RBF Quadrature for Neural Fields

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Summary

Goal: To create and test a neural field solver using radial basis function quadrature. The method should be

- High-order accurate
- Stable
- Geometrically flexible
- Fast (low complexity)

In the future, we will extend this to realistic curved 2D spatial domains.

Neural Field Models (NF)

- Tissue level models
- Integro-differential equation(s)
- Integral kernel represents neural network connectivity
- Non-linear firing rate function captures non-linear neural dynamics

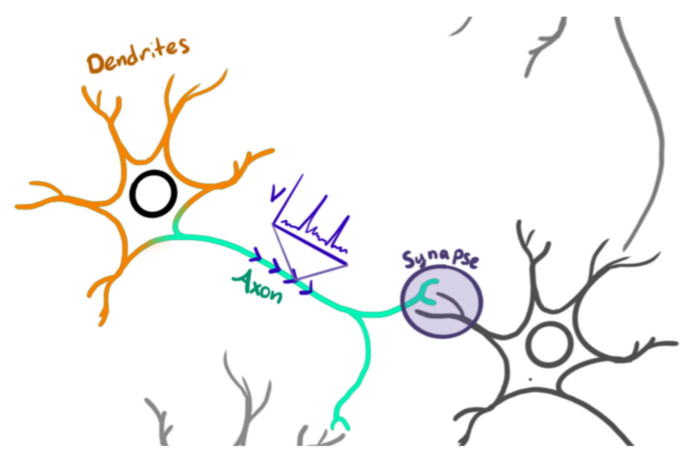
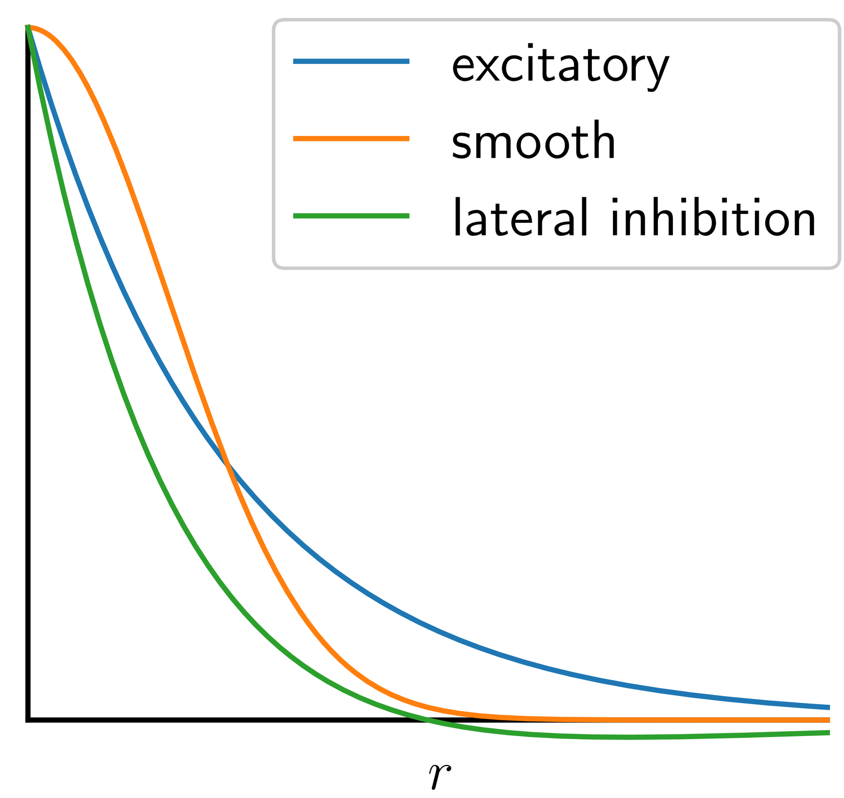


Image by Heather Cihak.

$$\partial_t u(t,\mathbf{x}) = -u + \iint_\Omega w(\mathbf{x},\mathbf{y}) f[u(t,\mathbf{y})]\ d\mathbf{y}$$

