Article

Portfolio Diversification Revisited

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Abstract: We relax several assumptions in Alexeev and Tapon (2012) to account for non-normally distributed, skewed, multi-regime, and leptokurtic asset return distributions. By calibrating a Markov-modulated Lévy process model to equity market data, we demonstrate that our approach not only effectively captures the empirical moments, including higher-order moments, but also accommodates regime-switching dynamics and path variation. We further argue that much of the literature on portfolio diversification relies on oversimplified assumptions that conflict with observable regularities in financial time series. Ignoring these complexities, particularly in terms of non-linear and multi-regime behaviour, may lead to a material underestimation of risk.

Keywords: Markov-modulated Lévy process; regime-switching models; Diversification ; Portfolio theory; Expected utility theory

0. Introduction

The motivation behind this study stems from the recognition that traditional portfolio diversification models often fall short in capturing the complexities of real-world financial markets. Conventional approaches, such as the Mean-Variance Optimization framework, typically assume normally distributed returns and static market conditions, which can lead to significant underestimations of risk, particularly during periods of economic turbulence or regime shifts. Our research seeks to address these limitations by introducing a Markov-modulated Lévy process model, which accounts for non-normally distributed, skewed, multi-regime, and leptokurtic asset return distributions.

Our primary contribution lies in demonstrating how this advanced modelling approach better aligns with empirical observations, offering a more accurate representation of market dynamics. By calibrating our model to equity market data, we show that it effectively captures higher-order moments and accommodates regime-switching behaviour, providing a more nuanced understanding of risk and return characteristics. This advancement not only enriches the academic literature by challenging oversimplified assumptions but also offers practical benefits for investment professionals seeking robust tools for risk management and portfolio optimization.

Academics will find our findings valuable as they push the boundaries of existing financial theories, opening new avenues for research into complex market behaviours. Practitioners, on the other hand, will benefit from a model that enhances decision-making processes by providing deeper insights into asset behaviour across different economic regimes. The originality of our study lies in its ability to integrate these sophisticated dynamics into a coherent framework, setting it apart from prior research and offering a compelling alternative to traditional models. Through this work, we aim to bridge the gap between theoretical advancements and practical applications, ultimately contributing to more resilient financial strategies.

An important issue in portfolio management is the development of strategies to mitigate risk. It is widely acknowledged that diversification is an effective method for reducing unsystematic (idiosyncratic) risk, provided that no insider information is available. Furthermore, idiosyncratic risk is well-established as a significant contributor to overall volatility. However, eliminating risk entirely is impossible, even with a broad range of investments. Specifically, systemic risk factors will always contribute to total risk, even in the presence of international diversification.

This remains a significant area of research among both investment professionals and academic circles. In simple terms, the objective is to invest an initial amount in financial assets to maximize the expected utility of terminal wealth. [1] was among the first to address this issue by developing the mean-variance approach to portfolio optimisation, introducing quantitative methods for selecting optimal portfolios. [2] and [3] extended these results, providing explicit formulations for portfolio asset selection and allocation in continuous time. While the techniques developed by Merton and colleagues led to substantial theoretical advancements, practical concerns remain. One key challenge stems from the implicit assumption that the value of a risky asset follows a geometric Brownian motion. Certain observable patterns in financial time series, such as asymmetry, heavy tails in return distributions, and fluctuating conditional volatility, are not fully consistent with this hypothesis.

Another limitation of the Black-Scholes-Merton framework is its assumption of static coefficients. This becomes a critical consideration for longer-term investment horizons, where macroeconomic conditions may shift multiple times, fundamentally altering investment opportunity sets. In this context, Markov-modulated, or *regime-switching*, models seem well-suited to represent such phenomena. For example, changes in economic conditions can reasonably be quantified using a Markov chain, which adjusts the model's parameters. These models can describe macroeconomic shifts, periods when market behaviour shifts dramatically during crises, or the various stages of business cycles.

When optimizing portfolios, numerous factors must be considered in determining the number of assets required for optimal diversification. Systematic risk measurement can vary depending on the size of the investment universe, investor characteristics, asset attributes (which may change over time), the chosen diversification model, data frequency, market conditions, and the investment horizon. Calculating the optimal number of assets for a fully diversified portfolio — within a specific market, time frame, or set of preferences — poses challenges when approached from the traditional Mean-Variance Optimization framework introduced by Markowitz. However, recent research suggests that the size of a well-diversified portfolio is now larger than ever, that this number is smaller in developing economies compared to established financial markets, and that it decreases as stock correlations with the market increase ([4–6]).

The objective of this paper is to critically examine these and related topics. By employing a Markov-modulated Lévy process (where a Lévy process can be described as a suitable combination of two processes: diffusions and jumps), we relax several assumptions found in studies like [4] to account for non-normally distributed, skewed, multi-regime, and leptokurtic asset return distributions. We fit a model to equity market data, demonstrating that these calibrated models align well with empirical moments. We contend that much of the literature on portfolio diversification relies on assumptions that conflict with observable patterns, potentially leading to significant underestimations of risk when these patterns are ignored.

In our study, we employed a Markov-modulated Lévy process model to capture the complexities of asset return distributions, which include non-normality, skewness, multi-regime behaviour, and leptokurtic characteristics. This approach stands in contrast to traditional models such as the Mean-Variance Optimization (MVO) framework and the Black-Scholes-Merton model, which often rely on assumptions of normally distributed returns and static coefficients. Traditional models, while foundational, may not fully account for the dynamic nature of financial markets, particularly during periods of economic instability or regime shifts.

Our results demonstrate that the Markov-modulated Lévy process provides a more realistic framework for modelling asset returns by accommodating regime-switching dynamics and capturing higher-order moments. This leads to a more accurate estimation of risk, especially in volatile market conditions where traditional models might underestimate potential risks due to their oversimplified assumptions. Furthermore, our approach allows for a more nuanced understanding of asset behaviour across different economic regimes, offering investors enhanced tools for portfolio diversification and risk management.

By integrating these complex dynamics into our model, we offer a significant advantage over traditional approaches: the ability to better align theoretical models with empirical observations. This alignment not only improves the precision of risk assessments but also supports more informed decision-making in portfolio management. Consequently, our methodology provides both academics and practitioners with a valuable alternative to traditional models, particularly in environments characterised by high volatility and structural changes.

The structure of this paper is as follows: Section 1 reviews relevant literature, Section 2 presents the theoretical foundation for our work, Section 3 presents a calibration exercise involving a Markov-modulated Lévy process applied to Nasdaq data, challenging several assumptions in [4] to accommodate non-normally distributed, skewed, multi-regime, and leptokurtic asset return distributions. Section 6 discusses the results, and Section 7 offers a conclusion.

1. Literature Review

Classic portfolio allocation theory is grounded in the mean-variance (MV) framework, which provides a method to analyse the trade-off between risk and return to achieve diversification benefits. Despite its limitations, the MV framework remains foundational in many asset allocation decisions, as numerous asset managers, consultants, and investment advisers utilise classical MV optimisation as a standard quantitative approach to portfolio construction. This framework was introduced by [1], laying the groundwork for subsequent developments in risk and return theories.

