Ambidexterity Seminar – The Chromatic Picture

November 23, 2017

1 A Quick Reminder to the Category of Spectra

?????????????? Do we need the following ??????????????

Definition 1. A *prespectrum* $E \in \mathrm{Sp}$ is series of CW-spaces $E_n \in \mathcal{S}$ together with structure maps $\Sigma E_n \to E_{n+1}$. A *map of degree* r between two spectra is $f_n : E_n \to F_{n-r}$, s.t. the structure maps commute with it. These form a category, called the *category of spectra*, denoted by Sp. We will work mainly with *finite spectra*, in which each E_n is a finite CW-space, which are exactly the compact objects, and we denote their full subcategory by $\mathrm{Sp}^{\mathrm{fin}}$.

Example. Given a space $X \in \mathcal{S}$, we can define the *suspension spectrum*, by $E_n = \Sigma^n X$, and $\Sigma E_n \to E_{n+1}$ the identity. In particular, we have the *sphere spectrum* $\mathbb{S}$. ???????????? Is this an embedding ???????????

?????????????? $[E, F]$, homotopy groups, cohomolgy, $K(A, n)$, HA, symmetric monoidal ???????????????

2 Motivation – Hopkins-Neeman and Balmer's Spectrum

Two short introductions to the topic are [5, 7] (note that they use the language of triangular categories, rather than ∞-categories.) In what follows, R is noetherian ring, $X = \mathrm{Spec}(R)$, and $\mathrm{Ch}(X)$ is the symmetric monoidal stable ∞-category of chain complexes over R.

Problem. Can we recover X from $\mathrm{Ch}(X)$?

The first partial answer to this question is given at [3, 8], which we state now.

Definition 2. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finite projective modules. These are the compact objects in the category, so that they can actually be defined categorically. Denote by $\mathrm{Ch}_{\mathrm{perf}}(X)$ the full subcategory of perfect complexes.

Definition 3. Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

1. $0 \in \mathcal{T}$
2. let $a \xrightarrow{f} b \to c$ cofiber sequence, if two out of $\{a, b, c\}$ are in $\mathcal{T}$, then so is the third (remember that cofiber and fiber sequences are the same)
3. it is closed under retracts

Example 4. Take $R = \mathbb{Z}$, thus $\mathrm{Ch}\,(X)$ are chain complexes of abelian groups, and $\mathrm{Ch}_{\mathrm{perf}}\,(X)$ are chain complexes with finitely-many non-zero entries, each of which is $\mathbb{Z}$ to some power. Let $K_\bullet \in \mathrm{Ch}\,(X)$, and define $\mathcal{T}_{K_\bullet} = \{A_\bullet \mid A_\bullet \otimes K_\bullet = 0\}$. Clearly $0 \in \mathcal{T}_{K_\bullet}$, in a pushout where 3 are 0 the fourth is 0, and if $A_\bullet \to B_\bullet \to A_\bullet$ is the identity and $B_\bullet \otimes K_\bullet = 0$ then $A_\bullet \otimes K_\bullet \to 0 \to A_\bullet \otimes K_\bullet$ is the identity thus 0. Therefore $\mathcal{T}_{K_\bullet}$ is thick.

Definition 5. A subset $V \subseteq X$ is called *specialization closed* if it is a union of closed sets. Equivalently, if $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in V$, then $\mathfrak{q} \in V$.

Theorem 6 (Hopkins–Neeman)**.** *There is an inclusion-preserving bijection of sets*

$$\{\textit{Thick subcategories of } \mathrm{Ch}_{\mathrm{perf}}\,(X)\} \rightleftarrows \{\textit{Specialization closed subsets of } X\}$$

Remark. They actually give an explicit way to define the functions, but we omit it for the sake of brevity.

Remark. The theorem was improved in [9] to any quasi-compact quasi-separated scheme X, and compact objects in its derived category.

Later on, in [1, 2] the result is improved further.

Definition 7. A thick subcategory $\mathcal{T}$ is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring, $\mathrm{Spc}\,(\mathcal{C}) = \{\mathcal{P} \text{ prime ideal}\}$, and for any family of objects $S \subseteq \mathcal{C}$ we define $V\,(S) = \{\mathcal{P} \in \mathrm{Spc}\,(\mathcal{C}) \mid S \cap \mathcal{P} = \emptyset\}$, and these are the closed subsets of the *Zariski topology* on $\mathrm{Spc}\,(\mathcal{C})$. We also denote $\mathrm{supp}\,(a) = V\,(\{a\})$.

