Solution of system of Linear Interval Equations for Portfolio Optimization

Project - Part II (MA57012) report submitted to
Indian Institute of Technology Kharagpur
in partial fulfilment for the award of the degree of
Master of Science
in
Mathematics and Computing

by

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Spring Semester, 2019-2020
May 27, 2020

DECLARATION

I certify that

1) The work contained in this report has been done by me under the guidance of my supervisor.
2) The work has not been submitted to any other Institute for any degree or diploma.
3) I have conformed to the norms and guidelines given in the Ethical Code of Conduct of the Institute.
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CERTIFICATE

This is to certify that the project report entitled "**Solution of system of Linear Interval Equations for Portfolio Optimization**" submitted by Shayan Shafquat (Roll No. 15MA20038) to Indian Institute of Technology Kharagpur towards partial fulfilment of requirements for the award of degree of Master of Science in Mathematics and Computing is a record of bona fide work carried out by him under my supervision and guidance during Spring Semester, 2019-2020.

Date: May 27, 2020
Place: Kharagpur

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Abstract

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Department: **Department of Mathematics**

Thesis title: **Solution of system of Linear Interval Equations for Portfolio Optimization**

Thesis supervisor: **Dr. Geetanjali Panda**

Month and year of thesis submission: **May 27, 2020**

This involves study of matrices in uncertain environment. The objective functions and constraints of problems in the field of portfolio management are assumed to be real numbers whereas considering an interval involving lower and upper bound gives us a range of values to be considered. So, a study involving basic operations like regularity, inverse of interval matrices is done to further develop a better methodology to solve a system of linear interval equations in portfolio management. Algorithms to check the regularity, finding a singular matrix and the inverse of an interval matrices is also studied.

Contents

Chapter 1

Introduction

Linear programming is a technique for the optimization of a linear objective function, subject to linear equality and linear inequality constraints. The coefficients of this model are in general assumed to be real numbers. But in real life situations due to the presence of various uncertainties in the domain of the optimisation model this assumption does not always holds true. Generally stochastic are applied to address these uncertainties which can referenced in many of the past papers. The parameters are assumed to be random variables whose probability distribution are known in stochastic approach. Dealing with the questionable parameters, one may utilize interim numbers to speak to the parameters; we generally use the past information to know about the lower and upper bound of these parameters and henceforth defeating our challenges. We consider a general interval vector optimization problem which addresses linear interval valued functions in the objective function as well as in the constraints. We propose a strategy for computing the exact bounds on the solution of a system of n linear equations in n variables whose coefficients and right-hand sides vary in some real intervals.

It is possible to have linear models for portfolio optimization. Can we measure the risk or safety in order to have a linear model? A first observation is that, in order to guarantee that a portfolio takes advantage of diversification, no risk or safety measure can be a linear function of the shares of the assets in the portfolio, that is of the variables x_j, $j = 1, \ldots, n$. Through discretization of the return random variables or, equivalently, through the concept of scenarios linear models can be obtained.

Motivation. The consideration of reward and risk is the basis for a general problem of optimizing the mean-variance-portfolio. The reward is based on the expected return on the portfolio and the risk on the variance of the portfolio. In the simplest form, the portfolio optimization model determines the proportion P of the total investment x_i of the assets of a portfolio $x = (x_1, x_2, . . . , x_n)$ where $\mathbf{\Sigma}\, x_i = 1$. Rate of expected returns of the assets of a portfolio are generally estimated from previous yearly/monthly/daily data. Presence of uncertainty in market does not allow a financial specialist to estimate the exact rate of expected return. If the investor finds the lower bound (r^L_i) and upper bound (r^R_i) of the return of the assets from previous data for a fixed time period then the expected rate of return of i^{th} asset lies in the interval $[r^L_i, r^R_i]$.

In a portfolio optimization problem, a financial specialist needs to boost the expected return of the portfolio with minimum risk. In this circumstance, a financial specialist needs to maximize the expected return as and to limit the variance. Suppose, the expected return and linear risk of portfolio are meant by R(x), δ(x) respectively, which can be defined as follows.

$$\hat{R}(x) = \sum_{i=1}^{n}[r_i^L, r_i^R]x_i$$

$$\delta(x) = \mathbb{E}\{|R_x - \mathbb{E}\{R_x\}|\} = \mathbb{E}\{\sum_{i=1}^{n}[r_i^L, r_i^R]x_i - \mathbb{E}\{\sum_{i=1}^{n}[r_i^L, r_i^R]x_i\}|\}$$

where $[r_i^L, r_i^R]$ denotes the interval return of the asset i from the past data, x_i is the proportion of the portfolio given to asset i and $\mathbb{E}$ is expectation of the given random variable.

