Students/NIU:

- Shayan Abbasi / 1596105
- Adria Brugarolas Soto / 1244445
- Javier Iván Gutiérrez / 1546795
- Yan Jiayu / 161965
- Hu Xuanyu / 1621613

Problem 1

- $f(x) = -3x^4 - 20x^3 + 144x^2 + 17$

$$f'(x) = -12x^3 - 60x^2 + 288x = 0$$

$$\Rightarrow f'(x) = x(x-3)(x+8) = 0 \to \begin{cases} x = 0 \to \text{local Min} \\ x = 3 \to \text{local max} \\ x = -8 \to \text{local max} \end{cases}$$

Due to concave sides of $f(x)$, $x=0$ is not a global max. $\to f(+\infty) = -\infty$, $f(-\infty) = -\infty$

$$f''(x) = -36x^2 - 120x + 288 \Rightarrow$$

x		-4.95		1.616	
$f''(x)$	$-$		$+$		$-$

which one is Global max?

$\underbrace{(-8, 7185)}_{\text{Global max}}$ $(3, 530)$

- $f(x) = -(x+13)^4$

$$f'(x) = -4(x+13)^3 = 0 \to (x+13)^3 = 0 \to x = -13 \to \text{Global max}$$

$$f''(x) = -12(x+13)^2 \to \text{Root} \to x = -13$$

x		-13	
$f''(x)$	$-$	$\bullet$	$-$

Problem 2

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}} = -2e^{-\sqrt{x}} \Rightarrow \int_1^b \frac{e^{-\sqrt{x}}}{\sqrt{x}} = \Big]_1^b -2e^{-\sqrt{x}} = -2e^{-\sqrt{b}} + 2e^{-1}$$

a.1) $$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b -2e^{-\sqrt{x}} = \left(\lim_{x\to\infty} -2e^{-\sqrt{x}}\right) - \left(\frac{-2}{e}\right)$$

$$= \lim_{x\to\infty}\left(\frac{-2}{e^{\sqrt{x}}}\right) + \frac{2}{e} = 0 + \frac{2}{e} = \frac{2}{e}$$

- $\int_1^\infty \frac{\ln x}{x^p}\,dx$ for $p > 1$

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \int_1^\infty \ln x \cdot x^{-p} = \frac{1}{1-p}\int_1^\infty \ln(x)\, d\,x^{1-p}$$

$$= \frac{1}{1-p}\ln(x) \cdot \frac{1}{x^{p-1}} - \frac{1}{1-p}\int_1^\infty \frac{1}{x^{p-1}}\, d\ln x$$

$$= \frac{1}{1-p} \ln x \cdot \frac{1}{x^{p-1}} - \frac{1}{1-p} \int_1^\infty \frac{1}{x^p}\, dx$$

$$= \frac{1}{1-p} \ln x \cdot \frac{1}{x^{p-1}} - \frac{1}{(1-p)^2} \cdot \frac{1}{x^{p-1}} \Big|_1^\infty$$

$$= \lim_{x\to\infty} \frac{1}{1-p} \cdot \frac{\ln x - \frac{1}{1-p}}{x^{p-1}} - \frac{1}{(1-p)^2}$$

$\frac{1}{1-p}$ and $-\frac{1}{(1-p)^2}$ are costants $\rightarrow$

if $\lim \frac{\ln x - \frac{1}{1-p}}{x^{p-1}}$ is convergent

$\Rightarrow \int_1^\infty \frac{\ln x}{x}\, dx$ converges.

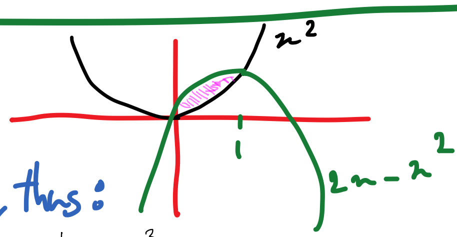
$$\Rightarrow \lim_{x\to\infty} \frac{\ln x - \frac{1}{1-p}}{x^{p-1}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{(p-1)x^{p-2}}$$

$$= \frac{1}{p-1} \lim_{x\to\infty} \frac{1}{x^{p-1}} = 0 \text{ for } p>1$$

$$\Rightarrow \int_1^\infty \frac{\ln x}{x^p}\, dx = -\frac{1}{(1-p)^2}$$

b) $y = 2x - x^2 \;,\; y = x^2$

The enclosed area looks like this:



And the purple area is the enclosed area, thus:

$$\text{enclosed area} = \int_0^1 2x - x^2 - \int_0^1 x^2 = \Big]_0^1 x^2 - \Big]_0^1 \frac{x^3}{3} - \Big]_0^1 \frac{x^3}{3}$$

$$= (1-0) - \left(\frac{1}{3} - 0\right) - \left(\frac{1}{3} - 0\right) = \frac{1}{3}$$

Problem 3

- $f(x_1, x_2) = x_1^2 - x_2^2$

$$\frac{\partial f}{\partial x_1} = 2x_1 \qquad \frac{\partial f}{\partial x_2} = -2x_2$$

$$\frac{\partial^2 f}{\partial x_1^2} = 2 \qquad \frac{\partial^2 f}{\partial x_2^2} = -2 \qquad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

critical points: $\longrightarrow x_1 = 0 ,\; x_2 = 0$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \qquad |H_1| = 2 > 0 \qquad |H_2| = -4 < 0$$

$\Rightarrow (x_1, x_2) = (0, 0) \rightarrow$ Global max

- $f(x_1, x_2) = 3x_1 e^{x_2} - x_1^3 - e^{3x_2}$

$$\frac{\partial f}{\partial x_1} = 3e^{x_2} - 3x_1^2 = 0 \longrightarrow e^{x_2} = x_1^2$$

$$\frac{\partial f}{\partial x_2} = 3x_1 e^{x_2} - 3e^{3x_2} = 0 \longrightarrow x_1 = \frac{e^{3x_2}}{e^{x_2}} = e^{2x_2}$$

$$\Rightarrow e^{4x_2} = e^{x_2} \longrightarrow x_2 = 0 ,\; x_1 = 1$$

$$\frac{\partial^2 f}{\partial x_1^2} = -6x_1 \qquad \frac{\partial^2 f}{\partial x_2^2} = 3x_1 e^{x_2} - 9e^{3x_2} \qquad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 3e^{x_2}$$

$$\Rightarrow H = \begin{bmatrix} -6x_1 & 3e^{x_2} \\ 3e^{x_2} & 3x_1 e^{x_2} - 9e^{3x_2} \end{bmatrix} \rightarrow \text{for } (x_1, x_2) = (1, 0) \Rightarrow H_{(1,0)} = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$$

$\Rightarrow (x_1, x_2) = (1, 0) \rightarrow$ Local max

$$R_1 \phi_1 - 1 = 0 \rightarrow R_1 \phi_1 = 1$$

$\boxed{\text{Problem 4}}$

$$R_1(\phi(r_0^*, r_T), k_m)\,\phi_1(r_0^*, r_T) - 1 = F(r_0^*, r_T, k_m)$$

$$\frac{\partial r_0^*}{\partial r_T} = -\frac{F'_{r_T}}{F_{r_0^*}} = -\frac{(\phi_2 R_{11} \phi_1 + \phi_{12} R_1)}{\phi_1 R_{11} \phi_1 + \phi_{11} R_1}$$

method 1

$$\left(R_{11}\phi_1 \frac{\partial r_0^*}{\partial r_T} + R_{11}\phi_2\right)\phi_1 + R_1\left(\phi_{11}\frac{\partial r_0^*}{\partial r_T} + \phi_{12}\right)$$

$$= R_{11}\phi_1\phi_1 \frac{\partial r_0^*}{\partial r_T} + R_{11}\phi_2\phi_1 + R_1\phi_{11}\frac{\partial r_0^*}{\partial r_T} + \phi_{12}R_1 = 0$$

$$\Rightarrow \frac{\partial r_0^*}{\partial r_T}\left(R_{11}\phi_1\phi_1 + R_1\phi_{11}\right) = -\left(R_{11}\phi_2\phi_1 + \phi_{12}R_1\right)$$

$$\Rightarrow \frac{\partial r_0^*}{\partial r_T} = -\frac{(R_{11}\phi_2\phi_1 + \phi_{12}R_1)}{(R_{11}\phi_1\phi_1 + R_1\phi_{11})}$$

