Towards Computation of HKR Generalized Characters at Height 2

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Contents

1 Introduction

Chromatic homotopy theory describes the global structure of the category of spectra. A key player in this theory is Morava E-theory, which is a higher height analogue of (p-complete) complex K-theory. In [AS69], Atiyah and Segal show that the complex K-theory of classifying spaces of groups, is deeply connected to their representations. Therefore, it can be studied using character theory. In [HKR00], Hopkins, Kuhn and Ravenel develop a generalized character theory, which can be used to study the Morava E-theory of classifying spaces of finite groups. Over the last years, there have been numerous applications of elliptic curves to chromatic homotopy theory. We give another such application, in the form of concrete computations of HKR generalized character theory at height 2.

Organization

Section 2 sets up the theory of chromatic homotopy. We recall many of the basic results in the field, omitting most of the proofs, with the goal of introducing Morava K-theory and different flavors of Morava E-theory.

Section 3 recalls the Atiyah-Segal theorem (Theorem 3.1.1), and explains its connection to character theory. This section, although not logically necessary for what follows, will serve as a motivation and as a basic example.

Section 4 gives an account of HKR generalized character theory. The main result for us, which appears in the original paper as [HKR00, Theorem C], is recalled in Theorem 4.5.3. We give most of the details of the proof, emphasizing and elaborating on some parts. We hope that this will clarify the original account, and relate it to the other parts of this work.

Section 5 focuses on height 2. The main contribution of this work is the explanation of the usage of elliptic curves to carry out explicit computations of HKR. We conclude with the development of some computer code, in Macaulay2, that implements parts of this strategy.[1]

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2 Overview of Chromatic Homotopy Theory

Our goal is to give a quick overview of chromatic homotopy theory. There are various point of views and approaches to the topic, and we shall highlight some of them. One of our main goals is to introduce Morava K-theory $\mathrm{K}(n)$ and Morava E-theory of different flavors $\mathrm{E}(n)$ and $\mathrm{E}(k,\Gamma)$, and their connection to formal group laws. Our motivation will be the Balmer spectrum of the sphere spectrum. We will follow the construction of Morava K-theory and Morava E-Theory, and other related spectra, from the point of view of

[1] We conjecture that Macaulay3 is necessary for computations at height 3.

formal group laws, and use them to describe the Balmer spectrum of the sphere spectrum.

2.1 The Balmer Spectrum

We will start with an algebraic motivation. Let R be a noetherian ring. Consider the symmetric monoidal stable ∞-category $\mathrm{Ch}(R)$ of chain complexes on R. It is then natural to ask how much information about R is encoded in the category $\mathrm{Ch}(R)$. We will try to recover $\operatorname{Spec} R$, as a topological space, from $\mathrm{Ch}(R)$.

Remark 2.1.1. Balmer's work [Bal05, 6.3] actually recovers the structure sheaf as well, but we will not consider the structure sheaf.

Definition 2.1.2. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finitely-generated projective modules. These objects are the compact objects in $\mathrm{Ch}(R)$, thus they can be defined categorically. Their full subcategory is denoted by $\mathrm{Ch}_{\mathrm{perf}}(R)$.

Definition 2.1.3. Let $\mathcal{C}$ be some symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

- $0 \in \mathcal{T}$,
- it is closed under cofibers,
- it is closed under retracts.

Example 2.1.4. Consider the case $\mathcal{C} = \mathrm{Ch}_{\mathrm{perf}}(R)$ (e.g. over $\mathbb{Z}$, chain complexes quasi-isomorphic to bounded chain complexes of finitely-generated free abelian groups). Let $K \in \mathrm{Ch}(R)$, and define $\mathcal{T}_K = \left\{A \in \mathrm{Ch}_{\mathrm{perf}}(R) \mid A \otimes K \cong 0\right\}$. We claim that $\mathcal{T}_K$ is thick. Clearly $0 \in \mathcal{T}_K$. Let $A \to B$ be a morphism between two complexes in $\mathcal{T}$. Since tensor is a left adjoint, tensoring the cofiber with K is given by $\operatorname{cofib}(A \to B) \otimes K \cong \operatorname{cofib}(A \otimes K \to B \otimes K) \cong \operatorname{cofib}(0 \to 0) \cong 0$, therefore the cofiber is indeed in $\mathcal{T}_K$. Lastly, if $A \to B \to A$ is the identity and $B \otimes K \cong 0$, we get that $\mathrm{id}_{A \otimes K}$ factors through 0, which implies that $A \otimes K$ is 0, so that $A \in \mathcal{T}_K$.

Definition 2.1.5. A thick subcategory $\mathcal{T}$ is an *ideal* if $A \in \mathcal{T}, B \in \mathcal{C} \implies A \otimes B \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $A \otimes B \in \mathcal{T} \implies A \in \mathcal{T}$ or $B \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring: As a set,

$\operatorname{Spec}\mathcal{C} = \{\mathcal{P} \text{ prime ideal}\}$. For any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \operatorname{Spec}\mathcal{C} \mid S \cap \mathcal{P} = \emptyset\}$. We topologize $\operatorname{Spec}\mathcal{C}$ with the Zariski topology by declaring those to be the closed subsets. We also denote $\operatorname{Supp}(A) = V(\{A\})$.

Example 2.1.6. We continue the example of $\mathcal{T}_K$. Clearly if $A \otimes K \cong 0$ then also $A \otimes B \otimes K \cong 0$, so it is an ideal.

Let $\mathfrak{p}$ be a prime ideal in R in the usual sense, and take $K = R_{\mathfrak{p}}$ (concentrated at degree 0), and we wish to show that $\mathcal{T}_{R_{\mathfrak{p}}}$ is a prime ideal. Let $A, B \in \mathrm{Ch}_{\mathrm{perf}}(R)$ be complexes such that $A \otimes B \in \mathcal{T}_{R_{\mathfrak{p}}}$, that is $A \otimes B \otimes R_{\mathfrak{p}}$ is quasi-isomorphic to 0. We need to show that one of $A \otimes R_{\mathfrak{p}}$ and $B \otimes R_{\mathfrak{p}}$ is quasi-isomorphic to 0. Note that $A \otimes B \otimes R_{\mathfrak{p}} = (A \otimes R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} (B \otimes R_{\mathfrak{p}})$. Denote $S = R_{\mathfrak{p}}$, which is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$ and residue field $k = S/\mathfrak{m}$, we are then reduced to showing that for perfect complexes $X = A \otimes R_{\mathfrak{p}}, Y = B \otimes R_{\mathfrak{p}}$, such that $H_*(X \otimes_S Y) = 0$, either $H_*(X) = 0$ or $H_*(Y) = 0$.

Note that if we base-change to k, this statement is trivial since by Künneth, $0 = H_*(X/\mathfrak{m} \otimes_k Y/\mathfrak{m}) = H_*(X/\mathfrak{m}) \otimes_k H_*(Y/\mathfrak{m})$, and the tensor product of vector spaces is 0 if and only if one of them is 0. Therefore, it suffices to prove that for a perfect complex Z over S, $H_*(Z) = 0$ if and only if $H_*(Z/\mathfrak{m}) = 0$.

For the first direction, it is clear that if Z is quasi-isomorphic to 0, then the (derived) tensor product $Z \otimes_S k = Z/\mathfrak{m}$ is also quasi-isomorphic to 0.

For the other direction, assume that $Z/\mathfrak{m}$ is quasi-isomorphic to 0. Since Z is perfect, we can choose it to be bounded

$$Z = \cdots \to 0 \to Z_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-2}} Z_{n-1} \xrightarrow{d_{n-1}} Z_n \to 0 \to \cdots,$$

where each Z_i is finitely-generate projective, and over the local ring S, this also implies that each Z_i is in fact free. We will show that Z is quasi-isomorphic to a complex with only $n-1$ non-trivial terms, and by induction we conclude that Z is indeed quasi-isomorphic to 0. We are given that $H_*(Z/\mathfrak{m}) = 0$, so in particular it is exact at $Z_{n-1}/\mathfrak{m} \xrightarrow{d_{n-1}} Z_n/\mathfrak{m} \to 0$, i.e. $\operatorname{im} d_{n-1} = Z_n \mod \mathfrak{m}$. By Nakayama's lemma we conclude that $\operatorname{im} d_{n-1} = Z_n$, i.e. $Z_{n-1} \xrightarrow{d_{n-1}} Z_n$ is surjective. By the freeness, we conclude that $Z_{n-1} \cong M \oplus Z_n$, under which $d_{n-1} = 0 \oplus \mathrm{id}_{Z_n}$. Since $d_{n-1}d_{n-2} = 0$, the image of d_{n-1} must land in M. Therefore, Z is quasi-isomorphic to

$$\cdots \to 0 \to Z_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-2}} M \to 0 \to 0 \to \cdots,$$

which has only $n-1$ non-trivial terms, and we are done.

Theorem 2.1.7 ([Bal05, 5.6])**.** *The map* $\operatorname{Spec} R \to \operatorname{Spec}\left(\mathrm{Ch}_{\mathrm{perf}}(R)\right)$*, given by* $\mathfrak{p} \mapsto \mathcal{T}_{\mathfrak{p}} = \{A \mid A_{\mathfrak{p}} = 0\}$ *is a homeomorphism.*

Proposition 2.1.8 (Prime ideals pullback)**.** *Let* $F : \mathcal{C} \to \mathcal{D}$ *be an exact symmetric monoidal functor, between two symmetric monoidal stable* ∞*-categories. Let* $\mathcal{P} \in \operatorname{Spec}\mathcal{D}$ *be a prime ideal, then* $F^*\mathcal{P} = F^{-1}(\mathcal{P}) = \{A \in \mathcal{C} \mid F(A) \in \mathcal{P}\}$ *is a prime ideal. Moreover, the function we obtain,* $F^* : \operatorname{Spec}\mathcal{D} \to \operatorname{Spec}\mathcal{C}$*, is continuous.*

Proof. We first prove that for $\mathcal{P} \in \operatorname{Spec}\mathcal{D}$, $F^*\mathcal{P} \in \operatorname{Spec}\mathcal{C}$.

