Completeness via Metacompleteness

Shawn Standefer

Abstract. We show that all logics in a certain class of modal relevant logics are complete with respect to their reduced frames. The proof uses a combination of the canonical frame method and metacompleteness results.

Keywords. Completeness, Metavaluations, Modal relevant logics, Reduced frames

This paper is dedicated to the memory of Mike Dunn. I had the good fortune of taking a class at Pitt with Mike, on relevant logics, although he preferred the term "relevance logics." That class was the first time I felt like I understood completeness proofs for relevant logics with respect to ternary relational frames. Mike was a wonderful teacher, and his work on relevant logics has influenced my research greatly. I hope he would have enjoyed this paper for connecting a few dots and answering an open question.

1. Introduction

The study of relevant logics has been concerned, from its early days, with modal elements. (Dunn and Restall [15] and Bimbó [7] are excellent overviews of the field of relevant logic, and the interested reader can also consult Anderson and Belnap [1], Read [46], Anderson et al. [2], Routley et al. [56], Brady [9], and Mares [34].) The logic E of [1] is the logic of relevance and *necessity*, and Meyer [38] introduced an alethic modal extension of R, which was algebraized by Dunn [13]. The modal aspect of E has been further investigated by Mares and Standefer [37], Standefer [64], and Standefer and Brady [68]. Meyer conjectured that E and the alethic extension of R would coincide under translation, but this was refuted by Maksimova [28].

Early work on models for relevant logics was concerned with modality, such as Urquhart [70, ch. 5] Routley and Meyer [53], and Fine [18, 359*ff.*]. This concern was revitalized several years later, as evidenced by Fuhrmann [19], Mares and Meyer [35; 36], Mares [29; 32], Meyer and Mares [40], and others. Certain approaches to negation in the setting of frames for relevant logics take negation to be a modal notion, such as Restall [49], Berto [3], and Berto and Restall [4]. Modal relevant logics are not just restricted to alethic modal logics, as demonstrated by Goble [21; 23], Wansing [71], Lokhorst [26; 27], Bilková et al. [5], Sedlár [58; 59], Punčochář and Sedlár [45], Standefer [65; 66], and Savić and Studer [57], for example. Nor are modal relevant logics restricted to entirely propositional concerns, as demonstrated by Ferenz [16]

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and Tedder and Ferenz [69]. Modal relevant logics are an active area of ongoing research.

Another theme in the area of frames for relevant logics is an interest in reduced frames, frames whose set of regular points is a singleton.[1] This goes back to the first papers by Routley and Meyer [53; 54; 55], although it recurs in later works, such as [56] and Slaney [63].[2]

The two themes come together in the work of [19]. Furhmann notes the interest in reduced frames, just before proving a result showing the incompleteness of an $\mathsf{S4}$-ish extension of R with respect to its reduced frames. The result is generalized by Standefer [67], extending the incompleteness to weaker base logics and more modal extensions.[3] This leads naturally to the question of whether any modal relevant logics are complete with respect to some class of reduced frames. In this paper, we will show that there are. En route to proving this result, we will highlight a slight simplification of Slaney's [63] completeness proof for relevant logics that lack the axiom (WI).

The plan of the paper is as follows. In §2, we will present an overview of the logics we are interested in and provide basic axiom systems. In §3, we will present an overview of the ternary relational frames for relevant logics and their modal extensions and we will define reduced frames. Then in §4, we will give an overview of the method of proving Completeness via the canonical model method, including the adjustments made for canonical reduced frames. Metavaluations are used in the latter construction as well as in the main result of this paper, so they are explained in §5. Finally, in §6, we bring the pieces together to prove that there are modal relevant logics that are complete with respect to their reduced frames.

2. Logics

There are many relevant logics, and there are different ways of distinguishing relevant and non-relevant logics. The logics of interest for this paper are the weaker relevant logics. The stronger relevant logics will not play a prominent role, since their modal extensions have been shown to be incomplete with respect to reduced modal frames.

We will work with a language $\mathcal{L}$ built from a countably infinite set of atoms and the connectives $\{\to, \wedge, \vee, \sim\}$ extended to include $\Box$.[4] We will use $\mathcal{L}$ to mean either the basic relevant language or the modal extension, leaving it to context to settle which is under discussion. The basic logic B is the smallest set of formulas containing the following axioms and closed under the following rules.

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[1] See [66] for some discussion of the interest in reduced frames.

[2] Reduced frames are in some work on simplified semantics, such as Priest and Sylvan [44] and Restall [48].

[3] There are incompleteness results for modal relevant logics that do not focus on reduced frames. Goble [22] and Mares [33] both obtain incompleteness results for modal extensions of relevant logics without restricting to reduced frames.

[4] To reduce parentheses, I will adopt the convention that the arrow binds least tightly, followed by conjunction and disjunction, with negation and necessity binding most tightly.

