

# Ambidexterity Seminar – The Chromatic Picture

Shay Ben Moshe

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## 1 Motivation – Hopkins-Neeman and Balmer’s Spectrum

Two short introductions to the topic are [5, 7] (note that they use the language of triangular categories, rather than  $\infty$ -categories.) In what follows,  $R$  is noetherian ring,  $X = \mathrm{Spec}(R)$ , and  $\mathrm{Ch}(X)$  is the symmetric monoidal stable  $\infty$ -category of chain complexes over  $R$ .

**Problem.** Can we recover  $X$  from  $\mathrm{Ch}(X)$ ?

The first partial answer to this question is given at [3, 8], which we state now.

**Definition 1.** A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finite projective modules. These are the compact objects in the category, so that they can actually be defined categorically. Denote by  $\mathrm{Ch}_{\mathrm{perf}}(X)$  the full subcategory of perfect complexes.

**Definition 2.** Let  $\mathcal{C}$  be a symmetric monoidal stable  $\infty$ -category. A full subcategory  $\mathcal{T}$  is *thick* if:

1.  $0 \in \mathcal{T}$
2. let  $a \xrightarrow{f} b \rightarrow c$  cofiber sequence, if two out of  $\{a, b, c\}$  are in  $\mathcal{T}$ , then so is the third (remember that cofiber and fiber sequences are the same)
3. it is closed under retracts

**Example 3.** Take  $R = \mathbb{Z}$ , thus  $\mathrm{Ch}(X)$  are chain complexes of abelian groups, and  $\mathrm{Ch}_{\mathrm{perf}}(X)$  are chain complexes with finitely-many non-zero entries, each of which is  $\mathbb{Z}$  to some power. Let  $K_{\bullet} \in \mathrm{Ch}(X)$ , and define  $\mathcal{T}_{K_{\bullet}} = \{A_{\bullet} \mid A_{\bullet} \otimes K_{\bullet} = 0\}$ . Clearly  $0 \in \mathcal{T}_{K_{\bullet}}$ , in a pushout where 3 are 0 the fourth is 0, and if  $A_{\bullet} \rightarrow B_{\bullet} \rightarrow A_{\bullet}$  is the identity and  $B_{\bullet} \otimes K_{\bullet} = 0$  then  $A_{\bullet} \otimes K_{\bullet} \rightarrow 0 \rightarrow A_{\bullet} \otimes K_{\bullet}$  is the identity thus 0. Therefore  $\mathcal{T}_{K_{\bullet}}$  is thick.

**Definition 4.** A subset  $V \subseteq X$  is called *specialization closed* if it is a union of closed sets. Equivalently, if  $\mathfrak{p} \subseteq \mathfrak{q}$  and  $\mathfrak{p} \in V$ , then  $\mathfrak{q} \in V$ .

**Theorem 5** (Hopkins–Neeman). *There is an inclusion-preserving bijection of sets*

$$\{\text{Thick subcategories of } \mathrm{Ch}_{\mathrm{perf}}(X)\} \rightleftarrows \{\text{Specialization closed subsets of } X\}$$

*Remark.* They actually give an explicit way to define the functions, but we omit it for the sake of brevity.

*Remark.* The theorem was improved in [9] to any quasi-compact quasi-separated scheme  $X$ , and compact objects in its derived category.

Later on, in [1, 2] the result is improved further.

**Definition 6.** A thick subcategory  $\mathcal{T}$  is an *ideal* if  $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$ . Furthermore, it is a *prime ideal* if it is a proper subcategory, and  $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$  or  $b \in \mathcal{T}$ . The *spectrum* of the category is defined similarly to the classical spectrum of a ring,  $\mathrm{Spc}(\mathcal{C}) = \{\mathcal{P} \text{ prime ideal}\}$ , and for any family of objects  $S \subseteq \mathcal{C}$  we define  $V(S) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) \mid S \cap \mathcal{P} = \emptyset\}$ , and these are the closed subsets of the *Zariski topology* on  $\mathrm{Spc}(\mathcal{C})$ . We also denote  $\mathrm{supp}(a) = V(\{a\})$ .