In addition to Markowitz, pioneering researchers such as [7], [8], [9], and [10], and [6] have made significant contributions to the field of portfolio risk diversification. Their key findings can be summarised as follows:

- A portfolio's risk can be reduced through diversification, which mitigates both systematic and unsystematic risk.
- A portfolio's overall risk increases when the number of stocks it holds approaches that of the entire market.
- Unsystematic risk can be minimised up to the point of achieving optimal portfolio diversification, at which point the total portfolio risk equals systematic risk.

Let us first examine some of the key issues surrounding the measurement of risk diversification. Various methods exist for measuring portfolio risk, each with its own set of advantages and disadvantages, and these have been the subject of ongoing debate. Prominent among these methods is standard deviation, which is widely used in many studies as a commonly accepted metric for risk. This is evident from earlier research ([7,9–11]) as well as more recent investigations ([12], [13]).

One major issue with using standard deviation as a risk metric is that it may produce erroneous and misleading results due to its sensitivity to extreme values and outliers. Specifically, it is well-known that standard deviation can lead to inaccurate estimates of extreme occurrences if returns are not normally distributed. Another limitation of the standard deviation approach is that it treats positive and negative deviations from the average return equally.

Given these challenges, a considerable amount of research has explored alternative risk measures, such as expected shortfall (ES) and terminal wealth standard deviation (TWSD), particularly in assessing the impact of financial crises on the optimal number of stocks in a portfolio ([12,13]). Additionally, the mean absolute deviation (MAD) approach [8], which uses absolute deviation rather than variance, and the unsystematic risk ratio (URR) [14], which provides a measure of diversification relative to its variation, have been proposed as alternative measures of portfolio risk. It is important to note that the portfolio structures generated by different risk measures can vary significantly.

The heterogeneity of viewpoints on what constitutes successful diversification significantly complicates the task of comparing research across the literature on optimal portfolio diversification. For instance, [5] claim that investing in a diversified portfolio of seven or ten evenly-weighted equities may reduce risk by as much as 85 percent or 90 percent, respectively. However, [15] argues that a portfolio of just 20 equities is needed to eliminate 95 percent of unsystematic risk, while a portfolio of 80 equities may reduce an additional 4 percent. [16] suggest that a differently-weighted portfolio could remove 97 percent of unsystematic risk with 25 stocks, whereas [17] found that in China, 67 percent of unsystematic risk could be mitigated with just 10 stocks. According to [18], investing in 20–25 stocks may minimise 90 percent of a portfolio's risk. They also find that risk-averse investors prefer portfolios with a 99 percent risk reduction, while more aggressive investors, who seek higher returns at the expense of more risk, may be satisfied with a 90 percent reduction.

[4] further argue that the required number of stocks in a well-diversified portfolio depends on the average correlations between stocks and the market, as well as the prevailing market conditions — whether distressed or stable — when analysing portfolio dynamics over multiple years. [19] suggest that the appropriate number of stocks is influenced by the investor's risk tolerance, desired level of confidence, and the portfolio's weighting structure. These findings align with intuition, as investors' actions typically vary according to their economic circumstances. Often, their decisions are dynamic and influenced by economic, cultural, and social factors, which may not always be entirely rational ([20]). Consequently, their values, preferences, assumptions, and perceptions shift alongside the economy ([21]). Moreover, an investor's location also affects both their optimal asset allocation and investment success ([22]).

It is also possible that the frequency of the data utilised may affect the optimal number of stocks in a portfolio. This could result in an exaggerated number of stocks, as demonstrated by [5]. They further note that this disparity is amplified during financial market crises. Early research based on (semi)annual and quarterly data suggests that the optimal number of equities for diversification ranges from 8 to 16 ([7,8,23]). However, [5] argue that high-frequency data significantly enhances risk assessment and decision-making. With higher frequency data, the number of stocks required to achieve the desired risk reduction decreases.

Data from various frequency intervals shows a minor variation in unsystematic risk during stable periods, but a substantial difference during times of high volatility. Risk estimates based on lower frequency data tend to be overstated, particularly during financial crises. Higher frequency data provides more accurate risk measurements, which suggests that holding large portfolios is not always necessary, as implied by lower frequency risk measures, especially in times of financial distress. Price fluctuations of financial instruments are also influenced by fundamental factors such as interest rates, economic growth, and currencies. Investors may uncover diversification benefits by constructing portfolios based on these criteria ([20]). Moreover, corporate bonds, in addition to equities, may offer significant risk reduction, especially during periods of financial turbulence.

Building on this reasoning, an aggregate stock index composed of the S&P 500's most prominent companies may reduce transaction costs while offering sufficient diversification, according to [24]. To prevent international financial contagion, investors and mutual fund managers must diversify their portfolios and utilise hedging strategies ([25]). Funds with less liquid equities in their portfolios tend to be more diversified, and as noted by [26], such funds are typically larger, cheaper, and trade more frequently. This represents a significant advancement in active portfolio management. Socially responsible investment (SRI) funds, according to [27], do not significantly affect idiosyncratic risk due to screening intensity, which is a crucial aspect of an investment strategy. Additionally, portfolio managers may opt to adjust their holdings during periods of political instability, favouring companies with more accurate reporting practices ([28,29]).

We note the foundational and pioneering works that have shaped the field of portfolio diversification. The seminal work by Markowitz (1952) introduced the Mean-Variance Optimization framework, laying the groundwork for modern portfolio theory by providing a quantitative method to balance risk and return. This approach has been further extended by Black, Scholes, and Merton, who developed continuous-time models for asset selection and allocation, contributing significantly to theoretical advancements in financial economics.

Building on these foundations, researchers such as [7], [8], and [10] explored the benefits of diversification in reducing unsystematic risk, establishing key principles that continue to influence portfolio management strategies. More recent studies, including those by [6] and [5], have highlighted the increasing volatility of individual stocks and the implications for portfolio size and composition, emphasizing the need for adaptive strategies in dynamic markets.

Our study also considers the limitations of traditional models, as discussed in works by [30] and [31], which underscore the importance of accounting for regime shifts and non-linear dynamics in financial time series. These insights have paved the way for alternative approaches, such as the use of Markov-switching models, to better capture the complexities of market behaviour.

By integrating these diverse strands of research, our literature review not only contextualizes our study within the broader academic discourse but also identifies gaps that our work seeks to address. We contribute to this evolving conversation by specifying a Markov-modulated Lévy process model, which offers a novel perspective on capturing higher-order moments and regime-switching dynamics in portfolio construction, thereby advancing both theoretical understanding and practical applications in risk diversification.

2. Theoretical background

There are several challenges in modelling financial time series, as observations can be influenced by unforeseen events. Events such as natural catastrophes, central bank statements, and government policy announcements may have a significant impact on the market. Consequently, the assumption of stationarity in financial data is often violated. As a result, traditional approaches to time series analysis may prove unsatisfactory. Markov-switching models are thus of interest, as they allow for the mitigation of issues related to suspected non-stationarity, under certain mild assumptions.

