Calculus-Based Optimization of the 200m Freestyle (LCM)

Hydrodynamics + stroke mechanics + lactate-constrained pacing optimization

By Shayan Doroudiani

Version 4 (layout improved, wrapped tables, boxed calculus equations emphasized)

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Designed for: elite / high-level age group swimmers and coaches (advanced calculus background)

All simulated numeric outputs are illustrative. The report includes an implementation workflow to fit the model to any athlete using race video, 15 m timing, stroke metrics, and optional lactate sampling.

Abstract

The 200 m freestyle (LCM) can be formulated as a constrained optimization problem: minimize race time while respecting hydrodynamic costs that grow convexly with speed (drag ~ v^2, power ~ v^3), stroke-mechanics constraints linking stroke rate (SR) and stroke length (SL), and physiology limits captured by a critical velocity (CV) anchor, a finite anaerobic reserve D′ (W′ analog), and lactate production/clearance dynamics that drive technique decay. We decompose the event into start, underwater phases (with the 15 m rule), surface segments, turns, and finish. Using calculus, we derive first-order optimality conditions for breakout distance, SR selection (dv/dSR=0), and pacing under state constraints. We then simulate three archetypal strategies (fast-out, even-ish, negative split), quantify split predictions, and run sensitivity analysis. Finally, we provide an implementation guide detailing what to measure, how to fit parameters, and how to validate race-plan predictions.

What you can extract from this report

- A phase-structured decision model: underwater distances x_u,i (start + 3 turns), lap surface speeds v_i, and SR targets per 50.

- Boxed calculus results: drag-power scaling, breakout first-order condition, SR* condition, lactate ODE, and D′ balance.

- 12+ figures: pacing curves, underwater decay, SR–SL optimum, lactate trajectories, D′ balance, contour trade-offs, sensitivity tornado, workflow diagram.

- Actionable coaching outputs: recommended underwater ranges per wall, measurable SR/SL targets, and a calibration checklist.

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Nomenclature

Symbols used throughout the report (units are typical; athlete-specific fitting is recommended).

Symbol	Meaning	Typical units
v(t), v(x)	Swimming speed (surface or instantaneous)	m/s
D(v)	Hydrodynamic drag force	N
P_mech	Mechanical power to overcome drag	W
k	Lumped drag-power coefficient (includes rho, Cd, A)	kg/m
SR	Stroke rate (cycles per second)	Hz
SL	Stroke length (distance per cycle)	m/cycle
CV	Critical velocity (aerobic steady-state anchor)	m/s
D′	Finite anaerobic distance capacity (W′ analog)	m
L(t)	Blood lactate (or proxy state)	mmol/L
alpha, gamma	Lactate production scaling and nonlinearity	-
k_clr	Lactate clearance rate constant	1/s
x_u	Underwater distance per wall (0–15 m)	m
v_u(x)	Underwater speed as a function of distance from wall	m/s

1. Introduction

The 200 m freestyle combines sprint-level velocities with middle-distance constraints. A strong swim requires (i) high-velocity impulses from the start and turns, (ii) efficient surface mechanics with low drag and stable SL, and (iii) physiological management so lactate accumulation and neuromuscular fatigue do not collapse technique in the final 50. Mathematically, the event is a time-minimization problem with nonlinear costs and state constraints. The role of calculus is to make marginal trade-offs explicit: each added underwater meter or each added 0.01 m/s has a time benefit but also a convex energetic cost and a physiological debt that impacts later segments.

1.1 Race decomposition and decision variables

We model the race as four 50 m laps. Each lap i contains an underwater distance $x_{u,i}$ (0–15 m constraint) and a surface distance (50-$x_{u,i}$). Decision variables (athlete-specific):

- $x_{u,0..3}$: underwater distances for start + turns (m), $0 \leq x_{u,i} \leq 15$.

- $v_{0..3}$: target surface speeds for lap surface segments (m/s).

- $SR_{0..3}$: stroke rate targets per lap (Hz or cycles/min), coupled to SL via SL(SR).

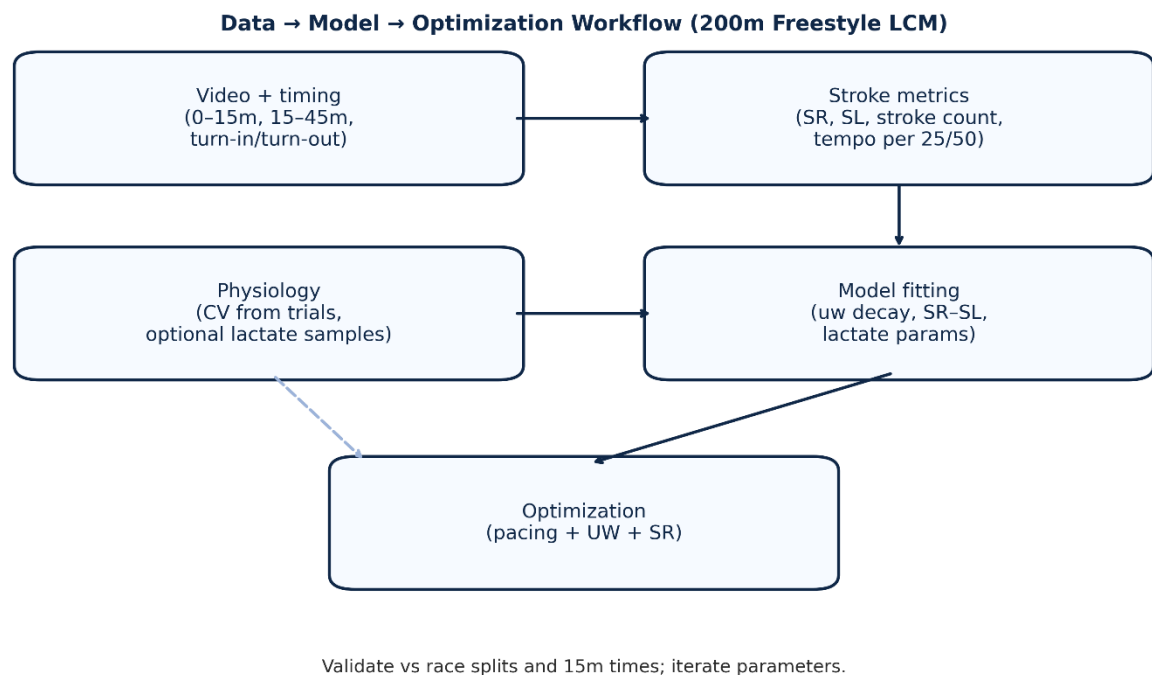- Optional: breath plan (breaths/length and timing) as a constraint or penalty.