Chapter 2

Literature Review

2.1 Notations specific for this text

All through the work, significant role is played by the set Y_m of all ± 1 vectors in R^m, i.e., $Y_m = \{y \in R^m; |y| = e\}$. The cardinality of Y_m is 2^m. For every $x \in R^m$ we characterize its sign vector sgn x by

$$sgn(x_i) = \begin{cases} 1, & x_i < 0 \\ -1, & x_i \geq 0 \end{cases} \qquad i = (1, \ldots, m)$$

so that sgn $x \in Y_m$. For a given vector $y \in R^m$ we denote

$$T_y = \begin{pmatrix} y_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & y_n \end{pmatrix} \tag{2.1}$$

With a couple of special cases we utilize the notation T_y for vectors $y \in Y_m$ only, in which case we have $T_{-y} = -T_y$, $T^{-1}_y = T_y$ and $|T_y| = I$. For every $x \in R^m$ we can compose $|x| = T_z x$, where $z = \text{sgn} x$. Notice that $T_z x = (z_i x_i)^m_{i=1} = z \circ x$.

2.2 Auxiliary results

The set Y_n: is the set of all ± 1-vectors in R^n (there are 2^n of them).
This calculation is utilized as a subroutine in thorough calculations that require to play out some activity for all $y \in Y_n$. The set Y_n itself isn't developed; the activity is applied to progressively created vectors.

The equation Ax+B|x| = b and the sign accord algorithm:

Given A, B $\in R_{n\times n}$ and b $\in R_n$, find a solution to the nonlinear equation

$$Ax + B|x| = b. \quad (2.2)$$

On the off chance that we knew the sign vector z = sgn x of the arrangement x of, we could revise as $(A+BT_z)x = b$ and explain it for x as $x = (A+BT_z)^{-1}b$. Notwithstanding, we know neither x, nor z; yet we do realize that they ought to fulfil $T_zx = |x| \geq 0$, i.e., $z_jx_j \geq 0$ for every j (a circumstance we call a sign accord of z and x). In its part structure the sign accord calculation registers the z's and x's over and again until a sign accord happens.

$z = \mathrm{sgn}(A^{-1}b)$;
$x = (A + BT_z)^{-1}b$;
while $z_jx_j < 0$ for some j
$k = \min\{j\, ;\, z_jx_j < 0\}$;
$z_k = -z_k$;
$x = (A+BT_z)^{-1}b$;
end

2.3 Interval Matrices

Definition. If $\overline{A}$, $\underline{A}$ are two matrices in $R_{m\times n}$ such that $\overline{A} \leq \underline{A}$, then the set of matrices

$$\mathbf{A} = [\underline{A}, \overline{A}\,] = \{A; \underline{A} \leq A \leq \overline{A}\,\}$$

is known an interval matrix, and the matrices $\underline{A}$, $\overline{A}$ are known as limits/bounds.

Hence, if $\underline{A} = (\underline{a_{ij}})$ and $\overline{A} = (\overline{a_{ij}})$, then A is the set of all matrices A = (a_{ij}) satisfying

$$\underline{a_{ij}} \leq a_{ij} \leq \overline{a_{ij}} \qquad (2.3)$$

for i = 1, . . ., m, j= 1, . . ., n.

It is increasingly favourable to express the information in terms of the center/middle matrix

$$A_c = 1/2(\underline{A}+\overline{A}) \qquad (2.4)$$

and of the radius matrix

$$\Delta = 1/2(\underline{A} - \overline{A}), \qquad (2.5)$$

which is always nonnegative (all elements greater than or equal to zero).
We easily obtain that

$$\underline{A} = A_c - \Delta,$$

$$\overline{A} = A_c + \Delta,$$

so that A can be given either as $[\overline{A}, \underline{A}]$, or as $[A_c - \Delta, A_c + \Delta]$.

Matrices A_{yz}. Given an m×n interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$, we define matrices $A_{yz} = A_c - T_y \Delta T_z$

for each $y \in Y_m$ and $z \in Y_n$ (T_y is given by (2.1)).

