Ambidexterity Seminar – The Chromatic Picture

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1 Motivation – Hopkins-Neeman and Balmer's Spectrum

Two short introductions to the topic are [7, 9] (note that they use the language of triangular categories, rather than ∞-categories). In what follows, R is noetherian ring, $X = \operatorname{Spec}(R)$, and $\operatorname{Ch}(X)$ is the symmetric monoidal stable ∞-category of chain complexes over R.

Problem. Can we recover X from $\operatorname{Ch}(X)$?

The first partial answer to this question is given at [5, 10], later on in [1, 2] the result is further improved, and we will state that version.

Definition 1. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finite projective modules. These are the compact objects in the category, so that they can actually be defined categorically. Denote by $\mathrm{Ch}_{\mathrm{perf}}(X)$ the full subcategory of perfect complexes.

Definition 2. Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

1. $0 \in \mathcal{T}$
2. let $a \xrightarrow{f} b \to c$ cofiber sequence, if two out of $\{a, b, c\}$ are in $\mathcal{T}$, then so is the third (remember that cofiber and fiber sequences are the same)
3. it is closed under retracts

Example 3. Considering the case of $\mathrm{Ch}(X)$ and $\mathrm{Ch}_{\mathrm{perf}}(X)$ (e.g. over $\mathbb{Z}$, chain complexes of abelian groups, and those with finitely-many non-zero entries, each of which is $\mathbb{Z}$ to some power, respectively). Let $K_\bullet \in \mathrm{Ch}(X)$, and define $\mathcal{T}_{K_\bullet} = \{A_\bullet \in \mathrm{Ch}$

Clearly $0 \in \mathcal{T}_{K_\bullet}$. Since tensor is left, it sends pushout to pushout, and three are 0 so the fourth is 0. Lastly, if $A_\bullet \to B_\bullet \to A_\bullet$ is the identity and $B_\bullet \otimes K_\bullet = 0$ then $\mathrm{id}_{A_\bullet \otimes K_\bullet}$ factors through 0, thus $A_\bullet \otimes K_\bullet = 0$. Therefore $\mathcal{T}_{K_\bullet}$ is thick.

Definition 4. A thick subcategory $\mathcal{T}$ is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring, $\mathrm{Spc}\,(\mathcal{C}) = \{\mathcal{P} \text{ prime ideal}\}$, and for any family of objects $S \subseteq \mathcal{C}$ we define $V\,(S) = \{\mathcal{P} \in \mathrm{Spc}\,(\mathcal{C}) \mid S \cap \mathcal{P} = \emptyset\}$, and these are the closed subsets of the *Zariski topology* on $\mathrm{Spc}\,(\mathcal{C})$. We also denote $\mathrm{spp}\,(a) = V\,(\{a\})$.

Theorem 5 (Balmer)**.** *There is a homeomorphism* $\varphi : X \to \mathrm{Spc}\,(\mathrm{Ch}_{\mathrm{perf}}\,(X))$ *given by* $\varphi\,(\mathfrak{p}) = \Big\{ A_\bullet \mid (A_\bullet)_{\mathfrak{p}}$
$\mathcal{T}_{R_\mathfrak{p}}$.

Remark. This was actually upgraded to an isomorphism of locally-ringed spaces.

Proof (sketch). First we note that $\varphi\,(\mathfrak{p})$ is indeed a prime ideal. It was shown to be thick. It is also clearly an ideal, since $A_\bullet \otimes B_\bullet \otimes R_\mathfrak{p} = A_\bullet \otimes 0 = 0$. Finally, if $0 = (A_\bullet \otimes B_\bullet)_\mathfrak{p} = (A_\bullet)_\mathfrak{p} \otimes_{R_\mathfrak{p}} (B_\bullet)_\mathfrak{p}$. Assume by negation that $(A_\bullet)_\mathfrak{p} \neq 0$ and $(B_\bullet)_\mathfrak{p} \neq 0$, i.e. $(A_n)_\mathfrak{p} \neq 0$ and $(B_m)_\mathfrak{p} \neq 0$ but $(A_n)_\mathfrak{p} \otimes_{R_\mathfrak{p}} (B_m)_\mathfrak{p} = 0$. Well, localization of projective is projective, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, so $(A_n)_\mathfrak{p} = 0$ and $(B_m)_\mathfrak{p} = 0$ which is a contradiction. Therefore $\varphi(\mathfrak{p})$ is indeed a prime ideal.

