Memo 6: Projection onto the tangent plane of a surface

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1 Tanget plane projections

Let $\mathcal{S}$ be a smooth embedded manifold of co-dimension one in $\mathbb{R}^d$. Consider a collection of points $X = \{\mathbf{x}_j\}_{j=1}^n \subset \mathcal{S}$ "surrounding" the point $\mathbf{x}_c \in \mathcal{S}$. Here we are assuming the points are expressed as coordinates in $\mathbb{R}^d$. Suppose we also have available the unit normal vectors to $\mathcal{S}$ at the points $\mathbf{x}_c$ and X, which we will denote by $\widehat{\mathbf{n}}_c$ and $\widehat{\mathbf{n}}_j$, $j = 1, \ldots, n$, respectively. The goal is to project the points in X to the tangent plane at $\mathbf{x}_c$. There are two types of projections we will consider. The first is the projection of X to the tangent plane along the normal direction $\widehat{\mathbf{n}}_c$ (we call this the stencil center normal projection). The second is the projection of each $\mathbf{x}_j$ along its normal $\widehat{\mathbf{n}}_j$ (we call this the stencil normals projection).

1.1 Stencil center normal projection

Let $\boldsymbol{\xi}_j$ denote the projection of $\mathbf{x}_j$ along $\widehat{\mathbf{n}}_c$ to the tangent plane. Then $\boldsymbol{\xi}_j = \mathbf{x}_j + t\widehat{\mathbf{n}}_c$, for some $t \in \mathbb{R}$. To determine t, we note that we want to make the line connecting $\boldsymbol{\xi}_j$ and $\mathbf{x}_c$ orthogonal to $\widehat{\mathbf{n}}_c$. This gives the result that

$$\begin{gathered}\widehat{\mathbf{n}}_c^T(\mathbf{x}_c - \boldsymbol{\xi}_j) = \widehat{\mathbf{n}}_c^T(\mathbf{x}_c - \mathbf{x}_j - t\widehat{\mathbf{n}}_c) = \widehat{\mathbf{n}}_c^T(\mathbf{x}_c - \mathbf{x}_j) - t = 0 \\ \Longrightarrow t = \widehat{\mathbf{n}}_c^T(\mathbf{x}_c - \mathbf{x}_j).\end{gathered}$$

So, the projection of the point $\mathbf{x}_j$ to the tangent plane is

$$\boldsymbol{\xi}_j = \mathbf{x}_j + \widehat{\mathbf{n}}_c^T(\mathbf{x}_c - \mathbf{x}_j)\widehat{\mathbf{n}}_c \;\text{ or }\; \boldsymbol{\xi}_j = (I - \widehat{\mathbf{n}}_c\widehat{\mathbf{n}}_c^T)\mathbf{x}_j + \widehat{\mathbf{n}}_c^T\mathbf{x}_c\widehat{\mathbf{n}}_c. \tag{1}$$

See Figure 1 (a) for an illustration.

1.2 Stencil normals projection

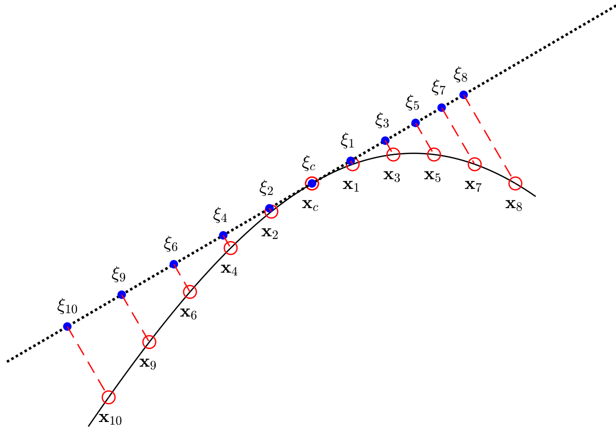
Let $\boldsymbol{\eta}_j$ denote the projection of $\mathbf{x}_j$ along $\widehat{\mathbf{n}}_j$ to the tangent plane. Then $\boldsymbol{\eta}_j = \mathbf{x}_j + s\widehat{\mathbf{n}}_j$, for some $s \in \mathbb{R}$. We can determine s by noting that points $\mathbf{x}_j$, $\boldsymbol{\xi}_j$, and $\boldsymbol{\eta}_j$ must form a right triangle with the hypotenuse being the vector $\mathbf{x}_j - \boldsymbol{\eta}_j$. This means that $t/s = \cos(\theta)$, where θ is the angle between $\widehat{\mathbf{n}}_j$ and $\widehat{\mathbf{n}}_c$, or written more succinctly

$$s = \frac{t}{\cos(\theta)} = \frac{t}{\widehat{\mathbf{n}}_j^T\widehat{\mathbf{n}}_c}.$$