$\boxed{\text{method 2}}$

Problem 5) Max $Q = 75\left(0.3\,K^{-0.4} + (1-0.3)L^{-0.4}\right)^{-1/0.4}$

s.t. $4K + 3L = 120$

$$\mathcal{L} = 75\left(0.3K^{-0.4} + 0.7L^{-0.4}\right)^{-2.5} - \lambda(4K + 3L - 120)$$

① $$\frac{\partial \mathcal{L}}{\partial K} = 75\left(0.3K^{-0.4} + 0.7L^{-0.4}\right)^{-3.5}\left((-0.4)\times 0.3\,K^{-1.4}\right) - 4\lambda = 0$$

② $$\frac{\partial \mathcal{L}}{\partial L} = 75\left(0.3K^{-0.4} + 0.7L^{-0.4}\right)^{-3.5}\left(0.7(-0.4)\,L^{-1.4}\right) - 3\lambda = 0$$

③ $$\frac{\partial \mathcal{L}}{\partial \lambda} = 4K + 3L - 120 = 0$$

①/② $\Rightarrow \frac{4\lambda}{3\lambda} = \frac{0.3K^{-1.4}}{0.7L^{-1.4}} \Rightarrow \left(\frac{L}{K}\right)^{1.4} = \frac{4}{3}\times\frac{0.7}{0.3} \rightarrow 1.4\ln\left(\frac{L}{K}\right) = \ln\left(\frac{4\times 0.7}{3\times 0.3}\right)$

$\Rightarrow \ln\left(\frac{L}{K}\right) \simeq 0.8107 \rightarrow \frac{L}{K} = e^{0.8107} \simeq 2.25$

③ $\Rightarrow 4K + 3(2.25K) - 120 = 0 \Rightarrow 10.75K = 120 \Rightarrow K \simeq \frac{120}{10.75}$

$\Rightarrow L = 2.25 \times \frac{120}{10.75} \simeq 25.12 \Rightarrow$ critical point:

$(K, L) \simeq (11.16, 25.12)$

Problem 6) Max $2x^2 - 3y^2 + z$

s.t. $x^2 + 2y^2 + 3z^2 = 2$

$$\mathcal{L} = 2x^2 - 3y^2 + z - \lambda(x^2 + 2y^2 + 3z^2 - 2)$$

$\frac{\partial \mathcal{L}}{\partial x} = 4x - 2x\lambda = 0 \rightarrow 2x(2-\lambda) = 0 \rightarrow x = 0$ or $\lambda = 2$

$\frac{\partial \mathcal{L}}{\partial y} = -6y - 4y\lambda = 0 \rightarrow -2y(3 + 2\lambda) = 0 \rightarrow y = 0$ or $\lambda = -3/2$

$\frac{\partial \mathcal{L}}{\partial z} = 1 - 6z\lambda = 0 \rightarrow \lambda = \frac{1}{6z} \rightarrow z = \frac{1}{6\lambda}$

if $\lambda = 2 \rightarrow y = 0,\ z = \frac{1}{12},\ x^2 = 2 - 2(0^2) - 3\left(\frac{1}{12^2}\right)$

$x^2 = \frac{285}{144} = \frac{95}{48} \rightarrow x = \pm\frac{\sqrt{95/3}}{4}$

$285 \,|\, 5$
57

$\Rightarrow (x, y, z) < \left(+\frac{\sqrt{95/3}}{4}, 0, \frac{1}{12}\right), \left(-\frac{\sqrt{95/3}}{4}, 0, \frac{1}{12}\right)$ } two critical points → Highest yield point ✱

based on Hesian below these are Global mins.

if $\lambda = \frac{3}{2} \to x = 0$, $z = -\frac{1}{9} \Rightarrow (0)^2 + 2y^2 + 3(\frac{1}{9})^2 = 2$

$\Rightarrow 2y^2 = 2 - \frac{1}{27} = \frac{54-1}{27} \Rightarrow y^2 = \frac{53}{54} \Rightarrow y = \pm \frac{\sqrt{\frac{53}{6}}}{3}$

$\Rightarrow (x,y,z) = \left(0, +\frac{\sqrt{53/6}}{3}, -\frac{1}{9}\right)$; $= \left(0, -\frac{\sqrt{53/6}}{3}, -\frac{1}{9}\right)$ two more critical points based on Hessian below, these are Global maxes