**Theorem 7** (Balmer). *There is a homeomorphism  $X \rightarrow \mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X))$ ,  $\mathfrak{p} \mapsto \mathcal{P} = \{M_{\bullet} \mid (M_{\bullet})_{\mathfrak{p}} = 0\}$ .*

*Remark.* This was actually upgraded to an isomorphism of locally-ringed spaces.

**Example 8.** Continuing the case  $R = \mathbb{Z}$ . We’ve seen that  $\mathcal{T}_{K_{\bullet}}$  is thick. Note that it is also an ideal, since  $A_{\bullet} \otimes B_{\bullet} \otimes K_{\bullet} = A_{\bullet} \otimes 0 = 0$ . Note that  $A_{\bullet} \in \mathcal{T}_{\mathbb{Z}_{(p)}}$  iff it is only  $q$ -torsion for  $q \neq p$ , and we can prove that it is indeed a prime ideal, and similarly for  $\mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_{\mathbb{Q}}$ . Clearly, if  $A_{\bullet} \in \mathcal{T}_{\mathbb{Z}_{(p)}}$  then  $A_{\bullet} \in \mathcal{T}_{\mathbb{Q}}$ , thus any  $S$  that doesn’t intersect  $\mathcal{T}_{\mathbb{Q}}$  doesn’t intersect any  $\mathcal{T}_{\mathbb{Z}_{(p)}}$ , so a closed set that contains  $\mathcal{T}_{\mathbb{Q}}$  includes all the others. Indeed by theorem we have  $p\mathbb{Z} \mapsto \{A_{\bullet} \mid (A_{\bullet})_{p\mathbb{Z}} = 0\} = \{A_{\bullet} \mid A_{\bullet} \otimes \mathbb{Z}_{(p)} = 0\} = \mathcal{T}_{\mathbb{Z}_{(p)}}$ , and similarly  $\mathbb{Z} \mapsto \mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_{\mathbb{Q}}$ . Therefore  $\mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X)) = \{\mathcal{T}_{\mathbb{Z}_{(2)}}, \mathcal{T}_{\mathbb{Z}_{(3)}}, \dots, \mathcal{T}_{\mathbb{Q}}\}$ . Note that the support of an element is all the prime ideals to which it *does not* belong, e.g.  $\mathcal{T}_{\mathbb{Z}_{(q)}} \in \mathrm{supp}(\mathbb{F}_p)$  iff  $\mathbb{F}_p \otimes \mathbb{Z}_{(q)} \neq 0$  which is only when  $q = p$ .

## 2 The Chromatic Picture

Similarly to the analysis in algebra/algebraic geometry, we are going to concentrate at a single prime  $p$ . Although the category of spectra doesn’t arise as the corresponding category for a scheme or a similar gadget, we can still try to “reconstruct the space  $X$ ” by applying this mechanism, and then try to use this decomposition. We remind ourselves that the compact objects are finite spectra.

## 2.1 Morava K-Theory

A good reference for this part is [6, lectures 22, 24]

**Definition 9.** Let  $R$  be an evenly graded ring.  $R$  is called a *graded field* if every non-zero homogenous is invertible, equivalently it is a field  $F$  concentrated at degree 0, or  $F[\beta^{\pm 1}]$  for  $\beta$  of positive even degree. An  $(A_\infty)$ -ring spectrum  $E$  is a *field* if  $\pi_* E$  is a field.

**Proposition 10.** A field  $E$  has *Kunneth*, i.e.  $E_*(X \otimes Y) \cong E_*(X) \otimes_{\pi_* E} E_*(Y)$ .

**Fact 11.** For each prime  $p$  and  $n = 1, 2, \dots$ , there exists a spectrum called Morava K-Theory of height  $n$ , denoted by  $K(p, n)$ , which has the following properties:

- $\pi_* K(p, n) \cong \mathbb{F}_p[v_n^{\pm 1}]$  where  $\deg v_n = 2(p^n - 1)$ .
- It is a field (and in particular,  $(A_\infty)$ -ring spectrum.)
- If  $E$  is a field, then it has the structure of a  $K(p, n)$ -module for unique  $p$  and  $n$ . In that sense  $K(p, n)$  is uniquely determined.

