Proof Theory: Logical and Philosophical Aspects

Class 5: Semantics and beyond

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To introduce *proof theory*, with a focus in its applications in philosophy, linguistics and computer science.

Examine the connections between proof theory and semantics, both formal *model theory*, and more general philosophical considerations concerning meaning.

Speech Acts and Norms

Proofs and Models

Beyond

SPEECH ACTS AND NORMS

Normative Pragmatics

An idea found in Brandom's *Making It Explicit* is that the *meaning* of linguistic items should first be understood in terms of their *use*

The linguistic (conceptual) practices of communities set up *norms* governing their behavior

These practices have features that we can make explicit through the introduction of new vocabulary

Rules as Definitions

The rules that govern a connective are taken to *define* the new connective

This appears to make it really easy to introduce new logical terms

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But, there's a problem

Arthur Prior pointed out that if a set of rules is enough to define a connective, then *tonk* is legitimate

$$\frac{X, A \succ C}{X, A \frownie B \succ C}\ [\text{tonkL}] \qquad \frac{X \succ B}{X \succ A \frownie B}\ [\text{tonkR}]$$

Tonk

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$$\dfrac{\dfrac{B \succ B}{B \succ A \frownie B}\,[\text{tonkR}] \qquad \dfrac{A \succ A}{A \frownie B \succ A}\,[\text{tonkL}]}{B \succ A}\,[\text{Cut}]$$

Responding to tonk

Nuel Belnap responded to Prior's article, saying that additional conditions need to be satisfied in order to define a connective

Connectives aren't introduced out of thin air, there is a *context of deducibility*, e.g. the full set of Gentzen's structural rules

In order to be a definition, an extension has to be *conservative*, while tonk manifestly is *not*

In order to be a definition, an addition has to be *uniquely specified*

These ideas have been taken up and developed by Dummett and others in discussions of *harmony*

Defining rules

Running with Belnap's idea, we can define connectives using the following double line rules

$$\dfrac{X \succ A, Y}{X, \neg A \succ Y}\ [\neg Df] \qquad \dfrac{X \succ A|^x_n, Y}{X \succ (\forall x)A, Y}\ [\forall Df]$$

$$\dfrac{X, A \succ B, Y}{X \succ A \to B, Y}\ [\to Df] \qquad \dfrac{X, A|^x_n \succ Y}{X, (\exists x)A \succ Y}\ [\exists Df]$$

$$\dfrac{X \succ A, B, Y}{X \succ A \vee B, Y}\ [\vee Df] \qquad \dfrac{X, Fs \succ Ft, Y}{X, (\exists x)A \succ Y}\ [= Df]$$

(Provided n and F are not present in X and Y)

Defining rules

Concepts defined using defining rules have the following features

- They're uniquely defined. Two concepts defined using the same rule are interderivable, given Identity and Cut.
- They can be used to generate the other introduction rules using Identity and Cut.
- They're conservatively extending.

Uniqueness

Suppose $\wedge$ and $\&$ both obey the defining rule for $\wedge$

$$\dfrac{\dfrac{\dfrac{}{A \wedge B \succ A \wedge B}\,[\mathrm{Id}]}{A, B \succ A \wedge B}\,[\wedge\mathrm{Df}]}{A \& B \succ A \wedge B}\,[\&\mathrm{Df}]$$

$$\dfrac{\dfrac{\dfrac{}{A \& B \succ A \& B}\,[\mathrm{Id}]}{A, B \succ A \& B}\,[\&\mathrm{Df}]}{A \wedge B \succ A \& B}\,[\wedge\mathrm{Df}]$$

Suppose we want to get the right conjunction rule

$$\frac{X \succ A, Y \qquad X \succ B, Y}{X \succ A \wedge B, Y}\,[\wedge R]$$

We proceed as follows

$$\dfrac{X \succ A, Y \qquad \dfrac{X \succ B, Y \qquad \dfrac{\dfrac{}{A \wedge B \succ A \wedge B}\,[Id]}{A, B \succ A \wedge B}\,[\wedge Df]}{X, A \succ A \wedge B, Y}\,[Cut]}{X \succ A \wedge B, Y}\,[Cut]$$

Many philosophers and logicians take *assertion* to be the primary speech act, which is used to define others

Others argue that *denial* should be understood as a primitive act on its own

We take logic, in particular valid sequents, as presenting normative relations between assertions and denials

$X \succ Y$ tells us that one should not assert everything in X while denying everything in Y