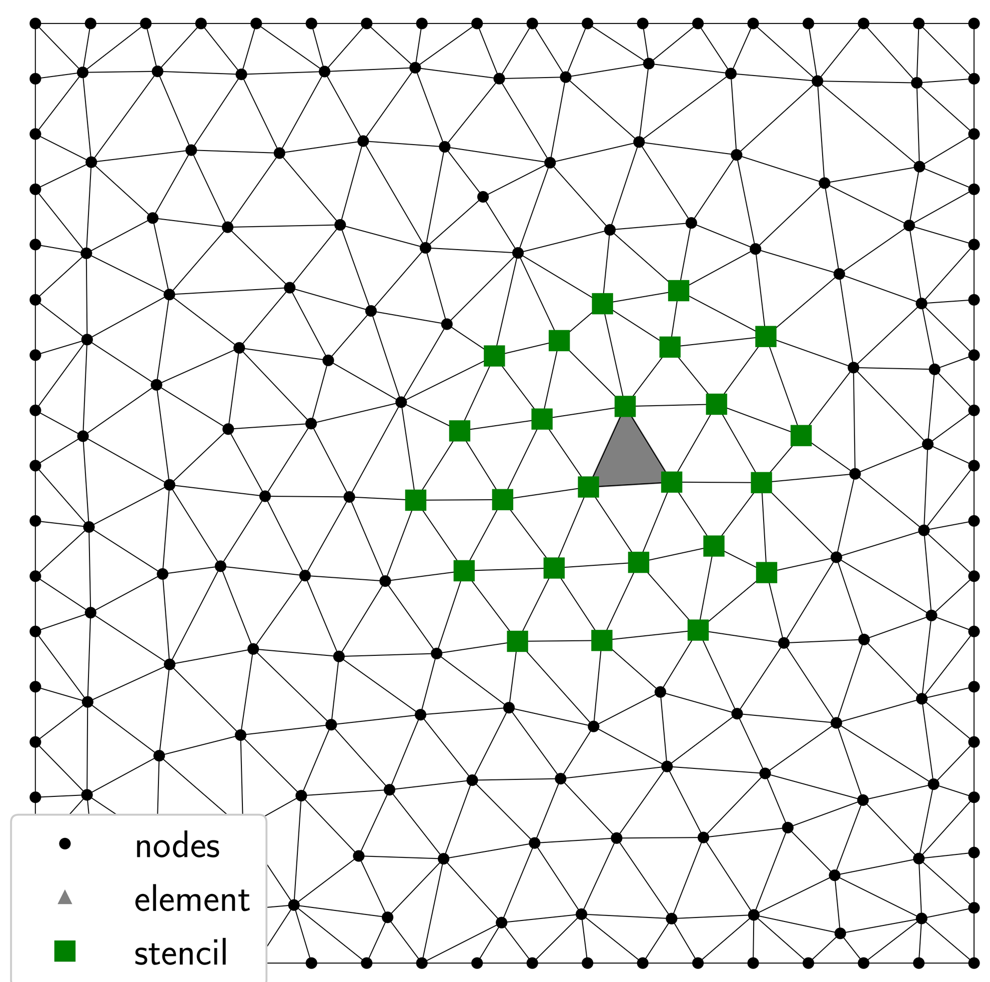
- $u(t,\mathbf{x})$ — Neural activity
- $w(\mathbf{x},\mathbf{y})$ — Connectivity kernel
- $f(\cdot)$ — Non-linear firing rate function
- $\Omega = [0,1]^2$ for now.

