Revisiting semilattice semantics

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The operational semantics of Urquhart [1972a,b] is a deep and important part of the development of relevant logics.[1] Formally, the operational semantics provided one of the first intuitive model-theoretic interpretations for the implication of relevant logics.[2] Philosophically, the operational models have a natural interpretation in terms of combining information: The elements of the domain are pieces of information, and a piece of information verifies an implication whenever combining it with any piece of information verifying the antecedent results in a piece of information verifying the consequent. This relation of verification extends naturally to conjunction and disjunction.[3] Completeness results are available as well.[4]

The operational frames come with a set of postulates, many of which can be dropped.[5] Dropping postulates, of course, results in different sets of validities.

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[1] See Dunn and Restall [2002] and Bimbó [2006] for more on the general area of relevant logics.

[2] The period when the operational semantics was developed was quite active for the area of models for relevant logics, with the publication of Maksimova [1969], Routley and Meyer [1972a,b, 1973], and Fine [1974]. See Bimbó and Dunn [2018] and Bimbó et al. [2018] for more on some of the early contributions to the area, including discussion of an early manuscript by Routley, published as Ferenz [2018]. Scott [1973, fn. 33] and Chellas [1975, 143, fn. 17-18] note that Scott had developed a version of ternary relational models earlier but had not published it. I thank Lloyd Humberstone for the references of the preceding sentence.

[3] The extension to conjunction is arguably more natural than the extension to disjunction, a point raised by Humberstone [1988]. Some information can reasonably verify a disjunction by exhaustively splitting into two portions, each of which verifies one of the disjuncts, as opposed to the standard clause used by Urquhart, namely that a disjunction is verified by some information when one or the other disjunct is. I will briefly return to Humberstone's approach to disjunction in the next section.

[4] See Fine [1976a] and Charlwood [1981]. It should be noted that Urquhart [1972b] already had completeness results for the implicational logics.

[5] Urquhart [1972b,a] also considered extending the frames with modal elements, adding a set

The full set of the standard postulates, which will be set out shortly, give the operational models the structure of a join semilattice. The models obeying the full set of postulates will be called *semilattice models*.

Operational models have been studied by others in other contexts. Došen [1988, 1989] studies general groupoid models of substructural logics and connects them with sequent systems. Buszkowski [1986] uses groupoid models to study Lambek calculus.

The goal of this paper is to set out another view on semilattice semantics. Restall and Standefer [20xx] provides a new approach to frame semantics for relevant logics. Our approach uses a binary relation between collections of points and points, rather than the standard ternary relation among points.[6] For this paper, I will focus on the case when the collections of interest are *sets* of points. In this paper, I will show that functional set models coincide with semilattice models in the sense that from a semilattice model one can define a functional set frame, and from a functional set frame, one can define a semilattice model, and repeating the process gets you the original model. Further, I will show that the logic of functional set frames properly extends the logic of (possibly non-functional) set frames. Before getting to these results, I will provide some background on operational and semilattice models and their logic, highlighting some features that are perhaps underappreciated. I will then briefly present an overview of set frames. In the final section, I will present the results, which will, I hope, add to our understanding of the logic of the semilattice models.

1 Semilattice frames

In this section I will define semilattice frames, and the more general operational frames, and provide some comments on their logic. Once the basic formal apparatus has been presented, I will briefly survey the work that has been done in the area, in order to highlight some underappreciated aspects of the semilattice and operational frames.

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of possible worlds and a modal accessibility relation on them in order to interpret the implication of the logic E of entailment. The modal accessibility relation for E obeys the usual $\mathsf{S4}$ conditions, namely reflexivity and transitivity. Urquhart raises some questions about different logics resulting from different conditions put on the modal accessibility relation. Fine [1976b] proves a completeness theorem for the $\mathsf{S5}$ analog of E. This idea is briefly discussed by Mares and Standefer [2017]. As far as I know, there has been no exploration of the modal expansions of the semilattice semantics, or more general operational semantics, with of a primitive modal operator, $\Box$, in addition to the non-modal implication of the underlying logic.

[6] For more on ternary relational frames, see Routley and Meyer [1972a,b, 1973], Routley et al. [1982], or Restall [2000], among others. For discussion of their philosophical significance, see Beall et al. [2012].

Definition 1 (Semilattice frame)**.** *A semilattice frame is a triple* $\langle P, \sqcup, 0\rangle$*, where* $0 \in P$ *and* $\sqcup : P \times P \mapsto P$ *obeys the following conditions.*

(S1) $0 \sqcup x = x$

(S2) $x \sqcup y = y \sqcup x$

(S3) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$

(S4) $x \sqcup x = x$

More general operational frames can be had by dropping any of the latter three conditions. The class of operational frames dropping postulate (S4) is one I will come back to briefly.

Definition 2 (Semilattice model)**.** *A semilattice model is a pair of a semilattice frame* $\langle P, \sqcup, 0\rangle$ *together with a valuation* $V : \mathsf{At} \mapsto \wp(P)$.