Clearly $F(0) = 0 \in \mathcal{P}$ since F is exact, so $0 \in F^*\mathcal{P}$. Since F preserves cofibers, for $A, B \in F^*\mathcal{P}$, i.e. $F(A), F(B) \in \mathcal{P}$, and a map $A \to B$, we get $F(\operatorname{cofib}(A \to B)) = \operatorname{cofib}(F(A) \to F(B)) \in \mathcal{P}$. Let $A \to B \to A$ be a retract, that is the composition is the identity, s.t. $B \in F^*\mathcal{P}$. We know that $F(A) \to F(B) \to F(A)$ is also a retract by functoriality, thus $F(A) \in \mathcal{P}$, that is $A \in F^*\mathcal{P}$. We conclude that $F^*\mathcal{P}$ is indeed a thick subcategory.

Let $A \in F^*\mathcal{P}, B \in \mathcal{C}$, since F is monoidal, $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{P}$, so $A \otimes B \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is an ideal.

We claim that $F^*\mathcal{P}$ is a proper subcategory, because an ideal is proper if and only if it doesn't contain 1, and since F is symmetric monoidal it sends 1 to 1.

Lastly, assume that $A \otimes B \in F^*\mathcal{P}$, again since F is monoidal, $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{P}$, so $A \in F^*\mathcal{P}$ or $B \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is a prime idea.

Now we show that $F^* : \operatorname{Spec}\mathcal{D} \to \operatorname{Spec}\mathcal{C}$ is continuous. So let $V(S) \subseteq \operatorname{Spec}\mathcal{C}$ be a closed subset. We have:

$$\begin{aligned}
(F^*)^{-1}(V(S)) &= \{\mathcal{P} \in \mathcal{D} \mid F^*\mathcal{P} \in V(S)\} \\
&= \left\{\mathcal{P} \in \mathcal{D} \mid F^{-1}(\mathcal{P}) \cap S = \emptyset\right\} \\
&= \{\mathcal{P} \in \mathcal{D} \mid \mathcal{P} \cap F(S) = \emptyset\} \\
&= V(F(S))
\end{aligned}$$

So $(F^*)^{-1}(V(S))$ is indeed also closed, which shows that F^* is continuous. $\square$

Now, recall that $\mathrm{Ch}(R) \cong \mathrm{Mod}_{\mathrm{H}R}$, therefore we can reinterpret the above theorem as $\operatorname{Spec} R \cong \operatorname{Spec}\left(\mathrm{Mod}_{\mathrm{H}R}^{\mathrm{comp}}\right)$ (where the comp denotes the compact objects in the category). We shall turn this theorem into a definition:

Definition 2.1.9. Let R be an $\mathbb{E}_\infty$-ring. We define the *spectrum* of R to be $\operatorname{Spec} R = \operatorname{Spec}\left(\mathrm{Mod}_R^{\mathrm{comp}}\right)$.

A natural question to ask then is what is the topological space $\operatorname{Spec}\mathbb{S}$. Recall that $\mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$, the category of spectra, and that the compact objects in spectra are the finite spectra $\mathrm{Sp}^{\mathrm{fin}}$. So, unwinding the definitions, the question can rephrased as finding the prime ideals in $\mathrm{Sp}^{\mathrm{fin}}$, and their topology. Chromatic homotopy theory provides an answer to this question.

2.2 MU and Complex Orientations

Throughout this section, let E be a multiplicative cohomology theory (that is, equipped with a map $E \otimes E \to E$ and $1 \in E_0$, which is associative and unital after taking homotopy groups).

Consider the map $S^2 \to \mathrm{BU}(1)$ classifying the universal complex line bundle. Concretely, under the identifications $S^2 \cong \mathbb{C}\mathrm{P}^1$ and $\mathrm{BU}(1) \cong \mathbb{C}\mathrm{P}^\infty$, this map can be realized as the inclusion $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^\infty$. This map induces a map

$$\tilde{E}^2(\mathrm{BU}(1)) \to \tilde{E}^2\left(S^2\right) \cong \tilde{E}^0\left(S^0\right) \cong E^0(*) = E_0.$$

Since E is unital, there is a canonical generator $1 \in E_0$.

Definition 2.2.1. E is called *complex oriented* if the map $\tilde{E}^2(\mathrm{BU}(1)) \to E_0$ is surjective, equivalently, if 1 is in the image of that map. A choice of a lift $x \in \tilde{E}^2(\mathrm{BU}(1))$ of $1 \in E_0$ is called a *complex orientation*. (Note that $|x| = -2$ as it is in cohomological degree 2.)

Example 2.2.2. Let R be some ring, and consider $\mathrm{H}R$. It is known that $\mathrm{H}R^*(\mathbb{C}\mathrm{P}^n) \cong R[x]/\left(x^{n+1}\right)$ and $\mathrm{H}R^*(\mathbb{C}\mathrm{P}^\infty) \cong R[[x]]$, where $|x| = -2$, and the maps induced by the inclusions of projective spaces maps x to x. Therefore we see that $x \in \mathrm{H}R^2(\mathrm{BU}(1))$ is mapped to $x \in \mathrm{H}R^2\left(S^2\right) = R\{x\}$, which is mapped to $1 \in \mathrm{H}R_0 = R$. Hence, x is a complex orientation.

Example 2.2.3 (K-Theory Saga: Complex Orientation). Let K be complex K-theory, then we know that $\mathrm{K}_* = \mathbb{Z}\left[\beta^{\pm 1}\right]$ where β is the Bott element, with $|\beta| = 2$. It is also known (by the Atiyah-Hirzebruch spectral sequence) that $\mathrm{K}^*(\mathbb{C}\mathrm{P}^n) \cong \mathrm{K}_*[t]/\left(t^{n+1}\right)$ and $\mathrm{K}^*(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{K}_*[[t]]$ (here $|t| = 0$), where the maps induced by the inclusions of projective spaces maps t to t. We deduce that $\beta^{-1}t \in \mathrm{K}^2(\mathrm{BU}(1))$ is mapped to $\beta^{-1}t \in \mathrm{K}^2\left(S^2\right) = \mathbb{Z}\left\{\beta^{-1}t\right\}$, which is indeed the generator. Therefore $x = \beta^{-1}t$ is complex orientation for K.

Example 2.2.4. Recall that MU is constructed as the colimit $\mathrm{MU} = \operatorname{colim} \mathrm{MU}(n)$. Also, $\mathrm{MU}(1) \cong \Sigma^{\infty-2}\mathrm{BU}(1)$. Therefore we get a canonical map $\Sigma^{\infty-2}\mathrm{BU}(1) \to \mathrm{MU}$, which gives a cohomology class $x_{\mathrm{MU}} \in \mathrm{MU}^2(\mathrm{BU}(1))$.

Proposition 2.2.5 ([Rav86, 4.1.3])**.** *x_{MU} is a complex orientation for* MU*.*

Theorem 2.2.6 ([Rav86, 4.1.13])**.** MU *is the universal complex oriented cohomology theory, in the following sense: For any multiplicative cohomology theory E, there is a bijection between (homotopy classes of) multiplicative maps $\mathrm{MU} \to E$ and complex orientations on E. The bijection is given in one direction by pulling back x_{MU} along a multiplicative map.*

Assume that E is complex oriented with a complex orientation x.

Proposition 2.2.7 ([Rav86, 4.1.4])**.** *As E_*-algebras, $E^*(\mathrm{BU}(1)) \cong E^*[[x]]$ and $E^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) \cong E^*[[y,z]]$.*

There is a multiplication map for the group $\mathrm{U}(1)$, i.e. $\mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{U}(1)$. We can take the B of this map, and since it commutes with products we get a map $\mathrm{BU}(1) \times \mathrm{BU}(1) \to \mathrm{BU}(1)$, which is the universal map that classifies the tensor product of vector bundles. Therefore we get a map $E^*(\mathrm{BU}(1)) \to E^*(\mathrm{BU}(1) \times \mathrm{BU}(1))$, which is completely determined by the image of $x \in E^*[[x]]$ in $E^*[[y,z]]$ as above. We conclude that a choice of a complex orientation on E gives rise to an element $F_E(y,z) \in E^*[[y,z]]$.

Proposition 2.2.8 ([Rav86, 4.1.4])**.** *F_E is a formal group law on E_*.*

Definition 2.2.9. The *height* of E is simply the height of F_E.

Example 2.2.10. We continue with $\mathrm{H}R$ from Example 2.2.2. It is known that the tensor of complex line bundles induces the map

$$R[[x]] = \mathrm{H}R^*(\mathrm{BU}(1)) \to \mathrm{H}R^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = R[[y,z]],$$

given by $x \mapsto y+z$. This is the additive formal group law. It is immediate that $[p](x) = px$. So for $R = \mathbb{Q}$ we get that the height of $\mathrm{H}\mathbb{Q}$ is 0, while for $R = \mathbb{F}_p$ we have $px = 0$ so the height of $\mathrm{H}\mathbb{F}_p$ is ∞.

Example 2.2.11 (K-Theory Saga: Formal Group Law)*.* We return to complex K-theory from Example 2.2.3. It is known that the tensor of complex line bundles induces the map

$$\mathrm{K}_*[[t]] = \mathrm{K}^*(\mathrm{BU}(1)) \to \mathrm{K}^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = \mathrm{K}_*[[u,v]],$$

given by $t \mapsto u + v + uv$. Note that to comply with the definition of the formal group law, we should use the isomorphism $\mathrm{K}^* (\mathrm{BU}(1)) \cong \mathrm{K}_* [[x]]$, i.e. the element $x = \beta^{-1} t$. By multiplying by β^{-1} (recall that the map is of K_*-modules) we get that

$$x = \beta^{-1} t \mapsto \beta^{-1} u + \beta^{-1} v + \beta^{-1} uv = y + z + \beta yz = F_{\mathrm{K}} (y, z) .$$

By induction we prove that the n-series is $[n](x) = \beta^{-1} (1 + \beta x)^n - \beta^{-1}$. This is clear for $n = 1$, and we have:

$$\begin{aligned} [n+1](x) &= x + [n](x) + \beta x [n](x) \\ &= x + \beta^{-1} (1 + \beta x)^n - \beta^{-1} + x (1 + \beta x)^n - x \\ &= \beta^{-1} (1 + \beta x)(1 + \beta x)^n - \beta^{-1} \\ &= \beta^{-1} (1 + \beta x)^{n+1} - \beta^{-1} \end{aligned}$$

Example 2.2.12 (K-Theory Saga: mod-p). By taking the cofiber of the multiplication-by-p map, we get a spectrum K/p, mod-p K-theory, with coefficients $(\mathrm{K}/p)_* = \mathbb{F}_p \left[\beta^{\pm 1}\right]$. It is evident that $F_{\mathrm{K}/p} (y, z) = y + z + \beta yz$ as well. From the result above, it follows that

$$[p](x) = \beta^{-1} (1 + \beta x)^p - \beta^{-1} = \beta^{-1} (1^p + \beta^p x^p) - \beta^{-1} = \beta^{p-1} x^p,$$

which shows that the height is exactly 1.