Data → Model → Optimization Workflow (200m Freestyle LCM)

Video + timing (0–15m, 15–45m, turn-in/turn-out) → Stroke metrics (SR, SL, stroke count, tempo per 25/50)

Physiology (CV from trials, optional lactate samples) → Model fitting (uw decay, SR–SL, lactate params)

Optimization (pacing + UW + SR)

Validate vs race splits and 15m times; iterate parameters.

Figure. Data-to-strategy workflow: how video, timing, stroke metrics, and physiology feed into model fitting and optimization.

2. Methods

2.1 Hydrodynamics and the v^3 power law

At race speeds, resistive force is well-approximated as quadratic in velocity: $D(v) \propto v^2$. Mechanical power is force times velocity, so $P\_mech \propto v^3$. This creates a convex cost of speed: increasing v by a small amount requires a disproportionately large increase in power. This convexity is a primary reason that extreme early surges are often suboptimal in the 200 m unless tactically required and funded by remaining D′.

Eq. (1) Drag model

$$D(v) = \tfrac{1}{2}\rho\, C_D\, A\, v^2$$

rho is water density; C_D and A are athlete- and posture-dependent.

Eq. (2) Power scaling

$$P_{\text{mech}}(v) = D(v)\, v = k v^3$$

k lumps hydrodynamic and efficiency terms (order-of-magnitude: $k \approx 0.5$ rho C_D A).

2.2 Even pacing as an optimal baseline (variational sketch)

Minimize time with a fixed energy-per-distance budget using a Lagrangian. If energy cost per unit distance scales approximately with v^2, the Euler-Lagrange condition yields v(x)=constant (even pacing) as an optimum.

Eq. (3) Even-pace optimality (sketch)

$$\min \int_0^{200} \left(\tfrac{1}{v(x)} + \lambda v(x)^2 \right) dx \Rightarrow v(x) = \text{const.}$$

Starts/turns and physiology force real races to deviate from perfect constancy.

2.3 Underwater phases: speed decay and breakout optimization

Underwaters are frequently the highest-speed part of each lap because (i) wall push-offs provide large initial velocity and (ii) wave drag is reduced. However, velocity decays as momentum dissipates and as

dolphin-kick propulsion competes with drag. We model underwater velocity as an exponential decay with distance: v_u(x)=v_inf + (v0-v_inf)exp(-x/d_c). A first-order breakout rule is: exit underwater when v_u(x) approaches the upcoming sustainable surface speed v_s. More formally, define lap time proxy T(x)= x/\bar v_u(x) + (50-x)/v_s; the optimal breakout satisfies dT/dx=0.

Eq. (4) Breakout objective

$$v_u(x^*) \approx v_s \Rightarrow x^* = \arg\min_X \left(\frac{x}{\bar{v}_u(x)} + \frac{50-x}{v_s} \right)$$

This objective separates underwater time and surface time for a lap.

Eq. (4a) Breakout first-order condition

$$\frac{d}{dx}\left(\frac{x}{\bar{v}_u(x)} + \frac{50-x}{v_s} \right) = 0 \Rightarrow \frac{1}{\bar{v}_u(x)} - \frac{x\,\bar{v}'_u(x)}{\bar{v}_u(x)^2} - \frac{1}{v_s} = 0$$

Interpretation: the marginal time change of extending underwater equals the marginal time change of shortening surface distance.
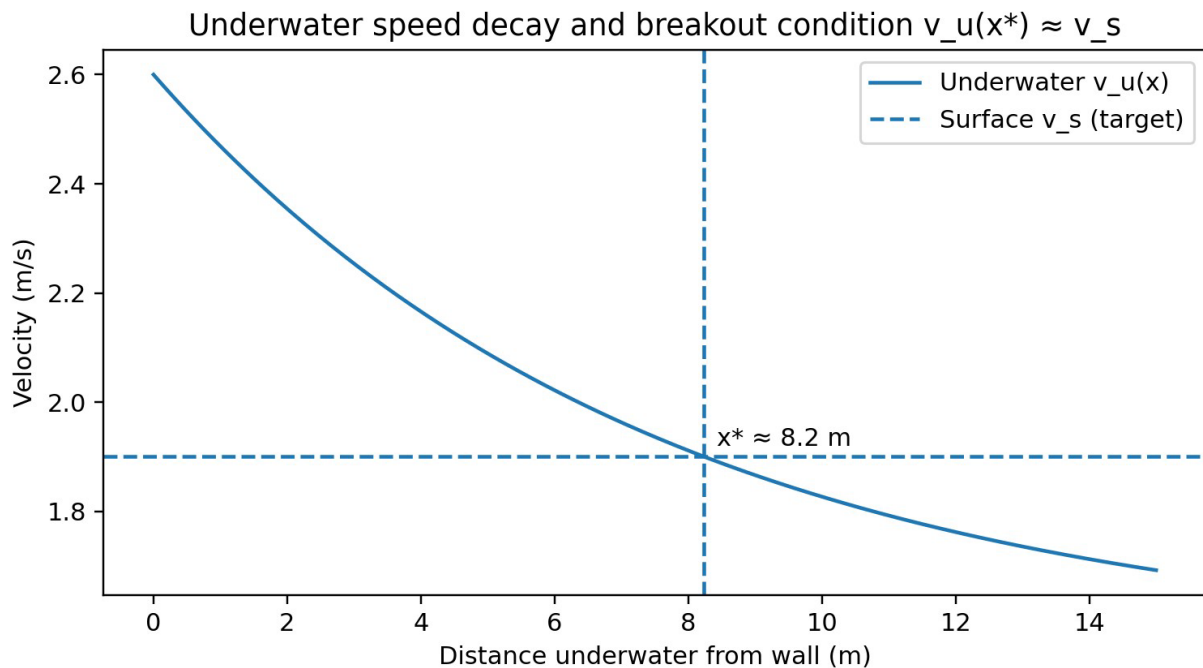

Figure. Underwater speed decay curve with a surface-speed reference. The intersection is a practical breakout estimate x*.

Phase	Recommended UW distance (m)	Kick-count range (≈2.0 m/kick)	Why / optimization rule

Start (0-15m)	13–15	6–8	Dive momentum + reduced wave drag; push until $v_u(x)$ falls toward v_s.
Turn 1 (50m)	9–12	4–6	Most swimmers still fresh; longer UW often time-positive if $v_u > v_s$.
Turn 2 (100m)	9–11	4–6	Balance speed with O2; maintain speed without pushing lactate too early.
Turn 3 (150m)	8–11 (often slightly shorter)	4–6	O2 debt highest; surface sooner if $v_u \approx v_s$ or breath timing dominates.

