Thesis

Shay Ben Moshe

?

Contents

1 Overview of Chromatic Homotopy Theory

Our goal is to motivate the introduction of Morava K-theory $\mathrm{K}(n)$ and Morava E-theory $\mathrm{E}(n)$, and other variants of Morava E-theory $\mathrm{E}(k,\Gamma)$, and their connection to formal group laws. There are different views on what chromatic homotopy theory is. **TODO write some more**

1.1 The Balmer Spectrum

We will start with an algebraic motivation. Let R be a noetherian ring. Consider the symmetric monoidal stable ∞-category $\mathrm{Ch}(R)$ of chain complexes on R. **TODO be more specific** It is then natural to ask how much information about R is encoded in the category $\mathrm{Ch}(R)$. We will try to recover $\operatorname{Spec} R$, as a topological space, from $\mathrm{Ch}(R)$.

Remark 1.1.1. Balmer's work actually recovers the structure sheaf as well. **TODO reference**

Definition 1.1.2. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finitely-generated projective modules. These objects are the compact objects in $\mathrm{Ch}(R)$, thus they can be defined categorically. Their full subcategory is denoted by $\mathrm{Ch}_{\mathrm{perf}}(R)$.

Definition 1.1.3. Let $\mathcal{C}$ be some symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

- $0 \in \mathcal{T}$,
- it is closed under cofibers (that is if $a \to b \to c$ is a cofiber sequence in $\mathcal{C}$ and $a, b \in \mathcal{T}$, then $c \in \mathcal{T}$),
- it is closed under retracts.

Example 1.1.4. Consider the case $\mathcal{C} = \mathrm{Ch}_{\mathrm{perf}}(R)$ (e.g. over $\mathbb{Z}$, bounded chain complexes of finitely-generated free abelian groups). Let $K_\bullet \in \mathrm{Ch}(R)$, and define $\mathcal{T}_{K_\bullet} = \{A_\bullet \in \mathrm{Ch}_{\mathrm{perf}}(R) \mid A_\bullet \otimes K_\bullet = 0\}$. We claim that $\mathcal{T}_{K_\bullet}$ is thick. Clearly $0 \in \mathcal{T}_{K_\bullet}$. Let $A_\bullet \to B_\bullet$ be a morphism between two complexes in $\mathcal{T}$. The cofiber of

$A_\bullet \to B_\bullet$ is the pushout $B_\bullet \times_{A_\bullet} 0$. Since tensor is left, tensoring the cofiber with $K_\bullet$ is given by the pushout $(B_\bullet \otimes K_\bullet) \times_{A_\bullet \otimes K_\bullet} (0 \otimes K_\bullet) = 0 \times_0 0 = 0$, therefore the cofiber is indeed in $\mathcal{T}$. Lastly, if $A_\bullet \to B_\bullet \to A_\bullet$ is the identity and $B_\bullet \otimes K_\bullet$, we get that $\mathrm{id}_{A_\bullet \otimes K_\bullet}$ factors through 0, which implies that $A_\bullet \otimes K_\bullet$ is 0, so that $A_\bullet \in \mathcal{T}$.

Definition 1.1.5. A thick subcategory $\mathcal{T}$ is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring: As a set, $\operatorname{Spec} \mathcal{C} = \{\mathcal{P} \text{ prime ideal}\}$. For any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \operatorname{Spec} \mathcal{C} \mid S \cap \mathcal{P} = \emptyset\}$. We topologize $\operatorname{Spec} \mathcal{C}$ with the Zariski topology by declaring those to be the closed subsets. We also denote $\operatorname{Supp}(a) = V(\{a\})$.

Example 1.1.6. We continue the example of $\mathcal{T}_{K_\bullet}$. Clearly if $A_\bullet \otimes K_\bullet = 0$ then also $A_\bullet \otimes B_\bullet \otimes K_\bullet = 0$, so it is an ideal. Let $\mathfrak{p}$ be a prime ideal in R in the usual sense, and take $K_\bullet = R_\mathfrak{p}$ (concentrated at degree 0), then $A_\bullet \otimes K_\bullet = (A_\bullet)_\mathfrak{p}$ (level-wise localization). Now, assume that $0 = (A_\bullet \otimes B_\bullet)_\mathfrak{p} = (A_\bullet)_\mathfrak{p} \otimes_{R_\mathfrak{p}} (B_\bullet)_\mathfrak{p}$ Assume by negation that $(A_\bullet)_\mathfrak{p}, (B_\bullet)_\mathfrak{p} \neq 0$, i.e. $(A_n)_\mathfrak{p}, (B_m)_\mathfrak{p} \neq 0$ but $(A_n)_\mathfrak{p} \otimes_{R_\mathfrak{p}} (B_m)_\mathfrak{p} = 0$ for some n, m. Well, localization of projective is projective, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, so $(A_n)_\mathfrak{p} = 0$ or $(B_m)_\mathfrak{p} = 0$, which is a contradiction. Therefore $\mathcal{T}_\mathfrak{p}$ is a prime ideal.

