Thesis

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1 Overview of Chromatic Homotopy Theory

Our goal is to motivate the introduction of Morava K-theory $\mathrm{K}(n)$ and Morava E-theory $\mathrm{E}(n)$, and other variants of Morava E-theory $\mathrm{E}(k, \Gamma)$, and their connection to formal group laws. There are different views on what chromatic homotopy theory is. **TODO write some more**

1.1 The Balmer Spectrum

We will start with an algebraic motivation. Let R be a noetherian ring. Consider the symmetric monoidal stable ∞-category $\mathrm{Ch}(R)$ of chain complexes on R. **TODO be more specific** It is then natural to ask how much information about R is encoded in the category $\mathrm{Ch}(R)$. We will try to recover $\operatorname{Spec} R$, as a topological space, from $\mathrm{Ch}(R)$.

Remark 1.1.1. Balmer's work actually recovers the structure sheaf as well. **TODO reference**

Definition 1.1.2. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finitely-generated projective modules. These objects are the compact objects in $\mathrm{Ch}(R)$, thus they can be defined categorically. Their full subcategory is denoted by $\mathrm{Ch}_{\mathrm{perf}}(R)$.

Definition 1.1.3. Let $\mathcal{C}$ be some symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

- $0 \in \mathcal{T}$,
- it is closed under cofibers,

- it is closed under retracts.

Example 1.1.4. Consider the case $\mathcal{C} = \mathrm{Ch}_{\mathrm{perf}}(R)$ (e.g. over $\mathbb{Z}$, chain complexes quasi-isomorphic to bounded chain complexes of finitely-generated free abelian groups). Let $K \in \mathrm{Ch}(R)$, and define $\mathcal{T}_K = \left\{A \in \mathrm{Ch}_{\mathrm{perf}}(R) \mid A \otimes K \cong 0\right\}$. We claim that $\mathcal{T}_K$ is thick. Clearly $0 \in \mathcal{T}_K$. Let $A \to B$ be a morphism between two complexes in $\mathcal{T}$. Since tensor is left, tensoring the cofiber with K is given by $\mathrm{cofib}(A \to B) \otimes K \cong \mathrm{cofib}(A \otimes K \to B \otimes K) \cong \mathrm{cofib}(0 \to 0) \cong 0$, therefore the cofiber is indeed in $\mathcal{T}_K$. Lastly, if $A \to B \to A$ is the identity and $B \otimes K \cong 0$, we get that $\mathrm{id}_{A \otimes K}$ factors through 0, which implies that $A \otimes K$ is 0, so that $A \in \mathcal{T}_K$.

Definition 1.1.5. A thick subcategory $\mathcal{T}$ is an *ideal* if $A \in \mathcal{T}, B \in \mathcal{C} \implies A \otimes B \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $A \otimes B \in \mathcal{T} \implies A \in \mathcal{T}$ or $B \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring: As a set, $\operatorname{Spec} \mathcal{C} = \{\mathcal{P} \text{ prime ideal}\}$. For any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \operatorname{Spec} \mathcal{C} \mid S \cap \mathcal{P} = \emptyset\}$. We topologize $\operatorname{Spec} \mathcal{C}$ with the Zariski topology by declaring those to be the closed subsets. We also denote $\mathrm{Supp}(A) = V(\{A\})$.

Example 1.1.6. We continue the example of $\mathcal{T}_K$. Clearly if $A \otimes K \cong 0$ then also $A \otimes B \otimes K \cong 0$, so it is an ideal. Let $\mathfrak{p}$ be a prime ideal in R in the usual sense, and take $K = R_{\mathfrak{p}}$ (concentrated at degree 0), then $A \otimes K = A_{\mathfrak{p}}$ (levelwise localization). **TODO consider actually proving** We will omit the proof that $\mathcal{T}_K$ is a prime, but we shall prove something weaker, namely only the case where A, B are bounded complexes of finitely generate projective modules (and not merely quasi-isomorphic to such complexes). Assume then that $0 = (A \otimes B)_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} B_{\mathfrak{p}}$. Assume by negation that $A_{\mathfrak{p}}, B_{\mathfrak{p}} \neq 0$, i.e. $(A_n)_{\mathfrak{p}}, (B_m)_{\mathfrak{p}} \neq 0$ but $(A_n)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (B_m)_{\mathfrak{p}} = 0$ for some n, m. Since the localization of a projective module is again a projective module, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, it follows that $(A_n)_{\mathfrak{p}} = 0$ or $(B_m)_{\mathfrak{p}} = 0$, which is a contradiction. Therefore $\mathcal{T}_{\mathfrak{p}}$ is a prime ideal.

Theorem 1.1.7. *The map* $\operatorname{Spec} R \to \operatorname{Spec}\left(\mathrm{Ch}_{\mathrm{perf}}(R)\right)$*, given by* $\mathfrak{p} \mapsto \mathcal{T}_{\mathfrak{p}} = \{A \mid A_{\mathfrak{p}} = 0\}$ *is a homeomorphism.*

TODO reference

Proposition 1.1.8. *Prime ideals pullback: Let* $F : \mathcal{C} \to \mathcal{D}$ *be a reduced symmetric monoidal functor that preserves cofibers, between two symmetric* *monoidal stable ∞-categories, and let $\mathcal{P}$ be a prime ideal in $\mathcal{D}$, then $F^*\mathcal{P} = \{A \in \mathcal{C} \mid F(A) \in \mathcal{P}\}$ is a prime ideal.*

Proof. Clearly $F(0) = 0 \in \mathcal{P}$ since F is reduced, so $0 \in F^*\mathcal{P}$. Since F preserves cofibers, for $A, B \in F^*\mathcal{P}$, i.e. $F(A), F(B) \in \mathcal{P}$, and a map $A \to B$ we get $F(\mathrm{cofib}(A \to B)) = \mathrm{cofib}(F(A) \to F(B)) = \mathrm{cofib}(F(A) \to F(B)) \in \mathcal{P}$. Let $A \to B \to A$ be a retract, that is the composition is the identity, s.t. $B \in F^*\mathcal{P}$. We know that $F(A) \to F(B) \to F(A)$ is also a retract by functoriality, thus $F(A) \in \mathcal{P}$, that is $A \in F^*\mathcal{P}$. We conclude that $F^*\mathcal{P}$ is indeed a thick subcategory.

Let $A \in F^*\mathcal{P}, B \in \mathcal{C}$, since F is monoidal, $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{P}$, so $A \otimes B \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is an ideal.

Lastly, assume that $A \otimes B \in F^*\mathcal{P}$, again since F is monoidal, $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{P}$, so $A \in F^*\mathcal{P}$ or $B \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is a prime ideal. $\square$

Now, recall that $\mathrm{Ch}(R) \cong \mathrm{Mod}_{\mathrm{H}R}$, therefore we can reinterpret the above theorem as $\mathrm{Spec}\, R \cong \mathrm{Spec}\left(\mathrm{Mod}_{\mathrm{H}R}^{\mathrm{comp}}\right)$ (where the comp denotes the compact objects in the category). We shall turn this theorem into a definition:

Definition 1.1.9. Let R be an $\mathbb{E}_\infty$-ring. We define the *spectrum* of R to be $\mathrm{Spec}\, R = \mathrm{Spec}\left(\mathrm{Mod}_R^{\mathrm{comp}}\right)$.

A natural question to ask then is what is the topological space $\mathrm{Spec}\, \mathbb{S}$. Recall that $\mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$, the category of spectra, and that the compact objects in spectra are the finite spectra $\mathrm{Sp}^{\mathrm{fin}}$. So, unwinding the definitions, the question can rephrased as finding the prime ideals in $\mathrm{Sp}^{\mathrm{fin}}$, and their topology. Chromatic homotopy theory provides an answer to this question.

1.2 MU and Complex Orientations

Throughout this section, let E be a multiplicative cohomology theory (that is, equipped with a map $E \otimes E \to E$ and $1 \in E_0$, which is associative and unital after taking homotopy groups).

Consider the map $S^2 \to \mathrm{BU}(1)$ classifying the universal complex line bundle. Concretely, under the identifications $S^2 \cong \mathbb{C}\mathrm{P}^1$ and $\mathrm{BU}(1) \cong \mathbb{C}\mathrm{P}^\infty$, this map can be realized as the inclusion $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^\infty$. This map induces a map

$$\tilde{E}^2(\mathrm{BU}(1)) \to \tilde{E}^2\left(S^2\right) \cong \tilde{E}^0\left(S^0\right) \cong E^0(*) = E_0.$$

Since E is unital, there is a canonical generator $1 \in E_0$.

Definition 1.2.1. E is called *complex oriented* if the map $\tilde{E}^2(\mathrm{BU}(1)) \to E_0$ is surjective, equivalently, if 1 is in the image of that map. A choice of a lift $x \in \tilde{E}^2(\mathrm{BU}(1))$ of $1 \in E_0$ is called a *complex orientation*.

Example 1.2.2. Let R be some ring, and consider $\mathrm{H}R$. It is known that $\mathrm{H}R^*(\mathbb{C}\mathrm{P}^n) \cong R[x]/\left(x^{n+1}\right)$ and $\mathrm{H}R^*(\mathbb{C}\mathrm{P}^\infty) \cong R[[x]]$, where $|x| = 2$, and the maps induced by the inclusions of projective spaces maps x to x. Therefore we see that $x \in \mathrm{H}R^2(\mathrm{BU}(1))$ is mapped to $x \in \mathrm{H}R^2\left(S^2\right) = R\{x\}$, which is mapped to the generator of the reduced part of $\mathrm{H}R^0\left(S^0\right) = R \oplus R$. Hence, x is a complex orientation for $\mathrm{H}R$.

Example 1.2.3. Let K be complex K-theory, then we know that $\mathrm{K}_* = \mathbb{Z}\left[\beta^{\pm 1}\right]$ where β is the Bott element, with $|\beta| = 2$. It is also known (by the Atiyah-Hirzebruch spectral sequence) that $\mathrm{K}^*(\mathbb{C}\mathrm{P}^n) \cong \mathrm{K}_*[t]/\left(t^{n+1}\right)$ and $\mathrm{K}^*(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{K}_*[[t]]$ (here $|t| = 0$), where the maps induced by the inclusions of projective spaces maps t to t. We deduce that $\beta^{-1}t \in \mathrm{K}^2(\mathrm{BU}(1))$ is mapped to $\beta^{-1}t \in \mathrm{K}^2\left(S^2\right) = \mathbb{Z}\left\{\beta^{-1}t\right\}$, and further mapped to $t \in \mathrm{K}^0\left(S^0\right) = \mathbb{Z}\{t\}$, which is indeed the generator of the reduced part. Therefore $x = \beta^{-1}t$ is complex orientation for K. **TODO write the reduced thing more clearly**

Example 1.2.4. Recall that MU is constructed as the colimit $\mathrm{MU} = \operatorname{colim} \mathrm{MU}(n)$. Also, $\mathrm{MU}(1) \cong \Sigma^{\infty-2}\mathrm{BU}(1)$. Therefore we get a canonical map $\Sigma^{\infty-2}\mathrm{BU}(1) \to \mathrm{MU}$, which gives a cohomology class $x_{\mathrm{MU}} \in \mathrm{MU}^2(\mathrm{BU}(1))$.

Proposition 1.2.5 ([Rav86, 4.1.3])**.** x_{MU} *is a complex orientation for* MU.

Theorem 1.2.6 (**TODO reference**)**.** MU *is the universal complex oriented cohomology theory, in the following sense: For any multiplicative cohomology theory E, then there is a bijection between (homotopy classes of) multiplicative maps* $\mathrm{MU} \to E$ *and complex orientations on E. The bijection is given in one direction by pulling back* x_{MU} *along a multiplicative map.*

Assume that E is complex oriented with a complex orientation x.

Proposition 1.2.7 ([Rav86, 4.1.4])**.** *As E_*-algebras,* $E^*(\mathrm{BU}(1)) \cong E^*[[x]]$ *and* $E^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) \cong E^*[[y, z]]$.

TODO maybe indicate the use of AHSS

There is a multiplication map for the group $\mathrm{U}(1)$, i.e. $\mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{U}(1)$. We can take the B of this map, and since it commutes with products we get a map $\mathrm{BU}(1) \times \mathrm{BU}(1) \to \mathrm{BU}(1)$, which is the universal map the classifies the tensor product of vector bundles. Therefore we get a map $E^*(\mathrm{BU}(1)) \to E^*(\mathrm{BU}(1) \times \mathrm{BU}(1))$, which is completely determined by the image of $x \in E^*[[x]]$ in $E^*[[y,z]]$ as above. We conclude that a choice of a complex orientation on E gives rise to an element $F_E(y,z) \in E^*[[y,z]]$.

Proposition 1.2.8 ([Rav86, 4.1.4])**.** *F_E is a formal group law on E_*.*

Definition 1.2.9. The *height* of E is simply the height of F_E.

Example 1.2.10. We continue with $\mathrm{H}R$ from 1.2.2. It is known that the tensor of complex line bundles induces the map

$$R[[x]] = \mathrm{H}R^*(\mathrm{BU}(1)) \to \mathrm{H}R^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = R[[y,z]],$$

given by $x \mapsto y+z$. This is the additive formal group law. It is immediate that $[p] = px$. So for $R = \mathbb{Q}$ we get that the height of $\mathrm{H}\mathbb{Q}$ is 0, while for $R = \mathbb{F}_p$ we have $px = 0$ so the height of $\mathrm{H}\mathbb{F}_p$ is ∞.

Example 1.2.11. We return to the example of complex K-theory 1.2.3. It is known that the tensor of complex line bundles induces the map

$$\mathrm{K}_*[[t]] = \mathrm{K}^*(\mathrm{BU}(1)) \to \mathrm{K}^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = \mathrm{K}_*[[u,v]],$$

given by $t \mapsto u+v+uv$. Note that to comply with the definition of the formal group law, we should use the isomorphism $\mathrm{K}^*(\mathrm{BU}(1)) \cong \mathrm{K}_*[[x]]$, i.e. the element $x = \beta t$. By multiplying by β^{-1} (recall that the map is of K_*-modules) we get that

$$x = \beta^{-1}t \mapsto \beta^{-1}u + \beta^{-1}v + \beta^{-1}uv = y + z + \beta yz = F_{\mathrm{K}}(y,z).$$

By induction we prove that the n-series is $[n](x) = \beta^{-1}(1+\beta x)^n - \beta^{-1}$. This is clear for $n=1$, and we have:

$$\begin{aligned}
[n+1](x) &= x + [n](x) + \beta x [n](x) \\
&= x + \beta^{-1}(1+\beta x)^n - \beta^{-1} + x(1+\beta x)^n - x \\
&= \beta^{-1}(1+\beta x)(1+\beta x)^n - \beta^{-1} \\
&= \beta^{-1}(1+\beta x)^{n+1} - \beta^{-1}
\end{aligned}$$

TODO consider discussing the computation of $\mathrm{BU}(1)$, maybe as part of complex K-theory example?

Example 1.2.12. By taking the cofiber of the multiplication-by-p map, we get a spectrum K/p, mod-p K-theory, with coefficients $(\mathrm{K}/p)_* = \mathbb{F}_p\left[\beta^{\pm 1}\right]$. It is evident that $F_{\mathrm{K}/p}(y,z) = y + z + \beta yz$ as well. From the result above, it follows that

$$[p](x) = \beta^{-1}(1+\beta x)^p - \beta^{-1} = \beta^{-1}(1^p + \beta^p x^p) - \beta^{-1} = \beta^{p-1}x^p,$$

which shows that the height is exactly 1.

A formal group law on E_* is the same data as a map from the Lazard ring L, so the complex orientation gives a map $L \to E_*$. In particular, since MU is complex oriented, there is a canonical map $L \to \mathrm{MU}_*$.

Theorem 1.2.13 (Quillen, [Rav86, 4.1.6]). *The canonical map $L \to \mathrm{MU}_*$ is an isomorphism.*

TODO consider adding something about the $X(n)$, and the obstruction to lifting to a complex orientation, with obstructions living in odd E_k, so even are automatically orientable

1.3 BP, Morava K-Theory and Morava E-Theory

A good principle in homotopy theory (and in many other areas in math) is to study it one prime at a time. This is possible in homotopy theory due to the arithmetic square **TODO reference**. So, let us fix a prime p. We can form $\mathrm{MU}_{(p)}$, the p-localization of MU.