$$w(x,y) = w(||x-y||) = w(r)$$



Radial Basis Function Quadrature Formuale

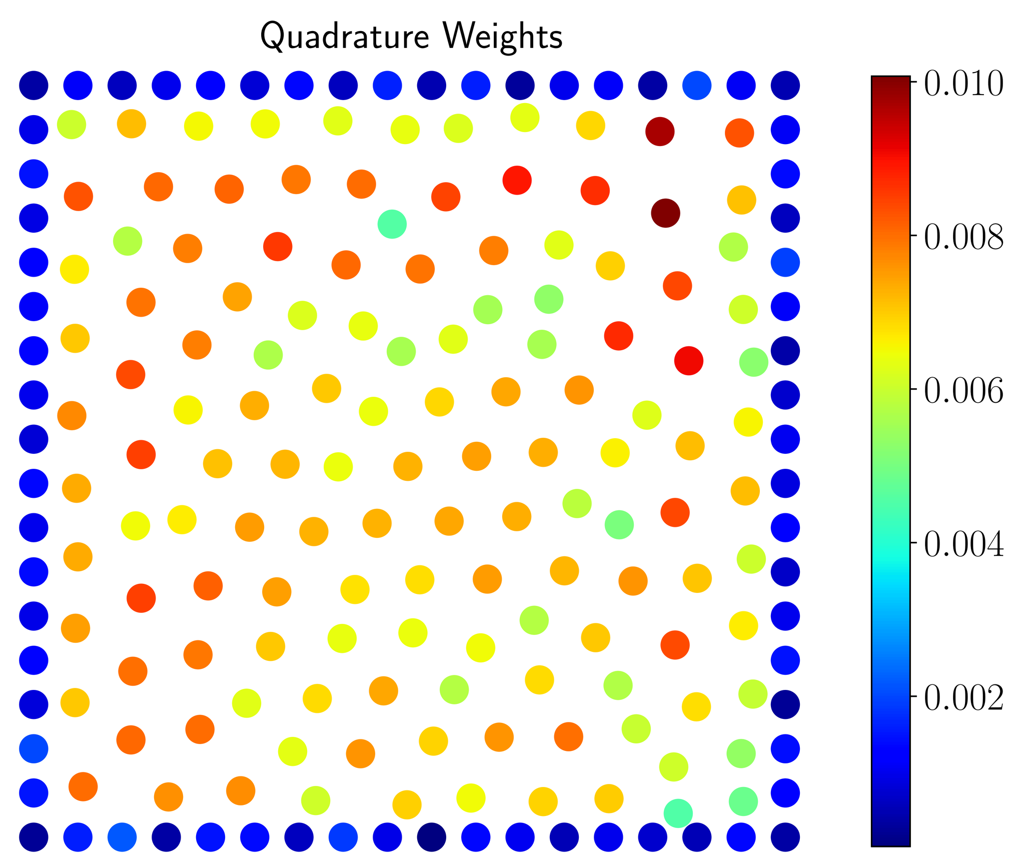
- Abreviated **RBF-QF**
- Place N nodes in Ω
- Partition Ω into elements
- For each element
  - select the k nearest nodes (the stencil),
  - interpolate Lagrange functions,
  - integrate over the element,
  - sum over interpolants
- sum over elements



The local interpolants have the form

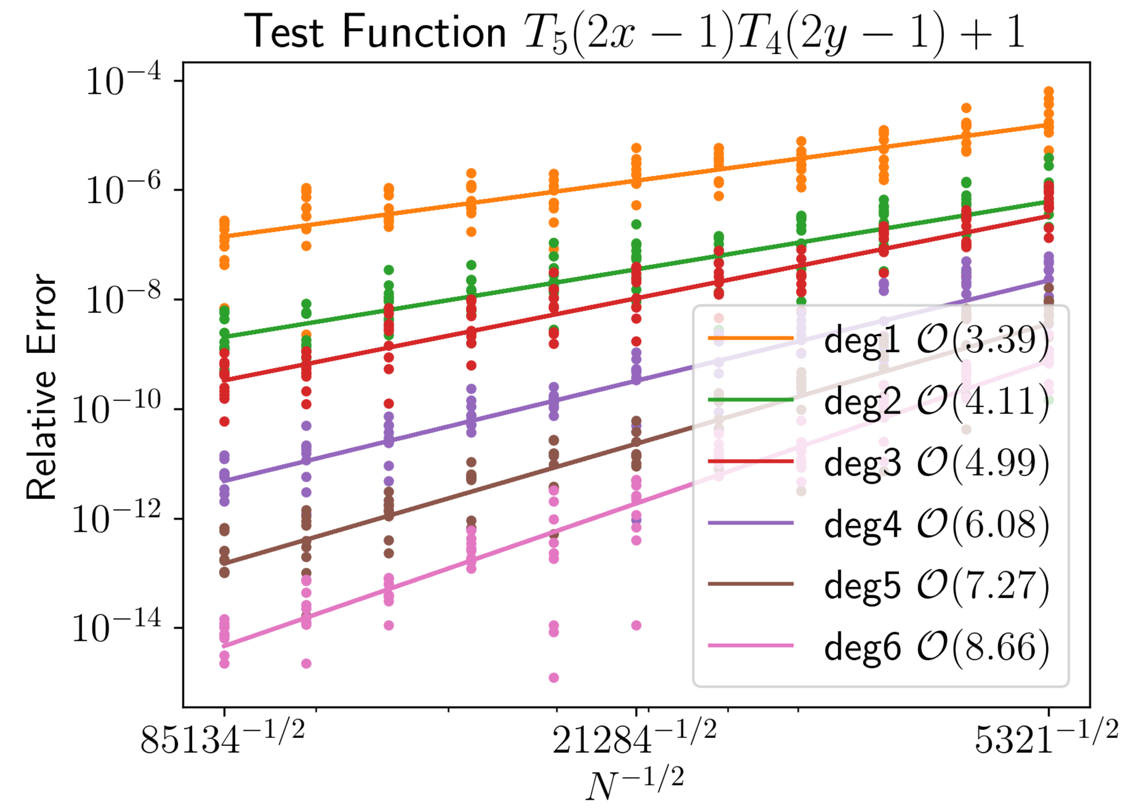
$$s(\mathbf{x}) = \sum_{i=1}^{k} c_i \phi(\|\mathbf{x}-\mathbf{x}_i\|) + \sum_{j=1}^{m} \gamma_i \pi_j(\mathbf{x})$$

