

# Ambidexterity Seminar – The Chromatic Picture

Shay Ben Moshe

December 03, 2017

## 1 Motivation – Hopkins-Neeman and Balmer's Spectrum

Two short introductions to the topic are [7, 9] (note that they use the language of triangular categories, rather than  $\infty$ -categories). In what follows,  $R$  is noetherian ring,  $X = \mathrm{Spec}(R)$ , and  $\mathrm{Ch}(X)$  is the symmetric monoidal stable  $\infty$ -category of chain complexes over  $R$ .

**Problem.** Can we recover  $X$  from  $\mathrm{Ch}(X)$ ?

The first partial answer to this question is given at [5, 10], later on in [1, 2] the result is further improved, and we will state that version.

**Definition 1.** A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finite projective modules. These are the compact objects in the category, so that they can actually be defined categorically. Denote by  $\mathrm{Ch}_{\mathrm{perf}}(X)$  the full subcategory of perfect complexes.

**Definition 2.** Let  $\mathcal{C}$  be a symmetric monoidal stable  $\infty$ -category. A full subcategory  $\mathcal{T}$  is *thick* if:

1.  $0 \in \mathcal{T}$
2. let  $a \xrightarrow{f} b \rightarrow c$  cofiber sequence, if two out of  $\{a, b, c\}$  are in  $\mathcal{T}$ , then so is the third (remember that cofiber and fiber sequences are the same)
3. it is closed under retracts

**Example 3.** Considering the case of  $\mathrm{Ch}(X)$  and  $\mathrm{Ch}_{\mathrm{perf}}(X)$  (e.g. over  $\mathbb{Z}$ , chain complexes of abelian groups, and those with finitely-many non-zero entries, each of which is  $\mathbb{Z}$  to some power, respectively). Let  $K_{\bullet} \in \mathrm{Ch}(X)$ , and define  $\mathcal{T}_{K_{\bullet}} = \{A_{\bullet} \in \mathrm{Ch}(X) \mid K_{\bullet} \oplus A_{\bullet} \in \mathrm{Ch}_{\mathrm{perf}}(X)\}$ .

Clearly  $0 \in \mathcal{T}_{K_\bullet}$ . Since tensor is left, it sends pushout to pushout, and three are 0 so the fourth is 0. Lastly, if  $A_\bullet \rightarrow B_\bullet \rightarrow A_\bullet$  is the identity and  $B_\bullet \otimes K_\bullet = 0$  then  $\text{id}_{A_\bullet \otimes K_\bullet}$  factors through 0, thus  $A_\bullet \otimes K_\bullet = 0$ . Therefore  $\mathcal{T}_{K_\bullet}$  is thick.

**Definition 4.** A thick subcategory  $\mathcal{T}$  is an *ideal* if  $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$ . Furthermore, it is a *prime ideal* if it is a proper subcategory, and  $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$  or  $b \in \mathcal{T}$ . The *spectrum* of the category is defined similarly to the classical spectrum of a ring,  $\text{Spc}(\mathcal{C}) = \{\mathcal{P} \text{ prime ideal}\}$ , and for any family of objects  $S \subseteq \mathcal{C}$  we define  $V(S) = \{\mathcal{P} \in \text{Spc}(\mathcal{C}) \mid S \cap \mathcal{P} = \emptyset\}$ , and these are the closed subsets of the *Zariski topology* on  $\text{Spc}(\mathcal{C})$ . We also denote  $\text{spp}(a) = V(\{a\})$ .

**Theorem 5 (Balmer).** *There is a homeomorphism  $\varphi : X \rightarrow \text{Spc}(\text{Ch}_{\text{perf}}(X))$  given by  $\varphi(\mathfrak{p}) = \{A_\bullet \mid (A_\bullet), \mathcal{T}_{R_{\mathfrak{p}}}\}$ .*

*Remark.* This was actually upgraded to an isomorphism of locally-ringed spaces.