A verification relation $\Vdash$ *is a binary relation between points and formulas defined inductively as follows.*

- $x \Vdash p$ *iff* $x \in V(p)$
- $x \Vdash B \wedge C$ *iff* $x \Vdash B$ *and* $x \Vdash C$
- $x \Vdash B \vee C$ *iff* $x \Vdash B$ *or* $x \Vdash C$
- $x \Vdash B \rightarrow C$ *iff for all* $y \in P$*, if* $y \Vdash B$*, then* $x \sqcup y \Vdash C$

Definition 3 (Holds, Validity)**.** *A formula* A *holds in a semilattice model* $\langle P, \sqcup, 0, V\rangle$ *iff* $0 \Vdash A$

A formula A *is valid for semilattice frames iff* A *holds in all semilattice models.*

Write $\models_{\mathsf{SL}} A$ *to mean that* A *is valid for semilattice frames.*

When discussing the operational semantics, the natural point of comparison is with the logic R^{+}, which is the "positive fragment" of R in the vocabulary $\{\rightarrow, \wedge, \vee\}$ and its subvocabularies.[7] R^{+} can be given a Hilbert-style axiomatization as follows.

(R1) $A \rightarrow A$

(R2) $A \wedge B \rightarrow A, A \wedge B \rightarrow B$

[7]The term "positive fragment" is somewhat misleading, since this is naturally taken to include at least the fusion connective, $\circ$, and the Ackermann constant, t, as these are usually included, with negation, in standard forms of the full axiomatization of R. For this paper, I will use "positive fragment" for what is better called "the implication-conjunction-disjunction fragment".

(R3) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(R4) $A \to A \vee B, A \to B \vee A$

(R5) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$

(R6) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

(R7) $(A \to B) \to ((B \to C) \to (A \to C))$

(R8) $A \to ((A \to B) \to B)$

(R9) $(A \to (A \to B)) \to (A \to B)$

(R10) $A, A \to B \Rightarrow B$

(R11) $A, B \Rightarrow A \wedge B$

The logic $\mathsf{RW}^+$is obtained by dropping axiom (R9). The logics T^+ and TW^+, which will figure only briefly below, can be obtained by dropping axiom (R8) from R^+ and RW^+, respectively, and adding $(A \to B) \to ((C \to A) \to (C \to B))$.

Let us call the logic generated by the semilattice semantics UR. There are a few related logics that get discussed in the literature. One of those, URW, is the set of formulas valid in the class of operational frames obtained by dropping the postulate $xx = x$ while retaining the others. The logic URW is most naturally compared with RW^+. Finally, the logics UT and UTW are obtained by adding a binary relation, $\preccurlyeq$, on points to the classes of operational frames for UR and URW, respectively, and modifying the verification clause for implication.[8] These four logics will be called the *operational logics*.

Let us say $\mathsf{L}_{\to}$ and $\mathsf{L}_{\to,\wedge}$ for the fragments of the logic L in the subscripted vocabularies. It turns out that the theorems of $\mathsf{UR}_{\to}$ coincide with those of $\mathsf{R}^+_{\to}$, and, similarly, $\mathsf{UR}_{\to,\wedge}$ coincides with $\mathsf{R}^+_{\to,\wedge}$.

With disjunction, a difference emerges. UR properly extends R^+. By way of example, both

$$(A \to B \vee C) \wedge (B \to D) \to (A \to D \vee C)$$

and

$$(A \to ((A \to A) \vee A)) \to (A \to A)$$

[8]The modification is: $x \Vdash B \to C$ iff for all $y \in P$ such that $x \preccurlyeq y$, if $y \Vdash B$, then $x \sqcup y \Vdash C$. The logics UT and UTW will not feature much below, so further comment on them will be relegated to footnotes.

are theorems of UR that are not theorems of R^+.[9] Humberstone [1988] shows how to modify the semilattice frames along with the verification condition for disjunction to yield frames for which R^+ is sound and complete. Humberstone does this by adding a second operation, $+$, on points to the frames, which operation is used in the verification condition for disjunction, as well as a distinguished unit element for this operation. Humberstone's verification condition for disjunction is the following.

- $x \Vdash B \vee C$ iff there are $y, z \in P$ such that $x = y + z$, $y \Vdash B$ and $z \Vdash C$.[10]

There are, additionally, a few conditions on the modified frames, for which the interested reader should see the cited paper.[11]

UR properly extends R^+, and further, the extension is not captured by a simple axiom scheme in the way that R extends T by the addition of axiom scheme (R8). Rather, the following additional rule is used, where the notation $[A_1, \ldots, A_k] \to B$ means $A_1 \to (\cdots (A_k \to B) \cdots)$: From

$$B \wedge ([A_1 \wedge q_1, \ldots, A_n \wedge q_n] \to C) \to ([B_1 \wedge q_1, \ldots, B_n \wedge q_n]) \to E)$$

and

$$B \wedge ([A_1 \wedge q_1, \ldots, A_n \wedge q_n] \to D) \to ([B_1 \wedge q_1, \ldots, B_n \wedge q_n] \to E),$$

to infer

$$B \wedge ([A_1, \ldots, A_n] \to C \vee D) \to ([B_1, \ldots, B_n] \to E),$$

where the q_i, $1 \leq i \leq n$, are distinct and occur only where displayed.

When viewed as a Hilbert-style axiom system, the charm of UR is, perhaps, not obvious. It adds to R^+ a complex rule, and one might wonder whether the additional theorems are really *that* appealing. The Hilbert-style axiomatization is, I think, not the logic's best side. Indeed, Dunn and Restall [2002, 69] remark, "We forbear taking cheap shots at such an ungainly rule, the true elegance of which is hidden in the details of the completeness proof that we shall not be looking into. Obviously Anderson and Belnap's R is to be preferred when the issue is simplicity of Hilbert-style axiomatisations." The models have a clear appeal, but there is more to say on a proof-theoretic front.