A key stylised feature of financial time series is regime switching, wherein economic conditions undergo periodic regime changes. The regular flow of economic activity may occasionally experience shocks significant enough to result in different observed dynamics. As a result, sampled time series data may exhibit not only periods of low and high volatility but also periods of slower and faster mean growth. The economy may oscillate between two states: (1) a stable, low-volatility state characterised by economic expansion, and (2) a panic-driven, high-volatility state defined by economic contraction. These dynamics are often described by deterministic or stochastic ordinary differential equations. Regime shifts indicate transitions between different dynamics, potentially involving changes in state space, the objective function, or both.

Regime changes can occur for various reasons: (i) exogenous changes in dynamics (e.g., due to sudden environmental disasters or social/political reform); (ii) unintentional internal changes in dynamics (e.g., due to human activity-related disasters or firm bankruptcy); (iii) intentional (controlled) shifts to new dynamics (e.g., technological innovations, mergers of firms); (iv) changes in preferences/objectives (e.g., environmental concerns). A single model may incorporate a combination of these triggers. Evidence of such regimes has been widely documented, and switching models have been applied across various domains, including but not limited to exchange rates, asset allocation, and equity markets. Surveys are provided by [32] and [31].

Following [33], we assume that the Lévy process L follows the Normal Inverse Gaussian (NIG) distribution, defined as a variance-mean mixture of a normal distribution with the inverse Gaussian as the mixing distribution (also see Barndorff-Nielsen et al. [34–37] for a more extensive discussion). Practical modelling of the aforementioned stylised features can be achieved using the General Hyperbolic Distribution. The specific case of the General Hyperbolic Distribution that we focus on is the Normal-Inverse Gaussian (NIG) distribution, a continuous probability distribution defined as a normal variance-mean mixture with an inverse Gaussian mixing density ([34,38]). The NIG density is characterised by a four-dimensional parameter vector $[\alpha, \beta, \mu, \delta]$, which determines its form. Due to its extensive parametrisation, the NIG density is well-suited for modelling a wide range of unimodal, positively kurtotic data.

The α parameter controls the steepness or peakedness of the density, increasing monotonically with larger values of α. A high α value indicates light tails, while a low α value corresponds to heavier tails. The β parameter governs the skewness of the distribution. When $\beta < 0$, the density is skewed to the left; when $\beta > 0$, it is skewed to the right; and when $\beta = 0$, the density is symmetric around the centrality parameter μ. The δ parameter is a scale parameter.

The class of NIG distributions is highly flexible, accommodating fat-tailed and skewed distributions, with the normal distribution, $N(\mu, \sigma^2)$, arising as a special case when $\beta = 0$, $\delta = \sigma^2/\alpha$, and $\alpha \to \infty$.

Consider the following generic Markov Regime Switching (MRS) model:

$$\begin{cases} y_t = f(S_t, \theta, \psi_{t-1}) \\ S_t = g(\tilde{S}_{t-1}, \psi_{t-1}) \\ S_t \in \Lambda \end{cases} \tag{1}$$

where θ is the vector of the parameters of the model, S_t is the state of the model at time t, $\psi_t := \{y_k : k = 1, \ldots, t\}$ is the set of all observations up to t, $\tilde{S}_t := \{S_1, ..., S_t\}$ is the set of all observed states up to t, $\Lambda = \{1, ..., M\}$ is the set of all possible states, and g is the function that regulates transitions between states. Function f indicates how observations at time t depend on $S_t, \theta,$ and ψ_{t-1} and finally, $t \in \{0, 1, ..., T\}$, where $T \in \mathbb{N}$, $T < +\infty$, is the terminal time.

Equations 1 enable us to address specific issues that may be difficult to capture in a single-state regime, which proves useful for time series applications. Although the literature on Markov-switching models is extensive, two general categories can be identified. The first category includes models with basic transition laws, such as a first-order Markov chain, but complex distributions for the data or a large number of states. For examples of research in this area, see [39–41]. The second category comprises models with more complex transition laws, but with simpler assumptions and a limited number of states, often restricted to two. See, for instance, [42–44].

2.1. NIG-type distribution

To motivate our modelling approach, we first outline the general structure of Lévy processes and then detail their properties, such as path variation and the Lévy-Khintchine theorem. The discussion highlights key properties of Markov chains, including irreducibility, aperiodicity, and ergodicity. This section also presents a framework for estimating a jump-robust model tempered by a Markov chain, which can be used to study the dependence relationships within financial returns.

The NIG distribution is a relatively novel process introduced by [35] as a model for the log returns of stock prices. It is a subclass of the broader class of hyperbolic Lévy processes. After its introduction, the NIG distribution was shown to provide an excellent fit to stock market log returns ([45]). Other studies have demonstrated the superior empirical fit of this distribution for various asset classes ([46–49]). [50] find that the Normal Inverse Gaussian distribution provides a better overall fit for financial data than other subclasses of Generalised Hyperbolic distributions and significantly outperforms Lévy-stable laws. [51] have successfully applied the NIG distribution to address well-known 'puzzles,' such as (i) the predictability of asset returns, (ii) the equity premium, and (iii) the volatility puzzle.

This type of heavy-tailed processes is of interest, particularly since the NIG distribution fulfills the fat-tails condition, is analytically tractable, yet is closed under convolution ([52]).

The density function of a $NIG(\alpha, \beta, \delta, \mu)$ is given by

$$f_{NIG}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha}{\pi} e^{\delta \sqrt{\alpha^2 - \beta^2 + \beta(x-\mu)}} \frac{K_1(\alpha\delta\sqrt{1 + (x-\mu)^2/\delta^2})}{\sqrt{1 + (x-\mu)^2/\delta^2}}, \tag{2}$$

where $\delta > 0, \alpha \geq 0$. The parameters in the Normal Inverse Gaussian distribution can be interpreted as follows: α is the tail heaviness of steepness, β is the skewness, δ is the scale, and μ is the location. The NIG distribution is the only member of the family of general hyperbolic distributions to be closed under convolution. K_v is the Hankel function with index v. This can be represented by

$$K_v(z) = \frac{1}{2} \int_0^\infty y^{v-1} e^{\left(-\frac{1}{2} z \left(y + \frac{1}{y}\right)\right)} dy \tag{3}$$

For a given real v, the function K_v satisfies the differential equation given by

$$v^2 y'' + xy' - (x^2 + v^2)y = 0. \tag{4}$$

The log cumulative function of a Normal Inverse Gaussian variable is given by

$$\phi^{NIG}(z) = \mu z + \delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}\right) \text{ for all } |\beta + z| < \alpha. \tag{5}$$

The first two moments are $\mathbb{E}[X] = \mu + \frac{\delta\beta}{\gamma}$, and $Var[X] = \frac{\delta\alpha^2}{\gamma^3}$, where $\gamma = \sqrt{\alpha^2 - \beta^2}$. The Lévy measure of a $NIG(\alpha, \beta, \delta, \mu)$ law is

$$F_{NIG}(dx) = e^{\beta x} \frac{\delta\alpha}{\pi|x|} K_1(\alpha|x|) dx. \tag{6}$$