Theorem 1.1.7. *The map* $\operatorname{Spec} R \to \operatorname{Spec}(\mathrm{Ch}_{\mathrm{perf}}(R))$*, given by* $\mathfrak{p} \mapsto \mathcal{T}_\mathfrak{p} = \left\{A_\bullet \mid (A_\bullet)_\mathfrak{p} = 0\right\}$ *is a homeomorphism.*

TODO reference

Proposition 1.1.8. *Prime ideals pullback: Let* $F : \mathcal{C} \to \mathcal{D}$ *be a reduced symmetric monoidal functor that preserves cofibers, between two symmetric monoidal stable* ∞*-categories, and let* $\mathcal{P}$ *be a prime ideal in* $\mathcal{D}$*, then* $F^*\mathcal{P} = \{a \in \mathcal{C} \mid F(a) \in \mathcal{P}\}$ *is a prime ideal.*

Proof. Clearly $F(0) = 0 \in \mathcal{P}$ since F is reduced, so $0 \in F^*\mathcal{P}$. Since F preserves cofibers, for $a, b \in F^*\mathcal{P}$, i.e. $F(a), F(b) \in \mathcal{P}$, and a map $a \to b$ we get $F(\operatorname{cofib}(a \to b)) = \operatorname{cofib}(F(a) \to F(b)) = \operatorname{cofib}(F(a) \to F(b)) \in \mathcal{P}$. Let $a \to b \to a$ be a retract, that is the composition is the identity, s.t. $b \in F^*\mathcal{P}$. We know that $F(a) \to F(b) \to F(a)$ is also a retract by functoriality, thus $F(a) \in \mathcal{P}$, that is $a \in F^*\mathcal{P}$. We conclude that $F^*\mathcal{P}$ is indeed a thick subcategory.

Let $a \in F^*\mathcal{P}, b \in \mathcal{C}$, since F is monoidal, $F(a \otimes b) = F(a) \otimes F(b) \in \mathcal{P}$, so $a \otimes b \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is an ideal.

Lastly, assume that $a \otimes b \in F^*\mathcal{P}$, again since F is monoidal, $F(a \otimes b) = F(a) \otimes F(b) \in \mathcal{P}$, so $a \in F^*\mathcal{P}$ or $b \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is a prime ideal. ☐

Now, recall that $\mathrm{Ch}(R) \cong \mathrm{Mod}_{\mathrm{H}R}$, therefore we can reinterpret the above theorem as $\operatorname{Spec} R \cong \operatorname{Spec}(\mathrm{Mod}_{\mathrm{H}R}^{\mathrm{comp}})$ (where the comp denotes the compact objects in the category). We shall turn this theorem into a definition:

Definition 1.1.9. Let R be an $\mathbb{E}_\infty$-ring. We define the *spectrum* of R to be $\operatorname{Spec} R = \operatorname{Spec}(\mathrm{Mod}_R^{\mathrm{comp}})$.

A natural question to ask then is what is $\operatorname{Spec} \mathbb{S}$. Recall that $\mathrm{Mod}_\mathbb{S} = \mathrm{Sp}$, the category of spectra, and that the compact objects in spectra are the finite spectra $\mathrm{Sp}^{\mathrm{fin}}$. So, unwinding the definitions, the question can rephrased as finding the prime ideals in $\mathrm{Sp}^{\mathrm{fin}}$, and their topology. Chromatic homotopy theory provides an answer to this question.

1.2 MU and Complex Orientations

Throughout this section, let E be a multiplicative cohomology theory (that is, equipped with a map $E \otimes E \to E$ and $1 \in E_0$, which is associative and unital after taking homotopy groups).

Consider the map $S^2 \to \mathrm{BU}(1)$ classifying the universal complex line bundle. Concretely, under the identifications $S^2 \cong \mathbb{C}\mathrm{P}^1$ and $\mathrm{BU}(1) \cong \mathbb{C}\mathrm{P}^\infty$, this map can be realized as the inclusion $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^\infty$. This map induces a map $\tilde{E}^2(\mathrm{BU}(1)) \to \tilde{E}^2(S^2) \cong \tilde{E}^0(S^0) \cong E^0(*) = E_0$. Since E is unital, there is a canonical generator $1 \in E_0$.

Definition 1.2.1. E is called *complex oriented* if the map $\tilde{E}^2(\mathrm{BU}(1)) \to E_0$ is surjective, equivalently, if 1 is in the image of that map. A choice of a lift $x \in \tilde{E}^2(\mathrm{BU}(1))$ of $1 \in E_0$ is called a *complex orientation*.

