Ambidexterity Seminar – The Chromatic Picture

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1 Motivation – Hopkins-Neeman and Balmer's Spectrum

Two short introductions to the topic are [7, 9] (note that they use the language of triangular categories, rather than ∞-categories.) In what follows, R is noetherian ring, $X = \mathrm{Spec}\,(R)$, and $\mathrm{Ch}\,(X)$ is the symmetric monoidal stable ∞-category of chain complexes over R.

Problem. Can we recover X from $\mathrm{Ch}\,(X)$?

The first partial answer to this question is given at [5, 10], later on in [1, 2] the result is further improved, and we will state that version.

Definition 1. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finite projective modules. These are the compact objects in the category, so that they can actually be defined categorically. Denote by $\mathrm{Ch}_{\mathrm{perf}}\,(X)$ the full subcategory of perfect complexes.

Definition 2. Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

1. $0 \in \mathcal{T}$

2. let $a \xrightarrow{f} b \to c$ cofiber sequence, if two out of $\{a, b, c\}$ are in $\mathcal{T}$, then so is the third (remember that cofiber and fiber sequences are the same)

3. it is closed under retracts

Example 3. Considering the case of $\mathrm{Ch}\,(X)$ and $\mathrm{Ch}_{\mathrm{perf}}\,(X)$ (e.g. over $\mathbb{Z}$, chain complexes of abelian groups, and those with finitely-many non-zero entries, each of which is $\mathbb{Z}$ to some power, respectively.) Let $K_\bullet \in \mathrm{Ch}\,(X)$, and define $\mathcal{T}_{K_\bullet} = \{A_\bullet \in \mathrm{Ch}_{\mathrm{perf}}\,(X) \mid A_\bullet \otimes K_\bullet = 0\}$. Clearly $0 \in \mathcal{T}_{K_\bullet}$. Since tensor is left, it sends pushout to pushout ????????? not sure ???????, and three are 0 so the fourth is 0. Lastly, if $A_\bullet \to B_\bullet \to A_\bullet$ is the identity and $B_\bullet \otimes K_\bullet = 0$ then $\mathrm{id}_{A_\bullet \otimes K_\bullet}$ factors through 0, thus $A_\bullet \otimes K_\bullet = 0$. Therefore $\mathcal{T}_{K_\bullet}$ is thick.

Definition 4. A thick subcategory $\mathcal{T}$ is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring, $\mathrm{Spc}(\mathcal{C}) = \{\mathcal{P} \text{ prime ideal}\}$, and for any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) \mid S \cap \mathcal{P} = \emptyset\}$, and these are the closed subsets of the *Zariski topology* on $\mathrm{Spc}(\mathcal{C})$. We also denote $\mathrm{spp}(a) = V(\{a\})$.

Theorem 5 (Balmer)**.** *There is a homeomorphism* $\varphi : X \to \mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X))$ *given by* $\varphi(\mathfrak{p}) = \left\{ A_\bullet \mid (A_\bullet)_{\mathfrak{p}} = 0 \right\} = \mathcal{T}_{R_\mathfrak{p}}$.

Remark. This was actually upgraded to an isomorphism of locally-ringed spaces.

Proof (sketch). First we note that $\varphi(\mathfrak{p})$ is indeed a prime ideal. It was shown to be thick. It is also clearly an ideal, since $A_\bullet \otimes B_\bullet \otimes R_\mathfrak{p} = A_\bullet \otimes 0 = 0$. Finally, since $(A_\bullet \otimes B_\bullet)_\mathfrak{p} = (A_\bullet)_\mathfrak{p} \otimes_{R_\mathfrak{p}} (B_\bullet)_\mathfrak{p}$, it is sufficient to prove that for any two finite projective modules A, B s.t. $A_\mathfrak{p} \otimes_{R_\mathfrak{p}} B_\mathfrak{p} = 0$, either $A_\mathfrak{p} = 0$ or $B_\mathfrak{p} = 0$. Well, localization of projective is projective, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes. Therefore $\varphi(\mathfrak{p})$ is indeed a prime ideal.

