercises

Exercises

RW: 1.4.4 (d)	All subsets of $\{a, b\}$: $\emptyset, \{a\}, \{b\}, \{a, b\}$
RW: 1.4.7 (a)	$A \oplus A = \emptyset$
RW: 1.4.7 (b)	$A \oplus \emptyset = A$

Cardinality

Number of elements in a set X (various notations):

$$|X| = \#(X) = \mathrm{card}(X)$$

Fact

Always $|Pow(X)| = 2^{|X|}$

Exercises

Exercises

- $|\emptyset| =$
- $\mathrm{Pow}(\emptyset) =$
- $|\mathrm{Pow}(\emptyset)| =$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) =$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| =$
- $|\{a\}| =$
- $\mathrm{Pow}(\{a\}) =$
- $|\mathrm{Pow}(\{a\})| =$
- $|[m, n]| =$

Exercises

Exercises

- $|\emptyset| = 0$
- $\mathrm{Pow}(\emptyset) =$
- $|\mathrm{Pow}(\emptyset)| =$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) =$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| =$
- $|\{a\}| =$
- $\mathrm{Pow}(\{a\}) =$
- $|\mathrm{Pow}(\{a\})| =$
- $|[m, n]| =$

Exercises

Exercises

- $|\emptyset| = 0$
- $\mathrm{Pow}(\emptyset) = \{\emptyset\}$
- $|\mathrm{Pow}(\emptyset)| = 1$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) =$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| =$
- $|\{a\}| =$
- $\mathrm{Pow}(\{a\}) =$
- $|\mathrm{Pow}(\{a\})| =$
- $|[m, n]| =$

Exercises

Exercises

- $|\emptyset| = 0$
- $\mathrm{Pow}(\emptyset) = \{\emptyset\}$
- $|\mathrm{Pow}(\emptyset)| = 1$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| = 2$
- $|\{a\}| =$
- $\mathrm{Pow}(\{a\}) =$
- $|\mathrm{Pow}(\{a\})| =$
- $|[m, n]| =$

Exercises

Exercises

- $|\emptyset| = 0$
- $\mathrm{Pow}(\emptyset) = \{\emptyset\}$
- $|\mathrm{Pow}(\emptyset)| = 1$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| = 2$
- $|\{a\}| = 1$
- $\mathrm{Pow}(\{a\}) =$
- $|\mathrm{Pow}(\{a\})| =$
- $|[m, n]| =$

Exercises

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- $|\emptyset| = 0$
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- $\mathrm{Pow}(\{a\}) = \{\emptyset, \{a\}\}$
- $|\mathrm{Pow}(\{a\})| =$
- $|[m, n]| =$

Exercises

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- $|\emptyset| = 0$
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- $|\{a\}| = 1$
- $\mathrm{Pow}(\{a\}) = \{\emptyset, \{a\}\}$
- $|\mathrm{Pow}(\{a\})| = 2$
- $|[m, n]| =$

Exercises

Exercises

- $|\emptyset| = 0$
- $\mathrm{Pow}(\emptyset) = \{\emptyset\}$
- $|\mathrm{Pow}(\emptyset)| = 1$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| = 2$
- $|\{a\}| = 1$
- $\mathrm{Pow}(\{a\}) = \{\emptyset, \{a\}\}$
- $|\mathrm{Pow}(\{a\})| = 2$
- $|[m, n]| = n - m + 1$

Exercises

$\boxed{\text{RW: 1.3.2}}$ Find the cardinalities of sets

1. $|\{\ \frac{1}{n} : n \in [1,4]\ \}| =$
2. $|\{\ n^2 - n : n \in [0,4]\ \}| =$
3. $|\{\ \frac{1}{n^2} : n \in \mathbb{N}_{>0}$ and $2|n$ and $n < 11\ \}| =$
4. $|\{\ 2 + (-1)^n : n \in \mathbb{N}\ \}| =$

Exercises

$\boxed{\text{RW: 1.3.2}}$ Find the cardinalities of sets

1. $|\{\ \frac{1}{n} : n \in [1,4]\ \}| = 4$
2. $|\{\ n^2 - n : n \in [0,4]\ \}| =$
3. $|\{\ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11\ \}| =$
4. $|\{\ 2 + (-1)^n : n \in \mathbb{N}\ \}| =$