Example 1.2.2. Let R be some ring, and consider $\mathrm{H}R$. It is known that $\mathrm{H}R^*\left(\mathbb{C}\mathrm{P}^n\right) \cong R[x]/\left(x^{n+1}\right)$ and $\mathrm{H}R^*\left(\mathbb{C}\mathrm{P}^\infty\right) \cong R[[x]]$, where $|x| = 2$, and the maps induced by the inclusions of projective spaces maps x to x. Therefore we see that $x \in \mathrm{H}R^2(\mathrm{BU}(1))$ is mapped to $x \in \mathrm{H}R^2\left(S^2\right) = \mathbb{Z}\{x\}$, which is mapped to the generator of the reduced part of $\mathrm{H}R^0\left(S^0\right) = R \oplus R$. Therefore x is a complex orientation for $\mathrm{H}R$.

Example 1.2.3. Let K be complex K-theory, then we know that $\mathrm{K}_* = \mathbb{Z}\left[\beta^{\pm 1}\right]$ where β is the Bott element, with $|\beta| = 2$. It is also known (by Atiyah-Hirzebruch spectral sequence) that $\mathrm{K}^*\left(\mathbb{C}\mathrm{P}^n\right) \cong \mathrm{K}_*[t]/\left(t^{n+1}\right)$ and $\mathrm{K}^*\left(\mathbb{C}\mathrm{P}^\infty\right) \cong \mathrm{K}_*[[t]]$ (here $|t| = 0$), where the maps induced by the inclusions of projective spaces maps t to t. We deduce that $\beta t \in \mathrm{K}^2(\mathrm{BU}(1))$ is mapped to $\beta t \in \mathrm{K}^2\left(S^2\right) = \mathbb{Z}\{\beta t\}$, which is mapped to $t \in \mathrm{K}^0\left(S^0\right) = \mathbb{Z}\{t\}$, which is indeed the generator of the reduced part. Therefore $x = \beta t$ is complex orientation for K. **TODO write the reduced thing more clearly**

Example 1.2.4. Recall that MU is constructed as the colimit $\mathrm{MU} = \operatorname{colim} \mathrm{MU}(n)$. Also, $\mathrm{MU}(1) \cong \Sigma^{\infty-2}\mathrm{BU}(1)$. Therefore we get a canonical map $\Sigma^{\infty-2}\mathrm{BU}(1) \to \mathrm{MU}$, which gives a cohomology class $x_{\mathrm{MU}} \in \mathrm{MU}^2(\mathrm{BU}(1))$.

Proposition 1.2.5 ([Rav86, 4.1.3])**.** *x_{MU} is a complex orientation for* MU*.*

Theorem 1.2.6. MU *is the universal complex oriented cohomology theory: Let E be a multiplicative cohomology theory, then there is a bijection between (homotopy classes of) multiplicative maps $\mathrm{MU} \to E$ and complex orientations on E. The bijection is given in one direction by pulling back x_{MU} along a multiplicative map.*

Assume that E is complex oriented with a complex orientation x.

Theorem 1.2.7 ([Rav86, 4.1.4])**.** *As E_*-algebras, $E^*(\mathrm{BU}(1)) \cong E^*[[x]]$ and $E^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) \cong E^*[[y,z]]$.*

TODO maybe indicate the use of AHSS

The tensor product of complex line bundles is classified by a universal map $\mathrm{BU}(1) \times \mathrm{BU}(1) \to \mathrm{BU}(1)$. Therefore we get a map $E^*(\mathrm{BU}(1)) \to E^*(\mathrm{BU}(1) \times \mathrm{BU}(1))$, which is completely determined by the image of $x \in E^*[[x]]$ in $E^*[[y,z]]$ as above. Therefore, a choice of a complex orientation on E gives rise to an element $F_E(y,z) \in E^*[[y,z]]$.

Proposition 1.2.8 ([Rav86, 4.1.4])**.** *F_E is a formal group law on E_*. We call the height of F_E the* height *of E.*

Example 1.2.9. Consider again $\mathrm{H}R$. It is known that the tensor of complex line bundles induces the map $R[[x]] = \mathrm{H}R^*(\mathrm{BU}(1)) \to \mathrm{H}R^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = R[[y,z]]$ given by $x \mapsto y + z$. This is the additive formal group law. It is immediate that $[p] = px$. So for $R = \mathbb{Q}$ we get that the height of $\mathrm{H}\mathbb{Q}$ is 0, while for $R = \mathbb{F}_p$ we have $px = 0$ so the height of $\mathrm{H}\mathbb{F}_p$ is ∞.

