→ We use numbers for counting

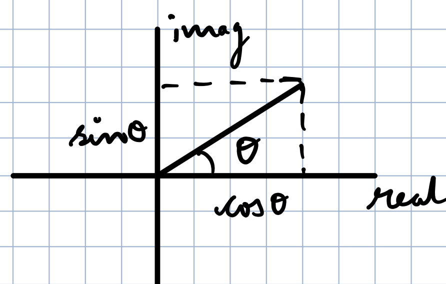
- 2 apples
- 3 students

→ Negative numbers → a way to signify opposite

I owe \$200 to my bank ⇒ bank balance = −\$200

→ Complex numbers → model rotation

$\cos\theta + i\sin\theta$ →

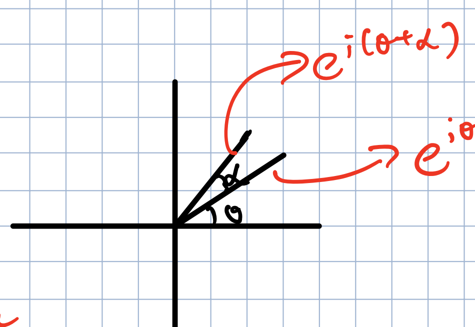


↓

Represents a vector of magnitude '1', making an angle θ above 'real' axis

$$\cos\theta + i\sin\theta = e^{i\theta} \quad \text{(Euler notation)}$$

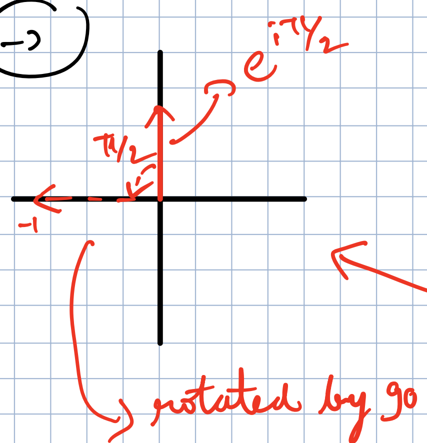
$$e^{i\theta} \cdot e^{i\alpha} = e^{i(\theta+\alpha)}$$



↓ multiply by $e^{i\alpha}$

↓ rotates a vector by an angle α

$\circlearrowleft$ →



Consider $e^{i\pi/2}$

Rotation by 90° (multiplying $e^{i\pi/2}$) should give -1

$$e^{i\pi/2}\, e^{i\pi/2} = -1$$

$$\left(e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i\right)$$

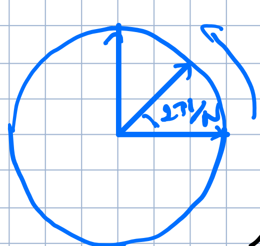
$$i \times i = -1$$

$$\boxed{i^2 = -1}$$

→

Consider a rotating complex number whose frequency of rotation is such that it finishes 1 cycle (2π radians) in N time-steps

$\Rightarrow$ Finishes $\frac{2\pi}{N}$ radians in 1 timesteps



The N points the complex number would trace is given by

$$f_1 = \left[e^{i0},\ e^{i\frac{2\pi}{N}},\ e^{i2\cdot\left(\frac{2\pi}{N}\right)} \ldots\ldots\ e^{i(N-1)\frac{2\pi}{N}} \right]$$

Similarly lets consider other frequencies f_k such that f_k moves faster than f_1 by k times (finishes k cycles in N timesteps)

