Ambidexterity Seminar – The Chromatic Picture

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1 Motivation – Hopkins-Neeman and Balmer's Spectrum

Two short introductions to the topic are [7, 9] (note that they use the language of triangular categories, rather than ∞-categories). In what follows, R is noetherian ring, $X = \mathrm{Spec}\,(R)$, and $\mathrm{Ch}\,(X)$ is the symmetric monoidal stable ∞-category of chain complexes over R.

Problem. Can we recover X from $\mathrm{Ch}\,(X)$?

The first partial answer to this question is given at [5, 10], later on in [1, 2] the result is further improved, and we will state that version.

Definition 1. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finitely-generated projective modules. These are the compact objects in the category, so they can actually be defined categorically. Denote by $\mathrm{Ch}_{\mathrm{perf}}(X)$ the full subcategory of perfect complexes.

Definition 2. Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

1. $0 \in \mathcal{T}$,
2. let $a \xrightarrow{f} b \to c$ cofiber sequence, if two out of $\{a, b, c\}$ are in $\mathcal{T}$, then so is the third (remember that cofiber and fiber sequences are the same),
3. it is closed under retracts.

Example 3. Consider the case $\mathcal{C} = \mathrm{Ch}_{\mathrm{perf}}(X)$ (e.g. over $\mathbb{Z}$, bounded chain complexes of finitely-generated free abelian groups). Let $K_\bullet \in \mathrm{Ch}(X)$, and define $\mathcal{T}_{K_\bullet} = \{A_\bullet \in \mathrm{Ch}_{\mathrm{perf}}(X) \mid A_\bullet \otimes K_\bullet = 0\}$. Clearly

$0 \in \mathcal{T}_{K_\bullet}$. Since tensor is left, it sends pushout to pushout, and three are 0 so the fourth is 0. Lastly, if $A_\bullet \to B_\bullet \to A_\bullet$ is the identity and $B_\bullet \otimes K_\bullet = 0$ then $\mathrm{id}_{A_\bullet \otimes K_\bullet}$ factors through 0, thus $A_\bullet \otimes K_\bullet = 0$. Therefore $\mathcal{T}_{K_\bullet}$ is thick.

Definition 4. A thick subcategory $\mathcal{T}$ is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring, $\mathrm{Spc}(\mathcal{C}) = \{\mathcal{P} \text{ prime ideal}\}$, and for any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) \mid S \cap \mathcal{P} = \emptyset\}$, and these are the closed subsets of the *Zariski topology* on $\mathrm{Spc}(\mathcal{C})$. We also denote $\mathrm{spp}(a) = V(\{a\})$.

Theorem 5 (Balmer). *There is a homeomorphism* $\varphi : X \to \mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X))$ *given by* $\varphi(\mathfrak{p}) = \Big\{A_\bullet \mid (A_\bullet)_{\mathfrak{p}}$
$\mathcal{T}_{R_\mathfrak{p}}$.

Remark. This was actually upgraded to an isomorphism of locally-ringed spaces.

Proof (sketch). First we note that $\varphi(\mathfrak{p})$ is indeed a prime ideal. It was shown to be thick. It is also clearly an ideal, since $A_\bullet \otimes B_\bullet \otimes R_\mathfrak{p} = A_\bullet \otimes 0 = 0$. Finally, if $0 = (A_\bullet \otimes B_\bullet)_\mathfrak{p} = (A_\bullet)_\mathfrak{p} \otimes_{R_\mathfrak{p}} (B_\bullet)_\mathfrak{p}$. Assume by negation that $(A_\bullet)_\mathfrak{p} \neq 0$ and $(B_\bullet)_\mathfrak{p} \neq 0$, i.e. $(A_n)_\mathfrak{p} \neq 0$ and $(B_m)_\mathfrak{p} \neq 0$ but $(A_n)_\mathfrak{p} \otimes_{R_\mathfrak{p}} (B_m)_\mathfrak{p} = 0$. Well, localization of projective is projective, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, so $(A_n)_\mathfrak{p} = 0$ and $(B_m)_\mathfrak{p} = 0$ which is a contradiction. (Note that I lied, we only know that $(A_\bullet \otimes B_\bullet)_\mathfrak{p}$ is quasi-isomorphic to 0, thus we need to work with maps, the correct proof is similar but messier). Therefore $\varphi(\mathfrak{p})$ is indeed a prime ideal.