(A1)	$A \to A$	(A7)	$\sim\sim A \to A$
(A2)	$A \wedge B \to A, \quad A \wedge B \to B$	(R1)	$A, A \to B \Rightarrow B$
(A3)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$	(R2)	$A, B \Rightarrow A \wedge B$
(A4)	$A \to A \vee B, \quad A \to B \vee A$	(R3)	$A \to \sim B \Rightarrow B \to \sim A$
(A5)	$(B \to A) \wedge (C \to A) \to (B \vee C \to A)$	(R4)	$A \to B \Rightarrow (B \to C) \to (A \to C)$
(A6)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	(R5)	$A \to B \Rightarrow (C \to A) \to (C \to B)$

There are many different axioms one can add to obtain other relevant logics. Meyer and Routley [41], [56, ch. 4] and Brady [10] provide some examples of common axioms to add to base relevant logics. For the main results of this paper, the upper bound for the strength of the base logic is marked by the addition of the following axioms.[5]

(A9) $A \wedge (A \to B) \to B$ (WI)
(A10) $(A \to B) \to ((B \to C) \to (A \to C))$ (B′)
(A11) $(A \to B) \to (\sim B \to \sim A)$ (Contra)

Adding these three axioms to B gives one the logic C of [56], who show this logic to be complete with respect to its reduced frames. C has the distinction of being the weakest logic to be shown complete with respect to reduced frames using the techniques of [56]. [63] showed how to obtain completeness results for weaker logics, notably those lacking (WI), with respect to their reduced frames.[6]

Given a base logic L, the minimal modal extension L.M is obtained by adding the following axiom and rule.

(Agg) $\Box A \wedge \Box B \to \Box(A \wedge B)$
(Mono) $A \to B \Rightarrow \Box A \to \Box B$

One gets further modal extensions by adding other modal axioms and rules. Some standard ones to be considered below are the following.

(T)	$\Box A \to A$	(B)	$A \to \Box\sim\Box\sim A$
(D)	$\Box A \to \sim\Box\sim A$	(5)	$\sim\Box A \to \Box\sim\Box A$
(4)	$\Box A \to \Box\Box A$	(K)	$\Box(A \to B) \to (\Box A \to \Box B)$
(Nec)	$A \Rightarrow \Box A$		

Adding a set X of the axioms and rule above to L.M will result in the logic L.MX. Below, "L" will at times be used in a way that is indifferent between a base relevant logic and a modal relevant logic, since many of the points do not depend on modal elements being absent. When a modal relevant logic is specifically under consideration, the "L.M" or "L.MX" notation will be used.

The last two items on the list deserve comment, since they are included in normal modal logics whose base logic is classical. The standard relational models for classically based modal logics ensure that (K) and (Nec) are valid. Despite the fact that

[5]This is not a common upper bound for logical strength, as it is properly weaker than T, perhaps the weakest of the well known strong relevant logics defended by Anderson and Belnap. To get T from C, one strengthens (A9) to $(A \to (A \to B)) \to (A \to B)$ (W).

[6]Giambrone [20] made a correction to Slaney's work, but the details do not matter for present purposes.

modal frames for relevant logics use a binary relation to interpret the necessity operator, as is done with relational models for normal modal logics, (K) and (Nec) are not valid.

We will say that a formula A is a theorem of the logic L just in case there is a proof using the axioms and rules of L ending in A. When this is the case, we write $\vdash_{\mathsf{L}} A$. It will be useful, at times, to identify a logic with its set of theorems.

Let us turn to the frames for relevant logics.

3. Frames

We will use ternary relational frames to define validity.[7] For that, we need some definitions.

Definition 1. A ternary relational frame is a quadruple $\langle K, N, R, ^* \rangle$, where $K \neq \emptyset$, $N \subseteq K$, $R \subseteq K \times K \times K$, $^*\colon K \mapsto K$, and which obeys the following conditions, where $a \leq b =_{\mathrm{Df}} \exists x \in N\, Rxab$:

(i) $\leq$ is a partial order (reflexive, transitive, and anti-symmetric);
(ii) if $a \in N$ and $a \leq b$, then $b \in N$;
(iii) if $a \leq b$, then $b^* \leq a^*$;
(iv) $a^{**} = a$; and
(v) if $Rabc$, $d \leq a$, $e \leq b$, and $c \leq f$, then $Rdef$.

The basic frames are for the logic B, defined in §2. Frames for stronger logics can be obtained by imposing frame conditions. We will return to these conditions later in this section.

The main result of the paper deals with modal extensions of relevant logics, so we will define modal frames.

Definition 2. A modal frame is a quintuple $\langle K, N, R, ^*, S \rangle$, where the first four components make up a ternary relational frame and $S \subseteq K \times K$ such that if Sbc and $a \leq b$, then Sac.

We have defined *modal frames* apart from *ternary relational frames* because there are a few points at which it will be useful to have the two notions separate. In particular, the completeness results with respect to reduced frames have mostly been proven for non-modal, reduced ternary relational frames. We will use "frame" indifferently for non-modal ternary relational frames and modal frames.

