Ambidexterity Seminar – The Chromatic Picture

Shay Ben Moshe

December 03, 2017

1 Motivation – Hopkins-Neeman and Balmer's Spectrum

Two short introductions to the topic are [5, 7] (note that they use the language of triangular categories, rather than ∞-categories.) In what follows, R is noetherian ring, $X = \mathrm{Spec}(R)$, and $\mathrm{Ch}(X)$ is the symmetric monoidal stable ∞-category of chain complexes over R.

Problem. Can we recover X from $\mathrm{Ch}(X)$?

The first partial answer to this question is given at [3, 8], which we state now.

Definition 1. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finite projective modules. These are the compact objects in the category, so that they can actually be defined categorically. Denote by $\mathrm{Ch}_{\mathrm{perf}}(X)$ the full subcategory of perfect complexes.

Definition 2. Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

1. $0 \in \mathcal{T}$
2. let $a \xrightarrow{f} b \to c$ cofiber sequence, if two out of $\{a, b, c\}$ are in $\mathcal{T}$, then so is the third (remember that cofiber and fiber sequences are the same)
3. it is closed under retracts

Example 3. Take $R = \mathbb{Z}$, then $\mathrm{Ch}(X)$ is chain complexes of abelian groups, and $\mathrm{Ch}_{\mathrm{perf}}(X)$ are those with finitely-many non-zero entries, each of which is $\mathbb{Z}$ to some power. Let $K_\bullet \in \mathrm{Ch}(X)$, and define $\mathcal{T}_{K_\bullet} = \{A_\bullet \in \mathrm{Ch}_{\mathrm{perf}}(X) \mid A_\bullet \otimes K_\bullet = 0\}$. Clearly

$0 \in \mathcal{T}_{K_\bullet}$. Since tensor is left, it sends pushout to pushout ????????? not sure ???????, and three are 0 so the fourth is 0. Lastly, if $A_\bullet \to B_\bullet \to A_\bullet$ is the identity and $B_\bullet \otimes K_\bullet = 0$ then $\mathrm{id}_{A_\bullet \otimes K_\bullet}$ factors through 0, thus $A_\bullet \otimes K_\bullet = 0$. Therefore $\mathcal{T}_{K_\bullet}$ is thick.

Definition 4. A subset $V \subseteq X$ is called *specialization closed* if it is a union of closed sets. Equivalently, if $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in V$, then $\mathfrak{q} \in V$.

Theorem 5 (Hopkins–Neeman). *There is an inclusion preserving bijection of sets*

$$\{\textit{Thick subcategories of } \mathrm{Ch}_{\mathrm{perf}}(X)\} \rightleftarrows \{\textit{Specialization}$$

Remark. They actually give an explicit way to define the functions, but we omit it for the sake of brevity.

Remark. The theorem was improved in [9] to any quasi-compact quasi-separated scheme X, and compact objects in its derived category.

Later on, in [1, 2] the result is improved further.

Definition 6. A thick subcategory $\mathcal{T}$ is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring, $\mathrm{Spc}(\mathcal{C}) = \{\mathcal{P} \text{ prime ideal}\}$, and for any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) \mid S \cap \mathcal{P} = \emptyset\}$, and these are the closed subsets of the *Zariski topology* on $\mathrm{Spc}(\mathcal{C})$. We also denote $\mathrm{supp}(a) = V(\{a\})$.

Theorem 7 (Balmer)**.** *There is a homeomorphism* $X \to \mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X))$, $\mathfrak{p} \mapsto \mathcal{P} = \left\{M_\bullet \mid (M_\bullet)_{\mathfrak{p}} = 0\right\}$.

Remark. This was actually upgraded to an isomorphism of locally-ringed spaces.

