The Riemann Hypothesis, Without Euler's Product, Through Divergence in a Dirichlet Series Multiplication

by Shawn James

Section 1 of 2: Theorem Motivation & Proof

The zeta function $\zeta(s)$ of Germany's Bernhard Riemann (1826-66) was defined by him, with $\zeta(0) = -½$, as an integral valid for all $s\neq 1$ of the complex plane. For all 1<Re s this integral coincides with the absolutely convergent Dirichlet series Σn^{-s}, and for all 0<Re $s\leq 1$ with the conditionally convergent Dirichlet series $\frac{1}{2^{1-s}-1}\zeta^A(s)$ where alternating zeta $\zeta^A(s) := \lim_{N\to\infty} \zeta_N^A(s)$ for $\zeta_N^A(s) = \sum_{n=1}^{N}(-1)^n / n^s$.

So $\zeta(0)\neq 0$, $\zeta^A(s)\neq 0$ for all ½<Re s <1, and

(i) $\zeta(s) \neq 0$ for all $s\neq 1$ with Re s =1,

(ii) $\zeta(s) = f(s)\zeta(1-s)$ with $f(s)\neq 0$ for all s not an integer, and

(iii) $\zeta(s_0) = 0$ for a s_0 with Re s_0=½,

imply the **Riemann Hypothesis (RH)** of 1859:

all $s\neq 1$, with 0≤Re $s\leq 1$, and such that $\zeta(s)$=0, have real part ½.

Condition (i) is long proven as part of the prime number theorem of 1896, (ii) is from the functional equation for zeta, proven for complex s by Riemann, and (iii) was more than shown by Hardy, 1914, who proved an infinite number of zeta's zeros have Re s=½ after Hadamard, 1893, proved Riemann's claim that an infinite number of non-integer zeros exist. In (ii) we see the want Re s=½ for then Re s=Re 1-s. Conditions (i) and (ii) are proven in any of [Ti], [Ed] or [In], and (iii) in [Ti] or [Ed]. Riemann originally conjectured RH, with (ii), as $\zeta(s)\neq 0$ for all ½<Re $s\leq 1$ ($s\neq 1$) but modern formulations include (iii). The influential Hardy championed RH as a worthy project almost all his career. Our task is to prove $\zeta^A(s)\neq 0$ for all ½<Re s <1 and then to note separately, without recourse to Euler's Product, for all Re s=1 that $\zeta(s)\neq 0$.

$\zeta(s)$ was originally studied for its connection to the prime numbers so it is surprising that our proof of RH does not use Euler's 1737 product-sum formula for zeta, with 1<s,

$$\text{EP}: \sum_n \frac{1}{n^s} = \prod_p \frac{1}{1-\frac{1}{p^s}} \tag{1}$$

where the product is over all primes p, i.e., RH's dependence on the primes in (1)'s context of the fundamental theorem of arithmetic can appear to be only in (i). From EP comes [Ti, pp 2, 51], with 1<Re s,

$$\log \zeta(s) = s\int_2^{\infty} \frac{\pi(x)dx}{x(x^s-1)} \tag{2}$$

where $\pi(x)$ denotes the number of primes ≤x. Riemann's original 1859 number theory paper gave a cumbersome, but unerring, analytic formula for $\pi(x)$ in terms of all the non-trivial zeros of zeta, those in the critical strip 0≤Re $s\leq 1$. RH is expressed without complex numbers in [Kl & Wa, pp 215-7, 307] in such a way as to suggest RH can be demonstrated without EP. We use for all σ:=Re $s \geq 0$, $s \neq 1$, that the absolute value of $H_N(s) := \sum_{n \leq N} n^{-s}$ is bounded by $|\frac{1}{1-s}| N^{1-\sigma}$ + a constant. For $B_N(s) := \sum_{n=1}^{N} b_n / n^s$, the partial sums of a divergent series which we shall define, our paper's main idea is no deeper than this: for all $½<\sigma<1$, when $|B_N|$ grows arbitrarily large as $N\to\infty$; $\zeta^A{}_N \to 0$ as $N\to\infty$ might allow that $B_N\zeta^A{}_N$ exist as a limit as $N\to\infty$, but non-oscillatory $\zeta^A{}_N \not\to 0$ as $N\to\infty$ prohibits that $B_N\zeta^A{}_N$ exist as a limit as $N\to\infty$. And, for all $s\neq\sigma=1$, if B_N oscillates finitely as $N\to\infty$; $\zeta^A{}_N \to 0$ as $N\to\infty$ requires that $B_N\zeta^A{}_N$ exist as the limit zero as $N\to\infty$, but non-oscillatory $\zeta^A{}_N \not\to 0$ as $N\to\infty$ prohibits $B_N\zeta^A{}_N$ exist as the limit zero as $N\to\infty$. Our (elementary, i.e., non-analytic) proof by contradiction of RH shows that such limits, on premise $\zeta^A=0$ for any $1>\sigma>½$, and for the same premise for all but some excluded points on $\sigma=1$, can not exist.