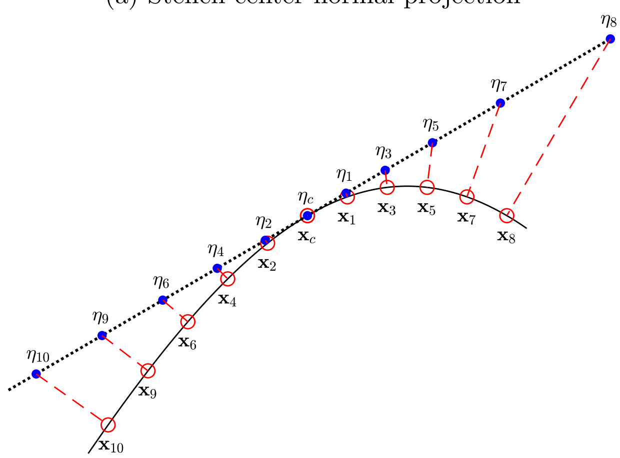
So, the projection of the point $\mathbf{x}_j$ along $\widehat{\mathbf{n}}_j$ to the tangent plane is

$$\boldsymbol{\eta}_j = \mathbf{x}_j + \frac{\widehat{\mathbf{n}}_c^T(\mathbf{x}_c - \mathbf{x}_j)}{\widehat{\mathbf{n}}_j^T\widehat{\mathbf{n}}_c}\widehat{\mathbf{n}}_j. \tag{2}$$

See Figure 1 (b) for an illustration.



(a) Stencil center normal projection



(b) Stencil normals projection

Figure 1: Illustrations of the two projection methods: (a) X projected along the normals $\widehat{\mathbf{n}}_c$, and (b) X projected along the normal $\widehat{\mathbf{n}}_j$, $j = 1, \ldots, n$.

1.3 Projections on the unit sphere

If $\mathcal{S} = \mathbb{S}^d$ (the d-dimensional unit sphere) then the above formulas simplify quite a bit since $\widehat{\mathbf{n}}_c = \mathbf{x}_c$ and $\widehat{\mathbf{n}}_j = \mathbf{x}_j$ on the sphere. Using these relationships in (1) and (2) gives

$$\boldsymbol{\xi}_j = \mathbf{x}_j + (1 - \mathbf{x}_c^T \mathbf{x}_j)\mathbf{x}_c, \tag{3}$$

$$\boldsymbol{\eta}_j = \frac{1}{\mathbf{x}_j^T \mathbf{x}_c} \mathbf{x}_j. \tag{4}$$

$$\tag{5}$$

2 Distance formulas on the tangent plane

2.1 Stencil center normal projection

Ultimately, we are interested in computing distances of the projected points in the tangent plane for RBF methods. In the case of stencil center normal projection (Figure 1(a)), the distances can be worked out to a compact expression that looks like a weighted ℓ_2-norm (Mahalanobis distance) in the ambient space.

Starting with the second expression of (1), we have

$$\|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\|^2 = \|(I - \widehat{\mathbf{n}}_c\widehat{\mathbf{n}}_c^T)(\mathbf{x}_i - \mathbf{x}_j)\|^2 = (\mathbf{x}_i - \mathbf{x}_j)^T \underbrace{(I - \widehat{\mathbf{n}}_c\widehat{\mathbf{n}}_c^T)}_{W}(\mathbf{x}_i - \mathbf{x}_j) := \|\mathbf{x}_i - \mathbf{x}_j\|_W^2. \tag{6}$$

This expression can be used directly as the distance function in an RBF expansion without ever needing to project the points.