Example 1.2.10. We return to the example of complex K-theory. It is known that the tensor of complex line bundles induces the map $\mathrm{K}_*[[t]] = \mathrm{K}^*(\mathrm{BU}(1)) \to \mathrm{K}^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = \mathrm{K}_*[[u,v]]$ given by $t \mapsto u+v+uv$. Note that to comply with the definition of the formal group law, we should use the isomorphism $\mathrm{K}^*(\mathrm{BU}(1)) \cong \mathrm{K}_*[[x]]$, i.e. the element $x = \beta t$. We get that $x = \beta t \mapsto \beta u + \beta v + \beta uv = y + z + \beta^{-1}yz = F_{\mathrm{K}}(y,z)$. By induction we prove that the n-series is $[n](x) = \beta\left(1 + \beta^{-1}x\right)^n - \beta$. This is clear for $n = 1$, and we have:

$$\begin{aligned}
[n+1](x) &= x + [n](x) + \beta^{-1}x[n](x) \\
&= x + \beta\left(1+\beta^{-1}x\right)^n - \beta + x\left(1+\beta^{-1}x\right)^n - x \\
&= \beta\left(1+\beta^{-1}x\right)\left(1+\beta^{-1}x\right)^n - \beta \\
&= \beta\left(1+\beta^{-1}x\right)^{n+1} - \beta
\end{aligned}$$

TODO consider discussing the computation of $\mathrm{BU}(1)$, maybe as part of complex K-theory example?

Example 1.2.11. By taking the cofiber of the multiplication-by-p map, we get a spectrum K/p, mod-p K-theory, with coefficients $(\mathrm{K}/p)_* = \mathbb{F}_p\left[\beta^{\pm 1}\right]$. It is evident that $F_{\mathrm{K}/p}(y,z) = y + z + \beta^{-1}yz$ as well. From the result above, it follows that $[p](x) = \beta\left(1+\beta^{-1}x\right)^p - \beta = \beta\left(1^p + \beta^{-p}x^p\right) - \beta = \beta^{-p}x^p$ which shows that the height is exactly 1.

A formal group law on E_* is the same data as a map from the Lazard ring L, so the complex orientation gives a map $L \to E_*$. In particular, since MU is complex oriented, there is a canonical map $L \to \mathrm{MU}_*$.

Theorem 1.2.12 (Quillen, [Rav86, 4.1.6])**.** *The canonical map $L \to \mathrm{MU}_*$ is an isomorphism.*

1.3 BP, Morava K-Theory and Morava E-Theory

A good principle in homotopy theory (and many other areas in math) is to do study it one prime at a time. This is possible in homotopy theory due to the arithmetic square **TODO reference**. So, let us fix a prime p. We can form $\mathrm{MU}_{(p)}$, the p-localization of MU.

Theorem 1.3.1 ([Ada74, II 15])**.** *There exists a map of ring spectra $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ (which depends on the prime p), which is an idempotent $\varepsilon^2 = \varepsilon$ (moreover, once the action on homotopy groups is specified, it is unique).*

The map $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ gives a cohomology operation, for every X we have $\varepsilon^* : \mathrm{MU}^*_{(p)}(X) \to \mathrm{MU}^*_{(p)}(X)$. Denote by $\mathrm{BP}^*_{(p)}(X)$ the image of ε^*.

Theorem 1.3.2 ([Ada74, II 16], [Rav86, 4.1.12])**.** BP *is a cohomology theory, represented by an associative commutative ring spectrum* BP *(which depends on the prime p), which is a retract of* $\mathrm{MU}_{(p)}$*. The homotopy groups of* BP *are* $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ *where* $|v_n| = 2(p^n - 1)$.

For convenience we denote $v_0 = p$ (and indeed $|v_0| = 2\left(p^0 - 1\right) = 0$). Since BP is a retract of MU it comes with a map $\mathrm{MU} \to \mathrm{BP}$, that is a complex orientation.

Proposition 1.3.3 (**TODO reference**)**.** *The p-series of the formal group law associated to* BP *is* $[p](x) = \sum v_n x^{p^n}$.

Once we have BP, we can turn to the definition of Morava K-theory and Morava E-theory

Definition 1.3.4. Let $0 < n < \infty$. *Morava K-theory* at height n and prime p, denoted by $\mathrm{K}(p,n)$ or $\mathrm{K}(n)$ when the prime is clear, is the spectrum obtained by killing $p = v_0, \dots, v_{n-1}, v_{n+1}, \dots$ in BP and inverting v_n. Therefore $\mathrm{K}(n)_* = \mathbb{F}_p\left[v_n^{\pm 1}\right]$. We also define $\mathrm{K}(0) = \mathrm{H}\mathbb{Q}$ and $\mathrm{K}(\infty) = \mathrm{H}\mathbb{F}_p$. Similarly, *Morava E-theory* at height n and prime p, denoted by $\mathrm{E}(p,n)$ or $\mathrm{E}(n)$, is the spectrum obtained by killing $v_{n+1}, v_{n+2}, \dots$ in BP and inverting v_n. Therefore $\mathrm{E}(n)_* = \mathbb{Z}_{(p)}\left[v_1, v_2, \dots v_{n-1}, v_n^{\pm 1}\right]$.