Theorem 1.3.1 ([Ada74, II 15]). *There exists a unique map of ring spectra $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ (depending on the prime p) satisfying:*

- *ε is an idempotent, i.e. $\varepsilon^2 = \varepsilon$,*
- *ε_* sends $[\mathbb{C}\mathrm{P}^n] \in \pi_*\left(\mathrm{MU}_{(p)}\right)$ to itself if $n = p^r - 1$ and to 0 otherwise.*

The map $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ gives a cohomology operation, for every X we have $\varepsilon^* : \mathrm{MU}^*_{(p)}(X) \to \mathrm{MU}^*_{(p)}(X)$. Denote by $\mathrm{BP}^*_{(p)}(X)$ the image of ε^*.

Theorem 1.3.2 ([Ada74, II 16], [Rav86, 4.1.12]). BP *is a cohomology theory, represented by an associative commutative ring spectrum* BP *(depending on the prime p), which is a retract of* $\mathrm{MU}_{(p)}$. *The homotopy groups of* BP *are* $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ *where* $|v_n| = 2(p^n - 1)$.

For convenience we denote $v_0 = p$ (and indeed $|v_0| = 2\left(p^0 - 1\right) = 0$). Since BP is a retract of MU, it comes with a map $\mathrm{MU} \to \mathrm{BP}$, that is, a complex orientation.

Proposition 1.3.3. *[**TODO reference**] The p-series of the formal group law associated to* BP *is* $[p](x) = \sum v_n x^{p^n}$.

Remark 1.3.4 ([Rav92, B.5]). The formal group law on BP has a similar interpretation to that of MU, namely it is the universal p-typical formal group law. Moreover, the idempotent $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ induces an idempotent on homotopy groups, which can be described as the map that takes a formal group law to the canonical p-typical formal group law isomorphic to it.

Once we have BP, we can turn to the definition of Morava K-theory and Morava E-theory

Definition 1.3.5. Let $0 < n < \infty$. *Morava K-theory* at height n and prime p, denoted by $\mathrm{K}(p, n)$ or $\mathrm{K}(n)$ when the prime is clear from the context, is the spectrum obtained by killing $p = v_0, \ldots, v_{n-1}, v_{n+1}, \ldots$ in BP and inverting v_n. Therefore $\mathrm{K}(n)_* = \mathbb{F}_p\left[v_n^{\pm 1}\right]$. We also define $\mathrm{K}(0) = \mathrm{H}\mathbb{Q}$ and $\mathrm{K}(\infty) = \mathrm{H}\mathbb{F}_p$. Similarly, *Morava E-theory* at height n and prime p, denoted by $\mathrm{E}(p, n)$ or $\mathrm{E}(n)$, is the spectrum obtained by killing $v_{n+1}, v_{n+2}, \ldots$ in BP and inverting v_n. Therefore $\mathrm{E}(n)_* = \mathbb{Z}_{(p)}\left[v_1, \ldots v_{n-1}, v_n^{\pm 1}\right]$.

Since Morava K-theory and E-theory are obtained from BP by cofibers and filtered colimits, they are equipped with a map from BP, hence also with a complex orientation. Then, from 1.3.3, we get:

Corollary 1.3.6. *The p-series associated to the formal group laws of* $\mathrm{K}(n)$ *and* $\mathrm{E}(n)$ *are* $v_n x^{p^n}$ *and* $v_0 x + \ldots v_n x^{p^n}$ *respectively. Therefore the height of* $\mathrm{K}(n)$ *is exactly* n. *(Note that by the example of* $\mathrm{H}R$*, this is also true for* $\mathrm{K}(0)$ *and* $\mathrm{K}(\infty)$*.)*

We want to describe some properties of Morava K-theory. To do so we first need some definitions.

Definition 1.3.7. Let R be an evenly graded ring. R is called a *graded field* if it satisfies one of the equivalent conditions:

- every non-zero homogenus element is invertible,
- it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree.

An $\mathbb{A}_\infty$-ring E is a *field* if E_* is a graded field.

Example 1.3.8. $K(n)$ is a field for $0 \leq n \leq \infty$.

Proposition 1.3.9. *A field E has Kunneth, i.e. $E_*(X \otimes Y) \cong E_*(X) \otimes_{E_*} E_*(Y)$ for any spectra X, Y.*

Proposition 1.3.10 ([Lur10, 24])**.** *Let $E \neq 0$ be a complex oriented cohomology theory, whose formal group law has height exactly n, then $E \otimes \mathrm{K}(n) \neq 0$. Let E be a field s.t. $E \otimes \mathrm{K}(n) \neq 0$, then E admits the structure of a $\mathrm{K}(n)$-module. (Here $0 \leq n \leq \infty$.)*

Example 1.3.11. As we have seen in 1.2.12, mod-p K-theory, K/p, has height exactly 1 and coefficients $(\mathrm{K}/p)_* = \mathbb{F}_p\left[\beta^{\pm 1}\right]$. It is also known that K and K/p, are A_∞ ring spectra, from which it follows that K/p is a field. We deduce that K/p is a $\mathrm{K}(1)$-module. Since $|\beta| = 2$ and $|v_1| = 2(p-1)$ it is free of rank $p-1$.

From this we also deduce some form of uniqueness for Morava K-theory:

Corollary 1.3.12. *Let E be a field with $E_* \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$, which is also complex oriented of height exactly n. Then $E \cong \mathrm{K}(n)$ (as spectra).*

1.4 $\operatorname{Spec} \mathbb{S}_{(p)}$ and $\operatorname{Spec} \mathbb{S}$

We are now in a position to describe the topological space $\operatorname{Spec} \mathbb{S}$. However, it will be easier to state it first for $\operatorname{Spec} \mathbb{S}_{(p)}$, and then pullback prime ideals. We know that $\mathrm{Mod}_{\mathbb{S}_{(p)}} = \mathrm{Sp}_{(p)}$, and its compact objects are $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$, the p-localizations of finite spectra.

Proposition 1.4.1. *Let $\mathcal{T}_E$ be the E-acyclics, i.e.*

$$\mathcal{T}_E = \ker E_* = \left\{X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid E_*(X) = 0\right\} = \left\{X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid X \otimes E_* = 0\right\}.$$

Then $\mathcal{T}_E$ is thick.

Proof. The proof follows the same lines of 1.1.4 for the case $\mathrm{Ch}_{\mathrm{perf}}(R)$. □

Definition 1.4.2. We define $\mathcal{C}_{p,n} = \mathcal{T}_{\mathrm{K}(n)}$, the $\mathrm{K}(n)$-acyclics. By the above proposition, it is thick. Also, $\mathcal{C}_{p,\infty} = \{0\}$, which is trivially thick. When the prime is clear from the context, we write $\mathcal{C}_n$ in place of $\mathcal{C}_{p,n}$.

Proposition 1.4.3 ([Lur10, 26])**.** *For $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$, if $\mathrm{K}(n)_*(X) = 0$ then $\mathrm{K}(n-1)_*(X) = 0$.*

Definition 1.4.4. We say that a spectrum $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$ is of *type n* (possibly ∞) if its first non-zero Morava K-theory homology is $\mathrm{K}(n)$.

Corollary 1.4.5. *$\mathcal{C}_n$ is the full subcategory of finite p-local spectra of type $> n$, that is $\mathcal{C}_n = \left\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid \forall m \le n : \mathrm{K}(m)_*(X) = 0\right\}$. Thus $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$.*

Proposition 1.4.6 (**TODO reference**)**.** *The inclusions $\mathcal{C}_{n+1} \subset \mathcal{C}_n$ are proper.*

Proposition 1.4.7. *If $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$ is not contractible, then is of finite type. Therefore $\bigcap_{n<\infty} \mathcal{C}_n = \{0\} = \mathcal{C}_\infty$.*

Proof. Let X be non-contractible. Then $\mathrm{H}\mathbb{Z}_*(X) \ne 0$. Let m be the first non-zero degree. Using the universal coefficient theorem and the fact that the spectrum is p-local we get that $(\mathrm{H}\mathbb{F}_p)_m(X) \ne 0$, thus $(\mathrm{H}\mathbb{F}_p)_*(X) \ne 0$. Since X is finite, $(\mathrm{H}\mathbb{F}_p)_*(X)$ is bounded. The Atiyah-Hirzebruch spectral sequence for X with cohomology $\mathrm{K}(n)$ has E^2-page given by $E^2_{p,q} = H_p\left(X; \mathrm{K}(n)_q\right)$. Since $\mathrm{K}(n)_q = \mathbb{F}_p$ for $q = 0 \mod 2(p^n-1)$ and 0 otherwise, we see that the rows $q = 0 \mod 2(p^n-1)$ are $(\mathrm{H}\mathbb{F}_p)_*(X)$, and the others are 0. Therefore if we take n such that the period $2(p^n-1)$ is larger then the bound on $(\mathrm{H}\mathbb{F}_p)_*(X)$, then all differentials have either source or target 0. Thus, the spectral sequence collapses at the E^2-page, and since $(\mathrm{H}\mathbb{F}_p)_*(X) \ne 0$, we ge that $\mathrm{K}(n)(X) \ne 0$, i.e. X has type $< n$. ☐

Proposition 1.4.8. *$\mathcal{C}_n$ is a prime ideal.*

Proof. For $X, Y \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$, by Kunneth we have

$$\mathrm{K}(n-1)_*(X \otimes Y) = \mathrm{K}(n-1)_*(X) \otimes \mathrm{K}(n-1)_*(Y).$$

Assume that $X \in \mathcal{C}_n$, that is $\mathrm{K}(n-1)_*(X) = 0$. It follows that $\mathrm{K}(n-1)_*(X \otimes Y) = 0$, i.e. $X \otimes Y \in \mathcal{C}_n$, so $\mathcal{C}_n$ is an ideal. Assume that $X \otimes Y \in \mathcal{C}_n$, that is $\mathrm{K}(n-1)_*(X \otimes Y)$, therefore one of the terms in the RHS of the equation must vanish (since they are graded vector spaces), so $\mathcal{C}_n$ is a prime ideal. ☐

Theorem 1.4.9 (Thick Subcategory Theorem [HS98])**.** *If $\mathcal{T}$ is a proper thick subcategory of $\mathrm{Sp}^{\mathrm{fin}}_{(p)}$, then $\mathcal{T} = \mathcal{C}_n$ for some $0 \le n \le \infty$.*

Remark 1.4.10. The proof relies on a major theorem called the Nilpotence Theorem.

Corollary 1.4.11. $\operatorname{Spec}\mathbb{S}_{(p)} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_\infty\}$*, and the closed subsets in the topology are chains* $\{\mathcal{C}_k, \mathcal{C}_{k+1}, \ldots, \mathcal{C}_\infty\}$ *for some* $0 \le k \le \infty$*.*

Proof. Follows immediately from the previous results. □

We now want to describe $\operatorname{Spec}\mathbb{S}$. Note that the p-localization functor $L_{(p)}$ is a Bousfield localization. As such, it is left (its right adjoint is the inclusion), and in particular preserves cofibers. It also clearly reduced, i.e. sends 0 to 0. Now, $L_{(p)}$ is smashing, that is $L_{(p)}X = X \otimes \mathbb{S}_{(p)}$, so it is also symmetric monoidal. As we have seen in 1.1.8, under these conditions we can pullback primes. Since $L_{(p)}$ is smashing and $\mathrm{K}(p,n)$ is p-local for every $0 \le n \le \infty$, we have that $\mathrm{K}(n)_*\left(X_{(p)}\right) = \mathrm{K}(n)_*(X)$. Therefore

$$\mathcal{P}_{p,n} = L^*_{(p)}\mathcal{C}_{p,n} = \left\{X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{K}(p,n)_*(X) = 0\right\}$$

and

$$\mathcal{P}_{p,\infty} = L^*_{(p)}\mathcal{C}_{p,\infty} = \left\{X \in \mathrm{Sp}^{\mathrm{fin}} \mid X_{(p)} = 0\right\}$$

are prime ideals. Moreover, by definition $\mathrm{K}(p,0) = \mathrm{H}\mathbb{Q}$, which implies that $\mathcal{P}_{p,0} = \left\{X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_*(X) = 0\right\}$. We see that $\mathcal{P}_{p,0}$ is independent of p, and we denote it by $\mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}}$.

Theorem 1.4.12. $\operatorname{Spec}\mathbb{S} = \left\{\mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}}\right\} \bigcup \left(\bigcup_p \{\mathcal{P}_{p,1}, \ldots, \mathcal{P}_{p,\infty}\}\right)$*, and the closed subsets in the topology are finite unions of chains* $\{\mathcal{P}_{p,k}, \mathcal{P}_{p,k+1}, \ldots, \mathcal{P}_{p,\infty}\}$

for some $0 \leq k \leq \infty$ *(i.e. they may include* $\mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}}$*).*

$$\begin{tikzcd}[column sep=small]
\mathcal{P}_{2,\infty} \arrow[d, dash] & \mathcal{P}_{3,\infty} \arrow[d, dash] & \cdots & \mathcal{P}_{p,\infty} \arrow[d, dash] & \cdots \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
\mathcal{P}_{2,n} \arrow[d, dash] & \mathcal{P}_{3,n} \arrow[d, dash] & \cdots & \mathcal{P}_{p,n} \arrow[d, dash] & \cdots \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
\mathcal{P}_{2,2} \arrow[d, dash] & \mathcal{P}_{3,2} \arrow[d, dash] & \cdots & \mathcal{P}_{p,2} \arrow[d, dash] & \cdots \\
\mathcal{P}_{2,1} \arrow[drr, dash] & \mathcal{P}_{3,1} \arrow[dr, dash] & \cdots & \mathcal{P}_{p,n} \arrow[dl, dash] & \cdots \\
& & \mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}} & &
\end{tikzcd}$$

TODO explain/reference TODO regarding the topology, maybe I should prove that the pullback is also continuous?

Remark 1.4.13. Thick subcategories are interesting for another reason, unrelated to the Balmer spectrum point of view, namely they give a very powerful proof method. Say we have a property that is satisfied by 0, and is closed under cofibers and retracts. It follows that the collection of objects that satisfy it is thick. Then, for example, by the thick subcategory theorem 1.4.9, it is enough to find one object in $\mathcal{C}_n \setminus \mathcal{C}_{n-1}$ that satisfies the property, to show that all objects in $\mathcal{C}_n$ satisfy it.

1.5 The Stacky Point of View and the Relationship Between Morava K-Theory and Morava E-Theory

First we will describe, without being precise, another point of view on what chromatic homotopy theory is about.

There is a stack of formal groups with strict isomorphisms, denoted by $\mathcal{M}^s_{\mathrm{fg}}$. It can be described as the stack that sends a ring R to the groupoid of formal group laws, with strict isomorphisms between them. Quillen theorem 1.2.13 tells us that MU_* is the Lazard ring, that is the universal ring that carries the universal formal group law. It turns out that this theorem has a second part, which says that $(\mathrm{MU} \otimes \mathrm{MU})_*$ is the universal ring that carries two formal group laws and a strict isomorphism between them. Therefore, $\mathcal{M}_{\mathrm{fg}}^{s}$ is represented by $(\mathrm{MU}_*, (\mathrm{MU}\otimes\mathrm{MU})_*)$.

The geometric points of the stack $\mathcal{M}_{\mathrm{fg}}^{s}$ are describe precisely the same as $\operatorname{Spec}\mathbb{S}$, that is because for an algebraically closed field of characteristic 0 there is a unique (up to isomorphism) formal group law which is of height 0 namely the additive formal group law, and for characteristic p there is a unique (up to isomorphism) formal group law of each height $1 \leq n \leq \infty$.

For a spectrum X, $\mathrm{MU}_*(X)$ is a $(\mathrm{MU}_*, (\mathrm{MU}\otimes\mathrm{MU})_*)$-comodule, which is the same as a sheaf over $\mathcal{M}_{\mathrm{fg}}^{s}$. From this point of view, chromatic homotopy theory lets us study a spectrum by decomposing it over the stack $\mathcal{M}_{\mathrm{fg}}^{s}$.

We can restrict ourselves to the stack only over rings of characteristic p, $\mathcal{M}_{\mathrm{fg},p}^{s}$, which is then represented by $\left(\left(\mathrm{MU}_{(p)}\right)_*, \left(\mathrm{MU}_{(p)}\otimes\mathrm{MU}_{(p)}\right)_*\right)$. **TODO I think it's true, is it?** Similarly to MU, BP is universal ring with the universal p-typical formal group law, and $(\mathrm{BP}\otimes\mathrm{BP})_*$ is the universal ring with two p-typical formal group laws and an isomorphism between them **TODO I didn't say this before**. Since every formal group law is isomorphic to a p-typical one, we know that the stack $\mathcal{M}_{\mathrm{fg},p}^{s}$ is also represented by $(\mathrm{BP}_*, (\mathrm{BP}\otimes\mathrm{BP})_*)$.