Note that

$$\varphi(\mathfrak{p}) \in \operatorname{spp}(A_\bullet) \iff A_\bullet \notin \varphi(\mathfrak{p}) \iff (A_\bullet)_\mathfrak{p} \neq 0 \iff$$

and their complements form bases for the topologies. Thus φ is continuous, and if it is invertible, the inverse is continuous as well. $\square$

2 The Chromatic Picture

Although the category of spectra doesn't arise as the corresponding category for a scheme or a similar gadget, we can still try to “reconstruct the space X” by applying this mechanism, and then try to use this decomposition.

We will concentrate at the p-local spectra, $\mathrm{Sp}_{(p)}$, for some fixed prime. Such localization is a mild operation, and actually all the statements that follow can be stated at the level of all spectra, but it is easier to state them at $\mathrm{Sp}_{(p)}$. We also remind ourselves that the compact objects in spectra are finite spectra $\mathrm{Sp}^{\mathrm{fin}}$, and in p-local spectra they are p-localizations of finite spectra $\mathrm{Sp}^{\mathrm{fin}}_{(p)}$.

2.1 Morava K-Theory

A good reference for this part is [8, lectures 22, 24, 25]

Definition 6. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenus element is invertible, equivalently it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree. An A_∞-ring spectrum E is a *field* if $\pi_* E$ is a graded field.

Proposition 7. *A field E has Kunneth, i.e.* $E_*\left(X \otimes Y\right) \cong$ $E_*\left(X\right) \otimes_{\pi_* E} E_*\left(Y\right)$ *for any spectra* X, Y.

Theorem 8 (Definition)**.** *For each prime p and $n = 1, 2, \ldots,$ there exists a spectrum called* Morava K-Theory of height n, *denoted by $K(p,n)$, which has the following properties:*

- $\pi_* K(p,n) \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$ *where* $\deg v_n = 2(p^n - 1)$,
- *It is a field (and in particular, an A_∞-ring spectrum).*

We also take $K(p,0) = H\mathbb{Q}$ and $K(p,\infty) = H\mathbb{F}_p$ and then we also have:

- *If E is a field, then it has the structure of a $K(p,n)$-module for some p and $n = 0, 1, 2, \ldots, \infty$. In that sense $K(p,n)$ is uniquely determined.*

Example. Remember that K (regular complex K-theory) has $\pi_* K = \mathbb{Z}\left[\beta^{\pm 1}\right]$ where $\deg \beta = 2$. Taking K/p we get a spectrum with homotopy groups $\mathbb{F}_p\left[\beta^{\pm 1}\right]$, and it can be shown that it is a module over $K(p,1)$, and since $\deg v_1 = 2(p-1)$ while $\deg \beta = 2$, K/p is a direct sum of $p-1$ copies of $K(p,1)$.

2.2 Localization at E

A reference for what follows is at [8, lecture 20]. Let E be a spectrum.

Definition 9. A spectrum Z is called *E-acyclic*, if $E_*(Z) = \pi_*(E \otimes Z) = 0$ (i.e. $E \otimes Z \simeq 0$). A spectrum Y is called *E-local*, if $[Z, Y]_* = 0$ (i.e. $\mathrm{Map}(Z, Y) \simeq 0$) for all E-acyclic Z. The E-local spectra form a full subcategory $\mathrm{Sp}_E \subset \mathrm{Sp}$.

Definition 10. Let X be a a spectrum, its *E-localization* is the universal E-local spectrum together with a map $\varphi : X \to L_E X$. I.e. s.t. for each map to an E-local spectrum $f : X \to Y$, there exists a unique $\tilde{f} : L_E X \to Y$ s.t. $f = \tilde{f}\varphi$. In other word, the E-localization is the left adjoint to the inclusion $\mathrm{Sp}_E \subset \mathrm{Sp}$ (and the map corresponds to $\mathrm{id} \in \mathrm{Map}(L_E X, L_E X) \cong \mathrm{Map}(X, L_E X)$).

Remark. The name localization might be confusing. We will use this mechanism for $K(p, n)$ which should be though of as a field. Analogously, the $\mathbb{F}_p$-localization of $\mathbb{Z}$ is $\mathbb{Z}_p$, i.e. the completion, not the localization (note that we actually want to work in complexes, but this is the result we would get after interpreting $\langle S \mid R \rangle$ as $\mathbb{Z}\langle R \rangle \to \mathbb{Z}\langle S \rangle$).

2.3 The Thick Subcategory Theorem and $\mathrm{Spc}\left(\mathrm{Sp}_{(p)}^{\mathrm{fin}}\right)$

Many of the results below can be found at [8, lecture 26]. The Balmer spectrum can be found at [2, corollary 9.5].

Proposition 11. *Let* $\mathcal{T}_E = \ker E_* = \left\{X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid E_*\right.$ *(equivalently* $X \otimes E \simeq 0$*) i.e. the* E*-acyclics, then* $\mathcal{T}_E$ *is thick.*

Proof. The exact same proof from $\mathrm{Ch}_{\mathrm{perf}}(X)$ works. $\square$

This leads us to the following definition.

Definition 12. We define $\mathcal{C}_{\geq n} = \mathcal{T}_{K(p,n-1)}$, the $K(p,n-1)$-acyclics. By the above it is thick. Also, $\mathcal{C}_{\geq 0} = \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ and $\mathcal{C}_{\geq \infty} = \{0\}$, which are trivially thick.

Proposition 13. *For* $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$*, if* $K(p,n)_*(X) = 0$ *then* $K(p,n-1)_*(X) = 0$.

Remark. This result is not true for any spectrum (e.g. for $H\mathbb{Q}$ whose $K(p,n)$-homology doesn't vanish at $n = 0$ but does at $n = 1$).

Definition 14. We say that a spectrum is of *type* n (possibly ∞), if its first non-zero Morava K-Theory-homology is $K(p,n)$.

Corollary. *$\mathcal{C}_{\geq n}$ is the full subcategory of finite p-local spectra of type $\geq n$ (i.e.* $\left\{X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid \forall m < n : K\right.$
Thus clearly $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$.

Proposition 15. *The inclusion is proper $\mathcal{C}_{\geq n+1} \subsetneq \mathcal{C}_{\geq n}$.*

Proposition 16. *If $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ is not contractible, then X has finite type. Therefore $\cap_{n<\infty} \mathcal{C}_{\geq n} = \{0\} = \mathcal{C}_{\geq \infty}$.*

Proof. $X \simeq 0$ iff $H_*(X;\mathbb{Z}) = 0$ iff $H_*(X;\mathbb{F}_p) = 0$. Assume that X is not contractible, then $H_*(X;\mathbb{F}_p)$ is bounded (since X is a finite spectrum), thus for large enough n, by Atiyah-Hirzebruch SS we have $K(p,n)_*(X) \cong H_*(X;\mathbb{F}_p)\left[v_n^{\pm 1}\right]$, i.e. X has finite type. We conclude that $\cap_{n<\infty} \mathcal{C}_{\geq n} = \{0\} = \mathcal{C}_{\geq \infty}$. $\square$

Theorem 17 (Thick Subcategory Theorem [6])**.** *If $\mathcal{T}$ is a thick subcategory of $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$, then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $n = 0, 1, 2, \ldots, \infty$.*

Remark. The proof relies on a major theorem called the Nilpotence Theorem.

Proposition 18. *$\mathcal{C}_{\geq n}$ is a prime ideal (note that $\mathcal{C}_{\geq 0}$ is not a proper subcategory, thus only for $n = 1, 2, \ldots, \infty$).*

Proof. For X, Y by Kunneth we have $K(p, n-1)_*(X$ $K(p, n-1)_*(X) \otimes K(p, n-1)_*(Y)$. Therefore, if $X \in \mathcal{C}_{\geq n}$, i.e. the homology vanishes, then so does the homology of $X \otimes Y$, i.e. $X \otimes Y \in \mathcal{C}_{\geq n}$, so $\mathcal{C}_{\geq n}$ is an ideal. If $X \otimes Y \in \mathcal{C}_{\geq n}$ then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces), so $\mathcal{C}_{\geq n}$ is a prime ideal. □

Corollary 19. $\mathrm{Spc}\left(\mathrm{Sp}_{(p)}^{\mathrm{fin}}\right) = \{\mathcal{C}_{\geq 1}, \mathcal{C}_{\geq 2}, \ldots, \mathcal{C}_{\geq \infty}\}$, *and the closed subsets are* $\{\mathcal{C}_{\geq k}, \mathcal{C}_{\geq k+1}, \ldots, \mathcal{C}_{\geq \infty}\}$.

Remark. The chromatic picture can be described for all $\mathrm{Sp}^{\mathrm{fin}}$ at once, which has all the primes above for each p with the above closed sets, except that all $\mathcal{C}_{\geq 1}$ for different p are the same ($H\mathbb{Q}$-acyclics.)

2.4 Morava E-Theory

Remark. There are many approaches and flavors to Morava E-Theory. The one we use is based on [3] and [11]. See also [4]. Another approach is via deformations of the formal group law of $K(p,n)$, which can found at [8].

The results above indicate that $K(p,n)$ "sees" $K(p,m)$ for $m < n$. (For example, we had the claim that $K(p,n)_*(X) = 0 \implies K(p,m)_*(X) = 0$ for $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$, which implied that any open set that contains $\mathcal{C}_{\geq n+1}$ contains $\mathcal{C}_{\geq m}$ as well). The localization $L_{K(p,n)}$ in some sense (which will be more precise later) determines the n-th chromatic level, and an infinitesimal neighbourhood around it, which will allow us to glue. We would like to find a spectrum that sees all $\leq n$ chromatic levels at once.

Remember that $\mathbb{S}$ is analogous to $\mathbb{Z}$, $K(p,n)$ is analogous to $\mathbb{F}_p$, so $L_{K(p,n)}$ is analogous to completion at p (localization at $\mathbb{F}_p$), thus the $K(p,n)$-local sphere $L_{K(p,n)}\mathbb{S}$ is analogous to $\mathbb{Z}_p = W(\mathbb{F}_p)$, which indeed sees infinitesimal neighbourhood around p.

It makes sense to try an investigate its Galois extensions. I will not give a precise definition, and definitely not for a general Galois Extension, but just to give an idea:

Definition 20 (kind of)**.** Let G be a finite group, and $f : A \to B$ a map between two E_∞-ring spectra s.t.:

1. f is equivariant w.r.t to the trivial G-action on A,
2. $A \to B^{hG}$ is an equivalence,
3. $B \otimes_A B \to \bigoplus_G B, x \otimes y \mapsto (x \cdot g.y)$ is an equivalence.

Then B is called a *Galois extension* of A with Galois group G.

Remark. If we think about extension of (classical) fields, the first condition means that G is acts on B as automorphisms over A, $B^G \subseteq B$ is always a Galois extension, and the second condition ensures that $A = B^G$, the third condition says that G is actually the Galois groups (it might not act faithfully for example).

It turns out that there is a spectrum called *Morava E-Theory of height n*, denoted by $E\,(p, n)$, which is the maximal Galois extension of $L_{K(p,n)}\mathbb{S}$ (and the Galois group, which is not finite but pro-finite, is called the Morava stabilizer group). It has coefficients $\pi_* E(p,n) \cong W\left(\overline{\mathbb{F}}_p\right)[\![u_1,\ldots,u_{n-1}]\!]\left[\beta^{\pm 1}\right]$ where $\deg u_i = 0$, $\deg\beta = 2$.

The following statement is a formalization of the idea that $E(p,n)$ sees all $\leq n$ chromatic levels at once.

Proposition 21. *We have:*

- *Being $E(p,n)$-acyclic and being $K(p,0)\vee\cdots\vee K(p,n)$-acyclic is the same,*
- *Being $E(p,n)$-local and being $K(p,0)\vee\cdots\vee K(p,n)$-local is the same,*
- *$E(p,n)$-localization and $K(p,0)\vee\cdots\vee K(p,n)$-localization coincide.*

Remark. In other words they are *Bousfield equivalent*, and clearly the first implies the rest.

2.5 Further Results

The ideas above lead to the idea of studying spectra one prime at a time, height-by-height. So given a spectrum we would like to know how to work out the original spectrum from its different localizations.

Definition 22. Let $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$. For each n we have a map $L_{E(p,n+1)}X \to L_{E(p,n)}X$, thus we can form *the chromatic tower*:

$$\ldots \to L_{E(p,2)}X \to L_{E(p,1)}X \to L_{E(p,0)}X$$

Theorem 23 (Chromatic Convergence Theorem [8, lecture 32])**.** *The limit of the chromatic tower is X.*

Theorem 24 (Chromatic Square [8, lecture 23])**.** *There is a pullback diagram:*

$$\begin{tikzcd}
L_{E(p,n)}X \arrow[r] \arrow[d] & L_{K(p,n)}X \arrow[d] \\
L_{E(p,n-1)}X \arrow[r] & L_{E(p,n-1)}L_{K(p,n)}X
\end{tikzcd}$$

The chromatic square gets its name from another relevant theorem (these theorems go under the name fracture theorems):

Theorem 25 (Arithmetic Square)**.** *Let $X \in$* Sp.

There is a pullback diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \prod L_{S\mathbb{F}_p} X \arrow[d] \\
L_{S\mathbb{Q}} X \arrow[r] & L_{S\mathbb{Q}} \left(\prod L_{S\mathbb{F}_p} X\right)
\end{tikzcd}$$

(where actually $L_{S\mathbb{F}_p} X = L_{S\mathbb{F}_p} X_{(p)}$, so it contains less information then $X_{(p)}$ [$X_{(p)} = L_{S\mathbb{Z}_{(p)}} X$ is the p-localization and $L_{S\mathbb{F}_p} X$ is the p-completion]).

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