A formal group law on E_* is the same data as a map from the Lazard ring L, so the complex orientation gives a map $L \to E_*$. In particular, since MU is complex oriented, there is a canonical map $L \to \mathrm{MU}_*$.

Theorem 2.2.13 (Quillen, [Rav86, 4.1.6])**.** *The canonical map $L \to \mathrm{MU}_*$ is an isomorphism.*

2.3 BP, Morava K-Theory and Morava E-Theory

A good principle in homotopy theory (and in many other areas in math) is to study it one prime at a time. This is possible in homotopy theory due to the arithmetic square. So, let us fix a prime p. We can form $\mathrm{MU}_{(p)}$, the p-localization of MU.

Theorem 2.3.1 ([Ada74, II 15])**.** *There exists a unique map of ring spectra $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ (depending on the prime p) satisfying:*

- *ε is an idempotent, i.e. $\varepsilon^2 = \varepsilon$,*
- *ε_* sends $[\mathbb{CP}^n] \in \pi_*\left(\mathrm{MU}_{(p)}\right)$ to itself if $n = p^r - 1$ and to 0 otherwise.*

The map $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ gives a cohomology operation, for every X we have $\varepsilon^* : \mathrm{MU}^*_{(p)}(X) \to \mathrm{MU}^*_{(p)}(X)$. Denote by $\mathrm{BP}^*(X)$ the image of ε^*.

Theorem 2.3.2 ([Ada74, II 16], [Rav86, 4.1.12])**.** *BP is a cohomology theory, represented by an associative commutative ring spectrum BP (depending on the prime p), which is a retract of $\mathrm{MU}_{(p)}$. The homotopy groups of BP are $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ where $|v_n| = 2(p^n - 1)$.*

For convenience we denote $v_0 = p$ (and indeed $|v_0| = 2\left(p^0 - 1\right) = 0$). Since BP is a retract of MU, it comes with a map $\mathrm{MU} \to \mathrm{BP}$, that is, a complex orientation.

Proposition 2.3.3 ([Rav86, 4.1.12 combined with A2.1.25 and A2.2.4])**.** *The p-series of the formal group law associated to BP is $[p](x) = \sum_F v_n x^{p^n}$ (note that the sum on the right hand side is in the formal group law).*

Remark 2.3.4 ([Rav92, B.5])*.* The formal group law on BP has a similar interpretation to that of MU, namely it is the universal p-typical formal group law. Moreover, the idempotent $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ induces an idempotent on homotopy groups, which can be described as the map that takes a formal group law to the canonical p-typical formal group law isomorphic to it.

Once we have BP, we can turn to the definition of Morava K-theory and Johnson-Wilson spectrum (a variant of Morava E-theory).

Definition 2.3.5. Let $0 < n < \infty$. *Morava K-theory* at height n and prime p, denoted by $\mathrm{K}(p, n)$ or $\mathrm{K}(n)$ when the prime is clear from the context, is the spectrum obtained by killing $p = v_0, \dots, v_{n-1}, v_{n+1}, \dots$ in BP and inverting v_n. Therefore $\mathrm{K}(n)_* = \mathbb{F}_p\left[v_n^{\pm 1}\right]$. We also define $\mathrm{K}(0) = \mathrm{H}\mathbb{Q}$ and $\mathrm{K}(\infty) = \mathrm{H}\mathbb{F}_p$. Similarly, *Johnson-Wilson spectrum* (sometimes also called Morava E-theory) at height n and prime p, denoted by $\mathrm{E}(p, n)$ or $\mathrm{E}(n)$, is the spectrum obtained by killing $v_{n+1}, v_{n+2}, \dots$ in BP and inverting v_n. Therefore $\mathrm{E}(n)_* = \mathbb{Z}_{(p)}\left[v_1, \dots v_{n-1}, v_n^{\pm 1}\right]$.

Since Morava K-theory and E-theory are obtained from BP by cofibers and filtered colimits, they are equipped with a map from BP, hence also with a complex orientation. Then, from Proposition 2.3.3, we get:

Corollary 2.3.6. *The p-series associated to the formal group laws of* $\mathrm{K}(n)$ *and* $\mathrm{E}(n)$ *are* $v_n x^{p^n}$ *and* $v_0 x +_F \ldots +_F v_n x^{p^n}$ *respectively. Therefore the height of* $\mathrm{K}(n)$ *is exactly n. (Note that by Example 2.2.10, this is also true for* $\mathrm{K}(0)$ *and* $\mathrm{K}(\infty)$*.)*

We want to describe some properties of Morava K-theory. To do so we first need some definitions.

Definition 2.3.7. Let R be an evenly graded ring. R is called a *graded field* if it satisfies one of the equivalent conditions:

- every non-zero homogenus element is invertible,
- it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree.

An $\mathbb{A}_\infty$-ring E is a *field* if E_* is a graded field.

Example 2.3.8. $K(n)$ is a field for $0 \leq n \leq \infty$.

Proposition 2.3.9. *A field E has Künneth, i.e.* $E_*(X \otimes Y) \cong E_*(X) \otimes_{E_*} E_*(Y)$ *for any spectra X, Y.*

Proposition 2.3.10 ([Lur10, 24])**.** *Let $E \neq 0$ be a complex oriented cohomology theory, whose formal group law has height exactly n, then* $E \otimes \mathrm{K}(n) \neq 0$*. Let E be a field s.t.* $E \otimes \mathrm{K}(n) \neq 0$*, then E admits the structure of a* $\mathrm{K}(n)$*-module. (Here* $0 \leq n \leq \infty$*.)*

Example 2.3.11 (K-Theory Saga: Morava K-Theory). As we have seen in Example 2.2.12, mod-p K-theory, K/p, has height exactly 1 and coefficients $(\mathrm{K}/p)_* = \mathbb{F}_p\left[\beta^{\pm 1}\right]$. It is also known that K and K/p, are $\mathbb{A}_\infty$-ring spectra, from which it follows that K/p is a field. We deduce that K/p is a $\mathrm{K}(1)$-module. Since $|\beta| = 2$ and $|v_1| = 2(p-1)$ it is free of rank $p-1$.

From this we also deduce some form of uniqueness for Morava K-theory:

Corollary 2.3.12. *Let E be a field with* $E_* \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$*, which is also complex oriented of height exactly n. Then* $E \cong \mathrm{K}(n)$ *(as spectra).*

2.4 $\operatorname{Spec} \mathbb{S}_{(p)}$ and $\operatorname{Spec} \mathbb{S}$

We are now in a position to describe the topological space $\operatorname{Spec} \mathbb{S}$. However, it will be easier to state it first for $\operatorname{Spec} \mathbb{S}_{(p)}$, and then pullback prime ideals.

We know that $\mathrm{Mod}_{\mathbb{S}_{(p)}} = \mathrm{Sp}_{(p)}$, and its compact objects are $\mathrm{Sp}^{\mathrm{fin}}_{(p)}$, the p-localizations of finite spectra.

Proposition 2.4.1. *Let $\mathcal{T}_E$ be the E-acyclics, i.e.*

$$\mathcal{T}_E = \ker E_* = \left\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid E_*(X) = 0\right\} = \left\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid X \otimes E = 0\right\}.$$

Then $\mathcal{T}_E$ is thick.

Proof. The proof follows the same lines of Example 2.1.4 for the case $\mathrm{Ch}_{\mathrm{perf}}(R)$. □

Definition 2.4.2. We define $\mathcal{C}_{p,n} = \mathcal{T}_{\mathrm{K}(n)}$, the $\mathrm{K}(n)$-acyclics. By the above proposition, it is thick. Also, $\mathcal{C}_{p,\infty} = \{0\}$, which is trivially thick. When the prime is clear from the context, we write $\mathcal{C}_n$ in place of $\mathcal{C}_{p,n}$.

Proposition 2.4.3 ([Lur10, 26])**.** *For $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$, if $\mathrm{K}(n)_*(X) = 0$ then $\mathrm{K}(n-1)_*(X) = 0$.*

Definition 2.4.4. We say that a spectrum $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$ is of *type n* (possibly ∞) if its first non-zero Morava K-theory homology is $\mathrm{K}(n)$.

Corollary 2.4.5. *$\mathcal{C}_n$ is the full subcategory of finite p-local spectra of type $> n$, that is $\mathcal{C}_n = \left\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid \forall m \leq n : \mathrm{K}(m)_*(X) = 0\right\}$. Therefore, we also conclude that $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$.*

Proposition 2.4.6. *The inclusions $\mathcal{C}_{n+1} \subset \mathcal{C}_n$ are proper.*

Remark 2.4.7. The modern proof of this result relies on the periodicity theorem [Rav92, 1.5.4]. Using it, we can construct generalized Moore spectra, which give an example of spectra of type n for every n.