It is now reasonable that $\mathrm{K}(n)$, obtained from BP by killing the v_m's for $m \neq n$ and inverting v_n, sees the n-th level, and that $\mathrm{E}(n)$ obtained in the same way but only killing v_m for $m > n$, sees the levels $\leq n$.

Let us now claim a precise statement, formalizing this description.

Theorem 1.5.1 (TODO reference). *$\mathrm{E}(n)$ and $\mathrm{K}(0)\oplus\cdots\oplus\mathrm{K}(n)$ are Bousfield equivalent. That is, they have the same acyclics, locals, and their localization functors are the same.*

TODO chromatic square and chromatic tower, maybe another subsection?

1.6 Landweber Exact Functor Theorem

As we have seen, a complex orientation on a cohomology theory, which is described by a map $\mathrm{MU} \to E$, has an associated formal group law, which is described by the map $L = \mathrm{MU}_* \to E_*$. Note that this formal group law is of degree -2, by virtue of the grading on $L = \mathrm{MU}_*$ **TODO say more?**. One can ask whether the converse is true, namely given a graded ring R and a formal group law F of degree -2 given by $L \to R$, is there a complex oriented cohomology theory whose coefficients are R and the associated formal group law is F.

A strategy is to define $(E_{R,F})_* (X) = \mathrm{MU}_* (X) \otimes_{\mathrm{MU}_*} R$. Unfortunately, this is not always a homology theory. However there is a condition that one can check, which guarantees that it is.

Definition 1.6.1. $L \to R$ is called *Landweber flat* if for every prime p, the image of the sequence $p = v_0, v_1, v_2, \dots$ in R, which are the coefficients of the p-series, is regular. That is, for each p and $n \geq 0$, v_n is not a zero divisor in $R/ (v_0, v_1, \dots, v_{n-1})$.

Remark 1.6.2. If p is invertible in R, then p is invertible, and R/p is already 0, so we don't need to check $v_1, v_2, \dots$.

Theorem 1.6.3 (Landweber Exact Functor Theorem (LEFT), [Lur10, 15, 16])**.** *If $L \to R$ is Landweber flat, then $E_{R,F}$ defined above is a homology theory. Moreover, there are no phantom maps between such spectra, so $E_{R,F}$ is represented by a spectrum. This spectrum is complex oriented, has coefficients R and associated formal group law F.*

Example 1.6.4. Morava E-theory is Landweber flat, since by 1.3.6, the p-series has coefficients $p = v_0, v_1, \dots, v_n$. p is not a zero divisor in $\mathrm{E}(n)_* = \mathbb{Z}_{(p)} \left[v_1, \dots v_{n-1}, v_n^{\pm 1}\right]$. Then v_i is not a zero divisor in $\mathrm{E}(n)_* / (p, v_1, \dots, v_{i-1}) \cong \mathbb{F}_p \left[v_i, \dots v_{n-1}, v_n^{\pm 1}\right]$. After v_n the ring becomes 0 and we are done. For other primes, by 1.6.2 we are done.

Example 1.6.5. Morava K-theory $\mathrm{K}(n)$ for $n > 0$ is not Landweber flat since $p = v_0$ is 0 in $\mathrm{K}(n)_* = \mathbb{F}_p \left[v_n^{\pm 1}\right]$.

Example 1.6.6. $\mathrm{H}\mathbb{Z}$ is not Landweber flat since although $p = v_0$ is invertible, as we have seen in 1.2.10 the p-series is px, so v_1 is 0 in $\mathbb{Z}/p = \mathbb{F}_p$.

We can also ask the following question: given complex oriented cohomology theory $\mathrm{MU} \to E$, such that $L \to E_*$ is Landweber flat, is $E_{R,F}$ equivalent to E? The answer is yes, at least in some cases.

Theorem 1.6.7. *Let E be as above, which is also evenly graded (i.e. E_* is an evenly graded ring), then there is an equivalence $E_{R,F} \to E$.*

Proof. This is a slight variation on [Lur10, 18, proposition 11]. First note that for every spectrum X we have $\mathrm{MU} \otimes X \to E \otimes X$, which induces $\mathrm{MU}_* (X) \to E_* (X)$, a map of MU_*-modules. Moreover, since $E_* \to E_* (X)$

is a map of E_*-module, the map $\mathrm{MU}_* \to E_*$ makes it a map of MU_*-modules. Together this gives a map $(E_{R,F})_*(X) = \mathrm{MU}_*(X) \otimes_{\mathrm{MU}_*} E_* \to E_*(X)$. This map is a map of homology theories. **TODO should I explain why?** By [Lur10, 17, theorem 6] **TODO does it follow?**, this map lifts to a map of spectra $E_{R,F} \to E$. Since by construction when $X = \mathbb{S}$ the map above is $E_* \to E_*$ which is an isomorphism, we see that the map $E_{R,F} \to E$ is an equivalence. $\square$

Example 1.6.8. We return to complex K-theory, from 1.2.3 and 1.2.11. We can take the completion at the element $p \in \mathrm{K}_*$, which gives the spectrum $\mathrm{K}_p^\wedge$. This spectrum has coefficients $\left(\mathrm{K}_p^\wedge\right)_* = (\mathrm{K}_*)_p^\wedge = \left(\mathbb{Z}\left[\beta^{\pm1}\right]\right)_p^\wedge = \mathbb{Z}_p\left[\beta^{\pm1}\right]$. The formal group law, as we have seen, is given by $F_{\mathrm{K}_p^\wedge}(y,z) = y + z + \beta yz$. We claim that $F_{\mathrm{K}_p^\wedge}/\mathrm{K}_p^\wedge$ is Landweber flat. Clearly $p = v_0$ is not a zero divisor in $\mathbb{Z}_p\left[\beta^{\pm1}\right]$. As we have seen in 1.2.12, mod-p the p-series is $\beta^{p-1}x^p$, so that $v_1 = \beta^{p-1}$ which is not a zero divisor $\mathbb{F}_p\left[\beta^{\pm1}\right]$. Modulo v_1 the ring is already 0, and we are done. For other primes, by 1.6.2 we are done. Therefore, by 1.6.7 we get that $\mathrm{K}_p^\wedge \cong E_{\mathrm{K}_p^\wedge, F_{\mathrm{K}_p^\wedge}}$.

1.7 Lubin-Tate Deformation Theory

The Morava E-theory we have considered until now $\mathrm{E}(n)$, also called Johnson-Wilson spectrum was constructed from BP. As we noted, it is Landweber flat, which indicates that there is another approach to constructing it. Indeed there is a way to construct a related spectrum, which will be called the Lubin-Tate spectrum.

To that end, we first define the category CompRing as the category of complete local rings. The objects are complete local rings $(R, \mathfrak{m})$, we also denote by $\pi : R \to R/\mathfrak{m}$ the projection. Morphisms $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ are local homomorphisms, i.e. a homomorphism $\varphi : R \to S$ s.t. $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. In particular it induces a homomorphism $\varphi/\mathfrak{m} : R/\mathfrak{m} \to S/\mathfrak{n}$, which satisfies $\varphi/\mathfrak{m} \circ \pi_R = \pi_S \circ \varphi$.

We fix k be a perfect field of characteristic p (i.e. the Frobenius is an isomorphism), and Γ a formal group law over k of height $n < \infty$. Lubin and Tate [LT66] considered a moduli problem associated to Γ/k, described by a functor $\mathrm{Def} : \mathrm{CompRing} \to \mathrm{Grpds}$.

Definition 1.7.1. Let $(R, \mathfrak{m})$ be a complete local ring and denote by $\pi : R \to R/\mathfrak{m}$ the quotient. A *deformation* of Γ/k to $(R, \mathfrak{m})$, is (G, i), where G is a formal group law over R, $i : k \to R/\mathfrak{m}$ is a homomorphism of fields, such that $i^*\Gamma = \pi^* G$. A $\star$-*isomorphism* between two deformations to $(R, \mathfrak{m})$, $f : (G_1, i_1) \to (G_2, i_2)$, is defined only when $i_1 = i_2$, and consists of an isomorphism $f : G_1 \to G_2$, such that $\pi^* f : i^*\Gamma = \pi^* G_1 \to \pi^* G_2 \to i^*\Gamma$ is the identity, i.e. $f(x) = x \mod \mathfrak{m}$. These assemble to a groupoid $\mathrm{Def}(R, \mathfrak{m})$, whose objects are deformations to $(R, \mathfrak{m})$, and morphisms are $\star$-isomorphisms.

Remark 1.7.2. $\mathrm{Def}(R, \mathfrak{m})$ can be seen as the pullback of the groupoids $\mathrm{FGL}(R)$ and $\coprod_{i:k\to R/\mathfrak{m}} \{\Gamma\}$ over $\mathrm{FGL}(R/\mathfrak{m})$ (where the maps are $G \mapsto q^* G$ and $i \mapsto i^*\Gamma$ respectively).

Proposition 1.7.3 (/definition)**.** *The construction* $\mathrm{Def}(R, \mathfrak{m})$ *is functorial.*

Proof. Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism.

For a deformation (G, i) to $(R, \mathfrak{m})$, we define $\mathrm{Def}(\varphi)(G, i) = (\varphi^* G, \varphi/\mathfrak{m} \circ i)$. Note that $\varphi^* G$ is a formal group law over S, and $\varphi/\mathfrak{m} \circ i : k \to R/\mathfrak{m} \to S/\mathfrak{n}$ is a homomorphism. Moreover, $(\varphi/\mathfrak{m} \circ i)^* \Gamma = (\varphi/\mathfrak{m})^* i^*\Gamma = (\varphi/\mathfrak{m})^* \pi_R^* G = (\varphi/\mathfrak{m} \circ \pi_R)^* G = (\pi_S \circ \varphi)^* G = \pi_S^* \varphi^* G$, which shows that $\mathrm{Def}(\varphi)(G, i)$ is a deformation to $(S, \mathfrak{n})$.

For a $\star$-isomorphism $f : (G_1, i) \to (G_2, i)$, which is the data of an isomorphism $f : G_1 \to G_2$ such that $\pi_R^* f = \mathrm{id}_{i^*\Gamma}$ is the identity, we need to define a $\star$-isomorphism $\mathrm{Def}(\varphi)(G, i_1) \to \mathrm{Def}(\varphi)(G, i_2)$. Take it to be the isomorphism $\varphi^* f : \varphi^* G_1 \to \varphi^* G_2$, which satisfies $\pi_S^* \varphi^* f = (\varphi/\mathfrak{m})^* \pi_R^* f = (\varphi/\mathfrak{m})^* \mathrm{id}_{i^*\Gamma} = \mathrm{id}_{(\varphi/\mathfrak{m})^* i^*\Gamma} = \mathrm{id}_{(\varphi/\mathfrak{m} \circ i)^*\Gamma}$. The identity $\mathrm{id}_G : (G, i) \to (G, i)$ is clearly sent to $\mathrm{id}_{\varphi^* G}$, and compositions are sent to compositions.

This shows that $\mathrm{Def}(\varphi) : \mathrm{Def}(R, \mathfrak{m}) \to \mathrm{Def}(S, \mathfrak{n})$ is indeed a functor. Moreover, it is clear that $\mathrm{Def}(\mathrm{id}_R)$ is the identity and compositions are sent to compositions, which shows that $\mathrm{Def} : \mathrm{CompRing} \to \mathrm{Grpds}$ is indeed a functor. $\square$

Remark 1.7.4. We recall quickly that the Witt vectors Wk is a ring of characteristic 0, with maximal ideal (p), and residue field $Wk/p \cong k$. For example, $W\mathbb{F}_p = \mathbb{Z}_p$.

Theorem 1.7.5 ([Rez98, 4.4, 5.10], originally due to [LT66])**.** *The functor* Def *lands in discrete groupoids (i.e.* $\mathrm{Def}(R, \mathfrak{m})$ *has* 0 *or* 1 *morphisms between objects). Furthermore the functor* Def *is co-represented, that is there exists a* universal deformation*, and the complete local ring can be chosen (non-canonically) to be* $Wk[[u_1, \ldots, u_{n-1}]]$.

Let us unravel what that means. First note that the quotient of $Wk\left[\left[u_1,\ldots,u_{n-1}\right]\right]$ by the maximal ideal $(p,u_1,\ldots,u_{n-1})$ is k. The universal deformation can be chosen such that the formal group law over it Γ_U over $Wk\left[\left[u_1,\ldots,u_{n-1}\right]\right]$ satisfies $\pi^*\Gamma_U$ is Γ. The universality means that for $(R,\mathfrak{m})$, the assignment

$$\hom_{\text{CompRing}}\left(Wk\left[\left[u_1,\ldots,u_{n-1}\right]\right],R\right)\to\operatorname{Def}\left(R,\mathfrak{m}\right),\quad\varphi\mapsto\varphi^*\Gamma_U$$

is an equivalence.

Now, we can form the graded ring $Wk\left[\left[u_1,\ldots,u_{n-1}\right]\right]\left[u^{\pm1}\right]$ where $|u|=2$. We can define the formal group law $(u\Gamma_U)(x,y)=u^{-1}\Gamma_U(uy,uz)$, which is of degree -2, thus we get a map $L\to Wk\left[\left[u_1,\ldots,u_{n-1}\right]\right]\left[u^{\pm1}\right]$.

Proposition 1.7.6 ([Rez98, 6.9])**.** *$L\to Wk\left[\left[u_1,\ldots,u_{n-1}\right]\right]\left[u^{\pm1}\right]$ is Landweber flat.*

Using LEFT 1.6.3, we immediately get:

Corollary 1.7.7. *There is a complex oriented cohomology theory $\mathrm{E}(k,\Gamma)=E_{Wk[[u_1,\ldots,u_{n-1}]][u^{\pm1}],u\Gamma_U}$, called* Lubin-Tate spectrum*, with coefficients $\mathrm{E}(k,\Gamma)_*=Wk\left[\left[u_1,\ldots,u_{n-1}\right]\right]\left[u^{\pm1}\right]$ and associated formal group law $u\Gamma_U$.*

Example 1.7.8. We continue the complex K-theory saga from 1.6.8. Take the field $k=\mathbb{F}_p$ and the formal group law $\Gamma(y,z)=y+z+yz$, of height $n=1$. By the above construction, the ring of the universal deformation is $W\mathbb{F}_p=\mathbb{Z}_p$. The universal formal group law of the universal deformation can be taken to be $\Gamma_U(y,z)=y+z+yz$ (this follows from the proof at [Rez98, 5.10], since here $n=1$ so there are no u_i's). We look at the ring $\mathbb{Z}_p\left[u^{\pm1}\right]$, and at the formal group law over it $(u\Gamma_U)(y,z)=u^{-1}\left(uy+uz+u^2yz\right)=y+z+uyz$. It is then clear that the isomorphism $\mathbb{Z}_p\left[u^{\pm1}\right]\to\mathbb{Z}_p\left[\beta^{\pm1}\right]$, sends $u\Gamma_U$ to $F_{\mathrm{K}_p^\wedge}$. It follows that $\mathrm{K}_p^\wedge\cong E_{\mathrm{K}_p^\wedge,F_{\mathrm{K}_p^\wedge}}\cong\mathrm{E}(\mathbb{F}_p,\Gamma)$, i.e. p-complete K-theory $\mathrm{K}_p^\wedge$ is a Lubin-Tate spectrum at height 1.

This concludes the construction of the Lubin-Tate variant of Morava E-theory. To compare this version with the previous, we define another variant of Morava E-theory.

Definition 1.7.9. Let $I\leq\mathrm{E}(n)_*=\mathbb{Z}_{(p)}\left[v_1,\ldots v_{n-1},v_n^{\pm1}\right]$ be the ideal $I=(p=v_0,v_1,\ldots v_{n-1})$. We define the spectrum $\widehat{\mathrm{E}(n)}=\mathrm{E}(n)_I^\wedge$, called completed Johnson-Wilson spectrum.

Theorem 1.7.10. *The following three forms of Morava E-theory are Bousfield equivalent:*

1. *Johnson-Wilson* $\mathrm{E}(n)$,
2. *completed Johnson-Wilson* $\widehat{\mathrm{E}(n)}$,
3. *Lubin-Tate* $\mathrm{E}(k, \Gamma)$.

TODO reference Hovey Strickland Morava K-Theories and Localization 5.3

2 Atiyah-Segal

We now leave the realm of chromatic homotopy theory. One aspect of algebraic topology is to try to capture properties of spaces using algebraic invariants. One of the most fruitful such invariants is complex K-theory, denoted K, and one of the most important spaces in homotopy theory is $\mathrm{B}G$, so it is natural to ask for a description of $\mathrm{K}(\mathrm{B}G)$ (by Bott periodicity, we will consider only $\mathrm{K} = \mathrm{K}^0$). Atiyah and Segal [AS69] gave a description of this, and more, in the case that G is a compact Lie group, in terms of representations.