Exercises

$\boxed{\text{RW: 1.3.2}}$ Find the cardinalities of sets

1. $|\{ \frac{1}{n} : n \in [1,4] \}| = 4$
2. $|\{ n^2 - n : n \in [0,4] \}| = 4$
3. $|\{ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11 \}| =$
4. $|\{ 2 + (-1)^n : n \in \mathbb{N} \}| =$

Exercises

$\boxed{\text{RW: 1.3.2}}$ Find the cardinalities of sets

1. $|\{\ \frac{1}{n} : n \in [1, 4]\ \}| = 4$
2. $|\{\ n^2 - n : n \in [0, 4]\ \}| = 4$
3. $|\{\ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11\ \}| = 5$
4. $|\{\ 2 + (-1)^n : n \in \mathbb{N}\ \}| =$

Exercises

$\boxed{\text{RW: 1.3.2}}$ Find the cardinalities of sets

1. $|\{ \frac{1}{n} : n \in [1,4] \}| = 4$
2. $|\{ n^2 - n : n \in [0,4] \}| = 4$
3. $|\{ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11 \}| = 5$
4. $|\{ 2 + (-1)^n : n \in \mathbb{N} \}| = 2$

Exercises

$\boxed{\text{RW: 1.4.8}}$ Relate the cardinalities to $|A \cap B|$, $|A|$, $|B|$

- $|A \cup B| =$
- $|A \setminus B| =$
- $|A \oplus B| =$

Exercises

$\boxed{\text{RW: 1.4.8}}$ Relate the cardinalities to $|A \cap B|$, $|A|$, $|B|$

- $|A \cup B| = |A| + |B| - |A \cap B|$
  hence $|A \cup B| + |A \cap B| = |A| + |B|$
- $|A \setminus B| = |A| - |A \cap B|$
- $|A \oplus B| = |A| + |B| - 2|A \cap B|$

Cartesian Product

$S \times T \stackrel{\text{def}}{=} \{ \, (s,t) : s \in S, \;\; t \in T \, \}$ where (s,t) is an **ordered** pair

$\bigtimes_{i=1}^{n} S_i \stackrel{\text{def}}{=} \{ \, (s_1, \ldots, s_n) : s_k \in S_k, \text{ for } 1 \leq k \leq n \, \}$

$S^2 = S \times S, \quad S^3 = S \times S \times S, \ldots, \quad S^n = \times_1^n S, \ldots$

$\emptyset \times S = \emptyset$, for every S
$|S \times T| = |S| \cdot |T|, \quad |\times_{i=1}^{n} S_i| = \prod_{i=1}^{n} |S_i|$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\ &= \{(0, a), (1, a), (0, b), (1, b)\} \end{aligned}$$

$$B \times A =$$

$$A^2 =$$

$$A^3 =$$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\ &= \{(0, a), (1, a), (0, b), (1, b)\} \\ B \times A &= \{(a, 0), (b, 0), (a, 1), (b, 1)\} \neq A \times B \\ A^2 &= \\ A^3 &= \end{aligned}$$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\ &= \{(0, a), (1, a), (0, b), (1, b)\} \end{aligned}$$

$$B \times A = \{(a, 0), (b, 0), (a, 1), (b, 1)\} \neq A \times B$$

$$A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

$$A^3 =$$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\ &= \{(0, a), (1, a), (0, b), (1, b)\} \end{aligned}$$

$$B \times A = \{(a, 0), (b, 0), (a, 1), (b, 1)\} \neq A \times B$$

$$A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

$$\begin{aligned} A^3 = \{&(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), \\ &(1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}. \end{aligned}$$

Exercise

Exercise

Let A, B, C be sets.

Is $A \times (B \times C) = (A \times B) \times C$?

Outline

- Introduction to Sets
- Formal Languages
- Set Equality
- Laws of Set Operations

Formal Languages: Symbols

Σ — **alphabet**, a finite, nonempty set

Examples (of various alphabets and their intended uses)

$\Sigma = \{a, b, \ldots, z\}$ for single words (in lower case)
$\Sigma = \{\mathrm{true}, -, a, b, \ldots, z\}$ for composite terms
$\Sigma = \{0, 1\}$ for binary integers
$\Sigma = \{0, 1, \ldots, 9\}$ for decimal integers

The above cases all have a natural ordering; this is not required in general, thus the set of all Chinese characters forms a (formal) alphabet.