From frames, whether ternary relational or modal, we obtain models by adding a valuation.

Definition 3. A model M is a pair of a ternary relational frame F and a valuation V, where V is a function from $\mathsf{At} \times K$ to $\{0, 1\}$ such that if $a \leq b$ and $V(p, a) = 1$ then $V(p, b) = 1$. The valuation is extended to the whole language as follows.

- $a \Vdash p$ iff $V(p, a) = 1$;
- $a \Vdash {\sim} B$ iff $a^* \nVdash B$;
- $a \Vdash B \wedge C$ iff $a \Vdash B$ and $a \Vdash C$;
- $a \Vdash B \vee C$ iff $a \Vdash B$ or $a \Vdash C$;

[7] For more on ternary relational frames, see, for example, Restall [50] or Bimbó and Dunn [8].

- $a \Vdash B \to C$ iff $\forall b, c \in K(Rabc \wedge b \Vdash B \Rightarrow c \Vdash C)$;
- $a \Vdash \Box B$ iff $\forall b \in K(Sab \Rightarrow b \Vdash B)$.

The heredity condition on $\leq$ is postulated only for atoms. One can show that in a given model, for all formulas A, if $a \Vdash A$ and $a \leq b$, then $b \Vdash A$. This heredity fact, while important for the overall development of the model theory, will be appealed to only implicitly in what follows.

With that background in place, we can define validity.

Definition 4. A formula A holds in a model M iff $\forall a \in N$, $a \Vdash A$.

A formula A is valid on a frame F iff A holds in every model M built on F.

A formula A is valid in a class of frames $\mathcal{C}$ iff A is valid on every frame F in $\mathcal{C}$.

As suggested by the definition of validity, we are interested in logics in the framework FMLA, that is, as sets of formulas.[8]

The goal of this paper is to demonstrate completeness with respect to reduced frames for a range of logics. So, we will define what it is for a frame to be reduced.

Definition 5. A frame F is *reduced* iff there is a unique $\leq$-minimal point $a \in N$ such that $N = \{b \in K : a \leq b\}$. We will denote the minimal element of N in a reduced frame by 0.

Where $\mathcal{C}$ is a class of frames, $\mathfrak{r}(\mathcal{C})$ is the class of reduced frames in $\mathcal{C}$.

We will say that a class of frames that does not satisfy the condition that all frames be reduced is *unreduced*.

In the definition of ternary relational frames, $\leq$ was required to be a partial order. This can be relaxed to be a pre-order, at the cost of adjusting the definition of being a reduced frame. If $\leq$ is a pre-order, then there may be multiple, $\leq$-equivalent minimal worlds, any one one of which could act as 0.[9]

In discussions of reduced frames, there is a different definition that is sometimes used, according to which N is a singleton, $\{0\}$.[10] In that context, a slightly different definition of heredity is used, one that involves only 0, namely, $a \preceq b$ iff $R0ab$. We can show that this definition of heredity agrees with the usual definition on reduced frames.

Lemma 6. *Suppose $\langle K, N, R, {}^* \rangle$ is a reduced frame. Let $\leq$ be the usual heredity ordering. Define $a \preceq b$ as $R0ab$. Then, $a \leq b$ iff $a \preceq b$.*

The lemma extends to modal frames, with the additional frame condition on S following immediately. There is no difference between the definition of reduced frame used here in terms of N having a $\leq$-least element and the other definition in terms of N being a singleton as far as validity and holding in a model go. This follows from the next lemma.

Lemma 7. *Let $\langle K, N, R, {}^*, S \rangle$ be a reduced frame. Then for any model on the frame, $0 \Vdash A$ iff for all $a \in N$, $a \Vdash A$.*

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[8] See Humberstone [25, 103*ff.*] for more on logical frameworks.

[9] I would like to thank Greg Restall for discussion of this point.

[10] The definition adopted here is used by [50, 304*ff.*].

Proof. The right to left direction is immediate. For the converse, suppose $0 \Vdash A$. Let $a \in N$ be arbitrary. From the definition of 0, $0 \leq a$, so $a \Vdash A$. Therefore, for all $a \in N$, $a \Vdash A$, as desired. ◁

For many logics L, the additional axioms and rules added to B, or to B.M, to obtain L have corresponding frame conditions. For example, the frame condition for (Contra) is $Rabc \Rightarrow Rac^{*}b^{*}$, and the frame condition for (4) is that S is transitive.[11] The class of frames obeying the frame conditions for the axioms and rules of L will be $\mathcal{C}_{\mathsf{L}}$. We will call these L-frames.