- ϕ - radial basis function (eg. $\phi(r) = r^3$)
- $\{\pi_j\}_{j=1}^m$ - polynomial basis
- k interpolation conditions
- m moment conditions



Quadrature Convergence

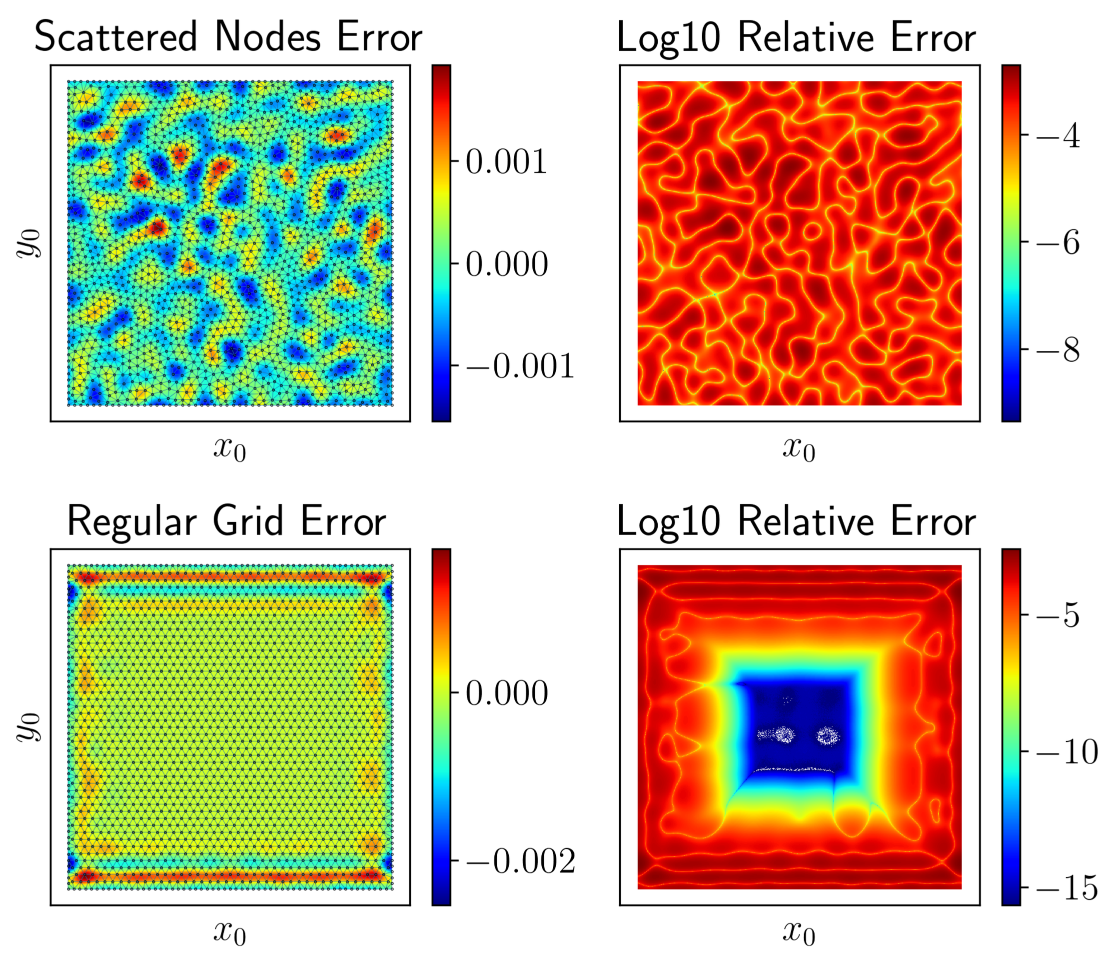
- No proven convergence rate.
- We expect at least $\mathcal{O}(d+1)$, where d is the degree of polynomial basis. (Think trapezoidal rule.)
- We test this on a product of Chebyshev polynomials.
- Area per point scaleds like $\mathcal{O}(N^{-1/2})$