Breath constraint extension: to include apnea cost, add a penalty term to $T(x)$ that increases with x (especially late race), or constrain $x_{u,3}$ to maintain a stable breathing rhythm into the finish.

2.4 Turns as time-impulses: separating approach, wall, and exit

A full turn can be decomposed into: (i) approach (last ~5 m), (ii) wall contact + rotation, (iii) push-off, and (iv) underwater kick to breakout. From an optimization perspective, turns inject a high-velocity impulse at a cost of (a) rotation time and (b) oxygen disruption. Practical measurement: capture 5 m-in and 15 m-out times for each turn; this lets you compute turn effectiveness independent of mid-pool speed. In a refined model, decision variables include turn-in speed, wall time, push-off angle/depth, and $x_{u,i}$. Here we focus on $x_{u,i}$ because it is the lever most directly tied to calculus breakout conditions and is measurable from video.

2.5 Stroke mechanics: SR–SL trade-off and calculus optimum

Surface speed is v = SR·SL. For most athletes, SL decreases as SR increases (less time to apply force and maintain a long effective catch). Fit SL(SR) from video-derived stroke counts and segment speeds; the speed-maximizing SR* satisfies dv/dSR=0. A 200 m plan should allow SR to rise across the race while actively protecting technical SL (head position, alignment, kick timing, breathing control).

Eq. (5) SR–SL relation

$$v = SR \cdot SL(SR), \quad \frac{dv}{dSR} = 0 \Rightarrow SR^*$$

SR and SL are measurable. SR is the control variable; SL is the technique outcome.

Eq. (5a) SR* condition

$$\frac{dv}{dSR} = 0 \Rightarrow SL(SR^*) + SR^* \frac{dSL}{dSR}\bigg|_{SR^*} = 0$$

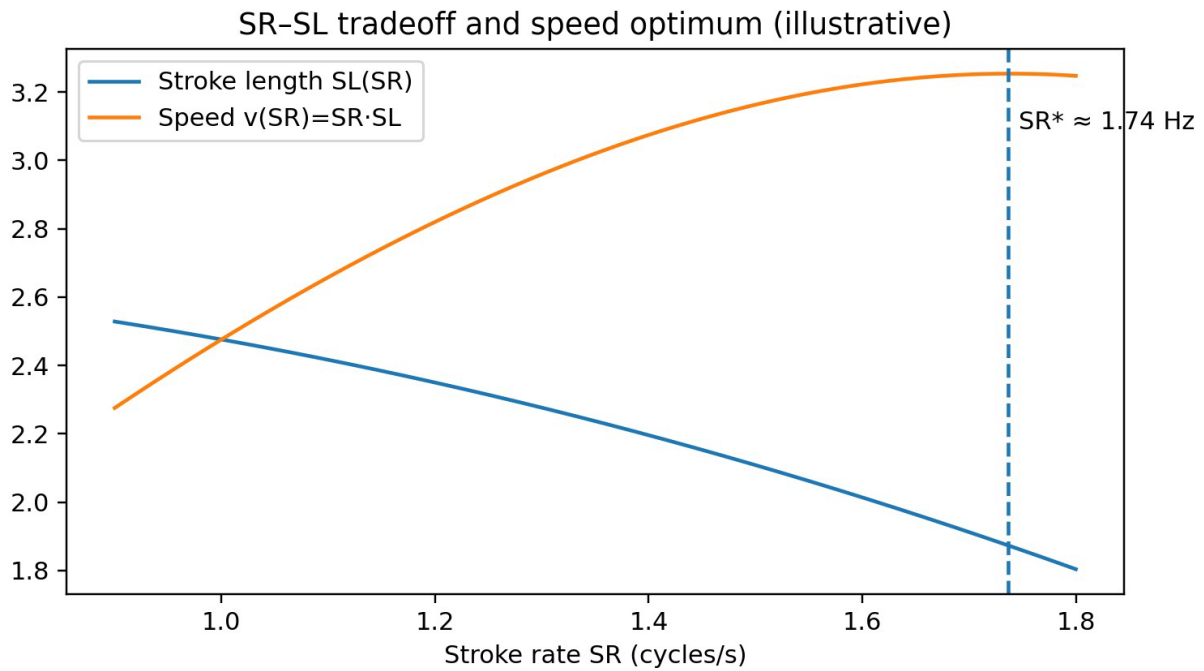Once SL(SR) is fit, solve for SR* per race phase (often lower early, higher late).

SR–SL tradeoff and speed optimum (illustrative)

Figure. Illustrative SR–SL curve and implied speed v(SR). The peak is SR* for that curve.

Measurable targets: for each lap, record stroke count per 25, compute SR from tempo (s/cycle), and infer SL = v/SR. A coach can then prescribe SR bands (e.g., 0.98–1.05 Hz early, 1.05–1.15 Hz late) while monitoring SL stability.

2.6 Physiology constraints: CV, D′, and lactate kinetics

We use two constraint layers: (1) critical velocity CV, representing a sustainable speed anchor; and (2) a finite anaerobic reserve D′ (distance analog of W′) spent when v>CV and partly restored when v≤CV. Lactate is modeled with a one-compartment ODE: production grows nonlinearly with intensity above CV; clearance decays toward baseline L0. We treat lactate as a proxy state that induces technique decay (e.g., SL loss and breathing instability) past a threshold.

Eq. (6) Lactate dynamics

$$\dot{L}(t) = \alpha \left[\max\left(0, \frac{v(t)}{CV} - 1\right) \right]^{\gamma} - k_{\text{clr}}(L - L_0)$$

Fit alpha, gamma, k_clr from standardized sets; use L(t) as a race-state proxy.