[9] It is worth noting that UT properly extends T^+, as shown by the same examples.

[10] This sort of condition for disjunction also occurs in work on dependence logic and inquisitive semantics. For the former, see Yang and Väänänen [2016]. For the latter, see Ciardelli et al. [2019], as well as Ciardelli and Roelofsen [2011], Punčochář [2015, 2016, 2019], and Holliday [forthcoming]. Humberstone [2019] discusses the issues in a general setting.

[11] The reader should also see the discussion of Humberstone [2011, 905ff.].

Charlwood [1978] presents a natural deduction system for UR.[12] The system uses subscripts, much like the Fitch systems of Anderson and Belnap [1975] and Brady [1984]. Charlwood shows that the natural deduction system for UR admits a normalization theorem. On the basis of that thoerem, he shows that a second additional rule used by Fine [1976a] is in fact admissible and, in light of a proved equivalence with the Hilbert-style axiomatization with the above rule, unnecessary. This is instrumental is showing that the Hilbert-style axiomatization is complete for the semilattice semantics. As far as I know, similar completeness results for Hilbert-style axiomatizations for the other operational logics have yet to be obtained. While completeness for UR has been settled, Urquhart [2016] points out that another important meta-theoretic question remains open, namely whether UR is decidable. Urquhart [1984] famously showed that R was undecidable, and a decidability result for semilattice logic would provide an important contrast.

The normalization theorem shows that the rules fit together in a natural way. Further evidence of the naturalness with which the rules fit together comes from the fact that distribution, $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$, is derivable without a special distribution rule, which is not the case in the Anderson-Belnap-Brady-style indexed Fitch systems. Distribution is, rather, a consequence of the introduction and elimination rules for the connectives involved, which the normal proof having the same form that it would in intuitionistic natural deduction. Indeed, Urquhart [1989] notes this as a point in favor of the semilattice logic.

Sequent systems have received much attention in the study of the logic of operational semantics. A sequent system was already provided by Urquhart [1972b, 31], along with a completeness proof for UR. This system uses indexed formulas and multiple conclusions. Giambrone and Urquhart [1987] presents two subscripted sequent systems for UR, as well as modifications to obtain systems for the other operational logics. These systems are proved equivalent to each other. Kashima [2003] presents cut-free, multiple conclusion, labelled sequent systems for the operational logics.

While the semilattice semantics has gotten a lot of attention, the more general operational semantics should not be ignored. In particular the operational frames that drop postulate (S4), $xx = x$, have a lot of appeal for logicians interested in non-contractive logics.[13] An alternative semantics, *disjoint semantics*, for the non-contractive logic URW was defined by Giambrone et al. [1987]. Disjoint frames keep all the postulates of the semilattice frame and add the postulate

[12] This system and variants for URW, UT, and UTW are presented by Giambrone and Urquhart [1987, 437-438]

[13] In the relevant logic tradition, one of the primary virtues of non-contractive logics is that they support a non-trivial naive set theory. For examples of work in this area, see Brady [1984, 1989, 2006, 2014, 2017] and Weber [2010a,b, 2012, 2013], among others.

- $x \leq y \sqcup z$ iff there are u, w such that $u \sqcup w = x$, $u \leq y$, and $w \leq z$, where $x \leq y$ iff $x \sqcup y = y$.

Two points x and y are said to be *disjoint*, Jxy, iff for all $z \in P$, $z \leq x$ and $z \leq y$ only if $z = 0$. The verification clause for the implication is then modified to the following.

- $x \Vdash B \to C$ iff for all $y \in P$, if Jxy and $y \Vdash B$, then $x \sqcup y \Vdash C$

Disjoint semantics for UTW is obtained by adding in a binary relation and adapting the verification clause, much the same as operational semantics for UT is obtained from the semilattice semantics.

Meyer et al. [1988] shows that over the vocabulary $\{\to, \wedge\}$, the disjoint and contraction-free operational semantics are equivalent and that $RW_{\to,\wedge}$ and $TW_{\to,\wedge}$ are complete with respect to the appropriate classes of operational models. Kashima [2003] shows that disjoint and operational semantics are equivalent even when disjunction is in the language. It is, as far as I know, an open question whether RW^{+} and TW^{+} are complete with respect to the appropriate classes of frames. As remarked by Giambrone and Urquhart [1987, 439], the standard examples where the semilattice semantics goes beyond the ternary relational semantics, or equivalently the standard axiomatizations of the contraction-less relevant logics, turn out not to be valid in contraction-free operational semantics.

This brief survey of work on semilattice semantics will conclude with some recent work. A logic has the *variable sharing property* when all theorems of the form $A \to B$ are such that A and B share a propositional variable.[14] The logic R enjoys the variable sharing property, and variable sharing is usually taken as a necessary condition on being a relevant logic. Weiss [2019] shows that UR has the variable sharing property, as does as an extension with an involutive negation, and he does this via a semilattice structure using arithmetic operations, as opposed to the matrix methods often used, such as the 8-valued algebra used by Anderson and Belnap [1975, 252-254]. Weiss [2020] shows how to conservatively extend the semilattice semantics with a constructive negation.