It is well-known that whatever value $\zeta^A_{2N}(s)$ converges to as N grows large, $\zeta^A_{2N+1}(s)$ converges to the same value, and does so [Ha & Riesz, p 67] as rapidly as $N^{-\sigma}$. To see this, the tail $T_{2N}(s) := |\zeta^A(s) - \zeta^A_{2N}(s)|$ is

$$|\sum_{n=N+1}^{\infty}(\frac{-1}{(2n-1)^s}+\frac{1}{(2n)^s})| = |\sum_{n=N+1}^{\infty}\frac{1-(1+\frac{1}{2n-1})^s}{(2n)^s}| \tag{3}$$

$$=|\sum_{n=N+1}^{\infty}(\frac{1-(1+\frac{s}{2n-1})}{(2n)^s}+O(\frac{1}{n^{2+\sigma}}))|$$

$$<\int_N^{\infty}(\frac{|s|}{(2y-1)^{1+\sigma}}+O(\frac{1}{y^{2+\sigma}}))dy$$

$$=\left[\frac{-|s|x^{-\sigma}}{2\sigma}+O(\frac{1}{(x+1)^{1+\sigma}})\right]_{2N-1}^{x\to\infty}$$

which is $\leq cN^{-\sigma}$, for c a constant, where the variable change is $x=2y-1$. Say $k_1 N^{1-\sigma} < |B_N(s)| < k_2 N^{1-\sigma}$ as $N\to\infty$ for $0<k_1$ and k_2 constants. Then, on assumption $\zeta^A(s)$ is zero in $T_{2N}(s)$, we have that $|B_N(s)||\zeta^A_N(s)|$, which is $|B_N(s)\zeta^A_N(s)|$ for every N, is a product $<k_2 c N^{1-2\sigma}$ which for $\sigma>½$ tends to a finite value (zero) as $N\to\infty$ (and which might bound an oscillatory value for $\sigma=½$.) The 'O' notation, originated by Bachmann and popularized by Landau, was not available to Riemann.

The formal multiplication of two ordinary Dirichlet series

$$\sum_{n=1}^{\infty}\frac{\alpha_n}{n^s}\sum_{n=1}^{\infty}\frac{\beta_n}{n^s}=\sum_{n=1}^{\infty}\frac{\sum_{0<d|n}\alpha_d\beta_{n/d}}{n^s}$$

$$=\sum_{n=1}^{\infty}\frac{\sum_{d|n,\, d<\sqrt{n}}\alpha_d\beta_{n/d}+\alpha_{n/d}\beta_d + \sum_{d|n,\, d=\sqrt{n}}\alpha_d\beta_d}{n^s}, \tag{4}$$

where $\sum_{d|n,\, d=\sqrt{n}}\alpha_d\beta_d = 0$ if n not a square, assumes existence (as zero) of the limit as $N\to\infty$ of

$$\sum_{n=N+1}^{N^2}\frac{\sum_{0<d|n,d<\sqrt{n},N<n/d}\alpha_d\beta_{n/d}+\alpha_{n/d}\beta_d}{n^s}=$$

$$\sum_{n=N+1}^{N^2}\frac{\sum_{d|n,d<\sqrt{n},N<n/d}\alpha_d\beta_{n/d}}{n^s}+\sum_{n=N+1}^{N^2}\frac{\sum_{d|n,d<\sqrt{n},N<n/d}\alpha_{n/d}\beta_d}{n^s}\qquad(5)$$

where we denote the first sum of (5)'s right-hand side as F_N, the second as G_N. This is because the product of two Dirichlet partial sums is, for $(\alpha_k,\beta_k)=(b_k,(-1)^k)$ where $b_k=O(1)$ for all k, by brute multiplication, $B_N(s)\zeta^A{}_N(s)$ or

$$\sum_{n=1}^{N}\frac{b_n}{n^s}\sum_{n=1}^{N}\frac{(-1)^n}{n^s}=\qquad(6)$$

$$\sum_{n=1}^{N^2}\frac{\sum_{d<\sqrt{n}}b_d(-1)^{n/d}+b_{n/d}(-1)^d}{n^s}+\sum_{k=1}^{N}\frac{b_k(-1)^k}{k^{2s}}-\sum_{n=N+1}^{N^2}\frac{\sum_{d<\sqrt{n},N<n/d}b_d(-1)^{n/d}+b_{n/d}(-1)^d}{n^s}.$$

Here we denote the right-hand side of (6)'s first sum by $\Sigma_N=\Sigma_N(s,B)$, the third by $-\Sigma'_N=-\Sigma'_N(s,B):=-(F_N+G_N)$, and we let the absolutely convergent (because $1<2\mathrm{Re}\,s$) second sum be $-C_N(B,s)\to -C$ as $N\to\infty$, where $B:=\{b_n\}_{n=1}^{\infty}$. Equation (6) works because the third sum is of terms included in the first sum, terms which can't be constructed by the multiplication $B_N(s)\zeta_N^A(s)$. (The denominators of Σ'_N are $(d\times\frac{n}{d})^s$ for $d<N<n/d$.) Even though $\zeta(1-s)=f(1-s)\zeta(s)$, we see formally that $0\le\sigma\le½$ typically allows of (4) no such separating out of C in (6) when $b_k\neq 0$ sufficiently often.

$$\begin{array}{llll} \text{Define } b_k &=0 & \text{for } k=N_i+1,\ldots,\lfloor N_i^2/2\rfloor & (7)\\ &=0 & k=\lfloor N_i^2/2\rfloor+1,\ldots,N_i^2 & (k\text{ even})\\ &=2 & k=\lfloor N_i^2/2\rfloor+1,\ldots,N_i^2 & (k\text{ odd})\end{array}$$

with $N_i^2=N_{i+1}$ for positive integers $1<N_i$ increasing with i=1, 2, ..., and $\lfloor x\rfloor$ the largest integer $\le x$. (The 'b' is meant to suggest binary-like.) For such B, we shall require partial sums $B_N(s)$ be shown to be such that $k_1N^{1-\sigma}<|B_N(s)|<k_2N^{1-\sigma}$ as $N\to\infty$ for $0<k_1$ and k_2 constants, i.e., that $|B_N(s)|$ diverges as if of magnitude $N^{1-\sigma}$. For $G_N=G_N(s,B)$ and $F_N=F_N(s,B)$, with $(\alpha_k,\beta_k)=(b_k,(-1)^k)$, $|B_N(s)\zeta^A{}_N(s)|$ is

$$|\Sigma_N-C_N-F_N-G_N|<k_2cN^{1-2\sigma}.\qquad(8)$$

Our sketch of our proof of RH, for $½<\sigma<1$ and $0\neq c_s$ a constant, is:

Lemma 1: $G_N=c_sN^{2(1-s)}$+conv. terms, and $|B_N(s)|$ diverges as if of magnitude $N^{1-\sigma}$.