Note that

$$\varphi(\mathfrak{p}) \in \mathrm{spp}(A_\bullet) \iff A_\bullet \notin \varphi(\mathfrak{p}) \iff (A_\bullet)_\mathfrak{p} = 0 \iff \mathfrak{p} \in \mathrm{supp}(A_\bullet)$$

and their complements form bases for the topologies. Thus φ is continous, and if it is invertible, the inverse is continous as well. $\qquad\square$

Example 6. The case $R = \mathbb{Z}$. Clearly, if $A_\bullet \in \mathcal{T}_{\mathbb{Z}_{(p)}}$ then $A_\bullet \in \mathcal{T}_\mathbb{Q}$, thus any S that doesn't intersect $\mathcal{T}_\mathbb{Q}$ doesn't intersect any $\mathcal{T}_{\mathbb{Z}_{(p)}}$, so a closed set that contains $\mathcal{T}_\mathbb{Q}$ contains all the others. This is in accordance with the theorem, indeed $p\mathbb{Z} \mapsto \left\{ A_\bullet \mid (A_\bullet)_{p\mathbb{Z}} = 0 \right\} = \left\{ A_\bullet \mid A_\bullet \otimes \mathbb{Z}_{(p)} = 0 \right\} = \mathcal{T}_{\mathbb{Z}_{(p)}}$ and $\mathbb{Z} \mapsto \mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_\mathbb{Q}$ are prime ideals, and we recovered the toplogy on $\mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X)) = \left\{ \mathcal{T}_{\mathbb{Z}_{(2)}}, \mathcal{T}_{\mathbb{Z}_{(3)}}, \ldots, \mathcal{T}_\mathbb{Q} \right\}$. Note that the support of an element is all the prime ideals to which it *does not* belong, e.g. $\mathcal{T}_{\mathbb{Z}_{(q)}} \in \mathrm{spp}(\mathbb{F}_p)$ iff $\mathbb{F}_p \otimes \mathbb{Z}_{(q)} \neq 0$ which is only when $q = p$, so $\mathrm{spp}(\mathbb{F}_p) = \left\{ \mathcal{T}_{\mathbb{Z}_{(p)}} \right\}$ as we'd expect.

2 The Chromatic Picture

Although the category of spectra doesn't arise as the corresponding category for a scheme or a similar gadget, we can still try to "reconstruct the space X" by applying this mechanism, and then try to use this decomposition.

We will concentrate at the p-local spectra, $\mathrm{Sp}_{(p)}$, for some fixed prime. Such localization is a mild operation, and actually all the statements that follow can be stated at the level of all spectra, but it is easier to state them at $\mathrm{Sp}_{(p)}$. We also remind ourselves that the compact objects are finite spectra.

2.1 Morava K-Theory

A good reference for this part is [8, lectures 22, 24]

Definition 7. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenus is invertible, equivalently it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree. An $(A_\infty\text{-})$ring spectrum E is a *field* if $\pi_* E$ is a field.

Proposition 8. *A field E has Kunneth, i.e. $E_*\left(X \otimes Y\right) \cong E_*\left(X\right) \otimes_{\pi_* E} E_*\left(Y\right)$ for any spectra X, Y.*

Fact 9. *For each prime p and $n = 1, 2, \ldots$, there exists a spectrum called Morava K-Theory of height n, denoted by $K\left(p, n\right)$, which has the following properties:*

- $\pi_* K\left(p, n\right) \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$ *where* $\deg v_n = 2\left(p^n - 1\right)$.

- *It is a field (and in particular, an A_∞-ring spectrum.)*

- *If E is a field, then it has the structure of a $K\left(p, n\right)$-module for unique p and n. In that sense $K\left(p, n\right)$ is uniquely determined.*

We also take $K\left(p, 0\right) = H\mathbb{Q}$.

Example. Remember that K (regular complex K-theory) has $\pi_*\left(K\right) = \mathbb{Z}\left[\beta^{\pm 1}\right]$ where $\deg \beta = 2$. Taking K/p we get a spectrum with homotopy groups $\mathbb{F}_p\left[\beta^{\pm 1}\right]$, and it can be shown that it is a module over $K\left(p, 1\right)$, and since $\deg v_1 = 2\left(p - 1\right)$ while $\deg \beta = 2$, K/p is a direct sum of $p - 1$ copies of $K\left(p, 1\right)$.

2.2 Localization at E

A reference for what follows is at [8, lecture 20]. Let E be a spectrum.

Definition 10. A spectrum Z is called *E-acyclic*, if $E_*\left(Z\right) = \pi_*\left(E \otimes Z\right) = 0$ (i.e. $E \otimes Z \simeq 0$.) A spectrum Y is called *E-local*, if $[Z, Y]_* = 0$ (i.e. equivalently $\operatorname{Map}\left(Z, Y\right) \simeq 0$) for all E-acyclic Z. The E-local spectra form a full subcategory $\operatorname{Sp}_E \subset \operatorname{Sp}$.

Definition 11. Let X be a a spectrum, its *E-localization* is the universal E-local spectrum together with a map $\varphi : X \to L_E X$. I.e. s.t. for each map to an E-local spectrum $f : X \to Y$, there exists a unique $\tilde{f} : L_E X \to Y$ s.t. $f = \tilde{f}\varphi$. In other word, the E-localization is the left adjoint to the inclusion $\operatorname{Sp}_E \subset \operatorname{Sp}$ (and the map corresponds to $\operatorname{id} \in \operatorname{Map}\left(L_E X, L_E X\right) \cong \operatorname{Map}\left(X, L_E X\right)$.)