Given the importance of the operational semantics, it is worth comparing any new semantics for relevant logics to it. In the remainder of the paper, I will provide enough background on collection frames to illuminate the connections and divergences between collection frames and operational frames.

[14] For a general characterization of the variable sharing property, see Robles and Méndez [2011, 2012].

2 Set frames

Let us turn to set frames. As a notational convention, where P is a non-empty set, $\mathcal{P}$ will be the set of all finite subsets of P.

Definition 4 (Set frames)**.** *A set frame is a pair* $\langle P, R\rangle$*, where* P *is a non-empty set of points and* R *is a binary relation on* $\mathcal{P} \times P$ *that obeys the conditions*

Reflexivity $\forall x \in P, \{x\}Rx$*, and*

Transitivity $\forall X, Y \in \mathcal{P}\forall y \in P$*, if* $\exists z(XRz \wedge (\{z\} \cup Y)Ry)$*, then* $(X \cup Y)Ry$*.*

Evaluation $\forall X, Y \in \mathcal{P}\forall y \in P$*, if* $(X \cup Y)Ry$*, then* $\exists z(XRz \wedge (\{z\} \cup Y)Ry)$*.*

The conjunction of Transitivity and Evaluation will be called Compositionality, and set relations obeying Compositionality will be called compositional.

In general, we do not have to impose the first condition,Reflexivity, although dropping it will require generalizing the definition of validity. In this paper there will not be a need to discuss non-reflexive set frames, since the desired equivalence appears to require the condition, so all set frames will be reflexive. Non-empty members of $\mathcal{P}$ will be called *inhabited*.[15] One can consider set frames only on inhabited sets, but I will not do so here. The conditions Transitivity and Evaluation appear to be required for collection frames to work properly, unlike the previous conditions. Their contributions in the development of the framework are many, including the verification of heredity for conditionals, $A \to B$, and validating structural rules in proof systems. An example of their contribution in the present work can be found in the proof of lemma 11, verifying that a frame has a certain property. The interested reader should consult Restall and Standefer [20xx] for details.

Some comments on set frames are in order. First, the binary relations of set frames are defined over finite sets of points because the binary relations of more general collection frames are defined over finite collections. This enables a straightforward connection with more familiar frames, such as ternary relational frames and operational frames, where one is possible. There appears to be no barrier to defining the binary relations over infinite collections, and this generalization will be left to future work.

Next is a comment on the interpretation of the binary relation R. If we think of the points as being bodies of information, we can think of XRy as saying that

[15] Restall and Standefer [20xx] consider many types of collections, not just sets, and there are empty versions of all of these. Especially in the general setting is useful to have a term for distinguishing the collection frames that exclude empty collections and those that include them. The term 'inhabited' is used here for terminological continuity with the cited paper and Restall [20xx].

the result of combining together all the information in X is contained in y. On this interpretation, Reflexivity is a sensible condition, as it is intuitive that the information obtained by combining all the information in $\{x\}$ is contained in x. After all, there is no other point that can supply information available. We can also use the informational interpretation to motivate the two parts of Compositionality. The two parts say that one can combine together the information in $X \cup Y$ in one go or break it into parts and combine together the information in X and combine that with the information in Y. These are especially natural conditions in the context of set frames, since most sets can be broken into parts in a variety of ways.

Finally, we will comment on the relation between set frames and the better known ternary relational frames for relevant logics. Every set frame, as defined above, induces a ternary relational frame, but not every ternary relational frame induces a set frame. This is a point that will come up again later. Set frames are interesting for at least two reasons. First, they permit generalizations that are not obvious with ternary relational frames, namely permitting non-reflexive and inhabited frames. Second, it is comparatively easy to verify whether a structure is a set frame, whereas it is somewhat more involved to verify that a structure is a ternary relational frame that verifies the frame conditions for R.

Compositional set relations are fairly common. For example, suppose that $P = \omega$ and XRy iff $y = \max(X)$, where $\max(\{\ \}) = 0$. This relation is compositional and reflexive. As another example, let $P = \omega^+$, the positive natural numbers, and XRy iff for some $x \in X$, x and y share a prime factor or $y = 1$, when $X \neq \{\ \}$, and $\{\ \}R1$. This relation is also compositional and reflexive. The interested reader should see Restall and Standefer [20xx] or Restall [20xx] for more. The first example is an example of a functional, compositional relation. I will put things more precisely in a definition, which will be important below.

Definition 5 (Functional)**.** *A set frame $\langle P, R\rangle$ is functional iff both*

- *for all $X \in \mathcal{P}$ there is $x \in P$ such that XRx, and*
- *if XRy and XRz, then $y = z$.*

Functional set frames are pleasantly common. Note that functional set frames obey a stronger form of Evaluation.[16]

Uniform Evaluation $\forall X \in \mathcal{P} \exists z \in P[XRz$ and $\forall Y \in \mathcal{P} \forall y \in P$, if $(X \cup Y)Ry$, then $(\{z\} \cup Y)Ry]$.

Uniform Evaluation differs from Evaluation in that the point to which X evaluates, z, is independent of the choice of Y.

[16] I thank Lloyd Humberstone for pointing this out.