Explanation. The definition implies that

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i \Delta_{ij} z_j = \begin{cases} a_{ij}, & y_i z_j = -1, \\ a_{ij}, & y_i z_j = \;\; 1 \end{cases}$$

(i= 1, . . ., m, j= 1, . . ., n), so that $A_{yz} \in \mathbf{A}$ for each $y \in Y_m$, $z \in Y_n$.

Special cases. We write A_{-yz} instead of $A_{-y,z}$. In particular, we have

$$A_{-yz} = A_c + T_y \Delta T_z,$$

$$A_{ye} = A_c - T_y \Delta,$$

$$A_{ez} = A_c - \Delta T_z,$$

A_{ee} = A and

A_{-ee} = A.

2.3.1 Properties of n * n interval matrices

Regularity: A square interval matrix A is called regular if each $A \in A^I$ is non-singular, and if it contains a singular matrix then it is said to be singular.

Necessary and sufficient conditions. For a square interval matrix, A =

$[A_c - \Delta, A_c + \Delta]$, the following are some assertions which are mutually equivalent:

(i) A is regular,

(ii) the inequality $|A_c x| \le \Delta|x|$ has only the trivial solution, (used in proving the theorem $|A_c x - b_c| \le \Delta|x| + \delta$ ahead)

(iii) $(\det A_{yz})(\det A_{y'z'}) > 0$ for each $y, z, y', z' \in Y_n$,

(iv) A_c is nonsingular and $\max_{y,z \in Y_n} \varrho(A_c^{-1} T_y \Delta T_z) < 1$, (used this as to check whether the central matrix in our system of equations is regular)

(v) for each $z \in Y_n$ the equation $QA_c - |Q|\Delta T_z = I$ has a unique matrix solution Q_z.

Sufficient regularity condition. An interval matrix $A = [A_c - \Delta, A_c + \Delta]$ is regular if

$$\varrho(|A_c^{-1}|\Delta) < 1 \qquad (2.6)$$

holds.

Sufficient singularity condition. An interval matrix $\mathbf{A} = [Ac - \Delta, Ac + \Delta]$ is singular if

$$\max_j (|A_c^{-1}|\Delta)_{jj} \ge 1 \text{ holds.}$$

2.4 Pseudo-inverse of a matrix

If A is a square matrix of full rank, then the inverse of A exists (A is referred to as an invertible matrix) and

$$Ax = b$$

has the solution

$$x = A^{-1}b$$

When the matrix may not be invertible, we use the generalized concept of matrix inverse i.e. Moore-Penrose pseudo inverse. If A is invertible, then the

Moore-Penrose pseudo inverse is equivalent to the matrix inverse. However, even when A is not invertible the Moore-Penrose pseudo inverse is defined.

Considering only the case where A consists of real numbers, the Moore-Penrose pseudo inverse, A^+, of an m-by-n matrix is defined by the unique n-by-m matrix satisfying the following four criteria:

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)' = AA^+$
4. $(A^+A)' = A^+A$

If A is an mxn matrix where $m > n$ and A is of full rank (= n), then

$$A^+ = (A'A)^{-1}A'$$

and the solution of $Ax = b$ hence comes out to be $x = A^+b$. Although, in this case the solution is not exact but it finds the solution that is closest in the least squares sense.

Chapter 3

Methodology

3.1 System of linear Interval Equations

A system of linear interval equations represents a family of ordinary linear systems in which the coefficients are fixed and the right-hand side values lie in the prescribed intervals. Assume that the coefficients and right-hand sides of an arrangement of n direct conditions in n factors are not decided precisely (are not real numbers), yet are just known to exist in some genuine intervals. This can be due to a lot of factors such as roundoff, truncation or information blunders. Under the assumption that regularity holds true, each of these systems has a unique solution, and all these solutions constitute a so-called solution set **X**.

$$\underline{x_i} = \min \{x_i; x \in X\},$$

$$\overline{x_i} = \max \{x_i; x \in X)$$

3.2 The solution set

Given an square n x n interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ and an interval n-vector $b^I = [b_c - \delta, b_c + \delta]$, we introduce for any vectors y, z that belongs to n-dimensional real space (i.e. $y, z \in R^n$) the notation

$$A_{yz} = A_c - T_y \Delta T_z,$$

$$b_y = b_c + T_y \delta$$

Since no values in A^I or b^I are preferred, each x satisfying $Ax = b$ for some $A \in A^I$, $b \in b^I$ is considered to be a solution of the system of linear interval equations.

$$A^I x = b^I$$

Hence, the solution set X is given by

$$\mathbf{X = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I\}}$$

- **X** is compact and connected when A is regular.
- Each component of **X** is unbounded for the case when A is singular.