3. Data and model

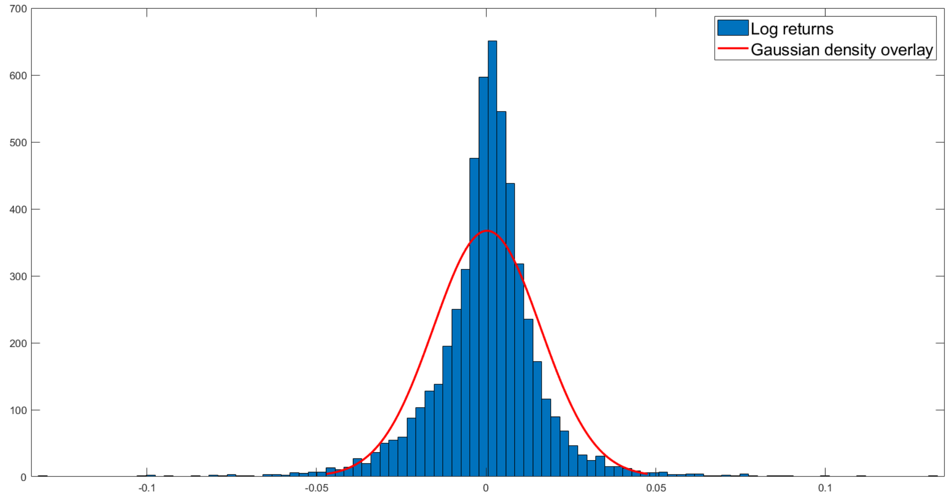
The present section aims to discuss the data used in our model. In our study, we utilise daily closing price data for the Nasdaq 100 index. The dataset spans from 4 January 2000 to 12 December 2024, encompassing 6,516 daily observations. This extensive time frame allows us to capture various market conditions, including periods of economic stability and volatility. The primary variables analysed include log returns, which are calculated as the natural logarithm of consecutive daily closing prices. These log returns serve as the basis for modelling asset return distributions and assessing the dynamics of financial markets.

To account for the complexities inherent in financial time series, we incorporate additional variables such as skewness, kurtosis, and regime indicators. Skewness measures the asymmetry of return distributions, while kurtosis captures the presence of fat tails or extreme values. Regime indicators, derived from our Markov-modulated Lévy process model, identify shifts between different market states, enabling us to examine how these transitions impact asset behaviour. By integrating these variables into our analysis, we provide a comprehensive framework for understanding the multifaceted nature of asset returns.

Figure 1 presents the plot of log returns for the Nasdaq 100 (blue) with a normal distribution overlay (red). The graph clearly demonstrates that Gaussian density underestimates random variation. This can be attributed to the fact that financial returns typically exhibit distributions with certain stylised features, such as excess kurtosis (i.e., fat or semi-heavy tails), skewness (tail asymmetries), and regime shifts.

We now focus on the estimation of a Markov-switching model augmented by jumps, represented as a Lévy process, to model financial returns.

Figure 1. Log returns for Nasdaq 100 (blue) with normal distribution overlay (red).



We follow the two-stage estimation strategy proposed by [33]. Figures 2 and 3 point to the presence of identifiable regimes.

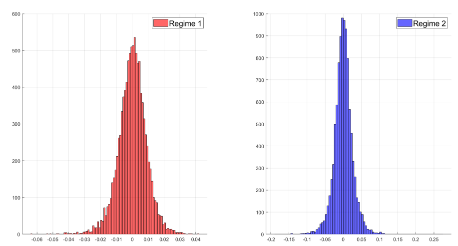


Figure 2. Nasdaq regimes, separated.

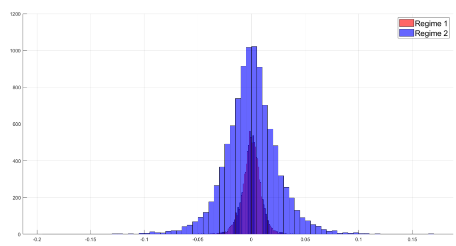


Figure 3. Regimes, superimposed.

4. Results

Figure 4 shows closing prices for the NASDAQ (top), together with the volatility of log-returns (middle), and the evolution of regimes in our MRS model over the extended period 2000-2024. The dataset comprises 6,516 observations, with prices ranging from 0 to 22,096.66 (mean: 5,247.18, standard deviation: 4,962.10). Tables 1 and 2 report the regime-switching parameters and the NIG distribution parameters of the Lévy jump process fitted to each regime, respectively.

Table 1. Two-State Regime-Switching Model Parameters

Parameter	Base Regime	Spike Regime
ϕ (Mean Reversion)	1.00160	1.00084
c (Level)	-2.08322	-5.19334
σ^2 (Variance)	724.09934	41556.20481
$E(Y_{t,i})$ (Expected Value)	1299.64104	6203.77304
q_{ii} (Transition Prob.)	0.98795	0.97397
$P(R = i)$ (Ergodic Prob.)	0.68359	0.31641

Note: LogL = -34711.719

The regime-switching model effectively identifies two distinct market states: a base regime and a spike regime. The base regime, characterized by lower volatility ($\sigma^2 = 724.09934$), dominates

Figure 4. NASDAQ close prices, log returns, and regimes (2000-2024)

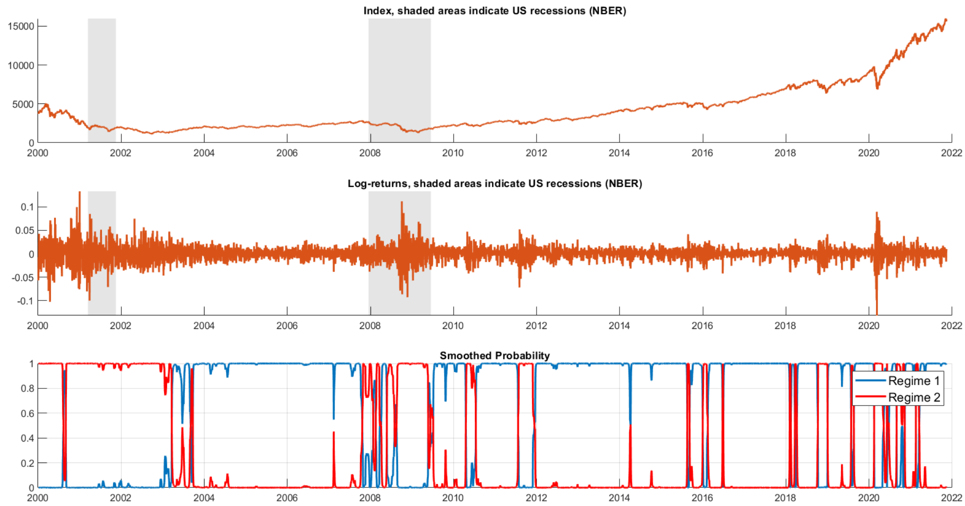


Table 2. NIG Distribution Parameters of the Lévy Jump Process

Parameter	Base Regime	Spike Regime
α (Tail Heaviness)	0.001330	0.001660
β (Asymmetry)	0.000757	0.000080
δ (Scale)	5.603221	69.380969
μ (Location)	-3.880216	-3.355581

Table 3. Model Classification and Fit Statistics

Metric	Value
RCM Statistic	5.574951
Classification Accuracy (%)	94.060773
Gaussian Case Log-Likelihood	-40267.312566
Gaussian Case BIC	80560.971180
Gaussian Case AIC	80540.625132

with an ergodic probability of 68.36%. The spike regime, exhibiting substantially higher volatility ($\sigma^2 = 41556.20481$), occurs 31.64% of the time. Both regimes show high persistence, with transition probabilities (q_{ii}) exceeding 0.97.

