Compact sets

Definition 1 (Covering) *Let (S,d) be a metric space, and $E \subseteq S$. We say that the family of open sets $\{G_\alpha\}_{\alpha \in A}$ is an open covering of E iff $\bigcup_\alpha G_a \supseteq E$.*

Example 2 *Consider (S,d) a metric space. $\{S\}$ is an open covering of any E.*

Example 3 *Consider (S,d) a metric space. $\{B(x,r)\}_{x\in E}$ is an open covering of E.*

Example 4 *Consider $(\mathbb{R}, |\cdot|)$ and $E = (0,1)$. $\{(\frac{1}{n}, 1)\}_{n\in\mathbb{N}}$ is an open covering of E.*

Definition 5 (Compact set) *A subset K of a metric space (S,d) is said to be compact if every open covering of K contains a finite subcovering.*

Example 6 *In $(\mathbb{R}, |\cdot|)$, let $A = (0,1]$, which is not compact. Consider the open covering $\{(\frac{1}{n}, 2)\}_{n\in\mathbb{N}}$. Note first that $\bigcup_{n\in\mathbb{N}}(\frac{1}{n}, 2) \supseteq A$. Now consider* any *finite subcovering. Then there exists some $n^* = \max\{n | (\frac{1}{n}, 2)$ is in the subcovering$\}$. Then:*

$$\bigcup_{subcovering} (\frac{1}{n}, 2) = (\frac{1}{n^*}, 2) \not\supseteq A$$

since it does not contain zero.

Example 7 *In $(\mathbb{R}, |\cdot|)$, let $A = [0, \infty)$, which is not compact. Take the open covering $\{(-1, n)\}_{n\in\mathbb{N}}$. However, for any n^* the subcover will only reach $(-1, n^*)$, which does not contain A.*

Definition 8 (Bounded sets) *Let (S,d) be a metric space. We say that a set $E \subseteq S$ is bounded iff there exists $x \in S$ and $R \in \mathbb{R}$ such that $B(x,R) \supseteq E$.*

Example 9 *In $(\mathbb{R}, |\cdot|)$,$(0,1)$ is bounded. $[0,1]$ is bounded. $[0,\infty)$ is not bounded. $\mathbb{N}$ is not bounded.*

Remark 10 *Boundedness does not imply finiteness. The set $A = [0,1]$ is bounded, but it has infinitely many elements.*

Theorem 11 *A compact set is always bounded.*

Proof. Consider K compact, and suppose that it is not bounded. Then $\forall x$, $\forall R$, $B(x,R) \not\supseteq K$. Consider the family $\{B(x,n)\}_{n\in\mathbb{N}}$. this is an open covering of K. For any finite subcovering $\{B(x,k_i)\}_{i=1,..,n}$:

$$\bigcup\nolimits_{i=1,\ldots,N} B\left(x, k_i\right) = B\left(x, k_n\right) \not\supseteq K$$

then K is not compact, which is a contradiction. Then the set K is bounded. ∎

Theorem 12 *A compact set is closed.*

Proof. Consider K compact. Take $x \in K^C$, and consider $y \in K$. We can find $V_Y = B(y, \varepsilon_y)$ and $W_Y = B(x, \varepsilon_y)$ such that $V_Y \cap W_Y = \emptyset$. (In fact, consider $\varepsilon_y = \frac{d(x,y)}{2}$, which allows for balls that never intersect). Then $\{V_Y\}_{y\in K}$ is an open covering of K. Then, since K is compact, there exists a finite set $\{y_1, y_2, ..., y_n\}$ such that $\{V_{Y_i}\}_{i=1,\ldots,n}$ is an open covering of K. Now consider $W = W_{y_1} \cap W_{y_2} \cap ... \cap W_{y_n}$. We now claim that $W \cap \left(\bigcup_{i=1,\ldots,n} V_{y_i}\right) = \emptyset$. To see that this is so, consider $z \in \bigcup_{i=1,\ldots,n} V_{y_i}$. Then $z \in V_{y_j}$ for some $j = 1, ..., n$. Then $z \notin W_{y_i}$. Then $z \notin W = W_{y_1} \cap W_{y_2} \cap ... \cap W_{y_n}$.