Theorem 8 (Balmer)**.** *There is a homeomorphism* $X \to \mathrm{Spc}\,(\mathrm{Ch}_{\mathrm{perf}}\,(X))$*,* $\mathfrak{p} \mapsto \mathcal{P} = \left\{M_\bullet \mid (M_\bullet)_\mathfrak{p} = 0\right\}$.

Remark. This was actually upgraded to an isomorphism of locally-ringed spaces.

Example 9. Continuing the case $R = \mathbb{Z}$. We've seen that $\mathcal{T}_{K_\bullet}$ is thick. Note that it is also an ideal, since $A_\bullet \otimes B_\bullet \otimes K_\bullet = A_\bullet \otimes 0 = 0$. Note that $A_\bullet \in \mathcal{T}_{\mathbb{Z}_{(p)}}$ iff it is only q-torsion for $q \neq p$, and we can prove that it is indeed a prime ideal, and similarly for $\mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_{\mathbb{Q}}$. Clearly, if $A_\bullet \in \mathcal{T}_{\mathbb{Z}_{(p)}}$ then $A_\bullet \in \mathcal{T}_{\mathbb{Q}}$, thus any S that doesn't intersect $\mathcal{T}_{\mathbb{Q}}$ doesn't intersect any $\mathcal{T}_{\mathbb{Z}_{(p)}}$, so a closed set that contains $\mathcal{T}_{\mathbb{Q}}$ includes all the others. Indeed by theorem we have $p\mathbb{Z} \mapsto \left\{A_\bullet \mid (A_\bullet)_{p\mathbb{Z}} = 0\right\} = \left\{A_\bullet \mid A_\bullet \otimes \mathbb{Z}_{(p)} = 0\right\} = \mathcal{T}_{\mathbb{Z}_{(p)}}$, and similarly $\mathbb{Z} \mapsto \mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_{\mathbb{Q}}$. Therefore $\mathrm{Spc}\left(\mathrm{Ch}_{\mathrm{perf}}(X)\right) = \left\{\mathcal{T}_{\mathbb{Z}_{(2)}}, \mathcal{T}_{\mathbb{Z}_{(3)}}, \ldots, \mathcal{T}_{\mathbb{Q}}\right\}$. Note that the support of an element is all the prime ideals to which it *does not* belong, e.g. $\mathcal{T}_{\mathbb{Z}_{(q)}} \in \mathrm{supp}\left(\mathbb{F}_p\right)$ iff $\mathbb{F}_p \otimes \mathbb{Z}_{(q)} \neq 0$ which is only when $q = p$.

3 The Chromatic Picture

We concentrate at a single prime p. Although the category of spectra doesn't arise as the corresponding category for a scheme or a similar gadget, we can still try to "reconstruct the space X" by applying this mechanism. We remind ourselves that the compact objects are finite spectra, and therefore we want to know what is $\mathrm{Spc}\left(\mathrm{Sp}^{\mathrm{fin}}_{(p)}\right)$.

???????????????? Morava E-Theory. There is a claim that $E(p,n)_*(X) = 0$ iff $K(p,m)_*(X) = 0$ for $0 \leq m \leq n$ but this is equiv to $K(p,n)_*(X) = 0$?????????????

???????????????? Chromatic convergence theorem (Lurie 32) ??????????????

???????????????? Smash product theorem (Lurie 23, and 30, 31) ?????????????

3.1 Morava K-Theory

A good reference for this part is [6, lectures 22, 24]

Definition 10. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenus is invertible, equivalently it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree. An (A_∞-)ring spectrum E is a *field* if $\pi_* E$ is a field.

Proposition 11. *A field E has Kunneth, i.e. $E_*(X \otimes Y) \cong E_*(X) \otimes_{\pi_* E} E_*(Y)$.*

Fact 12. *For each prime p and $n = 1, 2, \ldots,$ there exists a spectrum called* Morava K-Theory *of height* n*, denoted by $K(p,n)$, which has the following properties:*

- $\pi_* K(p,n) \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$ *where* $\deg v_n = 2\left(p^n - 1\right)$.
- *It is a field (and in particular, (A_∞-)ring spectrum.)*
- *If E is a field, then it has the structure of a $K(p,n)$-module for some p and n. In that sense it is uniquely determined.*

????????????? Is this a good characterization ???????