Curiously, W in (6) can also be derived by considering the following steps:

1. Rotate the stencil points so that the center point $\mathbf{x}_c$ is at $(0, 0, \|\mathbf{x}_c\|)$. This can be done using the rotation matrix

$$R = \begin{bmatrix}\widehat{\mathbf{t}}_1 & \widehat{\mathbf{t}}_2 & \widehat{\mathbf{n}}\end{bmatrix}, \tag{7}$$

   where $\widehat{\mathbf{t}}_1$ and $\widehat{\mathbf{t}}_2$ are orthonormal vectors that span the tangent plane at $\mathbf{x}_c$. The rotation of any of the stencil points can be computed as $R^T\mathbf{x}_i$.

2. Project off the z-coordinate of the rotated points (i.e. set the z-coordinate of the rotated point to zero). This can be done by multiplying the rotated points by the matrix $E = \mathrm{diag}(\begin{bmatrix}1 & 1 & 0\end{bmatrix})$.

3. Compute the pairwise distances between all the rotated and projected stencil points.

In the language of matrices and vectors, the above operations can be written as

$$\begin{aligned}\|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\|^2 = \|ER^T(\mathbf{x}_i - \mathbf{x}_j)\|^2 = (ER^T(\mathbf{x}_i - \mathbf{x}_j))^T(ER^T(\mathbf{x}_i - \mathbf{x}_j)) = (\mathbf{x}_i - \mathbf{x}_j)^T REER^T(\mathbf{x}_i - \mathbf{x}_j)\\ \Longrightarrow \|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\|^2 = (\mathbf{x}_i - \mathbf{x}_j)^T \underbrace{RER^T}_{W}(\mathbf{x}_i - \mathbf{x}_j) := \|\mathbf{x}_i - \mathbf{x}_j\|_W^2.\end{aligned} \tag{8}$$

The W in (8) is the same as the weight matrix W in (6), i.e. $W = I - \widehat{\mathbf{n}}_c\widehat{\mathbf{n}}_c^T = RER^T$. The reason this is important is that we can use the simpler construction of $W = I - \widehat{\mathbf{n}}_c\widehat{\mathbf{n}}_c^T$ to recover the rotation matrix R so that we never need to compute an orthonormal set of vectors $\widehat{\mathbf{t}}_1$ and $\widehat{\mathbf{t}}_2$ that span the tangent plane to the surface $\mathcal{S}$ at $\mathbf{x}_c$. The matrix R can be recovered from the singular value decomposition of W. Since W is symmetric, E has non-negative diagonals that are ordered from largest to smallest, and R is orthogonal, an SVD of W is just RER^T. Note that the first two columns of R are unique up to an arbitrary rotation by the same matrix and the third column is unique up to a sign change [TB97], so one may not recover the exact R from the two procedures. However, the first two columns will still span the tangent plane, i.e. we can extract $\widehat{\mathbf{t}}_1$ and $\widehat{\mathbf{t}}_2$ from R computed by the SVD.

It is important to have the rotation matrix R to implement the augmented polynomial terms in the RBF approximations.

2.2 Stencil normals projection

I could not work out a similar simple expression that looked like a weighted ℓ_2-norm for the stencil normals projection (Figure 1(b)), and I'm doubtful one exists. For this type of projection, one should probably just project the points to the tangent plane using (2), and then compute the distances. An orthonormal set of tangent vectors $\widehat{\mathbf{t}}_1$ and $\widehat{\mathbf{t}}_2$ for this plane can be computed as follows:

$$\widehat{\mathbf{t}}_1 = \frac{\mathbf{x}_c - \boldsymbol{\eta}_j}{\|\mathbf{x}_c - \boldsymbol{\eta}_j\|} \text{ and } \widehat{\mathbf{t}}_2 = \widehat{\mathbf{n}}_c \times \widehat{\mathbf{t}}_1,$$

where j is selected to be any of the integers $1, \ldots, n$. (Note that this approach also works in the stencil normal projection case, one just has to replace $\boldsymbol{\eta}_j$ with $\boldsymbol{\xi}_j$).

References

[TB97] Lloyd N. Trefethen and David Bau. *Numerical Linear Algebra.* SIAM, 1997.