Lemma 2: $F_N\to 0$ as $N\to\infty$.

Theorem: On $\zeta^A(s)=0$ for an s with $1>\sigma>½$, $\Sigma_N-C_N(B)\to 0$ as $N\to\infty$. The Theorem uses a Lemma 3, to do with an infinite sum transposition. We have contradiction because (8)'s right-hand side tends to 0 while its left-hand side does not. Note that (8) possibly fails if C_N diverges as $N\to\infty$, as may happen at $\sigma=½$.

Proof of Lemma 1: We omit the index i in N_i for which we write N. With this value for B in $G_N(B)$ of (5) and (8) we obtain, by summing there first for d=1, and then for d=2, 3, ... that

$$G_N(B, s) = \sum_{n=N+1}^{N^2} \frac{\sum_{d|n, d<\sqrt{n}, N<n/d} b_{n/d}(-1)^d}{n^s}$$
$$= -(B_{N^2}(s) - B_{\lfloor N^2/2\rfloor}(s)) + 0 + 0 + ... \tag{9}$$

for large enough N. To see (9)'s string of zeros result from B, we have to use $G_N(B)$'s constraint $N<n/d$ so that, for $d>1$, all $b_{n/d}$ are a b_k where $N+1 \le k \le \lfloor N^2/2 \rfloor$. Note (9), multiplied by -1, can be rewritten as

$$(H_{N^2}(s) - \zeta^A_{N^2}(s)) - (H_{\lfloor N^2/2\rfloor}(s) - \zeta^A_{\lfloor N^2/2\rfloor}(s)) = B_{N^2}(s) - B_{\lfloor N^2/2\rfloor}(s). \tag{10}$$

So our B_N's are similar to H_N's and much is known [Ti, pp 13-4, 49-50; In, pp 18-9, 26; Le, vol 2, pp 202-3] of the latter through the Abel partial summation formula, in particular, for all $s\neq 1$ and for all integers $1<a<b$

$$H_b(s) - H_a(s) = \frac{b^{1-s} - a^{1-s}}{1-s} + (\text{terms bounded as } b\to\infty). \tag{11}$$

Thus $|H_{N^2}(s) - H_{\lfloor N^2/2\rfloor}(s)|$ diverges as if of magnitude $N^{2(1-\sigma)}$ as $N\to\infty$, and so too will (10) approach $c_s N^{2(1-s)}$+conv. terms for a finite $c_s\neq 0$ not dependent on N, i.e., $k_1<c_s<k_2$ for suitable constants k_1 and k_2. The terms denoted as bounded in (11) will evidently imply new knowledge when we treat the case σ=1 in (8).

To see from B's definition and from (10) that $|B_N(s)|$ diverges as $N^{1-\sigma}$, or that $|B_{N^2}(s)|$ diverges as if of magnitude $N^{2(1-\sigma)}$, so B is a choice suitable for our requirements of (8), we write $B_{N^2}(s)$ as

$$B_N(s) + \sum_{k=N+1}^{N^2} \frac{b_k}{k^s} = B_N(s) + \sum_{k=\lfloor N^2/2\rfloor+1}^{N^2} \frac{b_k}{k^s} \tag{12}$$
$$= B_N(s) + (H_{N^2}(s) - H_{\lfloor N^2/2\rfloor}(s) + \text{conv. terms})$$
$$= B_N(s) - c_s N^{2(1-s)} + \text{conv. terms}$$

where $|B_N(s)|$ is dominated by $N^{2(1-\sigma)}$. This ends the proof of Lemma 1. □

Proof of Lemma 2: For large enough N, and for n as jm, define for every integer $j\ge 2$ a $L_j = \lfloor \frac{N+1}{j} \rfloor + \{j \nmid N+1\}$ and $U_j = \lfloor N^2/j \rfloor$, with $\{j \nmid N+1\}$=1 if true, 0 if false, so that $j(L_j-1) < N+1 \le j(L_j) < ... < ... < j(U_j) \le N^2 < j(U_j+1)$. Write $F_N(B, s)$ of (5) and (8) as

$$\sum_{n=N+1}^{N^2} \frac{\sum_{d|n, d<\sqrt{n}, d<n/N} b_d(-1)^{n/d}}{n^s} = b_1(\zeta^A_{N^2}(s) - \zeta^A_N(s)) + \sum_{j=2}^{N-1}\sum_{m=L_j}^{U_j} \frac{b_j(-1)^{jm/j}\delta_{j,m}\Delta_m}{(jm)^s} \tag{13}$$

where $\delta_{j,m}$ is 1 if $j<\sqrt{jm}$, i.e., $j<m$, 0 if $m\le j$ (14)

Δ_m is 1 if $j<jm/N$, i.e., $N<m$, 0 if $m\le N$.

