A Hamiltonian theory for whistler chorus wave in the magnetosphere

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Abstract

In this paper, we develop a novel Hamiltonian theory for the nonlinear resonant interactions between the energetic electrons and chorus wave in the magnetosphere. Our theory studies the dynamics of the deeply trapped resonant electrons and the slowly varying coherent wave envelope in the reference frames moving with the local resonance. We construct new canonical transformation to separates the motion scales, and the Hamiltonian of the resonant particle is transformed to those on each local reference frames. The Vlasov equation of the local distribution function is readily given and coupled with the slowly varying envelope equation derived from the Ampere's law self-consistently. We further discuss the onset situation in which we consider the early chirping stage with small frequency variation. For the onset stage, we can set the reference frame with the resonance velocity of the linear most unstable wave, with time independent thus simplifies the theoretic descriptions. We also consider the adiabatic regime and the waterbag approximation for the evolution of resonant phase space structure, it demonstrates the chirping behavior of the chorus wave packet. Based on our scale separated theoretic description, novel numerical simulation method can be readily constructed. It is able to provide high resolution in phase space with affordable computation cost, which is urgently required to reveal the underlying physics of the nonlinear frequency chirping problem.

I. INTRODUCTION

Whistler-mode chorus is a special electromagnetic emission in whistler range of frequency which are frequently observed in the planetary magnetosphere [1, 2]. The chorus is related with the energy transference between wave and energetic electrons, thus plays a key role in particle acceleration processes in the radiation belt [3–5] and pulsating and diffuse aurora in the atmosphere [6–8]. Thus it has received a significant research interest.

The physics of the chorus wave in essence are the nonlinear wave-particle interactions between the resonant trapped electrons and the whistler waves. The similar nonlinear wave particle interactions such as the Alfvén wave instabilities [9, 10], have been studies extensively in the fusion related plasma. Such instabilities also involve the mode frequency sweeping and lead to premature ejection of alpha particles that deteriorate plasma confinement [11]. The dynamics of nonlinear resonant particle can be modeled by Berk-Brizman

(BB) model [12], which is a general theoretic model based for the bump-on-tail (BOT) paradigm. The original version and the advanced versions [13] focus on the uniform regime, where the wave is treated as stationary with fixed wave number k. The spontaneous hole and clump structure and their evolution in the phase space were revealed and well explains the frequency chirping of the excited wave. For the chorus wave case, the phase space behavior of the resonant electrons also plays an important role for the chirping behavior. It has been long time discussed [14] and quantitatively explains the frequency chirping rate and nonlinear wave growth of chorus wave at various locations along the magnetic field [15, 16].

However, unlike the BB model for the BOT paradigm, a key feature for the chorus wave in the planetary magnetosphere is the involved inhomogeneous magnetic field. It was first noticed by Helliwell in his kinematic theory of the ”consistent-wave” concept in the 1960s [17]. It elucidates the change of chirping rate with respect to magnetic field inhomogeneity. In more recent studies [18, 19], it is found that the frequency chirping behavior can be controlled by changing the magnetic field configuration. Both downward chirping and bi-direction chirping were reproduced. And a phenomenological chirping model called Trap-Release-Amplify (TaRA) model was proposed and explained how frequency chirping occurs and why chirping direction with respect to the sign of magnetic field gradient. Although the model has successfully explained chirping behavior in various planetary magnetosphere, including that on Mars [20], it raises the intriguing question of how rapidly varying chirping elements and resonant electrons are influenced by the background magnetic field's inhomogeneity. This is despite the fact that the scale of the field's inhomogeneity is significantly larger than the characteristic scales of rapid changes and fine structure. Besides, numerical simulations of artificially triggered emissions show different mechanism for the formation of chorus wave [21, 22]. The wave source moves with a velocity that is a combination of the group velocity and resonance velocity, rather than the velocity alone. This suggests that the chirping may not be emitted by the released particle.

Anyhow, in current theoretical and numerical studies, many questions still remain regarding the fine structures of the chorus observed in the magnetosphere. [23]. The chirping mechanism and nonlinear wave particle interaction questions are also undercover as pointed by Tao, et al. in the recently studies [24, 25]. Additional information about the particle dynamics and wave evolution such as high resolution phase-space structure, fine wave number and frequency results, are still desired under current models. However, due to introduce background magnetic field inhomogeneity, the chorus is essentially a multiple problem. Finding an appropriate method to separate the scales of resonant particles is a complex task because of the intricate interactions between waves and particles along the inhomogeneous magnetic field. Challenges and the need have spurred the development of a new theoretical and numerical model.

In this paper, we go over the dynamics of resonant electrons and the evolution of whistler wave in Earth inner magnetosphere. The key feature in our theory is the scale separation of electron motion to the slowly one along the magnetic field and the fast wave interaction. To do so, we bulid a Hamiltonian theory in the reference frame moving with local resonance. We divide the continuous spatial domain of the single magnetic field line into a group of cells representing a group of local reference. By expanding the phase of wave and the resonance electrons, a canonical Hamiltonian on the cell is constructed which naturally separated the fast and slowly varying motion of resonant electrons. The canonical motion equations are then derived which governs the dynamics of resonant electrons. The evolution of whistler-mode wave is determined by the Ampere equation, in which the wave is expressed in its eikonal form, thus, the frequency and wave number can be given directly and accurately.

We organize our paper as follows. In section II, we present the Hamiltonian theory and derive the Vlasov equation for the resonant particle distribution, and the wave equation for the slowly varying wave envelope. In section III, we discuss the onset stage of the chirping, which gives a more brief form for the local Hamiltonian, and derives an integral of motion for the resonant electrons interacte with the chirping wave in an inhomogeneous magnetic field. We also discuss the adiabatic regime for the resonant electron phase space behavior, and give the chirping rate at typical parameters. Finally, the summary is presented in section IV.

II. THEORY

We consider a single electron with mass m_e gyrating in a weakly inhomogeneous magnetic field. It interacts with a small amplitude banded traveling wave with varying phase ϕ_q. The wave vector potential can be represented in Fourier integral

$$\mathbf{A}(s,t) = \int_{-\infty}^{\infty} \mathrm{d}q \ \mathbf{A}\left(q,t\right) \exp(\imath\phi_q) \ , \tag{1}$$

where ω is the wave frequency, s is the coordinates along the field line, q is the Fourier harmonic number, and $\phi_q \equiv \int^t \omega(q,\tau)d\tau - qs$. Here, we concern with the resonant interactions along a one-dimensional magnetic field line, thus s and q are in scalar form, for simplicity.

