Notes
on
Hidden Markov Models with Unknown States

Shawn Tan

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$$\begin{aligned}
T &= \text{length of the sequence of observations} \\
N &= \text{number of states} = |\Omega_X| \\
M &= \text{number of possible observations} = |\Omega_O| \\
\Omega_X &= \{q_1, \ldots, q_N\} \\
\Omega_O &= \{o_1, \ldots, o_N\} \\
X_t &= \text{The state at time } t \\
O_t &= \text{The observation at time } t \\
\sigma &= O_1, \ldots, O_T
\end{aligned}$$

Distribution parameters

$$\begin{aligned}
A &= \{a_{i,j} \mid a_{i,j} = P(X_{t+1} = q_j | X_t = q_i)\}\text{(Transitions)} \\
B &= \{b_i \mid b_i(k) = P(O_t = o_k | X_t = q_i)\}\text{(Observations)} \\
\pi &= \{\pi_k \mid \pi_k = P(X_0 = q_k)\}\text{(Initial)} \\
\lambda &= \{A, B, \pi\}
\end{aligned}$$

1. Find probability of $P(\sigma|\lambda)$ (backwards) Let α such that:

$$\begin{aligned}
\alpha_t(i) &= P(O_1, \ldots, O_t, X_t = q_i|\lambda) \\
\alpha_T(i) &= P(O_1, \ldots, O_T, X_T = q_i|\lambda) \\
&= P(\sigma, X_T = q_i|\lambda)
\end{aligned}$$

compute α values by:

$$\alpha_t(i) = \begin{cases} \pi_i b_i(O_1) & \text{if } t = 1 \\ \underbrace{b_i(o_{t+1})}_{\text{observation probabilty given } o_{t+1}} \sum_{j=1}^{N} \underbrace{a_{j,i}}_{\text{transition probability}} \alpha_t(j) & \text{if } t > 1 \end{cases}$$

marginalising,

$$
\begin{aligned}
P(\sigma|\lambda) &= \sum_{i=1}^{N} P(O_1, \ldots, O_T, X_T = q_i|\lambda) \\
&= \sum_{i=1}^{N} \alpha_T(i)
\end{aligned}
$$

Define β (forwards) such that

$$\beta_t(i) = P(O_{t+1}, \ldots, O_T | X_t = q_i, \lambda)$$

compute β values by:

$$
\beta_t(i) = \begin{cases} 1 & \text{if } t = T \\ \sum_{j=1}^{N} \underbrace{a_{i,j}}_{\text{transition probability}} \underbrace{b_j(o_{t+1})}_{\text{observation probabilty given } o_{t+1}} \beta_{t+1}(j) & \text{if } t < T \end{cases}
$$