Example 8. Continuing the case $R = \mathbb{Z}$. We've seen that $\mathcal{T}_{K_\bullet}$ is thick. Note that it is also an ideal, since $A_\bullet \otimes B_\bullet \otimes K_\bullet = A_\bullet \otimes 0 = 0$. Note that $A_\bullet \in \mathcal{T}_{\mathbb{Z}_{(p)}}$ iff it is only q-torsion for $q \neq p$, and we can prove that it is indeed a prime ideal, and similarly for $\mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_{\mathbb{Q}}$. Clearly, if $A_\bullet \in \mathcal{T}_{\mathbb{Z}_{(p)}}$ then $A_\bullet \in \mathcal{T}_{\mathbb{Q}}$, thus any S that doesn't intersect $\mathcal{T}_{\mathbb{Q}}$ doesn't intersect any $\mathcal{T}_{\mathbb{Z}_{(p)}}$, so a closed set that contains $\mathcal{T}_{\mathbb{Q}}$ includes all the others. Indeed by theorem we have $p\mathbb{Z} \mapsto \left\{A_\bullet \mid (A_\bullet)_{p\mathbb{Z}} = 0\right\} = \left\{A_\bullet \mid A_\bullet \otimes \mathbb{Z}_{(p)} = 0\right\} = \mathcal{T}_{\mathbb{Z}_{(p)}}$, and similarly $\mathbb{Z} \mapsto \mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_{\mathbb{Q}}$. Therefore $\mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X)) = \left\{\mathcal{T}_{\mathbb{Z}_{(2)}}, \mathcal{T}_{\mathbb{Z}_{(3)}}, \ldots, \mathcal{T}_{\mathbb{Q}}\right\}$. Note that the support of an element is all the prime ideals to which it *does not* belong, e.g. $\mathcal{T}_{\mathbb{Z}_{(q)}} \in \operatorname{supp}(\mathbb{F}_p)$ iff $\mathbb{F}_p \otimes \mathbb{Z}_{(q)} \neq 0$ which is only when $q = p$.

2 The Chromatic Picture

Similarly to the analysis in algebra/algebraic geometry, we are going to concentrate at a single prime p. Although the category of spectra doesn't arise as the corresponding category for a scheme or a similar gadget, we can still try to "reconstruct the space X" by applying this mechanism, and then try to use this decomposition. We remind ourselves that the compact objects are finite spectra.

2.1 Morava K-Theory

A good reference for this part is [6, lectures 22, 24]

Definition 9. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenus is invertible, equivalently it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree. An (A_∞-)ring spectrum E is a *field* if $\pi_* E$ is a field.

Proposition 10. *A field E has Kunneth, i.e. $E_*(X \otimes$ $E_*(X) \otimes_{\pi_* E} E_*(Y)$ for any spectra X, Y.*

Fact 11. *For each prime p and $n = 1, 2, \ldots$, there exists a spectrum called* Morava K-Theory of height *n, denoted by $K(p,n)$, which has the following properties:*

- $\pi_* K(p,n) \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$ *where* $\deg v_n = 2(p^n - 1)$.
- *It is a field (and in particular, (A_∞-)ring spectrum.)*
- *If E is a field, then it has the structure of a $K(p,n)$-module for unique p and n. In that sense $K(p,n)$ is uniquely determined.*

We also take $K(p,0) = H\mathbb{Q}$.

Example. Remember that K (regular K-theory) has $\pi_*(K) = \mathbb{Z}\left[\beta^{\pm 1}\right]$ where $\deg \beta = 2$. Taking K/p (i.e. mod p,) we get a spectrum with homotopy groups $\mathbb{F}_p\left[\beta^{\pm 1}\right]$ thus by the above it is a module over $K(p,1)$. Note that $\deg v_1 = 2(p-1)$ while $\deg \beta = 2$, thus K/p is a direct sum of $p-1$ copies of $K(p,1)$.

2.2 Localization

2.2.1 p-localization of an Abelian Group

Definition 12. An abelian group C is called *p-acyclic*, if it has only q-torsion for $q \neq p$, equivalently $\mathbb{Z}_{(p)} \otimes C = 0$. An abelian group B is called *p-local*, if all other primes are invertible (i.e. that the map $a \mapsto qa$ is an isomorphism for $q \neq p$,) equivalently $\mathbb{Z}_{(p)} \otimes B = B$, equivalently $\hom(C, B) = 0$ for all p-acyclic C. The p-local groups form a full subcategory $\mathrm{Ab}_{(p)} \subset \mathrm{Ab}$.

Definition 13. Let A be an abelian group, its *p-localization* is a p-local abelian group together with a map $\varphi : A \to A_{(p)}$ that is universal. I.e. s.t. for each map to a p-local group $f : A \to B$, there exists a unique $\tilde{f} : A_{(p)} \to B$ s.t. $f = \tilde{f}\varphi$. In other word, the p-localization is the left adjoint to the inclusion $\mathrm{Ab}_{(p)} \subset \mathrm{Ab}$ (and the map is $\mathrm{id} \in \hom\left(A_{(p)}, A_{(p)}\right) \cong \hom\left(A, A_{(p)}\right)$.)

Example. Given the abelian group $\mathbb{Z}$ we have $\mathbb{Z}_{(p)}$.

2.2.2 p-localization of a Spectrum

Analogously and using the case of abelian groups.

Definition 14. A spectrum Y is called *p-local*, if $\pi_*(Y)$ is a p-local abelian group. The p-local spectra form a full subcategory $\mathrm{Sp}_{(p)} \subset \mathrm{Sp}$.