Positions

$$X \succ Y$$

Positions

$$X \nvdash Y$$

Positions

$$[X : Y]$$

Invalid sequents can be viewed as *positions* in a discourse

Structural Rules

What do the structural rules say in terms of assertion and denial?

$$A \succ A$$

Asserting A clashes with denying A

$$\frac{X, Y \succ Z}{X, A, Y \succ Z}\,[KL]$$

$$\frac{X \succ Y, Z}{X \succ Y, A, Z}\,[KR]$$

If asserting X, Y clashes with denying Z, then asserting more stuff still clashes

Structural Rules

$$\frac{X, A, A Y \succ Z}{X, A, Y \succ Z}\,[WL]$$

$$\frac{X \succ Y, A, A, Z}{X \succ Y, A, Z}\,[WR]$$

If asserting or denying A twice results in a clash, then asserting or denying A just once results in a clash

Structural Rules

$$\frac{X, A, B, Y \succ Z}{X, B, A, Y \succ Z}\,[\mathrm{CL}]$$

$$\frac{X \succ Y, A, B, Z}{X \succ Y, B, A, Z}\,[\mathrm{CR}]$$

If some assertions and denials clash, then asserting and denying the same things in a different order still clashes

$$\frac{X \succ Y, A \qquad A, X \succ Y}{X \succ Y}\,[\mathrm{Cut}]$$

If asserting X and denying A and Y clashes, and asserting X and A while denying Y clashes, then asserting X and denying Y

Contrapositively, if asserting X and denying Y does not clash, then either asserting X and A while denying Y does not clash or asserting X while denying Y and A does not clash

Declaratives Are Not Enough

Belnap argued that a systematic logical treatment of language should give equal weight to imperatives and interrogatives

> 1. THE DECLARATIVE FALLACY
>
> My thesis is simple: systematic theorists should not only stop neglecting interrogatives and imperatives, but should begin to give them equal weight with declaratives. A study of the grammar, semantics, and pragmatics of all three types of sentence is needed for every single serious program in philosophy that involves giving important attention to language.[1]

Attempting to understand all linguistic behavior in terms of assertions commits the *Declarative Fallacy*

The hope is that the view of sequents and logic can be extended to other speech acts

PROOFS AND MODELS

Models as Ideal Positions

How might *truth* enter this picture?

Models are ways of systematically elaborating finite positions into ideal, infinite positions that settle every proposition

In the propositional case, valuations are generated by ideal positions

Positions to models

$$[X : Y]$$

The members of X are *true* and the members of Y are *false*

Positions to models

$$[X : Y]$$

The members of X are *true* and the members of Y are *false* (relative to $[X : Y]$).

Example

$$[p \vee q,\ r : \neg p]$$

Example

$$[p \vee q, r : \neg p]$$

$$\frac{\qquad\qquad}{p \vee q, r \qquad\qquad}$$

$$\overline{\qquad\qquad\qquad}$$

Example

$$[p \vee q,\ r : \neg p]$$

$$\dfrac{}{p \vee q,\ r \quad \textit{true}}$$

$$\overline{\phantom{p \vee q,\ r \quad \textit{true}}}$$

Example

$$[p \lor q,\ r : \neg p]$$

$$\begin{array}{ll} \hline p \lor q,\ r & \textit{true} \\ \neg p & \\ & \\ \hline \end{array}$$

Example

$$[p \vee q, r : \neg p]$$

$p \vee q, r$	*true*
$\neg p$	*false*

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$p \vee q, r$	*true*
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p	

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DEFINITION: A is *true* at $[X : Y]$ iff $X \succ A, Y$.

DEFINITION: A is *false* at $[X : Y]$ iff $X, A \succ Y$.

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$$[p \lor q, r : \neg p]$$

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p	*true*
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Example

$$[p \vee q, r : \neg p]$$

$p \vee q, r$	*true*
$\neg p$	*false*
p	*true*
$p \wedge r$	*true*

DEFINITION: A is *true* at $[X : Y]$ iff $X \succ A, Y$.

DEFINITION: A is *false* at $[X : Y]$ iff $X, A \succ Y$.

$A \wedge B$ is true at $[X : Y]$ iff A and B are true at $[X : Y]$.

$A \vee B$ is false at $[X : Y]$ iff A and B are false at $[X : Y]$.

$\neg A$ is true at $[X : Y]$ iff A is false at $[X : Y]$.

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Classical Logic

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Similarly, r is neither true nor false at $[p : q]$.