From now we fix a compact Lie group G. Also, a representation means a finite dimensional unitary representation. We should also note that beyond this part, we will be mostly interested in finite groups.

2.1 The Atiyah-Segal Theorem

We denote by $\mathrm{R}(G)$ the *representation ring* of G, that is the collection of virtual representations of G (which can be written as a formal difference $V - U$) up to isomorphism, where the addition is given by direct sum and the product is given by tensor product. This is an augmented ring $\varepsilon : \mathrm{R}(G) \to \mathbb{Z}$ by the virtual dimension (i.e. $\varepsilon(V - U) = \dim(V - U) = \dim V - \dim U$). The *augmentation ideal* is $I = \ker \varepsilon = \{V - U \in \mathrm{R}(G) \mid 0 = \dim(V - U)\}$.

Atiyah and Segal showed that one can describe $\mathrm{K}(\mathrm{B}G)$ in these terms, namely, it is the completion of $\mathrm{R}(G)$ at the ideal I:

Theorem 2.1.1 ([AS69])**.** $\mathrm{K}(\mathrm{B}G) \cong \mathrm{R}(G)^{\wedge}_{I}$.

We will not prove the theorem, but we will indicate some of the key ingredients.

First of all, to show that objects are isomorphic, we need a map. Before giving the map actually used in the proof, we describe an easier way to see where this map comes from. Recall that $\mathrm{K}(X) \cong [X, \mathrm{BU} \times \mathbb{Z}]$. The data of a representation of G is the same thing as a homomorphism $G \to \mathrm{U}(n)$. Since B is a functor, we get a map $\mathrm{B}G \to \mathrm{BU}(n)$, and by composing with the injection $\mathrm{BU}(n) \cong \mathrm{BU}(n) \times \{n\} \to \mathrm{BU} \times \mathbb{Z}$, we get a map $\mathrm{B}G \to \mathrm{BU} \times \mathbb{Z}$, that is, an element of $\mathrm{K}(\mathrm{B}G)$. Therefore we get a map $\mathrm{R}(G) \to \mathrm{K}(\mathrm{B}G)$. The theorem shows that it is a ring homomorphism which exhibits $\mathrm{K}(\mathrm{B}G)$ as the completion of $\mathrm{R}(G)$ at I.

There is an alternative description of this map. In [Seg68], Segal described a variant ofK theory, called equivariant K-theory K_G. This variant assigns to a G-space the ring of virtual G-bundles, that is, bundles equipped with an action of G which is compatible with the action on the base G-space. Note that K_G is no longer homotopy invariant, since it also takes into account the action of G. First we note the following:

Proposition 2.1.2. $\mathrm{K}_G(*) = \mathrm{R}(G)$ *(where $*$ is equipped with a trivial G-action).*

Proof. This follows from the definitions, since a vector bundle over a point is just a vector space, and it is equipped with a G-action over the point, which is just a G representation. □

For any G-space X, the projection map $\mathrm{pr} : X \to X/G$ allows us to pullback vector bundles on X/G to G-bundles on X. In other words, it induces a map $\mathrm{pr}^* : \mathrm{K}(X/G) \to \mathrm{K}_G(X)$.

Proposition 2.1.3 ([Seg68, 2.1])**.** *Suppose the action of G on X is* free. *Then* pr^* *admits an inverse, given by taking a bundle $E \to X$ to $E/G \to X/G$. In particular,* $\mathrm{K}(X/G) \cong \mathrm{K}_G(X)$.

Now, we have a map of G-spaces given by $\mathrm{E}G \to *$. By the above we get:

$$\mathrm{R}(G) \cong \mathrm{K}_G(*) \to \mathrm{K}_G(\mathrm{E}G) \cong \mathrm{K}(\mathrm{E}G/G) = \mathrm{K}(\mathrm{B}G)$$

It can be shown that this is the same map $\mathrm{R}(G) \to \mathrm{K}(\mathrm{B}G)$ described before, which exhibits $\mathrm{K}(\mathrm{B}G)$ as the I-completion of $\mathrm{R}(G)$. Atiyah and Segal use this map and variants to prove the theorem.

Here is a sketch of the proof given by Atiyah and Segal. First of all, we note that the theorem is proven for the entire K^* rather than just for

$\mathrm{K} = \mathrm{K}^0$. Also note that $\mathrm{R}^*(G) = \mathrm{K}^*_G(*)$ is a 2-periodic version of the representation ring (because K^*_G also satisfies Bott periodicity). We have the corresponding 2-periodic version of the augmentation ideal, which we denote by I^*. They use the Milnor join construction $\mathrm{E}G_n = \underbrace{G * \cdots * G}_{n \text{ times}}$ and $\mathrm{B}G_n = \mathrm{E}G_n/G$, which has the property that $\operatorname{colim} \mathrm{E}G_n \to \operatorname{colim} \mathrm{B}G_n$ is a model for $\mathrm{E}G \to \mathrm{B}G$. Then, for any compact G-space X there is a similar map to the map above: using $X \times \mathrm{E}G_n \to X$ we get a map $\mathrm{K}^*_G(X) \to \mathrm{K}^*_G(X \times \mathrm{E}G_n)$. All of these are $\mathrm{R}^*(G)$-modules, and Atiyah and Segal show that this map factors through the quotient by $(I^*)^n$, to give a map $\mathrm{K}^*_G(X) / (I^*)^n \to \mathrm{K}^*_G(X \times \mathrm{E}G_n)$. The two sides assemble into pro-rings, and the maps assemble to a map between the pro-rings:

$$\left\{\mathrm{K}^*_G(X) / (I^*)^n\right\}_n \to \left\{\mathrm{K}^*_G(X \times \mathrm{E}G_n)\right\}_n$$

What they actually prove is the strong form:

Theorem 2.1.4 ([AS69])**.** *If $\mathrm{K}^*_G(X)$ is finite over $\mathrm{R}^*(G)$, then the above map of pro-rings is an isomorphism.*

Their proof has another interesting aspect. Although it is a statement about the K_G of some class of G-spaces, for one specific group G, their proof involves the K_G's of several groups. In particular, to prove the result for example for a finite group, their proof involves more general compact Lie groups. The proof consists of four steps. In every step we show the theorem holds for a more general type of group:

- $G = \mathrm{U}(1)$ (circle group),
- $G = \mathrm{U}(1)^n$ (torus group),
- $G = \mathrm{U}(n)$,
- G a general compact Lie group (this step is proven by G embedding in $\mathrm{U}(n)$).

We note that the first formulation of the Atiyah-Segal theorem 2.1.1, is indeed a private case of the second formulation 2.1.4. Take the case $X = *$. By definition, $\mathrm{K}^*_G(*)$ is finite over $\mathrm{R}^*(G) = \mathrm{K}^*_G(*)$, so 2.1.4 holds and we have an isomorphism of pro-rings $\left\{\mathrm{K}^*_G(*) / (I^*)^n\right\}_n \to \left\{\mathrm{K}^*_G(\mathrm{E}G_n)\right\}_n$. In particular, after computing the limits $\lim : \mathrm{Pro\,Ring} \to \mathrm{Ring}$, we get an isomorphism of rings $\mathrm{K}_G^*(*)^\wedge_{I^*} \xrightarrow{\sim} \mathrm{K}_G^*(\mathrm{E}G)$. Taking only the 0-th cohomology gives the desired isomorphism:

$$\mathrm{R}(G)^\wedge_I \cong \mathrm{K}_G(*)^\wedge_I \xrightarrow{\sim} \mathrm{K}_G(\mathrm{E}G) \cong \mathrm{K}(\mathrm{E}G/G) = \mathrm{K}(\mathrm{B}G)$$

2.2 Examples

We compute a few examples in detail, to make the isomorphism more vivid.

2.2.1 $\mathrm{U}(1)$, the Circle Group

Take $G = \mathrm{U}(1)$, the circle group. It is known that the irreducible representations are of dimension 1 and labeled by an integer $m \in \mathbb{Z}$, i.e. they are homomorphisms $\rho_m = \mathrm{U}(1) \to \mathrm{U}(1)$ given by $\rho_m\left(e^{i\theta}\right) = e^{mi\theta}$. In particular, $\rho_0 = 1$ is the trivial representation. It is then clear that for $m \geq 0$, $\rho_1^{\otimes m} = \rho_m$ and $\rho_{-1}^{\otimes -m} = \rho_{-m}$. Therefore the representation ring generated under (virtual) direct sums and tensor products by ρ_1 and ρ_{-1}. Moreover, $\rho_1 \otimes \rho_{-1} = 1$. Therefore we conclude that $\mathrm{R}(\mathrm{U}(1)) = \mathbb{Z}\left[\rho, \rho^{-1}\right]$.

The augmentation map is the homomorphism $\varepsilon : \mathrm{R}(\mathrm{U}(1)) \to \mathbb{Z}$ which sends $1, \rho$ and ρ^{-1} to 1. Recall that the augmentation ideal is $I = \ker\varepsilon$. We set $t = \rho - 1$, which clearly belongs to I. We can also write then $\mathrm{R}(\mathrm{U}(1)) = \mathbb{Z}\left[t, (1+t)^{-1}\right]$. Note that ε factors to a map $\mathrm{R}(\mathrm{U}(1))/(t) \to \mathbb{Z}$, which is already an isomorphism, so by the first isomorphism theorem indeed $I = (t)$.

We compute the completion $\mathrm{R}(\mathrm{U}(1))^\wedge_I$. Note that in $\mathbb{Z}[t]/t^n$, $1+t$ is already invertible. The reason is that the formal power series for the inverse is finite since large enough powers of t are zero, $\frac{1}{1-(-t)} = \sum_{m=0}^{n-1}(-t)^m$ is an inverse. Therefore we see that $\mathrm{R}(\mathrm{U}(1))/I^n \cong \mathbb{Z}[t]/t^n$, and clearly the maps the in the limit diagram send t to t. We get that $\mathrm{R}(\mathrm{U}(1))^\wedge_I = \lim \mathbb{Z}[t]/t^n \cong \mathbb{Z}[[t]]$.

In [Hat17, proposition 2.24], it is shown that $\mathrm{K}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}[L]/(L-1)^{n+1}$, where L is the canonical line bundle on $\mathbb{C}\mathrm{P}^n$. In 1.2.3 we denoted $t = L - 1$ (warning: there we looked at K^*, now we focus on K), which allows us to rewrite this as $\mathrm{K}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}[t]/t^{n+1}$. As we noted, the limit is $\mathrm{K}(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}[[t]]$. We thus see that $\mathrm{K}(\mathrm{BU}(1)) \cong \mathbb{Z}[[t]]$ where $t = L-1$ is the canonical line bundle minus 1.

The identity map $\rho : \mathrm{U}(1) \to \mathrm{U}(1)$, is mapped to the identity $\mathrm{BU}(1) \to \mathrm{BU}(1)$ (by functoriality of B), which tautologically corresponds the universal line bundle L on $\mathrm{BU}(1)$. We therefore see that the Atiyah-Segal map $\mathrm{R}(\mathrm{U}(1)) \to \mathrm{K}(\mathrm{BU}(1))$, sends ρ to L, and therefore $t = \rho - 1$ to $t = L - 1$. Which indeed shows that the map admits $\mathrm{K}(\mathrm{BU}(1)) \cong \mathbb{Z}[[t]]$ as the $I = (t)$-completion of $\mathrm{R}(\mathrm{U}(1))$.

2.2.2 $\mathbb{Z}/2$, Cyclic Group of Order 2

Take $G = \mathbb{Z}/2$. Here we have only two irreducible representations, the trivial, and $\rho(0) = 1, \rho(1) = -1$. Also, it is clear that $\rho \otimes \rho$ is the trivial representation. Therefore, $\mathrm{R}(\mathbb{Z}/2) = \mathbb{Z}[\rho]/\left(\rho^2 - 1\right)$. Similarly to before, the augmentation $\varepsilon : \mathrm{R}(\mathbb{Z}/2) \to \mathbb{Z}$ sends $1, \rho$ to 1, so clearly $(\rho - 1) \subseteq I$, and for the same reasoning as in the previous example this is actually an equality. We change coordinates to $t = \rho - 1$, and we have $\mathrm{R}(\mathbb{Z}/2) = \mathbb{Z}[t]/\left(t^2 + 2t\right)$, and $I = (t)$.

We move to computing the completion $\mathrm{R}(\mathbb{Z}/2)^{\wedge}_I$. Modulo $t^2 + 2t$, i.e. $t^2 = -2t$, we have that $t^n = (-2)^{n-1} t$. Thus $I^n = \left((-2)^{n-1} t\right) = \left(2^{n-1} t\right)$, so $\mathrm{R}(\mathbb{Z}/2)/I^n = \mathbb{Z}[t]/\left(t^2 + 2t, 2^{n-1}t\right)$. We first compute the limit of $\mathrm{R}(\mathbb{Z}/2)/I^n$ in abelian groups. Since the forgetful functor from rings to abelian groups is right adjoint, it commutes with limits, so this will give us the abelian group structure. As an abelian group, $\mathrm{R}(\mathbb{Z}/2)/I^n$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2^{n-1}\{t\}$. It is then clear that as an abelian group, $\lim \mathrm{R}(\mathbb{Z}/2)/I^n$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2\{t\}$.

We now define a multiplication on $\mathbb{Z} \oplus \mathbb{Z}_2\{t\}$ abelian group, given by $(a + bt) * (c + dt) = ac + (ad + bc - 2bd)t$. It can be checked that it is associative and commutative. We have homomorphisms of groups $\mathbb{Z} \oplus \mathbb{Z}_2\{t\} \to \mathbb{Z}[t]/\left(t^2 + 2t, 2^{n-1}t\right)$, admitting it as the limit in groups, which are given by sending $a + bt$ to $a + \left(b \mod 2^{n-1}\right) t$. By construction this homomorphism is actually a homomorphism of rings (the $-2bdt$ term is explained by the relation $t^2 + 2t$). Therefore, by the universal property of the limit, we get a map $\mathbb{Z} \oplus \mathbb{Z}_2\{t\} \to \lim \mathrm{R}(\mathbb{Z}/2)/I^n$ in rings. After taking the forgetful we know that it becomes an isomorphism, but the forgetful reflects isomorphisms, so this is also an isomorphism in rings.

Using the Atiyah-Segal theorem we conclude that

$$\mathrm{K}(\mathbb{R}\mathrm{P}^\infty) = \mathrm{K}(\mathrm{B}\mathbb{Z}/2) \cong \mathbb{Z} \oplus \mathbb{Z}_2\{t\},$$

with multiplication given by $(a+bt)*(c+dt) = ac + (ad+bc-2bd)t$.

2.3 Recollections from Character Theory

We restrict ourselves to the case of finite groups G. We recall that representations of groups can be studied by their characters. Specifically the character map $\chi : \mathrm{R}(G) \to \mathbb{Z}[\chi_{\rho_i}]$, defined by $\chi_\rho = \operatorname{tr}\rho$, is an isomorphism, where the ring on the right is the ring of functions generated by the irreducible characters (the multiplication of two characters is a character so it is indeed closed under multiplication).

We also recall that characters are class functions, that is, they are constant on conjugacy classes. Let L be some field containing all the values of all characters. Then a natural place to study characters is in the ring of class functions with values in L, denote by $\mathrm{Cl}(G;L)$. Let us phrase this in a way that will be useful in the next section. G is equipped with a G-action by conjugation, $\gamma.g = \gamma g \gamma^{-1}$. Equip L with the trivial G-action. Then $\mathrm{Cl}(G;L) = \hom_{G\mathrm{Set}}(G, L)$.

We can of course extend the range of the character map to get an injection $\chi : \mathrm{R}(G) \to \mathrm{Cl}(G;L)$. The first classical theorem regarding the relationship between characters and class functions is:

Theorem 2.3.1. *After tensoring with L, the character map $\chi \otimes L : \mathrm{R}(G) \otimes L \xrightarrow{\sim} \mathrm{Cl}(G;L)$ becomes an isomorphism.*

Proof. Similarly to the proof in [Ser77, 9.1] for $L = \mathbb{C}$, we can view $\mathrm{Cl}(G;L)$ as a vector space over L, and the characters are linearly independent, so by counting them we see that the image of $\chi \otimes L$ has the dimension of the whole vector space and we are done. $\square$

By definition the value of a character is the trace of a linear transformation $\chi_\rho(g) = \operatorname{tr}\rho(g) = \sum \lambda_i$ where λ_i are the eigenvalues (which exist since the representation is unitary). Since $g^{|G|} = e$, we get $\rho\left(g^{|G|}\right) = \rho(e) = \mathrm{id}$, but then we get that the eigenvalues of $\rho\left(g^{|G|}\right)$ are on the one hand $\lambda_i^{|G|}$ and on the other hand they are all 1, which means that all the eigenvalues are roots of unity. Therefore $L = \mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\zeta_\infty)$ is always a valid choice for L (regardless of G). To be concrete, we will take this choice.