*Proof (sketch).* First we note that  $\varphi(\mathfrak{p})$  is indeed a prime ideal. It was shown to be thick. It is also

clearly an ideal, since  $A_{\bullet} \otimes B_{\bullet} \otimes R_{\mathfrak{p}} = A_{\bullet} \otimes 0 = 0$ . Finally, if  $0 = (A_{\bullet} \otimes B_{\bullet})_{\mathfrak{p}} = (A_{\bullet})_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (B_{\bullet})_{\mathfrak{p}}$ . Assume by negation that  $(A_{\bullet})_{\mathfrak{p}} \neq 0$  and  $(B_{\bullet})_{\mathfrak{p}} \neq 0$ , i.e.  $(A_n)_{\mathfrak{p}} \neq 0$  and  $(B_m)_{\mathfrak{p}} \neq 0$  but  $(A_n)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (B_m)_{\mathfrak{p}} = 0$ . Well, localization of projective is projective, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, so  $(A_n)_{\mathfrak{p}} = 0$  and  $(B_m)_{\mathfrak{p}} = 0$  which is a contradiction. Therefore  $\varphi(\mathfrak{p})$  is indeed a prime ideal.

Note that

$$\varphi(\mathfrak{p}) \in \text{spp}(A_{\bullet}) \iff A_{\bullet} \notin \varphi(\mathfrak{p}) \iff (A_{\bullet})_{\mathfrak{p}} \neq 0 \iff$$

and their complements form bases for the topologies. Thus  $\varphi$  is continuous, and if it is invertible, the inverse is continuous as well.  $\square$

**Example 6.** The case  $R = \mathbb{Z}$ . Clearly, if  $A_{\bullet} \in \mathcal{T}_{\mathbb{Z}_{(p)}}$  then  $A_{\bullet} \in \mathcal{T}_{\mathbb{Q}}$ , thus any  $S$  that doesn't intersect  $\mathcal{T}_{\mathbb{Q}}$  doesn't intersect any  $\mathcal{T}_{\mathbb{Z}_{(p)}}$ , so a closed set that contains  $\mathcal{T}_{\mathbb{Q}}$  contains all the others. This is in accordance with the theorem, indeed  $p\mathbb{Z} \mapsto \{A_{\bullet} \mid (A_{\bullet})_{p\mathbb{Z}} = 0\} = \{A_{\bullet} \mid A_{\bullet} \otimes \mathbb{Z}_{(p)} = 0\} = \mathcal{T}_{\mathbb{Z}_{(p)}}$  and  $\mathbb{Z} \mapsto \mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_{\mathbb{Q}}$  are prime ideals, and we recovered the topology on  $\text{Spc}(\text{Ch}_{\text{perf}}(X)) = \{\mathcal{T}_{\mathbb{Z}_{(2)}}, \mathcal{T}_{\mathbb{Z}_{(3)}}, \dots$

Note that the support of an element is all the prime ideals to which it *does not* belong, e.g.  $\mathcal{T}_{\mathbb{Z}_{(q)}} \in \text{spp}(\mathbb{F}_p)$  iff  $\mathbb{F}_p \notin \mathcal{T}_{\mathbb{Z}_{(q)}}$  iff  $\mathbb{F}_p \otimes \mathbb{Z}_{(q)} \neq 0$  which is only when  $q = p$ , so  $\text{spp}(\mathbb{F}_p) = \{\mathcal{T}_{\mathbb{Z}_{(p)}}\}$  as we'd expect.

## 2 The Chromatic Picture

Although the category of spectra doesn't arise as the corresponding category for a scheme or a similar gadget, we can still try to “reconstruct the space  $X$ ” by applying this mechanism, and then try to use this decomposition.

We will concentrate at the  $p$ -local spectra,  $\text{Sp}_{(p)}$ , for some fixed prime. Such localization is a mild operation, and actually all the statements that follow can be stated at the level of all spectra, but it is easier to state them at  $\text{Sp}_{(p)}$ . We also remind ourselves that the compact objects are finite spectra.