We also take  $K(p, 0) = H\mathbb{Q}$ .

**Example.** Remember that  $K$  (regular  $K$ -theory) has  $\pi_*(K) = \mathbb{Z}[\beta^{\pm 1}]$  where  $\deg \beta = 2$ . Taking  $K/p$  (i.e. mod  $p$ ), we get a spectrum with homotopy groups  $\mathbb{F}_p[\beta^{\pm 1}]$  thus by the above it is a module over  $K(p, 1)$ . Note that  $\deg v_1 = 2(p - 1)$  while  $\deg \beta = 2$ , thus  $K/p$  is a direct sum of  $p - 1$  copies of  $K(p, 1)$ .

## 2.2 Localization

### 2.2.1 $p$ -localization of an Abelian Group

**Definition 12.** An abelian group  $C$  is called  *$p$ -acyclic*, if it has only  $q$ -torsion for  $q \neq p$ , equivalently  $\mathbb{Z}_{(p)} \otimes C = 0$ . An abelian group  $B$  is called  *$p$ -local*, if all other primes are invertible (i.e. that the map  $a \mapsto qa$  is an isomorphism for  $q \neq p$ ), equivalently  $\mathbb{Z}_{(p)} \otimes B = B$ , equivalently  $\text{hom}(C, B) = 0$  for all  $p$ -acyclic  $C$ . The  $p$ -local groups form a full subcategory  $\text{Ab}_{(p)} \subset \text{Ab}$ .

**Definition 13.** Let  $A$  be an abelian group, its  *$p$ -localization* is a  $p$ -local abelian group together with a map  $\varphi : A \rightarrow A_{(p)}$  that is universal. I.e. s.t. for each map to a  $p$ -local group  $f : A \rightarrow B$ , there exists a unique  $\tilde{f} : A_{(p)} \rightarrow B$  s.t.  $f = \tilde{f}\varphi$ . In other word, the  $p$ -localization is the left adjoint to the inclusion  $\text{Ab}_{(p)} \subset \text{Ab}$  (and the map is  $\text{id} \in \text{hom}(A_{(p)}, A_{(p)}) \cong \text{hom}(A, A_{(p)})$ .)

**Example.** Given the abelian group  $\mathbb{Z}$  we have  $\mathbb{Z}_{(p)}$ .

### 2.2.2 $p$ -localization of a Spectrum

Analogously and using the case of abelian groups.

**Definition 14.** A spectrum  $Y$  is called  $p$ -local, if  $\pi_*(Y)$  is a  $p$ -local abelian group. The  $p$ -local spectra form a full subcategory  $\mathrm{Sp}_{(p)} \subset \mathrm{Sp}$ .

**Definition 15.** Let  $X$  be a spectrum, its  $p$ -localization is a  $p$ -local spectrum together with a map  $\varphi : X \rightarrow X_{(p)}$  that is universal. I.e. s.t. for each map to a  $p$ -local spectrum  $f : X \rightarrow Y$ , there exists a map  $\tilde{f} : X_{(p)} \rightarrow Y$ , unique up to homotopy, s.t.  $f = \tilde{f}\varphi$ . In other word, the  $p$ -localization is the left adjoint to the inclusion  $\mathrm{Sp}_{(p)} \subset \mathrm{Sp}$  (and the map is  $\mathrm{id} \in \mathrm{Map}(X_{(p)}, X_{(p)}) \cong \mathrm{Map}(X, X_{(p)})$ .)

**Example.** Given the spectrum  $\mathbb{S}$  we have  $\mathbb{S}_{(p)}$ , the  $p$ -local sphere.

*Remark.* This discussion carries word-for-word for finite spectra to give  $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$ .

