Simultaneous system of equations

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

$$\left.\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right\} \rightarrow \text{Matrix form}$$

Appears in many applications → MIMO, GPS etc

Typically solved by "Gaussian Elimination"

Example equation:

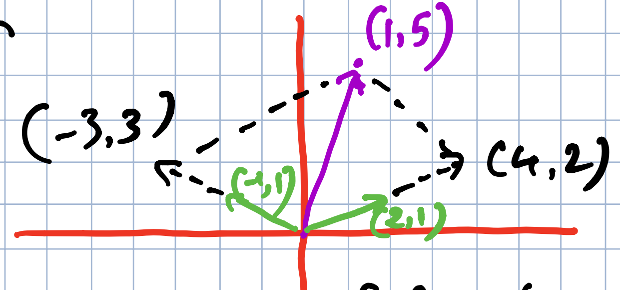
$$2x - y = 1$$
$$x + y = 5$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Geometric view

$\begin{bmatrix}1\\5\end{bmatrix}$ is a "linear combination" of $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$



$$\begin{bmatrix}1\\5\end{bmatrix} = 2\begin{bmatrix}2\\1\end{bmatrix} + 3\begin{bmatrix}-1\\1\end{bmatrix}$$

$x = 2, y = 3$

Linear Combination: Given "n" vectors $v_1, v_2, v_3, \ldots v_n$, and "n" scalars $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$

then any vector V_l which can be expressed as below is a linear combination

$$V_l = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_n v_n$$

Vector Space: Set of all vectors (V) that satisfy two properties

1) $\vec{u}, \vec{v} \in V$, $\vec{u} + \vec{v} \in V$
2) $\vec{u} \in V$, then $c\vec{u} \in V$ (c is a scalar)

Ex: R^2, set of all 2D vectors

Quiz: Vector space or not?

(1) 2D plane?

(2) one quadrant?

(3) line $x = y$?

(4) zero vector?

Column space of a matrix A ($C(A)$) is the set of all linear combinations of columns in A.

Ex: $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}$

$$C(A) = u\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + v\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

A linear combination of two vectors

$\rightarrow$ $C(A)$ is a plane

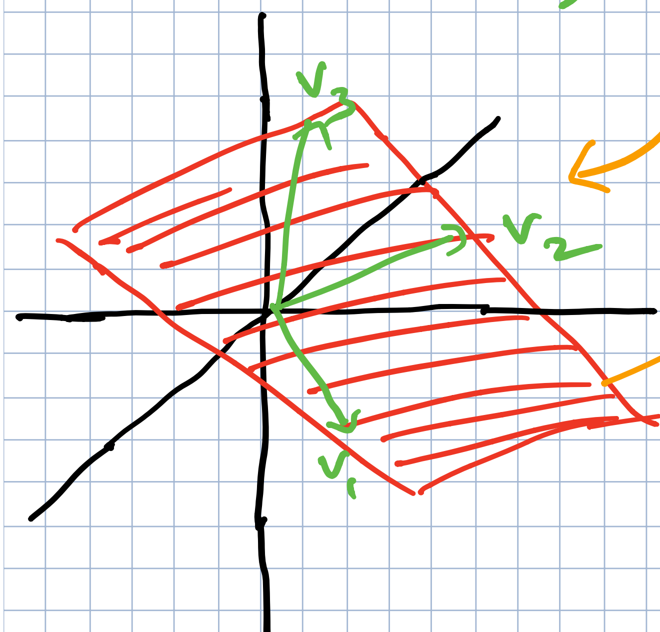
$B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $C(B)$ is R^2

$D = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $C(D)$ is a line since the second column is a multiple of first

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $C(E)$ is R^3, the 3D space

$$F = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix}$$
$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \end{array}$$

$C(F)$ is not R^3, since the three column vectors lie on a plane



$C(F)$ is a 2d plane

Any vector outside the plane is not in $C(F)$

Null Space of a matrix A $[N(A)]$ is the set of vectors $x_n \neq 0$ which satisfy $Ax_n = 0$

Null space is a vector space

→ $x_{n_1}, x_{n_2} \in N(A)$, then $x_{n_1} + x_{n_2} \in N(A)$
since $A(x_{n_1} + x_{n_2}) = Ax_{n_1} + Ax_{n_2}$
$= 0 + 0 = 0$