A. The local Hamiltonian theory

Considering the perturbed wave field is much smaller than the background field, we can treat the system perturbatively as the equilibrium part and the perturbed part. The former describes the particle motion in a slowly-varying magnetic field, and it is typically associated with three adiabatic invariants associated with the unperturbed equilibrium, namely the magnetic moment, the longitudinal invariant, and the flux invariants. They correspond to the gyromotion, the mirror bounce motion, and the drift motion perpendicular to the magnetic field. Since the timescale of perpendicular drift motion is very slow compare to the time period of the wave instabilities, we can neglect drift motion, and leave the equilibrium Hamiltonian as

$$H_0(\varphi, \mu; s, p_\parallel) = \frac{p_\parallel^2}{2m_e} + \mu\omega_{ce}(s) \ , \tag{2}$$

where φ is the particle gyroangle, $p_\parallel$ is the parallel momentum, ω_{ce} is the particle gyrofrequency, and $\mu \equiv mv_\perp^2/2\omega_{ce}$. While the latter describes the interaction with the banded wave, and the Hamiltonian can be similarly represented in a Fourier integral form

$$\delta H(\varphi, \mu; s, p_\parallel; t) = \frac{1}{2}\left(\int_{-\infty}^{\infty} \mathrm{d}q \ \delta H\left(\mu, p_\parallel, q; t\right) e^{\imath(\phi_q - \varphi)} + \mathrm{c.c}\right) \ , \tag{3}$$

where $\phi_q - \varphi$ is the difference between the wave angle and the gyroangel. Note that the perturbed field is much less than the equilibrium field, we still regard s and ϕ as canonical coordinates, and $p_\parallel$ and μ as canonical momentum for the total Hamiltonian.

The wave form and the Hamiltonian in Eqs. (1) and (3) involves wave phase ϕ_q and the particle gyrophase φ. Both of them vary fast in the laboratory frame of reference. While the wave-particle interaction scale and the adiabatic motion scales in the equilibrium magnetic field are considerablely slower. We here example several typical motion scales and their characteristic time and length. There has wave oscillation scale, as in frequency ω and wavenumber k; particle gyration scale, as in ω_{ce}; wave trapped particle bounce frequency ω_{b}; wave chirping scale, as in $\dot{\omega}/\omega$, $\nabla k/k$, and $\nabla a/a$; background spatial nonuniformity scale, as in $\nabla B/B$; particle parallel bouncing scale ω_B and L_{B}; particle transverse drift scale, as in

ω_D and $L_{\rm d}$. For the chorus wave in the Earth's dipole field, the relations of the characteristic scales are

$$\begin{aligned} \omega_{\rm ce} \sim \omega \gg \omega_{\rm b} \sim \dot{\omega}/\omega \gg \omega_{\rm B} \gg \omega_{\rm d} \ , \\ k \gg \nabla k/k \sim \nabla a/a \gg \nabla B/B \sim L_{\rm B}^{-1} \gg L_{\rm d}^{-1} \ . \end{aligned} \tag{4}$$

The last scale corresponds to the third adiabatic invariant, which is well-conserved in our studies. In contrast, the second to last one corresponds to the longitudinal invariant and is not constant of motion due to the perturbation in our study.

Our Hamiltonian theory aim to separate those motion scales, and focus on the resonant wave-particle interaction in such circumstance. Note that the resonant interaction indicates the particle and the wave stay enduringly and the motion scale seen by the particle is considerably slow. Therefore, if we choose a reference frame moving in the local resonant velocity, the resonance particles are at rest and see a nearly constant wave phase. For the inhomogeneous magnetic field, we can apply the local reference frames for a discretized finite length subdomains, referred as the "cell". We divide the continuous s into numerous cells centering at s_i where i is the index of the cell. The cell moves in the resonance velocity, i.e.,

$$\frac{{\rm d}s_i}{{\rm d}t} = v_r(s_i) \equiv \frac{\omega_i\left(k_i(t),t\right) - \omega_{ce}\left(s_i(t)\right)}{k_i(t)} \ , \tag{5}$$

Each cell represents a local resonance reference frame, and within a cell, we can locally expand the related quantities with respect to the cell center. The phase ϕ_q is then splits into the fast and slowly varying terms, i.e., $\phi_q \simeq \phi_f + \phi_s$. We recall the fast varying phase

$$\phi_q \equiv \int^t \omega(q,\tau)d\tau - qs \ . \tag{6}$$

For the triggering frequency chirping emission, although the background wave has a broad spectrum, the unstable mode is narrow in spectrum space. Thus we treat the wave spectrum as locally centered at $k_i(s_i,t)$ along the magnetic field line, and we can expand the phase with respect to the cell center. The Taylor expansion of the frequency in the first term up to the first-order is

$$\omega(q,\tau) \simeq \omega_i\left(k_i(\tau),\tau\right) + \frac{\partial\omega}{\partial k_i}\left(q - k_i(\tau)\right) \ . \tag{7}$$

We further expand qs and keep the terms up to the first-order,

$$\begin{aligned} qs &= (k_i + (q-k_i))(s_i + (s-s_i)) \\ &\simeq qs_i + k_i(s-s_i) \\ &= q\int^t v_r(\tau)d\tau + k_i(s-s_i) \ . \end{aligned} \tag{8}$$

The derivation is under the condition that the width of cell l is enough small, so that the phase difference due to finite spreading of spectrum satisfies $(q - k_i(t))\, l \ll 1$.

The expansion of ϕ_q then becomes

$$\phi_q \simeq \int^t \mathrm{d}\tau \left(\omega_i \left(k_i(\tau), \tau \right) + \frac{\partial \omega}{\partial k_i} \left(q - k_i(\tau) \right) \right) - q \int^t v_r(\tau) d\tau - k_i(t) \left(s - s_i(t) \right) \ . \tag{9}$$

Applying the first-order cyclotron resonant condition at cell center,

$$\omega_i - k_i v_r - \omega_{ce} = 0 \ , \tag{10}$$

and replacing ω_i, we have

$$\phi_q \simeq \int^t d\tau \left(\omega_{ce} \left(s_i(\tau) \right) + k_i(\tau) v_r(\tau) + \left(q - k_i(\tau) \right) \frac{\partial \omega}{\partial k_i} \right) - q \int^t v_r(\tau) d\tau - k_i(t) \left(s - s_i(t) \right) \ . \tag{11}$$

It is clear that the phase has been split into the fast and slowly varying scale, i.e.,

$$\begin{aligned} \phi_q &\simeq \phi_f + \phi_s \\ \phi_f &= \int^t \omega_{ce} \left(s_i(\tau) \right) d\tau - k_i(t) \left(s - s_i(t) \right) \\ \phi_s &= \int^t \left(q - k_i(\tau) \right) \left(\frac{\partial \omega}{\partial k_i} - v_i(\tau) \right) d\tau \ . \end{aligned} \tag{12}$$

For the slowly varying chirping wave, ϕ_s describes the slow variation of the wave kernel, which is related to the changing wave number and the velocity difference observed in the cell reference frame, i.e., $\int^t \Delta v \Delta k \mathrm{d}\tau$. On the other hand, the term ϕ_f is qualitatively equal to $\omega_i t - k_i s_i$ is the fast varying wave phase inside the envelope.