Definition 6. *A set model is a pair of a set frame* $\langle P, R\rangle$ *and a valuation* $V : \mathsf{At} \mapsto \wp(P)$ *satisfying the heredity property, if* $x \in V(p)$ *and* $\{x\}Ry$*, then* $y \in V(p)$*. Valuations with this property will be called hereditary. Such a model is said to be built on the set frame.*

A verification relation $\Vdash$ *is a binary relation between points and formulas defined inductively as follows.*

- $x \Vdash p$ *iff* $x \in V(p)$
- $x \Vdash B \wedge C$ *iff* $x \Vdash B$ *and* $x \Vdash C$
- $x \Vdash B \vee C$ *iff* $x \Vdash B$ *or* $x \Vdash C$
- $x \Vdash B \rightarrow C$ *iff for all* $y, z \in P$*, if* $\{x, y\}Rz$ *and* $y \Vdash B$*, then* $z \Vdash C$

As one might expect, preservation of verification along R extends from atoms to all formulas.

Theorem 7 (Heredity)**.** *If* $x \Vdash A$ *and* $\{x\}Ry$*, then* $y \Vdash A$.

Proof. The proof is by induction on the construction of the formula. It is routine. □

In the present setting, I will focus on valid formulas. This permits the use of the following definition for validity, which is a special case of the more general notion.[17]

Definition 8 (Holds, valid)**.** *A formula* A *holds on a set model iff for all* $x \in P$ *such that* $\{\ \}Rx$, $x \Vdash A$.

A formula A *is valid on a set frame iff* A *holds in all models built on that set frame.*

A formula A *is valid in a class of set frames iff* A *is valid on every set frame in that class.*

If A *is valid in the class of all set frames, we will write* $\models_{\mathsf{Set}} A$*. If* A *is valid in the class of all functional set frames, we will write* $\models_{\mathsf{Fun}} A$*.*

With the definition of validity in hand, we can talk about the logic of set frames.

The logic R^+ is sound for the class of set frames, which is to say that if A is a theorem of R^+then $\models A$. The question of completeness, whether whenever we have $\models A$ we also have that A is a theorem of R^+, is still open at the time of writing. In the next section, I will show that UR is sound and complete for the class of functional set frames. It is this contrast, between the logic of set frames, which may be R^+ or may extend it, and the logic of functional set frames, which coincides with UR, that is the main reason for focusing on set frames.

[17] Restall and Standefer [20xx] use a sequent presentation of R^+, and define validity for sequents. The present definition of validity is a special case of the definition they use.

An alternative that is not being pursued here is to use multiset frames, rather than set frames.[18] Multisets differ from sets in distinguishing the number of times an element is a member of that multiset, and multisets and sets are similar in not keeping track of the order.[19] The multisets $[a, a, b]$ and $[a, b, a]$ are identical, but they both differ from the multiset $[a, b]$, as the latter contains a only once and the former both contain it twice. Finite multisets are those that contain a finite number of elements a finite, non-zero number of times. Multiset frames and models are defined much as set frames and models, where the binary R relates finite multisets of points to points and the definition of verification trades sets for multisets. The technical details of the arguments to follow are slightly easier in the context of multiset frames, but the technical advance is in the context of set frames.[20] For that reason the focus is on set frames.

3 Another view on semilattice logic

With the necessary background in place, I can now turn to the task of connecting semilattice models and functional set models. There is a tight connection between them. Every semilattice frame induces a functional set frame, and each semilattice model induces a corresponding functional set model that agrees on all formulas. Similarly, every functional set frame induces a semilattice frame, and the models on those frames agree on all formulas. Broadening out to include non-functional set frames yields a counterexample to a theorem of UR.

Given a semilattice frame $\langle P, 0, \sqcup\rangle$, define $\bigsqcup : \mathcal{P} \mapsto P$ as follows.

$$\bigsqcup X = \begin{cases} 0 & X = \{\,\} \\ x & X = \{x\} \\ x_1 \sqcup (\cdots (x_{n-1} \sqcup x_n)) & X = \{x_1, \ldots, x_n\} \end{cases}$$

When $X = \{x, y\}$, I'll write $x \sqcup y$ for $\bigsqcup X$.

Lemma 9. *Let $\langle P, \sqcup, 0\rangle$ be a semilattice frame. Then $\langle P, R\rangle$ is a functional set frame, where R is defined as follows.*

- XRy *iff* $\bigsqcup X = y$

[18] Multiset frames that obey a contraction principle are similar to the definition of R-frame of Mares [2004, 210]. Given the conditions on R^{n+1}, for $n \geq 2$, the first n arguments can be viewed as forming a multiset related to the final argument.

[19] See Blizard [1988] for an overview of multiset theory. Meyer and McRobbie [1982a,b] uses multisets in an illuminating study of relevant logics.

[20] Every ternary relational frame induces a reflexive multiset frame. As mentioned above, some ternary relational frames can be shown not to induce a reflexive set frame.

Proof. The relation R is well-defined. If $X = Y$, then $\bigsqcup X = \bigsqcup Y$, so XRz iff YRz.

Reflexivity follows from the singleton case of the definition of $\bigsqcup$. It remains to check the two directions of compositionality, for which we show that $\bigsqcup(X \cup Y) = \bigsqcup X \sqcup \bigsqcup Y$, for all $X, Y \in \mathcal{P}$.