The Galois group of $\mathbb{Q}^{\mathrm{ab}}$ is $\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right) \cong \hat{\mathbb{Z}}^\times$. For every $m \in \hat{\mathbb{Z}}^\times$ we also denote by $m \in \mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)$ the corresponding element, which can be described as the homomorphism which raises a root of unity to the power of m. Similarly it acts on G by taking g to g^m. Then, for every such m and g we have that $\chi_\rho(g^m) = \operatorname{tr}\rho(g) = \sum \lambda_i^m = m.(\sum \lambda_i) = m.(\chi_\rho(g))$. We replace g with $g^{m^{-1}}$ (m is invertible), and rewrite this as $\chi_\rho(g) = m.\left(\chi_\rho\left(g^{m^{-1}}\right)\right)$. Similarly to this equality, we can define an action of $\operatorname{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)$ on $\operatorname{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)$, by taking a class function f to $m.f$ defined by $(m.f)(g) = m.\left(f\left(g^{m^{-1}}\right)\right)$.

Let us rewrite this action in another way, which will be helpful in the next section. We note that $G \cong \hom_{\mathrm{TopGrp}}\left(\hat{\mathbb{Z}}, G\right)$ (continuous homomorphisms). We get an action of $\operatorname{Aut}\left(\hat{\mathbb{Z}}\right) \cong \hat{\mathbb{Z}}^\times$ on G by pre-composition. Concretely, $m \in \hat{\mathbb{Z}}^\times$ acts by sending $g \in G$ to g^m. Since $\hat{\mathbb{Z}}^\times$ acts on $\mathbb{Q}^{\mathrm{ab}}$, we get an action on $\operatorname{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right) = \hom_{G\mathrm{Set}}\left(G,\mathbb{Q}^{\mathrm{ab}}\right)$ by acting with m^{-1} in the source and with m in the target. It is evident that this is the same action from the previous paragraph.

As we just saw, the characters are in the fixed points $\operatorname{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\operatorname{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$. Also, since the rationals are fixed by the action of the Galois group, rational linear combinations of characters are in the fixed points. We therefore conclude that the character map after tensoring with $\mathbb{Q}$ lands in the fixed points, i.e. $\chi\otimes\mathbb{Q} : \mathrm{R}(G)\otimes\mathbb{Q} \to \operatorname{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\operatorname{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$. Moreover, the second classical theorem is:

Theorem 2.3.2 ([Ser77, Theorem 25])**.** *The map* $\chi\otimes\mathbb{Q} : \mathrm{R}(G)\otimes\mathbb{Q} \xrightarrow{\sim} \operatorname{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\operatorname{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$ *is an isomorphism.*

To conclude, 2.3.1 tells us that $\mathrm{R}(G)\otimes\mathbb{Q}^{\mathrm{ab}} \cong \operatorname{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)$, and 2.3.2 tells us that $\mathrm{R}(G)\otimes\mathbb{Q} \cong \operatorname{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\operatorname{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$.

3 HKR Character Theory

As we have seen in the previous section, Atiyah and Segal gave a description of $\mathrm{K}(\mathrm{B}G)$ in terms of the representation ring. We have also seen in the section on chromatic homotopy theory, that complex K-theory is related to Morava K-theory at height 1 by 1.3.11, and to Morava E-theory at height 1 by 1.7.8. Representations can be studied using their characters, and one may wonder if a similar construction can be used to study higher analogues of complex K-theory evaluated at $\mathrm{B}G$.

Hopkins, Kuhn and Ravenel showed in [HKR00] that it is indeed possible. Their paper contains a lot of results, but we will concentrate on Theorem C. Fix some finite group G. Similarly to the proof Atiyah-Segal theorem, the actual proof of Theorem C involves a general construction, even to prove the specific case we are interested, but it will be easier to state it first for the specific case. Let $E = \mathrm{E}(k, \Gamma)$ be the Lubin-Tate spectrum from 1.7.7, for some field k of characteristic p, and Γ a formal group law over k of height n. We will also denote by F the formal group law on E_* (which is $u\Gamma_U$, as in 1.7.7). There is some ring $L = L(E^*)$ (which depends on the spectrum E). It is then possible define some generalized characters $\mathrm{Cl}_{n,p}(G; L)$, which are completely algebraic and combinatorial (besides the definition of the ring L). Lastly, there is a character map $\chi_{n,p}^G : E^*(\mathrm{B}G) \to \mathrm{Cl}_{n,p}(G; L)$. This character map has similar formal properties to the ordinary character map, namely, similarly to 2.3.1, after tensoring with L, the character map

$$\chi_{n,p}^G \otimes L : E^*(\mathrm{B}G) \otimes L \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G; L)$$

becomes an isomorphism. Similarly to 2.3.2, there is an action of $\mathrm{Aut}\left(\mathbb{Z}_p^n\right) \cong \left(\mathbb{Z}_p^\times\right)^n$ on $\mathrm{Cl}_{n,p}(G; L)$, and it turns out that the character map lands in the fixed points. Moreover, we can merely rationalize, which is given by inverting p, the source, rather tensoring with L. It turns out that after rationalization and restricting the codomain to the fixed points, the map becomes an isomorphism, that is

$$p^{-1}\chi_{n,p}^G : p^{-1}E^*(\mathrm{B}G) \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G; L)^{\mathrm{Aut}\left(\mathbb{Z}_p^n\right)}$$

is an isomorphism.

TODO write something about the structure of this section

3.1 Towards a Definition of the Character Map

Following [HKR00], we denote by $\Lambda_r = (\mathbb{Z}/p^r)^n$ and $\Lambda = \mathbb{Z}_p^n$.

An element $g \in G$ is called *p-power-torsion* if $g^{p^a} = e$ for some a. Note that a conjugation of a p-power-torsion element is again p-power-torsion. We also define $r_0 \in \mathbb{N}$ to be the minimal r s.t. every p-power-torsion element g is p^{r_0}-torsion, i.e. satisfies $g^{p^{r_0}} = e$.

Definition 3.1.1. We define $G_{n,p}$ to be the set of n-tuples $(g_1, \dots, g_n)$ of commuting p-power-torsion elements. It has a G-action by conjugation, $\gamma. (g_1, \dots, g_n) = (\gamma g_1 \gamma^{-1}, \dots, \gamma g_n \gamma^{-1})$.

Concretely, for $r \geq r_0$, we have $G_{n,p} = \hom_{\mathrm{Grp}} (\Lambda_r, G)$, with the G-action by conjugation at the values. In a similar fashion, $G_{n,p} = \hom_{\mathrm{TopGrp}} (\Lambda, G)$ (the homomorphisms are required to be continuous).

Let R be a ring. Equip it with the trivial G-action.

Definition 3.1.2. The *class functions* are $\mathrm{Cl}_{n,p} (G; R) = \hom_{G\mathrm{Set}} (G_{n,p}, R)$, that is functions from $G_{n,p}$ to R which are invariant under conjugation.

This is a ring, by defining the operations point-wise. Note that this is a purely combinatorial construction, just a copy of R for every orbit of $G_{n,p}/G$, that is $\mathrm{Cl}_{n,p} (G; R) \cong \bigoplus_{[\alpha] \in G_{n,p}/G} R$.

We would like to construct a character map $E^* (\mathrm{B}G) \to \mathrm{Cl}_{n,p} (G; R)$, for some R, which depends on $r \geq r_0$. We will try to unravel what this means, and find appropriate R's at the same time. By the above, this is a homomorphism $E^* (\mathrm{B}G) \to \bigoplus_{[\alpha] \in G_{n,p}/G} R$. That is, for every $[\alpha] \in G_{n,p}/G$ we need to provide a homomorphism $E^* (\mathrm{B}G) \to R$. Choose a representative $\alpha \in G_{n,p} = \hom_{\mathrm{Grp}} (\Lambda_r, G)$ (for $r \geq r_0$). Since B is a functor we get $\mathrm{B}\alpha : \mathrm{B}\Lambda_r \to \mathrm{B}G$, and then we can take E^*-cohomology to get a homomorphism $\mathrm{B}\alpha^* : E^* (\mathrm{B}G) \to E^* (\mathrm{B}\Lambda_r)$. If we had a homomorphism $E^* (\mathrm{B}\Lambda_r) \to R$, we would indeed get a well-defined character map.

3.2 The E^*-Cohomology of $\mathrm{B}A$ and Their Maps

We postpone the discussion of the rings, to give some properties of E^*-cohomology.

First we describe $E^* (\mathrm{B}\mathbb{Z}/m)$. Let $\psi_m : \mathbb{Z}/m \to \mathrm{U}(1)$ be the homomorphism $\psi(1) = e^{2\pi i/m}$ This induces $\mathrm{B}\psi^* : E^* (\mathrm{BU}(1)) \to E^* (\mathrm{B}\mathbb{Z}/m)$. Denote by $x \in E^2 (\mathrm{B}\mathbb{Z}/m)$ the cohomology class $\mathrm{B}\psi^* (x)$.

Proposition 3.2.1 ([HKR00, 5.8])**.** *The E^*-cohomology of $\mathrm{B}\mathbb{Z}/m$ is given by $E^* (\mathrm{B}\mathbb{Z}/m) = E^* [[x]] / ([m] (x))$. Write $m = sp^t$ with a coprime to p, then this is a free E^*-module of rank p^{nt}.*

TODO I don't understand the proof, try to understand and write here

Proposition 3.2.2. *Let Y be a space s.t. $E^*(Y)$ is a free E^*-module of finite rank. Then Y satisfies Kunneth with respect to any X, that is, the map $E^*(X) \otimes_{E^*} E^*(Y) \xrightarrow{\sim} E^*(X \times Y)$ is an isomorphism.*

Proof. Look at the functors $X \mapsto E^*(X) \otimes_{E^*} E^*(Y)$ and $X \mapsto E^*(X \times Y)$. Both of them are manifestly homotopy invariant. Since $E^*(Y)$ is free, it is also flat, and so both functors satisfy Mayer-Vietoris. Both functors send arbitrary wedges to arbitrary products, since tensor with a free finite rank modules commutes with arbitrary products. We conclude that they are both cohomology theories. Moreover, they agree on $X = *$, and therefore are isomorphic. $\square$

Using both propositions we can bootstrap to arbitrary finite abelian groups.

Proposition 3.2.3 ([HKR00, 5.8])**.** *Let A be an abelian group, and write $|A| = sp^t$ for s coprime to p. Then $E^*(\mathrm{B}A)$ is a free E^*-module of rank p^t, and $\mathrm{B}A$ satisfies Kunneth. Specifically, for $A = \mathbb{Z}/m_1 \oplus \cdots \oplus \mathbb{Z}/m_l$:*

$$E^*(\mathrm{B}\mathbb{Z}/m_1 \times \cdots \times \mathrm{B}\mathbb{Z}/m_l) \cong E^*[[x_1, \ldots, x_l]] / ([m_1](x_1), \ldots, [m_l](x_l)).$$

Proof. A finite abelian group is the product of finite cyclic groups. Since B commutes with products, we can induct on the number of components in the product. $\square$

TODO make the rest of the subsection more readable... maybe give a coordinate-free statement

Recall that the formal group law on E_* was defined by taking the $E^*(\mathrm{B}-)$ to the multiplication map $\mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{U}(1)$. That is, this map induces the map $x \mapsto F(y_1, y_2) = y_1 +_F y_2$ on the cohomology.

By pre-composing with the diagonal, and doing this for k-copies of $\mathrm{U}(1)$, we see that the multiplication-by-k map $\mathrm{U}(1) \xrightarrow{k} \mathrm{U}(1)$ induces the map $E^*[[x]] \to E^*[[y]]$ given by $x \mapsto [k](y)$. Moreover, it follows that $[l](x) \mapsto [kl](y)$.

Therefore, for a map $\bigoplus_{i=1}^t \mathrm{U}(1) \to \bigoplus_{j=1}^s \mathrm{U}(1)$, given on the i,j-th coordinate by multiplication-by-k_{ij}, induces a map $E^*[[x_1, \ldots, x_s]] \to E^*[[y_1, \ldots, y_t]]$

given by $x_j \mapsto \sum_F [k_{ij}](y_i)$. From this it follows that:

$$\begin{aligned}\sum_{j,F} [l_j](x_j) &\mapsto \sum_{j,F} [l_j] \left(\sum_{i,F} [k_{ij}](y_i) \right) \\ &= \sum_{j,F} \sum_{i,F} [k_{ij} l_j](y_i) \\ &= \sum_{i,F} \left[\sum_j k_{ij} l_j \right] (y_i)\end{aligned}$$

Now, let $k : \bigoplus_{i=1}^t \mathbb{Z}/m_i \to \bigoplus_{j=1}^s \mathbb{Z}/\mu_j$, given on the i,j-th coordinate by multiplication-by-k_{ij}. Recall the maps $\psi_m : \mathbb{Z}/m \to \mathrm{U}(1)$ given by $1 \mapsto e^{2\pi i/m}$. We look at the maps $\bigoplus_{i=1}^t \psi_{m_i} : \bigoplus_{i=1}^t \mathbb{Z}/m_i \to \bigoplus_{i=1}^t \mathrm{U}(1)$, and similarly $\bigoplus_{j=1}^s \psi_{\mu_j}$. The composition $\left(\bigoplus_{j=1}^s \psi_{\mu_j} \right) \circ k$ is given on the i,j-th coordinate by $1 \mapsto k_{ij} \mapsto e^{2\pi i k_{ij}/m_j}$. Define a map $k : \bigoplus_{i=1}^t \mathrm{U}(1) \to \bigoplus_{j=1}^s \mathrm{U}(1)$, by letting the i,j-th coordinate being the multiplication-by-k_{ij} map. We then get the commutative diagram:

$$\begin{tikzcd} \bigoplus_{i=1}^t \mathrm{U}(1) \arrow[r, "k"] & \bigoplus_{j=1}^s \mathrm{U}(1) \\ \bigoplus_{i=1}^t \mathbb{Z}/m_i \arrow[u, hook, "\bigoplus_{i=1}^t \psi_{\mu_i}"] \arrow[r, "k"] & \bigoplus_{j=1}^s \mathbb{Z}/\mu_j \arrow[u, hook, "\bigoplus_{j=1}^s \psi_{\mu_j}"] \end{tikzcd}$$

By taking $E^*(\mathrm{B}-)$ we get the commutative diagram:

$$\begin{tikzcd} E^*[[y_1, \ldots, y_t]] \arrow[d] & E^*[[x_1, \ldots, x_s]] \arrow[l] \arrow[d] \\ E^*[[y_1, \ldots, y_t]] / ([m_i](y_i)) & E^*[[x_1, \ldots, x_s]] / ([\mu_j](x_j)) \arrow[l] \end{tikzcd}$$

Where the vertical maps are given by $y_i \mapsto y_i$ and $x_j \mapsto x_j$. We have computed the upper map before, so we conclude:

Proposition 3.2.4. *Let $k : \bigoplus_{i=1}^t \mathbb{Z}/m_i \to \bigoplus_{j=1}^s \mathbb{Z}/\mu_j$ be given on the i,j-th coordinate by multiplication-by-k_{ij}. After taking $E^*(\mathrm{B}-)$, it induces the map given by $x_j \mapsto \sum_F [k_{ij}](y_i)$. Moreover, for integers $l_1, \ldots, l_s$, it gives $\sum_{j,F} [l_j](x_j) \mapsto \sum_{i,F} \left[\sum_j k_{ij} l_j \right] (y_i)$.*

Remark 3.2.5. Note that the k_{ij}'s are determined only modulo $\mathbb{Z}/\mu_j$, and the proposition implies that the value is independent of the their lifts to $\mathbb{Z}$. **TODO is that true?**

3.3 The Rings $L_r(E^*)$ and $L(E^*)$

As we have seen, to construct the character map we needed a ring R together with homomorphisms $E^*(\mathrm{B}\Lambda_r) \to R$. We will construct such a ring, $L_r = L_r(E^*)$. Moreover, we will take a colimit to construct a ring $L = L(E^*)$, which contains the information from all of those together.

Recall that $E^*(\mathrm{BU}(1)) = E^*[[x]]$, where x is the complex orientation. For any homomorphism $\alpha : \Lambda_r \to \mathrm{U}(1)$, we can take $E^*(\mathrm{B}-)$ to get $\mathrm{B}\alpha^* : E^*(\mathrm{U}(1)) \to E^*(\mathrm{B}\Lambda_r)$. Let $S_r = \{\mathrm{B}\alpha^*(x) \mid \alpha : \Lambda_r \to \mathrm{U}(1), \alpha \neq 1\} \subseteq E^*(\mathrm{B}\Lambda_r)$.