In $\mathcal{C}_{\mathsf{L}}$, the axioms and rules of L are sound, i.e., if A is a theorem of L then A is valid in $\mathcal{C}_{\mathsf{L}}$, and further, the logic is complete with respect to the class of frames obeying these conditions, that is, if A is valid in $\mathcal{C}_{\mathsf{L}}$, then A is a theorem of L. Not every axiom and rule has a frame condition to which it corresponds in the present sense. For the present paper, we are, for the most part, focusing on axioms and rules that do have corresponding frame conditions.

Soundness with respect to an unreduced class of frames $\mathcal{C}$ implies soundness with respect to a class of reduced frames, $\mathfrak{r}(\mathcal{C})$. Completeness, however, is another matter. It is a surprising feature of many non-modal relevant logics that they are complete with respect to their reduced frames. The primary result of this paper is that completeness extends to some, but not all, modal extensions of relevant logics.

The logic B is sound and complete with respect to $\mathcal{C}$, the class of all ternary relational frames. Completeness extends to $\mathfrak{r}(\mathcal{C})$.[12] The logic B.M is sound and complete with respect to $\mathcal{M}$, the class of all modal frames. If we let X be KT45Nec, for example, then the logic B.MX is complete with respect to $\mathcal{M}_{\mathsf{B.MX}}$ and in fact it is complete with respect to $\mathfrak{r}(\mathcal{M}_{\mathsf{B.MX}})$, a fact that follows from the results of §6. In §4 we will look at the relevant details of the completeness proof.

4. Canonical Model

The proof of Completeness for relevant logics proceeds via a Henkin-style canonical model method. In this method, one uses the logic L to obtain a large set of appropriate L-theories. One then defines the set of regular points as the set of L-theories containing all the theorems of the logic. One then defines the relations R and S and the operation * in terms of certain formulas being in, or not, certain L-theories. Let us look at the details.

First, let us define L-theories.

Definition 8. A set of formulas X is an L-theory iff (i) if $\vdash_{\mathsf{L}} A \to B$ and $A \in X$ then $B \in X$, and (ii) if $A, B \in X$, then $A \wedge B \in X$.

An L-theory X is *prime* iff $A \vee B \in X$ only if $A \in X$ or $B \in X$.

When proving Completeness, with respect to unreduced frames, one usually proves a lemma showing that there are enough prime L-theories, where there being enough

[11] The particular frame conditions will not matter for the results of this paper, so they will be omitted. For detailed lists of frame conditions, see [41], [56, ch. 4], [50, ch. 11], or Goldblatt and Kane [24], for example.

[12] The simplified semantics of [44] uses reduced frames. A version of the Completeness result is proved there.

implies that for any non-theorem A, there is a regular, prime L-theory that does not contain A. We will not recapitulate those details here, since detailed Completeness proofs for the logics we are interested in can be found elsewhere.[13]

The canonical frame for L is defined as follows, where S is omitted if a non-modal L is under consideration, in which case the language $\mathcal{L}$ is understood not to contain formulas with $\Box$.

Definition 9. The canonical frame for L is $\langle K, N, R, ^*, S\rangle$, where the components are defined as follows.

- K is the set of prime L-theories.
- $a \in N$ iff $\mathsf{L} \subseteq a$.
- $Rabc$ iff for all $B, C \in \mathcal{L}$, if $B \to C \in a$ and $B \in b$, then $C \in c$.
- $a^* = \{B \in \mathcal{L} \colon \sim B \notin a\}$.
- Sab iff $\{B \in \mathcal{L} \colon \Box B \in a\} \subseteq b$.

In the proof of Completeness, the canonical frame for L is shown to be in the class of L-frames, $\mathcal{C}_{\mathsf{L}}$. This holds for the logics whose axioms have been listed above, as well as other logics. The canonical valuation V is defined as $V(p,a) = 1$ iff $p \in a$, and one shows that for all formulas A, $V(A,a) = 1$ iff $A \in a$. This is often called the Truth Lemma, and it requires substantive proof, the details of which need not concern us here. One then uses the fact that there is a regular, prime L-theory a with $A \notin a$, for the target non-theorem A, to conclude that A is not valid in $\mathcal{C}_{\mathsf{L}}$. We then conclude that if A is valid in $\mathcal{C}_{\mathsf{L}}$, then A is a theorem of L.

To adapt the canonical model method to the case where reduced frames are being considered, using the method of Routley et al., one needs to consider some special L-theories. They use L-theories that are closed under T-implications, for some regular, prime L-theory T, where a theory X is closed under T-implication iff $A \to B \in T$ and $A \in X$ only if $B \in X$. L-theories closed under such a theory T are called T-theories. In the canonical frame, one defines K as the set of prime T-theories, with the remaining definitions unchanged.

The proofs demonstrating that the canonical model works, as developed by Routley et al, depend on the theoremhood of (WI), (B), and (Contra) in the target logic. This works for many of the logics stronger than their logic C, including some of the best known stronger relevant logics, such as Anderson and Belnap's R and T. This approach does not work for logics weaker than C, in particular for those lacking (WI), as discussed by Slaney. The question of completeness with respect to reduced frames was, then, left open for those weaker logics for many years. This was, perhaps, unfortunate, since those weaker logics have many virtues.