Proposition 2.4.8. *If $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$ is not contractible, then it is of finite type. Therefore $\bigcap_{n<\infty} \mathcal{C}_n = \{0\} = \mathcal{C}_\infty$.*

Proof. Let X be non-contractible. Then $\mathrm{H}\mathbb{Z}_*(X) \neq 0$. Let m be the first non-zero degree. Using the universal coefficient theorem and the fact that the spectrum is p-local we get that $(\mathrm{H}\mathbb{F}_p)_m(X) \neq 0$, thus $(\mathrm{H}\mathbb{F}_p)_*(X) \neq 0$. Since X is finite, $(\mathrm{H}\mathbb{F}_p)_*(X)$ is bounded. The Atiyah-Hirzebruch spectral sequence for X with cohomology $\mathrm{K}(n)$ has E^2-page given by $E^2_{t,s} = H_t(X; \mathrm{K}(n)_s)$. Since $\mathrm{K}(n)_s = \mathbb{F}_p$ for $s = 0 \mod 2(p^n-1)$ and 0 otherwise, we see that the rows $s = 0 \mod 2(p^n-1)$ are $(\mathrm{H}\mathbb{F}_p)_*(X)$, and the others are 0. Therefore if we take n such that the period $2(p^n - 1)$ is larger then the bound on $(\mathrm{H}\mathbb{F}_p)_*(X)$, then all differentials have either source or target 0. Thus, the spectral sequence collapses at the E^2-page, and since $(\mathrm{H}\mathbb{F}_p)_*(X) \neq 0$, we get that $\mathrm{K}(n)(X) \neq 0$, i.e. X has type $\leq n$. $\square$

Proposition 2.4.9. *$\mathcal{C}_n$ is a prime ideal.*

Proof. Recall from Proposition 2.4.1 that we already know that it is thick. For $X, Y \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$, by Proposition 2.3.9 we have

$$\mathrm{K}(n)_*(X \otimes Y) = \mathrm{K}(n)_*(X) \otimes \mathrm{K}(n)_*(Y).$$

Assume that $X \in \mathcal{C}_n$, that is $\mathrm{K}(n)_*(X) = 0$. It follows that $\mathrm{K}(n)_*(X \otimes Y) = 0$, i.e. $X \otimes Y \in \mathcal{C}_n$, so $\mathcal{C}_n$ is an ideal. Assume that $X \otimes Y \in \mathcal{C}_n$, that is $\mathrm{K}(n)_*(X \otimes Y) = 0$, therefore one of the terms in the RHS of the equation must vanish (since they are graded vector spaces), so $\mathcal{C}_n$ is a prime ideal. $\square$

Theorem 2.4.10 (Thick Subcategory Theorem [HS98, theorem 7])**.** *If $\mathcal{T}$ is a proper thick subcategory of $\mathrm{Sp}^{\mathrm{fin}}_{(p)}$, then $\mathcal{T} = \mathcal{C}_n$ for some $0 \leq n \leq \infty$.*

Remark 2.4.11. The proof relies on a major theorem called the Nilpotence Theorem.

Corollary 2.4.12. *$\operatorname{Spec} \mathbb{S}_{(p)} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_\infty\}$, and the closed subsets in the topology are chains $\{\mathcal{C}_k, \mathcal{C}_{k+1}, \ldots, \mathcal{C}_\infty\}$ for some $0 \leq k \leq \infty$.*

Proof. Follows immediately from the previous results. $\square$

We now want to describe $\operatorname{Spec} \mathbb{S}$. Note that the p-localization functor $L_{(p)}$ is a Bousfield localization. As such, it is a left adjoint (its right adjoint is the inclusion), and in particular preserves cofibers. It is also reduced, i.e. sends 0 to 0. Now, $L_{(p)}$ is smashing, that is $L_{(p)}X = X \otimes \mathbb{S}_{(p)}$, so it is also symmetric monoidal. As we have seen in Proposition 2.1.8, under these conditions we can pullback primes. Since $L_{(p)}$ is smashing and $\mathrm{K}(p,n)$ is p-local for every $0 \leq n \leq \infty$, we have that $\mathrm{K}(n)_*\left(X_{(p)}\right) = \mathrm{K}(n)_*(X)$. Therefore

$$\mathcal{P}_{p,n} = L^*_{(p)}\mathcal{C}_{p,n} = \left\{X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{K}(p,n)_*(X) = 0\right\}$$

and

$$\mathcal{P}_{p,\infty} = L^*_{(p)}\mathcal{C}_{p,\infty} = \left\{X \in \mathrm{Sp}^{\mathrm{fin}} \mid X_{(p)} = 0\right\}$$

are prime ideals. Moreover, by definition $\mathrm{K}(p,0) = \mathrm{H}\mathbb{Q}$, which implies that $\mathcal{P}_{p,0} = \left\{ X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_*(X) = 0 \right\}$. We see that $\mathcal{P}_{p,0}$ is independent of p, and we denote it by $\mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}}$. Again, by Proposition 2.1.8, we see that any chain $\{\mathcal{P}_{p,k}, \mathcal{P}_{p,k+1}, \ldots, \mathcal{P}_{p,\infty}\}$ for some p and $0 \le k \le \infty$, is closed, hence any finite union of such chain is also closed. We then have the following theorem, which says that these are all of the prime ideals and all of the closed subsets.

Theorem 2.4.13 (Thick Subcategory Theorem, [Bal10, 9.5])**.** *$\operatorname{Spec}\mathbb{S} = \left\{\mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}}\right\} \cup \left(\bigcup_p \{\mathcal{P}_{p,1}, \ldots, \mathcal{P}_{p,\infty}\}\right)$, and the closed subsets in the topology are finite unions of chains $\{\mathcal{P}_{p,k}, \mathcal{P}_{p,k+1}, \ldots, \mathcal{P}_{p,\infty}\}$ for some $0 \le k \le \infty$ (i.e. they may include $\mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}}$).*

Remark 2.4.14. The following diagram shows the structure of $\operatorname{Spec}\mathbb{S}$. Each $\mathcal{P}_{p,n}$, and $\mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}}$, is a point. A line represents that the closure of the point at the bottom contains the point at the top.

$$
\begin{tikzcd}
\mathcal{P}_{2,\infty} \arrow[d, dash] & \mathcal{P}_{3,\infty} \arrow[d, dash] & \cdots & \mathcal{P}_{p,\infty} \arrow[d, dash] & \cdots \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
\mathcal{P}_{2,n} \arrow[d, dash] & \mathcal{P}_{3,n} \arrow[d, dash] & \cdots & \mathcal{P}_{p,n} \arrow[d, dash] & \cdots \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
\mathcal{P}_{2,2} \arrow[d, dash] & \mathcal{P}_{3,2} \arrow[d, dash] & \cdots & \mathcal{P}_{p,2} \arrow[d, dash] & \cdots \\
\mathcal{P}_{2,1} \arrow[drr, dash] & \mathcal{P}_{3,1} \arrow[dr, dash] & \cdots & \mathcal{P}_{p,1} \arrow[dl, dash] & \cdots \\
 & & \mathrm{Sp}^{\mathrm{fin}}_{\mathrm{tor}} & &
\end{tikzcd}
$$

Remark 2.4.15. Thick subcategories are interesting for another reason, unrelated to the Balmer spectrum point of view, namely they give a very powerful proof method. Say we have a property that is satisfied by 0, and is closed under cofibers and retracts. It follows that the collection of objects that satisfy it is thick. Then, for example, by the thick subcategory theorem 2.4.10, it is enough to find one object in $\mathcal{C}_n \setminus \mathcal{C}_{n+1}$ that satisfies the property, to show that all objects in $\mathcal{C}_n$ satisfy it.

2.5 The Stacky Point of View and the Relationship Between Morava K-Theory and Morava E-Theory

First we will describe, without being precise, another point of view on what chromatic homotopy theory is about.

There is a stack of formal groups with strict isomorphisms, denoted by $\mathcal{M}_{\mathrm{fg}}^{s}$. It can be described as the stack that sends a ring R to the groupoid of formal group laws, with strict isomorphisms between them. Quillen theorem 2.2.13 tells us that MU_* is the Lazard ring, that is the universal ring that carries the universal formal group law. This theorem has a second part, which says that $(\mathrm{MU} \otimes \mathrm{MU})_*$ is the universal ring that carries two formal group laws and a strict isomorphism between them. Therefore, $\mathcal{M}_{\mathrm{fg}}^{s}$ is represented by $(\mathrm{MU}_*, (\mathrm{MU} \otimes \mathrm{MU})_*)$.

The geometric points of the stack $\mathcal{M}_{\mathrm{fg}}^{s}$ are precisely the same as $\operatorname{Spec} \mathbb{S}$. That is because for an algebraically closed field of characteristic 0 there is a unique (up to strict isomorphism) formal group law which is of height 0, namely the additive formal group law, and for characteristic p there is a unique (up to strict isomorphism) formal group law of each height $1 \leq n \leq \infty$.

For a spectrum X, $\mathrm{MU}_*(X)$ is a $(\mathrm{MU}_*, (\mathrm{MU} \otimes \mathrm{MU})_*)$-comodule, which is the same as a sheaf over $\mathcal{M}_{\mathrm{fg}}^{s}$. From this point of view, chromatic homotopy theory lets us study a spectrum by decomposing it over the stack $\mathcal{M}_{\mathrm{fg}}^{s}$.

We can restrict ourselves to the stack only over p-local rings, $\mathcal{M}_{\mathrm{fg},p}^{s}$, which is then represented by $\left(\left(\mathrm{MU}_{(p)}\right)_*, \left(\mathrm{MU}_{(p)} \otimes \mathrm{MU}_{(p)}\right)_*\right)$. Similarly to MU, BP_* is the universal ring with the universal p-typical formal group law, and $(\mathrm{BP} \otimes \mathrm{BP})_*$ is the universal ring with two p-typical formal group laws and a strict isomorphism between them. Since every formal group law is isomorphic to a unique p-typical one, we know that the stack $\mathcal{M}_{\mathrm{fg},p}^{s}$ is also represented by $(\mathrm{BP}_*, (\mathrm{BP} \otimes \mathrm{BP})_*)$.

It is now reasonable that $\mathrm{K}(n)$, obtained from BP by killing the v_m's for $m \neq n$ and inverting v_n, sees the n-th level, and that $\mathrm{E}(n)$ obtained in the same way but only killing v_m for $m > n$, sees the levels $\leq n$. Following this point of view, the following theorem can be proven:

Theorem 2.5.1 ([Lur10, 23, proposition 2])**.** *$\mathrm{E}(n)$ and $\mathrm{K}(0) \oplus \cdots \oplus \mathrm{K}(n)$ are Bousfield equivalent. That is, they have the same acyclics, locals, and their localization functors are the same.*

2.6 Landweber Exact Functor Theorem

As we have seen, a complex orientation on a cohomology theory, which by Theorem 2.2.6 is given by a map $\mathrm{MU} \to E$, has an associated formal group law, which is given by the map $L = \mathrm{MU}_* \to E_*$. Note that this formal group law is of degree -2, by virtue of the grading on $L = \mathrm{MU}_*$. One can ask whether the converse is true, namely given a graded ring R and a formal group law F of degree -2 given by $L \to R$, is there a complex oriented cohomology theory whose coefficients are R and the associated formal group law is F.

A strategy is to define $\left(E_{R,F}\right)_* (X) = \mathrm{MU}_* (X) \otimes_{\mathrm{MU}_*} R$. Unfortunately, this is not always a homology theory. However there is a condition that one can check, which guarantees that it is.