The model achieves strong classification performance with an RCM statistic of 5.57 and classification accuracy of 94.06%. Comparison of model fit statistics indicates that the regime-switching specification outperforms the Gaussian case, as evidenced by the lower BIC and AIC values. The NIG distribution parameters reveal distinct tail behaviors across regimes, with the spike regime showing notably higher scale (δ) and slightly different tail heaviness (α) compared to the base regime.

5. Backtest Results

Our empirical investigation is grounded in the application of a Markov-modulated Lévy process framework, which is employed to capture the complex, regime-dependent dynamics inherent in the NASDAQ index returns. The analysis utilises a dataset comprising 6,516 observations, partitioned into two distinct regimes via robust maximum likelihood estimation and supplementary validation tests. The base regime, representing conventional market behaviour, is characterised by lower volatility and relatively moderated risk measures, while the spike regime encapsulates episodes of heightened market stress, indicated by a pronounced surge in volatility and elevated risk estimates.

Table 4. Summary Statistics and Risk Measures by Market Regime

Measure	**Base Regime**	**Spike Regime**
Sample Size	4,451	2,065
Proportion (%)	68.31	31.69
Descriptive Statistics		
Mean	2,807.41	10,505.99
Volatility	1,954.67	5,382.77
Skewness	2.34	-0.0002
Kurtosis	9.31	-1.05
Risk Measures (95% Confidence)		
VaR	1,089.98	2,095.62
CVaR	969.32	1,681.11
Statistical Tests (p-values)		
Shapiro-Wilk	<0.001	<0.001
Kolmogorov-Smirnov	<0.001	<0.001
Ljung-Box	<0.001	<0.001

The regime classification, with approximately 68.31% of observations in the base regime and 31.69% in the spike regime, is statistically robust and aligns with theoretical expectations, wherein normal market conditions predominate. The considerable divergence in risk measures is particularly noteworthy; the spike regime exhibits a VaR nearly double that of the base regime, alongside a substantial elevation in conditional risk as evidenced by the CVaR values.

Further scrutiny of the descriptive statistics reveals distinctive distributional characteristics across regimes. The base regime is marked by significant positive skewness (2.34) and high kurtosis (9.31), indicative of heavy right tails and heightened sensitivity to extreme positive returns. Conversely, the spike regime displays near-zero skewness paired with negative kurtosis (-1.05), an anomalous finding that suggests potential misestimation or the need for refined tail risk modelling in periods of extreme market volatility.

Further statistical validation, employing Shapiro-Wilk, Kolmogorov-Smirnov, and Ljung-Box tests, unequivocally rejects the null hypotheses of normality and independence for both regimes ($p < 0.001$). The failure of the Augmented Dickey-Fuller test to confirm stationarity further underscores the inadequacy of conventional linear models. These findings collectively advocate for the adoption of more sophisticated methodologies that explicitly incorporate regime-switching dynamics and non-Gaussian distributions.

Established theory recognises many stylised facts, including non-linearity, non-Gaussianity, and volatility clustering. However, our findings underscore that these phenomena are not mere academic curiosities but have profound practical implications. In an era characterised by unprecedented market uncertainty, there is a compelling need to bridge the gap between theoretical models and real-world risk management. This study challenges practitioners to move beyond traditional linear frameworks and to consider portfolio management strategies that explicitly account for regime-dependent behaviours and extreme events. Embracing these enhanced methodologies could substantially improve risk assessment and investment decision making, fostering a more resilient and adaptive approach to asset allocation.

Thus, our results not only validate the presence of distinct market regimes but also highlight the critical importance of adopting alternative portfolio management frameworks that accommodate the heteroskedastic and non-linear characteristics of financial data. Future research should explore the integration of extreme value theory and advanced quantile estimation techniques to further refine risk assessments under varied market conditions.

6. Discussion

6.1. Dealing with Regimes and Higher Moments

Several factors must be considered when determining how many assets are necessary to achieve optimal diversification. These include the investment universe (size, asset classes, and features of the asset classes), the investor's characteristics, changes in asset features over time, the model adopted to measure diversification (e.g., equally weighted versus an "optimal" allocation), and the frequency of the data used. We argue that it is equally important to account for well-documented nuances in financial time series data, particularly the presence of non-normal distributions, skewness, multi-regime behaviour, leptokurtic characteristics, and non-linear or discontinuous price processes.

Additionally, the cost of overseas investment must be considered when evaluating the benefits of international diversification. Investors face several challenges when constructing portfolios outside their home market, including currency risks, political instability, limitations on money transfers, and differing regulatory environments across countries. In contrast, domestic investments are free from such complications, making them easier to manage from a diversification, security analysis, and asset allocation perspective.

There is a range of opinions in the academic literature on what constitutes successful diversification. Recent research—see, for example, [4], [19], [5], and the literature cited therein—suggests that the average size of a well-diversified portfolio has increased compared to earlier studies, which may be attributed in part to reduced transactional costs. This research also indicates that the ideal number of equities in a well-diversified portfolio has risen significantly when longer time horizons are considered. According to [53], U.S. unsystematic risk has grown markedly in recent decades relative to overall stock market volatility, underscoring the need for larger portfolios to reduce diversifiable risk as much as possible.

However, not all investors may achieve the same level of diversification by following uniform portfolio size guidelines. In making these decisions, it is important to consider several factors, including the frequency of data collection, the level of risk accounted for, the confidence of investors, and the potential diversification benefits of the chosen investment opportunity set, among others.

6.2. Simple Heuristic Diversification Rules

Our results also highlight the so-called $1/n$ investment puzzle, where some market participants allocate their contributions equally across all assets. Since this approach contradicts the core principles of modern portfolio theory, it is often regarded as a naive method. There is evidence (e.g. [54])

suggesting that many participants in defined contribution plans use basic heuristic diversification methods to distribute their contributions across different asset classes. The $1/n$ rule, commonly referred to as an "equally-weighted portfolio," is one such heuristic. However, it has been criticised for not lying on the efficient frontier, as it does not represent an optimal portfolio. Some academics argue that pension systems should be less flexible to prevent individuals from making suboptimal investment decisions.

Nevertheless, [55] presents arguments demonstrating that this behaviour may be less naive than it initially seems, showing that the $1/n$ rule offers certain advantages in terms of robustness. While we do not advocate for any heuristic diversification rule in this paper, we note that further work may be required to improve upon this simple diversification criterion. In this context, [56] observes that "naive formation rules such as the equal weight rule can outperform the Markowitz rule." Similarly, [57] notes that due to estimation risk, "an equally weighted portfolio may often be substantially closer to true mean-variance optimality than an optimised portfolio."

6.3. Attempts at a Unified Theory of Portfolio Management

Before concluding our discussion, we should briefly address the feasibility of applying the Mean-Variance Optimisation (MVO) framework to assets beyond equities, such as fixed income. Any attempt at diversification naturally involves diversifying across asset classes. However, when constructing bond portfolios, the established MVO approach poses certain challenges, making the development of a unified theory for bond portfolios non-trivial.

