Puzzle Compendium

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Abstract

I have always harbored a long-standing fascination with brainteasers. However, after one is solved, it becomes trivial (modulo forgetting the solution). As such, over time, it becomes necessary to find increasingly more difficult puzzles. This is my effort to consolidate those I have encountered which I consider to be some of the more challenging, high-quality, mathematically interesting, or otherwise noteworthy of the bunch.

1 Introduction

What makes a good puzzle? Solvability alone is not enough –"what is 2×3?" poses little challenge and thus little payoff. Difficulty alone is not enough – it may be hard to factorize large semiprimes, but this is also not a particularly interesting task (cryptographers may disagree). In fact, I find that a key theme from cryptography is a clean and concise way to express quality: A good puzzle is a problem which has a solution which is difficult to find, but makes sense (maybe with some hard thinking). The ratio of of difficulty to solve to cleverness of the actual answer is perhaps my best proxy for puzzle quality. The sense of insight one feels from truly understanding the solution to a puzzle is, in my opinion, the most significant indicator of puzzle quality. For some puzzles, I find that even being told the solution does not provide the level of satisfaction from understanding how it works. I attempt to highlight some puzzles which I have felt most satisfied this latter feeling. The puzzles are collected thematically, and sorted in order of my perception of relative difficulty – though this ranking is somewhat subjective and may not be entirely accurate. I make no claim to be the author of any of these puzzles, many of which are folklore. Unless explicitly stated otherwise, you may assume that any of the characters in the following puzzles are perfect logicians.

2 Hat Puzzles

These are puzzles often phrased to involve a sadistic prison warden lining up some number of prisoners and placing hats of various colors on their heads. The prisoners are instructed to each guess the color of their own hat(s), with an offer of freedom contingent upon some rate of successful guesses, and sometimes a punishment in the case of failure. However, they may not look at their own hats and are not allowed to communicate with each other. If the prisoners have advance knowledge of the warden's scheme, can they collectively plan ahead and devise a strategy to improve their chance of success?

There are some non-mathematical loopholes to this problem. For example, one could posit the strategy "everyone lines up, and speak your guess in less than 3 seconds if the next person's hat is blue, and more than 3 seconds if it's red." While these types of tactics may be effective in a real-life scenario, allowing them detracts from the puzzle's quality, so these will not be mentioned as "correct" solutions. The only permitted transfer of information is that which is part of the puzzle.

In the interest of somewhat reframing the power dynamics, I will pose these problems as games played between Papa Gnome and a set of his gnome children. This also has the benefit of appealing to gnomes' well-known penchant for both hats and mathematics.

1. Suppose n children play a game with Papa, where he places either a red or blue hat on each child (he has an infinite supply of any color of hat). They can each see every one else's hat, but not their own. The children guess sequentially (so they can hear the previous guesses), and win if every one of them is able to correctly guess their own hat color. What strategy can the children implement to maximize their chance of winning, and what is the probability?

   Solution:

2. This setup is the same as the previous puzzle, but now Papa has k different colors of hats.

   Solution:

3. One from reddit. Papa Gnome gathers n children and writes a positive whole number on their backs. Each child can see everyone else's numbers, but not their own, and all the numbers are different. Each child has a red hat and a blue hat in front of them. When Papa rings his bell, everyone as to pick a hat and put them on. Then Papa will line them up in ascending order of their numbers. If their hats alternate red-blue-red-blue-$\cdots$ or blue-red-blue-red-$\cdots$, then the children win. Otherwise Papa wins. What strategy should the children use to maximize their chance of winning?

   Solution:

In all further hat puzzles, at least one set involved may have infinite cardinality. We lay some ground rules for reasoning about the infinitary: You are allowed to freely use the axiom of choice (this is also a hint!). If you do not know what this is, you may find this next stretch of puzzles particularly challenging. You can assume that all actions happen instantaneously, so that gnomes are capable of performing infinitely many actions. Additionally, they have infinite memory (to help take advantage of any results of choice).

4. Papa plays the same game with countably infinitely many children, who each get either a red or blue hat (again, assume he has enough hats of each color for every child). The children

then all guess their own hat colors simultaneously. The children win if all but finitely many of them guess correctly.

Solution:

5. The same setup as the previous puzzle with a countably infinite number of children, but now the children answer in sequence. They win if all but one of them guess correctly.