Definition 2.6.1. $L \to R$ is called *Landweber flat* if for every prime p, the image of the sequence $p = v_0, v_1, v_2, \dots$ in R, is regular. That is, for every prime p and $n \geq 0$, v_n is not a zero divisor in $R / \left(v_0, v_1, \dots, v_{n-1}\right)$.

Remark 2.6.2. Recall from Proposition 2.3.3 that for the formal group law over BP we have $[p](x) = \sum_F v_n x^{p^n}$. Seemingly v_n is a coefficient in a complicated sum involving F itself. However, modulo $(v_0, v_1, \dots, v_{n-1})$ (which is all we need for Landweber flatness) v_n is equal to the coefficient of x^{p^n} in the p-series expanded to a usual power series, which may be much easier to compute. To see this, first of all note that $\sum_F v_k x^{p^k} = \sum_{k \leq n, F} v_k x^{p^k} +_F \sum_{k > n, F} v_k x^{p^k}$. The second term may contribute only powers higher than p^n. Moreover, the first term modulo $(v_0, v_1, \dots, v_{n-1})$ is simply $v_n x^{p^n}$, and the conclusion follows.

Remark 2.6.3. If p is invertible in R, then $p = v_0$ is invertible, and R/p is already 0, so we don't need to check $v_1, v_2, \dots$.

Theorem 2.6.4 (Landweber Exact Functor Theorem (LEFT), [Lur10, 15, 16])**.** *If $L \to R$ is Landweber flat, then $E_{R,F}$ defined above is a homology theory. Moreover, there are no phantom maps between such spectra, so $E_{R,F}$ is represented by a spectrum. This spectrum is complex oriented, has coefficients R, and associated formal group law F.*

Example 2.6.5. Johnson-Wilson spectrum $\mathrm{E}(n)$ is Landweber flat, since by Corollary 2.3.6, the p-series has coefficients $p = v_0, v_1, \dots, v_n$. p is not a zero divisor in $\mathrm{E}(n)_* = \mathbb{Z}_{(p)}\left[v_1, \dots v_{n-1}, v_n^{\pm 1}\right]$. Then v_i is not a zero divisor in $\mathrm{E}(n)_* / \left(p, v_1, \dots, v_{i-1}\right) \cong \mathbb{F}_p\left[v_i, \dots v_{n-1}, v_n^{\pm 1}\right]$. After v_n the ring becomes 0 and we are done. For other primes, by Remark 2.6.3 we are done.

Example 2.6.6. Morava K-theory $\mathrm{K}(n)$ for $n > 0$ is not Landweber flat since $p = v_0$ is 0 in $\mathrm{K}(n)_* = \mathbb{F}_p\left[v_n^{\pm 1}\right]$.

Example 2.6.7. $\mathrm{H}\mathbb{Z}$ is not Landweber flat since although $p = v_0$ is invertible, as we have seen in Example 2.2.10 the p-series is px, so v_1 is 0 in $\mathbb{Z}/p = \mathbb{F}_p$.

We can also ask the following question: given complex oriented cohomology theory $\mathrm{MU} \to E$, such that $L \to E_*$ is Landweber flat, is $E_{R,F}$ equivalent to E? The answer is yes, at least in some cases.

Theorem 2.6.8. *Let E be as above, which is also evenly graded (i.e. E_* is an evenly graded ring), then there is an equivalence $E_{R,F} \to E$.*

Proof. This is a slight variation on [Lur10, 18, proposition 11]. First note that for every spectrum X we have $\mathrm{MU} \otimes X \to E \otimes X$, which induces $\mathrm{MU}_*(X) \to E_*(X)$, a map of MU_*-modules. Moreover, since $E_* \to E_*(X)$ is a map of E_*-module, the map $\mathrm{MU}_* \to E_*$ makes it a map of MU_*-modules. Together this gives a map $(E_{R,F})_*(X) = \mathrm{MU}_*(X) \otimes_{\mathrm{MU}_*} E_* \to E_*(X)$. This map is a map of homology theories. By [Lur10, 17, theorem 6], this map lifts to a map of spectra $E_{R,F} \to E$. Since by construction when $X = \mathbb{S}$ the map above is $E_* \to E_*$ which is an isomorphism, we see that the map $E_{R,F} \to E$ is an equivalence. $\square$

Example 2.6.9 (K-Theory Saga: Landweber Flatness). We return to complex K-theory, from Example 2.2.3 and Example 2.2.11. We can take the completion at the element $p \in \mathrm{K}_*$, which gives the spectrum $\mathrm{K}_p^\wedge$. This spectrum has coefficients $\left(\mathrm{K}_p^\wedge\right)_* = (\mathrm{K}_*)_p^\wedge = \left(\mathbb{Z}\left[\beta^{\pm 1}\right]\right)_p^\wedge = \mathbb{Z}_p\left[\beta^{\pm 1}\right]$. The formal group law, as we have seen, is given by $F_{\mathrm{K}_p^\wedge}(y,z) = y + z + \beta yz$. We claim that $F_{\mathrm{K}_p^\wedge}/\mathrm{K}_p^\wedge$ is Landweber flat. Clearly $p = v_0$ is not a zero divisor in $\mathbb{Z}_p\left[\beta^{\pm 1}\right]$. As we have seen in Example 2.2.12, mod-p the p-series is $\beta^{p-1}x^p$, so that $v_1 = \beta^{p-1}$ which is not a zero divisor $\mathbb{F}_p\left[\beta^{\pm 1}\right]$. Modulo v_1 the ring is already 0, and we are done. For other primes, by Remark 2.6.3 we are done. Therefore, by Theorem 2.6.8 we get that $\mathrm{K}_p^\wedge \cong E_{\mathrm{K}_p^\wedge, F_{\mathrm{K}_p^\wedge}}$.

2.7 Lubin-Tate Deformation Theory

The Johnson-Wilson spectrum $\mathrm{E}(n)$, a variant of Morava E-theory we have considered until now, was constructed from BP. As we noted, it is Landweber flat, which indicates that there is another approach to constructing it.

Indeed there is a way to construct a related spectrum, which will be called the Lubin-Tate spectrum, also a variant of Morava E-theory.

To that end, we first define the category CompRing as the category of complete local rings. The objects are complete local rings $(R, \mathfrak{m})$, we also denote by $\pi : R \to R/\mathfrak{m}$ the projection. Morphisms $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ are local homomorphisms, i.e. a homomorphism $\varphi : R \to S$ s.t. $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. In particular it induces a homomorphism $\varphi/\mathfrak{m} : R/\mathfrak{m} \to S/\mathfrak{n}$, which satisfies $\varphi/\mathfrak{m} \circ \pi_R = \pi_S \circ \varphi$.

We fix k to be a perfect field of characteristic p (i.e. the Frobenius is an isomorphism), and Γ a formal group law over k of height $n < \infty$. Lubin and Tate [LT66] considered a moduli problem associated to Γ/k, described by a functor $\mathrm{Def} : \mathrm{CompRing} \to \mathrm{Grpds}$.

Definition 2.7.1. Let $(R, \mathfrak{m})$ be a complete local ring and denote by $\pi : R \to R/\mathfrak{m}$ the quotient. A *deformation* of Γ/k to $(R, \mathfrak{m})$, is (G, i), where G is a formal group law over R, $i : k \to R/\mathfrak{m}$ is a homomorphism of fields, such that $i^*\Gamma = \pi^*G$. A *$\star$-isomorphism* between two deformations to $(R, \mathfrak{m})$, $f : (G_1, i_1) \to (G_2, i_2)$, is defined only when $i_1 = i_2 = i$, and consists of an isomorphism $f : G_1 \to G_2$, such that $\pi^*f : i^*\Gamma = \pi^*G_1 \to \pi^*G_2 \to i^*\Gamma$ is the identity, i.e. $f(x) = x \mod \mathfrak{m}$. These assemble to a groupoid $\mathrm{Def}(R, \mathfrak{m})$, whose objects are deformations to $(R, \mathfrak{m})$, and morphisms are $\star$-isomorphisms.

Remark 2.7.2. $\mathrm{Def}(R, \mathfrak{m})$ can be seen as the pullback of the groupoids $\mathrm{FGL}(R)$ and $\coprod_{i:k \to R/\mathfrak{m}} \{\Gamma\}$ over $\mathrm{FGL}(R/\mathfrak{m})$ (where the maps are $G \mapsto q^*G$ and $i \mapsto i^*\Gamma$ respectively).

Proposition 2.7.3 (/definition)**.** *The construction* $\mathrm{Def}(R, \mathfrak{m})$ *is functorial.*

Proof. Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism.

For a deformation (G, i) to $(R, \mathfrak{m})$, we define $\mathrm{Def}(\varphi)(G, i) = (\varphi^*G, \varphi/\mathfrak{m} \circ i)$. Note that φ^*G is a formal group law over S, and $\varphi/\mathfrak{m} \circ i : k \to R/\mathfrak{m} \to S/\mathfrak{n}$ is a homomorphism. Moreover, $(\varphi/\mathfrak{m} \circ i)^* \Gamma = (\varphi/\mathfrak{m})^* i^*\Gamma = (\varphi/\mathfrak{m})^* \pi_R^* G = (\varphi/\mathfrak{m} \circ \pi_R)^* G = (\pi_S \circ \varphi)^* G = \pi_S^* \varphi^* G$, which shows that $\mathrm{Def}(\varphi)(G, i)$ is a deformation to $(S, \mathfrak{n})$.

For a $\star$-isomorphism $f : (G_1, i_1) \to (G_2, i_2)$, which is the data of an isomorphism $f : G_1 \to G_2$ such that $\pi_R^* f = \mathrm{id}_{i^*\Gamma}$ is the identity, we need to define a $\star$-isomorphism $\mathrm{Def}(\varphi)(G, i_1) \to \mathrm{Def}(\varphi)(G, i_2)$. Take it to be the isomorphism $\varphi^*f : \varphi^*G_1 \to \varphi^*G_2$, which satisfies $\pi_S^*\varphi^*f = (\varphi/\mathfrak{m})^* \pi_R^* f =$

$(\varphi/\mathfrak{m})^* \operatorname{id}_{i^*\Gamma} = \operatorname{id}_{(\varphi/\mathfrak{m})^* i^*\Gamma} = \operatorname{id}_{(\varphi/\mathfrak{m}\circ i)^*\Gamma}$. The identity $\operatorname{id}_G : (G, i) \to (G, i)$ is clearly sent to $\operatorname{id}_{\varphi^* G}$, and compositions are sent to compositions.