Extensions

FACT: If A is neither true nor false in $[X : Y]$
then both $[X, A : Y]$ and $[X : A, Y]$ is invalid,
and each sequent settles A — one as *true* and the other as *false*.

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In general, if $X \not\succ Y$ then either $X, A \not\succ Y$ or $X \not\succ A, Y$.

$$\frac{X \succ Y, A \qquad A, X \succ Y}{X \succ Y}\,[\text{Cut}]$$

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FACT: If $X \not\succ Y$, there's a *maximal* $[\mathcal{X} : \mathcal{Y}]$ extending $[X : Y]$.

Assign truth values relative to *maximal* positions.

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In a slogan, *truth value* $=$ *location in a maximal position*,

Variations

The ideal position construction handles classical logic

With some small adjustments, it can be used to provide models for intuitionistic logic

The system LJ is single-conclusion, but there is a intuitionistic sequent system that has multiple conclusions

The construction with these two systems yield *Kripke models* and *Beth models*

S5

The hypersequent system for S5 can be used to give a similar construction

Each component of a hypersequent describes a possible world

S5 hypersequents

$$\frac{\mathcal{H}[X \succ Y \mid X', A \succ Y']}{\mathcal{H}[X, \Box A \succ Y \mid X' \succ Y']}\ [\Box L] \qquad \frac{\mathcal{H}[X \succ Y \mid\ \succ A]}{\mathcal{H}[X \succ \Box A, Y]}\ [\Box R]$$

$$\frac{\mathcal{H}[X \succ Y \mid A \succ\ ]}{\mathcal{H}[\Diamond A, X \succ Y]}\ [\Diamond L] \qquad \frac{\mathcal{H}[X \succ Y \mid X' \succ A, Y']}{\mathcal{H}[X \succ \Diamond A, Y \mid X' \succ Y']}\ [\Diamond R]$$

Extending positions

Invalid *sequents* $[X : Y]$

Invalid *hypersequents* $[[X : Y], [X' : Y'], \ldots]$

Invalid *sequents* $[X : Y]$

Invalid *hypersequents* $[[X : Y], [X' : Y'], \ldots]$

Say one set of pairs $\mathcal{H}$ extends another $\mathcal{G}$, $\mathcal{G} \preceq \mathcal{H}$, just in case for each component $[X : Y]$ in $\mathcal{G}$, there is a component $[U : V]$ in $\mathcal{H}$ such that $X \subseteq U$ and $Y \subseteq V$

Example: $\{[p : q], [s : r]\}$ is extended by both $\{[p, s : r, q, t]\}$ and by $\{[p, t : q], [s : r, p]\}$

Where are the truth values *now*?

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Maximal *positions* $[\mathcal{X} : \mathcal{Y}]$

Maximal *modal positions* $[[\mathcal{X} : \mathcal{Y}], [\mathcal{X}' : \mathcal{Y}'], \ldots]$

Where are the truth values *now*?

Maximal *positions* $[\mathcal{X} : \mathcal{Y}]$

Maximal *modal positions* $[[\mathcal{X} : \mathcal{Y}], [\mathcal{X}' : \mathcal{Y}'], \ldots]$

A set of pairs $\mathcal{H}$ is a *modal position* iff there is no valid hypersequent $X_1 \succ Y_1 \mid \cdots \mid X_n \succ Y_n$ extended by $\mathcal{H}$

A modal position $\mathcal{H}$ is *maximal* iff there is no modal position $\mathcal{I}$ such that $\mathcal{H} \prec \mathcal{I}$

Building maximal modal positions

The process of expanding a modal position can add formulas to a *component* as well as adding *more components*

Some maximal modal positions, however, will contain finitely many components

The construction builds connected chunks of the S5 canonical model, taking the accessibility relation to be an equivalence relation rather than the universal relation

Building maximal modal positions

As in the classical case, the Cut rule adds new formulas to individual components

The modal rules can extend a position with new components

Building maximal modal positions

$$\frac{\mathcal{H}[X \succ Y \mid X', A \succ Y']}{\mathcal{H}[X, \Box A \succ Y \mid X' \succ Y']}\,[\Box L] \qquad \frac{\mathcal{H}[X \succ Y \mid \ \succ A]}{\mathcal{H}[X \succ \Box A, Y]}\,[\Box R]$$

If $[[X : \Box A, Y], [X_i : Y_i]]$ isn't derivable, then $[[X : \Box A, Y], [\ : A], [X_i : Y_i]]$ can't be either