## 2.1 Morava K-Theory

A good reference for this part is [8, lectures 22, 24, 25]

**Definition 7.** Let  $R$  be an evenly graded ring.  $R$  is called a *graded field* if every non-zero homogenus is invertible, equivalently it is a field  $F$  concentrated at degree 0, or  $F[\beta^{\pm 1}]$  for  $\beta$  of positive even degree. An  $A_\infty$ -ring spectrum  $E$  is a *field* if  $\pi_*E$  is a field.

**Proposition 8.** A field  $E$  has Kunneth, i.e.  $E_*(X \otimes Y) \cong E_*(X) \otimes_{\pi_*E} E_*(Y)$  for any spectra  $X, Y$ .

**Theorem 9** (Definition). For each prime  $p$  and  $n = 1, 2, \dots$ , there exists a spectrum called Morava K-Theory of height  $n$ , denoted by  $K(p, n)$ , which has the following properties:

- $\pi_*K(p, n) \cong \mathbb{F}_p[v_n^{\pm 1}]$  where  $\deg v_n = 2(p^n - 1)$ .
- It is a field (and in particular, an  $A_\infty$ -ring spectrum).
- If  $E$  is a field, then it has the structure of a  $K(p, n)$ -module for some  $p$  and  $n$ . In that sense  $K(p, n)$  is uniquely determined.

We also take  $K(p, 0) = H\mathbb{Q}$ .

**Example.** Remember that  $K$  (regular complex  $K$ -theory) has  $\pi_* K = \mathbb{Z}[\beta^{\pm 1}]$  where  $\deg \beta = 2$ . Taking  $K/p$  we get a spectrum with homotopy groups  $\mathbb{F}_p[\beta^{\pm 1}]$ , and it can be shown that it is a module over  $K(p, 1)$ , and since  $\deg v_1 = 2(p - 1)$  while  $\deg \beta = 2$ ,  $K/p$  is a direct sum of  $p - 1$  copies of  $K(p, 1)$ .

## 2.2 Localization at $E$

A reference for what follows is at [8, lecture 20]. Let  $E$  be a spectrum.

**Definition 10.** A spectrum  $Z$  is called  *$E$ -acyclic*, if  $E_*(Z) = \pi_*(E \otimes Z) = 0$  (i.e.  $E \otimes Z \simeq 0$ ). A spectrum  $Y$  is called  *$E$ -local*, if  $[Z, Y]_* = 0$  (i.e. equivalently  $\text{Map}(Z, Y) \simeq 0$ ) for all  $E$ -acyclic  $Z$ . The  $E$ -local spectra form a full subcategory  $\text{Sp}_E \subset \text{Sp}$ .

**Definition 11.** Let  $X$  be a spectrum, its  *$E$ -localization* is the universal  $E$ -local spectrum together with a map  $\varphi : X \rightarrow L_E X$ . I.e. s.t. for each map to an  $E$ -local spectrum  $f : X \rightarrow Y$ , there

exists a unique  $\tilde{f} : L_E X \rightarrow Y$  s.t.  $f = \tilde{f}\varphi$ . In other word, the  $E$ -localization is the left adjoint to the inclusion  $\mathrm{Sp}_E \subset \mathrm{Sp}$  (and the map corresponds to  $\mathrm{id} \in \mathrm{Map}(L_E X, L_E X) \cong \mathrm{Map}(X, L_E X)$ ).

*Remark.* The name localization might be confusing. We will use this mechanism for  $K(p, n)$  which should be though of as a field. Analogously, the  $\mathbb{F}_p$ -localization of  $\mathbb{Z}$  is  $\mathbb{Z}_p$ , i.e. the completion, not the localization (note that we actually want to work in complexes, but this is the result we would get after interpreting  $\langle S \mid R \rangle$  as  $\mathbb{Z} \langle R \rangle \rightarrow \mathbb{Z} \langle S \rangle$ ).

## 2.3 The Thick Subcategory Theorem and $\mathrm{Spc}(\mathrm{Sp}_{(p)}^{\mathrm{fin}})$

Many of the results below can be found at [8, lecture 26]. The Balmer spectrum can be found at [2, corollary 9.5].