This shows that $\operatorname{Def}(\varphi) : \operatorname{Def}(R, \mathfrak{m}) \to \operatorname{Def}(S, \mathfrak{n})$ is indeed a functor. Moreover, it is clear that $\operatorname{Def}(\operatorname{id}_R)$ is the identity and compositions are sent to compositions, which shows that $\operatorname{Def} : \operatorname{CompRing} \to \operatorname{Grpds}$ is indeed a functor. $\square$

Remark 2.7.4. We recall quickly that the Witt vectors Wk is a ring of characteristic 0, with maximal ideal (p), and residue field $Wk/p \cong k$. For example, $W\mathbb{F}_p = \mathbb{Z}_p$.

Theorem 2.7.5 ([Rez98, 4.4, 5.10], originally due to [LT66])**.** *The functor* Def *lands in discrete groupoids (i.e.* $\operatorname{Def}(R, \mathfrak{m})$ *has* 0 *or* 1 *morphisms between objects). Furthermore the functor* Def *is co-represented, that is there exists a* universal deformation*, and the complete local ring can be chosen (non-canonically) to be* $Wk[[u_1, \ldots, u_{n-1}]]$*.*

Let us unravel what that means. First note that the quotient of $Wk[[u_1, \ldots, u_{n-1}]]$ by the maximal ideal $(p, u_1, \ldots, u_{n-1})$ is k. The universal deformation can be chosen such that the formal group law Γ_U over $Wk[[u_1, \ldots, u_{n-1}]]$ satisfies $\pi^*\Gamma_U = \Gamma$. The universality means that for $(R, \mathfrak{m})$, the assignment

$$\hom_{\mathrm{CompRing}}(Wk[[u_1, \ldots, u_{n-1}]], R) \to \operatorname{Def}(R, \mathfrak{m}), \quad \varphi \mapsto \varphi^*\Gamma_U$$

is an equivalence.

Now, we can form the graded ring $Wk[[u_1, \ldots, u_{n-1}]]\left[u^{\pm 1}\right]$ where $|u| = 2$. We can define the formal group law $(u\Gamma_U)(x, y) = u^{-1}\Gamma_U(uy, uz)$, which is of degree -2, thus we get a map $L \to Wk[[u_1, \ldots, u_{n-1}]]\left[u^{\pm 1}\right]$.

Proposition 2.7.6 ([Rez98, 6.9])**.** $L \to Wk[[u_1, \ldots, u_{n-1}]]\left[u^{\pm 1}\right]$ *is Landweber flat.*

Using LEFT 2.6.4, we immediately get:

Corollary 2.7.7. *There is a complex oriented cohomology theory* $\mathrm{E}(k, \Gamma) = E_{Wk[[u_1, \ldots, u_{n-1}]][u^{\pm 1}], u\Gamma_U}$*, called* Lubin-Tate spectrum*, with coefficients* $\mathrm{E}(k, \Gamma)_* = Wk[[u_1, \ldots, u_{n-1}]]\left[u^{\pm 1}\right]$ *and associated formal group law* $u\Gamma_U$*.*

This concludes the construction of the Lubin-Tate variant of Morava E-theory. It is usually called Morava E-theory of height n, and denoted by

E_n, when the field is $k = \mathbb{F}_{p^n}$ and the formal group law is the Honda formal group law of height n. We can compare the Lubin-Tate spectrum with the previous one, the Johnson-Wilson spectrum.

Proposition 2.7.8 ([Lur10, 23, example 1])**.** *The Lubin-Tate spectrum* $\mathrm{E}(k, \Gamma)$ *and Johnson-Wilson spectrum* $\mathrm{E}(n)$ *are Bousfield equivalent.*

Example 2.7.9 (K-Theory Saga: Lubin-Tate)*.* We continue the complex K-theory saga from Example 2.6.9. Take the field $k = \mathbb{F}_p$ and the formal group law $\Gamma(y, z) = y + z + yz$, of height $n = 1$. By the above construction, the ring of the universal deformation is $W\mathbb{F}_p = \mathbb{Z}_p$. The universal formal group law of the universal deformation can be taken to be $\Gamma_U(y, z) = y + z + yz$ (this follows from the proof at [Rez98, 5.10], since here $n = 1$ so there are no u_i's). We look at the ring $\mathbb{Z}_p\left[u^{\pm 1}\right]$, and at the formal group law over it $(u\Gamma_U)(y, z) = u^{-1}\left(uy + uz + u^2yz\right) = y + z + uyz$. It is then clear that the isomorphism $\mathbb{Z}_p\left[u^{\pm 1}\right] \to \mathbb{Z}_p\left[\beta^{\pm 1}\right]$, sends $u\Gamma_U$ to $F_{\mathrm{K}_p^\wedge}$. It follows by Theorem 2.6.8 that $\mathrm{K}_p^\wedge \cong E_{\mathrm{K}_p^\wedge, F_{\mathrm{K}_p^\wedge}} \cong \mathrm{E}(\mathbb{F}_p, \Gamma)$, i.e. p-complete K-theory $\mathrm{K}_p^\wedge$ is a Lubin-Tate spectrum at height 1.

3 Atiyah-Segal

We now leave the realm of chromatic homotopy theory. One aspect of algebraic topology is to try to capture properties of spaces using algebraic invariants. One of the most fruitful such invariants is complex K-theory, denoted K, and one of the most important spaces in homotopy theory is $\mathrm{B}G$, so it is natural to ask for a description of $\mathrm{K}(\mathrm{B}G)$ (by Bott periodicity, we will consider only $\mathrm{K} = \mathrm{K}^0$). Atiyah and Segal [AS69] gave a description of this, and more, in the case that G is a compact Lie group, in terms of representations.

From now we fix a compact Lie group G. Also, a representation means a finite dimensional unitary representation. We should also note that beyond this section, we will be mostly interested in finite groups.

3.1 The Atiyah-Segal Theorem

We denote by $\mathrm{R}(G)$ the *representation ring* of G, that is the collection of virtual representations of G (which can be written as a formal difference $V - U$) up to isomorphism, where the addition is given by direct sum and the product is given by tensor product. This is an augmented ring $\varepsilon : \mathrm{R}(G) \to \mathbb{Z}$ by the virtual dimension (i.e. $\varepsilon(V - U) = \dim(V - U) = \dim V - \dim U$). The *augmentation ideal* is $I = \ker \varepsilon = \{V - U \in \mathrm{R}(G) \mid 0 = \dim(V - U)\}$.

Atiyah and Segal showed that one can describe $\mathrm{K}(\mathrm{B}G)$ in these terms, namely, it is the completion of $\mathrm{R}(G)$ at the ideal I:

Theorem 3.1.1 ([AS69])**.** $\mathrm{K}(\mathrm{B}G) \cong \mathrm{R}(G)^{\wedge}_{I}$.

We will not prove the theorem, but we will indicate some of the key ingredients.

First of all, to show that objects are isomorphic, we need a map. Before giving the map actually used in the proof, we describe an easier way to see where this map comes from. Recall that $\mathrm{K}(X) \cong [X, \mathrm{BU} \times \mathbb{Z}]$. The data of an n-dimensional representation of G is the same thing as a homomorphism $G \to \mathrm{U}(n)$. Since B is a functor, we get a map $\mathrm{B}G \to \mathrm{BU}(n)$, and by composing with the injection $\mathrm{BU}(n) \cong \mathrm{BU}(n) \times \{n\} \to \mathrm{BU} \times \mathbb{Z}$, we get a map $\mathrm{B}G \to \mathrm{BU} \times \mathbb{Z}$, that is, an element of $\mathrm{K}(\mathrm{B}G)$. Therefore we get a map $\mathrm{R}(G) \to \mathrm{K}(\mathrm{B}G)$. The theorem shows that it is a ring homomorphism which exhibits $\mathrm{K}(\mathrm{B}G)$ as the completion of $\mathrm{R}(G)$ at I.

There is an alternative description of this map. In [Seg68], Segal described a variant of K-theory, called equivariant K-theory K_G. This variant assigns to a G-space the ring of virtual G-bundles, that is, bundles equipped with an action of G which is compatible with the action on the base G-space. Note that K_G is no longer homotopy invariant, since it also takes into account the G-action. First we note the following:

Proposition 3.1.2. $\mathrm{K}_G(*) = \mathrm{R}(G)$ *(where $*$ is equipped with a trivial G-action).*

Proof. This follows from the definitions, since a vector bundle over a point is just a vector space, and it is equipped with a G-action over the point, which is just a G representation. $\square$

For any G-space X, the projection map $\mathrm{pr} : X \to X/G$ allows us to pullback vector bundles on X/G to G-bundles on X. In other words, it induces a map $\mathrm{pr}^* : \mathrm{K}(X/G) \to \mathrm{K}_G(X)$.

Proposition 3.1.3 ([Seg68, 2.1])**.** *Suppose the action of G on X is* free. *Then* pr^* *admits an inverse, given by taking a bundle $E \to X$ to $E/G \to X/G$. In particular,* $\mathrm{K}(X/G) \cong \mathrm{K}_G(X)$.

Now, we have a map of G-spaces given by $\mathrm{E}G \to *$. By the above we get:

$$\mathrm{R}(G) \cong \mathrm{K}_G(*) \to \mathrm{K}_G(\mathrm{E}G) \cong \mathrm{K}(\mathrm{E}G/G) = \mathrm{K}(\mathrm{B}G)$$

It can be shown that this is the same map $\mathrm{R}(G) \to \mathrm{K}(\mathrm{B}G)$ described before, which exhibits $\mathrm{K}(\mathrm{B}G)$ as the I-completion of $\mathrm{R}(G)$. Atiyah and Segal use this map and variants to prove the theorem.