Note that

$$\varphi(\mathfrak{p}) \in \mathrm{spp}(A_\bullet) \iff A_\bullet \notin \varphi(\mathfrak{p}) \iff (A_\bullet)_\mathfrak{p} \neq 0 \Leftarrow$$

and their complements form bases for the topologies. Thus φ is continous, and if it is invertible, the inverse is continous as well. $\square$

Example 6. The case $R = \mathbb{Z}$. Clearly, if $A_\bullet \in \mathcal{T}_{\mathbb{Z}_{(p)}}$ then $A_\bullet \in \mathcal{T}_\mathbb{Q}$, thus any S that doesn't intersect $\mathcal{T}_\mathbb{Q}$ doesn't intersect any $\mathcal{T}_{\mathbb{Z}_{(p)}}$, so a closed set that contains $\mathcal{T}_\mathbb{Q}$ contains all the others. This is in accordance with the theorem, indeed $p\mathbb{Z} \mapsto \left\{A_\bullet \mid (A_\bullet)_{p\mathbb{Z}} = 0\right\} = \left\{A_\bullet \mid A_\bullet \otimes \mathbb{Z}_{(p)} = 0\right\} = \mathcal{T}_{\mathbb{Z}_{(p)}}$ and $\mathbb{Z} \mapsto \mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_\mathbb{Q}$ are prime ideals, and we recovered the toplogy on $\mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X)) = \{\mathcal{T}_{\mathbb{Z}_{(2)}}, \mathcal{T}_{\mathbb{Z}_{(3)}}, \ldots$

Note that the support of an element is all the prime ideals to which it *does not* belong, e.g. $\mathcal{T}_{\mathbb{Z}_{(q)}} \in \operatorname{spp}(\mathbb{F}_p)$ iff $\mathbb{F}_p \notin \mathcal{T}_{\mathbb{Z}_{(q)}}$ iff $\mathbb{F}_p \otimes \mathbb{Z}_{(q)} \neq 0$ which is only when $q = p$, so $\operatorname{spp}(\mathbb{F}_p) = \left\{\mathcal{T}_{\mathbb{Z}_{(p)}}\right\}$ as we'd expect.

2 The Chromatic Picture

Although the category of spectra doesn't arise as the corresponding category for a scheme or a similar gadget, we can still try to "reconstruct the space X" by applying this mechanism, and then try to use this decomposition.

We will concentrate at the p-local spectra, $\mathrm{Sp}_{(p)}$, for some fixed prime. Such localization is a mild operation, and actually all the statements that follow can be stated at the level of all spectra, but it is easier to state them at $\mathrm{Sp}_{(p)}$. We also remind ourselves that the compact objects are finite spectra.

2.1 Morava K-Theory

A good reference for this part is [8, lectures 22, 24]

Definition 7. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenus is invertible, equivalently it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree. An A_∞-ring spectrum E is a *field* if $\pi_* E$ is a field.

Proposition 8. *A field E has Kunneth, i.e.* $E_*(X \otimes Y$ $E_*(X) \otimes_{\pi_* E} E_*(Y)$ *for any spectra* X, Y*.*

Theorem 9 (Definition)**.** *For each prime p and* $n = 1, 2, \ldots$*, there exists a spectrum called* Morava K-Theory of height n*, denoted by* $K(p, n)$*, which has the following properties:*

- $\pi_* K(p, n) \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$ *where* $\deg v_n = 2(p^n - 1)$.
- *It is a field (and in particular, an* A_∞*-ring spectrum).*
- *If* E *is a field, then it has the structure of a* $K(p, n)$*-module for unique* p *and* n*. In that sense* $K(p, n)$ *is uniquely determined.*

