Ambidexterity Seminar – The Chromatic Picture

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1 Motivation – Hopkins-Neeman and Balmer's Spectrum

Two short introductions to the topic are [5, 7] (note that they use the language of triangular categories, rather than ∞-categories.) In what follows, R is noetherian ring, $X = \operatorname{Spec}(R)$, and $\operatorname{Ch}(X)$ is the symmetric monoidal stable ∞-category of chain complexes over R.

Problem. Can we recover X from $\operatorname{Ch}(X)$?

The first partial answer to this question is given at [3, 8], which we state now.

Definition 1. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finite projective modules. These are the compact objects in the category, so that they can actually be defined categorically. Denote by $\operatorname{Ch}_{\mathrm{perf}}(X)$ the full subcategory of perfect complexes.

Definition 2. Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

1. $0 \in \mathcal{T}$
2. let $a \xrightarrow{f} b \to c$ cofiber sequence, if two out of $\{a, b, c\}$ are in $\mathcal{T}$, then so is the third (remember that cofiber and fiber sequences are the same)
3. it is closed under retracts

Example 3. Take $R = \mathbb{Z}$, then $\operatorname{Ch}(X)$ is chain complexes of abelian groups, and $\operatorname{Ch}_{\mathrm{perf}}(X)$ are those with finitely-many non-zero entries, each of which is $\mathbb{Z}$ to some power. Let $K_\bullet \in \operatorname{Ch}(X)$, and define $\mathcal{T}_{K_\bullet} = \{A_\bullet \in \operatorname{Ch}_{\mathrm{perf}}(X) \mid A_\bullet \otimes K_\bullet = 0\}$. Clearly $0 \in \mathcal{T}_{K_\bullet}$. Since tensor is left, it sends pushout to pushout ????????? not sure ???????, and three are 0 so the fourth is 0. Lastly, if $A_\bullet \to B_\bullet \to A_\bullet$ is the identity and $B_\bullet \otimes K_\bullet = 0$ then $\mathrm{id}_{A_\bullet \otimes K_\bullet}$ factors through 0, thus $A_\bullet \otimes K_\bullet = 0$. Therefore $\mathcal{T}_{K_\bullet}$ is thick.

Definition 4. A subset $V \subseteq X$ is called *specialization closed* if it is a union of closed sets. Equivalently, if $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in V$, then $\mathfrak{q} \in V$.

Theorem 5 (Hopkins–Neeman)**.** *There is an inclusion-preserving bijection of sets*

$$\{\textit{Thick subcategories of } \mathrm{Ch}_{\mathrm{perf}}(X)\} \rightleftarrows \{\textit{Specialization closed subsets of } X\}$$

Remark. They actually give an explicit way to define the functions, but we omit it for the sake of brevity.

Remark. The theorem was improved in [9] to any quasi-compact quasi-separated scheme X, and compact objects in its derived category.

Later on, in [1, 2] the result is improved further.

Definition 6. A thick subcategory $\mathcal{T}$ is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring, $\mathrm{Spc}(\mathcal{C}) = \{\mathcal{P} \text{ prime ideal}\}$, and for any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) \mid S \cap \mathcal{P} = \emptyset\}$, and these are the closed subsets of the *Zariski topology* on $\mathrm{Spc}(\mathcal{C})$. We also denote $\mathrm{supp}(a) = V(\{a\})$.

Theorem 7 (Balmer)**.** *There is a homeomorphism* $X \to \mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X))$, $\mathfrak{p} \mapsto \mathcal{P} = \left\{M_\bullet \mid (M_\bullet)_{\mathfrak{p}} = 0\right\}$.

Remark. This was actually upgraded to an isomorphism of locally-ringed spaces.