To obtain the dynamics in the local reference frame, we have to change the Hamiltonian on the global domain to that represented on each individual local reference frames. The form of the separated phase hints us to construct a new canonical transformation given by a generating function of the second kind,

$$F_i(s, \varphi; \Omega, \mathcal{J}; t) = \left(\varphi - \phi_f \right) \left(\Omega + \Pi_i(t) \right) + \left(\varphi - \int^t \omega_{ce} \left(s_i(\tau) \right) \mathrm{d}\tau \right) \mathcal{J} \ , \tag{13}$$

where s and φ are the old canonical coordinates, while Ω and $\mathcal{J}$ are the canonical momentum of the new Hamiltonian. $\Pi_i(t)$ is a to-be-determined function only depend on time. We will show that the choice of Π_i depends on the resonance velocity, which converts the dynamics to each local resonance reference frames. According to the definition of the generating function

[26], we can obtain the new canonical coordinates are

$$\begin{aligned} \xi &\equiv \frac{\partial F_i}{\partial \Omega} = \varphi - \phi_f \ , \\ \vartheta &\equiv \frac{\partial F_i}{\partial \mathcal{J}} = \varphi - \int^t \omega_{ce}\left(s_i(\tau)\right) d\tau \ , \end{aligned} \tag{14}$$

The generating function also gives

$$\begin{aligned} p_\parallel &\equiv \frac{\partial F_i}{\partial s} = k_i(t)(\Omega + \Pi_i(t)) \ , \\ \mu &\equiv \frac{\partial F_i}{\partial \varphi} = \mathcal{J} + \Omega + \Pi_i(t) \ , \end{aligned} \tag{15}$$

which then yields the new canonical momenta

$$\begin{aligned} \Omega &= \frac{p_\parallel}{k_i(t)} - \Pi_i(t) \ , \\ \mathcal{J} &= \mu - \Omega - \Pi_i(t) = \mu - \frac{p_\parallel}{k_i(t)} \ . \end{aligned} \tag{16}$$

The new variable ξ, Ω denoting the particle gyrophase and parallel momentum at the cell frame of reference. For the homogeneous magnetic field, ϑ is called the helical angle, and $\mathcal{J}$ is the conjugated helical invariant.

The generating function in Eq. (13) is unique for each local cell, it readily changes the Hamiltonian in the laboratory frame to the local resonance frame. The new Hamiltonian with canonical variables $(\vartheta, \mathcal{J}, \xi, \Omega)$ on the cell can be obtained from

$$H_i(\xi, \Omega, \vartheta, J; t) = H_0 + \delta H_i + \frac{\partial F_i}{\partial t} \ . \tag{17}$$

The equilibrium part in the first term is

$$\begin{aligned} H_0 &= \frac{k_i^2(t)\left(\Omega + \Pi_i(t)\right)^2}{2m_e} + \left(J + \Omega + \Pi_i(t)\right) \cdot \omega_{ce}(s) \\ &\simeq \frac{k_i^2(t)\left(\Omega + \Pi_i(t)\right)^2}{2m_e} + \left(J + \Omega + \Pi_i(t)\right) \cdot \left(\omega_{ce}(s_i) + \frac{\partial \omega_{ce}}{\partial s} \frac{\xi - \vartheta}{k_i(t)}\right) \end{aligned} \tag{18}$$

where we expand $\omega(s)$ with respect to the cell center s_i, and the displacement to the cell center is $s - s_i(t) \equiv (\xi - \vartheta)/k_i(t)$ according to the definition of the canonical variables in Eq. (14).

The time derivative of the generating function yields

$$\begin{aligned} \frac{\partial F_i}{\partial t} &= \frac{\mathrm{d}k_i}{\mathrm{d}t}(\Omega + \Pi(t)) \frac{\xi - \vartheta}{k_i(t)} \\ &\quad - k_i\left(\Omega + \Pi_i(t)\right) \frac{\mathrm{d}s_i}{\mathrm{d}t} + \frac{\mathrm{d}\Pi_i}{\mathrm{d}t} \xi - (\mathcal{J} + \Omega + \Pi_i)\omega_{ce}(s_i(t)) \end{aligned} \tag{19}$$

Collecting the terms in Eq. (18) and Eq. (19), we have

$$\begin{aligned}
H_0 + \frac{\partial F_i}{\partial t} &= \frac{k_i^2(t)\Omega^2}{2m_e} + \left(\frac{k_i^2(t)}{m_e}\Pi_i(t) - k_i(t)\frac{\mathrm{d}s_i}{\mathrm{d}t}\right)\Omega \\
&+ \frac{\mathrm{d}\Pi_i}{\mathrm{d}t}\xi + \left[\left(\frac{\mathrm{d}k_i}{\mathrm{d}t} + \frac{\mathrm{d}\omega_{ce}}{\mathrm{d}s_i}\right)(\Omega + \Pi_i(t)) + \frac{\mathrm{d}\omega_{ce}}{\mathrm{d}s_i}J\right]\left(\frac{\xi - \vartheta}{k_i(t)}\right) \\
&+ \frac{k_i^2}{2m_e}\Pi_i^2 - k_i\frac{\mathrm{d}s_i}{\mathrm{d}t}\Pi_i(t) \ .
\end{aligned} \tag{20}$$

Note that the last two terms in the above equation can be neglected. Those terms are canonical variables free, i.e., they do not contribute to the dynamics and can be eliminated readily by introducing arbitrary time dependent function in the F_i at the first place.

Besides, we have to eliminate the major linear terms of the new momentum Ω by setting

$$\Pi_i(t) = \frac{m_e}{k_i(t)}\frac{\mathrm{d}s_i}{\mathrm{d}t} = \frac{m_e}{k_i^2(t)}(\omega_i - \omega_{ce}(s_i)) \ . \tag{21}$$

Consequently, we have $\Omega \approx 0$ for particle near the resonance. Thus the second linear term of Ω can be neglected whenever the background parameters change slowly in one bounce period ω_b^{-1} of the particle trapped in the electromagnetic wave field, i.e.,

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} + \frac{\partial\omega_{ce}}{\partial s_i} \ll k_i(t)\omega_b \ . \tag{22}$$

Moreover, with the Π_i defined in Eq. (21), we have

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}t} = \frac{m_e}{k_i(t)}\frac{\mathrm{d}s_i^2}{\mathrm{d}t^2} - \frac{\mathrm{d}k_i}{\mathrm{d}t}\frac{\Pi_i}{k_i} \ . \tag{23}$$

