Proof Theory:
Logical and Philosophical Aspects

Class 5: Semantics and beyond

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Our Aim

To introduce *proof theory*, with a focus in its applications in philosophy, linguistics and computer science.

Examine the behaviour of quantifiers from the perspective of proof theory.

First-Order Predicate Logic

Free Logic

Quantified Modal Logic

Other Topics

FIRST-ORDER PREDICATE LOGIC

Quantifier Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{Fa \succ Fa}{\forall xFx \succ Fa}[\forall L] \qquad \dfrac{Ga \succ Ga}{\forall xGx \succ Ga}[\forall L]}{\forall xFx, \forall xGx \succ Fa \wedge Ga}[\wedge R]}{\forall xFx \wedge \forall xGx \succ Fa \wedge Ga}[\wedge L]}{\forall xFx \wedge \forall xGx \succ \forall x(Fx \wedge Gx)}[\forall R]$$

Quantifier Derivations

$$
\dfrac{\dfrac{\dfrac{\dfrac{Fa \succ Fa}{Fa \succ \exists x Fx}[\exists R] \qquad \dfrac{Ga \succ Ga}{Ga \succ \exists x Gx}[\exists R]}{Fa \lor Ga \succ \exists x Fx, \exists x Gx}[\lor L]}{Fa \lor Ga \succ \exists x Fx \lor \exists x Gx}[\lor R]}{\exists x(Fx \lor Gx) \succ \exists x Fx \lor \exists x Gx}[\exists L]
$$

Quantifier Derivations

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Fa \succ Fa \qquad Ga \succ Ga}{Fa \lor Ga \succ Fa, Ga}\,[\lor L]}{\forall x(Fx \lor Gx) \succ Fa, Ga}\,[\forall L]}{\forall x(Fx \lor Gx) \succ Fa, \exists x Gx}\,[\exists R]}{\forall x(Fx \lor Gx) \succ \forall x Fx, \exists x Gx}\,[\forall R]}{\forall x(Fx \lor Gx) \succ \forall x Fx \lor \exists x Gx}\,[\lor R]
$$

Quantifier Rules

$$\text{(any } t\text{)}\quad \frac{X, A(t) \succ Y}{X, \forall x A(x) \succ Y}\,[\forall L] \qquad\qquad \text{(}n \textit{ fresh}\text{)}\quad \frac{X, A(n) \succ Y}{X, \exists x A(x) \succ Y}\,[\exists L]$$

$$\text{(}n \textit{ fresh}\text{)}\quad \frac{X \succ A(n), Y}{X \succ \forall x A(x), Y}\,[\forall R] \qquad\qquad \text{(any } t\text{)}\quad \frac{X \succ A(t), Y}{X \succ \exists x A(x), Y}\,[\exists R]$$

In $[\forall R]$ and $[\exists L]$, the condition that n is fresh means that n is an *eigenvariable*, not occurring in X and Y.

An eigenvariable is a singular term n that is *inferentially general*.

That is, if $\pi(n)$ is a derivation of $X(n) \succ Y(n)$, then for any term t, $\pi(t)$ is a derivation of $X(t) \succ Y(t)$ (where we have replaced n by t through all the ancestors of each n occurring in $X(n) \succ Y(n)$ through $\pi(n)$.)

Isn't every term inferentially general?

- Function terms aren't.
  We can prove $(\exists x)(fx = f\underline{n})$, but not $(\exists x)(fx = t)$ for arbitrary t.

- In sequent presentations of *theories*, constants can be inferentially specific. For example, in PA we can prove $(\forall x)(0 \neq x')$ but not $(\forall x)(t \neq x')$.

The Point of Inferential Generality: Eliminating Cut

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta_l(n)\\ X \succ A(n), Y\end{array}}{X \succ \forall x A(x), Y}\,[\forall R] \qquad \dfrac{\begin{array}{c}\vdots\,\delta_r\\ X, A(t) \succ Y\end{array}}{X, \forall x A(x) \succ Y}\,[\forall L]}{X \succ Y}\,[Cut]$$

We need to transform $\delta_l(n)$ into a proof of $X \succ A(t), Y$ in order to simplify the Cut.

$$\dfrac{\begin{array}{c}\vdots\,\delta_l(t)\\ X \succ A(t), Y\end{array} \qquad \begin{array}{c}\vdots\,\delta_r\\ X, A(t) \succ Y\end{array}}{X \succ Y}\,[Cut]$$