[63] showed how to prove completeness with respect to reduced frames for many weaker logics, in fact for most of the better known weaker logics.[14] Slaney's approach is somewhat different from that of Routley et al. Rather than require that the target

[13] [50, ch. 5] is a good reference with the relevant details.

[14] Some of the weaker logics are known not to be complete with respect to reduced frames. An example is the logic obtained by adding $A \vee \sim A$ to B. It may be worth noting that $A \vee \sim A$ was included in a logic called "B" by [41], although that axiom was later dropped from the now standard B.

logic L contain the axioms (WI), (B), and (Contra), he defines properties of L-theories that will work in the axioms' absence:

- T is *detached* iff $A \to B \in T$ and $A \in T$ only if $B \in T$;
- T is *affixed* iff whenever $A \to B \in T$, both $(C \to A) \to (C \to B) \in T$ and $(B \to C) \to (A \to C) \in T$; and
- T is *transpositive* iff whenever $A \to B \in T$, $\sim B \to \sim A \in T$.

We then take a prime, detached, affixed, transpositive, regular L-theory T. We can use this theory to define the set K in the canonical frame as the set of all prime T-theories. The other parts of the canonical frame are defined as before.

The remaining issue is showing that there are enough prime theories, in particular that there is an appropriate regular prime theory. The construction of a regular, prime L-theory T, excluding the target non-theorem and obeying the conditions above, via Lindenbaum's lemma runs into a problem: the end result of the construction may not be prime. Slaney's second major innovation was to use metavaluations, to be explained in §5, to obtain an appropriate prime subtheory U of the constructed theory T, building on an idea of Meyer. The resulting theory, U, is regular, prime, obeys the conditions above, and excludes the target non-theorem. The construction of the canonical frame can proceed using U as 0. We are, then, left to provide the details of metavaluations, to which we now turn.

5. Metacompleteness

In this section, we will define metavaluations. We will not present any of the details of the history of metavaluations, an excellent overview of which is provided by Brady [12]. We will begin with the metavaluations presented by [63]. Slaney distinguishes two classes of logics, M1 and M2, depending on the additional axioms and rules included. The axioms and rules for each class of logics are displayed in Table 1.[15]

(B1)	$(A \to B) \wedge (B \to C) \to (A \to C)$	(B6)	$A \to (B \to A)$
(B2)	$(A \to \sim B) \to (B \to \sim A)$	(B7)	$A \Rightarrow B \to A$
(B3)	$(A \to B) \to ((C \to A) \to (C \to B))$	(B8)	$A \to ((A \to B) \to B)$
(B4)	$(A \to B) \to ((B \to C) \to (A \to C))$	(B9)	$(A \to (B \to C)) \to (B \to (A \to C))$
(B5)	$A \to (A \to A)$	(B10)	$A \Rightarrow (A \to B) \to B$

Table 1. Axioms and rules for the M1 and M2 logics.

The M1 logics are obtained by adding zero or more of (B1)–(B7) to B, and the M2 logics are obtained by adding (B10) and zero or more of (B2)–(B9) to B.

Definition 10. A *metavaluation* for L is a pair of functions $\mathfrak{m}(\cdot)$ and $\mathfrak{m}^\star(\cdot)$ from $\mathcal{L} \mapsto \{0,1\}$ such that

- $\mathfrak{m}(p) = 0, \quad \mathfrak{m}^\star(p) = 1, \quad$ for $p \in \mathsf{At}$;
- $\mathfrak{m}(t) = 1, \quad \mathfrak{m}^\star(t) = 1$;
- $\mathfrak{m}(\sim A) = 1 \quad$ iff $\quad \mathfrak{m}^\star(A) = 0, \qquad \mathfrak{m}^\star(\sim A) = 1 \quad$ iff $\quad \mathfrak{m}(A) = 0$;

[15] The metavaluations used here are the metavaluations for the logic. A more general definition is available, which is appropriate for applying metavaluations to arbitrary theories. The extra generality, while interesting and sometimes useful, is not needed here, so it is omitted. I will also note that the tables omit an axiom that does not have a corresponding frame condition for the ternary relational models.

- $\mathfrak{m}(A \wedge B) = 1$ iff $\mathfrak{m}(A) = 1$ and $\mathfrak{m}(B) = 1$,
  $\mathfrak{m}^\star(A \wedge B) = 1$ iff $\mathfrak{m}^\star(A) = 1$ and $\mathfrak{m}^\star(B) = 1$;
- $\mathfrak{m}(A \vee B) = 1$ iff $\mathfrak{m}(A) = 1$ or $\mathfrak{m}(B) = 1$,
  $\mathfrak{m}^\star(A \vee B) = 1$ iff $\mathfrak{m}^\star(A) = 1$ or $\mathfrak{m}^\star(B) = 1$;
- $\mathfrak{m}(A \to B) = 1$ iff (i) $\vdash_{\mathsf{L}} A \to B$, (ii) if $\mathfrak{m}(A) = 1$ then $\mathfrak{m}(B) = 1$, and (iii) if $\mathfrak{m}^\star(A) = 1$, then $\mathfrak{m}^\star(B) = 1$.
  For M1 logics, $\mathfrak{m}^\star(A \to B) = 1$.
  For M2 logics, $\mathfrak{m}^\star(A \to B) = 1$ iff $\mathfrak{m}(A) = 1$ only if $\mathfrak{m}^\star(B) = 1$.