→ $x_n \in N(A)$, then $cx_n \in N(A)$
since $A(cx_n) = c(Ax_n) = c * 0 = 0$

$$A = \begin{bmatrix} 1 & 4 & 9 & -1 \\ 2 & 6 & 14 & 0 \\ 3 & 11 & 25 & 2 \end{bmatrix}, \quad x_{n_1} = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$x_{n_1} \in N(A)$
$\because Ax_{n_1} = 0$

$$x_{n_2} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_{n_2} \in N(A)$$

All linear combinations
$ux_{n_1} + vx_{n_2} \in N(A)$

Column and null spaces have interference management applications in LTE (MIMO, Beamforming)

Existence of a solution to $A\vec{x}=b$

Consider the below set of equations

$$x + y + 2z = 1 \quad --- \; (1)$$
$$2x + y + 3z = 8 \quad --- \; (2)$$
$$3x + 4y + 7z = 10 \quad --- \; (3)$$

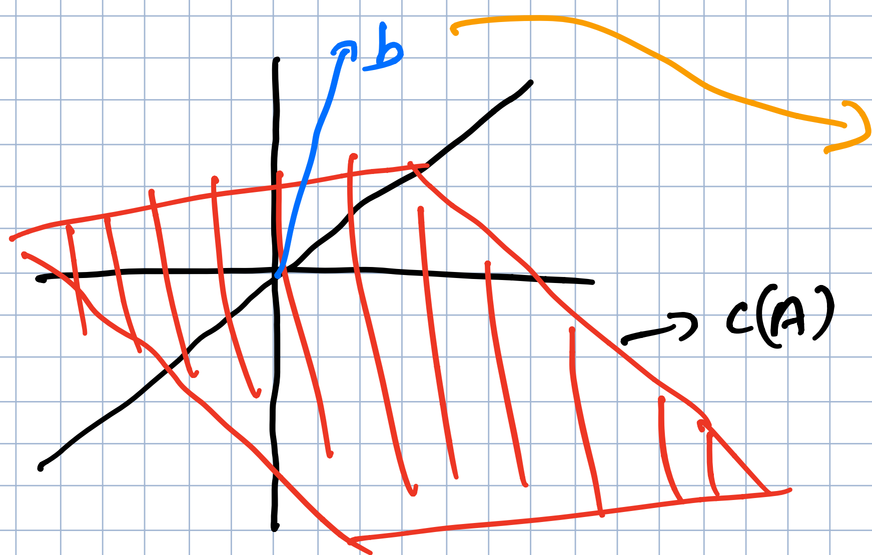
$$x + y + 2z = 1$$
$$\boxed{\begin{aligned} -y - z &= 6 \quad --- \; (2) - 2*(1) \\ y + z &= 7 \quad --- \; (3) - 3*(1) \end{aligned}}$$

$\hookrightarrow$ Inconsistent equations

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 4 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 8 \\ 10 \end{bmatrix}}_{b}$$

$$\underbrace{x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}1\\1\\4\end{bmatrix} + z\begin{bmatrix}2\\3\\7\end{bmatrix}}_{\text{lies in } C(A)} = \begin{bmatrix}1\\8\\10\end{bmatrix}$$

$C(A)$ is a plane in 3D since $\begin{bmatrix}2\\3\\7\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}1\\1\\4\end{bmatrix}$



b lies outside $C(A)$ hence no solution exists to $A\vec{x} = b$

Solution to $A\vec{x} = b$ exists if and only if $b \in C(A)$

Uniqueness of solution to $A\vec{x} = b$

If x_s is a solution, then

$Ax_s = b$

If $x_n \in N(A)$ $(Ax_n = 0)$, then

$$A(x_s + x_n) = Ax_s + Ax_n$$
$$= b + 0$$
$$A(x_s + x_n) = b$$

$\Rightarrow$ ✓ $x_s + x_n$ is also a solution

A non-empty $N(A)$ has infinite elements

$\Rightarrow$ ① If $N(A)$ is not empty and $b \in C(A)$, we have infinite solutions

② If $N(A)$ is empty, and $b \in C(A)$, we have a unique solution

[An empty $N(A)$ leads to a unique linear combination $\rightarrow$ more on this in the next lecture]