To elaborate, stocks and bonds differ in several key ways, the most significant being the fixed maturity date at which bonds expire and disappear from the market, whereas stock characteristics can fluctuate in response to business news or management decisions. In an unconstrained market, the maturity period of bonds can take on an infinite number of values, leading to an infinite number of distinct bonds. Consequently, the price of a stock is determined solely by its associated risks (e.g., market risk, idiosyncratic risk), while the price of a bond is influenced by both its risks and the time remaining until maturity.

The stochastic differential equations (SDEs) used to model stock prices are typically autonomous, meaning that the coefficients are time-independent functions of the prices, as seen in geometric Brownian motion or mean-reverting processes. However, any model for bond prices must account for the fact that volatility approaches zero as the time to maturity decreases—at least, this is how it is mathematically expressed.

A simple portfolio of stocks and bonds is made more analytically challenging by the fact that prices for each asset class follow different principles, even in the most basic scenario. Furthermore, due to the dependency on maturity, many strategies permissible in the stock market are not directly applicable in the bond market. For example, a simple buy-and-hold strategy results in bonds converting into cash upon maturity. In the context of fixed-income derivatives, it is natural to model interest rates with an uncountably large set of traded instruments.

While some notable attempts toward a unified theory of portfolio management incorporating both stocks and bonds have been made, the literature remains relatively sparse, and the problem of multi-asset portfolio optimisation is still an open research question. In this vein, [58] made strides toward introducing a bond portfolio management theory, building on foundations similar to stock portfolio management by modelling the discounted price curve as infinite-dimensional dynamics in a Banach space of continuous functions, driven by a cylindrical Wiener process. The interested reader is referred to [59], who extend these results using the HJM framework, and [60], who apply them to commodity futures markets. Extending these results presents a promising avenue for future research.

Our findings reveal that the Markov-modulated Lévy process model effectively captures the empirical characteristics of asset returns, including higher-order moments and regime-switching dynamics. The model's ability to accommodate non-normally distributed, skewed, multi-regime, and leptokurtic return distributions offers significant advantages over traditional approaches, which often rely on oversimplified assumptions.

One key inference from our study is the importance of considering regime shifts when assessing portfolio risk. Our results indicate that ignoring these shifts can lead to substantial underestimations of risk, particularly during periods of heightened market volatility. By identifying distinct regimes and their associated characteristics, our model provides valuable insights into the timing and nature of market transitions, informing more robust risk management strategies.

Furthermore, our analysis suggests that incorporating higher-order moments into portfolio optimisation can enhance diversification benefits by better capturing the true distribution of returns. This approach allows investors to construct portfolios that are more resilient to extreme market movements, ultimately improving long-term performance.

Overall, our study contributes to the literature by demonstrating the practical applicability of advanced modelling techniques in capturing complex market dynamics. These findings underscore the need for continuous innovation in financial modelling to address the evolving challenges faced by both academics and practitioners in the field of portfolio management.

7. Conclusion

This paper discusses some of the issues which arise in the portfolio allocation literature. However, we wish to focus the reader's attention on challenges of a different kind, which seem to be absent from the discussions in the optimum portfolio diversification literature. Specifically, there is merit in exploring certain stylized features of financial time series in an attempt to capture dynamics of non-normally distributed, skewed, multi-regime, and leptokurtic asset return distributions.

We promoted the estimation of a Markov-switching model augmented by jumps, under the form of a Lévy process. After setting up the general structure of Lévy processes, we outlined their properties concerning path variation and the Lévy-Khintchine theorem. The analysis highlighted the properties of Markov chains, such as irreducibility, aperiodicity, and ergodicity.

By deploying a Markov-modulated Lévy process, we relaxed several assumptions in [4] to account for account for non-normally distributed, skewed, multi-regime, and leptokurtic asset return distributions. We calibrated a regime-switching Lévy model to equity market data to demonstrate that such a model is a) analytically tractable and computationally effective, b) intuitive, and c) does a good job of matching the empirical moments, including those of higher order. Finally, we argue that a part of the related literature on portfolio diversification relies on assumptions that are in tension with certain observable regularities and which, if ignored, may lead to material underestimation of risk.

Code availability: Code is available at https://github.com/shawcharles/Portfolio-Diversification-Revisited

Appendix H Lévy processes

Lévy processes can be thought of as a combination of two distinct processes, namely diffusions and jumps. The attractive properties of such a combination can be demonstrated by sketching the connections between the two. A well-known pure diffusion process used in finance is the Wiener process, a continuous-time Markovian stochastic process with a.s. continuous sample paths. A well-known pure jump process is the Poisson process, which is a non-decreasing process that, unlike Wiener, does not have continuous paths. Whilst the Poisson process has paths of bounded variation over finite time horizons, the paths of a Wiener process exhibit unbounded variation over finite time horizons.

Appendix H.1 A regime-switching Lévy model

This subsection motivates the introduction of the regime-switching Lévy approach to the modelling of our time series. This part of the discussion follows the methodology proposed by [33], namely by combining a Lévy jump-diffusion model with a Markov-switching framework. The regime-switching Lévy model offers the possibility of identification of such stochastic jumps, together with disentangling different market regimes and capturing the regime-switching dynamics. We begin by introducing a set of key definitions and notations.

Definition H.1 (Stochastic Process)**.** *A stochastic process X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables $(X_t)_{0\leq t<\infty}$.*

If $X_t \in \mathcal{F}_t$, the process X is adapted to the filtration $\mathcal{F}$, or equivalently, $\mathcal{F}_t$-measurable.

Definition H.2 (Lévy Process)**.** *Let L be a stochastic process. Then L_t is a Lévy process if the following conditions are satisfied:*

1. $L_0 = 0$
2. *L has independent increments: $L_t - L_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$*
3. *L has stationary increments: $\mathbb{P}(L_t - L_s \leq x) = \mathbb{P}(L_{t-s} \leq x)$, $0 \leq s < t < \infty$*
4. *L_t is continuous in probability: $\lim_{t\to s} L_t = L_s$*

The class of Lévy Processes contains the subclasses of Brownian motion and Poisson processes. Reacall that a real-valued random variable Θ has an infinitely divisible distribution if for each positive integer n, there exists a *i.i.d.* sequence of random variables $\Theta_1, \ldots, \Theta_n$ such that

$$\Theta \overset{d}{=} \Theta_{1,n} + \ldots + \Theta_{n,n}$$

The full extent to which we may characterize infinitely divisible distributions is carried out via their characteristic function (or Fourier transform of their law) and the Lévy-Khintchine formula. By the Lévy-Khintchine formula (e.g. [61], Theorem 1.6) we can say that there exists a probability space where $L = L^{(1)} + L^{(2)} + L^{(3)}$. We can build intuition as follows: $L^{(1)}$ is standard Brownian motion with drift, $L^{(2)}$ is a compound Poisson process, and $L^{(3)}$ is a square-integrable martingale with countable number of jumps of magnitude less than unity (a.s.). This is the Lévy-Itô decomposition, which can be stated as follows

$$L_t = \eta t + \sigma W_t + \int_0^t \int_{|x|\geq 1} x\mu^L(ds, dx) + \int_0^t \int_{|x|<1} x(\eta^L - \Pi^L)(ds, dx). \tag{A7}$$

Definition H.3 (Markov-Switching)**.** *Let $(Z_t)_{t\in[0,T]}$ be a continuous time Markov chain on finite space $S := \{1, \ldots, K\}$. Let $\mathcal{F}_t^Z := \{\sigma(Z_s); 0 \leq s \leq t\}$ be the natural filtration generated by the continuous time Markov chain Z. The generator matrix of Z, denoted by Π^Z, is given by*

$$\Pi_{ij}^Z = \begin{cases} \geq 0, & \text{if } i \neq j \\ -\sum_{j\neq i} \Pi_{ij}^Z, & \text{otherwise} \end{cases} \tag{A8}$$

We can now define the Regime-switching Lévy model as follows.