$f\left(0, \frac{\sqrt{53/6}}{3}, \frac{1}{9}\right) = -3\left(\frac{\sqrt{53/6}}{3}\right)^2 - \frac{1}{9} = -\frac{\frac{53}{6}}{\frac{3}{1}} - \frac{1}{9} = \frac{-53-2}{18} = \frac{-55}{18}$

$f\left(0, -\frac{\sqrt{53/6}}{3}, \frac{1}{9}\right) = -\frac{55}{18}$

$f\left(\frac{\sqrt{95/3}}{4}, 0, \frac{1}{12}\right) = 2\left(\frac{\sqrt{95/3}}{4}\right)^2 + \frac{1}{12} = \frac{\frac{95}{3}}{8} + \frac{1}{12} = \frac{97}{24}$

$f\left(-\frac{\sqrt{95/3}}{4}, 0, \frac{1}{12}\right) = \frac{97}{24}$ highest yields

Is the Lagrange Function concave at $\left(0, \pm\frac{\sqrt{53/3}}{6}, -\frac{1}{9}\right)$?

Since $\lambda = \frac{3}{2}$, then $L = 2x^2 - 3y^2 + z + \frac{3}{2}(x^2 + 2y^2 + 3z^2 - 2)$

$= 2x^2 + \frac{3}{2}x^2 - 3y^2 + 3y^2 + z + 3z^2 - 2$

$= \frac{7}{2}x^2 + 0 + 4z^2 - 2$

$$H = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$|H_1| = +7$ $|H_2| = 0$ $|H_3| = 0$

$\Rightarrow$ positive semi-definite

$\Rightarrow$ The obtained local max points are Global max.

Problem (7) Max $-4x^2 + 100x - 2y^2 + 80y - 10$

s.t. $x + y \leq 50$

$L = -4x^2 + 100x - 2y^2 + 80y - 10 - \lambda(x + y - 50)$

$\frac{\partial L}{\partial x} = -8x + 100 - \lambda = 0 \Rightarrow \lambda = -8x + 100$ ①

$\frac{\partial L}{\partial y} = -4y + 80 - \lambda = 0 \Rightarrow \lambda = 80 - 4y$ ②

$8x - 4y = 20 \Rightarrow 2x - y = 5$ ③

$$\frac{\partial L}{\partial \lambda} = x + y - 50 = 0 \quad (4)$$

if $\lambda(x+y-50) = 0 \;<\; \begin{matrix} x+y-50 = 0 \;(5) \to \lambda \geq 0 \\ x+y-50 < 0 \;(6) \to \lambda = 0 \end{matrix}$

(1) First Scenario $\to$ $x+y=50$, $\lambda \geq 0$

(5) & (3) $\begin{cases} x+y=50 \\ 2x-y=5 \end{cases} \Rightarrow 3x = 55 \to x = \frac{55}{3}$, $y = \frac{110-15}{3} = \frac{95}{3}$

(1) $\to \lambda = -8\left(\frac{55}{3}\right) + 100 = \frac{-140}{3}$ Contradicts $\lambda \geq 0$

(2) $\to \lambda = 80 - 4\left(\frac{95}{3}\right) = -\frac{140}{3}$

(2) Second Scenario $x+y<50$, $\lambda = 0$

$\left.\begin{matrix} (1) \to \lambda = -8x+100 = 0 \to x = \frac{100}{8} \\ (2) \to \lambda = 80-4y = 0 \to y = 20 \end{matrix}\right\}$ (6) $\to x+y = \frac{100}{8} + 20 \leq 50$ ✓✓

one critical point $\to (x,y) = \left(\frac{100}{8}, 20\right)$ with $\lambda = 0$

The sufficient condition:

$$L = -4x^2 + 100x - 2y^2 + 80y - 10 - 0(x+y-50)$$

$$\Rightarrow L = -4x^2 + 100x - 2y^2 + 80y - 10$$

$$H = \begin{bmatrix} -8 & 0 \\ 0 & -4 \end{bmatrix} \qquad |H_1| = -8 \qquad |H_2| = 32$$

$\Rightarrow (x,y) = \left(\frac{100}{8}, 20\right)$ Global Max