We also take $K(p, 0) = H\mathbb{Q}$*.*

Example. Remember that K (regular complex K-theory) has $\pi_* K = \mathbb{Z}\left[\beta^{\pm 1}\right]$ where $\deg \beta = 2$. Taking K/p we get a spectrum with homotopy groups

$\mathbb{F}_p\left[\beta^{\pm 1}\right]$, and it can be shown that it is a module over $K(p,1)$, and since $\deg v_1 = 2(p-1)$ while $\deg \beta = 2$, K/p is a direct sum of $p-1$ copies of $K(p,1)$.

2.2 Localization at E

A reference for what follows is at [8, lecture 20]. Let E be a spectrum.

Definition 10. A spectrum Z is called E-*acyclic*, if $E_*(Z) = \pi_*(E \otimes Z) = 0$ (i.e. $E \otimes Z \simeq 0$). A spectrum Y is called E-*local*, if $[Z, Y]_* = 0$ (i.e. equivalently $\mathrm{Map}(Z, Y) \simeq 0$) for all E-acyclic Z. The E-local spectra form a full subcategory $\mathrm{Sp}_E \subset \mathrm{Sp}$.

Definition 11. Let X be a a spectrum, its E-*localization* is the universal E-local spectrum together with a map $\varphi : X \to L_E X$. I.e. s.t. for each map to an E-local spectrum $f : X \to Y$, there exists a unique $\tilde{f} : L_E X \to Y$ s.t. $f = \tilde{f}\varphi$. In other word, the E-localization is the left adjoint to the inclusion $\mathrm{Sp}_E \subset \mathrm{Sp}$ (and the map corresponds to $\mathrm{id} \in \mathrm{Map}(L_E X, L_E X) \cong \mathrm{Map}(X, L_E X)$).

Remark. The name localization might be confusing. We will use this mechanism for $K(p,n)$ which should be though of as a field. Analogously, the $\mathbb{F}_p$-localization of $\mathbb{Z}$ is $\mathbb{Z}_p$, i.e. the completion, not the localization (note that we actually want to work in complexes, but this is the result we would get after interpreting $\langle S \mid R\rangle$ as $\mathbb{Z}\langle R\rangle \to \mathbb{Z}\langle S\rangle$).

2.3 The Thick Subcategory Theorem and $\mathrm{Spc}\left(\mathrm{Sp}^{\mathrm{fin}}_{(p)}\right)$

Many of the results below can be found at [8, lecture 26]. The Balmer spectrum can be found at [2, corollary 9.5].

Proposition 12. *Let* $\mathcal{T}_E = \ker E_* = \left\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid E_*\right.$ *(equivalently* $X \otimes E \simeq 0$*) i.e. the* E*-acyclics, then* $\mathcal{T}_E$ *is thick.*

Proof. Clearly $0 \in \mathcal{T}_E$. Let be a cofiber sequence $X \to Y \to Z$, then we get a LES in E_* homology, in which every space is wrapped by the two others, therefore if two are 0, then so is the third:

$$\cdots \to E_{m-1}(Z) \to E_m(X) \to E_m(Y) \to E_m(Z) \to E$$

For a retract $X \to Y \to Y$, we get $E_m(X) \to E_m(Y) \to E_m(X)$, where the middle is 0, and the composition is identity, thus $E_m(X) = 0$. $\square$

This leads us to the following definition.

Definition 13. We define $\mathcal{C}_{\geq n} = \mathcal{T}_{K(p,n-1)}$, the $K(p, n-1)$-acyclics (equivalently $X \otimes K(p, n-1) \simeq 0$). By the above it is thick. Also, $\mathcal{C}_{\geq 0} = \mathrm{Sp}^{\mathrm{fin}}_{(p)}$ and $\mathcal{C}_{\geq \infty} = \{0\}$, which are trivially thick.