Now, if W does not intersect $\left(\bigcup_{i=1,\ldots,n} V_{y_i}\right)$, which is a complete covering of K, then for sure it does not intersect K iteslf. Then $W \cap K = \emptyset$, and $W \subseteq K^C \Rightarrow W \in B(x, \bar{r})$, $\bar{r} = \min\{\varepsilon_{y_1}, \varepsilon_{y_2}, ..., \varepsilon_{y_n}\} > 0$. Then K^C is open, and K is compact and closed. ∎

Theorem 13 *Consider (S,d) a metric space, $K \subseteq S$ compact. If $F \subseteq K$ and F is closed, then F is compact.*

Proof. Consider $F \subseteq K$ and F closed. Take $\{G_\alpha\}_{a \in A}$, an open covering of F. Then $\left\{\{G_\alpha\}_{\alpha \in A}, F^C\right\}$ is an open covering of K. Then there exists some finite subcovering of K, which can be either:

(a) $\{G_{\alpha_i}\}_{i=1,...,n}$

(b) $\left\{\{G_{\alpha_i}\}_{i=1,...,n}, F^C\right\}$

Then:

(a) $\bigcup_{i=1,...,n} G_{\alpha_i} \supseteq K$, but since $F \subseteq K$ then $\bigcup_{i=1,...,n} G_{\alpha_i} \supseteq F$, and the family $\{G_{\alpha_i}\}_{i=1,...,n}$ is a finite subcovering of F, so F is compact.

(b) Since $F^C \cup \{G_{\alpha_i}\}_{i=1,...,n} \supseteq K$, then $\bigcup_{i=1,...,n} G_{\alpha_i} \supseteq F$: if $x \in F$ then $x \in K$ because $F^C \cup \left\{\bigcup_{i=1,...,n} G_{\alpha_i}\right\} \supseteq K$, $x \in F^C$ or $x \in \left\{\bigcup_{i=1,...,n} G_{\alpha_i}\right\}$. Since $x \notin F^C$, it must be that $x \in \left\{\bigcup_{i=1,...,n} G_{\alpha_i}\right\}$. Then $\left\{\bigcup_{i=1,...,n} G_{\alpha_i}\right\} \supseteq F$ and $\{G_{\alpha_i}\}_{i=1,...,n}$ is an open subcovering of F, and therefore F is compact. ■

Corollary 14 *If K is compact and F is closed, then $F \cap K$ is compact.*

Proof. First note that $F \cap K \subseteq K$, and $F \cap K$ is closed, since K is compact and therefore closed. And since the intersection of closed sets is closed, then $F \cap K$ is compact. ■

Theorem 15 *Let (S, d) be a metric space. If $\{K_\alpha\}_{\alpha \in A}$ is a collection of compact sets such that the intersection of every finite subcollection of $\{K_\alpha\}_{\alpha \in A}$ is nonempty, then $\bigcap_{\alpha \in A} K_\alpha$ is nonempty.*

Proof. Consider K, any set in the collection $\{K_\alpha\}_{\alpha \in A}$. We claim that there exists $p \in K$ such that $p \in K_\alpha$ $\forall \alpha \in A$. Suppose not. Then $\forall p \in K$, there exists $\alpha \in A$ such that $p \in K_\alpha^C$. Then $\left\{K_\alpha^C\right\}_{\alpha \in A}$ is an open covering of K (since the K_α's are closed because they are compact, then the K_α^C's are open). Since K is compact, there exists a finite subcovering

$$\left\{K_{\alpha_i}^C\right\}_{i=1,...,n}.$$

But this implies that

$$K \subseteq K_{\alpha_1}^C \cup K_{\alpha_2}^C \cup ... \cup K_{\alpha_n}^C.$$

Then $K \cap K_{\alpha_1}^C \cap K_{\alpha_2}^C \cap ... \cap K_{\alpha_n}^C = \emptyset$. (If $x \in K$, then $x \in K_{\alpha_j}^C$ for some $j = 1, ..., n$. Then $x \notin K_{\alpha_j}$. It follows that $x \notin K \cap K_{\alpha_1}^C \cap K_{\alpha_2}^C \cap ... \cap K_{\alpha_n}^C$ and $K \cap K_{\alpha_1}^C \cap K_{\alpha_2}^C \cap ... \cap K_{\alpha_n}^C = \emptyset$). But this violates the assumption of every finite collection having a nonempty intersection, which is a contradiction.

