Appendix B: Resampling Algorithms

A common problem of all particle filters is the degeneracy of weights, which consists of the unbounded increase of the variance of the importance weights $\omega^{[i]}$ of the particles with time. The term "variance of the weights" must be understood as the potential variability of the weights among the possible different executions of the particle filter. In order to prevent this growth of variance, which entails a loss of particle diversity, one of a set of *resampling methods* must be employed, as it was explained in chapter 7.

The aim of resampling is to replace an old set of N particles by a new one, typically with the same population size, but where particles have been duplicated or removed according to their weights. More specifically, the expected duplication count of the $i-th$ particle, denoted by N_i, must tend to $N\omega^{[i]}$. After resampling, all the weights become equal to preserve the importance sampling of the target pdf. Deciding whether to perform resampling or not is most commonly done by monitoring the Effective Sample Size (ESS). As mentioned in chapter 7, the ESS provides a measure of the variance of the particle weights, e.g. the ESS tends to 1 when one single particle carries the largest weight and the rest have negligible weights in comparison. In the following we review the most common resampling algorithms.

1. REVIEW OF RESAMPLING ALGORITHMS

This section describes four different strategies for resampling a set of particles whose normalized weights are given by $\omega^{[i]}$, for $i = 1, \ldots, N$. All the methods will be explained using a visual analogy with a "wheel" whose perimeter is assigned to the different particles in such a way that the length of the perimeter associated to each particle is proportional to its weight. Therefore, picking a random direction in this "wheel" implies choosing a particle with a probability proportional to its weight. For a more formal description of the methods, please refer to the excellent reviews in (Arumlampalam, Maskell, Gordon, & Clapp, 2002; Douc, Capp, & Moulines, 2005). The four methods described here have $O(N)$ implementations, that is, their execution times can be made to be linear with the number of particles (Carpenter, Clifford, & Fearnhead, 1999; Arumlampalam, Maskell, Gordon, & Clapp, 2002).

Multinomial resampling: It is the most straightforward resampling method, where N independent random numbers are generated to pick a particle from the old set. In the "wheel" analogy, illustrated in Figure 1, this method consists of picking N independent random directions from the center of the wheel and taking the pointed particle. This method is named after the fact that the probability mass function for the duplication counts N_i is a multinomial distribution with the weights as parameters. A naïve implementation would have a time complexity of $O(N \log N)$, but applying the method of simulating order statistics (Carpenter, Clifford, & Fearnhead, 1999), it can be implemented in $O(N)$.

Figure 1. The multinomial resampling algorithm

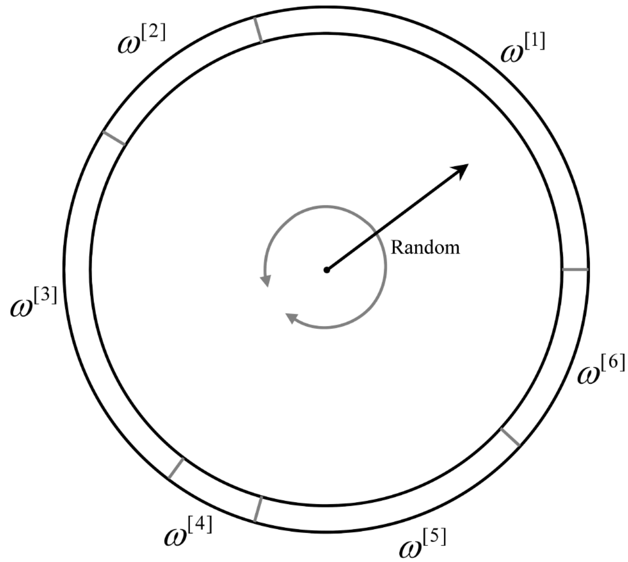


Figure 2. The residual resampling algorithm. The shaded areas represent the integer parts of $\omega^{[i]}/(1/N)$. *The residual parts of the weights, subtracting these areas, are taken as the modified weights* $\tilde{\omega}^{[i]}$.