Eq. (7) D′ balance

$$\dot{D}'(t) = -(v - CV)\,(v > CV); \quad \dot{D}'(t) = \frac{CV - v}{\tau_{\text{rec}}} \frac{D_0' - D'}{D_0'} \,(v \le CV)$$

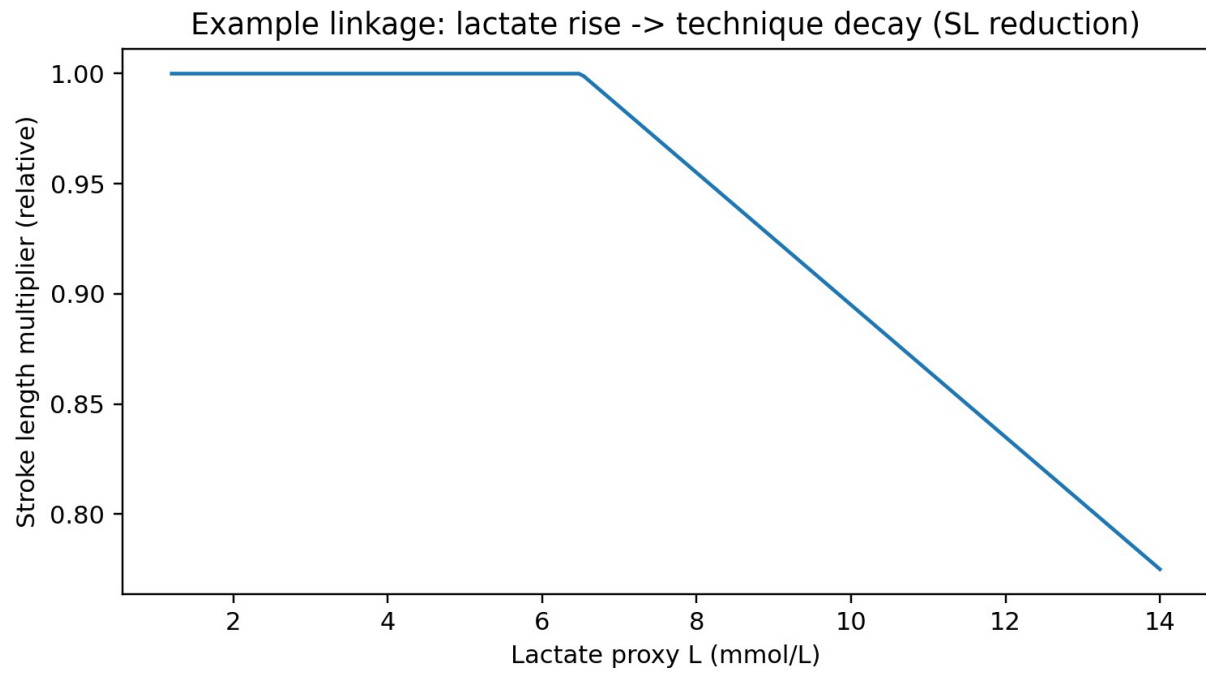D′ must stay nonnegative; spending too early forces late speed collapse.

Figure. Example linkage: higher lactate proxy reduces effective SL (technique decay model).

2.7 Pacing as constrained optimization (discrete optimal control)

Let v_i and x_u,i be decision variables per lap. Total time: T = Σ_i [t_u,i(x_u,i) + (50-x_u,i)/v_i].
Constraints include: 0 ≤ x_u,i ≤ 15; D′(t) ≥ 0; and a lactate/technique ceiling (e.g., maintain SL multiplier above 0.85 until the final 50). This defines a constrained optimal control problem. We solve it with a coarse-to-fine numerical search: grid over feasible x_u,i and then optimize v_i (or SR_i) subject to state evolution for D′ and L.

Eq. (8) Optimization statement

$$\min_{v(t),\, x_{u,i}} T \ \text{s.t.} \ \dot{L} = f(v), \ \dot{D}' = g(v), \ D'(t) \geq 0$$

In athlete-specific use, extend constraints to include breath counts, turn approach, and SR bounds.

3. Results and simulations (illustrative elite male profile)

We simulate three archetypal strategies with a representative elite parameter set (placeholders to demonstrate the workflow). The output is intended as a template for athlete-specific refitting.

3.1 Predicted splits and total time

Strategy	50 split (s)	100 split (s)	150 split (s)	200 split (s)	Total (s)
Fast-out	24.47	25.80	26.52	27.33	104.13
Even-ish	24.83	25.79	26.06	26.29	102.98
Negative-split	25.21	26.00	25.84	25.57	102.63

Model interpretation: in this parameterization, even-ish and negative-split strategies preserve D' and moderate lactate growth, enabling a stronger finish.
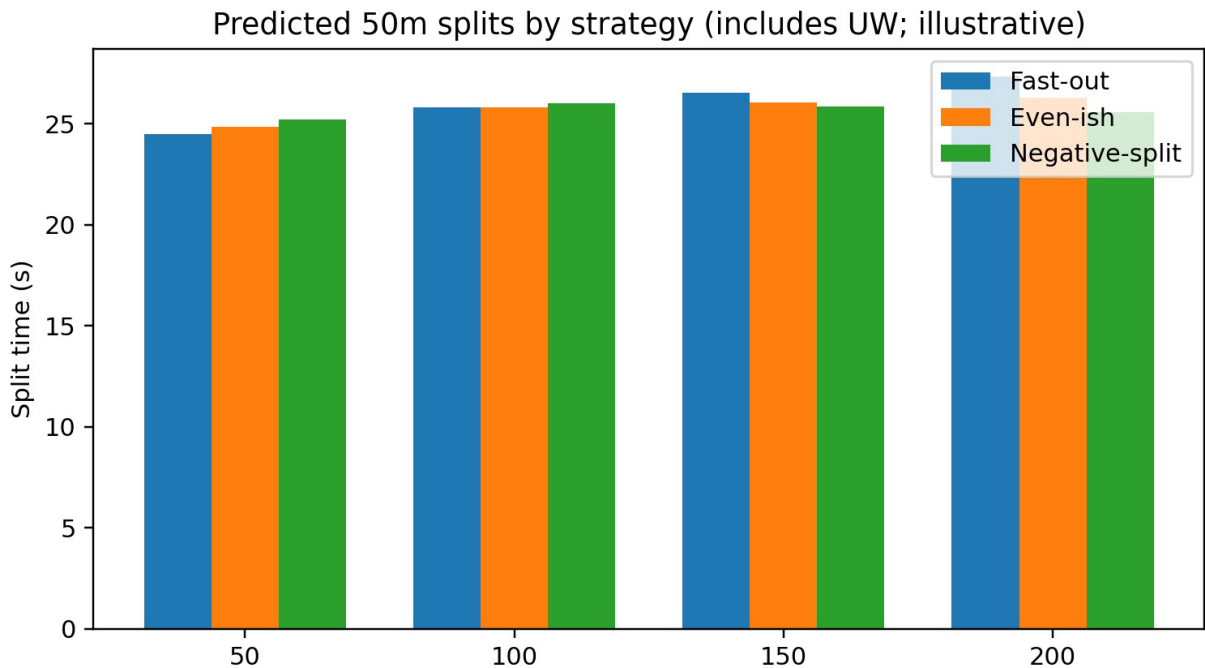