Then there exists some p such that $p \in \bigcap_{\alpha \in A} K_\alpha$, and $\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$. ■

Remark 16 *The compactness of K_α is critical. Consider $\{K_n\}_{n \in \mathbb{N}}$, $K_n = \left(0, \frac{1}{n}\right)$, which is not compact. Note that $\bigcap_{n=1,...,N} K_{j_n} = \left(0, \frac{1}{j_n}\right) \neq \emptyset$. But $\bigcap_{n \in \mathbb{N}} K_n = \emptyset$.*

Remark 17 *The Nested Interval Theorem (Theorem 17 in Sequences and convergence) is a particular case of Theorem 15.*

Theorem 18 *Let (S, d) be a metric space. If $K \subseteq S$ is compact, then for any sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq K$ there exists a subsequence $\{x_{k_n}\}_{n \in \mathbb{N}}$ such that $x_{k_n} \to x \in K$.*

Proof. Let E be the range of $\{x_n\}_{n \in \mathbb{N}}$, defined as

$$E = \{y | \text{there exists } n \in \mathbb{N} \text{ such that } x_n = y\}$$

(for example, for $x_n = \frac{1}{n}$, $E = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$). We have 2 possible cases:

(a) E is finite.

Then there must exist some element $\bar{x} \in E$ such that $x_n = \bar{x}$ infinitely many times: $x_{k_1} = \bar{x}$, $x_{k_2} = \bar{x}$, $x_{k_3} = \bar{x}$. Then $\{x_{k_n}\}_{n \in \mathbb{N}}$ converges to $\bar{x}$.

(b) E is not finite.

We claim that E has a limit point in K. Suppose not. Then every point $p \in K$ is not a limit point of E. Then for every $p \in K$, there exists a ball $B(x, \varepsilon_p)$ such that $B(p, \varepsilon_p) \cap E = \left\{\begin{array}{c} \emptyset \\ \{p\} \end{array}\right.$. Notice that $\{B(p, \varepsilon_p)\}_{p \in K}$ is an open covering of K (the most basic one). Now consider any finite subcovering. Then it does not cover E (since E is infinite and $B(p,\varepsilon_p)\cap E\subseteq\{p\}$, even less K, so K is not compact, which is a contradiction. So E has a limit point in K.

Consider $\bar{x}\in K$ that is a limit point of E. Then there is a sequence $\{y_n\}_{n\in\mathbb{N}}\subseteq E$ with $y_n\neq\bar{x}$, $y_n\to\bar{x}$. But $E=\{y|\text{there exists } n\in\mathbb{N} \text{ such that } x_n=y\}$. This is equivalent to $y_n=x_{k_n}$, and $\{x_{k_n}\}_{n\in\mathbb{N}}\to\bar{x}$. But $x_{k_n}\in K$, with K closed, then $\bar{x}\in K$. ■

Example 19 *Consider $x_n=(-1)^n$, and note that $\{x_n\}_{n\in\mathbb{N}}\subseteq[-1,1]$, which is compact. Then the sequences $\{1,1,1,...\}$ and $\{-1,-1,-1,...\}$ converge to 1 and -1, respectively.*

Theorem 20 *Let (S,d) be a metric space. If $K\subseteq S$ is such that $\forall\{x_n\}_{n\in\mathbb{N}}\subseteq K$ there exists some subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ with $x_{k_n}\to\bar{x}\in K$, then K is compact.*

Proof. Assigned for homework. ■

Theorem 21 *In $\left(\mathbb{R}^l,d_i\right)$, $i=1,2,3$, every closed and bounded set is compact.*