Formal Languages: Words

Definition

word — any finite string of symbols from Σ
empty word — λ

Example

$w = aba$, $w = 01101\ldots1$, etc.

length(w) — # of symbols in w
$\text{length}(aaa) = 3, \text{length}(\lambda) = 0$
The only operation on words (discussed here) is **concatenation**, written as juxtaposition $vw, wvw, abw, wbv, \ldots$

NB

$\lambda w = w = w\lambda$
$\text{length}(vw) = \text{length}(v) + \text{length}(w)$

Examples

Examples

Let $w = abb$, $v = ab$, $u = ba$

- $vw = ababb$
- $ww = abbabb = vubb$
- $w\lambda v = abbab$
- $\text{length}(vw) = \text{length}(ababb) = 5$

Formal Languages: Languages

Notation: Σ^k — set of all words of length k
We often identify $\Sigma^0 = \{\lambda\}$, $\Sigma^1 = \Sigma$
Σ^* — set of all words (of all lengths)
Σ^+ — set of all nonempty words (of any positive length)

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots; \quad \Sigma^{\leq n} = \bigcup_{i=0}^{n} \Sigma^i$$

$$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \ldots = \Sigma^* \setminus \{\lambda\}$$

Definition

A **language** is a subset of Σ^*.

Typically, only the subsets that can be formed (or described) according to certain rules are of interest. Such a collection of 'descriptive/formative' rules is called a **grammar**.

Example (Decimal numbers)

The "language" of all numbers written in decimal to at most two decimal places can be described as follows:

- $\Sigma = \{-, ., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Consider all words $w \in \Sigma^*$ which satisfy the following:
  - w contains at most one instance of $-$, and if it contains an instance then it is the first symbol.
  - w contains at most one instance of $.$, and if it contains an instance then it is preceeded by a symbol in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and followed by either one or two symbols in that set.
  - w contains at least one symbol from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

NB

According to these rules 123, 123.0 *and* 123.00 *are all (distinct) words in this language.*

Example (HTML documents)

Take $\Sigma = \{$“<html>”, “</html>”, “<head>”, “</head>”, “<body>”, $\ldots\}$.

The (language of) **valid HTML documents** is loosely described as follows:

- Starts with “<html>”
- Next symbol is “<head>”
- Followed by zero or more symbols from the set of HeadItems (defined elsewhere)
- Followed by “</head>”
- Followed by “<body>”
- Followed by zero or more symbols from the set of BodyItems (defined elsewhere)
- Followed by “</body>”
- Followed by “</html>”

Exercises

$\boxed{\text{RW: 1.3.10}}$ Number of elements in the sets (cont'd)

(e) Σ^* where $\Sigma = \{a, b, c\}$ —

(f) $\{ \ w \in \Sigma^* : \text{length}(w) \leq 4 \ \}$ where $\Sigma = \{a, b, c\}$
$|\Sigma^{\leq 4}| =$

Exercises

$\boxed{\text{RW: 1.3.10}}$ Number of elements in the sets (cont'd)

(e) Σ^* where $\Sigma = \{a, b, c\}$ — $|\Sigma^*| = \infty$

(f) $\{ w \in \Sigma^* : \text{length}(w) \leq 4 \}$ where $\Sigma = \{a, b, c\}$
$|\Sigma^{\leq 4}| =$

Exercises

$\boxed{\text{RW: 1.3.10}}$ Number of elements in the sets (cont'd)

(e) Σ^* where $\Sigma = \{a, b, c\}$ — $|\Sigma^*| = \infty$

(f) $\{\ w \in \Sigma^* : \text{length}(w) \leq 4\ \}$ where $\Sigma = \{a, b, c\}$
$|\Sigma^{\leq 4}| = 3^0 + 3^1 + \ldots + 3^4 = \frac{3^5-1}{3-1} = \frac{243-1}{2} = 121$

Set Operations for Languages

Languages are sets, so the standard set operations ($\cap$, $\cup$, $\setminus$, $\oplus$, etc) can be used to build new languages.

Two set operations that apply to languages uniquely:

- Concatenation (written as juxtaposition):
  $XY = \{xy \ : \ x \in X \text{ and } y \in Y\}$
- Kleene star: X^* is the set of words that are made up by concatenating 0 or more words in X
  - $X^0 = \{\lambda\}$; $X^{i+1} = XX^i$
  - $X^* = X^0 \cup X^1 \cup X^2 \cup \ldots$

NB

The set of all finite words over Σ is the Kleene star of Σ (hence notation).