Substituting it into Eq. (20), we have

$$\begin{aligned}
H_0 + \frac{\partial F_i}{\partial t} \approx &- \left[\left(\frac{\mathrm{d}k_i}{\mathrm{d}t} + \frac{\mathrm{d}\omega_{ce}}{\mathrm{d}s_i}\right)\Pi_i(t) + \frac{\mathrm{d}\omega_{ce}}{\mathrm{d}s_i}J\right]\frac{\vartheta}{k_i(t)} \\
&+ \frac{k_i^2(t)\Omega^2}{2m_e} + \left[m_e\frac{d^2 s_i}{\mathrm{d}t^2} + \frac{\mathrm{d}\omega_{ce}}{\mathrm{d}s_i}(J + \Pi_i(t))\right]\frac{\xi}{k_i(t)}
\end{aligned} \tag{24}$$

As to the perturbed Hamiltonian, according to the definition in Eq. (3), we first separate the phase ϕ_q according to equation (12), and obtain

$$\begin{aligned}
\delta H(\varphi, \mu; s, p_\parallel; t) &\simeq e^{\imath\left(\phi_f - \varphi\right)}\int_{-\infty}^{\infty}\mathrm{d}q\ \delta H\left(\mu, p_\parallel, q; t\right)e^{\imath\phi_s} \\
&= \delta H\left(\mu, p_\parallel, s_i; t\right)e^{\imath\left(\phi_f - \varphi\right)} \ .
\end{aligned} \tag{25}$$

The real part of the perturbed Hamiltonian $\delta H\left(\mu, p_\parallel, s_i; t\right)$ is given in terms of the perturbed wave envelope as $e|v_\perp| \cdot |a(s_i, t)|/c$. In the new canonical coordinates it becomes

$$\sqrt{\frac{2\omega_{ce}(s_i)(J + \Omega + \Pi_i(t))}{m_e}}\frac{e|a(s_i, t)|}{c} = m_e\frac{\omega_b^2}{k_i^2} \ , \tag{26}$$

where trapped particle bounce frequency $\omega_b \equiv e\delta B k_i v_\perp / m_e c$ in previous studies [14, 15, 24] now satisfies

$$\omega_b^2 = \frac{e}{m_e c}\sqrt{\frac{2\omega_{ce}\left(s_i\right)\left(\mathcal{J}+\Omega+\Pi_i\right)}{m_e}}k_i^2|a(s_i,t)| \ . \tag{27}$$

The phase term of the perturbed Hamiltonian simply written in new canonical angle as $\exp(\imath\left(\phi_f-\varphi\right)) = \exp\left(-\imath\xi\right)$.

The final perturbed Hamiltonian is

$$\delta H_i\left(\Omega, J; s_i, t\right) = \frac{1}{2}\left(\delta H_i e^{-\imath\xi} + \text{c.c}\right) = m_e\frac{\omega_b^2}{k_i^2}\cos\xi \ . \tag{28}$$

Merging equilibrium and the perturbed Hamiltonian we have

$$\begin{aligned} H_i\left(\xi,\Omega;\vartheta,J;s_i,t\right) = &\frac{k_i^2(t)\Omega^2}{2m_e} + \left[m_e\frac{d^2 s_i}{\mathrm{d}t^2} + \frac{\mathrm{d}\omega_{ce}}{\mathrm{d}s_i}\left(J+\Pi_i(t)\right)\right]\frac{\xi}{k_i(t)} \\ &-\left[\left(\frac{\mathrm{d}k_i}{\mathrm{d}t}+\frac{\mathrm{d}\omega_{ce}}{\mathrm{d}s_i}\right)\Pi_i(t)+\frac{\mathrm{d}\omega_{ce}}{\mathrm{d}s_i}J\right]\frac{\vartheta}{k_i(t)} \\ &+m_e\frac{\omega_b^2}{k_i^2}\cos\xi \ . \end{aligned} \tag{29}$$

Here, we introduce a dimensionless parameter α from shown in equation (29)

$$\alpha = \frac{1}{\omega_b^2}\left[k_i\frac{\mathrm{d}^2 s_i}{\mathrm{d}t^2} + \frac{k_i}{m_e}\frac{\partial\omega_{ce}}{\partial s_i}\left(\mathcal{J}+\Pi_i(t)\right)\right] \ . \tag{30}$$

We can further simplify α by expanding the second order derivative of s_i to show the physics meaning of α,

$$\begin{aligned} \frac{\mathrm{d}^2 s_i}{\mathrm{d}t^2} &= \frac{\mathrm{d}v_r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\omega_i-\omega_{ce}}{k_i} \\ &= \frac{1}{k_i}\frac{\mathrm{d}(\omega_i-\omega_{ce})}{\mathrm{d}t} - \frac{\omega_i-\omega_{ce}}{k_i^2}\frac{\mathrm{d}k_i}{\mathrm{d}t} \ . \end{aligned} \tag{31}$$

The exact derivatives are given along the resonant trajectory, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial s} \ . \tag{32}$$

For the frequency ω_i, applying the identity

$$\frac{\partial\omega_i}{\partial t} + v_g\frac{\partial\omega_i}{\partial s} = 0 \tag{33}$$

we have

$$\frac{\mathrm{d}\omega_i}{\mathrm{d}t} = \left(1-\frac{v_r}{v_g}\right)\frac{\partial\omega_i}{\partial t} \ . \tag{34}$$

For the derivative of wave number k_i, we additionally use

$$\frac{\partial k_i}{\partial t} \equiv -\frac{\partial \omega_i}{\partial s}, \tag{35}$$

and have

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = \frac{1}{v_g}\frac{\partial \omega_i}{\partial t} + v_r \frac{\partial k_i}{\partial s} \ . \tag{36}$$

The time derivative of gyrofrequency only depends on s, and

$$\frac{\mathrm{d}\omega_{ce}}{\mathrm{d}t} = v_r \frac{\partial \omega_{ce}}{\partial s} \ . \tag{37}$$

Substituting back to equation (30), we have the final expression of α

$$\alpha \equiv \frac{1}{\omega_b^2}\left[\left(1 - 2\frac{v_r}{v_g}\right)\frac{\partial \omega_i}{\partial t} - v_r^2 \frac{\partial k_i}{\partial s_i} + \frac{\partial \omega_{ce}}{\partial s_i}\frac{k_i}{m_e}\mathcal{J}\right]. \tag{38}$$

With the introduced parameter, we can further organize the Hamiltonian as

$$\begin{aligned} H(s_i, \vartheta, \mathcal{J}, \xi, \Omega) = \frac{k_i^2 \Omega^2}{2m_e} + m_e \frac{\omega_b^2}{k_i^2}(\cos\xi + \alpha\xi) \\ + \left[\left(\frac{\mathrm{d}k_i}{\mathrm{d}t} + \frac{\partial \omega_{ce}}{\partial s_i}\right)\Pi_i(t) + \frac{\partial \omega_{ce}}{\partial s_i}\mathcal{J}\right]\frac{\vartheta}{k_i} \ , \end{aligned} \tag{39}$$

Now, the motion scales have been separated in above Hamiltonian. The first two terms in Eq. (39) describes the fast varying scale in the cell. The Hamiltonian composed a modified pendulum Hamiltonian system. The α term acts as a noninertial force stemming from background inhomogeneity. It includes the mirror force from inhomogeneous magnetic field and force from wave chirping. The parameter itself, is a dimensionless parameter representing the ratio of the inertial force and wave restoring force, i.e., the ω_b. For the trapped particles, the ratio is from -1 to 1. The sign of α indicates the orientation of the hole. When $\alpha = 0$, it is readily reduced to a simple pendulum Hamiltonian, which describes the particle oscillation in the wave potential well solely. If $|\alpha|$ approaches to 1, it indicates that the inertial force destroys the entire wave trapping, and invalidate the adiabatic theory. When $|\alpha| > 1$, trapped particle would not exist.