Example 8. Continuing the case $R = \mathbb{Z}$. We've seen that $\mathcal{T}_{K_\bullet}$ is thick. Note that it is also an ideal, since $A_\bullet \otimes B_\bullet \otimes K_\bullet = A_\bullet \otimes 0 = 0$. Note that $A_\bullet \in \mathcal{T}_{\mathbb{Z}_{(p)}}$ iff it is only q-torsion for $q \neq p$, and we can prove that it is indeed a prime ideal, and similarly for $\mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_{\mathbb{Q}}$. Clearly, if $A_\bullet \in \mathcal{T}_{\mathbb{Z}_{(p)}}$ then $A_\bullet \in \mathcal{T}_{\mathbb{Q}}$, thus any S that doesn't intersect $\mathcal{T}_{\mathbb{Q}}$ doesn't intersect any $\mathcal{T}_{\mathbb{Z}_{(p)}}$, so a closed set that contains $\mathcal{T}_{\mathbb{Q}}$ includes all the others. Indeed by theorem we have $p\mathbb{Z} \mapsto \left\{A_\bullet \mid (A_\bullet)_{p\mathbb{Z}} = 0\right\} = \left\{A_\bullet \mid A_\bullet \otimes \mathbb{Z}_{(p)} = 0\right\} = \mathcal{T}_{\mathbb{Z}_{(p)}}$, and similarly $\mathbb{Z} \mapsto \mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_{\mathbb{Q}}$. Therefore $\mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X)) = \left\{\mathcal{T}_{\mathbb{Z}_{(2)}}, \mathcal{T}_{\mathbb{Z}_{(3)}}, \ldots, \mathcal{T}_{\mathbb{Q}}\right\}$. Note that the support of an element is all the prime ideals to which it *does not* belong, e.g. $\mathcal{T}_{\mathbb{Z}_{(q)}} \in \mathrm{supp}(\mathbb{F}_p)$ iff $\mathbb{F}_p \otimes \mathbb{Z}_{(q)} \neq 0$ which is only when $q = p$.

2 The Chromatic Picture

Although the category of spectra doesn't arise as the corresponding category for a scheme or a similar gadget, we can still try to "reconstruct the space X" by applying this mechanism, and then try to use this decomposition.

We wull concentrate at the p-local spectra, $\mathrm{Sp}_{(p)}$, for some prime, which is a mild operation. All the statements can actually be states at the level of all spectra, but it is easier to state them at $\mathrm{Sp}_{(p)}$. We remind ourselves that the compact objects are finite spectra.

2.1 Morava K-Theory

A good reference for this part is [6, lectures 22, 24]

Definition 9. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenus is invertible, equivalently it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree. An (A_∞-)ring spectrum E is a *field* if $\pi_* E$ is a field.

Proposition 10. *A field E has Kunneth, i.e. $E_*(X \otimes Y) \cong E_*(X) \otimes_{\pi_* E} E_*(Y)$ for any spectra X, Y.*

Fact 11. *For each prime p and $n = 1, 2, \ldots$, there exists a spectrum called* Morava K-Theory of height n*, denoted by $K(p, n)$, which has the following properties:*

- $\pi_* K(p, n) \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$ *where* $\deg v_n = 2(p^n - 1)$.
- *It is a field (and in particular, (A_∞-)ring spectrum.)*
- *If E is a field, then it has the structure of a $K(p, n)$-module for unique p and n. In that sense $K(p, n)$ is uniquely determined.*

We also take $K(p, 0) = H\mathbb{Q}$.

Example. Remember that K (regular complex K-theory) has $\pi_*(K) = \mathbb{Z}\left[\beta^{\pm 1}\right]$ where $\deg \beta = 2$. Taking K/p we get a spectrum with homotopy groups $\mathbb{F}_p\left[\beta^{\pm 1}\right]$. thus by the above it is a module over $K(p, 1)$. Note that $\deg v_1 = 2(p-1)$ while $\deg \beta = 2$, thus K/p is a direct sum of $p - 1$ copies of $K(p, 1)$.

2.2 Localization at E

A reference for what follows is at [6, lecture 20]. Let E be a spectrum.

Definition 12. A spectrum Z is called E*-acyclic*, if $E_*(Z) = \pi_*(E \otimes Z) = 0$, equivalently $E \otimes Z \simeq 0$. A spectrum Y is called E*-local*, if $[Z, Y]_* = 0$, equivalently $\mathrm{Map}(Z, Y) \simeq 0$ for all p-acyclic Z. The E-local spectra form a full subcategory $\mathrm{Sp}_E \subset \mathrm{Sp}$.