We also take $K(p,0) = H\mathbb{Q}$.

Example. ??????????? Something about $K(p,1)$?????????????

3.2 Localization

3.2.1 p-localization of an Abelian Group

Definition 13. An abelian group C is called *p-acyclic*, if it has only q-torsion for $q \neq p$, equivalently $\mathbb{Z}_{(p)} \otimes C = 0$. An abelian group B is called *p-local*, if all other primes are invertible (i.e. that the map $a \mapsto qa$ is an isomorphism for $q \neq p$,) equivalently $\mathbb{Z}_{(p)} \otimes B = B$, equivalently $\hom(C, B) = 0$ for all p-acyclic C. The p-local groups form a full subcategory $\mathrm{Ab}_{(p)} \subset \mathrm{Ab}$.

Definition 14. Let A be an abelian group, its *p-localization* is a p-local abelian group together with a map $\varphi : A \to A_{(p)}$ that is universal. I.e. s.t. for each map to a p-local group $f : A \to B$, there exists a unique $\tilde{f} : A_{(p)} \to B$ s.t. $f = \tilde{f}\varphi$. In other word, the p-localization is the left adjoint to the inclusion $\mathrm{Ab}_{(p)} \subset \mathrm{Ab}$ (and the map is $\mathrm{id} \in \hom\left(A_{(p)}, A_{(p)}\right) \cong \hom\left(A, A_{(p)}\right)$.)

Example. Given the abelian group $\mathbb{Z}$ we have $\mathbb{Z}_{(p)}$.

3.2.2 p-localization of a Spectrum

Analogously and using the case of abelian groups.

Definition 15. A spectrum Y is called *p-local*, if $\pi_*(Y)$ is a p-local abelian group. The p-local spectra form a full subcategory $\mathrm{Sp}_{(p)} \subset \mathrm{Sp}$.

Definition 16. Let X be a spectrum, its *p-localization* is a p-local spectrum together with a map $\varphi : X \to X_{(p)}$ that is universal. I.e. s.t. for each map to a p-local spectrum $f : X \to Y$, there exists a map $\tilde{f} : X_{(p)} \to Y$, unique up to homotopy, s.t. $f = \tilde{f}\varphi$. In other word, the p-localization is the left adjoint to the inclusion $\mathrm{Sp}_{(p)} \subset \mathrm{Sp}$ (and the map is $\mathrm{id} \in \mathrm{Map}\left(X_{(p)}, X_{(p)}\right) \cong \mathrm{Map}\left(X, X_{(p)}\right)$.)

Example. Given the spectrum $\mathbb{S}$ we have $\mathbb{S}_{(p)}$, the p-local sphere.

Remark. This discussion carries word-for-word for finite spectra to give $\mathrm{Sp}^{\mathrm{fin}}_{(p)}$.

3.2.3 E-localization of a Spectrum

Definition 17. A spectrum Z is called *E-acyclic*, if $E_*(Z) = \pi_*(E \otimes Z) = 0$, equivalently $E \otimes Z \simeq 0$. A spectrum Y is called *E-local*, if $[Z, Y]_* = 0$, equivalently $\mathrm{Map}(Z, Y) \simeq 0$ for all p-acyclic Z. The E-local spectra form a full subcategory $\mathrm{Sp}_E \subset \mathrm{Sp}$.

Definition 18. Let X be a a spectrum, its *E-localization* is a E-local spectrum together with a map $\varphi : X \to L_E X$ that is universal. I.e. s.t. for each map to a E-local spectrum $f : X \to Y$, there exists a unique $\tilde{f} : L_E X \to Y$ s.t. $f = \tilde{f}\varphi$. In other word, the E-localization is the left adjoint to the inclusion $\mathrm{Sp}_E \subset \mathrm{Sp}$ (and the map is $\mathrm{id} \in \mathrm{Map}(L_E X, L_E X) \cong \mathrm{Map}(X, L_E X)$.)