As is clear from the definition, the clause for the conditional differs depending on whether the logic is an M1 logic or an M2 logic.

[63] proves a metacompleteness theorem for the M1 and M2 logics.[16]

Theorem 11 (Metacompleteness)**.** *Let* L *be an* M1 *logic or an* M2 *logic. Then* $\vdash_{\mathsf{L}} A$ *iff* $\mathfrak{m}(A) = 1$.

Seki [61] extends Slaney's metavaluations to modal vocabulary. We will only use Seki's extension for necessity, although he provides clauses for a primitive possibility operator.

In addition to the distinction between M1 and M2 logics, Seki needs to distinguish Ms and Mt logics, depending on the modal axioms and rules that are included. The Ms/Mt distinction is independent of the M1/M2 distinction, so there are four classes of logics one can consider. The basic modal axioms and rules included in Ms and Mt logics are in Table 2.[17]

(C1)	$\Box A \Rightarrow A$	(CoNec)	(C3)	$\Box(A \to B) \to (\Box A \to \Box B)$	(K)
(C2)	$A \Rightarrow \Box A$	(Nec)	(C4)	$\Box(A \to B) \to ({\sim}\Box{\sim}A \to {\sim}\Box{\sim}B)$	
			(C5)	$A \Rightarrow {\sim}\Box{\sim}A$	(Poss)

Table 2. Axioms and rules for the Ms and Mt logics.

The Ms logics are obtained by adding zero or more of (C1)–(C4) to L.M, where L is an M1 or M2 logic, and the Mt logics are obtained by adding (C5) to an Ms logic.

Definition 12. A *modal metavaluation* for a logic L.MX is a pair of functions $\mathfrak{m}(\cdot)$ and $\mathfrak{m}^\star(\cdot)$ from $\mathcal{L}_\Box$ to $\{0,1\}$ that satisfy the conditions to be M1 or M2 metavaluations and also satisfy the following additional conditions.

- $\mathfrak{m}(\Box A) = 1$ iff $\vdash_{\mathsf{L}} \Box A$ and $\mathfrak{m}(A) = 1$.
  For Ms logics, $\mathfrak{m}^\star(\Box A) = 1$.
  For Mt logics, $\mathfrak{m}^\star(\Box A) = 1$ iff $\mathfrak{m}^\star(A) = 1$.

[61] proves a metacompleteness theorem for Ms and Mt logics. He goes on to show that several additional axioms can be added to the different logics, including various Sahlqvist axiom forms. Several of the more common modal axioms can be

[16] Slaney [62] proves metacompleteness for some particular M1 and M2 logics, but the later paper provides the general definition of the classes of logics.

[17] Seki includes a disjunctive meta-rule, if $A \Rightarrow B$, then $A \vee C \Rightarrow B \vee C$, as one of the optional additions to both the Ms and Mt classes. For more on the use of disjunctive meta-rules in reduced completeness proofs, see [9, pp. 7–9].

added to Ms and Mt logics while maintaining metacompleteness. Both Ms and Mt logics can be augmented with (4) ($\Box A \to \Box\Box A$) or (5) ($\sim\Box A \to \Box\sim\Box A$) and remain metacomplete. Mt logics can also be augmented with any of (D) ($\Box A \to \sim\Box\sim A$), (T) ($\Box A \to A$) and (B) ($A \to \Box\sim\Box\sim A$) and still be metacomplete.[18] These results hold regardless of whether the non-modal base logic is an M1 or M2 logic, so the modal element of the metavaluation has considerable freedom from the non-modal base logic.

Metacompleteness, whether for a base logic or for a modal logic, has many consequences. For our purposes, the primary consequence is that metacomplete logics are prime.

Corollary 13. *Suppose* L *is metacomplete. Then* $\vdash_{\mathsf{L}} A \lor B$ *only if* $\vdash_{\mathsf{L}} A$ *or* $\vdash_{\mathsf{L}} B$.

This is the crucial fact that we will appeal to in the completeness results, to which we now turn.

6. Completeness

We are now almost in a position to prove that there are modal relevant logics that are complete with respect to their reduced frames.