Proof. Consider K closed and bounded. Since K is bounded we know that $K\subseteq\bigotimes_{i=1}^{l}[a_i,b_i]$, a rectangle in $\mathbb{R}^l$. Now consider $\{x_n\}_{n\in\mathbb{N}}\subseteq K$. Then $\left\{x_n^1\right\}_{n\in\mathbb{N}}\subseteq[a_1,b_1]$. By the Bolzano-Weierstrass Theorem (Theorem 22 in Sequences and convergence) there exists some subsequence $\left\{x_{k_{n_1}}^1\right\}_{n\in\mathbb{N}}$ such that $x_{k_{n_1}}^1\to\bar{x}^1$. Consider now the sequence $\left\{x_{k_{n_1}}\right\}_{n\in\mathbb{N}}$. Looking at the second component of this subsequence, $\left\{x_{k_{n_1}}^2\right\}_{n\in\mathbb{N}}\subseteq[a_2,b_2]$, and by applying the Bolzano-Weierstrass Theorem we obtain a subsequence $x_{k_{n_{1_{n_2}}}}^2\to\bar{x}^2$.

Repeating the process until the l-th component, we get $\left\{x_{k_{n_{1_{n_{2\cdots n_l}}}}}\right\}_{n\in\mathbb{N}}$ such that $x_{k_{n_{1_{n_{2\cdots n_l}}}}}\to\bar{x}=\begin{pmatrix}\bar{x}^1 & \cdots & \bar{x}^l\end{pmatrix}$. Since $\{x_n\}_{n\in\mathbb{N}}\subseteq K$, then $\left\{x_{k_{n_{1_{n_{2\cdots n_l}}}}}\right\}_{n\in\mathbb{N}}\subseteq K$ is closed. Then $\bar{x}\in K$, and by Theorem 20, K is compact. ■

Theorem 22 *Let (S,d) be a metric space and $\{K_a\}_{\alpha\in A}$ a family of sets such that $K_\alpha\subseteq S$ and K_α is compact. Then:*
(a) $\bigcup_{\alpha\in A}K_\alpha$ is compact if A is finite.
(b) $\bigcap_{\alpha\in A}K_\alpha$ is compact.

Proof. (a)

Consider a sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq\bigcup_{\alpha\in A}K_\alpha$. Since A is finite, there exists $\bar{\alpha}\in A$ such that there are infinitely many elements of the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $K_{\bar{\alpha}}$. Now consider the subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ such that $x_{k_n}\in K_{\bar{\alpha}}$ $\forall n\in\mathbb{N}$. Since $K_{\bar{\alpha}}$ is compact, there exists a subsequence $\left\{x_{k_{l_n}}\right\}_{n\in\mathbb{N}}$ such that $x_{k_{l_n}}\to\bar{x}\in K_{\bar{\alpha}}\subseteq\bigcup_{\alpha\in A}K_\alpha$. Then $\bigcup_{\alpha\in A}K_\alpha$ is compact.

(b)

Consider a sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq\bigcap_{\alpha\in A}K_\alpha$. Then, in particular, $\{x_n\}_{n\in\mathbb{N}}\subseteq K_{\alpha_1}$. Since K_α is compact, there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ such that $x_{k_n}\to\bar{x}\in K_\alpha$. We claim that $\bar{x}\in\bigcap_{\alpha\in A}K_\alpha$. Suppose not. Then there exists $\bar{\alpha}\in A$ such that $\bar{x}\notin K_{\bar{\alpha}}$. But $\{x_{k_n}\}_{n\in\mathbb{N}}\subseteq\bigcap_{\alpha\in A}K_\alpha$ (since the sequence contained in the intersection, $\{x_n\}_{n\in\mathbb{N}}\subseteq\bigcap_{\alpha\in A}K_\alpha$). This is,$\{x_n\}_{n\in\mathbb{N}}\subseteq\bigcap_{\alpha\in A}K_\alpha\Rightarrow\{x_{k_n}\}_{n\in\mathbb{N}}\subseteq K_{\bar{\alpha}}\Rightarrow x_{k_n}\to\bar{x}$, but $\bar{x}\notin K_{\bar{\alpha}}$. Then $K_{\bar{\alpha}}$ is not closed, which is a contradiction since $K_{\bar{\alpha}}$ is compact, therefore closed. ■