Definition H.4 (Regime-switching Lévy model)**.** *For all $t \in [0, T]$, let Z_t be a continuous time Markov chain on finite space $S := \{1, \ldots, K\}$ defined as per Definition H.3. A regime-switching model is a stochastic process (X_t) which is a solution of the stochastic differential equation given by*

$$dX_t = k(Z_t)(\theta(Z_t) - X_t)dt + \sigma(Z_t)dY_t \tag{A9}$$

where $k(Z_t)$, $\theta(Z_t)$, and $\sigma(Z_t)$ are functions of the Markov chain Z. They are scalars which take values in $k(Z_t)$, $\theta(Z_t)$, and:

$$\begin{aligned} &\sigma(Z_t) : k(Z_t) := \{k(1), \ldots, k(K)\} \in \mathbb{R}^{K^*}, \\ &\theta(S) := \{\theta(1), \ldots, \theta(K)\}, \\ &\sigma(S) := \{\sigma(1), \ldots, \sigma(K)\} \in \mathbb{R}^{K^*}, \end{aligned}$$

where Y is a Wiener or a Lévy process. Here, k denotes the mean reverting rate, θ denotes the long-run mean, and σ denotes the volatility of X.

The above model exhibits two sources of randomness: the Markov chain Z, and the stochastic process Y which appears in the dynamics of X. In other words, there is stochasticity due to the Markov chain Z, $\mathcal{F}^Z$, and stochasticity due to the market information which is the initial continuous filtration $\mathcal{F}$ generated by the stochastic process Y.

Appendix I Technical supplement

The following arguments follow directly from [33].

Appendix I.1 Stage 1: The regime-switching model

We aim to estimate the set of parameters $\Theta = \hat{\Theta}_1 := (\hat{k}_i, \hat{\theta}_i, \hat{\sigma}_i, \hat{\Pi}_i)$ for $i \in S$.

1. Start with initial vector $\hat{\Theta}_1^{(0)} := \left(\hat{k}_i^{(0)}, \hat{\theta}_i^{(0)}, \hat{\sigma}_i^{(0)}, \hat{\Pi}_i^{(0)}\right)$ for $i \in S$. Let $N \in \mathbb{N}$ be the maximum number of iterations. Fix a positive constant ϵ as a convergence constant for the estimated log-likelihood function.
2. Assume that we are at the $n + 1 \leq N$ steps. The calculation in the previous iteration of the algorithm gives the following vector set $\hat{\Theta}_1^{(n)} := \left(\hat{k}_i^{(n)}, \hat{\theta}_i^{(n)}, \hat{\sigma}_i^{(n)}, \hat{\Pi}_i^{(n)}\right)$

Appendix I.2 EM-algorithm

Appendix I.2.1 Expectation step (E step)

We aim to estimate both filtered probability and smoothed probability. Optimality is achieved when a model can identify regimes sharply, such that smoothed probabilities approach either zero or one. Filtered probability is given by the probability such that the Markov chain Z is in regime $i \in S$ at time t with respect to $\mathcal{F}_T^X$:

For all $i \in S$ and $k = \{1, 2, \ldots, M\}$, estimate the following

$$\begin{aligned} P\Big(Z_{t_k} = i | \mathcal{F}_{t_k}^X; \hat{\Theta}_1^{(n)}\Big) &= \frac{P\Big(Z_{t_k}, X_{t_k} | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}\Big)}{f\big(X_{t_k} | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}\big)} \\ &= \frac{P\Big(Z_{t_k} = i | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}\Big) f\Big(X_{t_k} | Z_{t_k} = i; \mathcal{F}_{t_{k-1}}; \hat{\Theta}_1^{(n)}\Big)}{\sum_{j \in S} P\Big(Z_{t_k} = j | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}\Big) f\Big(X_{t_k} | Z_{t_k} = j; \mathcal{F}_{t_{k-1}}; \hat{\Theta}_1^{(n)}\Big)} \end{aligned} \tag{A10}$$

such that

$$\begin{aligned} P\Big(Z_{t_k} = i | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}\Big) &= \sum_{j \in S} P\Big(Z_{t_k} = i, Z_{t_{k-1}} = j | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}\Big) \\ &= \sum_{j \in S} P\Big(Z_{t_k} = i, Z_{t_{k-1}} = j | \hat{\Theta}_1^{(n)}\Big) P\Big(Z_{t_{k-1}} = j | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}\Big) \\ &= \sum_{j \in S} \Pi_{ij}^{(n)} P\Big(Z_{t_{k-1}} = j | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}\Big), \end{aligned} \tag{A11}$$

where $f\Big(X_{t_k} | Z_{t_k} = i; \mathcal{F}_{t_{k-1}}; \hat{\Theta}_1^{(n)}\Big)$ is the density of the process X at time t_k, conditional that the process is in regime $i \in S$. Using previous arguments we can observe that, given $\mathcal{F}_{t_{k-1}}^X$, the process X_{t_k} has a conditional Gaussian distribution $\sim N\Big(k_i^{(n)} \theta_i^{(n)} + (1 - k_i^{(n)}) X_{t_{k-1}}, \sigma_i^{2(n)}\Big)$. The density of this distribution is given by

$$f\Big(X_{t_k} | Z_{t_k} = i; \mathcal{F}_{t_{k-1}}; \hat{\Theta}_1^{(n)}\Big) = \frac{1}{\sqrt{2\pi\sigma_i^{(n)}}} \exp\left[\frac{X_{t_k} - \big(1 - k_i^{(n)}\big) X_{t_{k-1}} - \theta_i^{(n)} k_i^{(n)}\big)^2}{2\big(\sigma_i^{(n)}\big)^2}\right] \tag{A12}$$

On the other hand, to estimate smoothed probability we need to examine when Markov chain Z is in regime $i \in S$ at time t with respect to all the historical data $\mathcal{F}_T^X$. For all $i \in S$ and $k = \{M-1, M-2, \ldots, 1\}$ we obtain

$$P\Big(Z_{t_k} = i | \mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)}\Big) = \sum_{j \in S} \left(\frac{P\big(Z_{t_k} = i | \mathcal{F}_{t_k}^X; \hat{\Theta}_1^{(n)}\big) P\big(Z_{t_{k+1}} = j | \mathcal{F}_{t_M}; \hat{\Theta}_1^{(n)} | \Pi_{ij}^{(n)}\big)}{P\big(Z_{t_{k+1}} = j | \mathcal{F}_{t_k}^X; \hat{\Theta}_1^{(n)}\big)}\right) \tag{A13}$$

