Linear Independence: A set of vectors $(v_1, v_2, \ldots v_n)$ are linearly independent if and only if the below equation

$c_1 v_1 + c_2 v_2 + \ldots\ldots c_n v_n = 0$ has only one solution given by $c_1 = c_2 = \ldots\ldots = c_n = 0$

<u>Example</u>: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 8 \\ 3 \\ 2 \end{bmatrix}$ are not linearly independent because

$$\underbrace{5}_{\text{non-zero}} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 8 \\ 3 \\ 2 \end{bmatrix} = 0$$

The zero vector cannot be a part of a <u>linearly independent set</u>

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 10 \\ 6 \end{bmatrix}$ are not linearly independent since $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Two linearly dependent vectors lie on a line $\rightarrow$ $c_1 v_1 + c_2 v_2 = 0 \Rightarrow v_2 = \left(-\frac{c_1}{c_2}\right) v_1$

$(c_1, c_2 \neq 0)$

Three linearly dependent vectors lie on a plane $\rightarrow c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow$

$$v_3 = \left(-\frac{c_1}{c_3}\right) v_1 + \left(-\frac{c_2}{c_3}\right) v_2 = 0$$

Columns of below matrix are linearly independent

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

If columns of a matrix A are linearly independent, then $N(A) = \phi$ (Null set)

Inverse of a square matrix (A) is defined as below

$$A A^{-1} = I$$
$$A^{-1} A = I$$

A^{-1} is unique

suppose $A^{-1} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$ for a 3×3 matrix A

then $AA^{-1} = I$

$$A \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Existence and uniqueness of above three equations $\Rightarrow$ $N(A)$ is $\emptyset$

Inverse exists only if columns are linearly independent

Basis for a vector space V is a set of vectors with below properties

1) Vectors are linearly independent
2) They span V

Ex: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for R^3

$\begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}4\\5\\6\end{bmatrix}$ is not a basis of R^3 because it does not span R^3 (it only spans a 2D plane)

$\begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}4\\5\\6\end{bmatrix} \begin{bmatrix}10\\11\\9\end{bmatrix}$ is a basis for R^3

$\begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}4\\5\\6\end{bmatrix} \begin{bmatrix}10\\11\\9\end{bmatrix} \begin{bmatrix}7\\6\\5\end{bmatrix}$ spans R^3 but not a basis since they are linearly dependent

Basis for V is not unique

Ex: $\left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$ can both be basis of R^2

Dimension of a vector space is the number of vectors in the basis set

Ex: $\Rightarrow$ dimension of R^2 is 2

Dimension of a vector space is constant regardless of the basis chosen

Rank of a matrix A is the dimensions of its column space (Number of linearly independent columns)

Subspace is a subset of a vector space that satisfies all properties of a vector space

Ex: XY plane is a subspace of R^3

I quadrant is not a subspace of R^2 since it does not satisfy properties of a vector space $[(-1)\times(2,3) = (-2,-3) \notin \text{I Quad})$

Inner product of two $n\times 1$ vectors x & y is defined as follows

$x^T y$, here x^T is transpose of x

(x^T is a $1\times n$ vector)

$$x^T y = \sum_{i=1}^{n} x_i y_i$$ where x_i, y_i are i^{th} components of vectors x & y

$$[x = (x_1, x_2 ---- x_i --- x_n)]$$

Vectors x and y are **orthogonal** if

$$x^T y = 0$$

Ex: $(1,1)$ and $(1,-1)$ are orthogonal

Four fundamental subspaces

A $n \times m$ matrix A has following 4 subspaces

1) Column space $C(A)$ with dimension r (rank of matrix). Columns of A are members of R^n. $C(A)$ is a subspace of R^n

2) Null space $N(A)$. Its dimension is $m-r$. Vectors in $N(A)$ are members of R^m. $N(A)$ is a subspace of R^m.

3) Row space of A is the column space of A^T. Denoted by $C(A^T)$, it is spanned by the linear combination of rows of A. Its dimension is also r.
$C(A^T)$ is a subspace of R^m

4) **Left nullspace** of A is the nullspace of A^T $[N(A^T)]$. It contains all vectors y such that $A^T y = 0$. Its dimension is $n-r$. $N(A^T)$ is a subspace of R^n

The row space is orthogonal to nullspace:

<u>Proof</u>: If x is a vector in $N(A)$, then $Ax = 0$

$$Ax = \begin{bmatrix} \text{--- row1 ---} \\ \text{--- row2 ---} \\ \vdots \\ \text{--- rown ---} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

row1 is orthogonal to x

row2, row3 ... rown are all orthogonal to x. Therefore, x is orthogonal to all linear combinations of rows.