### 2.2.3 $E$ -localization of a Spectrum

**Definition 16.** A spectrum  $Z$  is called  $E$ -acyclic, if  $E_*(Z) = \pi_*(E \otimes Z) = 0$ , equivalently  $E \otimes Z \simeq 0$ . A spectrum  $Y$  is called  $E$ -local, if  $[Z, Y]_* = 0$ , equivalently  $\mathrm{Map}(Z, Y) \simeq 0$  for all  $p$ -acyclic  $Z$ . The  $E$ -local spectra form a full subcategory  $\mathrm{Sp}_E \subset \mathrm{Sp}$ .

**Definition 17.** Let  $X$  be a spectrum, its  $E$ -localization is a  $E$ -local spectrum together with a map  $\varphi : X \rightarrow L_EX$  that is universal. I.e. s.t. for each map to a  $E$ -local spectrum  $f : X \rightarrow Y$ , there exists a unique  $\tilde{f} : L_EX \rightarrow Y$  s.t.  $f = \tilde{f}\varphi$ . In other word, the  $E$ -localization is the left adjoint to the inclusion  $\mathrm{Sp}_E \subset \mathrm{Sp}$  (and the map is  $\mathrm{id} \in \mathrm{Map}(L_EX, L_EX) \cong \mathrm{Map}(X, L_EX)$ .)

## 2.3 The Thick Subcategory Theorem and $\mathrm{Spc}(\mathrm{Sp}_{(p)}^{\mathrm{fin}})$

Many of the results below can be found at [6, lecture 26]. The Balmer spectrum can be found at [2, corollary 9.5].

**Proposition 18.** Let  $\mathcal{T}_E = \ker E_* = \left\{ X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid E_*(X) = 0 \right\}$  (equivalently  $X \otimes E \simeq 0$ ) i.e. the  $E$ -acyclics, then  $\mathcal{T}_E$  is thick.

*Proof.* Let be a cofiber sequence  $X' \rightarrow X \rightarrow X''$ , then we get a LES in  $E_*$  homology, in which every space is wrapped by the two others, therefore if two are 0, then so is the third:

$$\cdots \rightarrow E_{m-1}(X'') \rightarrow E_m(X') \rightarrow E_m(X) \rightarrow E_m(X'') \rightarrow E_{m+1}(X') \rightarrow \cdots$$

For a retract  $i : X \rightarrow Y, r : Y \rightarrow X, ri = \mathrm{id}_X$ , we get  $E_m(X) \rightarrow E_m(Y) \rightarrow E_m(X)$ , where the middle is 0, and the composition is identity, thus  $E_m(X) = 0$ .  $\square$

This leads us to the following definition.

**Definition 19.** We define  $\mathcal{C}_{\geq n} = \mathcal{T}_{K(p, n-1)}$ , the  $K(p, n-1)$ -acyclics (equivalently  $A \otimes K(p, n-1) \simeq 0$ .) By the above it is thick. Also,  $\mathcal{C}_{\geq 0} = \mathrm{Sp}_{(p)}^{\mathrm{fin}}$  and  $\mathcal{C}_{\geq \infty} = \{0\}$ , which are trivially thick.

**Proposition 20.** *If  $K(p, n)_*(X) = 0$  then  $K(p, n-1)_*(X) = 0$ .*

**Definition 21.** We say that a spectrum is of *type  $n$*  (possibly  $\infty$ .) if the first non-zero Morava K-Theory is  $K(p, n)$ .

**Corollary.**  $\mathcal{C}_{\geq n}$  is the full subcategory of finite  $p$ -local spectra of type  $\geq n$ . Thus clearly  $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$ .