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} f_0 & f_1 & f_2 = 2f_1 & \cdots & f_m = mf_1 & \cdots \; f_{N-1} \end{array} \\
\begin{array}{c} n=0 \\ n=1 \\ n=2 \\ \vdots \\ n=N-1 \end{array} &
\left[\begin{array}{cccccc}
e^{i0} & e^{i0} & e^{i0} & \cdots & e^{i0} & \cdots \\
e^{i0} & e^{i\frac{2\pi}{N}} & e^{i\left(2\cdot\frac{2\pi}{N}\right)} & \cdots & e^{i\left(m\frac{2\pi}{N}\right)} & \cdots \\
e^{i0} & e^{i2\cdot\frac{2\pi}{N}} & e^{i2\cdot\left(2\cdot\frac{2\pi}{N}\right)} & \cdots & e^{i2\cdot\left(m\frac{2\pi}{N}\right)} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
e^{i0} & e^{i(N-1)\cdot\frac{2\pi}{N}} & e^{i(N-1)\left(2\cdot\frac{2\pi}{N}\right)} & \cdots & e^{i(N-1)\left(m\cdot\frac{2\pi}{N}\right)} & \cdots
\end{array}\right]
\end{array}
$$

Each column in the above matrix is orthogonal

$$\left(f_m^*\right)^T f_n = 0 \quad \forall \; m \neq n$$

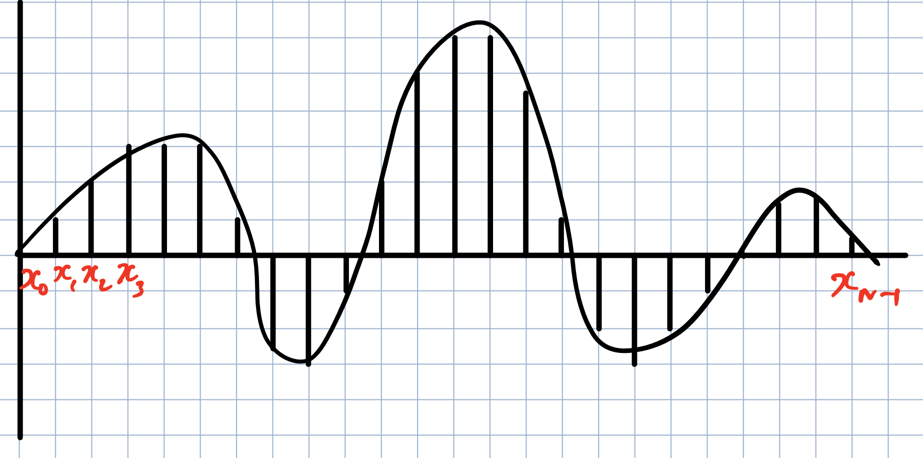
$f_m \perp f_n \Rightarrow$ The set of frequency vectors

$(f_0, f_1, f_2 \text{ ---- } f_{N-1})$

form a basis of C^N (N dim complex numbers)

They also form a basis of R^n (Since R^n is a subset of C^N)

$\rightarrow$ Now, a given data vector can be projected on any basis



$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & 1 & \vdots \\ \vdots & & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

$\langle x_0 \; x_1 \; \ldots \; x_{N-1} \rangle$ is typically represented as a linear combination of standard basis vectors (columns of $N \times N$ Identity matrix)

→ Similarly, the same vector can be represented as a linear combination of frequency basis

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ f_0 & f_1 & f_2 & \cdots & f_{N-1} \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \end{bmatrix}$$

→ frequency domain representation

$$\vec{x} = F_1 \; \vec{c}$$

$$\vec{x} = F_1 . \vec{c}$$

A scaling factor of $(1/N)$ is included in the above equation for normalization purposes

$$\vec{x} = \left(\frac{1}{N}\right) F_1 . \vec{c}$$

$\langle c_0, c_1 \ldots c_n \rangle$ is the frequency domain representation of $\langle x_0\, x_1 \ldots x_n \rangle$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \end{bmatrix} = \left\{ \left(\frac{1}{N}\right) \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ f_0 & f_1 & f_2 & \cdots & f_{N-1} \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} \right\}^{-1} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