Figure. Predicted 50 m splits by strategy. Bars include underwater contributions.
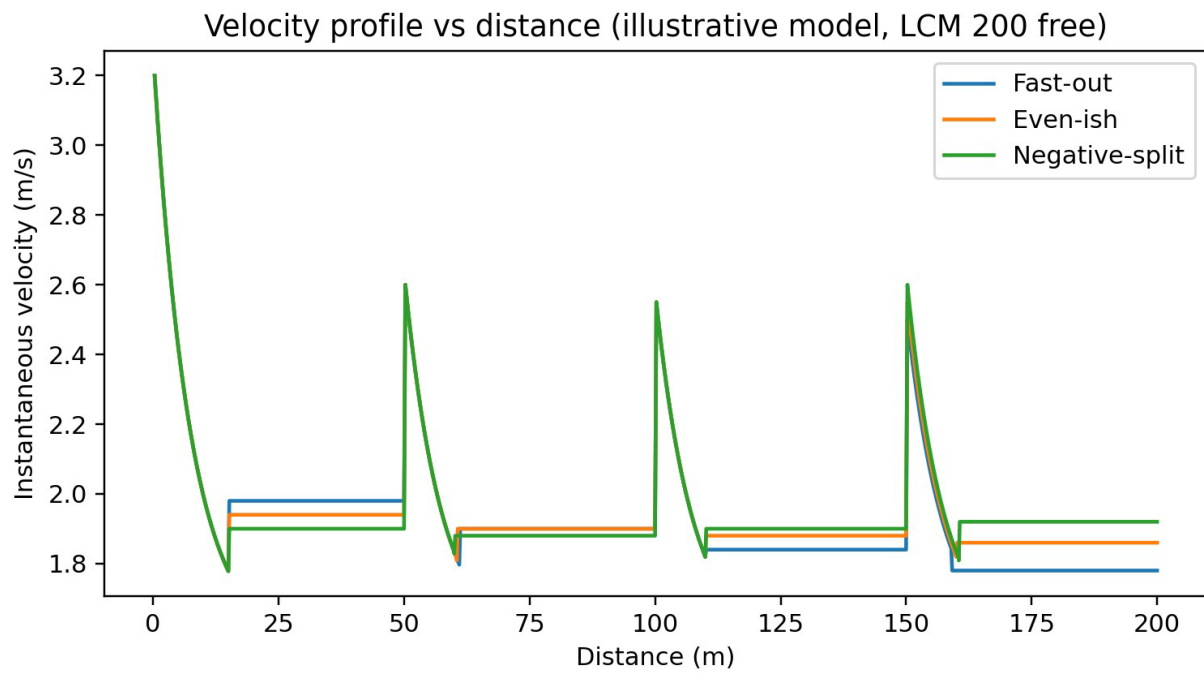
Figure. Velocity profile vs distance. Peaks cluster near walls (push-offs); plateaus reflect surface pacing choices.
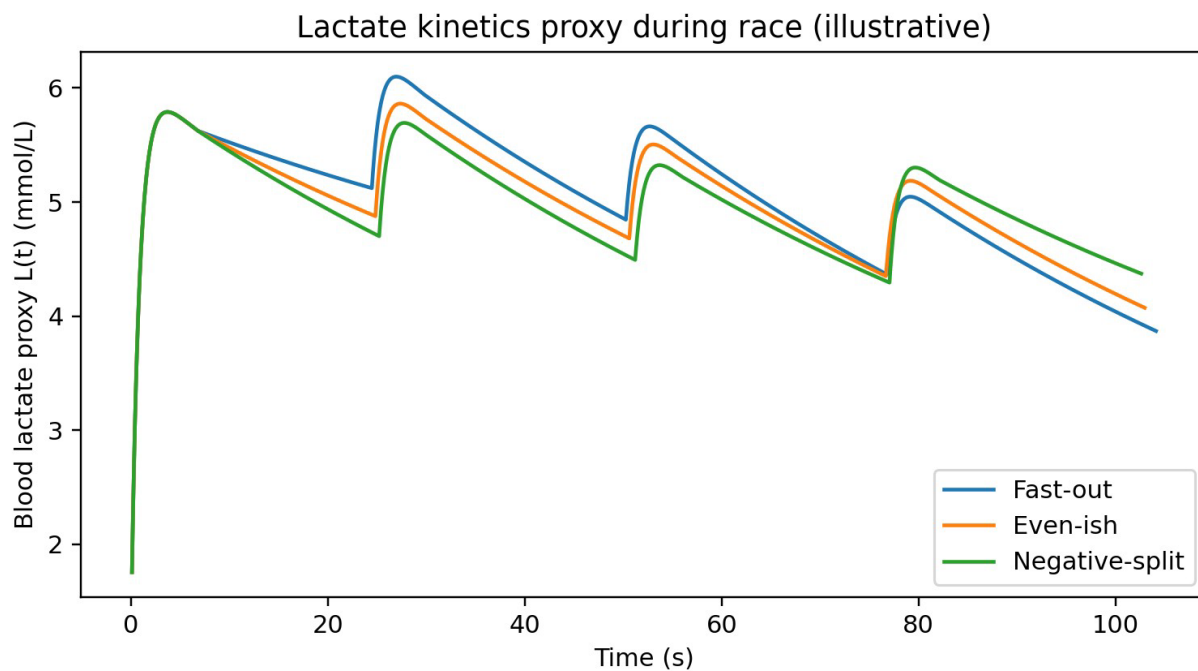
3.2 Lactate and anaerobic reserve trajectories

Lactate kinetics proxy during race (illustrative)

Figure. Lactate proxy L(t) during the race. Fast-out produces higher mid-race lactate and a steeper late rise.