- Better convergence than expected

Spatial Analysis

- Gaussian test functions centered at (x_0, y_0)
- Scattered Nodes vs Regular Hex Grid



- Error is continuous w.r.t. (x_0, y_0)
- Hex grid has spectral accuracy away from boundary (like trapezoidal rule)

Next Steps

Projection Method[1]

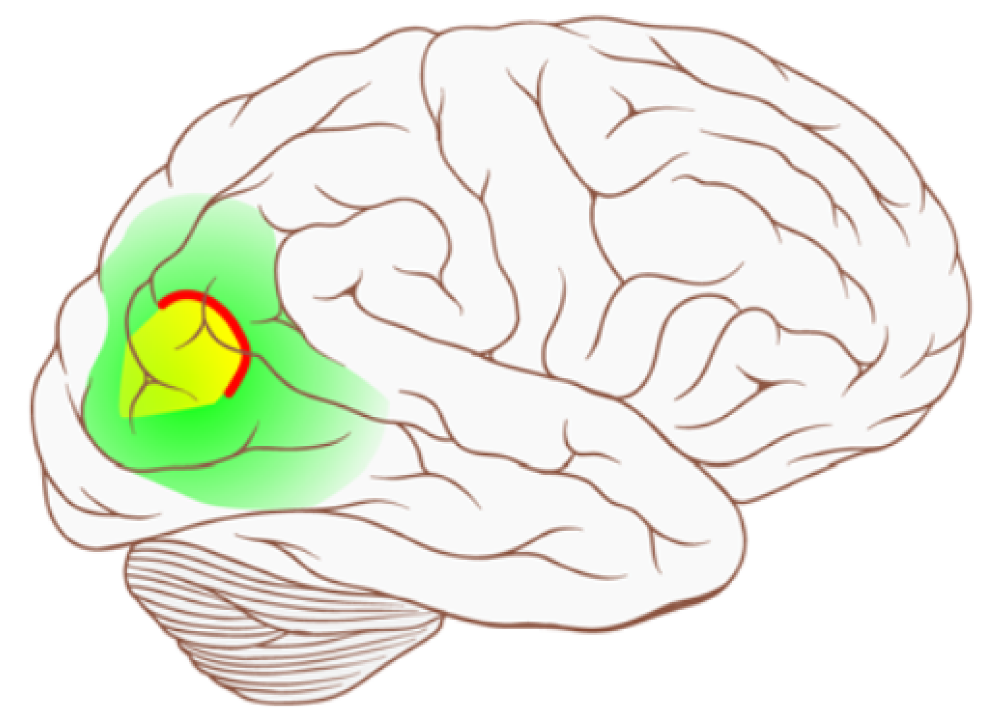
- Framework for error analysis of NF
- error = projection error + QF error
- We will unify these errors using RBF interpolation (projection) and RBF-QF.

Curved 2D Manifolds

- [2,3] Extended RBF-QF to 2D manifolds
- [4] Used FitzHugh–Nagumo (toy model) on torrus (toy surface) to study effects of cortical curvature.
- We will use RBF-QF on realistic cortical surfaces to study the effects of coticol curvature on *corticol spreading depression* (CSD).

Cortical Spreading Depression

- Slow moving chemical wave
- Causes depressed neural activity
- Causes *scintilating scotomas*



Cartoon of CSD wave[5]

Artistic Rendering of Scotomas (see QR code)

References and Funding

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`shawsa.github.io/presentations`

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