Since $(f_m^*)^T f_n = 0 \quad \forall\, m \neq n$,

the inverse of the matrix in the above equation simplifies as below

$$
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} \longleftarrow & f_0^* & \longrightarrow \\ \longleftarrow & f_1^* & \longrightarrow \\ \longleftarrow & f_2^* & \longrightarrow \\ & \vdots & \\ \longleftarrow & f_{N-1}^* & \longrightarrow \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}
$$

* denotes complex conjugate
$(x+iy)^* = x - iy$

$$\vec{c} = F\vec{x}$$

(Discrete Fourier Transform)

F is the DFT matrix, which transforms an input vector $\vec{x}$ into the frequency domain

Revisit equation (1)

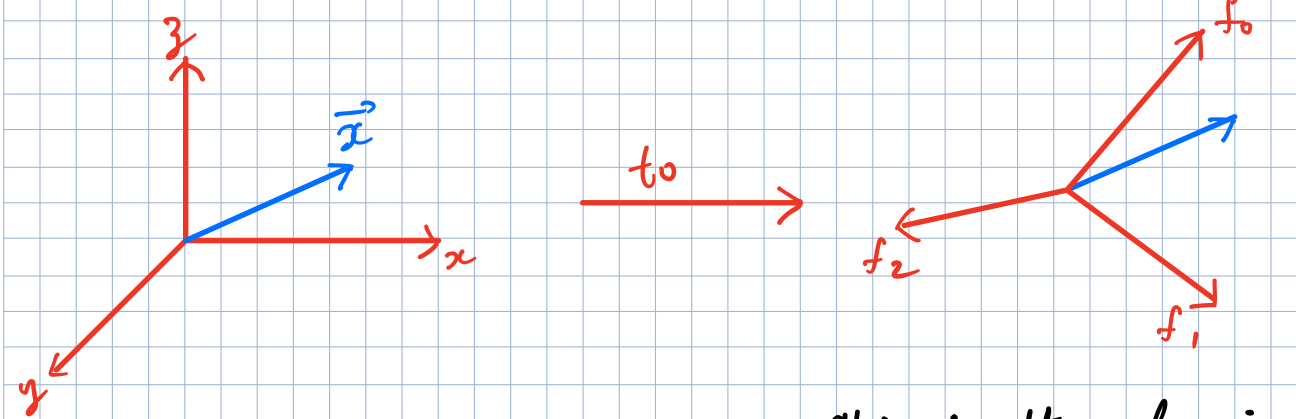
$$\vec{x} = \left(\frac{1}{N} F_1\right)\vec{c}$$

(IDFT matrix or Inverse DFT matrix)

$$\vec{x} = F^{-1}\vec{c}$$

Key take away message:

DFT is a change of basis



This basis is I matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

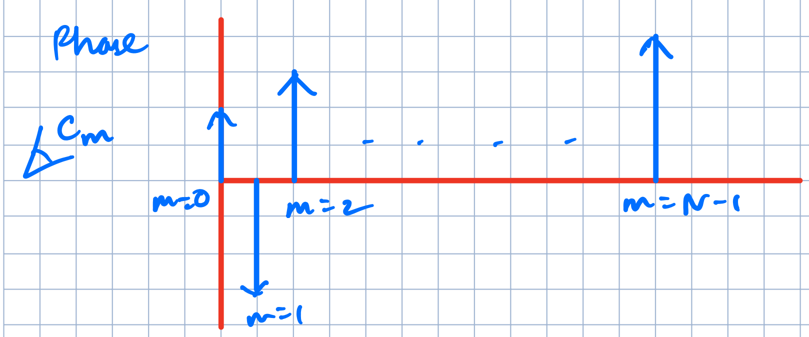
This is the fourier basis matrix

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ f_0 & f_1 & f_2 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

→ Frequency co-efficients $\langle c_0 \; c_1 \; - - - \; c_{N-1} \rangle$ are complex numbers, a plot of DFT consists of

(1) Magnitude plot

(2) Phase plot (Angle of c_m)