   Solution:

6. Papa Gnome has a countably infinite number of boxes, each containing a slip of paper with a real number written on it (they do not have to be distinct). He challenges Abby Gnome to the following game: She can look at as many of the numbers as she wants (potentially infinitely many) as long as she leaves at least one box untouched. She then has to guess the value of a number in a box that she has not opened. Prove that Abby can do this with probability $1 - \varepsilon$ for any $\varepsilon > 0$.

   Solution:

7. These next few, including solutions, are straight from reddit. The setup is as follows: Papa chooses a (possibly infinite) group of children and a (possibly infinite) pallet of hat colors, which are known to everybody. Colored hats get distributed among the children, with each color potentially appearing any number of times. Each child can see everyone else's hat but not their own. Everyone must simultaneously make a guess about the color of their own hat. Can you find a strategy that ensures:

   (a) If just one child guesses their hat color correctly, then everyone will guess correctly.

(b) Exactly one person guesses correctly, given that the cardinality of people is the same as the cardinality of possible hat colors.

Solution:

3 Probability Puzzles

This section contains riddles related to elementary probability. Though measure-theoretic results may be used, they should not be necessary.

8. Given a biased coin with probability p of heads, how can you simulate a fair coin using only a finite number of flips in expectation?

   Solution:

9. Given a fair coin, how can you simulate a biased coin with probability p of heads using only a finite number of flips in expectation?

   Solution:

4 Graph Theory Puzzles

This section contains riddles related to finite graph theory. Depending on your background, these may not be too difficult. Some graph conventions: $G = (V, E)$ for both directed and undirected graphs, and we typically use n to denote the number of vertices and m the number of edges. The neighbors $N(v)$ of a vertex v are v and all adjacent vertices.

10. Suppose that G is a graph with chromatic number 6. Prove that there are two vertex-disjoint odd cycles in G.

    Solution:

11. Let G be a graph on 3^n vertices. For each vertex v, count the number of vertices which are not adjacent to v, $|V \setminus N(v)|$. If the sum of these counts over all vertices is less than 3^k for some integer k, prove that there is a vertex with degree at least $3^n - 3^{k-n}$.

    Solution:

12. Let G be a directed tournament graph (a directed graph with exactly one edge between every pair of distinct vertices). We say a vertex u *dominates* another v if either: (1) $(u, v) \in E$, or (2) there exists a vertex w such that $(u, w), (w, v) \in E$. Prove that there is a vertex that dominates every other vertex.

    Solution:

13. Recall that $\chi(G)$ is the minimum number of colors needed to color an undirected graph G. G is called *color-critical* if $\chi(G - v) < \chi(G)$, for all $v \in V$. If $G = (V, E)$ is color-critical then show that $\deg(v) \geq \chi(G) - 1$, for all $v \in V$.

   Solution:

14. Let $k \in \mathbb{Z}^+$. Show that, for any graph G, it is always possible to k-color the vertices such that at least $\frac{k-1}{k}|E|$ edges have different-colored endpoints. (Equivalently, any graph contains a k-partite subgraph with at least $\frac{k-1}{k}|E|$ edges).

   Solution:

15. Let $k \in \mathbb{Z}^+$ and let G be a graph where, for every vertex v, $\deg(v) = \frac{1}{k} \sum\limits_{u \in N(v)} \deg(u)$. Prove that the maximum possible degree of any vertex in such a graph is $k^2 - k + 1$ and show that this bound is tight.

   Hint:

Solution:

5 Math Puzzles

This section consists of a collection of general mathematical riddles. The math required varies from high school to graduate-level, though the actual solution may be trickier than this suggests.

16. Prove that every real number can be written as a finite sum of real numbers, each of which contains only the digits 0 and 3.

    Solution:

17. Prove that in any set of n integers, there is either a number divisible by n, or there are two numbers whose difference is divisible by n.

    Solution:

18. Prove that in any set of $n+2$ integers, there are two numbers such that either their sum or difference is divisible by $2n$. (I think this is an old Putnam question but haven't been able to track it down)

Solution:

19. Prove, for any natural number n, that it is possible to select 2^n numbers from any arbitrary collection of 2^{n+1} integers (not necessarily distinct) such that that sum of the 2^n numbers is divisible by 2^n.

    Solution:

20. Let p be a three-digit prime. Prove that there exists repdigit (a number which consists only one digit repeated) which is divisible by p.