**Fact.** The inclusion is proper  $\mathcal{C}_{\geq n+1} \subsetneq \mathcal{C}_{\geq n}$ .

*Remark.*  $X \simeq 0$  iff  $H_*(X; \mathbb{Z}) = 0$  iff  $H_*(X; \mathbb{F}_p) = 0$ . Assume that  $X$  is not contractible, then  $H_*(X; \mathbb{F}_p)$  is bounded (since  $X$  is a finite spectrum,) thus for large enough  $n$ , by AHSS we have  $K(p, n)_*(X) \cong H_*(X; \mathbb{F}_p)[v_n^{\pm 1}]$ , i.e.  $X$  has finite type. We conclude that  $\bigcap_{n < \infty} \mathcal{C}_n = \{0\} = \mathcal{C}_{\geq \infty}$ .

**Theorem 22** (Thick Subcategory Theorem [4]). *If  $\mathcal{T}$  is a thick subcategory of  $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$ , then  $\mathcal{T} = \mathcal{C}_{\geq n}$  for some  $n = 0, 1, 2, \dots, \infty$ .*

*Remark.* The proof relies on a major theorem called the Nilpotence Theorem.

**Proposition 23.**  $\mathcal{C}_{\geq n}$  is a prime ideal (note that  $\mathcal{C}_{\geq 0}$  is not a proper subcategory, thus only for  $n = 1, 2, \dots, \infty$ .)

*Proof.* For  $X, Y$  by Kunneth we have  $K(p, n-1)_*(X \otimes Y) = K(p, n-1)_*(X) \otimes K(p, n-1)_*(Y)$ . Therefore, if  $X \in \mathcal{C}_{\geq n}$ , i.e. the homology vanishes, then so does the homology of  $X \otimes Y$ , i.e.  $X \otimes Y \in \mathcal{C}_{\geq n}$ , so  $\mathcal{C}_{\geq n}$  is an ideal. If  $X \otimes Y \in \mathcal{C}_{\geq n}$  then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces,) so  $\mathcal{C}_{\geq n}$  is a prime ideal.  $\square$

**Corollary 24.**  $\mathrm{Spc}\left(\mathrm{Sp}_{(p)}^{\mathrm{fin}}\right) = \{\mathcal{C}_{\geq 1}, \mathcal{C}_{\geq 2}, \dots, \mathcal{C}_{\geq \infty}\}$ , and the closed subsets are  $\{\mathcal{C}_{\geq k}, \mathcal{C}_{\geq k+1}, \dots, \mathcal{C}_{\geq \infty}\}$ .

*Remark.* The chromatic picture can be described for all  $\mathrm{Sp}^{\mathrm{fin}}$  at once, which has all the primes above for each  $p$  with the above closed sets, except that all  $\mathcal{C}_{\geq 1}$  for different  $p$  are the same ( $H\mathbb{Q}$ -acyclics.)

## 2.4 Morava E-Theory

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## 2.5 Further Consequences

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???????????????? Smash product theorem (Lurie 23, and 30, 31) ??????????????

???????????????? Arithmetic Square ????????????

### 2.5.1 The Chromatic Convergence Theorem

Can be found at [6, lecture 32].

**Definition 25.** For each  $n$  we have a map  $L_{E(p,n+1)}X \rightarrow L_{E(p,n)}X$ , thus we can form *the chromatic tower*  $\dots \rightarrow L_{E(p,2)}X \rightarrow L_{E(p,1)}X \rightarrow L_{E(p,0)}X$ .

**Theorem 26** (Chromatic Convergence Theorem). *The limit of the chromatic tower is  $X$ .*

### 2.5.2 The Chromatic Square

Can be found at [6, lecture 23].

**Theorem 27** (Chromatic Square). *There is a pullback diagram:*

$$\begin{array}{ccc} L_{E(p,n)}X & \longrightarrow & L_{K(p,n)}X \\ \downarrow & & \downarrow \\ L_{E(p,n-1)}X & \longrightarrow & L_{E(p,n-1)}L_{K(p,n)}X \end{array}$$

*Remark 28.* The chromatic square resembles the *arithmetic square*, which allows gluing data from different primes to get the original space. However, the pieces are not exactly the  $p$ -localizations, so we don't present it here. These theorems go under the name fracture theorems.

## References

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