Every $x \in N(A)$ is orthogonal to row space

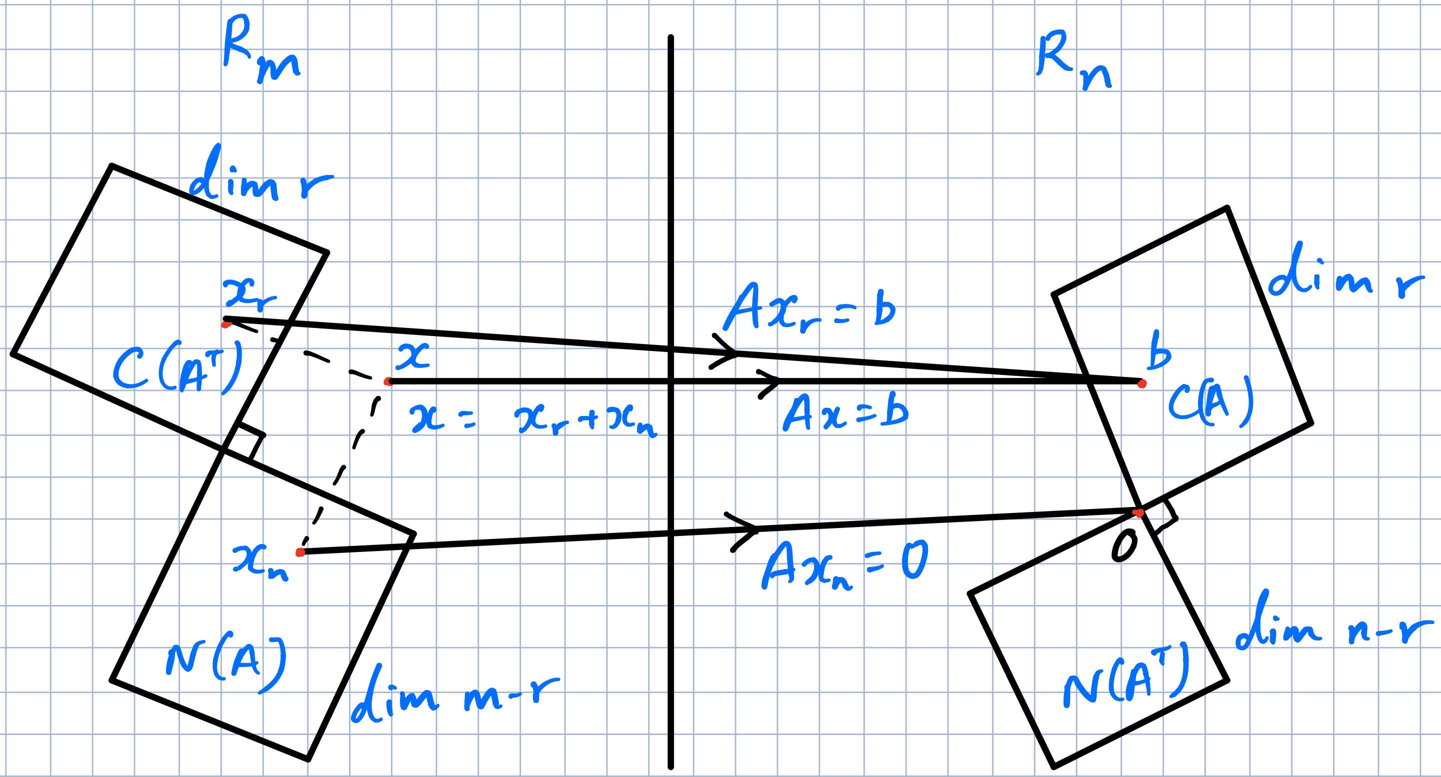
$$N(A) \perp C(A^T)$$

Similarly $N(A^T) \perp C(A)$

What does $N(A^T) \perp C(A)$ mean?

$C(A)$ is a subspace of R^n with dimension "r". Since $r \leq n$, the $C(A)$ may not span entire R^n. This uncovered gap $n-r$ is filled by $N(A^T)$ whose dimension is $n-r$. $N(A^T)$ spans the gap left by $C(A)$. Similarly $N(A)$ spans the gap left by $C(A^T)$

The figure below visualizes the 4 fundamental subspaces together with their relationship



A matrix A transforms a vector x in R_m to a vector b in R_n.

x can be decomposed into sum of its projections on row space (x_r) & null space (x_n)

$$Ax = b$$
$$x = x_r + x_n$$
$$Ax_r = b$$
$$Ax_n = 0$$
$$A(x_r + x_n) = b$$

The action of a matrix A to transform a vector x in R_m to vector b in $C(A)$ is an example of linear transformation

A linear transformation $T(x)$ satisfies two properties

$$T(x+y) = T(x) + T(y)$$
$$T(cx) = c\,T(x) \quad [c \text{ is a scalar}]$$

Linear transformations appear in several applications of computer graphics, smartphone gesture tracking, electromagnetics, MIMO, & quantum mechanics etc

Ex: A reflection matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflects a point about the line $x = y$ in 2D

$A = \begin{bmatrix} 2c^2-1 & 2cs \\ 2cs & 2s^2-1 \end{bmatrix}$ reflects a point about a line through origin making an angle θ with x-axis