Definition 3.3.1. We define $L_r = S_r^{-1}E^*(\mathrm{B}\Lambda_r)$. There is indeed a map $E^*(\mathrm{B}\Lambda_r) \to L_r$, namely the localization map.

We wish to describe the above construction with coordinates, to make it more explicit. Recall that $E^*(\mathrm{B}\Lambda_r) \cong E^*[[x_1,\dots,x_n]]/([p^r](x_1),\dots,[p^r](x_n))$ by 3.2.3. Let $\Lambda_r \xrightarrow{\alpha} S^1$ be a homomorphism. Since it lands in the p^r-torsion, it factors as $\Lambda_r \xrightarrow{k} \mathbb{Z}/p^r \xrightarrow{\psi_{p^r}} \mathrm{U}(1)$, where k is given on the i-th coordinate by multiplication-by-k_i. The condition $\alpha \neq 1$ amounts to the condition $(k_1,\dots,k_n) \neq 0 \mod p^r$. By 3.2.4, the induced map is given by $\mathrm{B}\alpha^*(x) = \sum_F [k_i](x_i)$. Therefore $S_r = \{\sum_F [k_i](x_i) \mid (k_1,\dots,k_n) \neq 0 \mod p^r\}$.

Proposition 3.3.2. *The map $E^*(\mathrm{B}\Lambda_r) \to E^*(\mathrm{B}\Lambda_{r+1})$ induced by the projection $\Lambda_{r+1} \to \Lambda_r$, lifts to a map $L_r \to L_{r+1}$.*

Proof. The projection $\Lambda_{r+1} \to \Lambda_r$ is given by the multiplication-by-1 on each coordinate, so again by 3.2.4 they induce the maps $E^*(\mathrm{B}\Lambda_r) \to E^*(\mathrm{B}\Lambda_{r+1})$, given by $x_i \mapsto x_i$. Moreover $\sum_F [k_i](x_i) \in S^r$ is mapped to $\sum_F [k_i](x_i) \in S^{r+1}$. Therefore, once we localize the target, the map lifts to the localization of the source. $\square$

Definition 3.3.3. We define $L = L(E^*) = \operatorname{colim} L_r$.

Clearly $\mathrm{Aut}(\Lambda_r)$ acts on $E^*(\mathrm{B}\Lambda_r)$, by its action on Λ_r and functoriality.

Proposition 3.3.4. *The* $\mathrm{Aut}(\Lambda_r)$*-action lifts to an action on L_r.*

Proof. Let $k : \Lambda_r \to \Lambda_r$ be an automorphism given by on the i,j-th coordinate by multiplication by k_{ij}. Once again, by 3.2.4, for integers $l_1,\dots,l_n$, the induced map sends $\sum_{j,F}[l_j](x_j)$ to $\sum_{i,F}\left[\sum_j k_{ij}l_j\right](x_i)$. Since k is an automorphism, the matrix (k_{ij}) is invertible. Therefore, if $(l_1, \ldots, l_n) \neq 0$, then also $\left(\sum_j k_{1j} l_j, \ldots, \sum_j k_{1j} l_j\right) \neq 0$, so if the source is in S^r, the result is in S^r as well. This shows that action lifts to an action on L_r. $\square$

TODO Don't forget to correct the thesis... I used $\sum$ rather than $\sum_F$.

Using the projection $\pi_r : \mathrm{Aut}(\Lambda) \to \mathrm{Aut}(\Lambda_r)$ we endow L_r with a $\mathrm{Aut}(\Lambda)$-action. By factoring the projection through $\mathrm{Aut}(\Lambda_{r+1})$, we see that the map $L_r \to L_{r+1}$ is equivariant with respect to that action. In conclusion:

Proposition 3.3.5. *The rings L_r have an $\mathrm{Aut}(\Lambda)$-action, and the maps $L_r \to L_{r+1}$ are equivariant with respect to this action. Therefore L has an $\mathrm{Aut}(\Lambda)$-action as well.*

One may wonder if the ring L_r is the zero ring. An argument in [HKR00] shows that this isn't the case, and even more is true.

Proposition 3.3.6 ([HKR00, 6.5, 6.6, 6.8])**.** *The element p is invertible in L, so L is a $p^{-1}E^*$-module. Furthermore, $L^{\mathrm{Aut}(\Lambda)} = p^{-1}E^*$, and L is faithfully flat over it. Moreover, this holds when L is replaced with L_r.*

Remark 3.3.7 (Algebro-geometric interpretation of L_r). First we wish to simplify the situation a little bit. Recall from 1.7.7 that we have a formal group law $F = u\Gamma_U$ over $\mathrm{E}(k,\Gamma)_* = Wk[[u_1, \ldots, u_{n-1}]]\left[u^{\pm 1}\right]$. This came from the computation $\mathrm{E}(k,\Gamma)_*(\mathrm{BU}(1)) = \mathrm{E}(k,\Gamma)_*[[x]]$ with $|x| = 2$. It will be more convenient to work with $t = u^{-1}x$, which lives in degree 0, similarly to 1.2.11. On these elements, the formal group law acts the same as Γ_U. By the invertibility of u, we can do the computation of L_r with these elements, and the results will not be affected. Moreover, since everything is defined already over $E^0 = Wk[[u_1, \ldots, u_{n-1}]]$, we can do all the computations over it, which will give the ring L_r^0, the 0-th degree part of L_r, and add $u^{\pm 1}$ at the end to get L_r.

The formal group law Γ_U over E^0, gives a formal group $\mathbb{G} = \mathrm{Spf}\, E^0[[t]]$ with multiplication $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$. By definition, the p^r-torision elements in $\mathbb{G}$, is the scheme-theoretic kernel of the multiplication-by-p^r map $[p^r] : \mathbb{G} \to \mathbb{G}$, that is $\mathbb{G}[p^r] = \mathrm{Spec}\left(R[[t]] / ([p^r](t))\right) = \mathrm{Spec}\, E^0(\mathrm{B}\mathbb{Z}/p^r)$. Since Γ_U is of height n, the leading term of the p^r-series, is $t^{p^{rn}}$, which shows that the number of points in $\mathbb{G}[p^r]$ is p^{rn}. We also see that $\mathrm{Spec}\, E^0(\mathrm{B}\Lambda_r) = (\mathbb{G}[p^r])^n$.

Inverting S_r is equivalent to inverting their 0-th graded analogues $S_r^0 = \left\{\sum_{\Gamma_U} [k_i](t_i) \mid (k_1, \ldots, k_n) \neq 0 \mod p^r\right\}$. Algebro-geometrically, this is equivalent to restricting to the open subset where all the functions $\sum_{\Gamma_U} [k_i](t_i)$ don't vanish (recall that the value depends only the values of k_i modulo $\mathbb{Z}/p^r$). That is, L_r^0 is the open subset of n-points in $\mathbb{G}[p^r]$, that is p^r-torsion, which are linearly independent. In fact, as a group $\mathbb{G}[p^r]$ is $(\mathbb{Z}/p^r)^n = \Lambda_r$, so the this means that the n-points are a basis for $\mathbb{G}[p^r]$.

Moreover, if $\sum_{\Gamma_U} [k_i](t_i) = 0$ then also $\sum_{\Gamma_U} [pk_i](t_i) = [p]\left(\sum_{\Gamma_U} [k_i](t_i)\right) = 0$. So, if any pk_i is not 0 modulo p^r, inverting $\sum_{\Gamma_U} [pk_i](t_i)$ already inverts $\sum_{\Gamma_U} [k_i](t_i)$. Well, $pk_i = 0 \mod p^r$ if and only if $k_i = 0 \mod p^{r-1}$, which shows that we can invert only those where k_i is a multiple of p^{r-1}. Since there are n k_i's, each of them can take any of p values (numbers which are a multiple of p^{r-1}), and not all 0, we need to invert only $p^n - 1$ elements.

The description of L_r^0 as a basis for Λ_r also shows clearly where the action of $\mathrm{Aut}(\Lambda_r)$ comes from, it just changes by multiplying by an invertible matrix.

Example 3.3.8. We continue with complex K-theory. Recall from 1.7.8 that $\mathrm{K}_p^\wedge \cong \mathrm{E}(\mathbb{F}_p, \Gamma)$, where Γ is the multiplicative formal group law, $\Gamma(y, z) = y + z$. That is, p-complete K-theory is a Lubin-Tate spectrum at height $n = 1$, so the the construction above applies to it. The computation that follows should be related to Atiyah-Segal, although there we considered K itself, and here $\mathrm{K}_p^\wedge$.

In 1.6.8, we saw that $\left(\mathrm{K}_p^\wedge\right)_* = \mathbb{Z}_p\left[\beta^{\pm 1}\right]$. As in 3.3.7, it is easier to work with the element $t = \beta^{-1}x$ and the formal group law $u + v + uv$ over $\left(\mathrm{K}_p^\wedge\right)_0 = \mathbb{Z}_p$. The n-series then is $[n](t) = (1+t)^n - 1$.

In our case $\Lambda_r = \mathbb{Z}/p^r$, and we have $\left(\mathrm{K}_p^\wedge\right)^0 (\mathrm{B}\mathbb{Z}/p^r) \cong \mathbb{Z}_p[[t]] / ([p^r](t)) \cong \mathbb{Z}_p[[t]] / \left((1+t)^{p^r} - 1\right)$. This shows that the homomorphism induced by the inclusion of the same ring with $[[t]]$ replaced by $[t]$ is an isomorphism, so the ring is isomorphic to $\mathbb{Z}_p[t] / \left((1+t)^{p^r} - 1\right)$. This is in accordance with the view taken in 3.3.7, since the scheme corresponding to this are the points where $(1+t)^{p^r} = 1$, i.e. when $1 + t$ is p^r-torsion under the multiplication $(1+u)(1+v) = 1 + u + v + uv$, and this is also clearly isomorphic to $\mathbb{Z}/p^r$ as a group.

To make the computation easier, we change variable $s = 1 + t$, to work with the ring $\mathbb{Z}_p[s] / \left(s^{p^r} - 1\right)$ Under this change of variables, $S_r = \left\{s^k - 1 \mid 0 < k < p^r\right\}$. By 3.3.6, p is invertible in the localization, so might as well invert it before inverting S_r. We then denote $R = \mathbb{Q}_p[s]/\left(s^{p^r}-1\right)$, and we are trying to compute ${S_r}^{-1}R$. Denote by $\Phi_k(s)$ the k-th cyclotomic polynomial, and by ζ_k a primitive k-th root of unity. We claim that the quotient map $R \to \mathbb{Q}_p[s]/(\Phi_{p^r}) \cong \mathbb{Q}_p(\zeta_{p^r})$, admits the target the S_r-localization of the source. First, note that s^k-1 is sent to $\zeta_{p^r}^k-1$, so for $0<k<p^r$ this is not zero (for ζ_{p^r} is a primitive root of unity of order p^r), and since the target is a field, this is invertible, so there is indeed a map ${S_r}^{-1}R \to \mathbb{Q}_p[s]/(\Phi_{p^r})$.

On the other hand, look at the map $\mathbb{Q}_p[s] \to {S_r}^{-1}R$. Recall that $s^{p^r}-1 = \Phi_{p^r}(s)\left(s^{p^{r-1}}-1\right)$, and $\left(s^{p^r}-1\right)$ is a unit in the target, multiplying by its inverse, we also get that $\Phi_{p^r}(s)$ is zero in ${S_r}^{-1}R$. Therefore, the map actually factors to a map $\mathbb{Q}_p[s]/(\Phi_{p^r}) \to {S_r}^{-1}R$, which is clearly an inverse to the map above.

We conclude that our ring is $L_r^0 = \mathbb{Q}_p(\zeta_{p^r})$. This is again in accordance with 3.3.7, where the elements $1+t$ are now taken to be roots of unity, that is, each of them forms a basis for the group $\mathbb{Z}/p^r$.

The whole graded ring is $L_r = \mathbb{Q}_p(\zeta_{p^r})\left[\beta^{\pm 1}\right]$. From this it is also easy to see that $L = \mathbb{Q}_p(\zeta_{p^\infty})\left[\beta^{\pm 1}\right]$.

3.4 The Generalized Class Functions Ring

As in the case of Atiyah-Segal, in order to establish Theorem C, that is, the result on the character map described in the introduction to this section, we need to formulate a more general theorem. The theorem will also depend on a G-space X, and the proof will use this freedom in a crucial way. Moreover, the proof will rely on passing to other subgroups as well, although unlike in Atiyah-Segal's proof, it will only use abelian subgroups of G. To this end, we define the generalized objects we need.

Let X be a finite G-CW complex. Recall that $r \geq r_0$, we have $G_{n,p} = \hom_{\mathrm{Grp}}(\Lambda_r, G)$, with the G-action by conjugation. Note that for $\alpha \in G_{n,p}$, $X^{\operatorname{im}\alpha}$ is simply the fixed points of X at the n-tuple $(g_1,\dots,g_n)$ represented by α, so the following is independent of $r \geq r_0$.

Definition 3.4.1. The *fixed point space* of X is $\operatorname{Fix}_{n,p}(G,X) = \coprod_{\alpha\in G_{n,p}} X^{\operatorname{im}\alpha}$. This space has a $G \times \operatorname{Aut}(\Lambda_r)$-action, described below.

$\operatorname{Fix}_{n,p}(G,X)$ admits a G-action, where $\gamma \in G$ sends $x \in X^{\operatorname{im}\alpha}$ to $\gamma x \in X^{\operatorname{im}\gamma\cdot\alpha}$. This is well defined, since x fixed by α, means that $g_i x = x$, which implies that $\gamma g_i \gamma^{-1} \gamma x = \gamma x$, so that γx is fixed by $\gamma.\alpha$. Moreover, it admits an $\mathrm{Aut}(\Lambda_r)$-action. Let $\varphi \in \mathrm{Aut}(\Lambda_r)$, for any $\alpha \in G_{n,p}$, clearly $\mathrm{im}\,\alpha = \mathrm{im}(\alpha \circ \varphi)$, so $x \in X^{\mathrm{im}\,\alpha}$ is mapped by φ to $x \in X^{\mathrm{im}(\alpha\circ\varphi)}$. Since G acts on α on the target, and φ acts on the source, the actions commute, therefore we have a $G \times \mathrm{Aut}(\Lambda_r)$-action.

The action on $\mathrm{Fix}_{n,p}(G,X)$ gives a $G \times \mathrm{Aut}(\Lambda_r)$-action of E^*-algebras on $E^*(\mathrm{Fix}_{n,p}(G,X))$. As we saw, L_r admits an $\mathrm{Aut}(\Lambda_r)$-action, define the trivial G-action on it, to get a $G \times \mathrm{Aut}(\Lambda_r)$-action. Take the diagonal $G \times \mathrm{Aut}(\Lambda_r)$-action on $L_r \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G,X))$. **TODO we probably need op action somewhere**.

Definition 3.4.2. The *class functions* are:

$$\mathrm{Cl}_{n,p}(G,X;L_r) = \left(L_r \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G,X))\right)^G$$

This E^*-algebra still has an $\mathrm{Aut}(\Lambda_r)$-action.

Note that for $X = *$, trivial G-space, $\mathrm{Fix}_{n,p}(G,*) = G_{n,p}$ as a G-space. Hence $E^*(\mathrm{Fix}_{n,p}(G,*)) \cong \hom_{\mathrm{Set}}(G_{n,p},E^*)$. Taking the G fixed points clearly gives $\hom_{G\mathrm{Set}}(G_{n,p},E^*)$. In conclusion, we get $\mathrm{Cl}_{n,p}(G,*;L_r) = \hom_{G\mathrm{Set}}(G_{n,p},L_r)$. Note that this agrees with our previous definition 3.1.2.

We give an alternative description of the algebra before taking the G fixed points. Simply by taking the coproduct out of the cohomology as a product, and out of the tensor product, we get:

Proposition 3.4.3. $L_r \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G,X)) \cong \prod_{\alpha \in G_{n,p}} \left(L_r \otimes_{E^*} E^*\left(X^{\mathrm{im}\,\alpha}\right)\right)$.