Since Morava K-theory and E-theory are obtained from BP by cofibers and filtered colimits, they are equipped with a map from BP, so they are also equipped with a complex orientation. It is then evident that the p-series associated to the formal group laws of $\mathrm{K}(n)$ and $\mathrm{E}(n)$ are $v_n x^{p^n}$ and $v_0 x + \dots v_n x^{p^n}$ respectively, and are therefore of height exactly n and height $\le n$ respectively. (Note that by the example of $\mathrm{H}R$, this is also true for $\mathrm{K}(0)$ and $\mathrm{K}(\infty)$.)

We want to describe some properties of Morava K-theory. To do so we first need some definitions.

Definition 1.3.5. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenus element is invertible, equivalently it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree. An $\mathbb{A}_\infty$-ring E is a *field* if E_* is a graded field.

Example 1.3.6. Clearly $K(n)$ for $0 \le n \le \infty$ is a field.

Proposition 1.3.7. *A field E is has Kunneth, i.e. $E_*(X \otimes Y) \cong E_*(X) \otimes_{E_*} E_*(Y)$ for any spectra X, Y.*

Proposition 1.3.8 ([Lur10, 24])**.** *Let $E \neq 0$ be a complex oriented cohomology theory, whose formal group law has height exactly n, then $E \otimes \mathrm{K}(n) \neq 0$. Let E be a field s.t. $E \otimes \mathrm{K}(n) \neq 0$, then E admits the structure of a $\mathrm{K}(n)$-module. (Here $0 \le n \le \infty$.)*

Example 1.3.9. As we have seen before mod-p K-theory, K/p, has height exactly 1 and coefficients $(\mathrm{K}/p)_* = \mathbb{F}_p\left[\beta^{\pm 1}\right]$. It is also known that K, and K/p, are A_∞ ring spectra, from which it follows that K/p is a field. Therefore, we deduce that K/p is a $\mathrm{K}(1)$-module. Since $|\beta| = 2$ and $|v_1| = 2(p-1)$ it is free of rank $p - 1$.

From this we also deduce some form of uniqueness for Morava K-theory:

Corollary 1.3.10. *Let E be a field with $E_* \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$, which is also complex oriented with height exactly n, then $E \cong \mathrm{K}(n)$ (as spectra).*

1.4 $\operatorname{Spec}\mathbb{S}_{(p)}$ and $\operatorname{Spec}\mathbb{S}$

We are now in a position to state the answer for $\operatorname{Spec}\mathbb{S}$. However, it will be easier to state it first for $\operatorname{Spec}\mathbb{S}_{(p)}$, and then pullback prime ideals. We know that $\mathrm{Mod}_{\mathbb{S}_{(p)}} = \mathrm{Sp}_{(p)}$, and the compact objects there are $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$, the p-localizations of finite spectra.

Proposition 1.4.1. *Let $\mathcal{T}_E = \ker E_* = \left\{X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid E_*(X) = 0\right\}$ (equivalently $X \otimes E = 0$) i.e. the E-acyclics, then $\mathcal{T}_E$ is thick.*

Proof. The exact same proof from $\mathrm{Ch}_{\mathrm{perf}}(R)$ works. □

Definition 1.4.2. We define $\mathcal{C}_{p,n} = \mathcal{T}_{\mathrm{K}(n)}$, the $\mathrm{K}(n)$-acyclics. By the above it is thick. Also, $\mathcal{C}_{p,\infty} = \{0\}$, which are trivially thick. When the prime is clear, we will denote by $\mathcal{C}_n$.

Proposition 1.4.3 ([Lur10, 26])**.** *For $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$, if $\mathrm{K}(n)_*(X) = 0$ then $\mathrm{K}(n-1)_*(X) = 0$.*

Definition 1.4.4. We say that a spectrum $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ is of *type n* (possibly ∞), if its first non-zero Morava K-theory homology is $\mathrm{K}(n)$.

Corollary 1.4.5. *$\mathcal{C}_n$ is the full subcategory of finite p-local spectra of type $> n$ (i.e. $\left\{X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid \forall m \le n : \mathrm{K}(m)_*(X) = 0\right\}$). Thus clearly $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$.*

Proposition 1.4.6 (TODO reference). *The inclusions are proper $\mathcal{C}_{n+1} \subsetneq \mathcal{C}_n$.*

Proposition 1.4.7. *If $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ is not contractible, then X has a finite type. Therefore $\bigcap_{n<\infty} \mathcal{C}_n = \{0\} = \mathcal{C}_\infty$.*