FREE LOGIC

Non-Denoting Terms

$$\frac{1}{0} \qquad \lim_{x \to 0} \frac{\sin x}{x} \qquad \sum_{n=0}^{\infty} f(n)$$

Non-Denoting Terms

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It is difficult to eliminate non-denoting terms as a matter of *syntax*.

$$(\forall x)(x < 0 \lor x = 0 \lor x > 0) \not\vdash (\tfrac{1}{0} < 0 \lor \tfrac{1}{0} = 0 \lor \tfrac{1}{0} > 0)$$

Non-Denoting Terms

$$\frac{1}{0} \qquad \lim_{x \to 0} \frac{\sin x}{x} \qquad \sum_{n=0}^{\infty} f(n) \qquad \textit{Pegasus}$$

It is difficult to eliminate non-denoting terms as a matter of *syntax*.

$$(\forall x)(x < 0 \lor x = 0 \lor x > 0) \not\succ (\tfrac{1}{0} < 0 \lor \tfrac{1}{0} = 0 \lor \tfrac{1}{0} > 0)$$

How can we modify the quantifier rules
to allow for non-denoting terms?

Pro and *Con* attitudes to Terms

To rule a term *in* is to take it as suitable
to substitute into a quantifier,
i.e., to take the term to *denote*.

To rule a term *out* is to take it as unsuitable
to substitute into a quantifier,
i.e., to take the term to *not denote*.

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i.e., to take the term to *not denote*.

We add terms to the LHS and RHS of sequents $X \succ Y$.

Structural Rules remain as before

$$\star \succ \star \ [Id]$$

$$\frac{X \succ Y}{X, \star \succ Y}\ [KL] \qquad \frac{X \succ Y}{X \succ \star, Y}\ [KR] \qquad \frac{X, \star, \star \succ Y}{X, \star \succ Y}\ [WL] \qquad \frac{X \succ \star, \star, Y}{X \succ \star, Y}\ [WR]$$

$$\frac{X \succ \star, Y \qquad X, \star \succ Y}{X \succ Y}\ [Cut]$$

Here $\star$ is either a *sentence* or a *term*.

Modified Quantifier Rules

$$\text{(any } t\text{)}\ \frac{X, A(t) \succ Y \qquad X \succ t, Y}{X, \forall x A(x) \succ Y}\,[\forall L] \qquad \text{(} n \textit{ fresh}\text{)}\ \frac{X, n, A(n) \succ Y}{X, \exists x A(x) \succ Y}\,[\exists L]$$

$$\text{(} n \textit{ fresh}\text{)}\ \frac{X, n \succ A(n), Y}{X \succ \forall x A(x), Y}\,[\forall R] \qquad \text{(any } t\text{)}\ \frac{X \succ A(t), Y \qquad X \succ t, Y}{X \succ \exists x A(x), Y}\,[\exists R]$$

In [∀R] and [∃L], the condition that n is fresh means that n is an *eigenvariable*, not occurring in X and Y.

Making Denotation Explicit

$$\frac{X, t \succ Y}{X, t{\downarrow} \succ Y}\,[{\downarrow}L] \qquad \frac{X \succ t, Y}{X \succ t{\downarrow}, Y}\,[{\downarrow}R]$$

$$
\begin{prooftree}
\AxiomC{$Fn, n \succ Fn$}
\AxiomC{$Gn \succ Gn$}
\RightLabel{[$\supset$L]}
\BinaryInfC{$Fn \supset Gn, n, Fn \succ Gn$}
\AxiomC{$Fn \supset Gn, n, Fn \succ n$}
\RightLabel{[$\forall$L]}
\BinaryInfC{$\forall x(Fx \supset Gx), n, Fn \succ Gn$}
\AxiomC{$\forall x(Fx \supset Gx), n, Fn \succ n$}
\RightLabel{[$\exists$R]}
\BinaryInfC{$\forall x(Fx \supset Gx), n, Fn \succ \exists x Gx$}
\RightLabel{[$\exists$L]}
\UnaryInfC{$\forall x(Fx \supset Gx), \exists x Fx \succ \exists x Gx$}
\RightLabel{[$\supset$R]}
\UnaryInfC{$\forall x(Fx \supset Gx) \succ \exists x Fx \supset \exists x Gx$}
\end{prooftree}
$$