Let us say that a frame condition is *reducible* iff it uses only $S, R, {}^*, \leq$, terms for points, existential quantifiers, universal quantifiers, conjunction, disjunction, and the material conditional. In particular, a frame condition is not reducible if it uses a restricted quantification over N that is not used in an instance of $\leq$.[19] For example, the frame condition for (Contra), $Rabc \Rightarrow Rac^*b^*$, is reducible, as is the frame condition for (T), that S is reflexive. By contrast, the frame condition for (Nec), $a \in N \land Sab \Rightarrow b \in N$, is not, since it uses "N" outside of "$\leq$." Similarly, $\exists x \in N\, Raxa$ is not a reducible condition, since it has a restricted quantification on N occurring outside of "$\leq$."

Many of the frame conditions for common axioms for relevant logics are reducible. Adding these axioms, individually or in a group, to B, or to B.M results in a logic that is sound and complete with respect to the class of frames obeying the corresponding conditions. The proof of Completeness for L with respect to L-frames, using the canonical model method, demonstrates that the canonical frame for L belongs to the class of L-frames. For our main result, we will record two lemmas, noting a connection between L-frames and reducible frame conditions.

Lemma 14. *Suppose* $\langle K, N, R, {}^*\rangle$ *is an* L*-frame, all the conditions on* L*-frames are reducible, and there is* $0 \in N$ *such that* $0 \leq a$*, for all* $a \in N$*. Then* $\langle K, N, R, {}^*\rangle$ *is a reduced* L*-frame.*

Proof. All the conditions on B-frames are satisfied in virtue of being an L-frame. Further, the additional frame conditions hold as the frame is unchanged. Therefore, it is a reduced L-frame. ◁

[18] NB: The axiom (B), "B" for "Brouwersche," is a modal axiom, not to be confused with the relevant logic B, "B" for "Basic," or the names for the implicational axioms (B) and (B′), whose designations come from combinatory logic.

[19] It should be clear that a frame condition may be reducible while a logically equivalent condition is not. A more general definition treats a condition as reducible iff it is equivalent to one of the highlighted form.

The lemma still holds if the frame is a modal frame, which we will state separately.

Lemma 15. *Suppose $\langle K, N, R, ^*, S\rangle$ is an L-frame, all the conditions on L-frames are reducible, and there is $0 \in N$ such that $0 \leq a$, for all $a \in N$. Then $\langle K, N, R, ^*, S\rangle$ is a reduced L-frame.*

The proof is the same as the previous lemma, so we omit it. These lemmas tell us that if the canonical frame for L has a $\leq$-minimal world and is an L-frame, then the canonical frame is a reduced L-frame.

We now come to our main lemma.

Lemma 16. *Suppose L is prime. If the canonical frame for L is in a class $\mathcal{C}$ of frames, then the canonical frame is in $\mathfrak{r}(\mathcal{C})$.*

Proof. Suppose L is prime and that the canonical frame for L is in the class $\mathcal{C}$. If L is a modal logic, then the frame includes S as well, defined as above. Since L is prime, $\mathsf{L} \in K$. In fact, $\mathsf{L} \in N$, and for all $a \in N$, $\mathsf{L} \subseteq a$, so $\mathsf{L} \leq a$. Thus, the canonical frame for L is reduced, with L acting as 0, and so the canonical frame is in $\mathfrak{r}(\mathcal{C})$. $\lhd$

The lemma's proof does not establish that there are any prime logics, but that is what the metavaluations do. In particular, the final bit of the argument we need is the corollary from the previous section, namely that metacomplete logics are prime.

Corollary 17. *If L is metacomplete and complete with respect to a class $\mathcal{C}$ of frames that includes the canonical frame for L, then L is complete with respect to $\mathfrak{r}(\mathcal{C})$.*

There are metacomplete relevant logics, so there are prime relevant logics. The axioms and rules for some of these metacomplete relevant logics have reducible frame conditions. Thus, the previous corollary applies to them and they are complete with respect to their reduced frames.

Theorem 18. *Let L be a metacomplete logic such that L is complete with respect to L-frames and its canonical frame is an L-frame. Then L is complete with respect to reduced L-frames.*

Proof. This follows from the previous corollary. $\lhd$

The logic TW is close to C, as their axiomatizations differ only in the presence of (WI). That difference makes a difference, as TW is metacomplete, so many of the modal extensions of TW are complete with respect to their reduced frames. In contrast, modal extensions of $\mathsf{C.M}$ are not complete with respect to their reduced frames, as shown by [67].

It is worth dwelling on the proof above to note a few points. First, we do not use metavaluations on (non-logical) theories, which is to say theories that are not themselves logics, whereas Slaney does, in general, in his construction of a canonical reduced frame.[20] Second, a logic being prime implies that it is in the set K of its canonical frame, which means that the canonical frame is, in fact, reduced. Since the theory L contains no non-theorems, by definition, every non-theorem is refuted at L, considered as 0, in the canonical model for L. As long as the frame conditions

[20] [50, ch. 5] also applies metavaluations to non-logical theories, as does Seki [60].

corresponding to the axioms for L are reducible, then the canonical frame for L is in fact a reduced L-frame. There is, then, no need to use a separate construction for the canonical reduced frame.[21] The canonical frame that is obtained from the more standard completeness proof for the unreduced frames does the job. This part of the argument applies equally to non-modal relevant logics as to modal relevant logics.