Proof. Let X be non-contractible. Then $\mathrm{H}\mathbb{Z}_*(X) \neq 0$. Let m be the first non-zero degree. Using the universal coefficient theorem and the fact that the spectrum is p-local we get that $(\mathrm{H}\mathbb{F}_p)_m(X) \neq 0$, thus $(\mathrm{H}\mathbb{F}_p)_*(X) \neq 0$. Since X is finite, $(\mathrm{H}\mathbb{F}_p)_*(X)$ is bounded. Atiyah-Hirzebruch spectral sequence for X with cohomology $\mathrm{K}(n)$ has E^2-page given by $E^2_{p,q} = H_p\left(X; \mathrm{K}(n)_q(*)\right)$. Since $\mathrm{K}(n)_q = \mathbb{F}_p$ for $q = 0 \mod 2(p^n-1)$ and 0 otherwise, we see that the rows $q = 0 \mod 2(p^n-1)$ are $(\mathrm{H}\mathbb{F}_p)_*(X)$, and the others are 0. Therefore if we take n such that the period $2(p^n-1)$ is larger then the bound on $(\mathrm{H}\mathbb{F}_p)_*(X)$, then all differentials have either source or target 0. Thus, the spectral sequence collapses at the E^2-page, and since $(\mathrm{H}\mathbb{F}_p)_*(X) \neq 0$, we ge that $\mathrm{K}(n)(X) \neq 0$, i.e. X has type $< n$. □

Proposition 1.4.8. *$\mathcal{C}_n$ is a prime ideal.*

Proof. For X,Y by Kunneth we have $\mathrm{K}(n-1)_*(X \otimes Y) = \mathrm{K}(n-1)_*(X) \otimes \mathrm{K}(n-1)_*(Y)$. Therefore, if $X \in \mathcal{C}_n$, i.e. the homology vanishes, then so does the homology of $X \otimes Y$, i.e. $X \otimes Y \in \mathcal{C}_n$, so $\mathcal{C}_n$ is an ideal. If $X \otimes Y \in \mathcal{C}_n$ then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces), so $\mathcal{C}_n$ is a prime ideal. □

Theorem 1.4.9 (Thick Subcategory Theorem [HS98])**.** *If $\mathcal{T}$ is a proper thick subcategory of $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$, then $\mathcal{T} = \mathcal{C}_n$ for some $0 \le n \le \infty$.*

Remark 1.4.10. The proof relies on a major theorem called the Nilpotence Theorem.

Corollary 1.4.11. *$\operatorname{Spec}\mathbb{S}_{(p)} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_\infty\}$, and the topology is such that the closed subsets are chains $\{\mathcal{C}_k, \mathcal{C}_{k+1}, \ldots, \mathcal{C}_\infty\}$ for some $0 \le k \le \infty$.*

Proof. Follows immediately from the previous results. □

We now want to move to finding $\operatorname{Spec}\mathbb{S}$. Note that the p-localization functor $L_{(p)}$ is a Bousfield localization. As such it is left (its right adjoint is the inclusion), and in particular preserves cofibers. It also clearly sends 0 to 0, i.e. reduced. Now, since $L_{(p)}$ is smashing, i.e. $L_{(p)}X = X \otimes \mathbb{S}_{(p)}$, we also get that it is symmetric monoidal. As we have seen in 1.1.8, under these conditions we can pullback primes. Therefore $\mathcal{P}_{p,n} = L_{(p)}^* \mathcal{C}_{p,n} = \left\{X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{K}(n)_*\left(X_{(p)}\right) = 0\right\}$ and $\mathcal{P}_{p,\infty} = L_{(p)}^* \mathcal{C}_{p,\infty} = \left\{X \in \mathrm{Sp}^{\mathrm{fin}} \mid X_{(p)} = 0\right\}$ are prime ideals. Note that

$\mathcal{P}_{p,0} = \left\{ X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_* \left(X_{(p)} \right) = 0 \right\} = \left\{ X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_* \left(X \right) = 0 \right\}$ **TODO explain** so it is independent of p, and we will denote it by $\mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}}$.

Theorem 1.4.12 (TODO explain/reference). $\operatorname{Spec} \mathbb{S} = \left\{ \mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}} \right\} \cup \bigcup_p \left\{ \mathcal{P}_{p,1}, \dots, \mathcal{P}_{p,\infty} \right\}$*, and the topology is such that the closed subsets finite unioins of chains* $\left\{ \mathcal{P}_{p,k}, \mathcal{P}_{p,k+1}, \dots, \mathcal{P}_{p,\infty} \right\}$ *for some* $0 \leq k \leq \infty$ *(i.e. they may include* $\mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}}$*).* ***TODO diagram***

TODO regarding the topology, maybe I should prove that the pullback is also continuous?

1.5 The Stacky Point of View and the Relationship Between Morava K-Theory and Morava E-Theory

First we will describe, without being precise, another point of view on what chromatic homotopy theory is about.

There is a stack of formal groups, denoted by $\mathcal{M}_{\mathrm{fg}}$. It can be described as the stack that sends a ring R to the groupoid of formal group laws, with isomorphisms between them. Quillen theorem 1.2.12 tells us that MU_* is the Lazard ring, that is the universal ring that carries the universal formal group law. It turns out that this theorem has a second part, which says that $\left(\mathrm{MU} \otimes \mathrm{MU}\right)_*$ is the universal ring that carries two formal group laws and an isomorphism between them. Therefore, $\mathcal{M}_{\mathrm{fg}}$ is represented by $\left(\mathrm{MU}_*, \left(\mathrm{MU} \otimes \mathrm{MU}\right)_*\right)$.