Theorem: We have $X = \{x;\ |A_c x - b_c| \leq \Delta|x| + \delta\}$

Proof. $|A_c x - b_c| = |(A_c - A)x + b - b_c| \leq \Delta|x| + \delta$

This theorem tells us that for regular A^I, the intersection of X with each orthant is a convex polyhedron (Beeck [10]); consequently X, being a union of convex polyhedral is generally non convex. Nonetheless, Conv X i.e. the convex hull of X which is the smallest convex polygon containing all the points is a convex polyhedron; consequently, equivalent to the convex hull of its vertices. Each vertex of Conv X fulfils the nonlinear equation

$$|A_c x - b_c| = \Delta|x| + \delta \qquad (3.1)$$

Let x solve (3.1); put $y = \text{sgn}(A_c x - b_c)$, then $y \in Y$, and from $|A_c x - b_c| = T_y(A_c x - b_c) = \Delta|x| + \delta$, using the fact that $T_y^{-1} = T_y$ the solution is further decomposed into 2^n simpler equations and hence we obtain,

$$A_c x - b_c = T_y(\Delta|x| + \delta) \qquad (3.2)$$

Substituting $|x| = T_z x$ and setting $z = \text{sgn}\ x$ in (3.2), we get equivalent form

$$\begin{aligned} &A_{yz} x = b_y, \\ &T_z x \geq 0, \qquad z \in Y, \qquad (3.3) \end{aligned}$$

which will be used further through least squares method for the formulation of a general algorithm to solve (3.2). Second, by simple rearrangement of (3.2) we have

$$x = D_y|x| + d_y \tag{3.4}$$

where $D_y = A_c^{-1}T_y\,\Delta$, $d_y = A_c^{-1}b_y$ which is a fixed-point equation to be used for solving (3.2) by an iterative method.

If A^I is regular. Then for each $y \in Y$, the equation (3.3) has exactly one solution $x_y \in X$, and Conv X = Conv$\{x_y; y \in Y\}$.

3.2.1 Finite method for computing x_y's using least squares

Computing x_y for a given $y \in Y$:

- Select a $z \in Y$ ($z = \text{sgn}\, d_y$)
- For every (y, z) compute A_{yz}, b_y, T_z to solve the given two equations:

$$A_{yz}x = b_y$$

$$T_z x - s = 0$$

- Objective function using least square method:

$$\min_{\substack{s \geq 0,\\ x \in \mathbb{R}^n}} \; \| C\, p - d \|^2$$

where, $C = \begin{pmatrix} A_{yz} & 0 \\ T_z & -1 \end{pmatrix}$,

$$p = \begin{matrix} x \\ s \end{matrix}\;, \qquad d = \begin{matrix} b_y \\ 0 \end{matrix}$$

For each $y \in Y$ we will get x_y

$$\underline{x} = \min \{ x_y \,;\, y \in Y \}$$

$$\overline{x} = \max \{ x_y \,;\, y \in Y \}$$

Objective Function:

$$\mathbf{Min} \sum_{i=1}^{n} [(\sum_{j=1}^{n} A_{yz}^{ij} x_j - b_y^i)^2 + (\sum_{j=1}^{n} T_z^{ij} x_j - s_j)^2$$

To unravel this we utilized Limited-memory BFGS-B (L-BFGS or LM-BFGS) which is an optimization algorithm in the family of quasi-Newton methods that approximates the (BFGS) utilizing a restricted measure of computer memory.

The algorithm starts with an initial estimate of the optimal value, x_0, and continues iteratively to refine that estimate with a sequence of better estimates x_1, x_2, …. We can also limit the iterative method to consume less time but with a trade-off to less efficient results. The derivatives of the function $g_k := \nabla f(x_k)$ are utilized as a key driver of the algorithm to recognize the direction of steepest descent, and also to form an estimate of the Hessian matrix of f(x).