Suppose that $X = Y = \{\,\}$. Then $\bigsqcup(X \cup Y) = \bigsqcup\{\,\} = 0 = 0 \sqcup 0 = \bigsqcup X \sqcup \bigsqcup Y$.

Suppose that exactly one of X and Y is $\{\,\}$, say X. Then $\bigsqcup(X \cup Y) = \bigsqcup Y = 0 \sqcup \bigsqcup Y = \bigsqcup X \sqcup \bigsqcup Y$.

Suppose that $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. Then, we have

$$\bigsqcup(X \cup Y) = \bigsqcup\{x_1, \ldots, x_n, y_1, \ldots, y_m\}.$$

In virtue of the semilattice frame conditions we have

$$\bigsqcup\{x_1, \ldots, x_n, y_1, \ldots, y_m\} = \bigsqcup\{x_1, \ldots, x_n\} \sqcup \bigsqcup\{y_1, \ldots, y_m\},$$

with (S2)–(S3) being used to separate out the x_i's from the y_j's, and (S4) being used to duplicate or collapse some elements in case $X \cap Y \neq \emptyset$. Finally, from definitions, we obtain

$$\bigsqcup\{x_1, \ldots, x_n\} \sqcup \bigsqcup\{y_1, \ldots, y_m\} = \bigsqcup X \sqcup \bigsqcup Y,$$

which suffices for the desired identity, $\bigsqcup(X \cup Y) = \bigsqcup X \sqcup \bigsqcup Y$.

For Transitivity, suppose that XRx and $(\{x\} \cup Y)Ry$. Then $\bigsqcup X = x$ and $x \sqcup \bigsqcup Y = y$. It follows that $\bigsqcup X \sqcup \bigsqcup Y = y$, so $\bigsqcup(X \cup Y) = y$, so $(X \cup Y)Ry$.

For Evaluation, suppose that $(X \cup Y)Ry$. Then $\bigsqcup X \sqcup \bigsqcup Y = y$. As $\bigsqcup X = z$, for some z, XRz and $z \sqcup \bigsqcup Y = y$, so $(\{z\} \cup Y)Ry$, as desired.

The functionality conditions are secured by the fact that $\bigsqcup$ is a function.

□

For a given semilattice frame, the *source frame*, say that the preceding construction *induces* the set frame defined, which will be called the *induced frame*. All semilattice frames induce functional set frames. What about the converse? Do all functional set frames induce semilattice frames? Yes, as will be shown. I will prove a lemma first.

Lemma 10. *Let $\langle P, R\rangle$ be a functional set frame. For the x such that $\{\,\}Rx$, $(\{x\} \cup X)Ry$ iff XRy.*

Proof. The left to right direction follows from Transitivity and the assumption that $\{\,\}Rx$. The right to left direction follows from Evaluation and the fact that $X = X \cup \{\,\}$. □

Lemma 11. *Let $\langle P, R\rangle$ be a functional set frame. Then $\langle P, \sqcup, 0\rangle$ is a semilattice frame, where 0 is the x such that $\{\,\}Rx$, and for $x, y \in P$, $x \sqcup y = z$ iff $\{x, y\}Rz$.*

Proof. We need to show that 0 and $\sqcup$ are well defined and obey the appropriate conditions. First, the uniqueness of 0 follows from the functionality of R.

Next, we show that $\sqcup$ is well-defined.

For all $x, y \in P$, there is a z such that $\{x, y\}Rz$, as R is functional. Suppose that $x \sqcup y = z$ and $x \sqcup y = z'$. Then $\{x, y\}Rz$ and $\{x, y\}Rz'$. As R is functional, this implies $z = z'$. We conclude $\sqcup$ is well-defined.

Finally, we show that $\sqcup$ satisfies the conditions on semilattice frames.

Since $\{x\}Rx$ and $\{x\} = \{x\} \cup \{\ \}$, from the preceding lemma, $\{0, x\}Rx$, so $0 \sqcup x = x$.

Since $\{x, y\} = \{y, x\}$, $\{x, y\}Rz$ iff $\{y, x\}Rz$, so $x \sqcup y = z = y \sqcup x$.

As $\{x, x\} = \{x\}$ and $\{x\}Rx$, by definition, $\{x, x\}Rx$, so $x \sqcup x = x$.

Let $\{x, y, z\}Rw$. By Compositionality, for some v, $\{x, y\}Rv$ and $\{v, z\}Rw$, so $x \sqcup y = v$ and $v \sqcup z = w$. By Compositionality again, for some v', $\{y, z\}Rv'$ and $\{x, v'\}Rw$, so $y \sqcup z = v'$ and $x \sqcup v' = w$. So, $x \sqcup (y \sqcup z) = w = (x \sqcup y) \sqcup z$. □

The preceding lemmas show that semilattice frames induce functional set frames and, conversely, functional set frames induce semilattice frames. The induced frames have a close connection with the source frames. I will prove two “round trip” theorems, showing that the constructions given above do not result in any changes when performed in succession.[21] They are, in a sense, inverses.

For the next results, it will be useful to define some notation. Given a source semilattice frame M, let M^σ be the induced functional set frame as defined in lemma 9. Given a source functional set frame N, let N^λ be the induced semilattice frame as defined in lemma 11.

Theorem 12. *Let* $M = \langle P_M, \sqcup_M, 0_M \rangle$ *be a semilattice frame. Then* $M = M^{\sigma\lambda}$.