**Proposition 12.** *Let  $\mathcal{T}_E = \ker E_* = \left\{ X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid E_* (X \otimes E) \simeq 0 \right\}$  i.e. the  $E$ -acyclics, then  $\mathcal{T}_E$  is thick.*

*Proof.* Clearly  $0 \in \mathcal{T}_E$ . Let be a cofiber sequence



$X \rightarrow Y \rightarrow Z$ , then we get a LES in  $E_*$  homology, in which every space is wrapped by the two others, therefore if two are 0, then so is the third:

$$\cdots \rightarrow E_{m-1}(Z) \rightarrow E_m(X) \rightarrow E_m(Y) \rightarrow E_m(Z) \rightarrow E_{m+1}(Z) \rightarrow \cdots$$

For a retract  $X \rightarrow Y \rightarrow X$ , we get  $E_m(X) \rightarrow E_m(Y) \rightarrow E_m(X)$ , where the middle is 0, and the composition is identity, thus  $E_m(X) = 0$ .  $\square$

This leads us to the following definition.

**Definition 13.** We define  $\mathcal{C}_{\geq n} = \mathcal{T}_{K(p,n-1)}$ , the  $K(p,n-1)$ -acyclics (equivalently  $X \otimes K(p,n-1) \simeq 0$ ). By the above it is thick. Also,  $\mathcal{C}_{\geq 0} = \mathrm{Sp}_{(p)}^{\mathrm{fin}}$  and  $\mathcal{C}_{\geq \infty} = \{0\}$ , which are trivially thick.

**Proposition 14.** For  $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ , if  $K(p,n)_*(X) = 0$  then  $K(p,n-1)_*(X) = 0$ .

*Remark.* This result is not true for any spectrum (e.g. for  $H\mathbb{Q}$  whose  $K(p,n)$  doesn't vanish at  $n = 0$  but does at  $n = 1$ ).

**Definition 15.** We say that a spectrum is of *type  $n$*  (possibly  $\infty$ ), if the first non-zero Morava K-Theory is  $K(p,n)$ .

**Corollary.**  $\mathcal{C}_{\geq n}$  is the full subcategory of finite  $p$ -local spectra of type  $\geq n$  (i.e.  $\left\{X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid \forall m < n : K(p, m)_*(X) \neq 0\right\}$ ). Thus clearly  $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$ .

**Proposition 16.** The inclusion is proper  $\mathcal{C}_{\geq n+1} \subsetneq \mathcal{C}_{\geq n}$ .

**Proposition 17.** If  $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$  is not contractible, then  $X$  has finite height. Therefore  $\bigcap_{n < \infty} \mathcal{C}_{\geq n} = \{0\} = \mathcal{C}_{\geq \infty}$ .

*Proof.*  $X \simeq 0$  iff  $H_*(X; \mathbb{Z}) = 0$  iff  $H_*(X; \mathbb{F}_p) = 0$ . Assume that  $X$  is not contractible, then  $H_*(X; \mathbb{F}_p)$  is bounded (since  $X$  is a finite spectrum), thus for large enough  $n$ , by Atiyah-Hirzebruch SS we have  $K(p, n)_*(X) \cong H_*(X; \mathbb{F}_p)[v_n^{\pm 1}]$ , i.e.  $X$  has finite type. We conclude that  $\bigcap_{n < \infty} \mathcal{C}_{\geq n} = \{0\} = \mathcal{C}_{\geq \infty}$ .  $\square$

**Theorem 18** (Thick Subcategory Theorem [6]). If  $\mathcal{T}$  is a thick subcategory of  $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$ , then  $\mathcal{T} = \mathcal{C}_{\geq n}$  for some  $n = 0, 1, 2, \dots, \infty$ .

*Remark.* The proof relies on a major theorem called the Nilpotence Theorem.

**Proposition 19.**  $\mathcal{C}_{\geq n}$  is a prime ideal (note that  $\mathcal{C}_{\geq 0}$  is not a proper subcategory, thus only for  $n = 1, 2, \dots, \infty$ .)