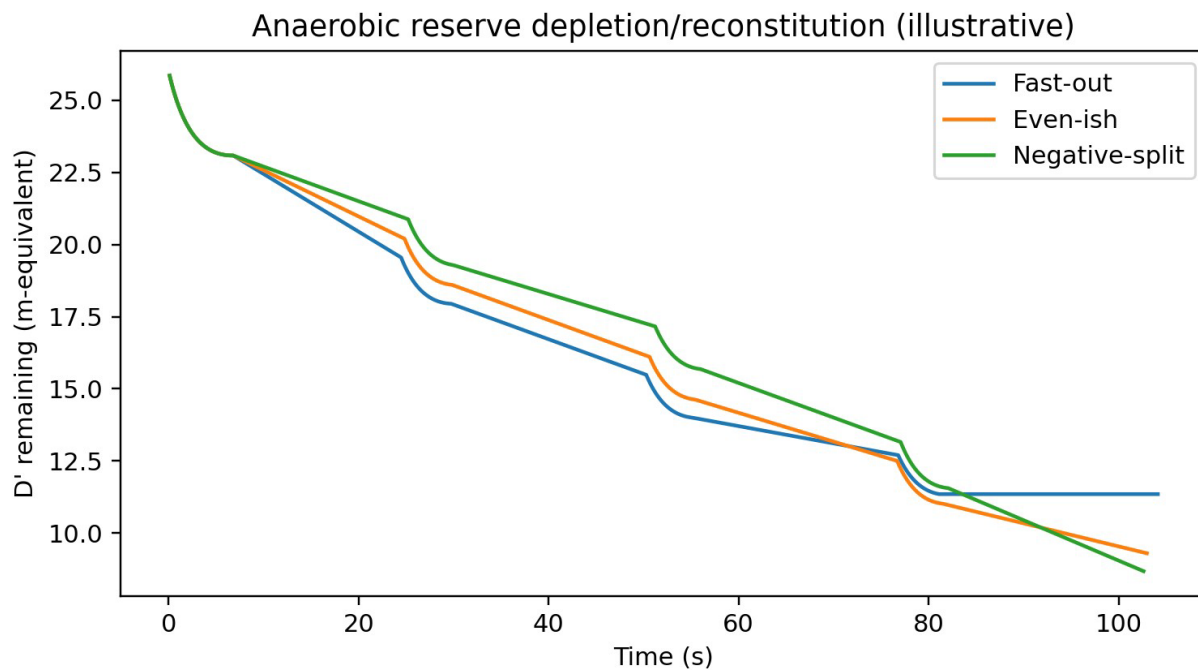### Anaerobic reserve depletion/reconstitution (illustrative)

Figure. Anaerobic reserve D′(t). Preserving D′ early can fund the last-50 speed increase.
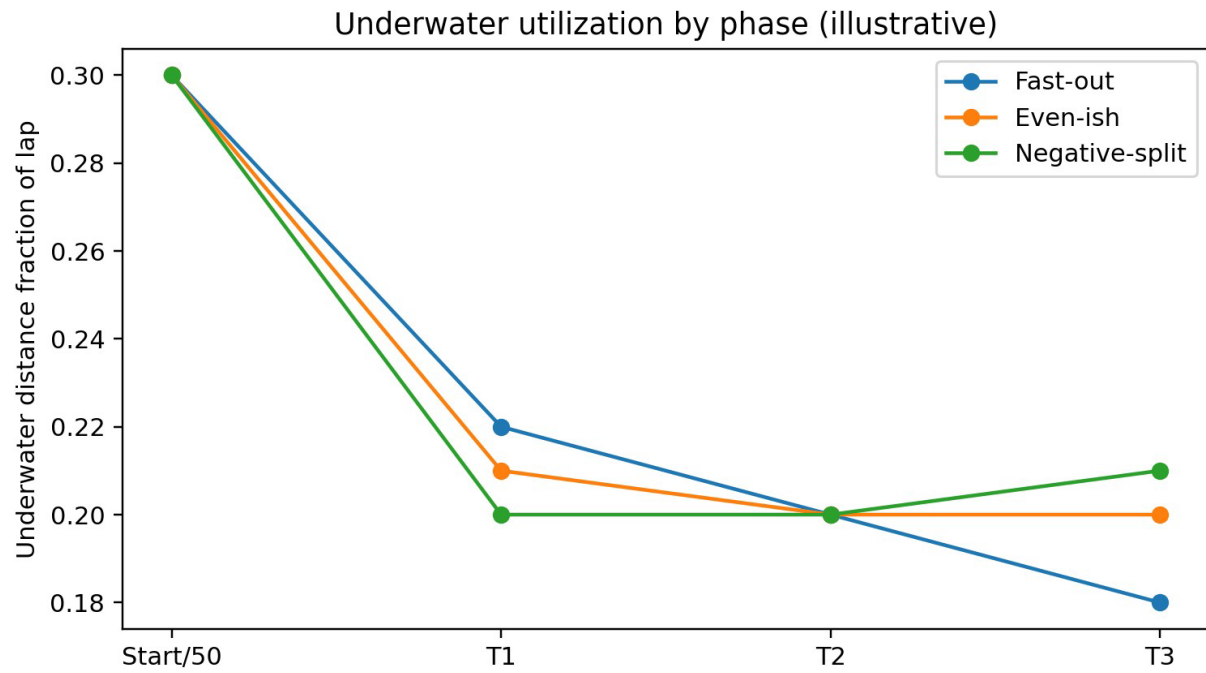
Figure. Underwater utilization by lap. Late-lap adjustments typically reflect oxygen debt and breath rhythm constraints.
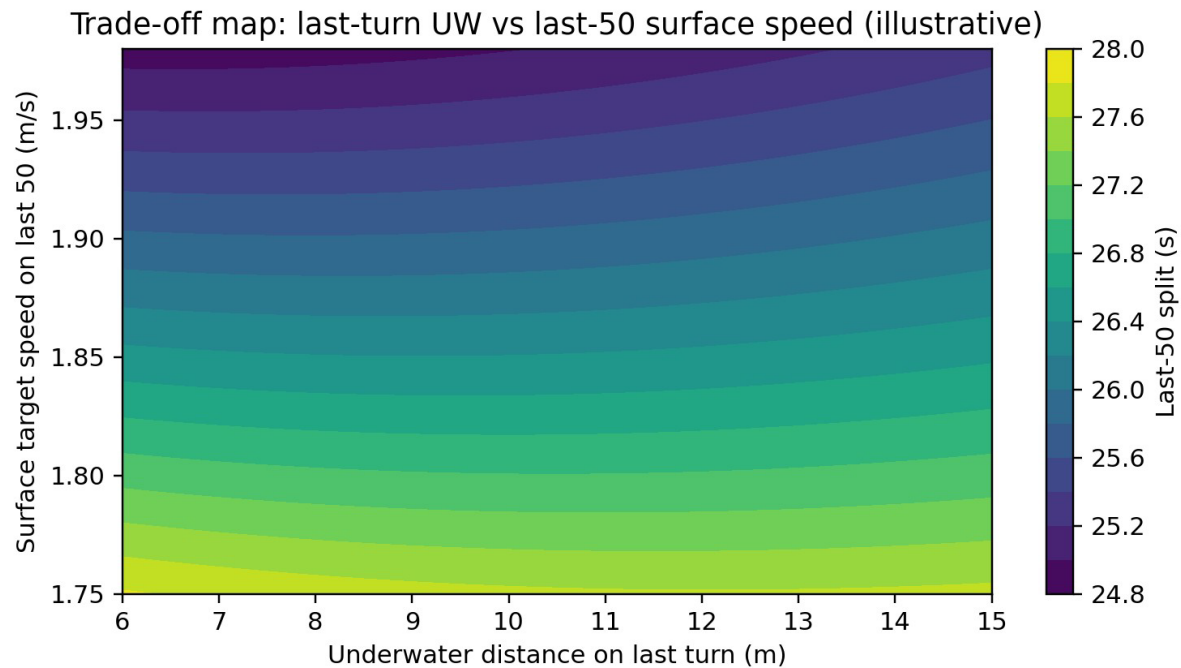
3.3 Trade-offs and sensitivity analysis