Proof. The constructions keep the set of points the same, so $P_M = P_{M^\sigma} = P_{M^{\sigma\lambda}}$.

By definition, $0_M = 0_{M^\sigma}$. As $\{\ \}R_{M^\sigma}0_{M^\sigma}$, $0_{M^{\sigma\lambda}} = 0_{M^\sigma}$, whence $0_M = 0_{M^{\sigma\lambda}}$.

Suppose $x \sqcup_M y = z$. This is the case iff $\{x, y\}R_{M^\sigma}z$, which is equivalent to $x \sqcup_{M^{\sigma\lambda}} y = z$. This suffices for the showing that $\sqcup_M = \sqcup_{M^{\sigma\lambda}}$. □

Theorem 13. *Let* $M = \langle P_M, R_M \rangle$ *be a functional set frame. Then* $M = M^{\lambda\sigma}$.

Proof. As in the proof of the previous theorem, the constructions do not change the sets of points, so $P_M = P_{M^\lambda} = P_{M^{\lambda\sigma}}$.

Let $X \in \mathcal{P}$ be arbitrary and suppose XR_My. There are three subcases depending on X.

Suppose $X = \{\ \}$. Then $y = 0_{M^\lambda}$, so $XR_{M^{\lambda\sigma}}y$.

[21] I thank Lloyd Humberstone for the suggestion of proving these theorems.

Suppose $X = \{x\}$. Then $\{x\}R_M x$ implies $x = y$. Then $\bigsqcup X = y$, so $XR_{M^{\lambda\sigma}}y$. The converse is similar.

Suppose $X = \{x_1, \dots, x_n\}$, for some $n \geq 2$. From repeated application of Evaluation, there are $z_1, \dots, z_{n-1}$ such that $\{x_1, x_2\}R_M z_1, \{z_1, x_3\}R_M z_2, \dots, \{z_{n-1}, x_n\}R_M y$. By definition, $x_1 \sqcup_{M^\lambda} x_2 = z_1$, ..., and $z_{n-1} \sqcup_{M^\lambda} x_n = y$. It then follows that $\bigsqcup_{M^\lambda}\{x_1, \dots, x_n\} = y$. Therefore $XR_{M^{\lambda\sigma}}y$, as desired. The converse is similar.

□

The final piece required for the connection between the logics of these two classes of frames is to show that the models built on a source frame and an induced frame agree on the evaluation of formulas. I will now prove that with two lemmas.

Lemma 14. *Let $M = \langle P, \sqcup, 0\rangle$ be a semilattice frame. If $\Vdash_{\mathsf{SL}}$ is a verification relation on M, then $\Vdash_{\mathsf{Set}}$ is a hereditary verification relation on M^σ, where $x \Vdash_{\mathsf{Set}} p$ iff $x \Vdash_{\mathsf{SL}} p$. Moreover, for all $x \in P$ and all formulas A, $x \Vdash_{\mathsf{Set}} A$ iff $x \Vdash_{\mathsf{SL}} A$*

Proof. The coherence of the definition is straightforward from the definition. Heredity then follows as $\{x\}Ry$ implies $x = y$ from the functionality of R.

The second part of the claim is proved by induction on formula structure. The base case holds by definition. The cases where A is of the form $B \wedge C$ or $B \vee C$ are immediate by the inductive hypothesis.

Suppose A is of the form $B \to C$. Then, $x \Vdash_{\mathsf{Set}} B \to C$ iff for all y, z, if $\{x, y\}Rz$ and $y \Vdash_{\mathsf{Set}} B$ then $z \Vdash_{\mathsf{Set}} C$. Let y be an arbitrary point such that $y \Vdash_{\mathsf{SL}} B$. As R is functional, there is a z such that $\{x, y\}Rz$. By the inductive hypothesis, $y \Vdash_{\mathsf{Set}} B$, so $z \Vdash_{\mathsf{Set}} C$. By the inductive hypothesis, $z \Vdash_{\mathsf{SL}} C$. As $\{x, y\}Rz$, $x \sqcup y = z$. Therefore, so $x \sqcup y \Vdash_{\mathsf{SL}} C$. Therefore, $x \Vdash_{\mathsf{SL}} B \to C$.

Suppose $x \Vdash_{\mathsf{SL}} B \to C$. Let y, z be arbitrary points such that $\{x, y\}Rz$ and suppose $y \Vdash_{\mathsf{Set}} B$. By the inductive hypothesis, $y \Vdash_{\mathsf{SL}} B$. Therefore, $x \sqcup y \Vdash_{\mathsf{SL}} C$. Since $\{x, y\}Rz$, $x \sqcup y = z$, so it follows that $z \Vdash_{\mathsf{SL}} C$. By the inductive hypothesis, $z \Vdash_{\mathsf{Set}} C$, which esablishes that $x \Vdash_{\mathsf{Set}} B \to C$.

□

From the preceding lemma, we can see that the logic of functional set frames is contained in the logic of semilattice frames.

Theorem 15. *For all formulas A, $\models_{\mathsf{Fun}} A$ only if $\models_{\mathsf{SL}} A$.*

As there are no conditions on verification relations in semilattice frames, we can prove the following lemma.