Example Derivations

$$\dfrac{\dfrac{Ft \succ Ft \qquad t \succ t}{\forall x Fx, t \succ Ft}[\forall L]}{\forall x Fx, t{\downarrow} \succ Ft}[{\downarrow}L]$$

Definedness Logic

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DEFINEDNESS

ABSTRACT. Questions of definedness are ubiquitous in mathematics. Informally, these involve reasoning about expressions which may or may not have a value. This paper surveys work on logics in which such reasoning can be carried out directly, especially in computational contexts. It begins with a general logic of "partial terms", continues with partial combinatory and lambda calculi, and concludes with an expressively rich theory of partial functions and polymorphic types, where termination of functional programs can be established in a natural way.

Erkenntnis 43: 295–320, 1995.

Definedness, function terms and predicates

$$\frac{t_i, X \succ Y}{f(t_1, \ldots, t_n), X \succ Y}\,[fL]$$

$$\frac{t_i, X \succ Y}{Ft_1 \cdots t_n, X \succ Y}\,[FL]$$

A MODEL for the logic DL is a structure $\mathfrak{M}$ consisting of

1. A *domain* D.
2. An n-ary predicate F is interpreted as a subset $F^{\mathfrak{M}}$ of D^n (as usual).
3. An n-ary function symbol f is interpreted as a *partial function* $f^{\mathfrak{M}} : D^n \rightharpoonup D$.

Assigning Values

- α is a (partial) assignment of values to variables.
- $[\![x]\!]_{\mathfrak{M},\alpha} = \alpha(x)$
- $[\![f(t_1, \ldots, t_n)]\!]_{\mathfrak{M},\alpha} = f^{\mathfrak{M}}([\![t_1]\!]_{\mathfrak{M},\alpha}, \ldots, [\![t_n]\!]_{\mathfrak{M},\alpha})$ if each $[\![t_i]\!]_{\mathfrak{M},\alpha}$ is defined, and $f^{\mathfrak{M}}$ is defined on the inputs $[\![t_1]\!]_{\mathfrak{M},\alpha}, \ldots, [\![t_n]\!]_{\mathfrak{M},\alpha}$.

Interpreting a Language

- $\mathfrak{M} \vDash_\alpha t{\downarrow}$ iff $[\![t]\!]_{\mathfrak{M},\alpha}$ is defined.
- $\mathfrak{M} \vDash_\alpha Ft_1 \cdots t_n$ iff for each i, the value $[\![t_i]\!]_{\mathfrak{M},\alpha}$ is defined, and the n-tuple $\langle [\![t_n]\!]_{\mathfrak{M},\alpha}, \ldots, [\![t_n]\!]_{\mathfrak{M},\alpha} \rangle \in F^{\mathfrak{M}}$
- $\mathfrak{M} \vDash_\alpha A \wedge B$ iff $\mathfrak{M} \vDash_\alpha A$ and $\mathfrak{M} \vDash_\alpha B$.
- $\mathfrak{M} \vDash_\alpha A \vee B$ iff $\mathfrak{M} \vDash_\alpha A$ or $\mathfrak{M} \vDash_\alpha B$.
- $\mathfrak{M} \vDash_\alpha A \supset B$ iff $\mathfrak{M} \nvDash_\alpha A$ or $\mathfrak{M} \vDash_\alpha B$.
- $\mathfrak{M} \vDash_\alpha \neg A$ iff $\mathfrak{M} \nvDash_\alpha A$.
- $\mathfrak{M} \vDash_\alpha (\forall x)A(x)$ iff $\mathfrak{M} \vDash_{\alpha[x:=d]} A(x)$ for every d in D.
- $\mathfrak{M} \vDash_\alpha (\exists x)A(x)$ iff $\mathfrak{M} \vDash_{\alpha[x:=d]} A(x)$ for some d in D.