The geometric points of the stack $\mathcal{M}_{\mathrm{fg}}$ are describe precisely the same as $\operatorname{Spec} \mathbb{S}$, that is because for an algebraically closed field of characteristic 0 there is a unique (up to isomorphism) formal group law which is of height 0 namely the additive formal group law, and for characteristic p there is a unique (up to isomorphism) formal group law of each height $1 \leq n \leq \infty$.

For a spectrum X, $\mathrm{MU}_* \left(X \right)$ is a $\left(\mathrm{MU}_*, \left(\mathrm{MU} \otimes \mathrm{MU}\right)_*\right)$-comodule, which is the same as a sheaf over $\mathcal{M}_{\mathrm{fg}}$. From this point of view, chromatic homotopy theory lets us study a spectrum by decomposing it over the stack $\mathcal{M}_{\mathrm{fg}}$.

We can restrict ourselves to the stack only over rings of characteristic p, $\mathcal{M}_{\mathrm{fg},p}$, which is then represented by $\left(\left(\mathrm{MU}_{(p)}\right)_*, \left(\mathrm{MU}_{(p)} \otimes \mathrm{MU}_{(p)}\right)_* \right)$. **TODO I think it's true, is it?** Similarly to MU, BP is universal ring with the universal p-typical formal group law, and $\left(\mathrm{BP} \otimes \mathrm{BP}\right)_*$ is the universal ring with two p-typical formal group laws and an isomorphism between them **TODO I didn't say this before**. Since every formal group law is isomorphic to a p-typical one, we know that the stack $\mathcal{M}_{\mathrm{fg},p}$ is also represented by $\left(\mathrm{BP}_*, \left(\mathrm{BP} \otimes \mathrm{BP}\right)_*\right)$.

It is now reasonable that $\mathrm{K}\left(n\right)$, obtained from BP by killing the v_m's for $m \neq n$ and inverting v_n, sees the n-th level, and that $\mathrm{E}\left(n\right)$ obtained in the same way but only killing v_m for $m > n$, sees the levels $\leq n$. **TODO I didn't interpret the v_n's before as coefficients in the p-series which determines the height**

Let us now claim a precise statement, formalizing this description.

Theorem 1.5.1 (TODO reference). $\mathrm{E}\left(n\right)$ *and* $\mathrm{K}\left(0\right) \oplus \cdots \oplus \mathrm{K}\left(n\right)$ *are Bousfield equivalent. That is, they have the same acyclics, locals, and their localization functors are the same.*

TODO chromatic square and chromatic tower, maybe another subsection?

1.6 Landweber Exact Functor Theorem and Deformation Theory

As we have seen, a complex orientation on a cohomology theory, which is described by a map $\mathrm{MU} \to E$, has an associated formal group law, which is described by the map $L = \mathrm{MU}_* \to E_*$. One can ask whether the converse is true, namely given a ring R and a formal group law F given by $L \to R$, is there a complex oriented cohomology theory whose coefficients are R and the associated formal group law is F.

A strategy is to define $E_{R,F}\left(X\right) = \mathrm{MU}\left(X\right) \otimes_{\mathrm{MU}_*} R$. Unfortunately, this is not always a homology theory. However there is a condition which one can check which guarantees that it is.

Definition 1.6.1. R is called *Landweber flat* if for every prime p, the sequence $p = v_0, v_1, v_2, \dots$ is regular, that is, for each n, v_n is not a zero divisor in $R / \left(v_0, v_1, \dots, v_{n-1}\right)$.

Theorem 1.6.2 (Landweber Exact Functor Theorem (LEFT) **TODO reference**)**.** *If R is Landweber flat, then $E_{R,F}$ defined above is a homology theory. Moreover, there are no phantom maps between such spectra, so $E_{R,F}$ is represented by a spectrum. This spectrum is complex oriented, has coefficients R and associated formal group law F.*

Example 1.6.3. Morava E-theory is Landweber flat. Morava K-theory and $\mathrm{H}\mathbb{Z}$ are not. **TODO elaborate? reference?**

The Morava E-theory we have considered until now $\mathrm{E}(n)$, also called Johnson-Wilson spectrum was constructed from BP. As we noted, it is Landweber flat, which indicates that there is another approach to constructing it. Indeed there is a way to construct a related spectrum, which will be called the Lubin-Tate spectrum.

To that end, we first define the category CompRing as the category of complete local rings. The objects are complete local rings $(R, \mathfrak{m})$, we also denote by $\pi : R \to R/\mathfrak{m}$ the projection. Morphisms $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ are local homomorphisms, i.e. a homomorphism $\varphi : R \to S$ s.t. $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. In particular it induces a homomorphism $\varphi/\mathfrak{m} : R/\mathfrak{m} \to S/\mathfrak{n}$, which satisfies $\varphi/\mathfrak{m} \circ \pi_R = \pi_S \circ \varphi$.