The L-BFGS-B algorithm stretches out L-BFGS is used in optimization problems dealing with simple bound constraints on variables; that is, limitations of the form $l_i \leq x_i \leq u_i$ where l_i and u_i are for each variable constant lower and upper bound, respectively. Either of the bounds can be omitted for any number of variables. This method works by identifying fixed and free variables at every step using a simple gradient method (i.e. moving in a direction opposite to the steepest direction) , and afterwards utilizing the L-BFGS strategy on the free factors only to get higher accuracy, and afterwards rehashing the process.

3.3 LP computable Risk functions (Portfolio Optimization)

The variance is the classical statistical quantity used to measure the dispersion of a random variable around its mean. Range, Gini mean absolute difference are some other ways to measure the dispersion of a random variable. The random variable, we are interested in, is the return of the portfolio i.e. $R\mathbf{x}$. Some of the linear programmable risk measures are-

1. Gini's mean difference (GMD)
2. Mean absolute deviation (MAD)

The Mean Absolute Deviation (MAD) is a dispersion measure which is defined as

$\delta(x) = \mathbb{E}\{|R_x - \mathbb{E}\{R_x\}|\} = \mathbb{E}\{| \sum_{i=1}^{n}[r_i^L, r_i^R]x_i - \mathbb{E}\{\sum_{i=1}^{n}[r_i^L, r_i^R]x_i\}|\}$

$\sum_{i=1}^{n} x_i = 1$

The MAD measures the average of the absolute value of the difference between the random variable and its expected value. Here in our problem the expected value is the mean return i.e. $[\mu^L, \mu^R]$. When calculating the variance, the MAD considers absolute values instead of squared values (which makes it non linear) and is denoted by $\delta(x)$ in our case.

$$[\mu^L, \mu^R] = \sum_{i=1}^{n}[r_i^L, r_i^R]\, x_i$$

$$\delta(x) = \{\mathbb{E}| \sum_{i=1}^{n}[r_i^L, r_i^R]x_i - [\mu^L, \mu^R]|\}$$

Chapter 4

Results

4.1 Example of system of linear interval equations

$$[1, 5]x_1 + [-3, -1]x_2 = [7, 12],$$

$$[4, 7]\, x_1 + [3, 4]\, x_2 = [6, 35]$$

First, we will check if the given interval matrix has regular central matrix. i.e.

$\begin{pmatrix} 3 & -2 \\ 5.5 & 3.5 \end{pmatrix}$ which is not singular. Hence, the given central matrix is regular.

Here for each y ϵ Y, starting from z = sgn d_y only one system must be solved as depicted below:

y_1	y_2	z_1	z_2	$(x_y)_1$	$(x_y)_2$
1	1	1	1	9.4598	-1.3521
-1	1	1	1	3.0178	6.7445
1	-1	1	-1	4.2194	-8.1185
-1	-1	1	-1	1.0779	-4.8587

Hence for the interval solution $x^I = [\underline{x}, \overline{x}]$ we obtain,

$\underline{x} = (1.0779, -8.1185)^T$, $\overline{x} = (9.4598, 6.7445)^T$

$$x_1 = [1.0779, 9.4598]$$
$$x_2 = [-8.1185, 6.7445]$$

4.2 Example of portfolio optimization

In the problem we are considering three assets namely A_1, A_2, A_3 and their proportion of investment as x_1, x_2, x_3 such that

$$\sum_{j=1}^{3}[1,1]x_j = 1 \qquad \text{(Utility equation)}$$

Here, x_1, x_2, x_3 are interval vectors which we need.

Returns of the asset A_1, A_2, A_3 are r_1, r_2, r_3 respectively given as below:

$$r_1 \in [r_1^L, r_1^U]$$

$$r_2 \in [r_2^L, r_2^U]$$

$$r_3 \in [r_3^L, r_3^U]$$

Total return of the portfolio is bounded to be between [20%, 30%] which is depicted below.

$$\sum_{j=1}^{3}[r_j^L, r_j^U]\, x_j \in [0.2, 0.3] \qquad \text{(Return Equation)}$$

Risk is calculated using MAD which is a linear computable sum of deviation of returns of asset with the mean of return. Hence, to fit in our problem scenario it is bounded in an interval too. i.e.

$$\sum_{j=1}^{3} |[r_j^L, r_j^U]\, x_j - [\mu^L, \mu^R]| \in [\theta^L, \theta^U] \qquad \text{(Risk Equation)}$$

where,

$$[\mu^L, \mu^R] = \sum_{i=1}^{n}[r_i^L, r_i^R]\, x_i$$

Introducing t_1, t_2, t_3 to make it linear in a way such that

abs([a, b]x + c)=abs((bt + (1 - t)a)x + c), where t lies between 0 and 1.