Remark 12. The name localization might be confusing. We will use this mechanism for $K\left(p, n\right)$ which should be though of as a field. Analogously, the $\mathbb{F}_p$-localization of $\mathbb{Z}$ is $\mathbb{Z}_p$, i.e. the completion, not the localization (note that we actually want to work in complexes, but this is the result we would get after interpreting $\langle S \mid R \rangle$ as $\mathbb{Z}\langle R \rangle \to \mathbb{Z}\langle S \rangle$.)

2.3 The Thick Subcategory Theorem and $\mathrm{Spc}\left(\mathrm{Sp}_{(p)}^{\mathrm{fin}}\right)$

Many of the results below can be found at [8, lecture 26]. The Balmer spectrum can be found at [2, corollary 9.5].

Proposition 13. *Let* $\mathcal{T}_E = \ker E_* = \left\{ X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid E_*\left(X\right) = 0 \right\}$ *(equivalently* $X \otimes E \simeq 0$*) i.e. the E-acyclics, then $\mathcal{T}_E$ is thick.*

Proof. Clearly $0 \in \mathcal{T}_E$. Let be a cofiber sequence $X \to Y \to Z$, then we get a LES in E_* homology, in which every space is wrapped by the two others, therefore if two are 0, then so is the third ??????????????????? can we use the same argument from complexes ????????????????:

$$\cdots \to E_{m-1}\left(Z\right) \to E_m\left(X\right) \to E_m\left(Y\right) \to E_m\left(Z\right) \to E_{m+1}\left(X\right) \to \cdots$$

For a retract $X \to Y \to Y$, we get $E_m\left(X\right) \to E_m\left(Y\right) \to E_m\left(X\right)$, where the middle is 0, and the composition is identity, thus $E_m\left(X\right) = 0$. $\qquad \square$

This leads us to the following definition.

Definition 14. We define $\mathcal{C}_{\geq n} = \mathcal{T}_{K(p,n-1)}$, the $K\left(p, n-1\right)$-acyclics (equivalently $X \otimes K\left(p, n-1\right) \simeq 0$.) By the above it is thick. Also, $\mathcal{C}_{\geq 0} = \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ and $\mathcal{C}_{\geq \infty} = \{0\}$, which are trivially thick.

Proposition 15. *For* $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$*, if* $K\left(p,n\right)_*\left(X\right) = 0$ *then* $K\left(p, n-1\right)_*\left(X\right) = 0$*.*

Remark. This result is not true for any spectrum (e.g. for $H\mathbb{Q}$ whose $K\left(p,n\right)$ doesn't vanish at $n = 0$ but does at $n = 1$,) and the fact that it doesn't vanish is important and has to do with Morava E-Theory and the way different levels glue.

Definition 16. We say that a spectrum is of *type n* (possibly ∞,) if the first non-zero Morava K-Theory is $K\left(p,n\right)$.

Corollary. $\mathcal{C}_{\geq n}$ *is the full subcategory of finite p-local spectra of type $\geq n$. Thus clearly* $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$*.*

Fact. *The inclusion is proper* $\mathcal{C}_{\geq n+1} \subsetneq \mathcal{C}_{\geq n}$*.*

Remark. $X \simeq 0$ iff $H_*\left(X; \mathbb{Z}\right) = 0$ iff $H_*\left(X; \mathbb{F}_p\right) = 0$. Assume that X is not contractible, then $H_*\left(X; \mathbb{F}_p\right)$ is bounded (since X is a finite spectrum,) thus for large enough n, by AHSS we have $K\left(p,n\right)_*\left(X\right) \cong H_*\left(X; \mathbb{F}_p\right)\left[v_n^{\pm 1}\right]$, i.e. X has finite type. We conclude that $\cap_{n < \infty} \mathcal{C}_{\geq n} = \{0\} = \mathcal{C}_{\geq \infty}$.

Theorem 17 (Thick Subcategory Theorem [6])**.** *If $\mathcal{T}$ is a thick subcategory of* $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$*, then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $n = 0, 1, 2, \ldots, \infty$.*

Remark. The proof relies on a major theorem called the Nilpotence Theorem.