The last term of Eq. (39) containing ϑ which gives the evolution of momentum $\mathcal{J}$ from the canonical Hamiltonian equation,

$$\frac{d\mathcal{J}}{dt} = -\frac{\partial H_i}{\partial \vartheta} = \frac{1}{k_i(t)}\left[\left(\frac{dk_i}{dt} + \frac{d\omega_{ce}}{ds_i}\right)\Pi_i(t) + \frac{d\omega_{ce}}{ds_i}\mathcal{J}\right] \ . \tag{40}$$

In fact, we can neglect the contribution from the perturbed Hamiltonian and have

$$\frac{d\vartheta}{dt} = \frac{\partial H_i}{\partial J} \approx \frac{d\omega_{ce}}{ds_i}\frac{\xi - \vartheta}{k_i(t)} = \frac{d\omega_{ce}}{ds_i}\left(s - s_i(t)\right) \ . \tag{41}$$

The canonical equation for the perturbed wave-particle interaction is then given as

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} \equiv -\frac{\partial H_i}{\partial \xi} = m_e \frac{\omega_b^2}{k_i^2}\left(\sin\xi - \alpha\right) \tag{42}$$

and the evolution of ξ is

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} \equiv \frac{\partial H_i}{\partial \Omega} = \frac{\sqrt{2\omega_{ce}}}{\sqrt{m_e(J+\Omega+\Pi)}}\frac{e|a|}{c}(\cos\xi + \alpha\xi) \ . \tag{43}$$

Equations (42)-(41) gives the evolution of the dynamics system inside the cell and with the cell.

B. The Vlasov equation and wave envelope on the cell

According to the Hamiltonian theory for dynamics of the resonance particle on the cell reference frame, we are now able to construct the Vlasov and the wave envelope equations which self-consistently describes the interaction of the resonant particles with the slowly varying wave envelope and the evolution of the wave.

The distribution function of resonant electrons is described separately on difference cells. Thus, the distribution function depends on the cell coordinate s_i. Besides, the dynamics concerning the angle variable ϑ in Eq. (41) can be neglected, since

$$\frac{\mathrm{d}\omega_{ce}}{\mathrm{d}s_i}\left(s - s_i(t)\right) \ll \omega_b \tag{44}$$

during the onset stage of the chorus emission. Thus, the distribution function depends on canonical variables $\mathcal{J}$,ξ, and Ω only, i.e., $f(s_i, \mathcal{J}, \xi, \Omega)$. Combining the canonical Hamiltonian equations, we can directly write the Vlasov equation as

$$\frac{\partial f}{\partial t} + \frac{ds_i}{dt}\frac{\partial f}{\partial s_i} - \frac{\mathrm{d}\mathcal{J}}{\mathrm{d}t}\frac{\partial f}{\partial \mathcal{J}} + \frac{\mathrm{d}\xi}{\mathrm{d}t}\frac{\partial f}{\partial \xi} - \frac{\mathrm{d}\Omega}{\mathrm{d}t}\frac{\partial f}{\partial \Omega} = 0 \ . \tag{45}$$

The time derivatives of s_i is given according to the local resonant velocity,

$$\frac{\mathrm{d}s_i}{\mathrm{d}t} = \frac{\omega_l - \omega_{ce}}{k_l}. \tag{46}$$

The variation of $\mathcal{J}$ is obtained from equation (77). The fast varying scale motions are given by equation (42) and (43) with simplified α in equation (73).

C. The wave envelope equation

The whistler waves are excited from the background cold plasma, where the plasma density and the magnetic field determines the cold whistler-mode wave numbers and frequencies. The waves then being driven unstably by energetic electrons, and from which the most unstable wave with wave number k_l and frequency ω_l could arise from the noise level. The wave is considered to be fully electromagnetic, i.e., $E \perp k$ and $B \perp k$, and has a pure circular polarization, thus all field components are presented in complex form, where the real and imaginary parts denote the x-axis direction and y-axis direction of the plane perpendicular to B_0. In the following discussion, the wave vector potential and current are represented in complex form to indicate the circular vectors, i.e., $a = a_x + \imath a_y$ and $j = j_x + \imath j_y$.

The evolution of whistler wave is governed by the Ampere's law,

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 A}{\partial s^2} = \frac{4\pi}{c}\left(j_w + j_p\right) \tag{47}$$

where the current includes the linear current j_w from bulk cold plasma and the current j_p from energetic electrons.

Similar to the Hamiltonian on the cell in Eq. (25), applying the phase expansion, we can write the wave potential in Eq. (1) as envelope form,

$$\begin{aligned} A(s,t) &\equiv \int_{-\infty}^{\infty} \mathrm{d}q \; A\left(q,t\right)\exp(\imath\phi_q) \\ &\simeq e^{\imath\phi_f}\int_{-\infty}^{\infty} \mathrm{d}q \; A\left(q,t\right)\exp(\imath\phi_s) \\ &\simeq e^{\imath\phi_f} a(s_i,t) \;, \end{aligned} \tag{48}$$

where $a(s_i,t) \equiv \int_{-\infty}^{\infty} \mathrm{d}q \; A(q,t)e^{\imath\phi_s}$ is the slowly varying wave envelope.

Substituting equation (48) into the wave equation (47), we have

$$\begin{aligned} \frac{\partial^2 A}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}(ae^{\imath\phi_f})\right) = \frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}e^{\imath\phi_f} + \imath a\frac{\partial \phi_f}{\partial t}e^{\imath\phi_f}\right) \\ &= \frac{\partial^2 a}{\partial t^2}e^{\imath\phi_f} + \frac{\partial a}{\partial t}(\imath\frac{\partial\phi_f}{\partial t}e^{\imath\phi_f}) + \imath\left(\frac{\partial a}{\partial t}\frac{\partial\phi_f}{\partial t}e^{\imath\phi_f} + a\frac{\partial^2\phi_f}{\partial t^2}e^{\imath\phi_f} + \imath a\left(\frac{\partial\phi_f}{\partial t}\right)^2 e^{\imath\phi_f}\right) \\ &= e^{\imath\phi_f}\left(\frac{\partial^2 a}{\partial t^2} + 2\imath\frac{\partial a}{\partial t}\frac{\partial\phi_f}{\partial t} + \imath a\frac{\partial^2\phi_f}{\partial t^2} - a\left(\frac{\partial\phi_f}{\partial t}\right)^2\right) \end{aligned} \tag{49}$$