*Proof.* For  $X, Y$  by Kunneth we have  $K(p, n-1)_*(X \otimes Y) = K(p, n-1)_*(X) \otimes K(p, n-1)_*(Y)$ . Therefore, if  $X \in \mathcal{C}_{\geq n}$ , i.e. the homology vanishes, then so does the homology of  $X \otimes Y$ , i.e.  $X \otimes Y \in \mathcal{C}_{\geq n}$ , so  $\mathcal{C}_{\geq n}$  is an ideal. If  $X \otimes Y \in \mathcal{C}_{\geq n}$  then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces), so  $\mathcal{C}_{\geq n}$  is a prime ideal.  $\square$

**Corollary 20.**  $\text{Spc}(\text{Sp}_{(p)}^{\text{fin}}) = \{\mathcal{C}_{\geq 1}, \mathcal{C}_{\geq 2}, \dots, \mathcal{C}_{\geq \infty}\}$ , and the closed subsets are  $\{\mathcal{C}_{\geq k}, \mathcal{C}_{\geq k+1}, \dots, \mathcal{C}_{\geq \infty}\}$ .

*Remark.* The chromatic picture can be described for all  $\text{Sp}^{\text{fin}}$  at once, which has all the primes above for each  $p$  with the above closed sets, except that all  $\mathcal{C}_{\geq 1}$  for different  $p$  are the same ( $H\mathbb{Q}$ -acyclics.)

## 2.4 Morava E-Theory

*Remark.* There are many approaches and flavors to Morava E-Theory. The one we use is based on [3]

and [11]. See also [4]. Another approach is via deformations of the formal group law of  $K(p, n)$ , this approach can be found at [8].

The results above indicate that  $K(p, n)$  “sees”  $K(p, n)$ . (For example, we had the claim that  $K(p, n)_*(X) = 0 \implies K(p, n-1)_*(X) = 0$  for  $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ , which implied that any open set containing  $\mathcal{C}_{\geq n+1}$  contained  $\mathcal{C}_{\geq n}$ .) The localization  $L_{K(p, n)}$  in some sense (which will be more precise later) determines the  $n$ -th chromatic level, and an infinitesimal neighbourhood around it, which will allow us to glue. We would like to find a spectrum that sees all  $\leq n$  chromatic levels at once.

Remember that  $\mathbb{S}$  is analogous to  $\mathbb{Z}$ ,  $K(p, n)$  is analogous to  $\mathbb{F}_p$ , so  $L_{K(p, n)}$  is analogous to completion at  $p$  (localization at  $\mathbb{F}_p$ ), thus the  $K(p, n)$ -local sphere  $L_{K(p, n)}\mathbb{S}$  is analogous to  $\mathbb{Z}_p = W(\mathbb{F}_p)$ , which indeed sees infinitesimal neighbourhood around  $p$ .

It makes sense to try and investigate its Galois extensions. I will not give a precise definition, and definitely not for a general Galois Extension, but just to give an idea:

**Definition 21** (kind of). Let  $G$  be a finite group,

and  $f : A \rightarrow B$  a map between two  $E_\infty$ -ring spectra s.t.:

1.  $f$  is equivariant w.r.t to the trivial  $G$ -action on  $A$ ,
2.  $A \rightarrow B^{hG}$  is an equivalence,
3.  $B \otimes_A B \rightarrow \bigoplus_G B, x \otimes y \mapsto (xg.y)$  is an equivalence.

Then  $B$  is called a *Galois extension* of  $A$  with Galois group  $G$ .

*Remark.* If we think about extension of (classical) fields, the first condition means that  $G$  acts on  $B$  as automorphisms over  $A$ ,  $B^G \subseteq B$  is always a Galois extension, and the second condition ensures that  $A = B^G$ , the third condition says that  $G$  is actually the Galois groups (it might not act faithfully for example).