We fix k be a perfect field of characteristic p (i.e. the Frobenius is an isomorphism), and Γ a formal group law over k of height $n < \infty$. Lubin and Tate [LT] considered a moduli problem associated to Γ/k, described by a functor Def : CompRing $\to$ Grpds.

Definition 1.6.4. Let $(R, \mathfrak{m})$ be a complete local ring and denote by $\pi : R \to R/\mathfrak{m}$ the quotient. A *deformation* of Γ/k to $(R, \mathfrak{m})$, is (G, i), where G is a formal group law over R, $i : k \to R/\mathfrak{m}$ is a homomorphism of fields, such that $i^*\Gamma = \pi^*G$. A $\star$*-isomorphism* between two deformations to $(R, \mathfrak{m})$, $f : (G_1, i_1) \to (G_2, i_2)$, is defined only when $i_1 = i_2$, and consists of an isomorphism $f : G_1 \to G_2$, such that $\pi^*f : i^*\Gamma = \pi^*G_1 \to \pi^*G_2 \to i^*\Gamma$ is the identity, i.e. $f(x) = x \mod \mathfrak{m}$. These assemble to a groupoid $\mathrm{Def}(R, \mathfrak{m})$, whose objects are deformations to $(R, \mathfrak{m})$, and morphisms are $\star$-isomorphisms.

Remark 1.6.5. $\mathrm{Def}(R, \mathfrak{m})$ can be seen as the pullback of the groupoids $\mathrm{FGL}(R)$ and $\coprod_{i:k \to R/\mathfrak{m}} \{\Gamma\}$ over $\mathrm{FGL}(R/\mathfrak{m})$ (where the maps are $G \mapsto q^*G$ and $i \mapsto i^*\Gamma$ respectively).

Proposition 1.6.6 (/definition)**.** *The construction* $\mathrm{Def}(R, \mathfrak{m})$ *is functorial.*

Proof. Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism.

For a deformation (G, i) to $(R, \mathfrak{m})$, we define $\mathrm{Def}(\varphi)(G, i) = (\varphi^*G, \varphi/\mathfrak{m} \circ i)$. Note that φ^*G is a formal group law over S, and $\varphi/\mathfrak{m} \circ i : k \to R/\mathfrak{m} \to S/\mathfrak{n}$ is a homomorphism. Moreover, $(\varphi/\mathfrak{m} \circ i)^* \Gamma = (\varphi/\mathfrak{m})^* i^*\Gamma = (\varphi/\mathfrak{m})^* \pi_R^* G = (\varphi/\mathfrak{m} \circ \pi_R)^* G = (\pi_S \circ \varphi)^* G = \pi_S^* \varphi^* G$, which shows that $\mathrm{Def}(\varphi)(G, i)$ is a deformation to $(S, \mathfrak{n})$.

For a $\star$-isomorphism $f : (G_1, i) \to (G_2, i)$, which is the data of an isomorphism $f : G_1 \to G_2$ such that $\pi_R^* f = \mathrm{id}_{i^*\Gamma}$ is the identity, we need to define a $\star$-isomorphism $\mathrm{Def}(\varphi)(G, i_1) \to \mathrm{Def}(\varphi)(G, i_2)$. Take it to be the isomorphism $\varphi^*f : \varphi^*G_1 \to \varphi^*G_2$, which satisfies $\pi_S^*\varphi^*f = (\varphi/\mathfrak{m})^* \pi_R^* f = (\varphi/\mathfrak{m})^* \mathrm{id}_{i^*\Gamma} = \mathrm{id}_{(\varphi/\mathfrak{m})^* i^*\Gamma} = \mathrm{id}_{(\varphi/\mathfrak{m} \circ i)^*\Gamma}$. The identity $\mathrm{id}_G : (G, i) \to (G, i)$ is clearly sent to id_{φ^*G}, and compositions are sent to compositions.

This shows that $\mathrm{Def}(\varphi) : \mathrm{Def}(R, \mathfrak{m}) \to \mathrm{Def}(S, \mathfrak{n})$ is indeed a functor. Moreover, it is clear that $\mathrm{Def}(\mathrm{id}_R)$ is the identity and compositions are sent to compositions, which shows that Def : CompRing $\to$ Grpds is indeed a functor. ◻

References

[Ada74] J. F. Adams. *Stable homotopy and generalised homology.* Chicago Press, Illinois and London, 1974.

[HS98] M. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory II. *Annals of Mathematics, 148(1), second series, 1-49*, 1998.

[LT] J. Lubin and J. Tate. Formal moduli for one-parameter formal lie groups. *Bull. Soc. Math. France 94 (1966), 49 59. MR 0238854.*

[Lur10] J. Lurie. Chromatic homotopy theory. *252x course notes,* `http://www.math.harvard.edu/~lurie/252x.html`, 2010.

[Rav86] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Academic Press, New York, 1986.