Theorem 23 *Let (S,d_S), (T,d_T) be two metric spaces, and $f:S\to T$ be a continuous function. If $K\subseteq S$ is compact, then $f[K]$ is also compact.*

Proof. Consider $\{y_n\}_{n\in\mathbb{N}}\subseteq f[K]$. Since $y_n\in f[K]$ $\forall n\in\mathbb{N}$, we know that there exists $x_n\in K$ with $f(x_n)=y_n$ $\forall n\in\mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}\subseteq K$. But since K is compact, there exists some subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ with $x_{k_n}\to\bar{x}\in K$. Since f is continuous, it follows that $f(x_{k_n})\to f(\bar{x})$. And since $f(x_{k_n})=y_{k_n}$, then $y_{k_n}\to f(\bar{x})\in f[K]$ (because $\bar{x}\in K$). Therefore, $f[K]$ is compact. ■

Example 24 *Let (S,d) be a metric space, and $f:S\to\mathbb{R}$ a continuous function. Then, the kernel of f, defined as $Z(f)=\{x\in S|f(x)=0\}$ is closed.*

Proof. Consider $\{x_n\}_{n\in\mathbb{N}} \subseteq Z(f)$ such that $x_n \to \bar{x}$. We now have to prove that $\bar{x} \in Z(f)$. Since $x_n \to \bar{x}$ and f is continuous, then $f(x_n) \to f(\bar{x})$. But since all the sequences $\{x_n\}_{n\in\mathbb{N}}$ are in $Z(f)$, then $f(x_n) = 0$. It follows that $f(\bar{x}) = 0$ and $\bar{x} \in Z(f)$. ■

Example 25 *Let (S, d) be a metric space, with $S = \{f | f : [0, 1] \to \mathbb{R},$ continuous$\}$, and the metric $d(f, g)$ defined as $d(f, g) = \max_{0\le x\le 1} |f(x) - g(x)|$. Then $\bar{B}(0, 1)$ is not compact (although it is closed and bounded).*

Proof. Consider the following sequence of functions $\{f_n\}_{n\in\mathbb{N}}$:

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \le x \le 2^{-n-1} \\ 2^{n+2}\left(x - 2^{-n-1}\right) & \text{for } 2^{-n-1} \le x \le 3\cdot 2^{-n-2} \\ -2^{n+2}\left(x - 2^{-n}\right) & \text{for } 3\cdot 2^{-n-2} \le x \le 2^{-n} \\ 0 & \text{for } 2^{-n} \le x \le 1 \end{cases}.$$

The proof is in 2 steps:
(a) Note first that $\{f_n\}_{n\in\mathbb{N}} \subseteq \bar{B}(0, 1)$, since $d(f_n, 0) = \max_{0\le x\le 1} |f_n(x) - 0| = 1$.
(b) Now,

$$d(f_n, f_m) = \max_{0\le x\le 1} |f_n(x) - f_m(x)| = 1.$$

Then any subsequence $\{f_{k_n}\}_{n\ge 1}$ has also the property that $d(f_{k_n}, f_{k_m}) = 1$. Then the subsequence is not Cauchy, so it does not converge. So the sequence $\{f_n\}_{n\in\mathbb{N}}$ does not have any convergent subsequence, and $\bar{B}(0, 1)$ is not compact. ■

Example 26 *Let (S, d) be a metric space. We say that $E \subseteq S$ is dense in S iff $\forall\varepsilon > 0$ and $\forall x \in S$, there exists $y \in E$ such that $d(x, y) < \varepsilon$.*

(a) Let $f : (S, d_S) \to (T, d_T)$ be a continuous function. If $E \subseteq S$ is dense in S, then $f[E]$ is dense in $f[S]$.

(b) Let $f, g : (S, d_S) \to (T, d_T)$ be two continuous functions such that $f(x) = g(x)$ $\forall x \in E$ and E is dense in S. Then $f(x) = g(x)$ $\forall x \in S$.

Proof. (a) We need to show that $f[E]$ is dense in $f[S]$, or that $\forall z \in f[S]$, $\forall\varepsilon > 0$, there exists some $w \in f[E]$ with $d(w, z) < \varepsilon$.