Eliminating Cut

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l(n)}{X, n \succ A(n), Y}}{X \succ \forall x A(x), Y}\,[\forall R] \qquad \dfrac{\overset{\vdots\,\delta_r}{X, A(t) \succ Y} \qquad \overset{\vdots\,\delta_r'}{X \succ t, Y}}{X, \forall x A(x) \succ Y}\,[\forall L]}{X \succ Y}\,[\mathrm{Cut}]$$

simplifies to

$$\dfrac{\overset{\vdots\,\delta_r'}{X \succ t, Y} \qquad \dfrac{\overset{\vdots\,\delta_l(t)}{X, t \succ A(t), Y} \qquad \overset{\vdots\,\delta_r}{X, A(t) \succ Y}}{X, t \succ Y}\,[\mathrm{Cut}]}{X \succ Y}\,[\mathrm{Cut}]$$

QUANTIFIED MODAL LOGIC

Flat hypersequents

A *flat hypersequent* is a non-empty multiset of sequents.

$$X_1 \succ Y_1 \ \mid \ X_2 \succ Y_2 \ \mid \ \cdots \ \mid \ X_n \succ Y_n$$

Modal Rules

$$\frac{\mathcal{H}[X \succ Y \mid X', A \succ Y']}{\mathcal{H}[X, \Box A \succ Y \mid X' \succ Y']}\,[\Box L] \qquad \frac{\mathcal{H}[X', A \succ Y']}{\mathcal{H}[X', \Box A \succ Y']}\,[\Box L]$$

$$\frac{\mathcal{H}[X \succ Y \ \mid\ X' \succ A, Y']}{\mathcal{H}[X \succ \Diamond A, Y \ \mid\ X' \succ Y']}\,[\Diamond R] \qquad \frac{\mathcal{H}[X' \succ A, Y']}{\mathcal{H}[X' \succ \Diamond A, Y']}\,[\Diamond R]$$

$\mathcal{H}[X \succ Y \ \mid\ X' \succ Y']$ is a hypersequent
in which $X \succ Y$ and $X' \succ Y'$ are components.

Deriving the Barcan Formula

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{Fn \succ Fn}{\Box Fn \succ \ \mid\ \succ Fn}\,[\Box L]}{\forall x \Box Fx \succ \ \mid\ \succ Fn}\,[\forall L]}{\forall x \Box Fx \succ \ \mid\ \succ \forall x Fx}\,[\forall R]}{\forall x \Box Fx \succ \Box \forall x Fx}\,[\Box R]}{\succ \forall x \Box Fx \supset \Box \forall x Fx}\,[\supset R]$$

This derivation is blocked if we use the rules from free logic

$$
\dfrac{\dfrac{\dfrac{Fn \succ Fn}{\Box Fn \succ \ \mid\ \succ Fn}[\Box L] \qquad n \succ n \ \mid\ \succ}{\forall x \Box Fx, n \succ \ \mid\ \succ Fn}[\forall L]}{\forall x \Box Fx \succ \ \mid\ \succ \forall x Fx}[??]
$$

The term n is in the wrong component for the derivation to go through

Weakening a copy of n into the right component will violate the eigenvariable condition on [$\forall$R]

Migration rules

The addition of these rules would permit the derivation of the Barcan formula

$$\frac{\mathcal{H}[t, X \succ Y \mid X' \succ Y']}{\mathcal{H}[X \succ Y \mid t, X' \succ Y']}\ [\textit{MigrateL}] \qquad \frac{\mathcal{H}[X \succ Y, t \mid X' \succ Y']}{\mathcal{H}[X \succ Y \mid X' \succ Y', t]}\ [\textit{MigrateR}]$$

But why think *these* rules should be adopted?

Unrestricted quantifiers

$$(\text{any } t)\ \frac{\mathcal{H}[X, A(t) \succ Y]}{\mathcal{H}[X, \Pi x A(x) \succ Y]}\ [\Pi L] \qquad (n\ \textit{fresh})\ \frac{\mathcal{H}[X, A(n) \succ Y]}{\mathcal{H}[X, \Sigma x A(x) \succ Y]}\ [\Sigma L]$$

$$(n\ \textit{fresh})\ \frac{\mathcal{H}[X \succ Y, A(n)]}{\mathcal{H}[X \succ Y, \Pi x A(x)]}\ [\Pi R] \qquad (\text{any } t)\ \frac{\mathcal{H}[X \succ Y, A(t)]}{\mathcal{H}[X \succ Y, \Sigma x A(x)]}\ [\Sigma R]$$

Unrestricted quantifiers

The derivation of the Barcan formula will go through with these rules

But, one can also derive $\succ \Sigma x(\frac{1}{0} = x)$

Does this vindicate Meinongian views about existence?