Figure. Trade-off map: last-50 split vs last-turn underwater distance and last-50 surface speed (illustrative).
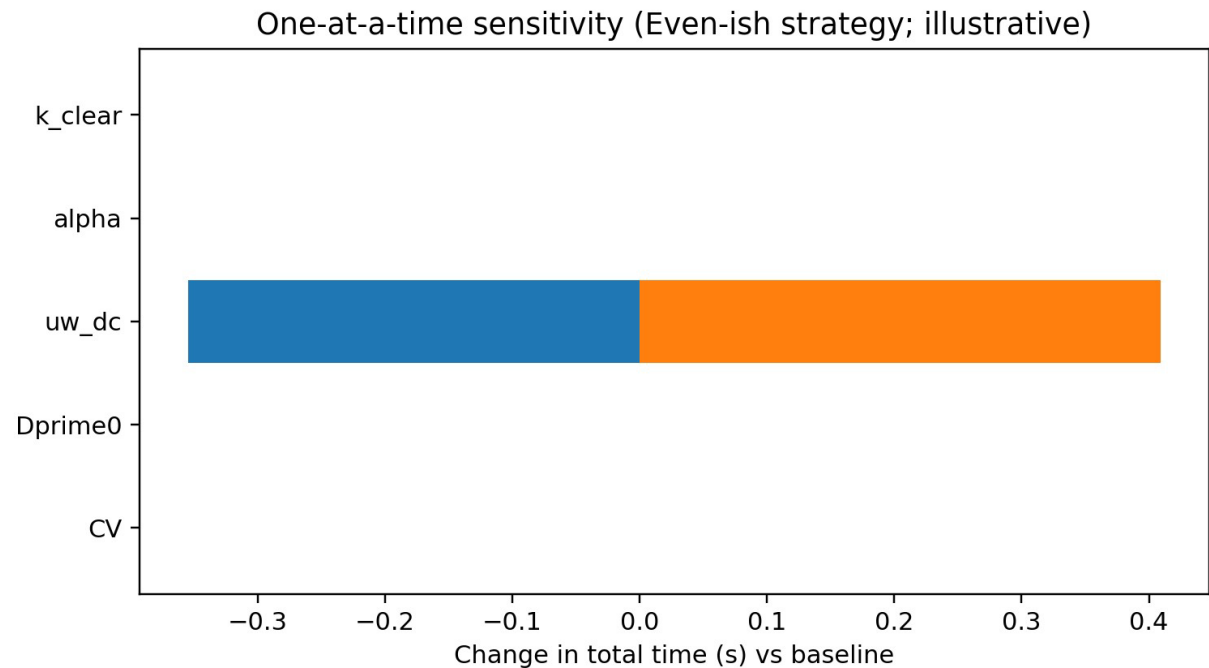

Figure. One-at-a-time sensitivity: which parameters move total time most (illustrative).

Parameter (±10%)	Δ time (s) at -10%	Δ time (s) at +10%
CV	+0.00	+0.00
Dprime0	+0.00	+0.00

uw_dc	-0.35	+0.41
alpha	+0.00	+0.00
k_clear	+0.00	+0.00

4. Discussion

The model formalizes several coaching truths: (1) Underwaters are high-leverage because they inject high speed with limited arm-fatigue cost. (2) Extreme first-100 surges are penalized by convex power cost (v^3) and amplified lactate growth that reduces SL later. (3) SR should be phase-dependent: protect SL early, then raise SR late when remaining D' can be spent and when the race horizon is short. The breakout intersection rule $v_u(x^*) \approx v_s$ is a strong baseline; the richer condition includes apnea and lactate effects: surface sooner if added underwater distance materially raises $L(t)$ or disrupts breathing into the first strokes after breakout.

4.1 Practical strategy recommendations (elite male baseline)

- Start: 13–15 m underwater unless your measured dolphin speed drops below your surface speed earlier than ~12 m.

- Turn 1–2: 9–11 m underwaters if $v_u(x)$ remains above v_s; measure 0–15 m time to validate your advantage.

- Turn 3: 8–11 m based on oxygen debt and breath rhythm; stabilize the first 6–8 strokes after breakout.

- Pacing: keep the first 100 controlled so D' is not depleted before 120–140 m; finish with an intentional increase in SR and controlled tempo.

- Technique: treat SL as the KPI. If SL collapses mid-race, reduce early speed targets and re-optimize under a stricter lactate ceiling.

4.2 Generalization to female athletes

The optimization framework is identical for female swimmers, but calibrated parameters often differ: typical differences include lower absolute speeds, potentially different SR–SL curves, and different lactate kinetics and recovery constants. In practice, refitting CV, D', and SL(SR) to the athlete automatically adapts the optimal pacing and underwater distances.

5. Implementation guide (athlete-specific fitting)

This report becomes fully actionable when refit to your athlete. The workflow below enables a coach to collect data, fit parameters, and output an individualized race plan (underwaters, SR targets, and 50 splits).

5.1 What to measure (minimum viable dataset)

- Video: side view (or above) with a visible clock; mark 0–15 m, 15–25 m, 25–45 m, and each turn-in/turn-out.

- 15 m timing off the start and each turn; record breakout distance $x_{u,i}$ (m).

- Stroke metrics: stroke count per 25 and per 50 in each lap; tempo (s/cycle) for at least one mid-pool segment per lap.

- CV estimation: 2–4 maximal trials (e.g., 200 and 400; or 100/200/400) with reliable timing.

- Optional lactate: post-rep samples from standardized sets to fit alpha, gamma, k_clr (follow safety/ethics procedures).