Similarly, the second order derivative of A with respect to s is

$$\frac{\partial^2 A}{\partial s^2} = e^{\imath\phi_f}\left(\frac{\partial^2 a}{\partial s^2} + 2\imath\frac{\partial a}{\partial s}\frac{\partial\phi_f}{\partial s} + \imath a\frac{\partial^2\phi_f}{\partial s^2} - a\left(\frac{\partial\phi_f}{\partial s}\right)^2\right) \tag{50}$$

According to the definition of fast varying phase in equation (12), we obtained its first order derivative as

$$\begin{aligned}\frac{\partial \phi_f}{\partial t} &= \omega_{ce}(s_i(t)) - \frac{\partial k_i(t)}{\partial t}(s - s_i(t)) + k_i v_i \\ &= \omega_i - \frac{\partial k_i(t)}{\partial t}(s - s_i(t)) \\ &\simeq \omega_i \ ,\end{aligned} \tag{51}$$

and the spatial derivative is

$$\frac{\partial \phi_f}{\partial s} \simeq -k_i \ . \tag{52}$$

While the second order derivative of the phase is vanished. Therefore, the left-hand-side of the wave equation is

$$\mathrm{L} = e^{\imath\phi_f}\left(\frac{1}{c^2}\frac{\partial^2 a}{\partial t^2} - \frac{\partial^2 a}{\partial s^2} + \frac{2\imath\omega_i}{c^2}\frac{\partial a}{\partial t} + 2\imath k_i \frac{\partial a}{\partial s} + (k_i^2 - \frac{\omega_i^2}{c^2})a\right) \tag{53}$$

For the right-hand-side of the wave equation, we first deal with the linear current j_w, with regard to the high frequency whistler mode, the ion is too heavy to respond the wave perturbation, thus j_w can be derived from the equation of motion of cold electrons [27],

$$\frac{\partial j_w}{\partial t} - \imath\omega_{ce}(s) j_w = -\frac{\omega_p^2(s)}{4\pi c}\frac{\partial A}{\partial t} \tag{54}$$

where ω_p is the plasma frequency of background electrons. The general solution of the above first order inhomogeneous differential equation is

$$j_w(t) = \mathrm{C}e^{\imath\omega_{ce}(s)t} + e^{\imath\omega_{ce}(s)t}\int_0^t -\frac{\omega_p^2}{4\pi c}\frac{\partial A}{\partial \tau}e^{-\imath\omega_{ce}(s)\tau}\mathrm{d}\tau \ . \tag{55}$$

Apply the initial condition $j_w(t=0) = 0$, the constant is vanished in the general solution, and gives the linear current as,

$$j_w = -\frac{\omega_p^2(s)}{4\pi c}\int_0^t \mathrm{d}\tau \frac{\partial A}{\partial \tau} e^{\imath\omega_{ce}(s)(t-\tau)} \ . \tag{56}$$

Also substituting the envelope form in Eq. (48) into the above equation, we have

$$j_w = -\frac{\omega_p^2(s)}{4\pi c}\int_0^t \mathrm{d}\tau \left(\frac{\partial a}{\partial \tau} + \imath\omega_i(\tau)a\right) e^{\imath\phi_f(s,\tau)}e^{\imath\omega_{ce}(t-\tau)} \ . \tag{57}$$

The last term can be approximately written as

$$e^{\imath\omega_{ce}(t-\tau)} \simeq e^{\imath\int_t^\tau \omega_{ce}(s_i(\tau'))\mathrm{d}\tau'} \tag{58}$$

As to the fast varying phase $\phi_f(s,\tau)$, similar to the wave eikonal, we can write it approximately as,

$$\phi_f(s,\tau) = \phi_f(s,t) + \int_t^\tau \omega_i(k_i(\tau'),\tau')\mathrm{d}\tau' \tag{59}$$

Thus, the linear current j_w becomes

$$j_w = -\frac{\omega_p^2(s)}{4\pi c}\int_0^t \mathrm{d}\tau \left(\frac{\partial a}{\partial \tau} + \imath\omega_i(\tau)a\right) e^{\imath\int_t^\tau(\omega_i(k_i(\tau'),\tau'))-\omega_{ce}(s_i(\tau'))\mathrm{d}\tau'} \ . \tag{60}$$

For the integral with the zeroth order derivative of a, we can use integration by parts, where

$$\begin{aligned}\int_0^t \frac{\omega_i(\tau)a}{\omega_i(k_i(\tau),\tau)-\omega_{ce}(s_i(\tau))}\mathrm{d}e^{\imath\int_t^\tau(\omega_i(k_i(\tau'),\tau'))-\omega_{ce}(s_i(\tau'))} \simeq \frac{\omega_i(\tau)a}{\omega_i(k_i(\tau),\tau)-\omega_{ce}(s_i(\tau))} - \\ \int_0^t \frac{\omega_i(\tau)}{\omega_i(k_i(\tau),\tau)-\omega_{ce}(s_i(\tau))}\frac{\partial a}{\partial \tau}e^{\imath\int_t^\tau(\omega_i(k_i(\tau'),\tau'))-\omega_{ce}(s_i(\tau'))}\mathrm{d}\tau \ .\end{aligned} \tag{61}$$

Substituting back we have

$$\begin{aligned}&\frac{1}{c^2}\frac{\partial^2 a}{\partial t^2} - \frac{\partial^2 a}{\partial s_i^2} + \frac{2\imath\omega_i}{c^2}\frac{\partial a}{\partial t} + 2\imath k_i\frac{\partial a}{\partial s_i} + \left(k_i^2(t) - \frac{\omega_i^2}{c^2} + \frac{\omega_p^2\left(s_i\right)\omega_i}{c^2\left(\omega_i-\omega_{ce}\right)}\right)a \\ &+ \frac{\omega_p^2\left(s_i(t)\right)}{c^2}\int_0^t d\tau\frac{\omega_{ce}\left(s_i(\tau)\right)}{\omega_{ce}\left(s_i(\tau)\right)-\omega_i\left(k_i(\tau),\tau\right)}\frac{\partial a}{\partial \tau}e^{-\imath\int_\tau^t(\omega_i(k_i(\tau'),\tau')-\omega_{ce}(s_i(\tau')))d\tau'} = \frac{4\pi}{c}j_p.\end{aligned} \tag{62}$$

Note that the linear current term has been analytic merged on the left-hand-side, and the right only contains the current term from energetic particle. It can be obtained through the integration over the phase space in the vicinity of the resonances of the perturbed distribution functions, i.e.,

$$j_p(s,t) = -en_{h0}\iiint f\left(s,p_\parallel,\mu,\varphi;t\right)\sqrt{\frac{2\omega_{ce}\mu}{m_e}}e^{\imath\varphi}dp_\parallel d\mu d\varphi \ , \tag{63}$$