We wish to emphasize the combinatorial nature of the $G \times \mathrm{Aut}(\Lambda_r)$-action. To that end, we first formulate a general combinatorial statement:

Proposition 3.4.4. *Let H be a group. Let I be some indexing H-set, and let $Y_{[i]}$ be a collection of H-spaces, i.e. indexed by I/H. Endow $\prod_{i\in I} Y_{[i]}$ an H-action by $h.(y_i)_{i\in I} = (h.y_{h.i})_{i\in I}$. Then $\left(\prod_{i\in I} Y_{[i]}\right)^H \cong \prod_{[i]\in I/H} Y_{[i]}^{\mathrm{St}_H(i)}$.*

Proof. First, the action is indeed well defined since $y_{h.i} \in Y_{[i]}$, for i and $h.i$ are in the same orbit. Requiring that $(y_i)_{i\in I}$ is a fixed point amounts to $y_i = h.y_{h.i}$. Note that this condition equates y_i only with values which are in the H-orbit of i. Moreover, the value of y_i determines the whole H-orbit $y_{h.i}$ (by $h^{-1}y_i$). Taking those values for the whole orbit, reduces the condition to $y_i = h.y_i$ when $h.i = i$, i.e. when h is in the stabilizer $\mathrm{St}_H(i)$. Therefore, the condition is simply that $y_i \in Y_{[i]}^{\mathrm{St}_H(i)}$, so the conclusion follows. □

We now want to apply this to our case.

Proposition 3.4.5. *We have:*

$$\begin{aligned}&\mathrm{Cl}_{n,p}\left(G,X;L_r\right)^{\mathrm{Aut}(\Lambda_r)}\\ &\qquad\cong \prod_{[\alpha]\in G_{n,p}/(G\times\mathrm{Aut}(\Lambda_r))}\left(L_r^{\mathrm{St}_{\mathrm{Aut}(\Lambda_r)}(\alpha)}\otimes_{E^*}E^*\left(X^{\mathrm{im}\,\alpha}\right)^{\mathrm{St}_G(\alpha)}\right)\end{aligned}$$

In particular, when $X=*$, we have:

$$\mathrm{Cl}_{n,p}\left(G;L_r\right)^{\mathrm{Aut}(\Lambda_r)}\cong \prod_{[\alpha]\in G_{n,p}/(G\times\mathrm{Aut}(\Lambda_r))} L_r^{\mathrm{St}_{\mathrm{Aut}(\Lambda_r)}(\alpha)}$$

Proof. We will take $G\times\mathrm{Aut}\left(\Lambda_r\right)$ fixed points of 3.4.3 in two steps, first by G then by $\mathrm{Aut}\left(\Lambda_r\right)$. Recall that L_r is fixed by G. The action on the cohomology part comes from the action on the space, taking $x\in X^{\mathrm{im}\,\alpha}$ to $\gamma x\in X^{\mathrm{im}\,\gamma.\alpha}$. Since this is invertible, this gives a homeomorphism between $X^{\mathrm{im}\,\alpha}$ and $X^{\mathrm{im}\,\gamma.\alpha}$, which shows that $E^*\left(X^{\mathrm{im}\,\alpha}\right)\cong E^*\left(X^{\mathrm{im}\,\gamma.\alpha}\right)$. Using this isomorphism, we indeed see that $E^*\left(X^{\mathrm{im}\,\alpha}\right)$ depends only on the G-orbit of α. By using 3.4.4 for 3.4.3, we get:

$$\mathrm{Cl}_{n,p}\left(G,X;L_r\right)\cong\prod_{[\alpha]\in G_{n,p}/G}\left(L_r\otimes_{E^*}E^*\left(X^{\mathrm{im}\,\alpha}\right)^{\mathrm{St}_G(\alpha)}\right)$$

We now have the $\mathrm{Aut}\left(\Lambda_r\right)$-action. Recall that $\mathrm{Aut}\left(\Lambda_r\right)$ didn't act on the space part, since it fixes $\mathrm{im}\,\alpha$. We are then in the situation of 3.4.4 again, and the conclusion follows for the general case. Note that when $X=*$, the trivial G-space, all fixed points $X^{\mathrm{im}\,\alpha}$ are again trivial, which shows that the E^*-cohomology is simply E^*, and tensoring with it does nothing, and the conclusion for the specific case follows. ☐

3.5 The General Character Map

We now construct the character map, that also depends on r, which is omitted from the notation:

$$\chi_{n,p}^G:E^*\left(\mathrm{E}G\times_G X\right)\to\mathrm{Cl}_{n,p}\left(G,X;L_r\right)$$

This map is given by a map $E^*\left(\mathrm{E}G\times_G X\right)\to L_r\otimes_{E^*}E^*\left(\mathrm{Fix}_{n,p}\left(G,X\right)\right)$ which lands in the G fixed points. By 3.4.3, this is the data of a map $E^*\left(\mathrm{E}G\times_G X\right)\to L_r\otimes_{E^*}E^*\left(X^{\mathrm{im}\,\alpha}\right)$ for each $\alpha\in G_{n,p}$.

Let $\alpha \in G_{n,p}$, that is $\alpha : \Lambda_r \to G$. By functoriality of E, this induces a map $\mathrm{E}\Lambda_r \to \mathrm{E}G$. Consider the inclusion $X^{\operatorname{im}\alpha} \to X$. The multiplication of these maps gives $\mathrm{E}\Lambda_r \times X^{\operatorname{im}\alpha} \to \mathrm{E}G \times X$. Since X and $\mathrm{E}G$ have a G-action, the map $\alpha : \Lambda_r \to G$ induces a Λ_r-action. Moreover, this Λ_r-action restricts to a trivial on $X^{\operatorname{im}\alpha}$. We equip both sides with the diagonal Λ_r-action, which makes the map equivariant. The Λ_r orbits of the source are $\mathrm{B}\Lambda_r \times X^{\operatorname{im}\alpha}$. We get a map between the Λ_r orbits, and since the action on the target was pulled form the diagonal G-action, we can further take the G orbits on the target, to get a map $\mathrm{B}\Lambda_r \times X^{\operatorname{im}\alpha} \to \mathrm{E}G \times_G X$.

Taking E^*-cohomology we get $E^*(\mathrm{E}G \times_G X) \to E^*\left(\mathrm{B}\Lambda_r \times X^{\operatorname{im}\alpha}\right)$. Since X was assumed to be a finite G-CW complex, we have Kunneth for the target by 3.2.3, so the map is equivalently a map $E^*(\mathrm{E}G \times_G X) \to E^*(\mathrm{B}\Lambda_r) \otimes_{E^*} E^*\left(X^{\operatorname{im}\alpha}\right)$. Using the localization $E^*(\mathrm{B}\Lambda_r) \to L_r$ we finally get the desired map $E^*(\mathrm{E}G \times_G X) \to L_r \otimes_{E^*} E^*\left(X^{\operatorname{im}\alpha}\right)$. This concludes the construction of the character map.

Proposition 3.5.1 ([HKR00, 6.9])**.** *The map* $E^*(\mathrm{E}G \times_G X) \to L_r \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X))$ *constructed above lands in the* $G \times \mathrm{Aut}(\Lambda_r)$ *fixed points.*

Therefore, by taking the G fixed points on the target, we indeed get the desired character map $\chi^G_{n,p} : E^*(\mathrm{E}G \times_G X) \to \mathrm{Cl}_{n,p}(G, X; L_r)$, which lands in the $\mathrm{Aut}(\Lambda_r)$ fixed points.

Proposition 3.5.2. *The character maps are compatible with the maps* $\mathrm{Cl}_{n,p}(G, X; L_r) \to \mathrm{Cl}_{n,p}(G, X; L_{r+1})$*, coming from the maps* $L_r \to L_{r+1}$*. Therefore, we have a character map for* L*, that is* $\chi^G_{n,p} : E^*(\mathrm{E}G \times_G X) \to \mathrm{Cl}_{n,p}(G, X; L)$*, which lands in the* $\mathrm{Aut}(\Lambda)$ *fixed points.*

Proof. We constructed the character map by constructing a map for each α. It is easy to see that these maps are compatible with the maps $L_r \to L_{r+1}$, coming from the projections. □

TODO write something about p being invertible in L_r, L Since p is invertible in L, and $\mathrm{Aut}(\Lambda)$ doesn't change p^{-1}, it also follows that after inverting p, that is, rationalizing, the map $p^{-1}\chi^G_{n,p} : p^{-1}E^*(\mathrm{E}G \times_G X) \to \mathrm{Cl}_{n,p}(G, X; L)$ still lands in the $\mathrm{Aut}(\Lambda)$ fixed points. (The same is true for L_r in place for L.)

We are now in position to state the main theorem. This should remind you of 2.3.1 and 2.3.2.

Theorem 3.5.3 ([HKR00, Theorem C]). *First, after tensoring with L, the character map $\chi_{n,p}^{G} \otimes L : E^{*}(\mathrm{E}G \times_{G} X) \otimes_{E^{*}} L \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)$ becomes an isomorphism. Second, the map $p^{-1}\chi_{n,p}^{G} : p^{-1}E^{*}(\mathrm{E}G \times_{G} X) \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)^{\mathrm{Aut}(\Lambda)}$ is an isomorphism. Moreover, these statements hold when L is replaced with L_r, for $r \geq r_0$.*

The first part will be proven later.

Proof (of the second part). Using the isomorphism from the first part, $E^{*}(\mathrm{E}G \times_{G} X) \otimes_{E^{*}} L \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)$. Endowing the source with $\mathrm{Aut}(\Lambda)$-action by acting only on L, makes it equivariant. Therefore, there is an isomorphism on the fixed points. Using $L^{\mathrm{Aut}(\Lambda)} = p^{-1}E^{*}$, from 3.3.6, the fixed points on the source are:

$$
\begin{aligned}
\left(E^{*}(\mathrm{E}G \times_{G} X) \otimes_{E^{*}} L\right)^{\mathrm{Aut}(\Lambda)} &= E^{*}(\mathrm{E}G \times_{G} X) \otimes_{E^{*}} L^{\mathrm{Aut}(\Lambda)} \\
&= E^{*}(\mathrm{E}G \times_{G} X) \otimes_{E^{*}} p^{-1}E^{*} \\
&= p^{-1}E^{*}(\mathrm{E}G \times_{G} X)
\end{aligned}
$$

So indeed $p^{-1}E^{*}(\mathrm{E}G \times_{G} X) \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)^{\mathrm{Aut}(\Lambda)}$ is an isomorphism. (The exact same proof works when L is replaced with L_r, for $r \geq r_0$.) ☐

3.6 The Idea of the Proof and Complex Oriented Descent

Our next goal is to proof the first part of 3.5.3. That is, for a finite group G and a finite G-CW complex X, the character map becomes an isomorphism after tensoring with L, i.e. $\chi_{n,p}^{G} \otimes L : E^{*}(\mathrm{E}G \times_{G} X) \otimes_{E^{*}} L \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)$ is an isomorphism, and the same with L replace with L_r, for $r \geq r_0$. One may wonder why we had to introduce the G-space X into the construction, in order to prove the case of interest, $X = *$. The reason is, that there is a trick, called *complex oriented descent*, that allows us to reduce to the of G-spaces X with *abelian stabilizers*. Using this and some further ideas we reduce to the case where G is abelian, and $X = *$. That is, introducing the space X into the construction, allows us to reduce the statement to abelian groups.

To be more explicit, this is the strategy. We will consider the character map as a natural transformation between functors of pairs (G, X), and then we will follow these steps:

- Use complex oriented descent to reduce to X with abelian stabilizers,
- Use Mayer-Vietoris to reduce to spaces $X = D^n \times G/A$ with A abelian,
- Use homotopy invariance to reduce to $X = G/A$,
- Use induction to reduce from $(G, G/A)$ to $(A, *)$,
- Prove for $(A, *)$.

This strategy will be formulated as a theorem later, after we introduce complex oriented descent. To introduce it, we need some definitions.

Definition 3.6.1. Let ξ be a d-dimensional complex vector bundle over a space X. The *flag bundle* $F(\xi) \to X$ is the bundle of complete flags in ξ.

The fiber over a point can be described as an (ordered) d-tuple $(\ell_1, \ldots, \ell_d)$ of orthogonal lines. To define it precisely, we can take the d-fold power of the projective bundle $P(\xi)$, and restrict to the sub-bundle of orthogonal lines. We note that for a trivial bundle $X \times V \to X$, we have $F(X \times V) \cong X \times F(V)$, i.e. the flags are computed fiber-wise.

Definition 3.6.2. Let $C^* : \mathcal{S}^{\mathrm{op}} \to \mathrm{GrAb}$ be a contra-variant functor from spaces to graded abelian groups. C^* is said to satisfy *complex oriented descent*, if for every space X and bundle ξ over X, the diagram $X \leftarrow F(\xi) \leftleftarrows F(\xi) \times_X F(\xi)$, is sent to an equalizer $C^*(X) \to C^*(F(\xi)) \rightrightarrows C^*(F(\xi) \times_X F(\xi))$.

We note that if ξ is a G-vector bundle over the G-space X, then $F(\xi) \to X$ is also a G-bundle, since G acts unitarily. We also recall that every finite group G has a faithful finite dimensional complex representation. The reason complex oriented descent is useful is the following.

Proposition 3.6.3. *Let X be a G-space, and let $\rho : G \to V$ be a faithful representations. Then the G-space $F(X \times V) \cong X \times F(V)$ has abelian stabilizer.*

Proof. Let $(x, (\ell_i))$ be a point in $X \times F(V)$. We wish to show that its stabilizer is abelian. Let $g, h \in G$ be two elements which fix the point, that is, $g.x = x = h.x$ and $\rho_g(\ell_i) = \ell_i = \rho_h(\ell_i)$. We see that the linear transformations $\rho_g : V \to V$ and $\rho_h : V \to V$ are simultaneously diagonalizable w.r.t to the decomposition (ℓ_i) of V. Therefore, by a classical result in linear algebra, they commute, $\rho_g \rho_h = \rho_h \rho_g$, i.e. $\rho_{gh} = \rho_{hg}$. Since ρ is faithful, we get that $gh = hg$. $\square$

Definition 3.6.4. Let $H \leq G$ be a subgroup, and let Y be an H-space. We define the G-space $G \times_H Y$ as follows. Define an H-action on G by $h.g = gh^{-1}$. This gives a diagonal action on $G \times Y$, i.e. $h.(g, y) = \left(gh^{-1}, h.y\right)$. The orbits are $G \times_H Y$. This space has a G-action by $\gamma.(g, y) = (\gamma g, y)$. This is well defined, since $\gamma. \left[gh^{-1}, h.y\right] = \left[\gamma gh^{-1}, h.y\right] = [\gamma g, y] = \gamma. [g, y]$.

Definition 3.6.5. Let $\mathcal{C}$ be the category whose objects are pairs (G, X) where G is a finite group and X is a finite G-CW complex. The morphisms in $\mathcal{C}$ are generated from the following: First, the usual morphisms $(G, X) \to (G, Y)$. Secondly, for $H < G$ and Y, we add a morphism $(H, Y) \to (G, G \times_H Y)$.

Definition 3.6.6. Let $C^* : \mathcal{C}^{\mathrm{op}} \to \mathrm{GrAb}$, be a contra-variant functor. We define the following properties of C^*:

- *Homotopy invariance* - $C^*(G, -)$ is G-homotopy invariant, for every G.
- *Mayer-Vietories* - $C^*(G, -)$ satisfies Mayer-Vietories, for every G.
- *Complex oriented descent* - $C^*(G, -)$ satisfies complex oriented descent, for every G.
- *Induction* - for every, $H \leq G$ and H-space Y, the morphism $(H, Y) \to (G, G \times_H Y)$ induces an isomorphism $C^*(G, G \times_H Y) \xrightarrow{\sim} C^*(H, Y)$. G.

Theorem 3.6.7 ([HKR00, 6.10])**.** *Let $C^*, D^* : \mathcal{C}^{\mathrm{op}} \to \mathrm{GrAb}$ be functors satisfying the above properties. Let $\tau : C^* \to D^*$ be a natural transformation between them. Suppose that τ commutes with the connecting morphisms of Mayer-Vietories, and that $\tau(A, *)$ is an isomorphism for all abelian groups A. Then τ is a natural isomorphism.*

Proof. We will follow the steps of the strategy outlined before (although we will describe it in the opposite order).

Let G be a group, and $A \leq G$ an abelian subgroup. The morphism $(A, *) \to (G, G \times_A *) \cong (G, G/A)$, by naturality of τ, induces a commutative square:

$$\begin{array}{ccc} C^*(G, G/A) & \longrightarrow & C^*(A, *) \\ \big\downarrow{\scriptstyle \tau(G,G/A)} & & \big\downarrow{\scriptstyle \tau(A,*)} \\ D^*(G, G/A) & \longrightarrow & D^*(A, *) \end{array}$$

By induction, the horizontal morphisms are isomorphisms. By assumption, $\tau(A,*)$ is an isomorphism. We conclude that $\tau(G,G/A)$ is an isomorphism.