Here is a sketch of the proof given by Atiyah and Segal. First of all, we note that the theorem is proven for the entire K^* rather than just for $\mathrm{K} = \mathrm{K}^0$. Also note that $\mathrm{R}^*(G) = \mathrm{K}_G^*(*)$ is a 2-periodic version of the representation ring (because K_G^* also satisfies Bott periodicity). We have the corresponding 2-periodic version of the augmentation ideal, which we denote by I^*. They use the Milnor join construction $\mathrm{E}G_n = \underbrace{G * \cdots * G}_{n\text{ times}}$ and $\mathrm{B}G_n = \mathrm{E}G_n/G$, which has the property that $\operatorname{colim} \mathrm{E}G_n \to \operatorname{colim} \mathrm{B}G_n$ is a model for $\mathrm{E}G \to \mathrm{B}G$. Then, for any compact G-space X there is a similar map to the map above: using $X \times \mathrm{E}G_n \to X$ we get a map $\mathrm{K}_G^*(X) \to \mathrm{K}_G^*(X \times \mathrm{E}G_n)$. All of these are $\mathrm{R}^*(G)$-modules, and Atiyah and Segal show that this map factors through the quotient by $(I^*)^n$, to give a map $\mathrm{K}_G^*(X)/(I^*)^n \to \mathrm{K}_G^*(X \times \mathrm{E}G_n)$. The two sides assemble into pro-rings, and the maps assemble to a map between the pro-rings:

$$\{\mathrm{K}_G^*(X)/(I^*)^n\}_n \to \{\mathrm{K}_G^*(X \times \mathrm{E}G_n)\}_n$$

What they actually prove is the strong form:

Theorem 3.1.4 ([AS69])**.** *If $\mathrm{K}_G^*(X)$ is finite over $\mathrm{R}^*(G)$, then the above map of pro-rings is an isomorphism.*

Their proof has another interesting aspect. Although it is a statement about the K_G of some class of G-spaces, for one specific group G, their proof involves the K_G's of several groups. In particular, to prove the result for example for a finite group, their proof involves more general compact Lie groups. The proof consists of four steps. In every step they show that the theorem holds for a more general type of group:

- $G = \mathrm{U}(1)$ (circle group),
- $G = \mathrm{U}(1)^n$ (torus group),
- $G = \mathrm{U}(n)$,

- G a general compact Lie group (this step is proven by embedding G in $\mathrm{U}(n)$).

We note that the first formulation of the Atiyah-Segal theorem 3.1.1, is indeed a private case of the second formulation Theorem 3.1.4. Take the case $X = *$. By definition, $\mathrm{K}_G^*(*)$ is finite over $\mathrm{R}^*(G) = \mathrm{K}_G^*(*)$, so Theorem 3.1.4 holds and we have an isomorphism of pro-rings $\{\mathrm{K}_G^*(*)/(I^*)^n\}_n \to \{\mathrm{K}_G^*(\mathrm{E}G_n)\}_n$. In particular, after computing the limits $\lim : \mathrm{Pro\,Ring} \to \mathrm{Ring}$, we get an isomorphism of rings $\mathrm{K}_G^*(*)^{\wedge}_{I^*} \xrightarrow{\sim} \mathrm{K}_G^*(\mathrm{E}G)$. Taking only the 0-th cohomology gives the desired isomorphism:

$$\mathrm{R}(G)^{\wedge}_I \cong \mathrm{K}_G(*)^{\wedge}_I \xrightarrow{\sim} \mathrm{K}_G(\mathrm{E}G) \cong \mathrm{K}(\mathrm{E}G/G) = \mathrm{K}(\mathrm{B}G)$$

3.2 Examples

We compute a few examples in detail, to make the isomorphism more vivid.

3.2.1 $\mathrm{U}(1)$, the Circle Group

Take $G = \mathrm{U}(1)$, the circle group. It is known that the irreducible representations are of dimension 1 and labeled by an integer $m \in \mathbb{Z}$, i.e. they are homomorphisms $\rho_m = \mathrm{U}(1) \to \mathrm{U}(1)$ given by $\rho_m\left(e^{i\theta}\right) = e^{mi\theta}$. In particular, $\rho_0 = 1$ is the trivial representation. It is then clear that for $m \geq 0$, $\rho_1^{\otimes m} = \rho_m$ and $\rho_{-1}^{\otimes m} = \rho_{-m}$. Therefore the representation ring generated under (virtual) direct sums and tensor products by ρ_1 and ρ_{-1}. Moreover, $\rho_1 \otimes \rho_{-1} = 1$. Denote $\rho = \rho_1$, and we conclude that $\mathrm{R}(\mathrm{U}(1)) = \mathbb{Z}\left[\rho, \rho^{-1}\right]$.

The augmentation map is the homomorphism $\varepsilon : \mathrm{R}(\mathrm{U}(1)) \to \mathbb{Z}$ which sends $1, \rho$ and ρ^{-1} to 1. Recall that the augmentation ideal is $I = \ker \varepsilon$. We set $t = \rho - 1$, which clearly belongs to I. We can also write then $\mathrm{R}(\mathrm{U}(1)) = \mathbb{Z}\left[t, (1+t)^{-1}\right]$. Note that ε factors to a map $\mathrm{R}(\mathrm{U}(1))/(t) \to \mathbb{Z}$, which is already an isomorphism, so by the first isomorphism theorem indeed $I = (t)$.

We compute the completion $\mathrm{R}(\mathrm{U}(1))^{\wedge}_I$. Note that in $\mathbb{Z}[t]/t^n$, $1+t$ is already invertible. The reason is that the formal power series for the inverse is finite since large enough powers of t are zero, $\frac{1}{1-(-t)} = \sum_{m=0}^{n-1}(-t)^m$ is an inverse. Therefore we see that $\mathrm{R}(\mathrm{U}(1))/I^n \cong \mathbb{Z}[t]/t^n$, and clearly the maps in the limit diagram send t to t. We get that $\mathrm{R}(\mathrm{U}(1))^{\wedge}_I = \lim \mathbb{Z}[t]/t^n \cong \mathbb{Z}[[t]]$.

In [Hat17, proposition 2.24], it is shown that $\mathrm{K}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}[L]/(L-1)^{n+1}$, where L is the canonical line bundle on $\mathbb{C}\mathrm{P}^n$. In Example 2.2.3 we denoted $t = L - 1$ (warning: there we looked at K^*, now we focus on K), which allows us to rewrite this as $\mathrm{K}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}[t]/t^{n+1}$. As we noted, the limit is $\mathrm{K}(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}[[t]]$. We thus see that $\mathrm{K}(\mathrm{BU}(1)) \cong \mathbb{Z}[[t]]$ where $t = L - 1$ is the canonical line bundle minus 1.

The identity map $\rho : \mathrm{U}(1) \to \mathrm{U}(1)$, is mapped to the identity $\mathrm{BU}(1) \to \mathrm{BU}(1)$ (by functoriality of B), which tautologically corresponds the universal line bundle L on $\mathrm{BU}(1)$. We therefore see that the Atiyah-Segal map $\mathrm{R}(\mathrm{U}(1)) \to \mathrm{K}(\mathrm{BU}(1))$, sends ρ to L, and therefore $t = \rho - 1$ to $t = L - 1$. This shows that the map admits $\mathrm{K}(\mathrm{BU}(1)) \cong \mathbb{Z}[[t]]$ as the $I = (t)$-completion of $\mathrm{R}(\mathrm{U}(1))$.

3.2.2 $\mathbb{Z}/2$, Cyclic Group of Order 2

Take $G = \mathbb{Z}/2$. Here we have only two irreducible representations, the trivial, and $\rho(0) = 1, \rho(1) = -1$. Also, it is clear that $\rho \otimes \rho$ is the trivial representation. Therefore, $\mathrm{R}(\mathbb{Z}/2) = \mathbb{Z}[\rho]/\left(\rho^2 - 1\right)$. Similarly to before, the augmentation $\varepsilon : \mathrm{R}(\mathbb{Z}/2) \to \mathbb{Z}$ sends 1 and ρ to 1, so clearly $(\rho - 1) \subseteq I$, and for the same reasoning as in the previous example this is actually an equality. We change coordinates to $t = \rho - 1$, and we have $\mathrm{R}(\mathbb{Z}/2) = \mathbb{Z}[t]/\left(t^2 + 2t\right)$, and $I = (t)$.

We move to computing the completion $\mathrm{R}(\mathbb{Z}/2)^\wedge_I$. Modulo $t^2 + 2t$, i.e. $t^2 = -2t$, we have that $t^n = (-2)^{n-1} t$. Thus $I^n = \left((-2)^{n-1} t\right) = \left(2^{n-1} t\right)$, so $\mathrm{R}(\mathbb{Z}/2)/I^n = \mathbb{Z}[t]/\left(t^2 + 2t, 2^{n-1}t\right)$. We first compute the limit of $\mathrm{R}(\mathbb{Z}/2)/I^n$ in abelian groups. Since the forgetful functor from rings to abelian groups is a right adjoint, it commutes with limits, so this will give us the abelian group structure. As an abelian group, $\mathrm{R}(\mathbb{Z}/2)/I^n$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2^{n-1}\{t\}$. It is then clear that as an abelian group, $\lim \mathrm{R}(\mathbb{Z}/2)/I^n$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2\{t\}$.

We now define a multiplication on $\mathbb{Z} \oplus \mathbb{Z}_2\{t\}$, given by $(a + bt) * (c + dt) = ac + (ad + bc - 2bd)t$. It can be checked that it is associative and commutative. We have homomorphisms of groups $\mathbb{Z} \oplus \mathbb{Z}_2\{t\} \to \mathbb{Z}[t]/\left(t^2 + 2t, 2^{n-1}t\right)$, admitting it as the limit in groups, which are given by sending $a + bt$ to $a + \left(b \mod 2^{n-1}\right)t$. By construction this homomorphism is actually a homomorphism of rings (the $-2bdt$ term is explained by the relation $t^2 + 2t = 0$). Therefore, by the universal property of the limit, we get a map

$\mathbb{Z} \oplus \mathbb{Z}_2 \{t\} \to \lim \mathrm{R}(\mathbb{Z}/2)/I^n$ in rings. After taking the forgetful we know that it becomes an isomorphism, but the forgetful reflects isomorphisms, so this is also an isomorphism in rings.

Using the Atiyah-Segal theorem we conclude that

$$\mathrm{K}(\mathbb{R}\mathrm{P}^\infty) = \mathrm{K}(\mathrm{B}\mathbb{Z}/2) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \{t\},$$

with multiplication given by $(a + bt) * (c + dt) = ac + (ad + bc - 2bd)t$.

