

Ambidexterity Seminar – The Chromatic Picture

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1 Motivation – Hopkins-Neeman and Balmer’s Spectrum

Two short introductions to the topic are [7, 9] (note that they use the language of triangular categories, rather than ∞ -categories). In what follows, R is noetherian ring, $X = \mathrm{Spec}(R)$, and $\mathrm{Ch}(X)$ is the symmetric monoidal stable ∞ -category of chain complexes over R .

Problem. Can we recover X from $\mathrm{Ch}(X)$?

The first partial answer to this question is given at [5, 10], later on in [1, 2] the result is further improved, and we will state that version.

Definition 1. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finite projective modules. These are the compact objects in the category, so that they can actually be defined categorically. Denote by $\mathrm{Ch}_{\mathrm{perf}}(X)$ the full subcategory of perfect complexes.

Definition 2. Let \mathcal{C} be a symmetric monoidal stable ∞ -category. A full subcategory \mathcal{T} is *thick* if:

1. $0 \in \mathcal{T}$
2. let $a \xrightarrow{f} b \rightarrow c$ cofiber sequence, if two out of $\{a, b, c\}$ are in \mathcal{T} , then so is the third (remember that cofiber and fiber sequences are the same)
3. it is closed under retracts

Example 3. Considering the case of $\mathrm{Ch}(X)$ and $\mathrm{Ch}_{\mathrm{perf}}(X)$ (e.g. over \mathbb{Z} , chain complexes of abelian groups, and those with finitely-many non-zero entries, each of which is \mathbb{Z} to some power, respectively). Let $K_{\bullet} \in \mathrm{Ch}(X)$, and define $\mathcal{T}_{K_{\bullet}} = \{A_{\bullet} \in \mathrm{Ch}_{\mathrm{perf}}(X) \mid A_{\bullet} \otimes K_{\bullet} = 0\}$. Clearly $0 \in \mathcal{T}_{K_{\bullet}}$. Since tensor is left, it sends pushout to pushout, and three are 0 so the fourth is 0. Lastly, if $A_{\bullet} \rightarrow B_{\bullet} \rightarrow A_{\bullet}$ is the identity and $B_{\bullet} \otimes K_{\bullet} = 0$ then $\mathrm{id}_{A_{\bullet} \otimes K_{\bullet}}$ factors through 0, thus $A_{\bullet} \otimes K_{\bullet} = 0$. Therefore $\mathcal{T}_{K_{\bullet}}$ is thick.

Definition 4. A thick subcategory \mathcal{T} is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring, $\mathrm{Spc}(\mathcal{C}) = \{\mathcal{P} \text{ prime ideal}\}$, and for any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) \mid S \cap \mathcal{P} = \emptyset\}$, and these are the closed subsets of the *Zariski topology* on $\mathrm{Spc}(\mathcal{C})$. We also denote $\mathrm{spp}(a) = V(\{a\})$.

Theorem 5 (Balmer). *There is a homeomorphism $\varphi : X \rightarrow \mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X))$ given by $\varphi(\mathfrak{p}) = \{A_\bullet \mid (A_\bullet)_{\mathfrak{p}} = 0\} = \mathcal{T}_{R_{\mathfrak{p}}}$.*

Remark. This was actually upgraded to an isomorphism of locally-ringed spaces.

Proof (sketch). First we note that $\varphi(\mathfrak{p})$ is indeed a prime ideal. It was shown to be thick. It is also clearly an ideal, since $A_\bullet \otimes B_\bullet \otimes R_{\mathfrak{p}} = A_\bullet \otimes 0 = 0$. Finally, if $0 = (A_\bullet \otimes B_\bullet)_{\mathfrak{p}} = (A_\bullet)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (B_\bullet)_{\mathfrak{p}}$. Assume by negation that $(A_\bullet)_{\mathfrak{p}} \neq 0$ and $(B_\bullet)_{\mathfrak{p}} \neq 0$, i.e. $(A_n)_{\mathfrak{p}} \neq 0$ and $(B_m)_{\mathfrak{p}} \neq 0$ but $(A_n)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (B_m)_{\mathfrak{p}} = 0$. Well, localization of projective is projective, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, so $(A_n)_{\mathfrak{p}} = 0$ and $(B_m)_{\mathfrak{p}} = 0$ which is a contradiction. Therefore $\varphi(\mathfrak{p})$ is indeed a prime ideal.

Note that

$$\varphi(\mathfrak{p}) \in \mathrm{spp}(A_\bullet) \iff A_\bullet \notin \varphi(\mathfrak{p}) \iff (A_\bullet)_{\mathfrak{p}} \neq 0 \iff \mathfrak{p} \in \mathrm{supp}(A_\bullet)$$

and their complements form bases for the topologies. Thus φ is continuous, and if it is invertible, the inverse is continuous as well. \square

Example 6. The case $R = \mathbb{Z}$. Clearly, if $A_\bullet \in \mathcal{T}_{\mathbb{Z}_{(p)}}$ then $A_\bullet \in \mathcal{T}_{\mathbb{Q}}$, thus any S that doesn't intersect $\mathcal{T}_{\mathbb{Q}}$ doesn't intersect any $\mathcal{T}_{\mathbb{Z}_{(p)}}$, so a closed set that contains $\mathcal{T}_{\mathbb{Q}}$ contains all the others. This is in accordance with the theorem, indeed $p\mathbb{Z} \mapsto \{A_\bullet \mid (A_\bullet)_{p\mathbb{Z}} = 0\} = \{A_\bullet \mid A_\bullet \otimes \mathbb{Z}_{(p)} = 0\} = \mathcal{T}_{\mathbb{Z}_{(p)}}$ and $\mathbb{Z} \mapsto \mathcal{T}_{\mathbb{Z}_{(0)}} = \mathcal{T}_{\mathbb{Q}}$ are prime ideals, and we recovered the topology on $\mathrm{Spc}(\mathrm{Ch}_{\mathrm{perf}}(X)) = \{\mathcal{T}_{\mathbb{Z}_{(2)}}, \mathcal{T}_{\mathbb{Z}_{(3)}}, \dots, \mathcal{T}_{\mathbb{Q}}\}$. Note that the support of an element is all the prime ideals to which it *does not* belong, e.g. $\mathcal{T}_{\mathbb{Z}_{(q)}} \in \mathrm{spp}(\mathbb{F}_p)$ iff $\mathbb{F}_p \notin \mathcal{T}_{\mathbb{Z}_{(q)}}$ iff $\mathbb{F}_p \otimes \mathbb{Z}_{(q)} \neq 0$ which is only when $q = p$, so $\mathrm{spp}(\mathbb{F}_p) = \{\mathcal{T}_{\mathbb{Z}_{(p)}}\}$ as we'd expect.

2 The Chromatic Picture

Although the category of spectra doesn't arise as the corresponding category for a scheme or a similar gadget, we can still try to "reconstruct the space X " by applying this mechanism, and then try to use this decomposition.

We will concentrate at the p -local spectra, $\mathrm{Sp}_{(p)}$, for some fixed prime. Such localization is a mild operation, and actually all the statements that follow can be stated at the level of all spectra, but it is easier to state them at $\mathrm{Sp}_{(p)}$. We also remind ourselves that the compact objects are finite spectra.

2.1 Morava K-Theory

A good reference for this part is [8, lectures 22, 24]

Definition 7. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenous is invertible, equivalently it is a field F concentrated at degree 0, or $F[\beta^{\pm 1}]$ for β of positive even degree. An A_∞ -ring spectrum E is a *field* if $\pi_* E$ is a field.

Proposition 8. A field E has *Kunneth*, i.e. $E_*(X \otimes Y) \cong E_*(X) \otimes_{\pi_* E} E_*(Y)$ for any spectra X, Y .

Theorem 9 (Definition). For each prime p and $n = 1, 2, \dots$, there exists a spectrum called Morava K-Theory of height n , denoted by $K(p, n)$, which has the following properties:

- $\pi_* K(p, n) \cong \mathbb{F}_p[v_n^{\pm 1}]$ where $\deg v_n = 2(p^n - 1)$.
- It is a field (and in particular, an A_∞ -ring spectrum).
- If E is a field, then it has the structure of a $K(p, n)$ -module for unique p and n . In that sense $K(p, n)$ is uniquely determined.

We also take $K(p, 0) = H\mathbb{Q}$.

Example. Remember that K (regular complex K-theory) has $\pi_*(K) = \mathbb{Z}[\beta^{\pm 1}]$ where $\deg \beta = 2$. Taking K/p we get a spectrum with homotopy groups $\mathbb{F}_p[\beta^{\pm 1}]$, and it can be shown that it is a module over $K(p, 1)$, and since $\deg v_1 = 2(p - 1)$ while $\deg \beta = 2$, K/p is a direct sum of $p - 1$ copies of $K(p, 1)$.

2.2 Localization at E

A reference for what follows is at [8, lecture 20]. Let E be a spectrum.

Definition 10. A spectrum Z is called *E -acyclic*, if $E_*(Z) = \pi_*(E \otimes Z) = 0$ (i.e. $E \otimes Z \simeq 0$). A spectrum Y is called *E -local*, if $[Z, Y]_* = 0$ (i.e. equivalently $\text{Map}(Z, Y) \simeq 0$) for all E -acyclic Z . The E -local spectra form a full subcategory $\text{Sp}_E \subset \text{Sp}$.

Definition 11. Let X be a spectrum, its *E -localization* is the universal E -local spectrum together with a map $\varphi : X \rightarrow L_E X$. I.e. s.t. for each map to an E -local spectrum $f : X \rightarrow Y$, there exists a unique $\tilde{f} : L_E X \rightarrow Y$ s.t. $f = \tilde{f}\varphi$. In other word, the E -localization is the left adjoint to the inclusion $\text{Sp}_E \subset \text{Sp}$ (and the map corresponds to $\text{id} \in \text{Map}(L_E X, L_E X) \cong \text{Map}(X, L_E X)$).

Remark 12. The name localization might be confusing. We will use this mechanism for $K(p, n)$ which should be thought of as a field. Analogously, the \mathbb{F}_p -localization of \mathbb{Z} is \mathbb{Z}_p , i.e. the completion, not the localization (note that we actually want to work in complexes, but this is the result we would get after interpreting $\langle S \mid R \rangle$ as $\mathbb{Z}\langle R \rangle \rightarrow \mathbb{Z}\langle S \rangle$).

2.3 The Thick Subcategory Theorem and $\mathrm{Spc}(\mathrm{Sp}_{(p)}^{\mathrm{fin}})$

Many of the results below can be found at [8, lecture 26]. The Balmer spectrum can be found at [2, corollary 9.5].

Proposition 13. *Let $\mathcal{T}_E = \ker E_* = \left\{ X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid E_*(X) = 0 \right\}$ (equivalently $X \otimes E \simeq 0$) i.e. the E -acyclics, then \mathcal{T}_E is thick.*

Proof. Clearly $0 \in \mathcal{T}_E$. Let be a cofiber sequence $X \rightarrow Y \rightarrow Z$, then we get a LES in E_* homology, in which every space is wrapped by the two others, therefore if two are 0, then so is the third:

$$\cdots \rightarrow E_{m-1}(Z) \rightarrow E_m(X) \rightarrow E_m(Y) \rightarrow E_m(Z) \rightarrow E_{m+1}(X) \rightarrow \cdots$$

For a retract $X \rightarrow Y \rightarrow Y$, we get $E_m(X) \rightarrow E_m(Y) \rightarrow E_m(X)$, where the middle is 0, and the composition is identity, thus $E_m(X) = 0$. \square

This leads us to the following definition.

Definition 14. We define $\mathcal{C}_{\geq n} = \mathcal{T}_{K(p, n-1)}$, the $K(p, n-1)$ -acyclics (equivalently $X \otimes K(p, n-1) \simeq 0$). By the above it is thick. Also, $\mathcal{C}_{\geq 0} = \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ and $\mathcal{C}_{\geq \infty} = \{0\}$, which are trivially thick.

Proposition 15. *For $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$, if $K(p, n)_*(X) = 0$ then $K(p, n-1)_*(X) = 0$.*

Remark. This result is not true for any spectrum (e.g. for $H\mathbb{Q}$ whose $K(p, n)$ doesn't vanish at $n = 0$ but does at $n = 1$), and the fact that it doesn't vanish is important and has to do with Morava E-Theory and the way different levels glue.

Definition 16. We say that a spectrum is of *type n* (possibly ∞), if the first non-zero Morava K-Theory is $K(p, n)$.

Corollary. $\mathcal{C}_{\geq n}$ is the full subcategory of finite p -local spectra of type $\geq n$. Thus clearly $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$.

Proposition 17. *The inclusion is proper $\mathcal{C}_{\geq n+1} \subsetneq \mathcal{C}_{\geq n}$.*

Proposition 18. *If $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ is not contractible, then X has finite height. Therefore $\bigcap_{n < \infty} \mathcal{C}_{\geq n} = \{0\} = \mathcal{C}_{\geq \infty}$.*

Proof. $X \simeq 0$ iff $H_*(X; \mathbb{Z}) = 0$ iff $H_*(X; \mathbb{F}_p) = 0$. Assume that X is not contractible, then $H_*(X; \mathbb{F}_p)$ is bounded (since X is a finite spectrum), thus for large enough n , by Atiyah-Hirzebruch SS we have $K(p, n)_*(X) \cong H_*(X; \mathbb{F}_p)[v_n^{\pm 1}]$, i.e. X has finite type. We conclude that $\bigcap_{n < \infty} \mathcal{C}_{\geq n} = \{0\} = \mathcal{C}_{\geq \infty}$. \square

Theorem 19 (Thick Subcategory Theorem [6]). *If \mathcal{T} is a thick subcategory of $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$, then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $n = 0, 1, 2, \dots, \infty$.*

Remark. The proof relies on a major theorem called the Nilpotence Theorem.

Proposition 20. $\mathcal{C}_{\geq n}$ is a prime ideal (note that $\mathcal{C}_{\geq 0}$ is not a proper subcategory, thus only for $n = 1, 2, \dots, \infty$.)

Proof. For X, Y by Kunneth we have $K(p, n-1)_*(X \otimes Y) = K(p, n-1)_*(X) \otimes K(p, n-1)_*(Y)$. Therefore, if $X \in \mathcal{C}_{\geq n}$, i.e. the homology vanishes, then so does the homology of $X \otimes Y$, i.e. $X \otimes Y \in \mathcal{C}_{\geq n}$, so $\mathcal{C}_{\geq n}$ is an ideal. If $X \otimes Y \in \mathcal{C}_{\geq n}$ then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces), so $\mathcal{C}_{\geq n}$ is a prime ideal. \square

Corollary 21. $\text{Spc}(\text{Sp}_{(p)}^{\text{fin}}) = \{\mathcal{C}_{\geq 1}, \mathcal{C}_{\geq 2}, \dots, \mathcal{C}_{\geq \infty}\}$, and the closed subsets are $\{\mathcal{C}_{\geq k}, \mathcal{C}_{\geq k+1}, \dots, \mathcal{C}_{\geq \infty}\}$.

Remark. The chromatic picture can be described for all Sp^{fin} at once, which has all the primes above for each p with the above closed sets, except that all $\mathcal{C}_{\geq 1}$ for different p are the same ($H\mathbb{Q}$ -acyclics.)

2.4 Morava E-Theory

Remark. There are many approaches to Morava E-Theory. The one we use is based on [3]. See also [4]. A more standard approach is via deformations of the formal group law of $K(p, n)$, this approach can be found at [8].

The results above indicate that $K(p, n)$ “sees” $K(p, n-1)$. (For example, we had the claim that $K(p, n)_*(X) = 0 \implies K(p, n-1)_*(X) = 0$ for $X \in \text{Sp}_{(p)}^{\text{fin}}$. And first the is in the latter’s closure.) In some sense (which will be more precise later), $K(p, n)$ determines the n -th chromatic level. We would like to find a spectrum that sees all $\leq n$ chromatic levels at once.

Remember that \mathbb{S} is analogous to \mathbb{Z} , and $L_{K(p, n)}$ is analogous to completion at p (localization at \mathbb{F}_p), so the $K(p, n)$ -local sphere $L_{K(p, n)}\mathbb{S}$ is analogous to \mathbb{Z}_p , and it makes sense to try and investigate its Galois extensions. I will not give a precise definition, and definitely not for a general Galois Extension, but just to give an idea:

Definition 22 (kind of). Let G be a finite group, and $f : A \rightarrow B$ a map between two E_{∞} -ring spectra s.t.:

1. f is equivariant w.r.t to the trivial G -action on A ,
2. $A \rightarrow B^{hG}$ is an equivalence,
3. $B \otimes_A B \rightarrow \bigoplus_G B, x \otimes y \mapsto (x\sigma_i(y))$ is an equivalence.

Then B is called a *Galois extension* of A with Galois group G .

Remark. If we think about extension of (classical) fields, the first condition means that G acts on B as automorphisms over A , $B^G \subseteq B$ is always a Galois extension, and the second condition ensures that $A = B^G$, the third condition says that G is actually the Galois groups (it might not act faithfully for example).

It turns out that there is a spectrum called *Morava E -Theory*, denoted by $E(p, n)$, which is the maximal Galois extension of $L_{K(p, n)}\mathbb{S}$ (and the Galois group, which is not finite, is called the Morava stabilizer group). It has coefficients $\pi_* E(p, n) \cong W(\mathbb{F}_{p^n})[[v_1, \dots, v_{n-1}]][\beta^{\pm 1}]$.

The following statement is a formalization of the idea that $E(p, n)$ sees all $\leq n$ chromatic levels at once.

Proposition 23. *For $E = K(p, 0) \vee \dots \vee K(p, n)$ and for $E = E(p, n)$, being E -acyclic, being E -local and L_E are the same.*

Remark. In other words they are *Bousfield equivalent*, and clearly the first implies the rest.

2.5 Further Results

The ideas above lead to the idea of studying spectra one prime at a time, height-by-height. We would like to know how to work out the original spectrum.

Definition 24. For each n we have a map $L_{E(p, n+1)}X \rightarrow L_{E(p, n)}X$, thus we can form the *chromatic tower* $\dots \rightarrow L_{E(p, 2)}X \rightarrow L_{E(p, 1)}X \rightarrow L_{E(p, 0)}X$.

Theorem 25 (Chromatic Convergence Theorem [8, lecture 32]). *The limit of the chromatic tower is X .*

Theorem 26 (Chromatic Square [8, lecture 23]). *There is a pullback diagram:*

$$\begin{array}{ccc} L_{E(p, n)}X & \longrightarrow & L_{K(p, n)}X \\ \downarrow & & \downarrow \\ L_{E(p, n-1)}X & \longrightarrow & L_{E(p, n-1)}L_{K(p, n)}X \end{array}$$

The chromatic square gets its name from another relevant theorem (these theorems go under the name fracture theorems):

Theorem 27 (Arithmetic Square). *There is a pullback diagram:*

$$\begin{array}{ccc} X & \longrightarrow & \prod L_{S\mathbb{F}_p}X \\ \downarrow & & \downarrow \\ L_{S\mathbb{Q}}X & \longrightarrow & L_{S\mathbb{Q}}(\prod L_{S\mathbb{F}_p}X) \end{array}$$

(where actually $L_{S\mathbb{F}_p}X = L_{S\mathbb{F}_p}X_{(p)}$, so it contains less information than $X_{(p)}$ [$X_{(p)} = L_{S\mathbb{Z}_{(p)}}X$ is the p -localization and $L_{S\mathbb{F}_p}$ is the p -completion]).

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