3.3 The Thick Subcategory Theorem and $\mathrm{Spc}\left(\mathrm{Sp}^{\mathrm{fin}}_{(p)}\right)$

Many of the results below can be found at [6, lecture 26]. The Balmer spectrum can be found at [2, corollary 9.5].

Proposition 19. *Let $\mathcal{T}_E = \ker E_* = \left\{A \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid E_*(A) = 0\right\}$ (equivalently $A \otimes E \simeq 0$) i.e. the E-acyclics, then $\mathcal{T}_E$ is thick.*

Proof. Let be a cofiber sequence $X' \to X \to X''$, then we get a LES in E_* homology, in which every space is wrapped by the two others, therefore if two are 0, then so is the third:

$$\cdots \to E_{m-1}(X'') \to E_m(X') \to E_m(X) \to E_m(X'') \to E_{m+1}(X') \to \cdots$$

For a retract $i : X \to Y, r : Y \to X, ri = \mathrm{id}_X$, we get $E_m(X) \to E_m(Y) \to E_m(X)$, where the middle is 0, and the composition is identity, thus $E_m(X) = 0$. □

This leads us to the following definition.

Definition 20. We define $\mathcal{C}_{\geq n} = \mathcal{T}_{K(p,n-1)}$, the $K(p, n-1)$-acyclics (equivalently $A \otimes K(p, n-1) \simeq 0$.) By the above it is thick. Also, $\mathcal{C}_{\geq 0} = \mathrm{Sp}^{\mathrm{fin}}_{(p)}$ and $\mathcal{C}_{\geq\infty} = \{0\}$, which are trivially thick.

Proposition 21. *If $K(p,n)_*(X) = 0$ then $K(p, n-1)_*(X) = 0$.*

Definition 22. We say that a spectrum is of *type n* (possibly ∞,) if the first non-zero Morava K-Theory is $K(p, n)$.

Corollary. *$\mathcal{C}_{\geq n}$ is the full subcategory of finite p-local spectra of type $\geq n$. Thus clearly $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$.*

Fact. *The inclusion is proper $\mathcal{C}_{\geq n+1} \subsetneq \mathcal{C}_{\geq n}$.*

Remark. $X \simeq 0$ iff $H_*(X; \mathbb{Z}) = 0$ iff $H_*(X; \mathbb{F}_p) = 0$. Assume that X is not contractible, then $H_*(X; \mathbb{F}_p)$ is bounded, thus for large enough n, by AHSS we have $K(n)_*(X) \cong H_*(X; \mathbb{F}_p)\left[v_n^{\pm 1}\right]$, i.e. X has finite type, thus $\mathcal{C}_{\geq\infty} = \{0\}$.

Theorem 23 (Thick Subcategory Theorem [4])**.** *If $\mathcal{T}$ is a thick subcategory of $\mathrm{Sp}^{\mathrm{fin}}_{(p)}$, then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $n = 0, 1, 2, \ldots, \infty$.*

Remark. The proof relies on a major theorem called the Nilpotence Theorem.

Proposition 24. *$\mathcal{C}_{\geq n}$ is a prime ideal (note that $\mathcal{C}_{\geq 0}$ is not a proper subcategory, thus only for $n = 1, 2, \ldots, \infty$.)*

Proof. For X, Y by Kunneth we have $K(n-1)_*(X \otimes Y) = K(n-1)_*(X) \otimes K(n-1)_*(Y)$. Therefore, if $X \in \mathcal{C}_{\geq n}$, i.e. the homology vanishes, then so does the homology of $X \otimes Y$, i.e. $X \otimes Y \in \mathcal{C}_{\geq n}$, so $\mathcal{C}_{\geq n}$ is an ideal. If $X \otimes Y \in \mathcal{C}_{\geq n}$ then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces,) so $\mathcal{C}_{\geq n}$ is a prime ideal. □

Corollary 25. $\mathrm{Spc}\left(\mathrm{Sp}^{\mathrm{fin}}_{(p)}\right) = \{\mathcal{C}_{\geq 1}, \mathcal{C}_{\geq 2}, \ldots, \mathcal{C}_{\geq \infty}\}$*, and the closed subsets are* $\{\mathcal{C}_{\geq k}, \mathcal{C}_{\geq k+1}, \ldots, \mathcal{C}_{\geq \infty}\}$.

???????????????????? Should I say something about the global picture, and mention $\mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}}$???????????????????

References

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