$(c = \cos\theta,\ s = \sin\theta)$

A rotation matrix $R = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

$(c = \cos\theta,\ s = \sin\theta)$

rotates a point by an angle θ around an axis passing through z-axis

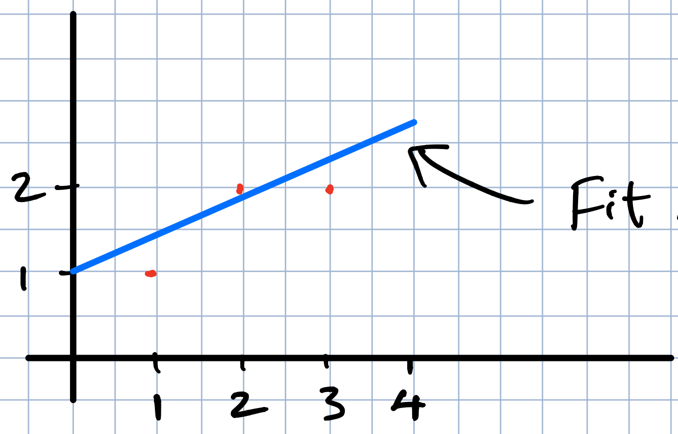
- A smartphone gyroscope provides 3D rotation matrices corresponding to changes in phone orientation.

Phone orientation tracking is a building block of several gesture tracking applications.

Least squares

$Ax = b$ has a valid solution only if b lies in $C(A)$.

In real world, we encounter many applications where this is not the case



m and c are unknown

$x = 1$: $m + c = 1$

$x = 2$: $2m + c = 2$

$x = 3$: $3m + c = 2$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} m \\ c \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

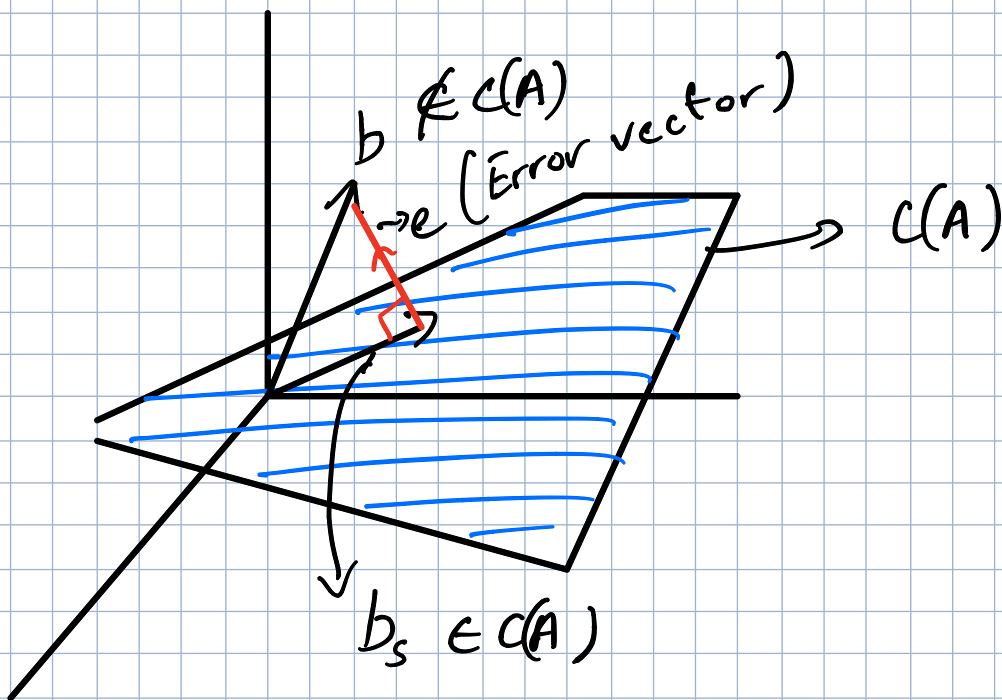
The above equation does not have a solution since $b \notin C(A)$

Suppose x_s is an approximate solution, then $Ax_s = b_s \in C(A)$

The error vector of approximation

$e = b_s - b$

A least squares solution minimizes the norm of e ($||e||$)



$||e||$ is minimized when

$e \perp C(A)$

$$\begin{bmatrix} — & c_1 & — \\ — & c_2 & — \\ & \vdots & \\ — & c_m & — \end{bmatrix} \begin{bmatrix} e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$A^T \quad e = 0$$

$$A^T e = 0$$

$$A^T (b_s - b) = 0$$

$$A^T (A x_s - b) = 0$$

$$A^T A x_s = A^T b$$

$$\boxed{x_s = (A^T A)^{-1} A^T b}$$

x_s is the least squares solution to $Ax = b$

$$b_s = A x_s = A (A^T A)^{-1} A^T b$$

is the projection of b on $C(A)$

$$b_s = Pb$$

$$\boxed{P = A(A^TA)^{-1}A^Tb}$$

P projects a vector into $C(A)$

Quiz: $P^2 = P$. why?