Definition 13. Let X be a a spectrum, its E*-localization* is an E-local spectrum together with a map $\varphi : X \to L_E X$ that is universal. I.e. s.t. for each map to a E-local spectrum $f : X \to Y$, there exists a unique $\tilde{f} : L_E X \to Y$ s.t. $f = \tilde{f}\varphi$. In other word, the E-localization is the left adjoint to the inclusion $\mathrm{Sp}_E \subset \mathrm{Sp}$ (and the map is $\mathrm{id} \in \mathrm{Map}(L_E X, L_E X) \cong \mathrm{Map}(X, L_E X)$.)

Remark 14. The name localization might be confusing. We will use this mechanism for $K(p, n)$ which should be though of as a field. In analogy, the $\mathbb{F}_p$-localization of $\mathbb{Z}$ is $\mathbb{Z}_p$, i.e. the completion, not the localization (note that we actually want to work in complexes, but this is the result we would get after interpreting abelian groups $\mathbb{Z}\langle R \rangle \to \mathbb{Z}\langle S \rangle$.)

2.3 The Thick Subcategory Theorem and $\mathrm{Spc}\left(\mathrm{Sp}^{\mathrm{fin}}_{(p)}\right)$

Many of the results below can be found at [6, lecture 26]. The Balmer spectrum can be found at [2, corollary 9.5].

Proposition 15. *Let $\mathcal{T}_E = \ker E_* = \left\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid E_*(X) = 0\right\}$ (equivalently $X \otimes E \simeq 0$) i.e. the E-acyclics, then $\mathcal{T}_E$ is thick.*

Proof. Clearly $0 \in \mathcal{T}_E$. Let be a cofiber sequence $X \to Y \to Z$, then we get a LES in E_* homology, in which every space is wrapped by the two others, therefore if two are 0, then so is the third ??????????????????? can we use the same argument from complexes ????????????????:

$$\cdots \to E_{m-1}(Z) \to E_m(X) \to E_m(Y) \to E_m(Z) \to E_{m+1}(X) \to \cdots$$

For a retract $X \to Y \to Y$, we get $E_m(X) \to E_m(Y) \to E_m(X)$, where the middle is 0, and the composition is identity, thus $E_m(X) = 0$. ☐

This leads us to the following definition.

Definition 16. We define $\mathcal{C}_{\geq n} = \mathcal{T}_{K(p,n-1)}$, the $K(p, n-1)$-acyclics (equivalently $X \otimes K(p, n-1) \simeq 0$.) By the above it is thick. Also, $\mathcal{C}_{\geq 0} = \mathrm{Sp}^{\mathrm{fin}}_{(p)}$ and $\mathcal{C}_{\geq \infty} = \{0\}$, which are trivially thick.

Proposition 17. *For $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$, if $K(p,n)_*(X) = 0$ then $K(p, n-1)_*(X) = 0$.*

Remark. This result is not true for any spectrum (e.g. for $H\mathbb{Q}$ whose $K(p, n)$ doesn't vanish at $n = 0$ but does at $n = 1$,) and the fact that it doesn't vanish is important and has to do with Morava E-Theory and the way different levels glue.

Definition 18. We say that a spectrum is of *type* n (possibly ∞,) if the first non-zero Morava K-Theory is $K(p, n)$.

Corollary. *$\mathcal{C}_{\geq n}$ is the full subcategory of finite p-local spectra of type $\geq n$. Thus clearly $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$.*

Fact. *The inclusion is proper $\mathcal{C}_{\geq n+1} \subsetneq \mathcal{C}_{\geq n}$.*

Remark. $X \simeq 0$ iff $H_*(X; \mathbb{Z}) = 0$ iff $H_*(X; \mathbb{F}_p) = 0$. Assume that X is not contractible, then $H_*(X; \mathbb{F}_p)$ is bounded (since X is a finite spectrum,) thus for large enough n, by AHSS we have $K(p,n)_*(X) \cong H_*(X; \mathbb{F}_p)\left[v_n^{\pm 1}\right]$, i.e. X has finite type. We conclude that $\cap_{n<\infty} \mathcal{C}_n = \{0\} = \mathcal{C}_{\geq \infty}$.