Since $\zeta_N^A(s)$ finite, we have, for large enough N, with $b_1 := 0$, that $|F_N(B, s)|$ is

$$\left|\sum_{j=2}^{N-1}\frac{b_j}{j^s}\sum_{m=\max[j+1,L_j]}^{\lfloor N^2/j\rfloor}\frac{(-1)^{jm/j}\Delta_m}{m^s}\right| \le \sum_{j=2}^{N-1}\left|\frac{b_j}{j^s}\right|\left|\sum_{m=N+1}^{\lfloor N^2/j\rfloor}\frac{(-1)^m}{m^s}\right| < k_2 cN^{1-\sigma}N^{-\sigma} \qquad (15)$$

where positive k_2 is the value of some function of (11)'s $1/(1-s)$. Eq. (15)'s right-hand side $\to 0$ as $N\to\infty$ for $\sigma > ½$. This ends the proof of Lemma 2, and we note it needs only $|B_N(s)| = O(N^{1-\sigma})$, not the particulars of B in (7). □

Proof of Theorem: Suppose $\zeta^A(s)=0$ for a $½<\sigma<1$. We wish to show Σ_N of (6) and (8) converges as $N\to\infty$, and does so to the absolutely convergent series $C=\sum_{k=1}^{\infty} b_k / k^{2s}$. Let $b_d^{(k)}$ be b_d for $d=k$ and be 0 for $d\neq k$. So for $K<n$, $\sum_{k=1}^{K}\sum_{d|n} b_d^{(k)} \neq \sum_{d|n} b_d$ for non-trivial b_i, and for $n\le K$, including when K arbitrarily large, $\sum_{k=1}^{K}\sum_{d|n} b_d^{(k)} = \sum_{d|n} b_d$. Let M_k, with $k<M_k$, be a sequence of integers, and later we will define a similar N_k. Then, for odd $k>1$,

$$\begin{aligned}\frac{b_k}{k^{2s}} &= \frac{b_k}{k^s}(0-\frac{(-1)^k}{k^s})\\ &= \frac{b_k}{k^s}(\sum_{m=1}^{k-1}\frac{(-1)^m}{m^s}+\sum_{m=k+1}^{M_k}\frac{(-1)^m}{m^s}+O(M_k{}^{-\sigma}))\\ &= \sum_{m=1}^{k-1}\frac{b_k(-1)^m}{(km)^s}+0+\sum_{m=k+1}^{M_k}\frac{b_k(-1)^m}{(km)^s}+O((kM_k)^{-\sigma}) \;. \end{aligned} \qquad (16)$$

Let $S(k,mk,B) := \sum_{d|mk,\, d<\sqrt{mk}} [b_d^{(k)}(-1)^{mk/d} + b_{mk/d}^{(k)}(-1)^d]$ so (16) is

$$\sum_{m=1}^{k-1}\frac{S(k,mk,B)}{(km)^s}+\sum_{m=k}\frac{S(k,mk,B)}{(km)^s}+\sum_{m=k+1}^{M_k}\frac{S(k,mk,B)}{(km)^s}+O((kM_k)^{-\sigma}) \,. \qquad (17)$$

We now explain the passage from (16) to (17). In the sum in the numerators of the first sum of (17), every d is less than $\sqrt{(k-1)k}<k$ so, for every d there, $d\neq k$ so $b_d^{(k)}=0$; and for every m, $b_{mk/d}^{(k)}$ has superscript = subscript precisely once, at $d=m$. Thus the first sum of (16) is the first sum of (17). In the sum in the numerator of the second sum of (17), $d<\sqrt{mk}$ where $m=k$ so $d\neq k$ so $b_d^{(k)}=0$; and mk/d is k^2/d so $d\neq k$ also means $mk/d\neq k$ so $b_{mk/d}^{(k)}=0$. Thus $S(k,k^2,B)=0$. In the sum in the numerators of the third sum of (17), for every m, $b_d^{(k)}$ has superscript = subscript precisely once, at $d=k$; and $b_{mk/d}^{(k)}$ has superscript and subscript different for every m because $d=m$ is prevented by the constraint $d<\sqrt{mk}$ when m is such that $k<m$. Thus (16) as (17) is justified and, also, $k<m$ allows $m=M_k$ may be infinite. Written explicitly, we have

$$\frac{b_k}{k^{2s}}=\sum_{m=1}^{M_k}\frac{\sum_{d|mk,\, d<\sqrt{mk}} b_d^{(k)}(-1)^{mk/d}+b_{mk/d}^{(k)}(-1)^d}{(km)^s}+O((kM_k)^{-\sigma}) \,. \qquad (18)$$

For example, m an even number allows d to be even. If d even then $b_d^{(k)}$ is 0 while $b_{mk/d}^{(k)}$ might well be non-zero. If $k\nmid n$ then, for any divisor d of n, $b_d^{(k)}=0$

as $d \neq k$, and (18) is

$$\frac{b_k}{k^{2s}} = \sum_{n=1}^{kM_k} \frac{\sum_{d|n,\, d<\sqrt{n}} b_d^{(k)}(-1)^{n/d} + b_{n/d}^{(k)}(-1)^d}{n^s} + r_k(M_k) \quad \text{for } r_k(M_k) := \sum_{n=M_k+1}^{\infty} \frac{b_k(-1)^n}{(kn)^s} \qquad (19)$$

where $|r_k(M_k)| = O(b_k(kM_k)^{-\sigma})$. Summing (19) for every k (as we may for absolutely convergent C,) including k's trivial even values, and adding $b_1/1^{2s}$ which is $b_1/1^s$, lets C be

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sum_{d|n,\, d<\sqrt{n}} b_d^{(k)}(-1)^{n/d} + b_{n/d}^{(k)}(-1)^d}{n^s}, \text{ this writeable as } \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{S(k,n,B)}{n^s}. \qquad (20)$$

Here the denominators are independent of k, and we've treated M_k as infinite. On premise $\zeta^A(s)=0$ for a $\frac{1}{2}<\sigma<1$, C has been constructed from scratch.