Proposition 14. *For $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$, if $K(p,n)_*(X) = 0$ then $K(p, n-1)_*(X) = 0$.*

Remark. This result is not true for any spectrum (e.g. for $H\mathbb{Q}$ whose $K(p,n)$ doesn't vanish at $n = 0$ but does at $n = 1$).

Definition 15. We say that a spectrum is of *type n* (possibly ∞), if the first non-zero Morava K-Theory is $K(p,n)$.

Corollary. *$\mathcal{C}_{\geq n}$ is the full subcategory of finite p-local spectra of type $\geq n$ (i.e. $\Big\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid \forall m < n : K$ Thus clearly $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$.*

Proposition 16. *The inclusion is proper $\mathcal{C}_{\geq n+1} \subsetneq \mathcal{C}_{\geq n}$.*

Proposition 17. *If $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ is not contractible, then X has finite height. Therefore $\cap_{n<\infty}\mathcal{C}_{\geq n} = \{0\} = \mathcal{C}_{\geq\infty}$.*

Proof. $X \simeq 0$ iff $H_*(X;\mathbb{Z}) = 0$ iff $H_*(X;\mathbb{F}_p) = 0$. Assume that X is not contractible, then $H_*(X;\mathbb{F}_p)$ is bounded (since X is a finite spectrum), thus for large enough n, by Atiyah-Hirzebruch SS we have $K(p,n)_*(X) \cong H_*(X;\mathbb{F}_p)\left[v_n^{\pm1}\right]$, i.e. X has finite type. We conclude that $\cap_{n<\infty}\mathcal{C}_{\geq n} = \{0\} = \mathcal{C}_{\geq\infty}$. $\square$

Theorem 18 (Thick Subcategory Theorem [6])**.** *If $\mathcal{T}$ is a thick subcategory of $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$, then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $n = 0, 1, 2, \ldots, \infty$.*

Remark. The proof relies on a major theorem called the Nilpotence Theorem.

Proposition 19. *$\mathcal{C}_{\geq n}$ is a prime ideal (note that $\mathcal{C}_{\geq 0}$ is not a proper subcategory, thus only for $n = 1, 2, \ldots, \infty$.)*

Proof. For X, Y by Kunneth we have $K(p,n-1)_*(X$
$K(p,n-1)_*(X) \otimes K(p,n-1)_*(Y)$. Therefore, if $X \in \mathcal{C}_{\geq n}$, i.e. the homology vanishes, then so does

the homology of $X \otimes Y$, i.e. $X \otimes Y \in \mathcal{C}_{\geq n}$, so $\mathcal{C}_{\geq n}$ is an ideal. If $X \otimes Y \in \mathcal{C}_{\geq n}$ then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces), so $\mathcal{C}_{\geq n}$ is a prime ideal. $\square$

Corollary 20. $\operatorname{Spc}\left(\mathrm{Sp}_{(p)}^{\mathrm{fin}}\right) = \{\mathcal{C}_{\geq 1}, \mathcal{C}_{\geq 2}, \ldots, \mathcal{C}_{\geq \infty}\}$, *and the closed subsets are* $\{\mathcal{C}_{\geq k}, \mathcal{C}_{\geq k+1}, \ldots, \mathcal{C}_{\geq \infty}\}$.

Remark. The chromatic picture can be described for all $\mathrm{Sp}^{\mathrm{fin}}$ at once, which has all the primes above for each p with the above closed sets, except that all $\mathcal{C}_{\geq 1}$ for different p are the same ($H\mathbb{Q}$-acyclics.)

2.4 Morava E-Theory

Remark. There are many approches and flavors to Morava E-Theory. The one we use is based on [3] and [11]. See also [4]. Anoter approch is via deformations of the formal group law of $K(p,n)$, this approch can found at [8].