Theorem 19 (Thick Subcategory Theorem [4])**.** *If $\mathcal{T}$ is a thick subcategory of $\mathrm{Sp}^{\mathrm{fin}}_{(p)}$, then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $n = 0, 1, 2, \ldots, \infty$.*

Remark. The proof relies on a major theorem called the Nilpotence Theorem.

Proposition 20. $\mathcal{C}_{\geq n}$ *is a prime ideal (note that* $\mathcal{C}_{\geq 0}$ *is not a proper subcategory, thus only for* $n = 1, 2, \ldots, \infty$*.)*

Proof. For X, Y by Kunneth we have $K(p, n-1)_*(X \otimes Y) = K(p, n-1)_*(X) \otimes K(p, n-1)_*(Y)$. Therefore, if $X \in \mathcal{C}_{\geq n}$, i.e. the homology vanishes, then so does the homology of $X \otimes Y$, i.e. $X \otimes Y \in \mathcal{C}_{\geq n}$, so $\mathcal{C}_{\geq n}$ is an ideal. If $X \otimes Y \in \mathcal{C}_{\geq n}$ then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces,) so $\mathcal{C}_{\geq n}$ is a prime ideal. $\square$

Corollary 21. $\mathrm{Spc}\left(\mathrm{Sp}^{\mathrm{fin}}_{(p)}\right) = \{\mathcal{C}_{\geq 1}, \mathcal{C}_{\geq 2}, \ldots, \mathcal{C}_{\geq \infty}\}$*, and the closed subsets are* $\{\mathcal{C}_{\geq k}, \mathcal{C}_{\geq k+1}, \ldots, \mathcal{C}_{\geq \infty}\}$.

Remark. The chromatic picture can be described for all $\mathrm{Sp}^{\mathrm{fin}}$ at once, which has all the primes above for each p with the above closed sets, except that all $\mathcal{C}_{\geq 1}$ for different p are the same ($H\mathbb{Q}$-acyclics.)

2.4 Morava E-Theory

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2.5 Further Results

The ideas above lead to the idea of studying spectra height-by-height, and investigate further the connection between Morava E-Theory and Morava K-Theory.

???????????????? Smash product theorem (Lurie 23, and 30, 31) ?????????????

Definition 22. For each n we have a map $L_{E(p,n+1)}X \to L_{E(p,n)}X$, thus we can form *the chromatic tower* $\ldots \to L_{E(p,2)}X \to L_{E(p,1)}X \to L_{E(p,0)}X$.

Theorem 23 (Chromatic Convergence Theorem [6, lecture 32])**.** *The limit of the chromatic tower is* X*.*

Theorem 24 (Chromatic Square [6, lecture 23])**.** *There is a pullback diagram:*

$$\begin{tikzcd} L_{E(p,n)}X \arrow[r] \arrow[d] & L_{K(p,n)}X \arrow[d] \\ L_{E(p,n-1)}X \arrow[r] & L_{E(p,n-1)}L_{K(p,n)}X \end{tikzcd}$$

Remark 25. The chromatic square resembles the *arithmetic square*, which allows gluing data from different primes to get the original spectrum. However, the pieces are not exactly the p-localizations, so we don't present it here. These theorems go under the name fracture theorems.

References

[1] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *arXiv:math/0409360*, 2004.

[2] P. Balmer. Spectra, spectra, spectra – tensor triangular spectra versus zariski spectra of endomorphism rings. *Algebraic and Geometric Topology 10, 1521–1563*, 2010.

[3] M. Hopkins. Global methods in homotopy theory. *Homotopy Theory – Proc. Durham Symp. 1985. Cambridge University Press. Cambridge*, 1987.

[4] M. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory II. *Annals of Mathematics, 148(1), second series, 1-49*, 1998.

[5] S. B. Iyengar. Thick subcategories of perfect complexes over a commutative ring. 2006.

[6] J. Lurie. Chromatic homotopy theory. *252x course notes*, 2010.

[7] T. Murayama. The classification of thick subcategories and balmer's reconstruction theorem. 2015.

[8] A. Neeman. The chromatic tower of $\mathcal{D}(R)$. *Topology 31*, 1992.

[9] R. W. Thomason. The classification of triangulated subcategories. *Compositio Math. 105.1*, 1997.