Consider $\varepsilon > 0$ and $z \in f[S]$. Then there exists $x \in S$ such that $f(x) = z$. By continuity of f, there exists some $\delta > 0$ such that if $d_S(x, y) < \delta$, then $d_T(f(x), f(y)) < \varepsilon$. Since E is dense in S, there exists $\bar{y} \in E$ such that $d_S(x, \bar{y}) < \delta \Rightarrow d_T(f(x), f(\bar{y})) < \varepsilon \Rightarrow d_T(z, w) < \varepsilon$, where $w \in f[E]$.

(b) We need to show that $f(x) = g(x)$ $\forall x \in S$.

Note that

$$d_T(f(x), g(x)) \le d_T(f(x), f(y)) + d_T(f(y), g(y)) + d_T(g(y), g(x)). \tag{1}$$

Consider any $\varepsilon > 0$. There exists δ_1 such that $d_S(x, y) < \delta_1 \Rightarrow d_T(f(x), f(y)) < \frac{\varepsilon}{2}$. There exists δ_2 such that $d_S(x, y) < \delta_2 \Rightarrow d_T(g(x), g(y)) < \frac{\varepsilon}{2}$ (by continuity of f and g). Take $\delta = \min\{\delta_1, \delta_2\}$, and the first and third terms in the right-hand side of (1) are less than $\frac{\varepsilon}{2}$. Since E is dense in S, it is possible to find Y with $d(y, x) < \delta$ and $y \in E$. Then $d_T(f(y), g(y)) = 0$ and

$$d_T(f(x), g(x)) < \varepsilon.$$

Since this is true $\forall\varepsilon > 0$, we have that $d_T(f(x), g(x)) = 0$, then $f(x) = g(x)$. ■

Example 27 *Let (S, d_S) and (T, d_T) be metric spaces, $f : E \subseteq S \to T$, E compact. In $S \times T$ define the metric*

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_S(x_1, x_2), d_T(y_1, y_2)\}.$$

We define the graph of f as

$$gr(f) = \{(x, y) \in S \times T | x \in E \text{ and } y = f(x)\}.$$

Then f is continuous iff $gr(f)$ is compact.

Proof. "$\Rightarrow$"

Assume that f is continuous, and consider $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq gr(f)$. Then the sequence can be expressed as $\{(x_n, f(x_n))\}_{n\in\mathbb{N}}$, where the first components are in E. Since $\{x_n\}_{n\in\mathbb{N}} \subseteq E$, and E is compact, we know that there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ with $x_{k_n} \to \bar{x} \in E$.

Then $f(x_{k_n}) \to f(\bar{x})$, and $(x_{k_n}, f(x_{k_n})) \to (\bar{x}, f(\bar{x})) \in gr(f)$ since $\bar{x} \in E$ and $f(\bar{x})$ is the preimage of $\bar{x}$. But $gr(f) = \{(x,y) \in S \times T | x \in E \text{ and } y = f(x)\}$, then the graph is compact.

"$\Leftarrow$"

Suppose that $gr(f)$ is compact. Then f is continuous. Suppose not. Then there exists some point $\bar{x}$ with a discontinuity. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to \bar{x}$ but $f(x_n)$ does not converge to $f(\bar{x})$. If $f(x_n)$ does not converge to $f(\bar{x})$, then there exists some $\varepsilon > 0$, such that $\forall N \in \mathbb{N}$, there exists some $n > N$ and $d(f(x_n), f(\bar{x})) \geq \varepsilon$. It follows that we can construct a subsequence $\{f(x_{k_n})\}_{n\in\mathbb{N}}$ with the property that