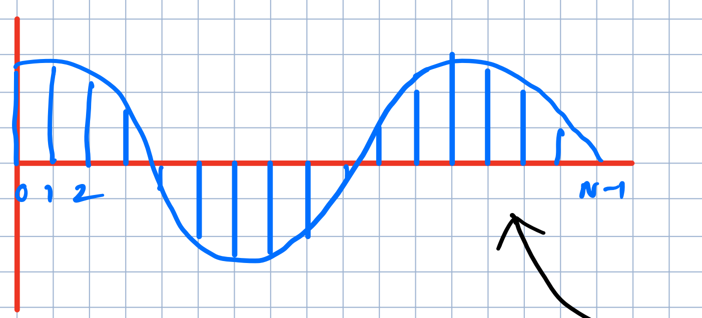
(→) Examples

Consider $\cos 2\pi f_m t$

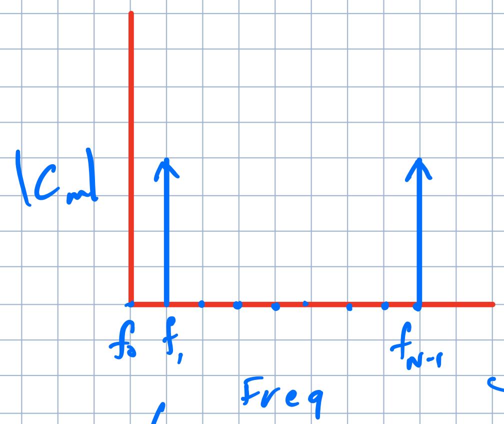
Discrete sampled signal

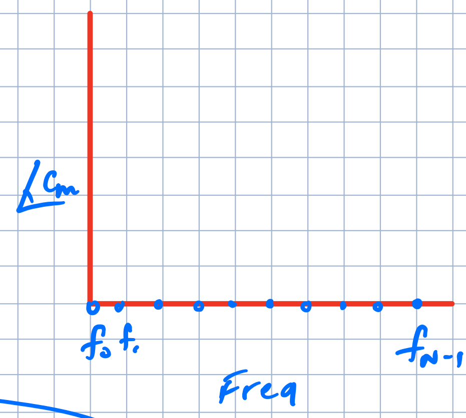
$x(n) = \cos 2\pi f_m n t_s$, where $t_s = \frac{1}{f_s}$

is the sampling interval, f_s is sampling freq



(→) what is the DFT of $x(n) = \cos 2\pi f_1 n t_s$





rotates anti-clockwise with frequency f_1

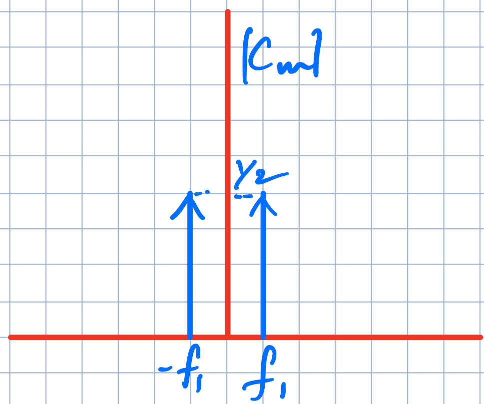
rotates anti-clockwise with frequency f_{N-1}

$\boxed{\text{or}}$

rotates clockwise with frequency f_1

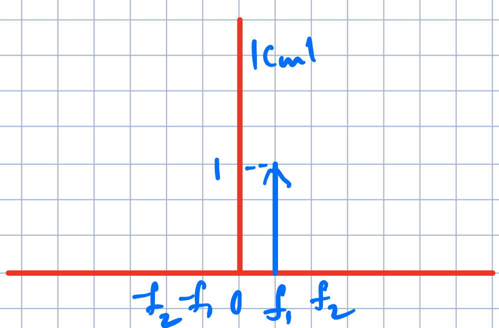
Thus, f_{N-1} can be interpreted as a negative frequency $-f_1$ which rotates in the opposite direction of f_1 but at the same frequency