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$
- $A^* = \{\lambda, aa, bb, aaaa, aabb, bbaa, bbbb, aaaaaa, \ldots\}$
- $B^* = \{\lambda, c, cc, ccc, cccc, \ldots\}$

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$
- $A^* = \{\lambda, aa, bb, aaaa, aabb, bbaa, bbbb, aaaaaa, \ldots\}$
- $B^* = \{\lambda, c, cc, ccc, cccc, \ldots\}$
- $\{\lambda\}^* =$

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$
- $A^* = \{\lambda, aa, bb, aaaa, aabb, bbaa, bbbb, aaaaaa, \ldots\}$
- $B^* = \{\lambda, c, cc, ccc, cccc, \ldots\}$
- $\{\lambda\}^* = \{\lambda\}$

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$
- $A^* = \{\lambda, aa, bb, aaaa, aabb, bbaa, bbbb, aaaaaa, \ldots\}$
- $B^* = \{\lambda, c, cc, ccc, cccc, \ldots\}$
- $\{\lambda\}^* = \{\lambda\}$
- $\emptyset^* =$

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$
- $A^* = \{\lambda, aa, bb, aaaa, aabb, bbaa, bbbb, aaaaaa, \ldots\}$
- $B^* = \{\lambda, c, cc, ccc, cccc, \ldots\}$
- $\{\lambda\}^* = \{\lambda\}$
- $\emptyset^* = \{\lambda\}$

Outline

- Introduction to Sets
- Formal Languages
- Set Equality
- Laws of Set Operations

Set Equality

Two sets are **equal** ($A = B$) if they contain the same elements

To show equality:

- Examine all the elements
- Show $A \subseteq B$ and $B \subseteq A$
- Use the Laws of Set Operations

NB

Venn diagrams can help visualize, but are not rigorous.

Examples

Example

Show $\{3, 2, 1\} = (0, 4)$.

Examples

Example

Show $\{3, 2, 1\} = (0, 4)$.

$(0, 4) = \{1, 2, 3\} = \{3, 2, 1\}$.

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$

$$\begin{aligned}\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} &= \{-2, -1, 0, 1, 2\} \\ &= \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}\end{aligned}$$

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 > 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| > 2\}$

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 > 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| > 2\}$

Show:

- For all $n \in \mathbb{Z}$, if $n^2 > 5$ then $|n| > 2$; and
- For all $n \in \mathbb{Z}$, if $|n| > 2$ then $n^2 > 5$.

That is, show:

$$\text{For all } n \in \mathbb{Z}: \quad n^2 > 5 \quad \text{if, and only if} \quad |n| > 2$$

Outline

- Introduction to Sets
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Laws of Set Operations

For all sets A, B, C:

Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distribution	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	$A \cup \emptyset = A$
	$A \cap \mathcal{U} = A$
Complementation	$A \cup (A^c) = \mathcal{U}$
	$A \cap (A^c) = \emptyset$

Substitution

Because the laws hold for all sets, we can substitute complex expressions for each set symbol.

Example

Commutativity	$A \cup B = B \cup A$

Substitution

Because the laws hold for all sets, we can substitute complex expressions for each set symbol.

Example

Commutativity $$A \cup B = B \cup A$$

Therefore: $$(C \cap D) \cup (D \oplus E) = (D \oplus E) \cup (C \cap D)$$

Example

Example

Show that for all sets $A \cap (B \cap C) = C \cap (B \cap A)$:

Example

Example

Show that for all sets $A \cap (B \cap C) = C \cap (B \cap A)$:

$$\begin{aligned} A \cap (B \cap C) &= (A \cap B) \cap C && \text{[Associativity]} \\ &= C \cap (A \cap B) && \text{[Commutativity]} \\ &= C \cap (B \cap A) && \text{[Commutativity]} \end{aligned}$$

Important!

(Aim to) limit each step to a single application of a single rule

Other useful set laws

The following are all derivable from the previous 10 laws.