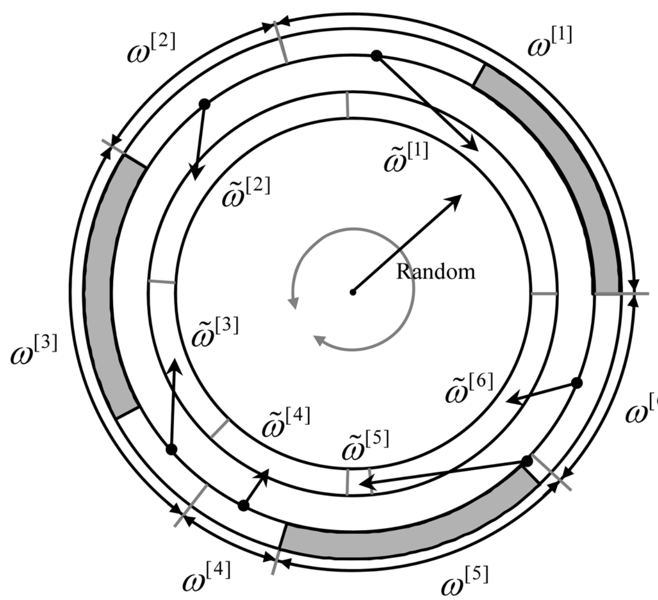


Figure 3. The stratified resampling algorithm. The entire circumference is divided into N equal parts, represented as the N circular sectors of $1/N$ perimeter lengths each.