Definition 15. Let X be a spectrum, its *p-localization* is a p-local spectrum together with a map $\varphi : X \to X_{(p)}$ that is universal. I.e. s.t. for each map to a p-local spectrum $f : X \to Y$, there exists a map $\tilde{f} : X_{(p)} \to Y$, unique up to homotopy, s.t. $f = \tilde{f}\varphi$. In other word, the p-localization is the left adjoint to the inclusion $\mathrm{Sp}_{(p)} \subset \mathrm{Sp}$ (and the map is $\mathrm{id} \in \mathrm{Map}\left(X_{(p)}, X_{(p)}\right) \cong \mathrm{Map}\left(X, X_{(p)}\right)$.)

Example. Given the spectrum $\mathbb{S}$ we have $\mathbb{S}_{(p)}$, the p-local sphere.

Remark. This discussion carries word-for-word for finite spectra to give $\mathrm{Sp}^{\mathrm{fin}}_{(p)}$.

2.2.3 E-localization of a Spectrum

Definition 16. A spectrum Z is called *E-acyclic*, if $E_*(Z) = \pi_*(E \otimes Z) = 0$, equivalently $E \otimes Z \simeq$

0. A spectrum Y is called *E-local*, if $[Z,Y]_* = 0$, equivalently $\mathrm{Map}(Z,Y) \simeq 0$ for all p-acyclic Z. The E-local spectra form a full subcategory $\mathrm{Sp}_E \subset \mathrm{Sp}$.

Definition 17. Let X be a a spectrum, its *E-localization* is a E-local spectrum together with a map $\varphi : X \to L_E X$ that is universal. I.e. s.t. for each map to a E-local spectrum $f : X \to Y$, there exists a unique $\tilde{f} : L_E X \to Y$ s.t. $f = \tilde{f}\varphi$. In other word, the E-localization is the left adjoint to the inclusion $\mathrm{Sp}_E \subset \mathrm{Sp}$ (and the map is $\mathrm{id} \in \mathrm{Map}(L_E X, L_E X) \cong \mathrm{Map}(X, L_E X)$.)

2.3 The Thick Subcategory Theorem and $\mathrm{Spc}\left(\mathrm{Sp}^{\mathrm{fin}}_{(p)}\right)$

Many of the results below can be found at [6, lecture 26]. The Balmer spectrum can be found at [2, corollary 9.5].

Proposition 18. *Let* $\mathcal{T}_E = \ker E_* = \left\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid E_*\right.$ *(equivalently $X \otimes E \simeq 0$) i.e. the E-acyclics, then $\mathcal{T}_E$ is thick.*

Proof. Clearly $0 \in \mathcal{T}_E$. Let be a cofiber sequence

$X \to Y \to Z$, then we get a LES in E_* homology, in which every space is wrapped by the two others, therefore if two are 0, then so is the third ??????????????????? can we use the same argument from complexes ????????????????:

$$\cdots \to E_{m-1}(Z) \to E_m(X) \to E_m(Y) \to E_m(Z) \to E$$

For a retract $X \to Y \to Y$, we get $E_m(X) \to E_m(Y) \to E_m(X)$, where the middle is 0, and the composition is identity, thus $E_m(X) = 0$. $\square$

This leads us to the following definition.

Definition 19. We define $\mathcal{C}_{\geq n} = \mathcal{T}_{K(p,n-1)}$, the $K(p,n-1)$-acyclics (equivalently $X \otimes K(p,n-1) \simeq 0$.) By the above it is thick. Also, $\mathcal{C}_{\geq 0} = \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ and $\mathcal{C}_{\geq \infty} = \{0\}$, which are trivially thick.

Proposition 20. *For $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$, if $K(p,n)_*(X) = 0$ then $K(p,n-1)_*(X) = 0$.*

Remark. This result is not true for any spectrum (e.g. for $H\mathbb{Q}$ whose $K(p,n)$ doesn't vanish at $n = 0$ but does at $n = 1$,) and the fact that it doesn't vanish is important and has to do with Morava E-Theory and the way different levels glue.

Definition 21. We say that a spectrum is of *type n* (possibly ∞,) if the first non-zero Morava K-Theory is $K(p,n)$.