Appendix I.2.2 Maximization step (M step)

We can obtain an explicit formula of the maximum likelihood estimator of the initial subset of parameters $\hat{\Theta}_1$. The maximum likelihood estimates $\hat{\Theta}_1^{(n+1)}$ for all parameters, for all $i \in S$, can be obtained by

$$
\begin{aligned}
\theta_i^{(n+1)} &= \frac{\sum_{k=2}^{M}\left[P(Z_{t_k}=i|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\,(X_{t_k}-(1-k_i^{(n+1)})X_{t_{k-1}}\right]}{k_i^{n+1}\sum_{k=2}^{M}\left[P(Z_{t_k}=i|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\right]} \\
k_i^{(n+1)} &= \frac{\sum_{k=2}^{M}\left[P(Z_{t_k}=i|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\,X_{t_{k-1}}B_1\right]}{\sum_{k=2}^{M}\left[P(Z_{t_k}=i|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\,X_{t_{k-1}}B_2\right]} \qquad \text{(A14)} \\
\sigma_i^{(n+1)} &= \frac{\sum_{k=2}^{M}\left[P(Z_{t_k}=i|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\,(X_{t_k}-k_i^{(n+1)}\theta_i^{(n+1)}(1-k_i^{(n+1)})X_{t_{k-1}})^2\right]}{\sum_{k=2}^{M}\left[P(Z_{t_k}=i|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\right]}
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= X_{t_k}-X_{t_{k-1}} = \frac{\sum_{k=2}^{M}\left[P(Z_{t_k}=i|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\,(X_{t_k}-X_{t_{k-1}}\right)}{\sum_{k=2}^{M}\left[P(Z_{t_k}=i|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\right]} \\
B_2 &= \frac{\sum_{k=2}^{M}\left[P(Z_{t_k}=i|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\,X_{t_{k-1}}\right]}{\sum_{k=2}^{M}\left[P(Z_{t_k}=i|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\right]}X_{t_{k-1}}.
\end{aligned}
$$

We then obtain the transition probabilities:

$$
\Pi_{ij}^{(n+1)} = \frac{\sum_{k=2}^{M}\left[P(Z_{t_k}=j|\mathcal{F}_{t_M};\hat{\Theta}_1^{(n)})\frac{\Pi_{ij}^{n}P(Z_{t_{k-1}}=i|\mathcal{F}_{t_{k-1}};\hat{\Theta}_1^{(n)})}{P(Z_{t_k}=j|\mathcal{F}_{t_{k-1}};\hat{\Theta}_1^{(n)})}\right]}{\sum_{k=2}^{M}\left[P(Z_{t_{k-1}}=i|\mathcal{F}_{t_{k-1}};\hat{\Theta}_1^{(n)})\right]} \qquad \text{(A15)}
$$

Let $\hat{\Theta}_1^{(n+1)} := (k_i^{(n+1)}, \theta_i^{(n+1)}, \sigma_i^{(n+1)}, \Pi_i^{(n+1)})$ be the new parameters of the algorithm. These are iterated in step 2 until convergence of the EM algorithm is achieved. The procedure can be stopped if either:

a) the procedure has been performed N times; or
b) the difference between the log-likelihood at step $n+1 \leq N$ and the log-likelihood at step n, satisfies the equation $\log L(n+1) - \log L(n) < \epsilon$.

Proof of consistency of the (quasi) maximum likelihood estimators is provided in [62]; see also [63].

Appendix I.3 Stage 2: Lévy distribution fitted to each regime

We have estimated the regime-switching model A9 using the EM algorithm. Now, we estimate the set of parameters $\hat{\Theta}_2$ by fitting a NIG distribution for each regime.

$$X(\text{Regime } 1) - L_1(\alpha^1, \beta^1, \delta^1, \mu^1) \qquad \text{(A16)}$$

$$X(\text{Regime } 2) - L_2(\alpha^2, \beta^2, \delta^2, \mu^2) \qquad \text{(A17)}$$

where L_1 and L_2 relate to relates to a separate set of Normal Inverse Gaussian distribution parameters of the Lévy jump process. Estimation of the distribution parameters is done by maximum likelihood, where $\Phi^1 = (\alpha^1, \beta^1, \delta^1, \mu^1)$ and $\Phi^1 = (\alpha^2, \beta^2, \delta^2, \mu^2)$. Directly following from [33], initialization of the algorithm is performed by the method of moments.

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Review Reports:

Academic Editor Comments:

This paper offers a fresh and rigorous perspective on modern portfolio analysis, demonstrating a level of sophistication that is commendable, particularly coming from an independent researcher. The author's focus on Markov-modulated Lévy processes and regime-switching dynamics represents a valuable contribution to portfolio diversification theory, challenging traditional models by addressing the limitations of assuming normally distributed, static market conditions.

The paper is particularly strong in its calibration of the proposed model to equity market data, effectively capturing higher-order moments and accounting for empirical complexities such as skewness, heavy tails, and multi-regime behavior. These are critical aspects often neglected in conventional approaches like the Mean-Variance Optimization framework or the Black-Scholes-Merton model.

That said, my primary concern lies in the lack of backtesting of the optimal portfolio suggested by the model. While the theoretical advancements are significant, it would greatly strengthen the paper if the proposed framework were tested in "portfolio horse races" or competitive empirical settings against existing methods. My experience in research suggests that this addition would likely yield positive outcomes, further solidifying the paper's findings.

I am pleasantly surprised and highly impressed by the quality of research demonstrated in this work. It is rare to see such a high-caliber contribution from an independent researcher, and I wholeheartedly congratulate the author for this achievement. I strongly encourage continuation with the peer-review process, as this paper holds considerable potential to advance both theoretical and practical aspects of portfolio optimization.

Overall, this is a highly valuable contribution that merits serious consideration for publication.

Reply to Academic Editor:

Dear Editor,

Thank you very much for your thoughtful and encouraging review of our manuscript. I am delighted that you found our analysis of Markov-modulated Lévy processes and regime-switching dynamics to be both novel and rigorous, and I appreciate your acknowledgement of the contributions made toward addressing limitations in "conventional" portfolio models.

I have taken note of your primary concern regarding the lack of backtesting of the optimal portfolio derived from our model. In response, I am currently extending the empirical evaluation section to include a comparative backtesting analysis. This additional study will directly illustrate the practical benefits of our approach in real-world portfolio allocation practices.

I believe that integrating this comparative analysis will not only bolster the theoretical advancements outlined in the manuscript but will also reinforce the empirical robustness of our proposed framework. I am confident that the results from the backtesting exercise will further validate the model's capacity to capture higher-order moments, heavy-tailed behavior, and regime-dependent market dynamics in a competitive empirical setting.

Thank you again for your positive evaluation and for your suggestion, which I believe will significantly contribute to the overall strength of the paper. I look forward to submitting the revised manuscript for further consideration.

Sincerely,

Charles Shaw