We wish to transpose (20)'s infinite sums and show this remains C, it then (6)'s & (8)'s Σ_N as $N \to \infty$. So we write (20), for finite M_k and N_k each $>k$, as: for all $\epsilon>0$ there is K_ϵ such that for all $K>K_\epsilon$, with $M_K = N_K$, and with $R_K = \sum_{k=1}^{K} r_k(M_k)$, and $R'_K = \sum_{k=1}^{K} r_k(N_k)$,

$$\varepsilon > \left|\sum_{k=1}^{K} \frac{b_k}{k^{2s}} - C\right| = \left|\sum_{k=1}^{K}\sum_{n=1}^{kM_k} \frac{S(k,n,B)}{n^s} + R_K - C\right| = \left|\sum_{k=1}^{K}\sum_{n=1}^{kN_k} \frac{S(k,n,B)}{n^s} + R'_K - C\right| =$$

$$\left|\sum_{k=1}^{K}\sum_{n=1}^{KM_K} \frac{S(k,n,B)}{n^s}\delta_{n\le kM_k} + R_K - C\right| = \left|\sum_{k=1}^{K}\sum_{n=1}^{KN_K} \frac{S(k,n,B)}{n^s}\delta_{n\le kN_k} + R'_K - C\right| =$$

$$\left|\sum_{n=1}^{KN_K}\sum_{k=1}^{K} \frac{S(k,n,B)}{n^s}\delta_{n\le kN_k} + R'_K - C\right| = \left|\sum_{n=1}^{KN_K}\sum_{k=1}^{K} \frac{S(k,n,B)}{n^s} + R'_K - C\right|. \qquad (21)$$

Here (a) Kronecker's delta symbol $\delta_X = 1$ if statement X true, 0 if false, and (b) for any K and any finite $N_K = M_K$, we've chosen N_k for every $k=1$ to $K-1$ such that $kN_k \ge KM_K \ge n$. I claim that RH false, a sufficiently small ϵ (and hence large enough K,) and (21) with a provoking use of N_K infinite then all say

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{S(k,n,B)}{n^s} = C \text{ if } R'_K \to 0 \text{ as } K \to \infty. \qquad (22)$$

And indeed, when $k<M_k$ finite, $O(k^{-2\sigma}) \ge |r_k(M_k)|$ which, summed over all k, is convergent for $\frac{1}{2}<\sigma$. If for every k, M_k and N_k as large as we please, then the sums R_K and R'_K are as small as we please.

Lemma 3: (22)'s double-sum is not some D with $|D| \nless \infty$ if $\zeta^A(s)=0$ for a $1 \ge \text{Re } s > \frac{1}{2}$ and $b_i = O(1)$.

Proof of Lemma 3 (by contradiction:) Ignoring R'_K and using $S(k,n,B)=0$ for all $k>n$, the last absolute value of (21) may be written as

$$\varepsilon > \left|\sum_{n=1}^{KN_K}\sum_{k=1}^{\infty} \frac{S(k,n,B)}{n^s} - C - \sum_{n=1}^{KN_K}\sum_{k=K+1}^{KN_K} \frac{S(k,n,B)}{n^s}\right| =: |D_K| \qquad (23)$$

where the interiors of the absolute values equal. Since N_K may grow arbitrarily large for fixed K, (23) is the difference of two divergent double-sums if D indeed divergent. Say it is. After a few steps of elementary algebra which would be masked if C were divergent, for $K'>K$ and with or without $KN_K=K'N_{K'}$, (23) is that ϵ bounds the absolute value of

$$D_K = D_{K'} + (\text{double-sums with indices } KN_K+1 \text{ to } K'N_{K'}) - \sum_{n=1}^{K'N_{K'}} \sum_{k=K+1}^{K'} . \quad (24)$$

We choose for every K and K' that N_K and $N_{K'}$ are such that $KN_K=K'N_{K'}$. Since $|D_{K'}|$ ϵ-bounded for $K'>K$, the last double-sum of (24) tends to zero for $N_{K'}$ finite but diverges for $N_{K'}$ infinite if also $K'\to\infty$. Yet we can contradict this by choosing $K'=K+w_K$, for w_K a positive integer, for example $w_K=1$, to, using (20)'s definition, make of (24)'s last double-sum

$$\sum_{n=1}^{\infty}\frac{S(K+1,n,B)}{n^s} = \sum_{n=1}^{\infty}\frac{\sum_{d|n,d<\sqrt{n}} b_d^{(K+1)}(-1)^{n/d}}{n^s} + \sum_{n=1}^{\infty}\frac{\sum_{d|n,d<\sqrt{n}} b_{n/d}^{(K+1)}(-1)^{d}}{n^s}$$

$$= \sum_{n'=K+2}^{\infty}\frac{b_{K+1}(-1)^{n/(K+1)}}{(K+1)^s(n')^s} + \sum_{n'=1}^{K}\frac{b_{K+1}(-1)^{n'}}{(K+1)^s(n')^s} \quad \text{for } n=(K+1)n'. \quad (25)$$