Corollary. *$\mathcal{C}_{\geq n}$ is the full subcategory of finite p-local spectra of type $\geq n$. Thus clearly $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$.*

Fact. *The inclusion is proper $\mathcal{C}_{\geq n+1} \subsetneq \mathcal{C}_{\geq n}$.*

Remark. $X \simeq 0$ iff $H_*(X;\mathbb{Z}) = 0$ iff $H_*(X;\mathbb{F}_p) = 0$. Assume that X is not contractible, then $H_*(X;\mathbb{F}_p)$ is bounded (since X is a finite spectrum,) thus for large enough n, by AHSS we have $K(p,n)_*(X) \cong H_*(X;\mathbb{F}_p)\left[v_n^{\pm 1}\right]$, i.e. X has finite type. We conclude that $\cap_{n<\infty}\mathcal{C}_n = \{0\} = \mathcal{C}_{\geq\infty}$.

Theorem 22 (Thick Subcategory Theorem [4])**.** *If $\mathcal{T}$ is a thick subcategory of $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$, then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $n = 0,1,2,\ldots,\infty$.*

Remark. The proof relies on a major theorem called the Nilpotence Theorem.

Proposition 23. *$\mathcal{C}_{\geq n}$ is a prime ideal (note that $\mathcal{C}_{\geq 0}$ is not a proper subcategory, thus only for $n = 1,2,\ldots,\infty$.)*

Proof. For X, Y by Kunneth we have $K(p,n-1)_*(X$ $K(p,n-1)_*(X) \otimes K(p,n-1)_*(Y)$. Therefore, if $X \in \mathcal{C}_{\geq n}$, i.e. the homology vanishes, then so does the homology of $X \otimes Y$, i.e. $X \otimes Y \in \mathcal{C}_{\geq n}$, so $\mathcal{C}_{\geq n}$ is an ideal. If $X \otimes Y \in \mathcal{C}_{\geq n}$ then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces,) so $\mathcal{C}_{\geq n}$ is a prime ideal. $\square$

Corollary 24. $\mathrm{Spc}\left(\mathrm{Sp}_{(p)}^{\mathrm{fin}}\right) = \{\mathcal{C}_{\geq 1}, \mathcal{C}_{\geq 2}, \ldots, \mathcal{C}_{\geq \infty}\}$, *and the closed subsets are* $\{\mathcal{C}_{\geq k}, \mathcal{C}_{\geq k+1}, \ldots, \mathcal{C}_{\geq \infty}\}$.

Remark. The chromatic picture can be described for all $\mathrm{Sp}^{\mathrm{fin}}$ at once, which has all the primes above for each p with the above closed sets, except that all $\mathcal{C}_{\geq 1}$ for different p are the same ($H\mathbb{Q}$-acyclics.)

2.4 Morava E-Theory

??????????????????????????

2.5 Further Results

The ideas above lead to the idea of studying spectra height-by-height, and investigate further the connection between Morava E-Theory and Morava K-Theory.

???????????????? Smash product theorem (Lurie 23, and 30, 31) ????????????

Definition 25. For each n we have a map $L_{E(p,n+1)}X \to L_{E(p,n)}X$, thus we can form *the chromatic tower* $\ldots \to L_{E(p,2)}X \to L_{E(p,1)}X \to L_{E(p,0)}X$.

Theorem 26 (Chromatic Convergence Theorem [6, lecture 32])**.** *The limit of the chromatic tower is X.*

Theorem 27 (Chromatic Square [6, lecture 23])**.** *There is a pullback diagram:*

$$\begin{tikzcd}
L_{E(p,n)}X \arrow[r] \arrow[d] & L_{K(p,n)}X \arrow[d] \\
L_{E(p,n-1)}X \arrow[r] & L_{E(p,n-1)}L_{K(p,n)}X
\end{tikzcd}$$

Remark 28. The chromatic square resembles the *arithmetic square*, which allows gluing data from different primes to get the original spectrum. However, the pieces are not exactly the p-localizations, so we don't present it here. These theorems go under the name fracture theorems.

References

[1] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *arXiv:math/0409360*, 2004.

[2] P. Balmer. Spectra, spectra, spectra – tensor triangular spectra versus zariski spectra of endomorphism rings. *Algebraic and Geometric Topology 10, 1521–1563*, 2010.

[3] M. Hopkins. Global methods in homotopy theory. *Homotopy Theory – Proc. Durham Symp. 1985. Cambridge University Press. Cambridge*, 1987.

[4] M. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory II. *Annals of Mathematics, 148(1), second series, 1-49*, 1998.

[5] S. B. Iyengar. Thick subcategories of perfect complexes over a commutative ring. 2006.

[6] J. Lurie. Chromatic homotopy theory. *252x course notes*, 2010.

[7] T. Murayama. The classification of thick subcategories and balmer's reconstruction theorem. 2015.

[8] A. Neeman. The chromatic tower of $\mathcal{D}(R)$. *Topology 31*, 1992.

[9] R. W. Thomason. The classification of triangulated subcategories. *Compositio Math. 105.1*, 1997.