$$d(f(x_n), f(\bar{x})) \geq \varepsilon \ \forall n \in \mathbb{N}. \tag{2}$$

Now consider, in $gr(f)$, the sequence $\{(x_{k_n}, f(x_{k_n}))\}_{n\in\mathbb{N}}$. We claim that there is no subsequence of $\{(x_{k_n}, f(x_{k_n}))\}_{n\in\mathbb{N}}$ that converges to a point in $gr(f)$. Note that any subsequence $\left\{\left(x_{k_{l_n}}, f\left(x_{k_{l_n}}\right)\right)\right\}_{n\in\mathbb{N}}$ has the property that $x_{k_{l_n}} \to \bar{x}$ and $f\left(x_{k_{l_n}}\right)$ does not converge to $f(\bar{x})$ (because of (2)). Then, if $\left(x_{k_{l_n}}, f\left(x_{k_{l_n}}\right)\right) \to (\bar{x}, \bar{y})$ then $\bar{y} \in f(\bar{x})$, and $(\bar{x}, \bar{y}) \notin gr(f)$. It follows that $gr(f)$ is not compact, which is a contradiction. ■

Definition 28 (Upper and lower bounds) *Consider $\mathbb{R}$ or a subset of $\mathbb{R}$. We say that X is an upper bound of $S \subseteq \mathbb{R}$ iff $s \leq X$ $\forall s \in S$. We say that X is a lower bound of $S \subseteq \mathbb{R}$ iff $s \geq X$ $\forall s \in S$.*

Example 29 *Consider $S = [0, 1)$. 1 is an upper bound of S. But so is 2, or 3, or π,... On the other hand, -1, 0, -100,... are lower bounds of S.*

Definition 30 (Supremum) *We say that $X = \sup S$ iff*

(a) X is an upper bound of S.

(b) If Y is an upper bound of S, then $X \leq Y$.

Definition 31 (Infimum) *We say that $X = \inf S$ iff*

(a) X is a lower bound of S.

(b) If Y is a lower bound of S, then $X \geq Y$.

Example 32 *Consider $S = [0, 1)$. Then $\sup S = 1$. Also, $\inf S = 0$. Now consider $S = (0, 1]$. Then, as before, $\sup S = 1$ and $\inf S = 0$.*

Remark 33 $\sup S$ *and* $\inf S$ *do not necessarily belong to the set.*

Remark 34 *If S has no upper bound, we say that $\sup S = \infty$. If S has no lower bound, we say that $\inf S = -\infty$.*

Axiom 35 *If $S \subseteq \mathbb{R}$, $S \neq \emptyset$ and S is bounded from above, then $\sup S$ exists (and it belongs to $\mathbb{R}$).*

Proposition 36 *If $S \subseteq \mathbb{R}$, $S \neq \emptyset$ and S is bounded from below,then $\inf S$ exists (and it belongs to $\mathbb{R}$).*

Proof. Assigned for homework. ■

Definition 37 *If $\sup S \in S$, we say that $\sup S = \max S$ (and also we say that the maximum exists). If $\inf S \in S$, we say that $\inf S = \min S$ (and also we say that the minimum exists).*

Lemma 38 *If $L = \sup S$, then $\forall \varepsilon > 0$ there exists some $x \in S$ such that $x > L - \varepsilon$.*

Proof. Suppose not. Then there exists some $\varepsilon > 0$ such that $\forall x \in S$, $x \leq L - \varepsilon$. Then $L - \varepsilon$ is an upper bound. But this is a contradiction, since L is the supremum, so it is the lowest upper bound. ■

Proposition 39 *Consider $S \subseteq \mathbb{R}$ and the usual metric in $\mathbb{R}$. If $\sup S$ exists, then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq S$ such that $x_n \to \sup S$.*

Proof. By Lemma 38, there exists $x_1 > L - 1$, $x_1 \in S$. Also, there exists some $x_2 > L - \frac{1}{2}$, $x_2 \in S$. This reasoning can be repeated up to $x_n > L - \frac{1}{n}$, $x_n \in S$. Then $\{x_n\}_{n \in \mathbb{N}} \subseteq S$ and

$$L - \frac{1}{n} < x_n \leq L$$

and by Theorem 13 in Sequences and convergence, since $L - \frac{1}{n} \to L$ and $L \to L$, then $x_n \to L$. ■

Theorem 40 *Let (S, d) be a metric space, $X \subseteq S$ compact and $X \neq \emptyset$. If $f : X \to \mathbb{R}$ is continuous, then there exists x^* such that $f(x^*) = \sup_{x \in X} f(x)$, y^* such that $f(y^*) = \inf_{x \in X} f(x)$. In this case we say that*
$f(x^*) = \max_{x \in X} f(x)$,
$f(y^*) = \min_{x \in X} f(x)$.