Idempotence	$A \cap A = A$
	$A \cup A = A$
Double complementation	$(A^c)^c = A$
Annihilation	$A \cap \emptyset = \emptyset$
	$A \cup \mathcal{U} = \mathcal{U}$
de Morgan's Laws	$(A \cap B)^c = A^c \cup B^c$
	$(A \cup B)^c = A^c \cap B^c$

Example (Idempotence of $\cup$)

$$A = A \cup \emptyset \qquad \text{(Identity)}$$

Example (Idempotence of $\cup$)

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \end{aligned}$$

Example (Idempotence of $\cup$)

$$
\begin{aligned}
A &= A \cup \emptyset && \text{(Identity)} \\
&= A \cup (A \cap A^c) && \text{(Complementation)} \\
&= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)}
\end{aligned}
$$

Example (Idempotence of $\cup$)

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\ &= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \end{aligned}$$

Example (Idempotence of $\cup$)

$$
\begin{aligned}
A &= A \cup \emptyset && \text{(Identity)} \\
&= A \cup (A \cap A^c) && \text{(Complementation)} \\
&= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\
&= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \\
&= (A \cup A) && \text{(Identity)}
\end{aligned}
$$

Two useful results

Definition

If A is a set defined using ^{c}, $\cap$, $\cup$, $\emptyset$ and $\mathcal{U}$, then dual(A) is the expression obtained by replacing $\cap$ with $\cup$ (and vice-versa) and $\emptyset$ with $\mathcal{U}$ (and vice-versa).

Theorem (Principle of Duality)

If you can prove $A_1 = A_2$ using the Laws of Set Operations then you can prove $dual(A_1) = dual(A_2)$

Example

Absorption law: $A \cup (A \cap B) = A$

Dual: $A \cap (A \cup B) = A$

Application (Idempotence of $\cap$)

Recall Idempotence of $\cup$:

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\ &= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \\ &= (A \cup A) && \text{(Identity)} \end{aligned}$$

Application (Idempotence of $\cap$)

Invoke the dual laws!

$$\begin{aligned} A &= A \cap \mathcal{U} && \text{(Identity)} \\ &= A \cap (A \cup A^c) && \text{(Complementation)} \\ &= (A \cap A) \cup (A \cap A^c) && \text{(Distributivity)} \\ &= (A \cap A) \cup \emptyset && \text{(Complementation)} \\ &= (A \cap A) && \text{(Identity)} \end{aligned}$$

Two useful results

Theorem (Uniqueness of complement)

$A \cap B = \emptyset$ *and* $A \cup B = \mathcal{U}$ *if, and only if,* $B = A^c$.

Two useful results

Theorem (Uniqueness of complement)

$A \cap B = \emptyset$ *and* $A \cup B = \mathcal{U}$ *if, and only if,* $B = A^c$.

Proof (Only if).

$$
\begin{aligned}
B &= B \cap \mathcal{U} && \text{(Identity)}\\
&= B \cap (A \cup A^c) && \text{(Complement)}\\
&= (B \cap A) \cup (B \cap A^c) && \text{(Distributivity)}\\
&= (A \cap B) \cup (A^c \cap B) && \text{(Commutativity)}\\
&= \emptyset \cup (A^c \cap B) && \text{(Given)}\\
&= (A \cap A^c) \cup (A^c \cap B) && \text{(Complement)}\\
&= (A^c \cap A) \cup (A^c \cap B) && \text{(Commutativity)}\\
&= A^c \cap (A \cup B) && \text{(Distributivity)}\\
&= A^c \cap \mathcal{U} && \text{(Given)}\\
&= A^c && \text{(Identity)}
\end{aligned}
$$

$\square$

Application (Double complement)

Take $A = X^c$ and $B = X$:

$$\begin{aligned} X^c \cap X &= X \cap X^c && \text{(Commutativity)} \\ &= \emptyset && \text{(Identity)} \\ \\ X^c \cup X &= \mathcal{U} && \text{(Principle of duality)} \end{aligned}$$

By the uniqueness of complement, $(X^c)^c = X$.

Exercises

Exercises

Show the following for all sets A, B, C:

- $B \cup (A \cap \emptyset) = B$
- $(C \cup A) \cap (B \cup A) = A \cup (B \cap C)$
- $(A \cap B) \cup (A \cup B^c)^c = B$

Exercises

Give counterexamples to show the following do not hold for all sets:

- $A \setminus (B \setminus C) = (A \setminus B) \setminus C$
- $(A \cup B) \setminus C = A \cup (B \setminus C)$
- $(A \setminus B) \cup B = A$