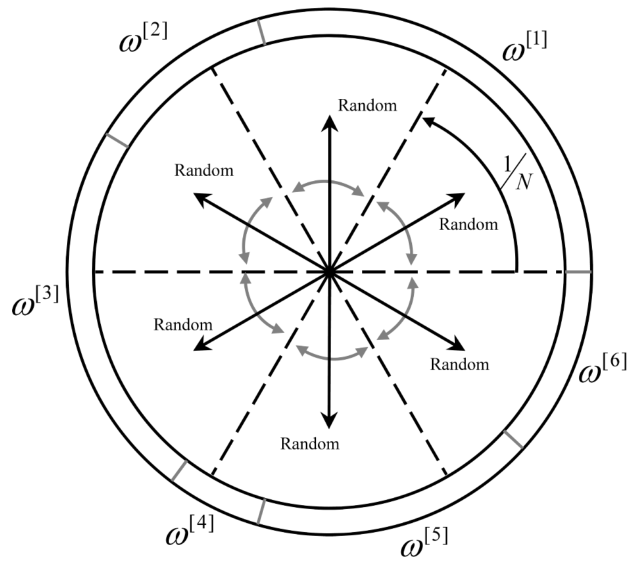


Figure 4. The systematic resampling algorithm

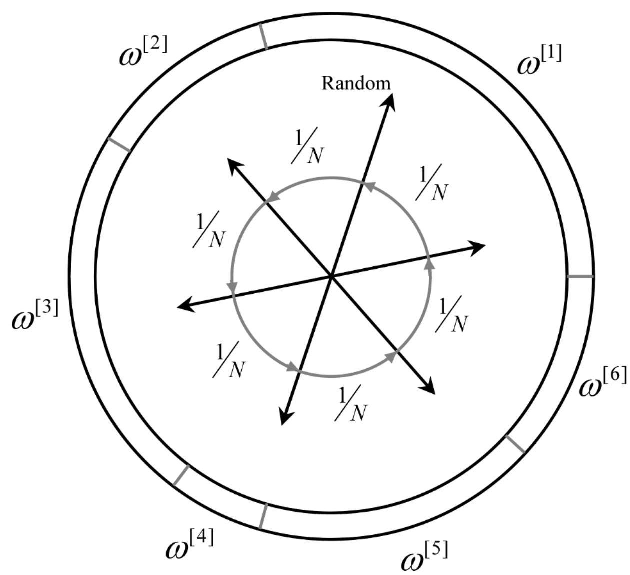
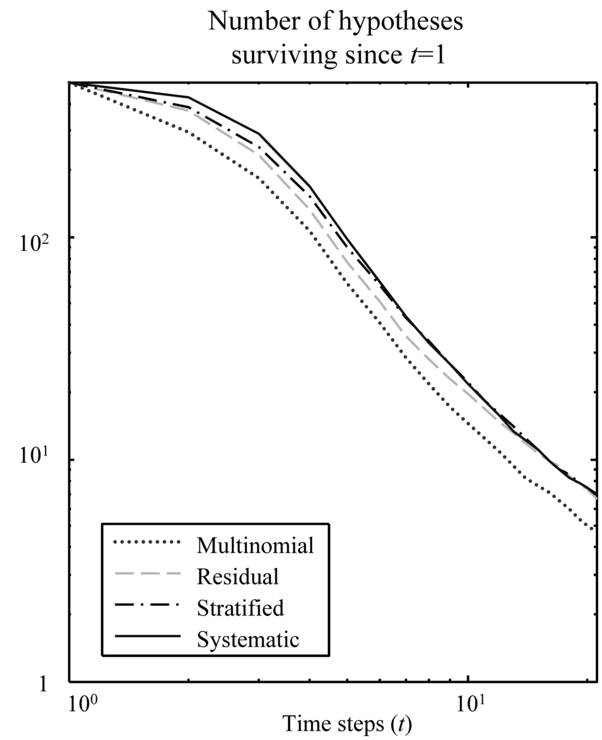


Figure 5. A simple benchmark to measure the loss of hypothesis diversity with time in an RBPF for the four different resampling techniques discussed in this appendix. The multinomial method clearly emerges as the worst choice.



Residual resampling: This method comprises two stages, as can be seen in Figure 1. Firstly, particles are resampled deterministically by picking $N_i = \lfloor N\omega^{[i]} \rfloor$ copies of the $i-th$ particle—where $\lfloor x \rfloor$ stands for the floor of x, the largest integer above or equal to x. Then, multinomial sampling is performed with the residual weights: $\tilde{\omega}^{[i]} = \omega^{[i]} - N_i / N$ (see Figure 1-4).

Stratified resampling: In this method, the "wheel" representing the old set of particles is divided into N equally-sized segments, as represented in Figure 3. Then, N numbers are independently generated from a uniform distribution like in multinomial sampling, but instead of mapping each draw to the entire circumference, they are mapped within its corresponding partition out of the N ones.

Systematic resampling: Also called *universal sampling*, this popular technique draws only one random number, i.e., one direction in the "wheel," with the others $N-1$ directions being fixed at $1/N$ increments from that randomly picked direction.

2. COMPARISON OF THE DIFFERENT METHODS

In the context of Rao-Blackwellized Particle Filters (RBPF), where each particle carries a hypothesis of the complete history of the system state evolution, resampling becomes a crucial operation that reduces the diversity of the PF estimate for past states. We saw the application of those filters to SLAM in chapter 9.

In order to evaluate the impact of the resampling strategy on this loss, the four different resampling methods discussed above have been evaluated in a benchmark that measures the diversity of different states remaining after t time steps, assuming all the states were initially different. The results, displayed in Figure 5, agree with the theoretical conclusions in Douc, Capp, and Moulines (2005), stating that multinomial resampling is the worst of the four methods in terms of variance of the sample weights. Therefore, due to its simple implementation and good results, the systematic method is recommended when using a static number of particles in all the iterations. If a dynamic number of samples is desired, things get more involved and it is recommended to switch to a specific particle filter algorithm which simultaneously takes into account this particularity while also aiming at optimal sampling (Blanco, González, & Fernandez-Madrigal, 2010).

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