=max{ (bt + (1 - t)a)x + c, - (bt + (1 - t)a)x + c}

Here, our risk equation changes to,

max{ $(r_1^U t_1 + (1 - t_1)\, r_1^L)x_1 + [\,\mu^L, \mu^R]$, - $(r_1^U t_1 + (1 - t_1)\, r_1^L)x_1 + [\,\mu^L, \mu^R]$ } +
max{ $(r_2^U t_2 + (1 - t_2)\, r_2^L)x_2 + [\,\mu^L, \mu^R]$, - $(r_2^U t_2 + (1 - t_2)\, r_2^L)x_2 + [\,\mu^L, \mu^R]$ } +
max{ $(r_3^U t_3 + (1 - t_3)\, r_3^L)x_3 + [\,\mu^L, \mu^R]$, - $(r_3^U t_3 + (1 - t_3)\, r_3^L)x_3 + [\,\mu^L, \mu^R]$ } ϵ
$[\theta^L, \theta^U]$

Combining, the three equations we can solve for x_1, x_2, x_3 using the least square method developed above to solve for $A^I x = b^I$ giving us the proportion of investment.

Problems involving assets more than the number of equations i.e. 3 will require more equations otherwise can be done using the principle of pseudo inverse of rectangular system. Hence, we need to develop the concept of pseudo inverse of a rectangular matrix for a interval system.

Chapter 5

Conclusion

We are thorough with the theory involving interval system. First task was to get used to with the concepts of interval matrices, secondly using them to solve the system of linear interval equations. We have used least squares method to fit the system of linear interval equations as error and further optimized that using a iterative gradient descent algorithm. The codes have been written in Python with making use of packages like Scipy (for optimization), Numpy (for matrix computations). Finally designing a problem of portfolio optimization where the developed algorithm for solving the system of linear interval equation can be used to get the proportion of portfolio. With our method being constrained to regular central matrix (A_c of A^I should not be a singular matrix) and equal number of equations to the number of assets we can stretch out this project involving rectangular interval matrices.

Chapter 6

References

[1] **Rohn, J.** A sufficent condition for an interval matrix to have a full column rank

[2] **Jana, Kumar, P., Panda, G.** Efficient Portfolio for Interval Sharpe Ratio Model

[3] **Orgyczak, W., Speranza, M., Mansini, R.** Linear models for portfolio optimization

[4] **Rohn, J.** A handbook of results on interval linear problems: Ebook. 2005. 80p. Available at: http://www.nsc.ru/interval/Library/Surveys?IlinProblems.pdf

[5] **Jana, Panda, G.** Solution of nonlinear interval vector optimization problem

[6] **G. E. Coxson,** Computing exact bounds on elements of an inverse interval matrix isNP-hard, Reliable Computing, 5 (1999), pp. 137–142.

[7] **M. Baumann,** A regularity criterion for interval matrices, in Collection of Scientific Papers Honouring Prof. Dr. K. Nickel on Occasion of his 60th Birthday, Part I,J. Garloff et al., eds., Freiburg, 1984, Albert-Ludwigs-Universiẗat, pp. 45–50.

[8] **J. Rohn,** Systems of linear interval equations, Linear Algebra and Its Applications, 126 (1989), pp.39–78. http://www.cs.cas.cz˜/rohn/publist/47.doc

[9] **J. Rohn,** Interval matrices: Singularity and real eigenvalues, SIAMJournal on Matrix Analysis and Applications, 14 (1993), pp. 82–91.http://www.cs.cas.cz˜/rohn/publist/61.doc

[10] **J. Rohn,** Systems of interval linear equations and inequalities (rectangular case), Technical Report 875, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, September 2002.

[11] **J. Rohn and V. Kreinovich,** Computing exact component wise bounds on solutions of linear systems with interval data is NP-hard, SIAM Journal on Matrix Analysis and Applications, 16 (1995), pp. 415–420.http://www.cs.cas.cz/rohn/publist/72.ps

[12] **H. Beeck, Zur Problematik der Hienbestimmung von Intervallgleichungssystemen**, in Interval Mathematics, L.echxe Notes in Comput. Sci. 29 (K. Nickel, Ed.), Springer-Verlag, 1975