Proposition 18. $\mathcal{C}_{\geq n}$ *is a prime ideal (note that* $\mathcal{C}_{\geq 0}$ *is not a proper subcategory, thus only for* $n = 1, 2, \ldots, \infty$.*)*

Proof. For X, Y by Kunneth we have $K(p, n-1)_* (X \otimes Y) = K(p, n-1)_* (X) \otimes K(p, n-1)_* (Y)$. Therefore, if $X \in \mathcal{C}_{\geq n}$, i.e. the homology vanishes, then so does the homology of $X \otimes Y$, i.e. $X \otimes Y \in \mathcal{C}_{\geq n}$, so $\mathcal{C}_{\geq n}$ is an ideal. If $X \otimes Y \in \mathcal{C}_{\geq n}$ then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces,) so $\mathcal{C}_{\geq n}$ is a prime ideal. $\square$

Corollary 19. $\mathrm{Spc}\left(\mathrm{Sp}_{(p)}^{\mathrm{fin}}\right) = \{\mathcal{C}_{\geq 1}, \mathcal{C}_{\geq 2}, \ldots, \mathcal{C}_{\geq \infty}\}$, *and the closed subsets are* $\{\mathcal{C}_{\geq k}, \mathcal{C}_{\geq k+1}, \ldots, \mathcal{C}_{\geq \infty}\}$.

Remark. The chromatic picture can be described for all $\mathrm{Sp}^{\mathrm{fin}}$ at once, which has all the primes above for each p with the above closed sets, except that all $\mathcal{C}_{\geq 1}$ for different p are the same ($H\mathbb{Q}$-acyclics.)

2.4 Morava E-Theory

Remark. There are many approches to Morava E-Theory. The one we use is based on [3]. See also [4]. A more standard approch is via deformations of the formal group law of $K(p, n)$, this approch can found at [8].

The results above indicate that $K(p, n)$ "sees" $K(p, n-1)$ (for we had the claim, if $K(p, n)_* (X) = 0 \implies K(p, n-1)_* (X) = 0$ for $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$. And first the is in the latter's closure.) In some sense (which will be more precise later,) $K(p, n)$ determines the n-th chromatic level. We would like to find a spectrum that sees all $\leq n$ chromatic levels at once.

Remember that $\mathbb{S}$ is analogus to $\mathbb{Z}$, and $L_{K(p,n)}$ is analogus to completion at p (localization at $\mathbb{F}_p$,) so the $K(p, n)$-local sphere $L_{K(p,n)}\mathbb{S}$ is analogus to $\mathbb{Z}_p$, and it makes sense to try an investigate its Galois extensions. It turns out that there is a spectrum called *Morava E-Theory*, denoted by $E(p, n)$, which is the maximal Galois extension of $L_{K(p,n)}\mathbb{S}$ (and the Galois group is called the Morava stabilizer group.) It has coefficients $\pi_* E(p, n) \cong W(\mathbb{F}_{p^n}) [\![v_1, \ldots, v_{n-1}]\!] [\beta^{\pm 1}]$.

The following statement is a formalization of the idea that $E(p, n)$ sees all $\leq n$ chromatic levels at once.

Proposition 20. *For* $E = K(p, 0) \vee \cdots \vee K(p, n)$ *and for* $E = E(p, n)$, *being* E-*acyclic, being* E-*local and* L_E *are the same.*

Remark. In other words they are *Bousfield equivalent*, and clearly the first implies the rest.

2.5 Further Results

The ideas above lead to the idea of studying spectra one prime at a time, height-by-height. We would like to know how to work out the original spectrum.

Definition 21. For each n we have a map $L_{E(p,n+1)}X \to L_{E(p,n)}X$, thus we can form *the chromatic tower* $\dots \to L_{E(p,2)}X \to L_{E(p,1)}X \to L_{E(p,0)}X$.

Theorem 22 (Chromatic Convergence Theorem [8, lecture 32])**.** *The limit of the chromatic tower is X.*

Theorem 23 (Chromatic Square [8, lecture 23])**.** *There is a pullback diagram:*

$$
\begin{array}{ccc}
L_{E(p,n)}X & \longrightarrow & L_{K(p,n)}X \\
\downarrow & & \downarrow \\
L_{E(p,n-1)}X & \longrightarrow & L_{E(p,n-1)}L_{K(p,n)}X
\end{array}
$$

The chromatic square gets its name from another relevant theorem (these theorems go under the name fracture theorems):

Theorem 24 (Arithmetic Square)**.** *There is a pullback diagram:*

$$
\begin{array}{ccc}
X & \longrightarrow & \left(\prod L_{S\mathbb{F}_p}X\right) \\
\downarrow & & \downarrow \\
L_{S\mathbb{Q}}X & \longrightarrow & L_{S\mathbb{Q}}\left(\prod L_{S\mathbb{F}_p}X\right)
\end{array}
$$

(where actually $L_{S\mathbb{F}_p}X = L_{S\mathbb{F}_p}X_{(p)}$, so it contains less information then $X_{(p)}$ [$X_{(p)} = L_{S\mathbb{Z}_{(p)}}X$ is the p-localization and $L_{S\mathbb{F}_p}$ is the p-completion.])

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