Lemma 16. *Let $\Vdash_{\mathsf{Set}}$ be a verification relation on a functional set frame $N = \langle P, R\rangle$ and let $\langle P, 0, \sqcup\rangle$ be N^λ. Define a semilattice verification $\Vdash_{\mathsf{SL}}$ as $x \Vdash_{\mathsf{SL}} p$ iff $x \Vdash_{\mathsf{Set}} p$. The result is a semilattice model. Moreover, for every $x \in P$ and formula A, $x \Vdash_{\mathsf{SL}} A$ iff $x \Vdash_{\mathsf{Set}} A$.*

Proof. The initial portion of the corollary is immediate from the preceding lemma. The moreover portion follows from a straightforward induction on formula complexity. We will present the $B \to C$ case, as it is the only non-trivial one.

Suppose $x \Vdash_{\mathsf{Set}} B \to C$. Then, for all y, z such that $\{x, y\}Rz$, if $y \Vdash_{\mathsf{Set}} B$, then $z \Vdash_{\mathsf{Set}} C$. Let y be arbitrary and suppose $y \Vdash_{\mathsf{SL}} B$. By the inductive hypothesis, $y \Vdash_{\mathsf{Set}} B$. Since R is functional, for some z, $\{x, y\}Rz$, so $z \Vdash_{\mathsf{Set}} C$. By the inductive hypothesis again, $z \Vdash_{\mathsf{SL}} C$. Since $\{x, y\}Rz$, $x \sqcup y = z$, so $x \sqcup y \Vdash_{\mathsf{SL}} C$, which suffices for $x \Vdash_{\mathsf{SL}} B \to C$.

Suppose $x \Vdash_{\mathsf{SL}} B \to C$. Then for all y, if $y \Vdash_{\mathsf{SL}} B$ then $x \sqcup y \Vdash_{\mathsf{SL}} C$. Let y, z be arbitrary points such that $\{x, y\}Rz$. Suppose $y \Vdash_{\mathsf{Set}} B$. By the inductive hypothesis, $y \Vdash_{\mathsf{SL}} B$, so $x \sqcup y \Vdash_{\mathsf{SL}} C$. Since $\{x, y\}Rz$, $x \sqcup y = z$, so $z \Vdash_{\mathsf{SL}} C$. By the inductive hypothesis $z \Vdash_{\mathsf{Set}} C$, which suffices to establish $x \Vdash_{\mathsf{Set}} B \to C$. □

This corollary suffices for the following theorem.

Theorem 17. *For all formulas* A, $\models_{\mathsf{SL}} A$ *only if* $\models_{\mathsf{Fun}} A$.

There is, then, a match between the valid formulas of semilattice frames and those of functional set frames.

One more theorem remains to be proved, showing that the logic of functional set frames properly extends the logic of set frames.

Lemma 18. *There is a formula* A *such that* A *is valid in the class of functional set frames but not valid in the class of set frames.*

Proof. For the formula, we take $(p \to (q \vee r)) \wedge (q \to r) \to (p \to r)$, which is valid in semilattice frames but is not a theorem of R^{+}, as noted by Urquhart [1972a, 163] who attributes it to Dunn and Meyer. A simple non-functional set frame counterexample to this in the class of all set frames can be found. For this counterexample, let $P = \{a, b, c\}$ and R defined as in Table 1. R so defined is a reflexive, compositional set relation. The valuation given in Table 1 is trivially hereditary. It

R	
$\{\}$	b
$\{a\}$	a
$\{b\}$	b
$\{c\}$	c

R	
$\{a, b\}$	a
$\{a, c\}$	a, b, c
$\{b, c\}$	c
$\{a, b, c\}$	a, b, c

$\Vdash$	
a	r
b	q
c	p, r

Table 1: Counterexample

suffices to refute $(p \to (q \vee r)) \wedge (q \to r) \to (p \to r)$ to find a point x at which the antecedent of the implication is true but the consequent is not, for which we will use a. Since $\{a, c\}Rb$, while $c \Vdash p$ and $b \nVdash r$, $a \nVdash p \to r$. It remains to verify that $a \Vdash p \to (q \vee r)$ and $a \Vdash q \to r$. For the former, note that $q \vee r$ is true at all points, although it is in virtue of the q disjunct at b and in virtue of the r disjunct at the other points. For the latter, the only point at which q is true is b, and the only point that $\{a, b\}$ bears R to is a, which has r true. □

This lemma suffices for the desired theorem.

Theorem 19. *The set of formulas valid in the class of all set frames is a proper subset of the set of all formulas valid in the class of all functional set frames, which is* UR*. In symbols, there is a formula* A *such that* $\models_{\mathsf{Fun}} A$ *but* $\not\models_{\mathsf{Set}} A$.

Proof. Immediate from the preceding lemma. □

The results of this section situate set frames with respect to the well known semilattice frames. Functional set models and semilattice models line up neatly. They generate the logic UR. Further, the procedure of inducing one frame type from the other takes you back to where you started after two steps. We can see points in a functional set frame as pieces of information, as suggested in the context of semilattice frames by Urquhart [1972a], and sets of points are collections of information. Combining these collections of information is done via set union, which pleasantly coincides with Urquhart's original notation.

Stepping back, we see that UR is not sound for the class of all set frames. R^+ is sound for the class of all set frames but it is currently unknown whether it is complete with respect to that class.[22] Finally, I will note that essentially the same arguments show the same fit between functional *multiset* frames and operational frames that drop postulate (S4), $xx = x$, but retain the others.

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