There are other possible ways to adjust the quantifier rules, such as permitting the term n in $[\forall R]$ to appear in *some* zone

OTHER TOPICS

Identity

We have not introduced rules that deal much with terms occurring in formulas

Let's look at Identity rules

Left rules

The left rules for identity are fairly straightforward

$$\frac{X, \phi(a) \succ Y}{X, a = b, \phi(b) \succ Y}\,[= L] \qquad \frac{X \succ Y, \phi(a)}{X, a = b \succ Y, \phi(b)}\,[= L]$$

The identity on the right rule is a bit trickier

Right rule

The natural idea is

$$\frac{X, \phi(a) \succ \phi(b), Y \qquad X, \phi(b) \succ \phi(a), Y}{X \succ Y, a = b}\ [= R?]$$

This presupposes that if a is not identical to b, then there is formula, ϕ, that can distinguish them

One's present vocabulary may not be up to that task

Right rule done right

The solution to the problem is to recognize that identity is a higher-order notion

Use schematic letters, here G, as a way to consider possible extensions of the language

$$\frac{X, G(a) \succ G(b), Y \qquad X, G(b) \succ G(a), Y}{X \succ Y, a = b}\,[{=}R] \quad (F\ \textit{fresh})$$

As with term eigenvariables, predicate eigenvariables need to be inferentially general

Eliminating Cut

$$
\dfrac{\dfrac{\dfrac{\vdots\,\delta_l(G)}{X, G(a) \succ G(b), Y} \quad \dfrac{\vdots\,\delta_r}{X, G(b) \succ G(a), Y}}{X \succ Y, a = b}\,[= R] \quad \dfrac{\dfrac{\vdots\,\delta'}{X \succ \phi(a), Y}}{X, a = b \succ \phi(b), Y}\,[= L]}{X \succ \phi(b), Y}\,[\mathrm{Cut}]
$$

simplifies to

$$
\dfrac{\dfrac{\vdots\,\delta'}{X \succ \phi(a), Y} \quad \dfrac{\vdots\,\delta_l(\phi)}{X, \phi(a) \succ \phi(b), Y}}{X \succ \phi(b), Y}\,[\mathrm{Cut}]
$$

Alternative rules may make it easier to eliminate Cut

$$\frac{a = a, X \succ Y}{X \succ Y}\ [= R]$$

$$\frac{s = t, A(t), A(s), X \succ Y}{s = t, A(t), X \succ Y}\ [= L]$$

One can use the Dragalin-style proof to show that Cut can be eliminated

Higher-Order Quantifiers

The treatment of first-order quantifiers extends to second-order quantifiers in a fairly straightforward way

We use predicate eigenvariables, which must be inferentially general

Second-order quantifiers range over properties, and there may be a property that is not expressed by any formula in the language

Second-Order Quantifier rules

$$\text{(any } F) \quad \frac{X, F(t) \succ Y}{X, (\forall Z)Z(x) \succ Y}\,[\forall^2 L] \qquad \text{(} G \textit{ fresh}) \quad \frac{X, G(t) \succ Y}{X, (\exists Z)Z(t) \succ Y}\,[\exists^2 L]$$

$$\text{(} G \textit{ fresh}) \quad \frac{X \succ G(t), Y}{X \succ (\forall Z)Z(t), Y}\,[\forall^2 R] \qquad \text{(any } F) \quad \frac{X \succ F(t), Y}{X \succ (\exists Z)Z(t), Y}\,[\exists^2 R]$$

In $[\forall^2 R]$ and $[\exists^2 L]$, the condition that G is fresh means that G is an *eigenvariable*, not occurring in X and Y.

Uniqueness

Suppose that we had two second-order universal quantifiers, $(\forall X)$ and (X), both governed by the rules $[\forall^2 L]$ and $[\forall^2 R]$

They can be shown to be interderivable

$$\dfrac{\dfrac{Ga \succ Ga}{(\forall Z)Za \succ Ga}\,[\forall^2 L]}{(\forall Z)Za \succ (Z)Za}\,[\forall^2 R] \qquad \dfrac{\dfrac{Ga \succ Ga}{(Z)Za \succ Ga}\,[\forall^2 L]}{(Z)Za \succ (\forall Z)Za}\,[\forall^2 R]$$

This appears to be a sense in which the rules uniquely pick out the second-order universal quantifier

This may be surprising, given that there is some leeway on what the interpretations for second-order quantifiers can be

Semantics and beyond

Speech Acts and Norms

Proofs and Models

Where to go from here

THANK YOU!

`https://standefer.net/teaching/`

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