Eq. (25)'s right-hand side tends to 0 as K increases if $\zeta^A(s)=0$ for the s with $1\geq\text{Re}\, s>½$ if $b_{K+i}=O(1)$. Writing similarly for $w_K>1$, we make of (24)'s last double-sum

$$\sum_{n=1}^{\infty}\sum_{i=1}^{w_K}\frac{S(K+i,n,B)}{n^s} = \quad (26)$$

$$\left[\sum_{n'=K+2}^{\infty}\frac{b_{K+1}(-1)^{n'}}{(K+1)^s(n')^s}+\ldots+\sum_{n'=K+w_K+1}^{\infty}\frac{b_{K+w_K}(-1)^{n'}}{(K+w_K)^s(n')^s}\right]+\left[\sum_{n'=1}^{K}\frac{b_{K+1}(-1)^{n'}}{(K+1)^s(n')^s}+\ldots+\sum_{n'=1}^{K+w_K-1}\frac{b_{K+w_K}(-1)^{n'}}{(K+w_K)^s(n')^s}\right].$$

By grouping the i-th member of the first square brackets with the i-th of the second (as (25)'s order of summation indicates we may,) and by then using $\zeta^A(s)=0$, we see (26)'s absolute value is

$$\left|0-\sum_{i=1}^{w_K}\frac{b_{K+i}(-1)^{K+i}}{(K+i)^{2s}}\right| \quad (27)$$

which, since $b_{K+i}=O(1)$, does not diverge for $1<2\sigma$, and which $\to 0$ as $K\to\infty$. So D finite and Lemma 3 proved. □

Next, (22)'s double-sum is the first double-sum in the right-hand side of identity (28) at M_L infinite, the second double-sum there is zero by $S(k,n,B)=0$ for all $k>n$, while the third double-sum there is, since D exists, zero when $L\to\infty$:

$$\sum_{n=1}^{LM_L}\sum_{k=1}^{L} = \sum_{n=1}^{LM_L}\sum_{k=1}^{LM_L} - \sum_{n=1}^{L}\sum_{k=L+1}^{LM_L} - \sum_{n=L+1}^{LM_L}\sum_{k=L+1}^{LM_L} . \quad (28)$$

So we may make a triangle inequality argument using (21)'s last absolute value and (21)'s ϵ. Suppose there is for each $\epsilon'>0$ an increasing $L_i=L$ such that $|\sum_{n=1}^{LM_L}\sum_{k=1}^{L}S(k,n,B)/n^s-D|<\varepsilon'$ for all $L_i>L_{\epsilon'}>K_\epsilon$ where M_L infinite. Then

$$\varepsilon>|\sum_{n=1}^{\infty}\sum_{k=1}^{L}-D+D+R'_L-C|>|D-C+R'_L|-|D-\sum_{n=1}^{\infty}\sum_{k=1}^{L}|>|D-C+R'_L|-\varepsilon' \quad , (29)$$

a contradiction for sufficiently small $\epsilon+\epsilon'$ if $D\neq C$. This ends the proof of the Theorem, and we note it needs not the particulars of (7)'s B but, rather, only b_n be 0 for all n even, $b_n\in\{0,2\}$ for all n odd, and not even that $|B_N(s)|$ be $O(N^{1-\sigma})$. ■

Eq. (8) is thus contradicted and RH proved, the contradiction evidently stemming from (16)'s and Lemma 3's use of $\zeta^A(s)=0$. We now note that at $s\neq\sigma=1$ the partial sum $H_N(s)$ diverges by oscillating finitely as $N\to\infty$; if in eq. (10) & (12) B_N oscillates it will still be dominated by $O(1)$. At $s\neq\sigma=1$ the $N^{1-\sigma}$ in (15) may be replaced by $O(\log N)$ as, at $\sigma=1$, $0<2H_N(1)-2\log N$ bounded as $N\to\infty$. Thus even condition (i) of RH, at our paper's outset, does not rely on EP. We should, however, note that in the Theorem we may exclude values of s which are [Son pp 435-7] zeros of ζ^A on the line 1, namely $s_1=1+2k\pi i/\log 2$ for all integers $k\neq 0$ where $i^2=-1$. We can make such exclusion through the identity, for $1<\mathrm{Re}\, s$, $\zeta^A(s)/(2^{1-s}-1)=\Sigma n^{-s}$ (which we consider as 0/0, i.e., non-zero as RH requires, at these $s=s_1$) because when $s\to s_1$, for $1<\mathrm{Re}\, s$, $\zeta(s)\to\Sigma n^{-s}$, the latter divergent at $s=s_1$. (That the zeta function is analytic and finite at s_1 is non-trivial, asked 1909 in a particular way by Landau, shown 1946 by Widder.) Finally, we note our use of $B_N(s)$ has shown no contradiction for $1<\sigma$, when $B_N(s)$ converges absolutely, and $\zeta(s)\neq 0$ for $1<\sigma$ is shown, using EP, in [Ti p 2; Ed p18; In p 26]. Passage from (18) to (19) was so facile, because $\sum_{m=1}^{M_k}\frac{S(k,mk,B)}{(km)^s}=\sum_{n=1}^{kM_k}\frac{S(k,n,B)}{n^s}$ for M_k finite or infinite, that it left behind as if truism the following: consider the k's and m's of (18) to form $\{(k,m)\}_{k=1}^{W}{}_{,m=1}^{W}$, with $k\neq m$, a set of pairs of non-equal positive integers each $\leq W$ finite or infinite; then consider (19)'s denominators n as $\frac{n}{m}\times m$ to form $\{(\frac{n}{m},m)\}_{n=1}^{W}{}_{,m|n}$, with $0<m\neq\sqrt{n}$, a set of every factorization into two non-equal positive factors of every positive integer $n\leq W$; and then consider that these two sets may be identical, for $W>2$, only if W infinite. While our RH proof proceeds, i.e., while $K\to\infty$ in $\sum_{k=1}^{K} b_k/k^{2s}$, this equality of sets is inevitably achieved and seems to be a requirement of number much less demanding than the fundamental theorem of arithmetic which is required by, or which is, EP.

