Computing ℓ_1 distance to origin on a Spiral over the Natural Numbers

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1 Abstract

We present a coordinate space for addressing elements on the Integer Spiral which is rotationally symmertric 90° about the origin (1). We develop and present useful supporting formulae, in terms of some integer n, for computing the coordinates of n in this space, with a view to computing the ℓ_1 (taxicab) distance from n to the origin.

2 Introduction

We define a Spiral over some ordered set S to be an arrangement of the elements of S in an $N \times N$ grid - where N is odd - such that the least element of S is in the centre, with the elements arranged in a counter-clockwise fashion, spiralling outward from the centre.

An example of one such Spiral is the Spiral over the Natural numbers, which we will denote as $\mathcal{N}$:

`DIAGRAM OF THE SPIRAL`

As with any ordered set, it will prove useful to develop a distance metric for different elements of $\mathcal{N}$, not least for solving part one of day three of the 2017 edition of the *Advent of Code*. Per the problem, we will concern ourselves with the ℓ_1 of some element $n \in \mathcal{N}$ to the origin, 1.

2.1 Shells in the Spiral

When dealing with elements in $\mathcal{N}$, it will prove useful to partition $\mathcal{N}$ into subsets radiating outward from the origin, which we will call *shells*, denoted as S_n, where n is the number of shells between S_n and the origin. See, for example, S_0, S_1, and S_5, below, highlighted each in red, white and green, respectively:

`Highlighted DIAGRAM OF THE SPIRAL`

2.2 Boundary numbers

We define the *boundary number* of a shell to be the greatest number in that shell, so-called as it marks the boundary between two successive shells when following the spiral from the center.

We can see by inspection that boundary numbers of S_0, S_1, and S_3 are 0, 9, 25, and 49. Noting that this is simply the sequence of squares of odd numbers, in general, the boundary number for S_n is simply

$$b(n) = (2n + 1)^2$$

The inverse [1] of this function

$$b^{-1}(S) = \frac{1}{2}(\sqrt{x} - 1)$$

will, given a boundary number, yield its shell number.

Further treatment of this function is required should we wish to find the shell number of *any* number, not just a boundary number.

A commonly known method of finding the *next closest* square of some real number x is to compute $\lceil \sqrt{x} \rceil^2$.

[1] b is invertible as of course we are ignoring the case where n is negative, as the notion of "shell -1" is clearly absurd