If the latter were derivable then the former would be by $[\Box R]$

Similarly but for $[\Box L]$, if, e.g. $[[X, \Box A : Y], [X' : Y'], [X_i : Y_i]]$ isn't derivable, then $[[X, \Box A : Y], [X', A : Y'], [X_i : Y_i]]$ can't be

Necessity in maximal modal positions

For a maximal modal position $\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\}$,
$\Box A$ is true at $[\mathcal{X}_i : \mathcal{Y}_i]$ iff A is true at each $[\mathcal{X}_j : \mathcal{Y}_j]$, $j \in I$

($\Rightarrow$) If $\Box A$ is true at $[\mathcal{X}_i : \mathcal{Y}_i]$ and A were not true at some component $[\mathcal{X} : \mathcal{Y}]$, then since $[\mathcal{X} : \mathcal{Y}]$ is a maximal position, we would have $A \in \mathcal{Y}$ but $\Box A \succ \mid \succ A$ is a valid sequent (by $[\Box L]$ from the axiom $\succ \mid A \succ A$), so $[[\mathcal{X}_i : \mathcal{Y}_i], [\mathcal{X}_j : \mathcal{Y}_j]]$ would not be a position, as $\Box A \in \mathcal{X}_i$ and $A \in \mathcal{Y}_j$, so $\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\}$ isn't a position. As it is, whenever $\Box A \in [\mathcal{X}_i : \mathcal{Y}_i]$, A is true at every component.

Necessity in maximal modal positions

For a maximal modal position $\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\}$,
$\Box A$ is true at $[\mathcal{X}_i : \mathcal{Y}_i]$ iff A is true at each $[\mathcal{X}_j : \mathcal{Y}_j]$, $j \in I$

($\Leftarrow$) Suppose $\Box A$ isn't true at $[\mathcal{X}_i : \mathcal{Y}_i]$. So we have $\Box A \in \mathcal{Y}_i$. Take $\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\} \cup [\ : A]$. This is a position. Suppose that it is not. Then there is a derivable hypersequent $\succ A \mid X \succ Y \mid \mathcal{H}$, where $X \subseteq \mathcal{X}_i$, $Y \subseteq \mathcal{Y}_i$ and $\mathcal{H}$ is extended by the other components of the modal position. If that were the case, then by [$\Box$R], we could derive $X \succ \Box A, Y \mid \mathcal{H}$, but that is extended by the original modal position. It is, then, not valid. So, $\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\} \cup [\ : A]$ is a position, so it is extended by a maximal modal position, which must be $\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\}$, as that is not extended by any modal positions. Therefore, for some $j \in I$, $A \in \mathcal{Y}_j$.

A modal position extended by a finite, maximal modal position

$\succ \Box p \vee \Box \neg p$ is not valid

So $[\ : \Box p \vee \Box \neg p]$ is a position

Using the rules, one obtains

$$[[\ :\ ], [\ :\ ], [\ :\ \Box p \vee \Box \neg p]]$$

One can then choose extensions in such a way that no additional components are needed

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Using the rules, one obtains

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One can then choose extensions in such a way that no additional components are needed

Maximality Facts

Each modal position can be extended to a maximal modal position

Each component of a maximal modal position is a maximal position

Each maximal modal position corresponds to a simple Kripke model: $\Box A$ is true at $\{[\mathcal{X}_i; \mathcal{Y}_i] : i \in I\}$ iff A is true in *every* position in the modal position

BEYOND

There are many directions one could go from here

One could add other *connectives* and *predicates*

One could add axioms to obtain *theories*

Truth

$$\frac{A, X \succ Y}{T\langle A\rangle, X \succ Y}\,[TL] \qquad \frac{X \succ Y, A}{X \succ Y, T\langle A\rangle}\,[TR]$$

These rules are inconsistent in classical logic, so one will need to go non-classical to hang onto them

They take complex formulas to atomic formulas, which leads to complications for showing that Cut can be eliminated

Arithmetic

Take a language with =, 0, $'$, $+$, $\times$

$$\succ x + 0 = x$$

$$\succ x + y' = (x + y)'$$

$$\succ x \times 0 = 0$$

$$\succ x \times y' = (x \times y) + x$$

$$x' = y' \succ x = y$$

$$0 = x' \succ$$

$$\frac{X \succ A(0), Y \qquad X, A(x) \succ A(x'), Y}{X \succ A(x), Y}$$

$$\frac{X, A(x') \succ A(x), Y \qquad A(0), X \succ Y}{A(x), X \succ Y}$$

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THANK YOU!

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