The results above indicate that $K(p,n)$ "sees" $K(p,n$ (For example, we had the claim that $K(p,n)_*(X) = 0 \implies K(p,n-1)_*(X) = 0$ for $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$. And first the is in the latter's closure.) In some sense (which will be more precise later), $K(p,n)$ determines the n-th chromatic level, and an infinitesimal neighbourhood around it, so that we can glue data from lower height. We would like to find a spectrum that sees all $\leq n$ chromatic levels at once.

Remember that $\mathbb{S}$ is analogus to $\mathbb{Z}$, and $L_{K(p,n)}$ is analogus to completion at p (localization at $\mathbb{F}_p$), so the $K(p,n)$-local sphere $L_{K(p,n)}\mathbb{S}$ is analogus to $\mathbb{Z}_p = W(\mathbb{F}_p)$, and it makes sense to try an investigate its Galois extensions. I will not give a precise definition, and definitely not for a general Galois Extension, but just to give an idea:

Definition 21 (kind of)**.** Let G be a finite group, and $f : A \to B$ a map between two E_∞-ring spectra s.t.:

1. f is equivariant w.r.t to the trivial G-action on A,
2. $A \to B^{hG}$ is an equivalence,
3. $B \otimes_A B \to \bigoplus_G B, x \otimes y \mapsto (xg.y)$ is an equivalence.

Then B is called a *Galois extension* of A with Galois group G.

Remark. If we think about extension of (clasical) fields, the first condition means that G is acts on B as automorphisms over A, $B^G \subseteq B$ is always a Galois extension, and the second condition ensures that $A = B^G$, the third condition says that G is actually the Galois groups (it might not act faithfully for example).

It turns out that there is a spectrum called *Morava E-Theory*, denoted by $E(p,n)$, which is the maximal Galois extension of $L_{K(p,n)}\mathbb{S}$ (and the Galois group, which is not finite, is called the Morava stabilizer group). It has coefficients $\pi_* E(p,n) \cong W\left(\overline{\mathbb{F}}_p\right)[\![u_1, \ldots$ where $\deg u_i = 0$ and $\deg \beta = 2$.

The following statement is a formalization of the idea that $E(p,n)$ sees all $\leq n$ chromatic levels at once.

Proposition 22. *For $E = K(p,0) \vee \cdots \vee K(p,n)$ and for $E = E(p,n)$, being E-acyclic, being E-local and L_E are the same.*

Remark. In other words they are *Bousfield equivalent*, and clearly the first implies the rest.

2.5 Further Results

The ideas above lead to the idea of studying spectra one prime at a time, height-by-height. We would like to know how to work out the original spectrum.

Definition 23. For each n we have a map $L_{E(p,n+1)}X$ $L_{E(p,n)}X$, thus we can form *the chromatic tower* $\ldots \to L_{E(p,2)}X \to L_{E(p,1)}X \to L_{E(p,0)}X$.

Theorem 24 (Chromatic Convergence Theorem [8, lecture 32])**.** *The limit of the chromatic tower is X.*

Theorem 25 (Chromatic Square [8, lecture 23])**.** *There is a pullback diagram:*

$$\begin{tikzcd}
L_{E(p,n)}X \arrow[r] \arrow[d] & L_{K(p,n)}X \arrow[d] \\
L_{E(p,n-1)}X \arrow[r] & L_{E(p,n-1)}L_{K(p,n)}X
\end{tikzcd}$$

The chromatic square gets its name from another relevant theorem (these theorems go under the name fracture theorems):

Theorem 26 (Arithmetic Square)**.** *There is a pull-*

back diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod L_{S\mathbb{F}_p} X \arrow[d] \\ L_{S\mathbb{Q}} X \arrow[r] & L_{S\mathbb{Q}} \left(\prod L_{S\mathbb{F}_p} X\right) \end{tikzcd}$$

(where actually $L_{S\mathbb{F}_p} X = L_{S\mathbb{F}_p} X_{(p)}$, so it contains less information then $X_{(p)}$ [$X_{(p)} = L_{S\mathbb{Z}_{(p)}} X$ is the p-localization and $L_{S\mathbb{F}_p} X$ is the p-completion]).

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