Section 2 of 2: Historical Context

Except for $\sigma=½$ in $|B_N(s)\zeta_N^A(s)|=O(N^{1-2\sigma})$, or for conditions laid on (6)'s C, our proof $\zeta_N^A(s)\not\to 0$ as $N\to\infty$ for $½<\mathrm{Re}\, s\leq 1$ $(s\neq 1)$ seems to give no inkling that zeta even has non-trivial zeros, let alone which values $\gamma:=\mathrm{Im}\, s$ make $\zeta(½+i\gamma)=0$, where again i^2 is -1. These γ have been abundantly found, the first few by Riemann. Before him, through investigations by Euler, Legendre, Gauss and Chebyshev, $\pi(x)$ was known to be approximated by

$$Li(x):=\int_2^x\frac{dt}{\log t}=\frac{x}{\log x}+O(\frac{x}{(\log x)^2}), \tag{30}$$

the Logarithmic integral, and von Koch [Ed p 88; In p 83] showed in 1901, with EP used, that the approximation's accuracy is as good as

$$|Li(x)-\pi(x)|\leq O(x^{\sigma}\log x), \tag{31}$$

where σ is an upper bound for the real part of the non-trivial zeta roots. RH now proven, we put $\sigma=½$. Riemann's paper, which can today seem obscure even to the initiated, is [Ed]'s appendix as English translation from German; its reading is helped with [Riese, §2], for example, as complement. Condition (i) was, using EP, independently proven [In, pp 17, 29] by Hadamard and de la Vallée Poussin; before (31), they showed [In, p 33], this time EP necessarily used, that (i) and an 1895 von Mangoldt formula proved the long-sought prime number theorem:

$$\lim_{x\to\infty}\frac{Li(x)}{\pi(x)} = \lim_{x\to\infty}\frac{x/\log x}{\pi(x)} \text{ is } 1. \tag{32}$$

A summary of (32) is that average distance between consecutive primes $q<q'$ is $O(\log q)$. Cramér, 1920, showed RH and the von Mangoldt formula imply $q'-q\leq O(\sqrt{q}\log q)$. Belief RH has practical cryptanalytic worth, by helping to factor large composites into primes, has lately been eclipsed, for example, by quantum science's new mastery of Bell's theorem and its accompanying secure message transmission via entangled photon pair production. Yet RH was known for other reasons of beauty, in a way to Riemann, indicated in [In, pp 80-1, 92-3, 101] using von Mangoldt's formula [Ed, pp 48-54; In, pp 17-18, 73-80] which relies on EP, and elaborated in [Gr]: with $\lfloor x\rfloor$, as before, the greatest integer $\leq x$, and with π=3.14159..., the well-known Fourier series for non-integer x

$$\frac{1}{2}\frac{[\lfloor x\rfloor+\frac{1}{2}]-x}{x} = \sum_{n=1}^{\infty}\frac{\sin(2n\pi x)}{2n\pi x} \tag{33}$$

contrasts with, up to a factor, and for $x\geq 2$, the approximation's relative accuracy

$$\frac{\pi(x)}{2\sqrt{x}}\frac{Li(x)-\pi(x)}{\pi(x)} \approx \frac{1/2}{\log x} + \sum_{n=1}^{\infty}\frac{\sin(\gamma_n \log x)}{\gamma_n \log x} \tag{34}$$

where $\zeta(\frac{1}{2}\pm i\gamma_n)=0$ shows every non-integer root and $0<\gamma_n$ increases with n. Had RH been false, and hence $|\gamma_n|$ non-decreasing with n, (34)'s coefficients $1/\gamma_n$ would have been replaced by complicated functions of x and so the contrast with (33) arguably spoiled. The contrast goes quite far, for just as for 50% of the values of x is $\lfloor x\rfloor+\frac{1}{2}-x<0$, a fact long unsuspected, even by Gauss, and not yet divined empirically from small x, is that $Li(x)-\pi(x)$ is also infinitely often <0. This latter is a great theorem of Hardy's peer and collaborator Littlewood, 1914, who much later -perhaps to posture- opined RH false. As (31) shows at $\sigma=\frac{1}{2}$, $Li(x)-\pi(x)$ is not 'too much' <0; there is no constant $\delta>0$ with $Li(x)-\pi(x)<-x^{\frac{1}{2}+\delta}$ for an infinite set of values x; [La] explains how this used with 1984 work of Robin allows RH is:

$$\text{for every } N>1,\ \sum_{d|N} d < H_N(1) + e^{H_N(1)}\log H_N(1)\ , \tag{35}$$