Since C^*, D^* are homotopy invariant, for every disk D^n equipped with a trivial action, the map $(G, G/A \times D^n) \to (G, G/A)$ induces an isomorphism. Similarly to before, by naturality $\tau(G, G/A \times D^n)$ is an isomorphism.

Now, let X a finite G-CW complex, s.t. the stabilizer of every point is abelian. All the cells are then of the form $G/A \times D^n$ for some abelian subgroup $A \leq G$ and disk D^n. By an induction on the number of cells, using Mayer-Vietories and the fact that τ commutes with the connecting morphisms, (G, X) is an isomorphism.

Lastly, let X be an arbitrary finite G-CW complex. Let V be a faithful G representation, and consider the bundle $X \times V \to V$. By naturality, the diagram $X \leftarrow X \times F(V) \leftleftarrows X \times F(V) \times F(V)$ induces a commutative diagram:

$$\begin{tikzcd}
C^*(X) \arrow[r, "f"] \arrow[d, "\alpha=\tau(G,X)"] & C^*(X \times F(V)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\beta=\tau(G,X\times F(V))"] & C^*(X \times F(V) \times F(V)) \arrow[d, "\tau(G,X\times F(V)\times F(V))"] \\
D^*(X) \arrow[r, "g"] & D^*(X \times F(V)) \arrow[r, shift left] \arrow[r, shift right] & D^*(X \times F(V) \times F(V))
\end{tikzcd}$$

By complex oriented descent, the two rows are equalizer diagrams. By 3.6.3, $X \times F(V)$ has abelian stabilizers, hence we already know that $\beta = \tau(G, X \times F(V))$ is an isomorphism. We can then construct the a map $\beta^{-1}g : D^*(X) \to C^*(X \times F(V))$ making the diagram commute. By the universal property of the equalizer, we get a map $\alpha' : D^*(X) \to C^*(X)$ s.t. the diagram is commutative. The composition $\alpha'\alpha : C^*(X) \to C^*(X)$ makes the diagram commute, and since $C^*(X)$ is the equalizer, by uniqueness, $\alpha'\alpha = \mathrm{id}_{C^*(X)}$. Similarly $\alpha\alpha' : D^*(X) \to D^*(X)$ makes the diagram commute, so $\alpha\alpha' = \mathrm{id}_{D^*(X)}$. This shows that $\alpha = \tau(G,X)$, which completes the proof. $\square$

3.7 Proof of the Main Theorem

We are going to use the previous results to prove the main theorem, 3.5.3. The whole proof will work for L replaced by L_r, for $r \geq r_0$, without a change. Recall that we have already proved the second part. Therefore, what is left to prove is that $\chi^G_{n,p} \otimes L : E^*(\mathrm{E}G \times_G X) \otimes_{E^*} L \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)$ is an isomorphism. We will do this using 3.6.7.

Denote $C^*(G,X) = E^*(\mathrm{E}G \times_G X) \otimes_{E^*} L$, and $D^*(G,X) = \mathrm{Cl}_{n,p}(G,X;L) = (L_r \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G,X)))^G$. Their definition on morphisms $(G,X) \to (G,Y)$ is clear, simply by functoriality of all constructions when G is fixed. The definition on morphisms for induction will be given below, together with the proof that they satisfy induction. We also denote by $\tau(G,X)$ the character map $\chi^G_{n,p} \otimes L$ for X.

Lemma 3.7.1. *Both functors C^* and D^* are homotopy invariant.*

Proof. This is immediate since E^* is homotopy invariant, the Borel construction $X \mapsto \mathrm{E}G \times_G X$ is G-homotopy invariant, and the fixed points of a G-CW complex are also G-homotopy invariant. □

Lemma 3.7.2. *Both functors C^* and D^* satisfy Mayer-Vietories, and τ commutes with the connecting morphisms.*

Proof. The Borel construction $X \mapsto \mathrm{E}G \times_G X$ is a limit, and so are fixed points, so they commute with pushouts. Therefore, the usual pushouts that induce Mayer-Vietories, give Mayer-Vietories for our functors. Moreover, the definition makes it clear that the character map commutes with the connecting morphisms. □

Lemma 3.7.3. *Both functors C^* and D^* satisfy complex oriented descent.*

Proof. Hopkins, Kuhn and Ravenel actually prove in [HKR00, 2.5] that any complex oriented cohomology theory (and not only the cohomology theories of interest to us, namely Lubin-Tate) satisfies complex oriented descent. **TODO consider proving**

Let ξ be a G-vector bundle over X.

Then $\mathrm{E}G \times_G \xi$ is a G-vector bundle on $\mathrm{E}G \times_G X$, and it satisfies $F(\mathrm{E}G \times_G \xi) \cong \mathrm{E}G \times_G F(\xi)$. Then the fact that E^* satisfies complex oriented descent, and that L is flat, imply that C^* satisfies complex oriented descent.

Moreover, in [HKR00, 2.6], they prove that for an abelian subgroup, $A \leq G$, the diagram $X^A \leftarrow F(\xi)^A \leftleftarrows F(\xi)^A \times_{X^A} F(\xi)^A$ gives an equalizer diagram in E^*-cohomology. In the situation of D^*, we use the result for $A = \operatorname{im} \alpha$ which is indeed abelian by the fact they are n-commuting elements (equivalently, by the fact that it is the image of an abelian group). Equalizers, which are limits, commute with limits, and therefore commute with products and taking G-fixed points. Using this, and the flatness of L again, we deduce, by 3.4.3, that D^* satisfies complex oriented descent as well. $\square$

Lemma 3.7.4. *Both functors C^* and D^* satisfy induction.*

Proof. Let $H \leq G$ be a subgroup, and Y an H-space.

We have $\mathrm{E}G \times_G (G \times_H Y) \cong \mathrm{E}H \times_H Y$. Taking E^*-cohomology and tensoring with L gives an isomorphism $C^*(G, G \times_H Y) \xrightarrow{\sim} C^*(H, Y)$, which shows the functoriality for this sort of morphisms, and the fact that it is an isomorphism show that C^* satisfies induction.

We now claim that there is a homeomorphism $\varphi : G \times_H \mathrm{Fix}_{n,p}(H, Y) \xrightarrow{\sim} \mathrm{Fix}_{n,p}(G, G \times_H Y)$.

By definition:

$$G \times_H \mathrm{Fix}_{n,p}(H, Y) = G \times_H \coprod_{\alpha \in H_{n,p}} Y^{\mathrm{im}\,\alpha}$$

An element here is the data of $g \in G$, $\alpha \in H_{n,p}$ and $y \in Y^{\mathrm{im}\,\alpha}$. We will denote its H-orbit by $[g, \alpha, y]$. For an $h \in H$, the relation we get is $[g, \alpha, y] = \left[gh^{-1}, h.\alpha, h.y\right]$. An element $\gamma \in G$ acts by $\gamma. [g, \alpha, y] = [\gamma g, \alpha, y]$.

Similarly, by definition:

$$\mathrm{Fix}_{n,p}(G, G \times_H Y) = \coprod_{\alpha \in G_{n,p}} (G \times_H Y)^{\mathrm{im}\,\alpha}$$

An element here is the data of $\alpha \in G_{n,p}$, $[g, y] \in (G \times_H Y)^{\mathrm{im}\,\alpha}$. We will denote this by $(\alpha, [g, y])$. For an $h \in H$, the relation we get is $(\alpha, [g, y]) = \left(\alpha, \left[gh^{-1}, hy\right]\right)$. An element $\gamma \in G$ acts by $\gamma. (\alpha, [g, y]) = (\gamma.\alpha, [\gamma g, y])$.

Define the map $\varphi : G \times_H \mathrm{Fix}_{n,p}(H, Y) \to \mathrm{Fix}_{n,p}(G, G \times_H Y)$ by $\varphi([g, \alpha, y]) = (g.\alpha, [g, y])$.

We need to show that it doesn't depend on the H-orbit representative in the source, and indeed,

$$\begin{aligned}
\varphi\left(\left[gh^{-1}, h.\alpha, h.y\right]\right) &= \left(gh^{-1}.h.\alpha, \left[gh^{-1}, h.y\right]\right) \\
&= \left(g.\alpha, \left[gh^{-1}, h.y\right]\right) \\
&= (g.\alpha, [g, y]) \\
&= \varphi([g, \alpha, y]).
\end{aligned}$$

We need to show that the element defined lands in the target, $(g.\alpha, [g,y]) \in \mathrm{Fix}_{n,p}(G, G\times_H Y)$, i.e. $[g,y] \in (G\times_H Y)^{g.\operatorname{im}\alpha}$. Since α comes from $H_{n,p}$, we have $g.\operatorname{im}\alpha \leq gHg^{-1}$, so let $ghg^{-1} \in \operatorname{im}\alpha$, and we check the $[g,y]$ is invariant under it (via the G-action). Remember the $y \in Y^{\operatorname{im}\alpha}$, and we get, $ghg^{-1}.[g,y] = \left[ghg^{-1}g, y\right] = [gh, y] = [g, hy] = [g,y]$.

We show that it is G-equivariant. So let $\gamma \in G$, and indeed,

$$\begin{aligned}\varphi(\gamma.[g,\alpha,y]) &= \varphi([\gamma g,\alpha,y]) \\ &= (\gamma g.\alpha, [\gamma g, y]) \\ &= \gamma.(g.\alpha,[g,y]) \\ &= \gamma.\varphi([g,\alpha,y])\end{aligned}$$

We show that it is one-to-one. Assume $\varphi([g,\alpha,y]) = \varphi([g',\alpha',y'])$, i.e. $(g.\alpha,[g,y]) = (g'.\alpha',[g',y'])$. In particular, $[g,y] = [g',y']$. It follows that $y' = h.y$ and $g' = gh^{-1}$ for some $h \in H$. Then, $g.\alpha = g'.\alpha' = gh^{-1}.\alpha'$, so $\alpha = h^{-1}.\alpha'$, equivalently $\alpha' = h.\alpha$. We therefore conclude that they are indeed in the same H-orbit, $[g,\alpha,y] = \left[gh^{-1}, h.\alpha, h.y\right] = [g',\alpha',y']$.

Lastly, we show that it is surjective, which is the only step which is not routine. Let $(\alpha,[g,y])$, i.e. $\alpha \in G_{n,p}$, $[g,y] \in (G\times_H Y)^{\operatorname{im}\alpha}$. We claim that if such a triple exists, i.e. the fixed points are not empty, then necessarily $g^{-1}.\alpha \in H_{n,p}$. Assume by negation that there is some $\gamma \in \operatorname{im} g^{-1}.\alpha$ which is not in H. Since it is in $g\gamma g^{-1}\operatorname{im}\alpha$, it fixes $[g,y]$, but then $[g,y] = g\gamma g^{-1}.[g,y] = \left[g\gamma g^{-1}g, y\right] = [g\gamma, y]$, therefore, for some $h \in H$, $\left(gh^{-1}, hy\right) = (g\gamma, y)$, in particular $\gamma = h^{-1}$, which contradicts that $g \notin H$. It follows that $g^{-1}.\alpha \in H_{n,p}$. We now claim that $y \in Y^{\operatorname{im} g^{-1}.\alpha}$. So let $\eta \in \operatorname{im} g^{-1}.\alpha$, which we now know is in H as well. Since $[g,y] \in (G\times_H Y)^{\operatorname{im}\alpha}$, it is fixed by $g\eta g^{-1}$, i.e. there is some $h \in H$, s.t. $\left(gh^{-1}, hy\right) = g\eta g^{-1}.(g,y) = \left(g\eta g^{-1}g, y\right) = (g\eta, y)$, and in particular $y = hy$. In conclusion, we do have a well defined element $\left[g, g^{-1}\alpha, y\right]$, which is mapped to $(\alpha,[g,y])$.

It is clear that the map φ is continuous, and so is its inverse $\left[g, g^{-1}\alpha, y\right] \mapsto (\alpha,[g,y])$, which shows that indeed D^* satisfies induction. $\square$

Lemma 3.7.5. $\tau(A,*)$ *is an isomorphism for all abelian groups* A*.*

Proof. We need to verify that $\chi^A_{n,p} : E^*(\mathrm{B}A)\otimes_{E^*} L \to \mathrm{Cl}_{n,p}(A;L)$ is an isomorphism for abelian groups A.

By 3.2.3, we have Kunneth for $E^*(\mathrm{B}A)$, so it takes direct sums in the A to tensor products.

For an abelian group, the A-action on $A_{n,p}$ is trivial (since it is by conjugation). Therefore, $\mathrm{Cl}_{n,p}(A; L) = \hom_{A\mathrm{Set}}(A_{n,p}, L) = \hom_{\mathrm{Set}}(A_{n,p}, L) = \hom_{\mathrm{Set}}(\hom_{\mathrm{Ab}}(\Lambda, A), L)$. The $\hom_{\mathrm{Ab}}(\Lambda, A)$ commutes with direct sums in the second coordinate, and the outer $\hom_{\mathrm{Set}}$ takes them to the tensor product.

We see that both functors take direct sums to tensor products, so we by the structure theorem for abelian groups, we reduce to the case $\mathbb{Z}/q^k$ for a prime q.

First we handle the case $q \neq p$. Again by 3.2.3, $E^*(\mathrm{B}\mathbb{Z}/q^k)$ is a free module of rank 1, so the source of $\chi_{n,p}^{\mathbb{Z}/q^k}$ is L. Moreover, $\hom_{\mathrm{Ab}}(\Lambda, \mathbb{Z}/q^k)$ has only the trivial homomorphism, so $\mathrm{Cl}_{n,p}(\mathbb{Z}/q^k; L) = L$. It is easy to see that this is indeed an isomorphism then.

Now for the case $q = p$. In this case, $(\mathbb{Z}/p^k)_{n,p}$, i.e. n-commuting p-power-torsion elements, is just n elements, so it is canonically isomorphic to $(\mathbb{Z}/p^k)^n = \Lambda_k$. Then $\chi_{n,p}^{\mathbb{Z}/p^k}$ is the canonical morphism $E^*(\mathrm{B}\Lambda_k) \otimes L \to \hom_{\mathrm{Set}}(\Lambda_k, L)$. □

Proof (of the first part of 3.5.3). Follows immediately by combining the previous lemmas and 3.6.7. □

4 Elliptic Curves

At this point, one may wonder how we can find interesting pairs (k, Γ), of a perfect field and a formal group law over it, to obtain Lubin-Tate spectra. Two simple examples which have already seen are the additive formal group law over $\mathbb{F}_p$, of height ∞, which gives rise to $\mathrm{H}\mathbb{F}_p$, and the multiplicative formal group law over $\mathbb{F}_p$, of height 1, which gives rise to $\mathrm{K}_p^\wedge$. Elliptic curves are another source for formal group laws.

4.1 Formal Group Laws From Elliptic Curves

Let C be an elliptic curve over a ring R, with O the point at infinity. In [Sil09, IV], there is a construction of a formal group from C, obtained by considering the infinitesimal neighborhood of O. Choosing coordinates gives coordinates to the formal group, i.e. a formal group law denoted by Γ_C.

Now, assume that $R = k$ is a finite field of characteristic p. We denote by $C[p^r]$ the p^r-torsion, i.e. the kernel of the multiplication-by-p^r map. By [Sil09, IV.7.5] and [Sil09, V.3.1], we have:

Proposition 4.1.1. *The height of* Γ_C *is either* 1 *or* 2. *Moreover, the height is* 2 *if and only if* $C[p^r] = \{O\}$ *for all* $r \geq 1$.

In fact, there are many more equivalent conditions to the above, and we make this into a definition.

Definition 4.1.2. C is called *supersingular* if Γ_C is of height 2.

Similarly to the Lubin-Tate deformation theory of formal group laws described in 1.7, there is a deformation theory for elliptic curves. Denote by C_U the universal deformation of C, which is an elliptic curve over some ring R_U. This has a corresponding formal group law Γ_{C_U}. We then have the following theorem, implied by the Serre-Tate theorem [Kat81, 1.2.1].

Theorem 4.1.3 ("$\Gamma_{C_U} = (\Gamma_C)_U$")**.** *The formal group law* Γ_{C_U} *over* R_U*, is a universal deformation of* Γ_C *over* k*, in the sense of 1.7.*

In this case we get a Lubin-Tate spectrum $E = \mathrm{E}(k, \Gamma_C)$ of height 2. We recall from 1.7.7 that the coefficients can be taken to be $E_* = Wk[[u_1]]\left[u^{\pm 1}\right]$ where $|u| = 2$, and the formal group law is $u(\Gamma_C)_U$, which by Serre-Tate can be described by $u\Gamma_{C_U}$.

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