Proof. The proof is done in 3 steps.
(a) Note that

$$\sup_{x \in X} f(x) = \sup f[X]$$

where $f[X] = \{y \in \mathbb{R} | \text{there exists } x \in X \text{ with } f(x) = y\}$.
(b) Since f is continuous and X is compact, we know that $f[X]$ is compact. In particular, it is bounded and it is nonempty. It follows that $\sup f[X]$ exists.
(c) By Proposition 39, there exists $\{y_n\}_{n \in \mathbb{N}} \subseteq f[X]$ such that $y_n \to \sup f[X]$. Since $f[X]$ is closed (since it's compact), we know that $\sup f[X] \in f[X]$. Then, by definition of $f[X]$, there exists some x^* such that $f(x^*) = \sup f[X] = \sup_{x \in X} f(x) = \max_{x \in X} f(x)$. ■

Remark 41 *The hypotheses are important. If f is not continuous, then it does not attain a maximum. The same applies when X is not closed, or when X is not bounded.*

Example 42 *Consider the problem of maximizing a function $U(x_1, x_2, ..., x_L)$ subject to $p_1 x_1 + p_2 x_2 + ... + p_L x_L \leq I$, $x_i \geq 0$ $\forall i \in \{1, 2, ..., L\}$. If $U : \mathbb{R}^L_+ \to \mathbb{R}$ is continuous and $p_1, p_2, ..., p_L > 0$, then there exists $(x_1^*, x_2^*, ..., x_L^*)$ such that $U(x_1^*, x_2^*, ..., x_L^*) = \max U(x_1, x_2, ..., x_L)$.*
We are working in $\mathbb{R}^L_+$ with U continouous. We only need to prove that the set

$$\{(x_1, ..., x_L) | p_1 x_1 + ... + p_L x_L \leq I \text{ and } x_i \geq 0 \ \forall i \in \{1, ..., L\}\}$$

is compact. So we need to check that it is bounded and closed.
(a) Bounded
Consider $d_3(x, y) = \max_{i=1,...,L} |x_i - y_i|$. If $(x_1, ..., x_L) \in X$ then $0 \leq x_1 \leq \frac{I}{p_1}$, $0 \leq x_2 \leq \frac{I}{p_2}$, ..., $0 \leq x_L \leq \frac{I}{p_L}$. But note that $0 \leq x_1 \leq \frac{I}{p_1} \leq \frac{I}{\min\{p_1,...,p_L\}}$, ..., $0 \leq x_L \leq \frac{I}{p_L} \leq \frac{I}{\min\{p_1,...,p_L\}}$. Then:

$$d(x, 0) \leq \frac{I}{\min\{p_1, ..., p_L\}}$$

It follows that $\forall x \in X$,

$$x \in B\left(0, \frac{I}{\min\{p_1, ..., p_L\}} + 1\right)$$

So it is bounded.
(b) Closed
Note that the set X can be expressed as

$$X = \mathbb{R}^L_+ \cap \{(x_1, ..., x_L) | p_1 x_1 + ... + p_L x_L \leq I\}.$$

And note also that $\mathbb{R}^L_+$ is closed, so it remains to prove that $\{\cdot\}$ is closed for X to be closed (since the intersection of closed sets is closed). Consider $g : \mathbb{R}^L_+ \to \mathbb{R}$, $(x_1, ..., x_L) \to p_1 x_1 + ... + p_L x_L$, g continuous. But $\{\cdot\}$ is $g^{-1}[(-\infty, I]]$. Since $(-\infty, I]$ is closed, then $g^{-1}[(-\infty, I]]$ is closed and X is closed.
It follows that X is compact and by Theorem 40, there exists $(x_1^*, x_2^*, ..., x_L^*) \in X$ such that $U(x_1^*, x_2^*, ..., x_L^*) = \max_{(x_1,...,x_L) \in X} U(x_1, x_2, ..., x_L)$.