It turns out that there is a spectrum called *Morava E-Theory*, denoted by  $E(p, n)$ , which is the maximal Galois extension of  $L_{K(p,n)}\mathbb{S}$  (and the Galois group, which is not finite, is called the Morava stabilizer

group). It has coefficients  $\pi_* E(p, n) \cong W(\overline{\mathbb{F}}_p) \llbracket u_1, \dots \rrbracket$  where  $\deg u_i = 0$ ,  $\deg \beta = 2$ .

The following statement is a formalization of the idea that  $E(p, n)$  sees all  $\leq n$  chromatic levels at once.

**Proposition 22.** *For  $E = K(p, 0) \vee \dots \vee K(p, n)$  and for  $E = E(p, n)$ , being  $E$ -acyclic, being  $E$ -local and  $L_E$  are the same.*

*Remark.* In other words they are *Bousfield equivalent*, and clearly the first implies the rest.

## 2.5 Further Results

The ideas above lead to the idea of studying spectra one prime at a time, height-by-height. We would like to know how to work out the original spectrum.

**Definition 23.** For each  $n$  we have a map  $L_{E(p, n+1)} X \rightarrow L_{E(p, n)} X$ , thus we can form *the chromatic tower*  $\dots \rightarrow L_{E(p, 2)} X \rightarrow L_{E(p, 1)} X \rightarrow L_{E(p, 0)} X$ .

**Theorem 24** (Chromatic Convergence Theorem [8, lecture 32]). *The limit of the chromatic tower is  $X$ .*

**Theorem 25** (Chromatic Square [8, lecture 23]). *There is a pullback diagram:*

$$\begin{array}{ccc} L_{E(p,n)}X & \longrightarrow & L_{K(p,n)}X \\ \downarrow & & \downarrow \\ L_{E(p,n-1)}X & \longrightarrow & L_{E(p,n-1)}L_{K(p,n)}X \end{array}$$

The chromatic square gets its name from another relevant theorem (these theorems go under the name fracture theorems):

**Theorem 26** (Arithmetic Square). *There is a pullback diagram:*

$$\begin{array}{ccc} X & \longrightarrow & \prod L_{S\mathbb{F}_p}X \\ \downarrow & & \downarrow \\ L_{S\mathbb{Q}}X & \longrightarrow & L_{S\mathbb{Q}}\left(\prod L_{S\mathbb{F}_p}X\right) \end{array}$$

(where actually  $L_{S\mathbb{F}_p}X = L_{S\mathbb{F}_p}X_{(p)}$ , so it contains less information than  $X_{(p)}$  [ $X_{(p)} = L_{S\mathbb{Z}_{(p)}}X$  is the  $p$ -localization and  $L_{S\mathbb{F}_p}X$  is the  $p$ -completion]).

# References

- [1] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *arXiv:math/0409360*, 2004.
- [2] P. Balmer. Spectra, spectra, spectra – tensor triangular spectra versus zariski spectra of endomorphism rings. *Algebraic and Geometric Topology* 10, 1521–1563, 2010.
- [3] E. Devinatz and M. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology* 43, no.1, 1-47, 2004.
- [4] H.-W. Henn. A mini-course on Morava stabilizer groups and their cohomology. *arXiv:1702.05033*, 2017.
- [5] M. Hopkins. Global methods in homotopy theory. *Homotopy Theory – Proc. Durham Symp. 1985. Cambridge University Press. Cambridge*, 1987.
- [6] M. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory II. *Annals of Mathematics*, 148(1), second series, 1-49, 1998.



- [7] S. B. Iyengar. Thick subcategories of perfect complexes over a commutative ring. 2006.
- [8] J. Lurie. Chromatic homotopy theory. *252x course notes*, 2010.
- [9] T. Murayama. The classification of thick subcategories and balmer’s reconstruction theorem. 2015.
- [10] A. Neeman. The chromatic tower of  $\mathcal{D}(R)$ . *Topology* 31, 1992.
- [11] J. Rognes. Galois extensions of structured ring spectra. *arXiv:math/0502183*, 2005.