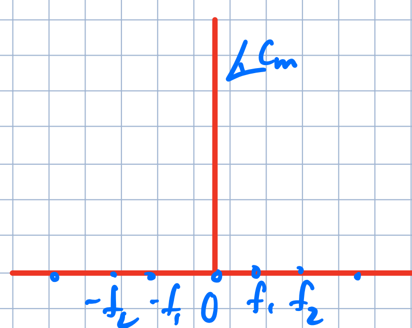
DFT of $\cos 2\pi f_1 n t_s$



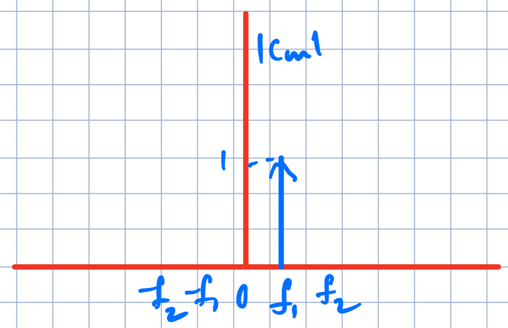
→ Real signals always have symmetric DFT magnitude because negative frequencies are needed to eliminate imaginary parts of frequency basis

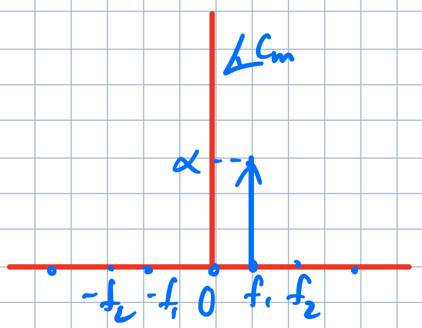
→ What is the DFT of $x[n] = e^{i 2\pi f_1 n t_s}$



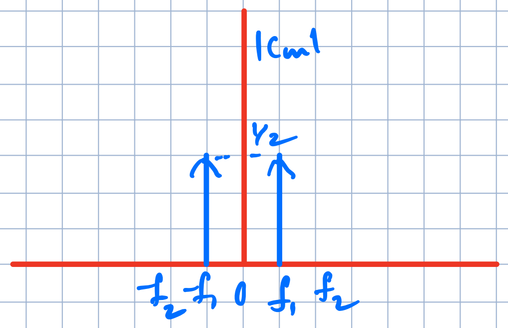


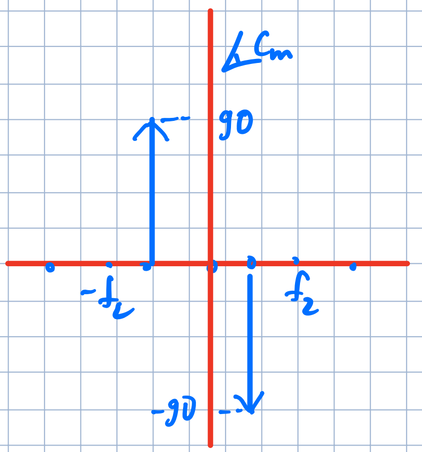
→ What is the DFT of $x[n] = e^{i(2\pi f_1 n t_s + \alpha)}$





→ What is the DFT of $x[n] = \sin 2\pi f_1 n t_s$





For real signals, the phases are anti-symmetric

(→) Relationship between DFT, f_s and N

f_1 finishes 1 cycle in N samples

" " " in Nt_s seconds

(each sample is t_s second long, t_s is sample time)

f_1 finishes $\frac{1}{Nt_s}$ cycles in 1 second

$$f_1 = \frac{1}{Nt_s} = \frac{f_s}{N} \text{ Hz} \quad \left(f_s = \frac{1}{t_s} \text{, sampling freq} \right)$$

$$f_2 = 2 \cdot f_1 = \frac{2f_s}{N}$$

$$f_m = mf_1 = \frac{mf_s}{N} \text{ Hz}$$