In the definition of the Ms and Mt logics, the (Nec) rule is one of the optional extras that can be added included in a metacomplete logic. While the frame condition for (Nec) is not reducible, an alternative condition, $0 \leq a \wedge Sab \Rightarrow 0 \leq b$, can be shown to work in the context of reduced models. This condition is obtained from the condition for unreduced frames by replacing $x \in N$ with $0 \leq x$. In fact, an equivalent, simplified condition can be used instead, $S0a \Rightarrow 0 \leq a$.[22]

The rule (CoNec), $\Box A \Rightarrow A$, is also an optional extra for the Ms and Mt logics. Unlike (Nec), it does not appear to have a corresponding frame condition that the canonical frame is guaranteed to satisfy. Some Ms and Mt logics may not be complete with respect to any class of reduced frames, because of the lack of a suitable frame condition for (CoNec).

Since the axioms (Contra) and (B′) are both available in the M1 and M2 logics, the axiom (WI) presents a stark boundary for completeness with respect to reduced frames for modal relevant logics. There are non-modal relevant logics containing (WI) that are complete with respect to their reduced frames, so the issue, or at least *this issue*, with (WI) only emerges when one looks at modal relevant logics.[23]

The results above imply that there are many modal relevant logics that are complete with respect to their reduced frames, even S4-ish and S5-ish logics. [19] and [67] show that S4-ish and S5-ish extensions of R are incomplete with respect to their reduced frames. When the base logic is weakened to an M1 or M2 logic whose axioms have appropriate frame conditions, the S4-ish and S5-ish extensions are complete with respect to their reduced frames. Since many of the more common weaker logics are metacomplete, including B, DJ, TW, and RW, many of their modal extensions, including S4-ish and S5-ish extensions will be complete with respect to their reduced frames. Thus, the question left open by [67] has a positive answer. Completeness with respect to reduced frames can be had by modal relevant logics, if the logic is metacomplete. This covers a wide range of modal relevant logics, although it does not

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[21] There is no need as far as completeness goes. [63, pp. 405–406] notes a reason to prefer one's reduced models satisfy some contingent truths.

[22] The simplified condition is consequence of the other in virtue of the fact that $0 \leq 0$, setting a in the condition to be 0 and b to be a. The equivalence is obtained by noting that $0 \leq a$ and Sab yields $S0b$, from a frame condition for modal frames, whence $0 \leq b$, by the simplified condition.

[23] (WI) is known to lead to other problems. For example, Meyer et al. [42] show how it leads to triviality in combination with naive set theory via a variation on Curry's paradox. Indeed, Brady [11] uses metavaluations to show that for many weaker relevant logics, the addition of the naive set theory axioms is consistent. It is known that (WI) is not unique for leading to triviality in combination with naive set theory axioms. Restall [47], Rogerson and Restall [52], Bimbó [6], Robles and Méndez [51], Øgaard [43], and Field et al. [17] all discuss different routes to triviality via contraction-like axioms, of which (WI) is a prominent instance.

include all, and generally does not include logics containing the modal confinement axiom, (MC) $\Box(A \vee B) \to {\sim}\Box{\sim}A \vee \Box B$.[24]

Finally, there are some multi-modal relevant logics in the literature. For example, logics of alethic necessity and actuality are suggested by [66]. Seki's metavaluations can be extended with clauses to cover the additional modalities, with the Ms and Mt distinction being duplicated for the new modalities. This opens the doorway for metacompleteness results for these multi-modal logics.[25] Some metacompleteness results will be straightforward consequences of the extensions. Whether logics with interaction axioms, such as $\mathbb{A}B \to \Box B$, are metacomplete will depend on the details of the metavaluations and the classes of logics. For logics that are metacomplete, however, completeness with respect to reduced frames will be available, provided the axioms added to B.M, or its adaptation to a multi-modal setting, have reducible frame conditions.

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[24] Logics containing (MC) have been studied by Dunn [14] and Mares [30, 31, 32], among others.

[25] It also opens the doorway for other sorts of results that use metacompleteness, such as γ-admissibility arguments, showing that if $\vdash_{\mathsf{L}} A$ and $\vdash_{\mathsf{L}} {\sim}A \vee B$, then $\vdash_{\mathsf{L}} B$. For such arguments, see [60]. Although modal relevant logics are not considered by Meyer et al. [39], it would be interesting to see whether any distinctively modal principles result in failures of γ.

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DEPARTMENT OF PHILOSOPHY, NATIONAL TAIWAN UNIVERSITY, TAIPEI, TAIWAN
Email: `standefer@ntu.edu.tw`