which is quite striking when each side of (35) is graphed as a function of N. Germany's Hilbert, who died in 1943, four years before Hardy, gave RH emphasis by including it in1900 in his famous Parisian list of unsolved problems. Folklore originating with Pólya, and which may later have been embellished, says that late in life Hilbert claimed his first question upon resurrection would be 'Is RH proven?' RH was by no means celebrated or encouraged in every quarter; a severe view of its worth was given 1985 [Alb & Ale, p 179] by a then leading historian of mathematics. Dirichlet, even before Riemann, and then Dedekind, Hurwitz, and Lerch, followed in the 20th century by Hasse, Hecke, Selberg, Weil, and many other mathematicians, made use of their own zeta function generalizations, each of these with its own RH and Euler product. For a welcoming, but not easy, discussion of these steep heights, see [Bo & Sar] where England's Hardy is quoted as predicting that an RH proof for the Dirichlet L function,

$$L(s,\chi) := \sum_n \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \tag{36}$$

associated to any Dirichlet character function χ, would soon arise after the RH proof for zeta. (The present author has not made such a proof for L but believes it now simple enough that all the tools necessary to the task are in [Le].) And then the Millennium Problems program of L.&L.Clay's Mathematics Institute added to RH's notoriety by attaching to it in the year 2000 a b☺unty, n☹t inflation-indexed, of $1,000,000 US, a prize to be awarded two years after proof publication in a refereed journal. Dyson and Montgomery [Gr, Bo & Sar], about 1974, and Odlyzko, 1987, confirming a hunch of Hilbert, Landau and Pólya, found quantum energy physicality in the spacing of γ_n on the line ½. So perhaps there is use in having b_k be 0 for even k, 2 or 0 randomly for odd k, as $N\to\infty$ in

$$\sum_{k=1}^{N} \frac{b_k^2}{k^{2s}} + (B_N(s) + \zeta_N^A(s))^2 - B_N(s)^2\,, \tag{37}$$

an example, for some values of B, of the difference of divergent series being itself convergent. Dirichlet series (lately through zeta function regularization in quantum field theory) have long association with the periphery of physics, arguably since 1687, through Newton's second law of motion $F = m\ddot{x}$, where $\ddot{x}$ is position x's second derivative with respect to time. For zeta's functional equation (ii), known 1749 to Euler for real s, uses the gamma function $\Gamma(s)/n^s := \int_0^\infty t^{s-1}e^{-nt}dt$, known 1729 to Euler, and valid for Re $s>0$. As Riemann noted and exploited, this integral summed for integers $0<n\le N$ is $\Gamma(s)H_N(s)$. Students may today discover Γ [Sok & Sok, pp 272-3] while seeking the time a point-mass m takes traveling in a straight line from rest to a center of attraction when the attractive force varies inversely with distance, $F \propto -1/x$.

Hardy's eccentric and charming conduct, which was sometimes deliberately very public, often used RH as a prominent prop [Sau]. If we speculate, darkly, but also whimsically, that he knew all along, or at some point, a proof of RH but wished to keep it from his charge Ramanujan, as a teacher can be wont to do when guiding a student, or to keep it from him and others for reasons of rivalry (or cryptographic security,) then we should also consider that Hardy -and others, such as Hilbert- may have had solely the motive which Fermat himself oft quoted from the Bible's *Daniel 12.4*: "*... many shall run to and fro and knowledge shall increase.*" Withholding an RH proof, even while sharing such noble motive, might have been Hardy's way of resenting Hilbert's success at setting the agenda for 20th century mathematics. Hardy and Littlewood of course knew the former's proof of (iii) did not use the fundamental theorem of arithmetic, i.e., Euler's Product, except in the inconsiderable showing that zeta has no zeros right of σ=1. (See [Ti, p 45] for the agenda of extending EP left of σ=1.) Did they at least know RH true a few years after knowing (iii) for an infinite number of zeta zeros? It's doubtful as by then Hardy's proof had been examined by many, simplified by Landau, and had inspired many useful likenesses [Ti, §X]. To continue to conceal an RH proof would be to insult these many people. So if we proceed with our whimsy, we're now up to an agreement, or a tacit agreement, involving at least four top mathematicians: Hilbert, Hardy, Littlewood and Landau. And then maybe Pólya decided to keep the game going, these men thinking, perhaps, that RH was a good way to out from the next generation of mathematicians those who were not humble and those who were 'too' reliant on past investigations not their own. (Such a quartet is of course unlikely -its exclusion of Hadamard, for example, is almost arbitrary- and the addition of Pólya much more so for by all accounts he was a warm and friendly man, unreported to be even occasionally ruthless with others, as Hardy sometimes could be [Kr, p 237]. At Cambridge, Hardy, Littlewood and Pólya wrote a text together; at Göttingen, Hilbert chose Landau to replace the prematurely deceased Minkowski.)

All members of the number theory community have benefitted so much from Hardy's career-long RH-enthusiasm, quite a few of us financially, that such conspiracy theory speculation might be taken as unnecessary and ungentlemanly, and so needs good reason, and it is this: the proof of RH just presented uses surprisingly slight mathematics, math which is much less than what so many for so long expected to be, or were taught to expect as, an advance. (The author too put many years into RH.) One thousand years ago the concept of zero was deep and sophisticated; today every schoolchild is expected to know that zero multiplied by anything finite is a product equal still to zero. One would hope that a proof of RH is more than the fact that infinity multiplied by any non-zero real number remains infinity, but it is apparently (except possibly for our infinite sum transposition) not much more than this. Riemann died seven years after his one and only number theory paper. He wrote of his own hypothesis, now a theorem: "*One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation.*" How right he was to do so. Let us conclude upbeat. We can all, this and past generations of number theorists, take solace, if any is even needed, in what has evidently always been both the underlying strength and persistent allure of the Riemann Hypothesis: that so much of arithmetic is informed by it, yet that arithmetic herself is virtually bypassed to obtain it.

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shawnjames.riemann@yahoo.ca

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