    Solution:

21. It is well-known that the derivative of x^2 is $2x$. But if we rewrite the product $x \cdot x$ as a sum, the following happens:

$$\begin{aligned}
\frac{d}{dx}(x^2) &= \frac{d}{dx}(x + x + \cdots + x, x \text{ times}) \\
&= \frac{d}{dx}(x) + \frac{d}{dx}(x) + \cdots + \frac{d}{dx}(x), x \text{ times} \\
&= 1 + 1 + \cdots + 1, x \text{ times} \\
&= x
\end{aligned}$$

What went wrong?

Solution:

22. Alice and Bob play the following game: Alice chooses a polynomial f with positive integer coefficients. Bob can adaptively query the value of f on some points $x \in \mathbb{Z}$. After k guesses, he must guess the identity of f and wins if he is correct; otherwise, Alice wins. What is the minimum k for which Bob has a winning strategy?

Solution:

23. The same game as above, but now Bob is allowed to query the value of f on $x \in \mathbb{R}$.

Solution:

24. Another from reddit. Alice and Bob play a game on the reals. Alice starts by selecting an uncountable subset $S_0 \subseteq \mathbb{R}$. Then they alternate selecting $S_1, S_2, \ldots$, each of which is uncountable, such that $S_i \supseteq S_{i+1}$. They play for a countably infinite number of steps. Alice wins if $\bigcap_{i \in \mathbb{N}} S_i$ is empty; otherwise Bob wins. Who has a winning strategy?

Solution:

25. Suppose you are given a k-coloring of the d-dimensional integer lattice $\mathbb{Z}^d$. Prove that there exists a d-hyperrectangle such that all of its vertices are the same color. (This is a generalization of an old USAMTS problem)

Solution:

26. The coast guard is trying to track down a rogue pirate ship somewhere in $\mathbb{R}^2$. The only thing they know is the last time the ship was seen, and at that time, it was at an unknown lattice point (a element of $\mathbb{Z}^2$) and has set a course to travel along a straight line with that passes through at least one other lattice point at an integer speed (in units/day). The coast guard has one plane they can send to check a single location (anywhere they want) every day, with a vision radius of 1. Can they devise a strategy to guarantee that they find the ship in a finite amount of time?

Solution:

6 Other Puzzles

Though mathematical knowledge may be helpful, it is less explicitly required for this more general selection.

27. A wealthy man dies and leaves, among his other assets, his 17 horses to his children. They are to be divided in the following ratio: The eldest child receives 1/2, the middle child receives 1/3, and the youngest receives 1/9. How are the children to divide the horses if none of them feel comfortable with the idea of a fractional horse?

    Solution:

28. Alice and Bob play a game on a large circular table. Starting with Alice, they take turns placing identical coins on the table. The coins must be placed flat, cannot overlap other coins, and must be completely contained in the table. The winner is the last player who is able to place a coin. Who has a winning strategy?

    Solution:

29. On a certain island live 100 villagers, 50 with brown eyes and 50 with blue eyes. The village has a peculiar rule: anyone who knows that they have blue eyes must leave the island at sunset. Luckily, there are no mirrors in the village, and nobody knows for certain the color of their own eyes. There's also no restrictions on sight – nobody is blind, and everybody can see everyone else's eyes. One day, a shipwrecked sailor stumbles upon the village and is nursed back to health. Before he departs, the village holds a festival in his honor, where he announces to the crowd that "at least one villager has blue eyes." Now, this is information that all the villagers already know – they can see that there are blue-eyed villagers among them. What effect, if any, does this announcement have?

    Hint:

    Solution:

30. You are placed in a dark room with two identical tables. One table contains 100 coins of various denominations and the other is empty. You are told that exactly 10 of the coins are heads-up, but you cannot see anything. Your task is to move some of the coins from the full table to the empty one so that at the end, both tables have the same number of coins with heads up. You are forced to wear gloves, so you cannot feel the surfaces of the coins at all. What is your best strategy?

    Hint:

    Solution:

31. There is a magic trick performed by two magicians, Alice and Bob, with one regular, shuffled deck of 52 cards (assume that the cards are rotationally symmetric). Alice asks Carol to randomly select 5 cards out of a deck and hand the 5 cards back to Alice. After looking at the 5 cards, Alice picks one of the 5 cards and gives it back to Carol. Alice then arranges the other four cards in some way, and gives those 4 cards face down, in a neat pile, to Bob. Bob looks at these 4 cards and then announces the card that Carol is holding. How does this trick work?

Solution: