

Thesis

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1 Overview of Chromatic Homotopy Theory

Our goal is to motivate the introduction of Morava K-theory $K(n)$ and Morava E-theory $E(n)$, and other variants of Morava E-theory $E(k, \Gamma)$, and their connection to formal group laws. There are different views on what chromatic homotopy theory is. **TODO write some more**

1.1 The Balmer Spectrum

We will start with an algebraic motivation. Let R be a noetherian ring. Consider the symmetric monoidal stable ∞ -category $\mathrm{Ch}(R)$ of chain complexes on R . **TODO be more specific** It is then natural to ask how much information about R is encoded in the category $\mathrm{Ch}(R)$. We will try to recover $\mathrm{Spec} R$, as a topological space, from $\mathrm{Ch}(R)$.

Remark 1.1.1. Balmer's work actually recovers the structure sheaf as well. **TODO reference**

Definition 1.1.2. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finitely-generated projective modules. These objects are the compact objects in $\mathrm{Ch}(R)$, thus they can be defined categorically. Their full subcategory is denoted by $\mathrm{Ch}_{\mathrm{perf}}(R)$.

Definition 1.1.3. Let \mathcal{C} be some symmetric monoidal stable ∞ -category. A full subcategory \mathcal{T} is *thick* if:

- $0 \in \mathcal{T}$,
- it is closed under cofibers,
- it is closed under retracts.

Example 1.1.4. Consider the case $\mathcal{C} = \mathrm{Ch}_{\mathrm{perf}}(R)$ (e.g. over \mathbb{Z} , bounded chain complexes of finitely-generated free abelian groups). Let $K \in \mathrm{Ch}(R)$, and define $\mathcal{T}_K = \{A \in \mathrm{Ch}_{\mathrm{perf}}(R) \mid A \otimes K = 0\}$. We claim that \mathcal{T}_K is thick.

Clearly $0 \in \mathcal{T}_K$. Let $A \rightarrow B$ be a morphism between two complexes in \mathcal{T} . The cofiber of $A \rightarrow B$ is the pushout $B \times_A 0$. Since tensor is left, tensoring the cofiber with K is given by the pushout $(B \otimes K) \times_{A \otimes K} (0 \otimes K) = 0 \times_0 0 = 0$, therefore the cofiber is indeed in \mathcal{T} . Lastly, if $A \rightarrow B \rightarrow A$ is the identity and $B \otimes K$, we get that $\text{id}_{A \otimes K}$ factors through 0, which implies that $A \otimes K$ is 0, so that $A \in \mathcal{T}$.

Definition 1.1.5. A thick subcategory \mathcal{T} is an *ideal* if $A \in \mathcal{T}, B \in \mathcal{C} \implies A \otimes B \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $A \otimes B \in \mathcal{T} \implies A \in \mathcal{T}$ or $B \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring: As a set, $\text{Spec } \mathcal{C} = \{\mathcal{P} \text{ prime ideal}\}$. For any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \text{Spec } \mathcal{C} \mid S \cap \mathcal{P} = \emptyset\}$. We topologize $\text{Spec } \mathcal{C}$ with the Zariski topology by declaring those to be the closed subsets. We also denote $\text{Supp}(A) = V(\{A\})$.

Example 1.1.6. We continue the example of \mathcal{T}_K . Clearly if $A \otimes K = 0$ then also $A \otimes B \otimes K = 0$, so it is an ideal. Let \mathfrak{p} be a prime ideal in R in the usual sense, and take $K = R_{\mathfrak{p}}$ (concentrated at degree 0), then $A \otimes K = A_{\mathfrak{p}}$ (level-wise localization). **TODO consider actually proving** We will omit the proof that \mathcal{T}_K is a prime, but we shall prove something weaker, namely only the case where A, B are bounded complexes of finitely generate projective modules (and not merely quasi-isomorphic to such complexes). Assume then that $0 = (A \otimes B)_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} B_{\mathfrak{p}}$. Assume by negation that $A_{\mathfrak{p}}, B_{\mathfrak{p}} \neq 0$, i.e. $(A_n)_{\mathfrak{p}}, (B_m)_{\mathfrak{p}} \neq 0$ but $(A_n)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (B_m)_{\mathfrak{p}} = 0$ for some n, m . Well, localization of projective is projective, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, so $(A_n)_{\mathfrak{p}} = 0$ or $(B_m)_{\mathfrak{p}} = 0$, which is a contradiction. Therefore $\mathcal{T}_{\mathfrak{p}}$ is a prime ideal.

Theorem 1.1.7. *The map $\text{Spec } R \rightarrow \text{Spec}(\text{Ch}_{\text{perf}}(R))$, given by $\mathfrak{p} \mapsto \mathcal{T}_{\mathfrak{p}} = \{A \mid A_{\mathfrak{p}} = 0\}$ is a homeomorphism.*

TODO reference

Proposition 1.1.8. *Prime ideals pullback: Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a reduced symmetric monoidal functor that preserves cofibers, between two symmetric monoidal stable ∞ -categories, and let \mathcal{P} be a prime ideal in \mathcal{D} , then $F^*\mathcal{P} = \{A \in \mathcal{C} \mid F(A) \in \mathcal{P}\}$ is a prime ideal.*

Proof. Clearly $F(0) = 0 \in \mathcal{P}$ since F is reduced, so $0 \in F^*\mathcal{P}$. Since F preserves cofibers, for $A, B \in F^*\mathcal{P}$, i.e. $F(A), F(B) \in \mathcal{P}$, and a map $A \rightarrow B$ we get $F(\text{cofib}(A \rightarrow B)) = \text{cofib}(F(A) \rightarrow F(B)) = \text{cofib}(F(A) \rightarrow F(B)) \in \mathcal{P}$. Let $A \rightarrow B \rightarrow A$ be a retract, that is the composition is the identity, s.t. $B \in F^*\mathcal{P}$. We know that $F(A) \rightarrow F(B) \rightarrow F(A)$ is also a retract by functoriality, thus $F(A) \in \mathcal{P}$, that is $A \in F^*\mathcal{P}$. We conclude that $F^*\mathcal{P}$ is indeed a thick subcategory.

Let $A \in F^*\mathcal{P}, B \in \mathcal{C}$, since F is monoidal, $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{P}$, so $A \otimes B \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is an ideal.

Lastly, assume that $A \otimes B \in F^*\mathcal{P}$, again since F is monoidal, $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{P}$, so $A \in F^*\mathcal{P}$ or $B \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is a prime ideal. \square

Now, recall that $\text{Ch}(R) \cong \text{Mod}_{\text{HR}}$, therefore we can reinterpret the above theorem as $\text{Spec } R \cong \text{Spec}(\text{Mod}_{\text{HR}}^{\text{comp}})$ (where the comp denotes the compact objects in the category). We shall turn this theorem into a definition:

Definition 1.1.9. Let R be an \mathbb{E}_{∞} -ring. We define the *spectrum* of R to be $\text{Spec } R = \text{Spec}(\text{Mod}_R^{\text{comp}})$.

A natural question to ask then is what is $\text{Spec } \mathbb{S}$. Recall that $\text{Mod}_{\mathbb{S}} = \text{Sp}$, the category of spectra, and that the compact objects in spectra are the finite spectra Sp^{fin} . So, unwinding the definitions, the question can be rephrased as finding the prime ideals in Sp^{fin} , and their topology. Chromatic homotopy theory provides an answer to this question.

1.2 MU and Complex Orientations

Throughout this section, let E be a multiplicative cohomology theory (that is, equipped with a map $E \otimes E \rightarrow E$ and $1 \in E_0$, which is associative and unital after taking homotopy groups).

Consider the map $S^2 \rightarrow \text{BU}(1)$ classifying the universal complex line bundle. Concretely, under the identifications $S^2 \cong \mathbb{CP}^1$ and $\text{BU}(1) \cong \mathbb{CP}^{\infty}$, this map can be realized as the inclusion $\mathbb{CP}^1 \subseteq \mathbb{CP}^{\infty}$. This map induces a map $\tilde{E}^2(\text{BU}(1)) \rightarrow \tilde{E}^2(S^2) \cong \tilde{E}^0(S^0) \cong E^0(*) = E_0$. Since E is unital, there is a canonical generator $1 \in E_0$.

Definition 1.2.1. E is called *complex oriented* if the map $\tilde{E}^2(\mathrm{BU}(1)) \rightarrow E_0$ is surjective, equivalently, if 1 is in the image of that map. A choice of a lift $x \in \tilde{E}^2(\mathrm{BU}(1))$ of $1 \in E_0$ is called a *complex orientation*.

Example 1.2.2. Let R be some ring, and consider HR . It is known that $HR^*(\mathbb{CP}^n) \cong R[x]/(x^{n+1})$ and $HR^*(\mathbb{CP}^\infty) \cong R[[x]]$, where $|x| = 2$, and the maps induced by the inclusions of projective spaces maps x to x . Therefore we see that $x \in HR^2(\mathrm{BU}(1))$ is mapped to $x \in HR^2(S^2) = \mathbb{Z}\{x\}$, which is mapped to the generator of the reduced part of $HR^0(S^0) = R \oplus R$. Therefore x is a complex orientation for HR .

Example 1.2.3. Let K be complex K-theory, then we know that $K_* = \mathbb{Z}[\beta^{\pm 1}]$ where β is the Bott element, with $|\beta| = 2$. It is also known (by Atiyah-Hirzebruch spectral sequence) that $K^*(\mathbb{CP}^n) \cong K_*[t]/(t^{n+1})$ and $K^*(\mathbb{CP}^\infty) \cong K_*[[t]]$ (here $|t| = 0$), where the maps induced by the inclusions of projective spaces maps t to t . We deduce that $\beta t \in K^2(\mathrm{BU}(1))$ is mapped to $\beta t \in K^2(S^2) = \mathbb{Z}\{\beta t\}$, which is mapped to $t \in K^0(S^0) = \mathbb{Z}\{t\}$, which is indeed the generator of the reduced part. Therefore $x = \beta t$ is complex orientation for K . **TODO write the reduced thing more clearly**

Example 1.2.4. Recall that MU is constructed as the colimit $\mathrm{MU} = \mathrm{colim} \mathrm{MU}(n)$. Also, $\mathrm{MU}(1) \cong \Sigma^{\infty-2}\mathrm{BU}(1)$. Therefore we get a canonical map $\Sigma^{\infty-2}\mathrm{BU}(1) \rightarrow \mathrm{MU}$, which gives a cohomology class $x_{\mathrm{MU}} \in \mathrm{MU}^2(\mathrm{BU}(1))$.

Proposition 1.2.5 ([Rav86, 4.1.3]). x_{MU} is a complex orientation for MU .

Theorem 1.2.6. MU is the universal complex oriented cohomology theory: Let E be a multiplicative cohomology theory, then there is a bijection between (homotopy classes of) multiplicative maps $\mathrm{MU} \rightarrow E$ and complex orientations on E . The bijection is given in one direction by pulling back x_{MU} along a multiplicative map.

Assume that E is complex oriented with a complex orientation x .

Theorem 1.2.7 ([Rav86, 4.1.4]). As E_* -algebras, $E^*(\mathrm{BU}(1)) \cong E^*[[x]]$ and $E^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) \cong E^*[[y, z]]$.

TODO maybe indicate the use of AHSS

The tensor product of complex line bundles is classified by a universal map $\mathrm{BU}(1) \times \mathrm{BU}(1) \rightarrow \mathrm{BU}(1)$. Therefore we get a map $E^*(\mathrm{BU}(1)) \rightarrow E^*(\mathrm{BU}(1) \times \mathrm{BU}(1))$, which is completely determined by the image of $x \in E^*[[x]]$ in $E^*[[y, z]]$ as above. Therefore, a choice of a complex orientation on E gives rise to an element $F_E(y, z) \in E^*[[y, z]]$.

Proposition 1.2.8 ([Rav86, 4.1.4]). F_E is a formal group law on E_* . We call the height of F_E the height of E .

Example 1.2.9. Consider again HR . It is known that the tensor of complex line bundles induces the map $R[[x]] = HR^*(\mathrm{BU}(1)) \rightarrow HR^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = R[[y, z]]$ given by $x \mapsto y + z$. This is the additive formal group law. It is immediate that $[p] = px$. So for $R = \mathbb{Q}$ we get that the height of $H\mathbb{Q}$ is 0, while for $R = \mathbb{F}_p$ we have $px = 0$ so the height of $H\mathbb{F}_p$ is ∞ .

Example 1.2.10. We return to the example of complex K-theory 1.2.3. It is known that the tensor of complex line bundles induces the map $K_*[[t]] = K^*(\mathrm{BU}(1)) \rightarrow K^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = K_*[[u, v]]$ given by $t \mapsto u + v + uv$. Note that to comply with the definition of the formal group law, we should use the isomorphism $K^*(\mathrm{BU}(1)) \cong K_*[[x]]$, i.e. the element $x = \beta t$. We get that $x = \beta t \mapsto \beta u + \beta v + \beta uv = y + z + \beta^{-1}yz = F_K(y, z)$. By induction we prove that the n -series is $[n](x) = \beta(1 + \beta^{-1}x)^n - \beta$. This is clear for $n = 1$, and we have:

$$\begin{aligned} [n+1](x) &= x + [n](x) + \beta^{-1}x[n](x) \\ &= x + \beta(1 + \beta^{-1}x)^n - \beta + x(1 + \beta^{-1}x)^n - x \\ &= \beta(1 + \beta^{-1}x)(1 + \beta^{-1}x)^n - \beta \\ &= \beta(1 + \beta^{-1}x)^{n+1} - \beta \end{aligned}$$

TODO consider discussing the computation of $\mathrm{BU}(1)$, maybe as part of complex K-theory example?

Example 1.2.11. By taking the cofiber of the multiplication-by- p map, we get a spectrum K/p , mod- p K-theory, with coefficients $(K/p)_* = \mathbb{F}_p[\beta^{\pm 1}]$. It is evident that $F_{K/p}(y, z) = y + z + \beta^{-1}yz$ as well. From the result above, it follows that $[p](x) = \beta(1 + \beta^{-1}x)^p - \beta = \beta(1^p + \beta^{-p}x^p) - \beta = \beta^{-(p-1)}x^p$ which shows that the height is exactly 1.

A formal group law on E_* is the same data as a map from the Lazard ring L , so the complex orientation gives a map $L \rightarrow E_*$. In particular, since MU is complex oriented, there is a canonical map $L \rightarrow \text{MU}_*$.

Theorem 1.2.12 (Quillen, [Rav86, 4.1.6]). *The canonical map $L \rightarrow \text{MU}_*$ is an isomorphism.*

1.3 BP, Morava K-Theory and Morava E-Theory

A good principle in homotopy theory (and many other areas in math) is to do study it one prime at a time. This is possible in homotopy theory due to the arithmetic square **TODO reference**. So, let us fix a prime p . We can form $\text{MU}_{(p)}$, the p -localization of MU .

Theorem 1.3.1 ([Ada74, II 15]). *There exists a map of ring spectra $\varepsilon : \text{MU}_{(p)} \rightarrow \text{MU}_{(p)}$ (which depends on the prime p), which is an idempotent $\varepsilon^2 = \varepsilon$ (moreover, once the action on homotopy groups is specified, it is unique).*

The map $\varepsilon : \text{MU}_{(p)} \rightarrow \text{MU}_{(p)}$ gives a cohomology operation, for every X we have $\varepsilon^* : \text{MU}_{(p)}^*(X) \rightarrow \text{MU}_{(p)}^*(X)$. Denote by $\text{BP}_{(p)}^*(X)$ the image of ε^* .

Theorem 1.3.2 ([Ada74, II 16], [Rav86, 4.1.12]). *BP is a cohomology theory, represented by an associative commutative ring spectrum BP (which depends on the prime p), which is a retract of $\text{MU}_{(p)}$. The homotopy groups of BP are $\text{BP}_* = \mathbb{Z}_{(p)}[v_1, \dots]$ where $|v_n| = 2(p^n - 1)$.*

For convenience we denote $v_0 = p$ (and indeed $|v_0| = 2(p^0 - 1) = 0$). Since BP is a retract of MU it comes with a map $\text{MU} \rightarrow \text{BP}$, that is a complex orientation.

Proposition 1.3.3 (TODO reference). *The p -series of the formal group law associated to BP is $[p](x) = \sum v_n x^{p^n}$.*

Once we have BP, we can turn to the definition of Morava K-theory and Morava E-theory

Definition 1.3.4. Let $0 < n < \infty$. *Morava K-theory at height n and prime p , denoted by $K(p, n)$ or $K(n)$ when the prime is clear, is the spectrum obtained by killing $p = v_0, \dots, v_{n-1}, v_{n+1}, \dots$ in BP and inverting v_n . Therefore $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$. We also define $K(0) = \text{H}\mathbb{Q}$ and $K(\infty) = \text{H}\mathbb{F}_p$. Similarly, *Morava E-theory at height n and prime p , denoted by $E(p, n)$ or $E(n)$, is the spectrum obtained by killing v_{n+1}, v_{n+2}, \dots in BP and inverting v_n . Therefore $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$.**

Since Morava K-theory and E-theory are obtained from BP by cofibers and filtered colimits, they are equipped with a map from BP, so they are also equipped with a complex orientation. It is then evident that the p -series associated to the formal group laws of $K(n)$ and $E(n)$ are $v_n x^{p^n}$ and $v_0 x + \dots v_n x^{p^n}$ respectively, and are therefore of height exactly n and height $\leq n$ respectively. (Note that by the example of $\text{H}R$, this is also true for $K(0)$ and $K(\infty)$.)

We want to describe some properties of Morava K-theory. To do so we first need some definitions.

Definition 1.3.5. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenous element is invertible, equivalently it is a field F concentrated at degree 0, or $F[\beta^{\pm 1}]$ for β of positive even degree. An \mathbb{A}_∞ -ring E is a *field* if E_* is a graded field.

Example 1.3.6. Clearly $K(n)$ for $0 \leq n \leq \infty$ is a field.

Proposition 1.3.7. *A field E has Kunneth, i.e. $E_*(X \otimes Y) \cong E_*(X) \otimes_{E_*} E_*(Y)$ for any spectra X, Y .*

Proposition 1.3.8 ([Lur10, 24]). *Let $E \neq 0$ be a complex oriented cohomology theory, whose formal group law has height exactly n , then $E \otimes K(n) \neq 0$. Let E be a field s.t. $E \otimes K(n) \neq 0$, then E admits the structure of a $K(n)$ -module. (Here $0 \leq n \leq \infty$.)*

Example 1.3.9. As we have seen before mod- p K-theory, K/p , has height exactly 1 and coefficients $(K/p)_* = \mathbb{F}_p[\beta^{\pm 1}]$. It is also known that K , and K/p , are A_∞ ring spectra, from which it follows that K/p is a field. Therefore, we deduce that K/p is a $K(1)$ -module. Since $|\beta| = 2$ and $|v_1| = 2(p - 1)$ it is free of rank $p - 1$.

From this we also deduce some form of uniqueness for Morava K-theory:

Corollary 1.3.10. *Let E be a field with $E_* \cong \mathbb{F}_p[v_n^{\pm 1}]$, which is also complex oriented with height exactly n , then $E \cong K(n)$ (as spectra).*

1.4 $\text{Spec } \mathbb{S}_{(p)}$ and $\text{Spec } \mathbb{S}$

We are now in a position to state the answer for $\text{Spec } \mathbb{S}$. However, it will be easier to state it first for $\text{Spec } \mathbb{S}_{(p)}$, and then pullback prime ideals. We know that $\text{Mod}_{\mathbb{S}_{(p)}} = \text{Sp}_{(p)}$, and the compact objects there are $\text{Sp}_{(p)}^{\text{fin}}$, the p -localizations of finite spectra.

Proposition 1.4.1. *Let $\mathcal{T}_E = \ker E_* = \left\{ X \in \text{Sp}_{(p)}^{\text{fin}} \mid E_*(X) = 0 \right\}$ (equivalently $X \otimes E = 0$) i.e. the E -acyclics, then \mathcal{T}_E is thick.*

Proof. The exact same proof from $\text{Ch}_{\text{perf}}(R)$ works. \square

Definition 1.4.2. We define $\mathcal{C}_{p,n} = \mathcal{T}_{K(n)}$, the $K(n)$ -acyclics. By the above it is thick. Also, $\mathcal{C}_{p,\infty} = \{0\}$, which are trivially thick. When the prime is clear, we will denote by \mathcal{C}_n .

Proposition 1.4.3 ([Lur10, 26]). *For $X \in \text{Sp}_{(p)}^{\text{fin}}$, if $K(n)_*(X) = 0$ then $K(n-1)_*(X) = 0$.*

Definition 1.4.4. We say that a spectrum $X \in \text{Sp}_{(p)}^{\text{fin}}$ is of *type n* (possibly ∞), if its first non-zero Morava K -theory homology is $K(n)$.

Corollary 1.4.5. \mathcal{C}_n is the full subcategory of finite p -local spectra of type $> n$ (i.e. $\left\{ X \in \text{Sp}_{(p)}^{\text{fin}} \mid \forall m \leq n : K(m)_*(X) = 0 \right\}$). Thus clearly $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$.

Proposition 1.4.6 (TODO reference). *The inclusions are proper $\mathcal{C}_{n+1} \subsetneq \mathcal{C}_n$.*

Proposition 1.4.7. *If $X \in \text{Sp}_{(p)}^{\text{fin}}$ is not contractible, then X has a finite type. Therefore $\bigcap_{n < \infty} \mathcal{C}_n = \{0\} = \mathcal{C}_\infty$.*

Proof. Let X be non-contractible. Then $\text{HZ}_*(X) \neq 0$. Let m be the first non-zero degree. Using the universal coefficient theorem and the fact that the spectrum is p -local we get that $(\text{HF}_p)_m(X) \neq 0$, thus $(\text{HF}_p)_*(X) \neq 0$. Since X is finite, $(\text{HF}_p)_*(X)$ is bounded. Atiyah-Hirzebruch spectral sequence for X with cohomology $K(n)$ has E^2 -page given by $E_{p,q}^2 = H_p(X; K(n)_q)$. Since $K(n)_q = \mathbb{F}_p$ for $q = 0 \pmod{2(p^n - 1)}$ and 0 otherwise, we see that the rows $q = 0 \pmod{2(p^n - 1)}$ are $(\text{HF}_p)_*(X)$, and the others are 0. Therefore if we take n such that the period $2(p^n - 1)$ is larger then the bound on $(\text{HF}_p)_*(X)$, then all differentials have either source or target 0. Thus, the spectral sequence collapses at the E^2 -page, and since $(\text{HF}_p)_*(X) \neq 0$, we get that $K(n)_*(X) \neq 0$, i.e. X has type $< n$. \square

Proposition 1.4.8. \mathcal{C}_n is a prime ideal.

Proof. For X, Y by Kunneth we have $K(n-1)_*(X \otimes Y) = K(n-1)_*(X) \otimes K(n-1)_*(Y)$. Therefore, if $X \in \mathcal{C}_n$, i.e. the homology vanishes, then so does the homology of $X \otimes Y$, i.e. $X \otimes Y \in \mathcal{C}_n$, so \mathcal{C}_n is an ideal. If $X \otimes Y \in \mathcal{C}_n$ then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces), so \mathcal{C}_n is a prime ideal. \square

Theorem 1.4.9 (Thick Subcategory Theorem [HS98]). *If \mathcal{T} is a proper thick subcategory of $\text{Sp}_{(p)}^{\text{fin}}$, then $\mathcal{T} = \mathcal{C}_n$ for some $0 \leq n \leq \infty$.*

Remark 1.4.10. The proof relies on a major theorem called the Nilpotence Theorem.

Corollary 1.4.11. $\text{Spec } \mathbb{S}_{(p)} = \{\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_\infty\}$, and the topology is such that the closed subsets are chains $\{\mathcal{C}_k, \mathcal{C}_{k+1}, \dots, \mathcal{C}_\infty\}$ for some $0 \leq k \leq \infty$.

Proof. Follows immediately from the previous results. \square

We now want to move to finding $\text{Spec } \mathbb{S}$. Note that the p -localization functor $L_{(p)}$ is a Bousfield localization. As such it is left (its right adjoint is the inclusion), and in particular preserves cofibers. It also clearly sends 0 to 0, i.e. reduced. Now, since $L_{(p)}$ is smashing, i.e. $L_{(p)}X = X \otimes \mathbb{S}_{(p)}$, we also get that it is symmetric monoidal. As we have seen in 1.1.8, under these conditions we can pullback primes. Therefore $\mathcal{P}_{p,n} = L_{(p)}^* \mathcal{C}_{p,n} = \left\{ X \in \text{Sp}^{\text{fin}} \mid K(n)_*(X_{(p)}) = 0 \right\}$ and $\mathcal{P}_{p,\infty} = L_{(p)}^* \mathcal{C}_{p,\infty} = \left\{ X \in \text{Sp}^{\text{fin}} \mid X_{(p)} = 0 \right\}$ are prime ideals. Note that

$\mathcal{P}_{p,0} = \left\{ X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_*(X_{(p)}) = 0 \right\} = \left\{ X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_*(X) = 0 \right\}$ **TODO explain** so it is independent of p , and we will denote it by $\mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}}$.

Theorem 1.4.12 (TODO explain/reference). $\mathrm{Spec} \mathbb{S} = \left\{ \mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}} \right\} \cup \bigcup_p \{ \mathcal{P}_{p,1}, \dots, \mathcal{P}_{p,\infty} \}$, and the topology is such that the closed subsets finite unions of chains $\{ \mathcal{P}_{p,k}, \mathcal{P}_{p,k+1}, \dots, \mathcal{P}_{p,\infty} \}$ for some $0 \leq k \leq \infty$ (i.e. they may include $\mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}}$). **TODO diagram**

TODO regarding the topology, maybe I should prove that the pullback is also continuous?

1.5 The Stacky Point of View and the Relationship Between Morava K-Theory and Morava E-Theory

First we will describe, without being precise, another point of view on what chromatic homotopy theory is about.

There is a stack of formal groups, denoted by $\mathcal{M}_{\mathrm{fg}}$. It can be described as the stack that sends a ring R to the groupoid of formal group laws, with isomorphisms between them. Quillen theorem 1.2.12 tells us that MU_* is the Lazard ring, that is the universal ring that carries the universal formal group law. It turns out that this theorem has a second part, which says that $(\mathrm{MU} \otimes \mathrm{MU})_*$ is the universal ring that carries two formal group laws and an isomorphism between them. Therefore, $\mathcal{M}_{\mathrm{fg}}$ is represented by $(\mathrm{MU}_*, (\mathrm{MU} \otimes \mathrm{MU})_*)$.

The geometric points of the stack $\mathcal{M}_{\mathrm{fg}}$ describe precisely the same as $\mathrm{Spec} \mathbb{S}$, that is because for an algebraically closed field of characteristic 0 there is a unique (up to isomorphism) formal group law which is of height 0 namely the additive formal group law, and for characteristic p there is a unique (up to isomorphism) formal group law of each height $1 \leq n \leq \infty$.

For a spectrum X , $\mathrm{MU}_*(X)$ is a $(\mathrm{MU}_*, (\mathrm{MU} \otimes \mathrm{MU})_*)$ -comodule, which is the same as a sheaf over $\mathcal{M}_{\mathrm{fg}}$. From this point of view, chromatic homotopy theory lets us study a spectrum by decomposing it over the stack $\mathcal{M}_{\mathrm{fg}}$.

We can restrict ourselves to the stack only over rings of characteristic p , $\mathcal{M}_{\mathrm{fg},p}$, which is then represented by $((\mathrm{MU}_{(p)})_*, (\mathrm{MU}_{(p)} \otimes \mathrm{MU}_{(p)})_*)$. **TODO I think it's true, is it?** Similarly to MU , BP is universal ring with the universal p -typical formal group law, and $(\mathrm{BP} \otimes \mathrm{BP})_*$ is the universal ring with two p -typical formal group laws and an isomorphism between them **TODO I didn't say this before**. Since every formal group law is isomorphic to a p -typical one, we know that the stack $\mathcal{M}_{\mathrm{fg},p}$ is also represented by $(\mathrm{BP}_*, (\mathrm{BP} \otimes \mathrm{BP})_*)$.

It is now reasonable that $\mathrm{K}(n)$, obtained from BP by killing the v_m 's for $m \neq n$ and inverting v_n , sees the n -th level, and that $\mathrm{E}(n)$ obtained in the same way but only killing v_m for $m > n$, sees the levels $\leq n$. **TODO I didn't interpret the v_n 's before as coefficients in the p -series which determines the height**

Let us now claim a precise statement, formalizing this description.

Theorem 1.5.1 (TODO reference). $\mathrm{E}(n)$ and $\mathrm{K}(0) \oplus \dots \oplus \mathrm{K}(n)$ are Bousfield equivalent. That is, they have the same acyclics, locals, and their localization functors are the same.

TODO chromatic square and chromatic tower, maybe another subsection?

1.6 Landweber Exact Functor Theorem

As we have seen, a complex orientation on a cohomology theory, which is described by a map $\mathrm{MU} \rightarrow E$, has an associated formal group law, which is described by the map $L = \mathrm{MU}_* \rightarrow E_*$. Note that this formal group law is of degree -2 , by virtue of the grading on $L = \mathrm{MU}_*$ **TODO say more?**. One can ask whether the converse is true, namely given a graded ring R and a formal group law F of degree -2 given by $L \rightarrow R$, is there a complex oriented cohomology theory whose coefficients are R and the associated formal group law is F .

A strategy is to define $(E_{R,F})_*(X) = \mathrm{MU}_*(X) \otimes_{\mathrm{MU}_*} R$. Unfortunately, this is not always a homology theory. However there is a condition which one can check which guarantees that it is.

Definition 1.6.1. $L \rightarrow R$ is called *Landweber flat* if for every prime p , the sequence $p = v_0, v_1, v_2, \dots$ is regular, that is, for each n , v_n is not a zero divisor in $R/(v_0, v_1, \dots, v_{n-1})$.

Theorem 1.6.2 (Landweber Exact Functor Theorem (LEFT), [Lur10, 15, 16]). *If $L \rightarrow R$ is Landweber flat, then $E_{R,F}$ defined above is a homology theory. Moreover, there are no phantom maps between such spectra, so $E_{R,F}$ is represented by a spectrum. This spectrum is complex oriented, has coefficients R and associated formal group law F .*

Example 1.6.3. We return to complex K-theory, from 1.2.3 and 1.2.10. We can take the completion at the element $p \in K_*$, which gives the spectrum K_p^\wedge . This spectrum has coefficients $(K_p^\wedge)_* = (K_*)_p^\wedge = (\mathbb{Z}[\beta^{\pm 1}])_p^\wedge = \mathbb{Z}_p^\wedge[\beta^{\pm 1}]$. The formal group law, as we have seen, is given by $F_{K_p^\wedge}(y, z) = y + z + \beta^{-1}yz$. We claim that $F_{K_p^\wedge}/K_p^\wedge$ is Landweber flat. Clearly $p = v_0$ is invertible in $\mathbb{Z}_p^\wedge[\beta^{\pm 1}]$, thus not a zero divisor. As we have seen in 1.2.11, mod- p the p -series is $\beta^{-(p-1)}x^p$, so that $v_1 = \beta^{-(p-1)}$ which is not a zero divisor. Modulo v_1 the ring is already 0, and we are done. Therefore, by 1.6.5 we get that $K_p^\wedge \cong E_{K_p^\wedge, F_{K_p^\wedge}}$.

Example 1.6.4. Morava E-theory is Landweber flat. Morava K-theory and $H\mathbb{Z}$ are not. **TODO elaborate? reference?**

We can also ask the following question: given complex oriented cohomology theory $MU \rightarrow E$, such that $L \rightarrow E_*$ is Landweber flat, is $E_{R,F}$ equivalent to E ? The answer is yes, at least in some cases.

Theorem 1.6.5. *Let E be as above, which is also evenly graded (i.e. E_* is an evenly graded ring), then there is an equivalence $E_{R,F} \rightarrow E$.*

Proof. This is a slight variation on [Lur10, 18, proposition 11]. First note that for every spectrum X we have $MU \otimes X \rightarrow E \otimes X$, which induces $MU_*(X) \rightarrow E_*(X)$, a map of MU_* -modules. Moreover, since $E_* \rightarrow E_*(X)$ is a map of E_* -module, the map $MU_* \rightarrow E_*$ makes it a map of MU_* -modules. Together this gives a map $(E_{R,F})_*(X) = MU_*(X) \otimes_{MU_*} E_* \rightarrow E_*(X)$. This map is a map of homology theories. **TODO should I explain why?** By [Lur10, 17, theorem 6] **TODO does it follow?**, this map lifts to a map of spectra $E_{R,F} \rightarrow E$. Since by construction when $X = \mathbb{S}$ the map above is $E_* \rightarrow E_*$ which is an isomorphism, we see that the map $E_{R,F} \rightarrow E$ is an equivalence. \square

1.7 Lubin-Tate Deformation Theory

The Morava E-theory we have considered until now $E(n)$, also called Johnson-Wilson spectrum was constructed from BP. As we noted, it is Landweber flat, which indicates that there is another approach to constructing it. Indeed there is a way to construct a related spectrum, which will be called the Lubin-Tate spectrum.

To that end, we first define the category CompRing as the category of complete local rings. The objects are complete local rings (R, \mathfrak{m}) , we also denote by $\pi : R \rightarrow R/\mathfrak{m}$ the projection. Morphisms $\varphi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ are local homomorphisms, i.e. a homomorphism $\varphi : R \rightarrow S$ s.t. $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. In particular it induces a homomorphism $\varphi/\mathfrak{m} : R/\mathfrak{m} \rightarrow S/\mathfrak{n}$, which satisfies $\varphi/\mathfrak{m} \circ \pi_R = \pi_S \circ \varphi$.

We fix k be a perfect field of characteristic p (i.e. the Frobenius is an isomorphism), and Γ a formal group law over k of height $n < \infty$. Lubin and Tate [LT] considered a moduli problem associated to Γ/k , described by a functor $\text{Def} : \text{CompRing} \rightarrow \text{Grpds}$.

Definition 1.7.1. Let (R, \mathfrak{m}) be a complete local ring and denote by $\pi : R \rightarrow R/\mathfrak{m}$ the quotient. A *deformation* of Γ/k to (R, \mathfrak{m}) , is (G, i) , where G is a formal group law over R , $i : k \rightarrow R/\mathfrak{m}$ is a homomorphism of fields, such that $i^*\Gamma = \pi^*G$. A \star -*isomorphism* between two deformations to (R, \mathfrak{m}) , $f : (G_1, i_1) \rightarrow (G_2, i_2)$, is defined only when $i_1 = i_2$, and consists of an isomorphism $f : G_1 \rightarrow G_2$, such that $\pi^*f : i^*\Gamma = \pi^*G_1 \rightarrow \pi^*G_2 \rightarrow i^*\Gamma$ is the identity, i.e. $f(x) = x \pmod{\mathfrak{m}}$. These assemble to a groupoid $\text{Def}(R, \mathfrak{m})$, whose objects are deformations to (R, \mathfrak{m}) , and morphisms are \star -isomorphisms.

Remark 1.7.2. $\text{Def}(R, \mathfrak{m})$ can be seen as the pullback of the groupoids $\text{FGL}(R)$ and $\coprod_{i:k \rightarrow R/\mathfrak{m}} \{\Gamma\}$ over $\text{FGL}(R/\mathfrak{m})$ (where the maps are $G \mapsto \pi^*G$ and $i \mapsto i^*\Gamma$ respectively).

Proposition 1.7.3 (/definition). *The construction $\text{Def}(R, \mathfrak{m})$ is functorial.*

Proof. Let $\varphi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a local homomorphism.

For a deformation (G, i) to (R, \mathfrak{m}) , we define $\text{Def}(\varphi)(G, i) = (\varphi^*G, \varphi/\mathfrak{m} \circ i)$. Note that φ^*G is a formal group law over S , and $\varphi/\mathfrak{m} \circ i : k \rightarrow R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is a homomorphism. Moreover, $(\varphi/\mathfrak{m} \circ i)^*\Gamma = (\varphi/\mathfrak{m})^*i^*\Gamma =$

$(\varphi/\mathfrak{m})^* \pi_R^* G = (\varphi/\mathfrak{m} \circ \pi_R)^* G = (\pi_S \circ \varphi)^* G = \pi_S^* \varphi^* G$, which shows that $\text{Def}(\varphi)(G, i)$ is a deformation to (S, \mathfrak{n}) .

For a \star -isomorphism $f : (G_1, i) \rightarrow (G_2, i)$, which is the data of an isomorphism $f : G_1 \rightarrow G_2$ such that $\pi_R^* f = \text{id}_{i^* \Gamma}$ is the identity, we need to define a \star -isomorphism $\text{Def}(\varphi)(G, i_1) \rightarrow \text{Def}(\varphi)(G, i_2)$. Take it to be the isomorphism $\varphi^* f : \varphi^* G_1 \rightarrow \varphi^* G_2$, which satisfies $\pi_S^* \varphi^* f = (\varphi/\mathfrak{m})^* \pi_R^* f = (\varphi/\mathfrak{m})^* \text{id}_{i^* \Gamma} = \text{id}_{(\varphi/\mathfrak{m})^* i^* \Gamma} = \text{id}_{(\varphi/\mathfrak{m} \circ i)^* \Gamma}$. The identity $\text{id}_G : (G, i) \rightarrow (G, i)$ is clearly sent to $\text{id}_{\varphi^* G}$, and compositions are sent to compositions.

This shows that $\text{Def}(\varphi) : \text{Def}(R, \mathfrak{m}) \rightarrow \text{Def}(S, \mathfrak{n})$ is indeed a functor. Moreover, it is clear that $\text{Def}(\text{id}_R)$ is the identity and compositions are sent to compositions, which shows that $\text{Def} : \text{CompRing} \rightarrow \text{Grpds}$ is indeed a functor. \square

Theorem 1.7.4 ([Rez, 4.4, 5.10], originally due to [LT]). *The functor Def lands in discrete groupoids (i.e. $\text{Def}(R, \mathfrak{m})$ has 0 or 1 morphisms between objects). Furthermore the functor Def is co-represented, that is there exists a universal deformation, and the complete local ring can be chosen (non-canonically) to be $Wk[[u_1, \dots, u_{n-1}]]$.*

Let us unravel what that means. First note that the quotient of $Wk[[u_1, \dots, u_{n-1}]]$ by the maximal ideal (p, u_1, \dots, u_{n-1}) is k . The universal deformation can be chosen such that the formal group law over it Γ_U over $Wk[[u_1, \dots, u_{n-1}]]$ satisfies $\pi^* \Gamma_U$ is Γ . The universality means that for (R, \mathfrak{m}) , the assignment

$$\text{hom}_{\text{CompRing}}(Wk[[u_1, \dots, u_{n-1}]], R) \rightarrow \text{Def}(R, \mathfrak{m}), \quad \varphi \mapsto \varphi^* \Gamma_U$$

is an equivalence.

TODO more about the witt vectors?

Now, we can form the graded ring $Wk[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$ where $|u| = 2$. We can define the formal group law $(u\Gamma_U)(x, y) = u^{-1} \Gamma_U(uy, uz)$, which is of degree -2 , thus we get a map $L \rightarrow Wk[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$.

Proposition 1.7.5 ([Rez, 6.9]). *$L \rightarrow Wk[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$ is Landweber flat.*

Using LEFT 1.6.2, we immediately get:

Corollary 1.7.6. *There is a complex oriented cohomology theory $E(k, \Gamma) = E_{Wk[[u_1, \dots, u_{n-1}]] [u^{\pm 1}], u\Gamma_U}$, called Lubin-Tate spectrum, with coefficients $E(k, \Gamma)_* = Wk[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$ and associated formal group law $u\Gamma_U$.*

Example 1.7.7. We continue the complex K-theory saga from 1.6.3. Take the field $k = \mathbb{F}_p$ and the formal group law $\Gamma(y, z) = y + z + yz$, of height $n = 1$. By the above construction, the ring of the universal deformation is $W\mathbb{F}_p = \mathbb{Z}_p^\wedge$. The universal formal group law of the universal deformation is again $\Gamma_U(y, z) = y + z + yz$. **TODO show** We look at the ring $\mathbb{Z}_p^\wedge [u^{\pm 1}]$, and at the formal group law over it $(u\Gamma_U)(y, z) = u^{-1} (uy + uz + u^2 yz) = y + z + uyz$. It is clear that the isomorphism $\mathbb{Z}_p^\wedge [u^{\pm 1}] \rightarrow \mathbb{Z}_p^\wedge [\beta^{\pm 1}]$, sends $u\Gamma_U$ to $F_{K_p^\wedge}$. It follows that $K_p^\wedge \cong E_{K_p^\wedge, F_{K_p^\wedge}} \cong E(\mathbb{F}_p, \Gamma)$, i.e. p -complete K-theory K_p^\wedge is a Lubin-Tate spectrum at height 1.

This concludes the construction of the Lubin-Tate variant of Morava E-theory. To compare this version with the previous, we define another variant of Morava E-theory.

Definition 1.7.8. Let $I \leq E(n)_* = \mathbb{Z}_{(p)} [v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ be the ideal $I = (p = v_0, v_1, \dots, v_{n-1})$. We define the spectrum $\widehat{E(n)} = E(n)_I^\wedge$, called completed Johnson-Wilson spectrum.

Theorem 1.7.9. *The following three forms of Morava E-theory are Bousfield equivalent:*

1. Johnson-Wilson $E(n)$,
2. completed Johnson-Wilson $\widehat{E(n)}$,
3. Lubin-Tate $E(k, \Gamma)$.

TODO reference Hovey Strickland Morava K-Theories and Localization 5.3

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