Since the determinant of the Jacobian matrix of the canonical variables is

$$\frac{\partial(p_\parallel,\mu,\varphi)}{\partial(\xi,\Omega,\mathcal{J})} = \frac{1}{k} \ , \tag{64}$$

Besides, similar to the slowly varying kernel in Eq. (48), the current envelope on the cell is

$$j_p(s_i,t) = \exp(\imath(\xi-\varphi))j_p(s,t) \ , \tag{65}$$

which gives

$$j_p(s_i,t) = -\frac{\omega_{h0}^2 k_i(t)}{4\pi e}\iiint \sqrt{2m_e\omega_{ce}(s)(\mathcal{J}+\Omega+\Pi_i(t))}f(\xi,\Omega,\mathcal{J};s_i(t),t)e^{\imath\xi}\mathrm{d}\xi\mathrm{d}\Omega\mathrm{d}\mathcal{J} \ , \tag{66}$$

where $\omega_{h0}^2 = 4\pi n_{h0}e^2/m_e$ is the plasma frequency of the energetic electrons.

Using the onset condition, i.e., set the frame center at the most unstable frequency ω_l we have the final second order reduced equation of the wave envelope,

$$\begin{aligned}&\frac{1}{c^2}\frac{\partial^2 a}{\partial t^2}-\frac{\partial^2 a}{\partial s_i^2}+\frac{2\imath\omega_l}{c^2}\frac{\partial a}{\partial t}+2\imath k_l\frac{\partial a}{\partial s_i}+\frac{\omega_p^2\omega_{ce}}{c^2(\omega_{ce}-\omega_l)}\int_0^t d\tau\frac{\partial a}{\partial\tau}e^{-\imath(\omega_l-\omega_{ce})(t-\tau)}=\\&-\frac{\omega_{h0}^2k_l}{ec}\iiint\sqrt{2m_e\omega_{ce}(s)(\mathcal{J}+\Omega+\Pi_i(t))}f(\xi,\Omega,\mathcal{J};s_i(t),t)e^{\imath\xi}\mathrm{d}\xi\mathrm{d}\Omega\mathrm{d}\mathcal{J}\ .\end{aligned} \tag{67}$$

III. DISCUSSION

A. The onset stage of frequency chirping

Recalling cell definition, it is a local reference frame that tracks the local resonant particle at rest. However, when the chirping begins, it is hard to follow the resonance particle precisely since the local resonance changes all the time. To reduce the complexity of the dynamics system without losing any generality, we here consider the initial time stage of the chirping problem, i.e., the onset problem. The variation of wave frequency is considerably small in the onset stage of the instability. Thus the resonance frame is approximately centered at the most unstable frequency ω_l and the corresponding wave number k_l obtained from the linear wave dispersion. By simply replacing k_i, ω_i by ω_l and k_l in the related parameters and equations, we can greatly reduce the motion equation. For the perturbed motion, the parameter *alpha* is greatly reduced. Since the most unstable frequency is a temporospatial constant, $\partial\omega/\partial t$ is vanished in Eq. (73). While the exact time derivative of k_l still has dependence on s, i.e.,

$$\frac{\mathrm{d}k_l}{\mathrm{d}t}=v_r\frac{\partial k_l}{\partial s}\ . \tag{68}$$

According to the linear dispersion

$$c^2k_l^2=\omega_l^2+\frac{\omega_{pe}^2\omega_l}{\omega_{ce}-\omega_l}\ , \tag{69}$$

do differentiate on both sides with respect to s yields

$$\begin{aligned}2c^2k_l\frac{\partial k_l}{\partial s}&=\left(2\omega_l+\frac{\omega_{ce}\omega_{pe}^2}{(\omega_{ce}-\omega_l)^2}\right)\frac{\partial\omega_l}{\partial s}-\frac{\omega_l\omega_{pe}^2}{(\omega_{ce}-\omega_l)^2}\frac{\partial\omega_{ce}}{\partial s}\\&=-\frac{\omega_l\omega_{pe}^2}{(\omega_{ce}-\omega_l)^2}\frac{\partial\omega_{ce}}{\partial s}\ ,\end{aligned} \tag{70}$$

i.e.,

$$\frac{\partial k_l}{\partial s}=-\frac{\omega_l\omega_{pe}^2}{2c^2k_l\left(\omega_{ce}-\omega_l\right)^2}\frac{\partial\omega_{ce}}{\partial s} \tag{71}$$

Note that we neglect the dependence of ω_{pe} on s.

$$\begin{aligned}\frac{\partial k_l}{\partial s} &= -\frac{c^2k_l^2-\omega_l^2}{2c^2k_l^2}\frac{k_l}{(\omega_{ce}-\omega_l)}\frac{\partial \omega_{ce}}{\partial s}\\ &= -\frac{1}{v_r}(\frac{1}{2}-\frac{\omega_l^2}{2c^2k_l^2})\frac{\partial \omega_{ce}}{\partial s}\\ &\simeq \frac{1}{2v_r}\frac{\partial \omega_{ce}}{\partial s}\ .\end{aligned} \tag{72}$$

The phase velocity is close to the speed of light for the whistler dispersion at the frequency of interest, thus we have the above approximation.

The onset condition finally gives

$$\alpha \simeq \frac{k_l}{\omega_b^2}\left(\mathcal{J}-\frac{\Pi_i}{2}\right)\frac{\mathrm{d}\omega_{ce}}{\mathrm{d}s} \tag{73}$$

In addition, with the onset condition, the motion equation (40) for $\mathcal{J}$ reduces to

$$\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}t} \simeq \frac{m_e(\omega_l-\omega_{ce})}{k_l^3}\frac{\mathrm{d}k_l}{\mathrm{d}t}+\frac{m_e(\omega_l-\omega_{ce})}{k_l^3}\frac{\partial \omega_{ce}}{\partial s_i}+\frac{\mathcal{J}}{k_l}\frac{\partial \omega_{ce}}{\partial s_i}\ . \tag{74}$$

Multiplying $\omega_l-\omega_{ce}$ on both sides to construct v_r term on the spatial derivative of ω_{ce} that change the spatial derivative to exact time derivative, i.e.,

$$(\omega_l-\omega_{ce})\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}t} \simeq \frac{m_e(\omega_l-\omega_{ce})^2}{k_l^3}\frac{\mathrm{d}k_l}{\mathrm{d}s_i}+\frac{m_e(\omega_l-\omega_{ce})}{k_l^2}\frac{\mathrm{d}\omega_{ce}}{\mathrm{d}t}+\mathcal{J}\frac{\mathrm{d}\omega_{ce}}{\mathrm{d}t}\ . \tag{75}$$

Further using the identity $\mathrm{d}\omega_l/\mathrm{d}t \equiv 0$, we obtain the following relation

$$(\omega_l-\omega_{ce})\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}t}+\mathcal{J}\frac{\mathrm{d}}{\mathrm{d}t}(\omega_l-\omega_{ce}) \simeq -m_e\left((\omega_l-\omega_{ce})^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2k_l^2}+\frac{1}{2k_l^2}\frac{\mathrm{d}}{\mathrm{d}t}(\omega_l-\omega_{ce})^2\right)\ . \tag{76}$$

The equation is clearly to be analytically integrated, and yields

$$(\omega_l-\omega_{ce})\mathcal{J}+\frac{m_e(\omega_l-\omega_{ce})^2}{2k_l^2} = \mathrm{Const.} \tag{77}$$

The constant for each cell is determined by the initial choice of $\mathcal{J}$, which give the entire information of the dynamics on the slowly varying scale along the magnetic field line.