5.2 Parameter fitting recipes

Underwater decay: fit $v\_u(x)=v\_inf+(v0-v\_inf)exp(-x/d\_c)$ from 0–15 m time and measured breakout distance. SR–SL curve: compute SR and SL for each segment; fit SL(SR) (quadratic often sufficient). CV and D′: fit a critical speed regression from time trials to estimate CV and an anaerobic-capacity proxy. Lactate: fit the ODE by matching model intensity to post-set lactate across intensities; treat L(t) as a proxy state.
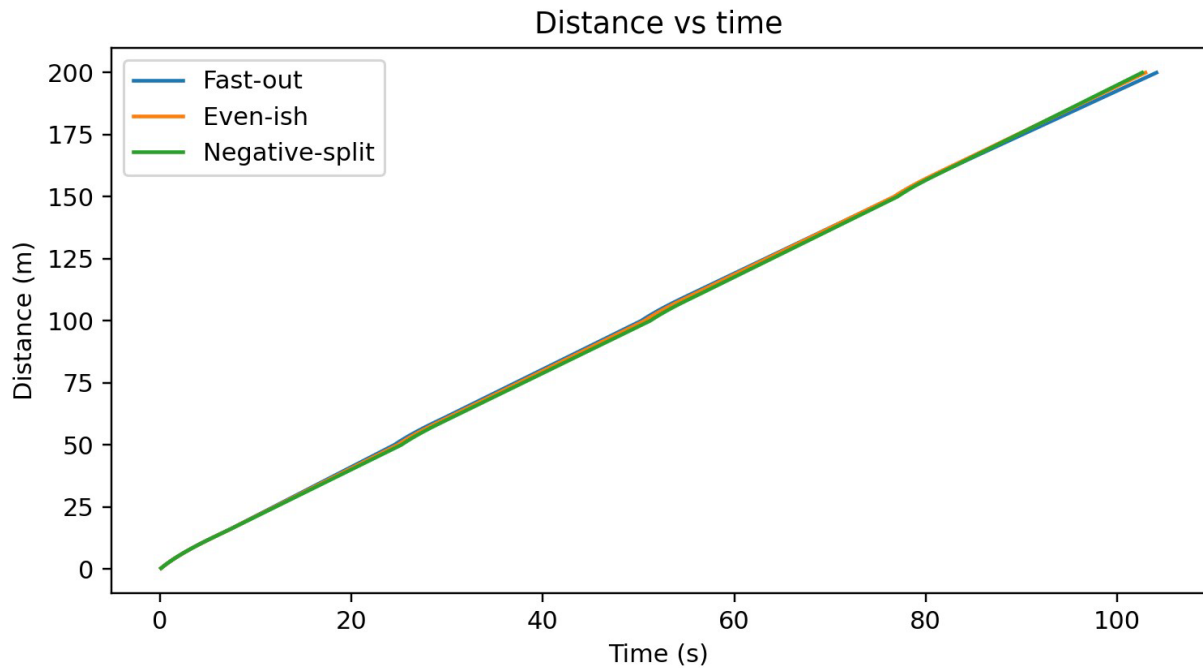

Figure. Distance vs time curves in simulation. In athlete fitting, replace with your trial curves and race video splits.

6. Limitations

1) Simulated parameters are illustrative and not athlete-specific. 2) Lactate in-race is not directly observed; L(t) is a proxy capturing intensity and fatigue. 3) Underwater speed depends on depth, kick technique, and streamline; the exponential decay model is simplified. 4) Turn approach and wall-contact time are not explicitly optimized here; incorporating them increases fidelity. 5) Psychological/tactical dynamics are not modeled.

7. Conclusion

The 200 m freestyle LCM can be modeled as a calculus-driven optimization problem where velocity, underwaters, and stroke mechanics are selected to minimize time under physiological constraints. The v^3 power law penalizes early surges; breakout optimality identifies when to surface; SR–SL calculus formalizes tempo choices; and lactate + D′ dynamics explain late-race technique collapse. The provided workflow and code allow coaches and athletes to refit the model from video and timing data and generate an individualized, testable race plan.

References (representative)

- Skiba, P. F. (2012). Modeling the expenditure and reconstitution of work capacity above critical power.

- Wakayoshi, K. et al. (1992–1993). Critical speed concept applied to swimming performance (series).

- Toussaint, H. M. and colleagues. Swimming hydrodynamics and efficiency (review works).

- Seifert, L. and Chollet, D. Stroke mechanics and coordination across distances.

- General exercise physiology sources on lactate production/clearance kinetics and fatigue modeling.

- World Aquatics rule: 15 m underwater limit after start/turns (LCM).

Appendix A — Model parameters used for figures

These values are placeholders chosen to generate coherent visuals; refit to your athlete.

Parameter	Value
CV	1.78 m/s
D′0	26.0 m-equivalent
tau_rec	40.0 s
L0	1.2 mmol/L
alpha	10.0, gamma
k_clr	0.015 1/s
Underwater model: v_inf	1.55 m/s, d_c

Appendix B — Minimal Python simulation code

```
# Minimal reproducible model (illustrative). # Replace parameters
with athlete-specific fitted values and re-run. import numpy as
np, math

def v_uw(x, v0, v_inf, d_c):      return v_inf +
(v0 - v_inf)*math.exp(-x/d_c)

def simulate(strategy, params, dt=0.1):    CV=params["CV"]; D0=params["Dprime0"];
tau=params["tau_rec"]     L0=params["L0"]; alpha=params["alpha"];
gamma=params["gamma"]; k=params["k_clear"]     v_inf=params["uw_vinf"];
d_c=params["uw_dc"]

    t=0.0; x_total=0.0; Dp=D0; L=L0     split_times=[]
for lap in range(4):         # underwater
uw=strategy["uw_dist"][lap]; v0=strategy["uw_v0"][lap]
x=0.0        while x < uw-1e-6:
        v=v_uw(x, v0, v_inf,
d_c)         dx=v*dt
if x+dx>uw:
            dt_eff=(uw-x)/v; dx=uw-x
else:
            dt_eff=dt             # D' balance
if v>CV: Dp -= (v-CV)*dt_eff         else: Dp +=
(CV-v)*(D0-Dp)/D0*(dt_eff/tau)
Dp=max(0.0, min(D0, Dp))
        # lactate
I=v/CV
        prod=alpha*max(0.0, I-1.0)**gamma
L += (prod - k*(L-L0))*dt_eff          t
+= dt_eff; x_total += dx; x += dx

    # surface        surf=50.0-uw;
v_s=strategy["seg_v"][lap]        x=0.0
while x < surf-1e-6:
        v=v_s
dx=v*dt          if
x+dx>surf:
            dt_eff=(surf-x)/v; dx=surf-x
else:
            dt_eff=dt            if v>CV: Dp -=
(v-CV)*dt_eff           else: Dp += (CV-v)*(D0-
Dp)/D0*(dt_eff/tau)          Dp=max(0.0, min(D0,
Dp))         I=v/CV
        prod=alpha*max(0.0, I-1.0)**gamma
L += (prod - k*(L-L0))*dt_eff           t
+= dt_eff; x_total += dx; x += dx

    split_times.append(t)
return split_times[-1], split_times
```

Suggested extension: grid-search x_u,i and v_i under D'(t)≥0 and a lactate/technique ceiling; export a race plan (underwaters + SR targets + splits).