B. The adiabatic theory

In the previous section, we have shown the nonlinear current as the velocity momentum of perturbed distribution function. The current dominates the nonlinear behavior of the chorus chirping process, and under certain condition, we can simplify the integral and show qualitatively how does the frequency chirp. The distribution function of the trapped electrons forms a hole in the phase space at the nonlinear stage, and we consider the deviation from the unperturbed distribution Δf, i.e., the depth of the hole. The nonlinear current is directly determined by Δf, since the equilibrium distribution does not contribute the current.

For those electrons trapped by the slowly varying wave envelope and circling around in the phase space, we can define the adiabatic invariant at a given $s_i, \mathcal{J}$,

$$\mathcal{I} = \frac{1}{2\pi} \oint \Omega(H_i, \xi, t) \mathrm{d}\xi \ , \tag{78}$$

where the momentum Ω is the function of local Hamiltonian. The deviation can be written as the function of the adiabatic invariant, i.e., $\Delta f(s_i, \mathcal{J}, \mathcal{I}, \xi, t)$.

We replace f by Δf in the current integral in Eq. (66) and directly change the integral to

$$j_p(s_i, t) \approx -\sqrt{2m_e\omega_{ce}(s)(\mathcal{J} + \Pi_i(t))} \int \mathrm{d}\mathcal{J} \int_0^{I_{\mathrm{spx}}} \mathrm{d}\mathcal{I} \int_0^{2\pi} \mathrm{d}\psi \Delta f(s_i, \mathcal{J}, \mathcal{I}, \xi, t) e^{\imath\xi} \ , \tag{79}$$

since $\mathrm{d}\xi\mathrm{d}\Omega = \mathrm{d}\mathcal{I}\mathrm{d}\psi$, where the ψ is the angle variable of $\mathcal{I}$.

Note that $\mathrm{d}\mathcal{I} = \Omega\mathrm{d}\xi$, thus differential element $\mathrm{d}\psi = 1/\Omega\mathrm{d}\xi$, and we define

$$\int f \mathrm{d}\psi = \int \frac{f}{\Omega} \mathrm{d}\xi \equiv 2\pi\tau\langle f \rangle \tag{80}$$

where $\tau = \langle 1 \rangle / 2\pi$ and the bracket $\langle ... \rangle$ denotes the bounce average. Thus the current integral becomes

$$j_p(s_i, t) \approx -2\pi\tau\sqrt{2m_e\omega_{ce}(s)(\mathcal{J} + \Pi_i(t))} \int \mathrm{d}\mathcal{J} \int_0^{I_{\mathrm{spx}}} \mathrm{d}\mathcal{I} \langle \Delta f(s_i, \mathcal{J}, \mathcal{I}, \xi, t) e^{\imath\xi} \rangle \ . \tag{81}$$

For the slowly varying wave field, Δf can be expanded in powers of small scale ϵ satisfies condition [28]

$$\epsilon \equiv \mathrm{spx}\left(\ddot{\delta\omega}/\omega_b^3, \dot{\delta\omega}/\omega_b^2, \omega_b/\delta\omega\right) \ll 1 \ , \tag{82}$$

and follows the derivation in ref. [28], we keep to the first order term, the average integrals are

$$\begin{aligned} \langle \Delta f \sin\xi \rangle &\simeq \alpha \Delta f_0 \ , \\ \langle \Delta f \cos\xi \rangle &\simeq \Delta f_0 \langle \cos\xi \rangle \ . \end{aligned} \tag{83}$$

Moreover, we further assume that Δf_0 is independent with $\mathcal{I}$, which indicates the depth of the hole is flat with in the enclosed phase space region, i.e., the water bag approximation

[15, 29]. Therefore, the integral over $\mathcal{I}$ only depend on the $I_{\Updownarrow\dashv\S}$ which is the one on the separatrix. According to equation (78), the integral becomes

$$\int_0^{\mathcal{I}_{spx}} \mathrm{d}\mathcal{I} = \mathcal{I}_{spx} \equiv \oint_{spx} \Omega(\xi) d\xi. \tag{84}$$

where the boundary trapped particle phase space hole can be analytically given as

$$\Omega(\xi) = \pm \frac{\omega_b}{k^2} \sqrt{2(e_{spx} - \cos\xi - \alpha\xi)}\ , \tag{85}$$

e_{spx} is the Hamiltonian on the separatrix.

Here we define two function of (α)

$$\begin{aligned} m_{spx}(\alpha) &\equiv \langle \cos\xi \rangle \mathcal{I}_{spx} = \frac{\sqrt{2}\omega_b}{\kappa^2} \oint_{spx} d\xi \cos\xi \sqrt{e_{spx} - \cos\xi - \alpha\xi} \\ n_{spx}(\alpha) &\equiv \alpha \mathcal{I}_{spx} = \frac{\sqrt{2}\omega_b}{\kappa^2} \oint_{spx} d\xi \sqrt{e_{spx} - \cos\xi - \alpha\xi}\ . \end{aligned} \tag{86}$$

The current can be directly obtained as

$$j_p \approx \frac{2\omega_b}{\kappa^2} \sqrt{\omega_{ce}(s)(J+\Pi)}\, (m_{spx} + \imath\, n_{spx}) \int d\mathcal{J} \Delta f(\mathcal{J}, s_i, t)\ . \tag{87}$$

IV. CONCLUSION

In this paper, we explore the dynamics of resonant electrons and the evolution of whistler chorus wave within the Earth's inner magnetosphere, focusing on the scale separation of electron motion and fast wave interaction. We developed a Hamiltonian theory describes the dynamics of the resonant particles on the reference frame moving with the local resonance. The slowly varying motion along the weakly inhomogeneous magnetic field and the fast varying wave-particle interaction can be separately managed. Our work also provides several new view angle for the chorus chirping problem. In the onset stage of the chorus chirping, our theoretical description can be greatly reduced, which is beneficial to compose of numerical solver without losing any key physics for the generation mechanism of the chorus emission. The discussed adiabatic regime also shows the potential chirping behavior with respect to the inhomogeneity parameter α.

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