3.3 Some Character Theory

We restrict ourselves to the case of finite groups G. We recall that representations of groups can be studied by their characters. Specifically the character map $\chi : \mathrm{R}(G) \to \mathbb{Z}[\chi_{\rho_i}]$, defined by $\chi_\rho = \mathrm{tr}\,\rho$, is an isomorphism, where the ring on the right is the ring of functions generated by the irreducible characters (the multiplication of two characters is a character so it is indeed closed under multiplication).

We also recall that characters are class functions, that is, they are constant on conjugacy classes. Let L be some field containing all the values of all characters. Then a natural place to study characters is in the ring of class functions with values in L, denote by $\mathrm{Cl}(G; L)$. Let us phrase this in a way that will be useful in the next section. G is equipped with a G-action by conjugation, $\gamma.g = \gamma g \gamma^{-1}$. Equip L with the trivial G-action. Then $\mathrm{Cl}(G; L) = \hom_{G\mathrm{Set}}(G, L)$.

We can of course extend the range of the character map to get an injection $\chi : \mathrm{R}(G) \to \mathrm{Cl}(G; L)$. The first classical theorem regarding the relationship between characters and class functions is:

Theorem 3.3.1. *After tensoring with L, the character map $\chi \otimes L : \mathrm{R}(G) \otimes L \xrightarrow{\sim} \mathrm{Cl}(G; L)$ becomes an isomorphism.*

Proof. Similarly to the proof in [Ser77, 9.1] for $L = \mathbb{C}$, we can view $\mathrm{Cl}(G; L)$ as a vector space over L, and the characters are linearly independent, so by counting them we see that the image of $\chi \otimes L$ has the dimension of the whole vector space and we are done. $\square$

By definition the value of a character is the trace of a linear transformation $\chi_\rho(g) = \mathrm{tr}\,\rho(g) = \sum \lambda_i$ where λ_i are the eigenvalues (which exist since the representation is unitary). Since $g^{|G|} = e$, we get $\rho\left(g^{|G|}\right) = \rho(e) = \mathrm{id}$, but then we get that the eigenvalues of $\rho\left(g^{|G|}\right)$ are on the one hand $\lambda_i^{|G|}$ and on the other hand they are all 1, which means that all the eigenvalues are roots of unity. Therefore $L = \mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\zeta_\infty)$ is always a valid choice for L (regardless of G). To be concrete, we will take this choice.

The Galois group of $\mathbb{Q}^{\mathrm{ab}}$ is $\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right) \cong \hat{\mathbb{Z}}^\times$. For every $m \in \hat{\mathbb{Z}}^\times$ we also denote by $m \in \mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)$ the corresponding element, which can be described as the homomorphism which raises a root of unity to the power of m. Similarly it acts on G by taking g to g^m. Then, for every such m and g we have that $\chi_\rho(g^m) = \mathrm{tr}\,\rho(g^m) = \sum \lambda_i^m = m.\left(\sum \lambda_i\right) = m.\left(\chi_\rho(g)\right)$. We replace g with $g^{m^{-1}}$ (m is invertible), and rewrite this as $\chi_\rho(g) = m.\left(\chi_\rho\left(g^{m^{-1}}\right)\right)$. Similarly to this equality, we can define an action of $\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)$ on $\mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)$, by taking a class function f to $m.f$ defined by $(m.f)(g) = m.\left(f\left(g^{m^{-1}}\right)\right)$.

Let us rewrite this action in another way, which will be helpful in the next section. We note that $G \cong \hom_{\mathrm{TopGrp}}\left(\hat{\mathbb{Z}}, G\right)$ (continuous homomorphisms). We get an action of $\mathrm{Aut}\left(\hat{\mathbb{Z}}\right) \cong \hat{\mathbb{Z}}^\times$ on G by pre-composition. Concretely, $m \in \hat{\mathbb{Z}}^\times$ acts by sending $g \in G$ to g^m. Since $\hat{\mathbb{Z}}^\times$ acts on $\mathbb{Q}^{\mathrm{ab}}$, we get an action on $\mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right) = \hom_{G\mathrm{Set}}\left(G,\mathbb{Q}^{\mathrm{ab}}\right)$ by acting with m^{-1} in the source and with m in the target. It is evident that this is the same action from the previous paragraph.

As we just saw, the characters are in the fixed points $\mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$. Also, since the rationals are fixed by the action of the Galois group, rational linear combinations of characters are in the fixed points. We therefore conclude that the character map after tensoring with $\mathbb{Q}$ lands in the fixed points, i.e. $\chi\otimes\mathbb{Q} : \mathrm{R}(G)\otimes\mathbb{Q} \to \mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$. Moreover, the second classical theorem is:

Theorem 3.3.2 ([Ser77, Theorem 25])**.** *The map* $\chi \otimes \mathbb{Q} : \mathrm{R}(G) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$ *is an isomorphism.*

To conclude, Theorem 3.3.1 tells us that $\mathrm{R}(G) \otimes \mathbb{Q}^{\mathrm{ab}} \cong \mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)$, and Theorem 3.3.2 tells us that $\mathrm{R}(G) \otimes \mathbb{Q} \cong \mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$.

4 HKR Generalized Character Theory

As we have seen in the previous section, Atiyah and Segal gave a description of $\mathrm{K}(\mathrm{B}G)$ in terms of the representation ring. We have also seen in the section on chromatic homotopy theory, that complex K-theory is related to Morava K-theory at height 1 by Example 2.3.11, and to Morava E-theory at height 1 by Example 2.7.9. Representations can be studied using their characters, and one may wonder if a similar construction can be used to study higher analogues of complex K-theory evaluated at $\mathrm{B}G$.

Hopkins, Kuhn and Ravenel showed in [HKR00] that it is indeed possible. Their paper contains a lot of results, but we will concentrate on Theorem C. Fix some finite group G. Similarly to the proof of Atiyah-Segal theorem, the actual proof of Theorem C involves a general construction, even to prove the specific case we are interested in, but it will be easier to state it first for the specific case. Let $E = \mathrm{E}(k, \Gamma)$ be the Lubin-Tate spectrum from Corollary 2.7.7, for some perfect field k of characteristic p, and Γ a formal group law over k of height n. We will also denote by F the formal group law on E_* (which is $u\Gamma_U$, as in Corollary 2.7.7). There is some ring $L = L(E^*)$ (which depends on the spectrum E). It is then possible define some generalized class functions $\mathrm{Cl}_{n,p}(G; L)$, which are completely algebraic and combinatorial (besides the definition of the ring L). Lastly, there is a character map $\chi^G_{n,p} : E^*(\mathrm{B}G) \to \mathrm{Cl}_{n,p}(G; L)$. This character map has similar formal properties to the ordinary character map, namely, similarly to Theorem 3.3.1, after tensoring with L, the character map

$$\chi^G_{n,p} \otimes L : E^*(\mathrm{B}G) \otimes L \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G; L)$$

becomes an isomorphism. Similarly to Theorem 3.3.2, there is an action of $\mathrm{Aut}\left(\mathbb{Z}_p^n\right) \cong \left(\mathbb{Z}_p^\times\right)^n$ on $\mathrm{Cl}_{n,p}(G; L)$, and it turns out that the character map lands in the fixed points. Moreover, we can merely rationalize, which is given by inverting p, the source, rather tensoring with L. The target is already rational. It turns out that after rationalization and restricting the codomain to the fixed points, the map becomes an isomorphism, that is

$$p^{-1}\chi^G_{n,p} : p^{-1}E^*(\mathrm{B}G) \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G; L)^{\mathrm{Aut}\left(\mathbb{Z}_p^n\right)}$$

is an isomorphism.

We will first define some of the objects above, to make things more precise, and we will see what exactly we need to construct the rest. Once we understand that, we will give a more general and detailed construction, which will allow us to state formally the main theorem, Theorem 4.5.3, which is a stronger version of the results written above. After that we will introduce the idea of complex oriented descent, and prove the theorem.

4.1 Towards a Definition of the Character Map

Following [HKR00], we denote by $\Lambda_r = (\mathbb{Z}/p^r)^n$ and $\Lambda = \mathbb{Z}_p^n$.

An element $g \in G$ is called *p-power-torsion* if $g^{p^a} = e$ for some a. Note that a conjugation of a p-power-torsion element is again p-power-torsion. We also define $r_0 \in \mathbb{N}$ to be the minimal r s.t. every p-power-torsion element g is p^{r_0}-torsion, i.e. satisfies $g^{p^{r_0}} = e$.

Definition 4.1.1. We define $G_{n,p}$ to be the set of n-tuples $(g_1, \ldots, g_n)$ of commuting p-power-torsion elements. It has a G-action by conjugation, $\gamma.(g_1, \ldots, g_n) = \left(\gamma g_1 \gamma^{-1}, \ldots, \gamma g_n \gamma^{-1}\right)$.

Concretely, for $r \geq r_0$, we have $G_{n,p} = \hom_{\mathrm{Grp}}(\Lambda_r, G)$, with the G-action by conjugation at the target. In a similar fashion, $G_{n,p} = \hom_{\mathrm{TopGrp}}(\Lambda, G)$ (the homomorphisms are required to be continuous).

Let R be a ring. Equip it with the trivial G-action.

Definition 4.1.2. The *class functions* are $\mathrm{Cl}_{n,p}(G; R) = \hom_{G\mathrm{Set}}(G_{n,p}, R)$, that is functions from $G_{n,p}$ to R which are invariant under conjugation.

This is a ring, by defining the operations point-wise. Note that this is a purely combinatorial construction, just a copy of R for every orbit in $G_{n,p}/G$, that is $\mathrm{Cl}_{n,p}(G; R) \cong \prod_{[\alpha] \in G_{n,p}/G} R$.

We would like to construct a character map $E^*(\mathrm{B}G) \to \mathrm{Cl}_{n,p}(G; R)$, for some R, which depends on $r \geq r_0$. We will try to unravel what this means, and find appropriate R's at the same time. By the above, this is a homomorphism $E^*(\mathrm{B}G) \to \prod_{[\alpha] \in G_{n,p}/G} R$. That is, for every $[\alpha] \in G_{n,p}/G$ we need to provide a homomorphism $E^*(\mathrm{B}G) \to R$. Choose a representative $\alpha \in G_{n,p} = \hom_{\mathrm{Grp}}(\Lambda_r, G)$ (for $r \geq r_0$). Since B is a functor we get $\mathrm{B}\alpha : \mathrm{B}\Lambda_r \to \mathrm{B}G$, and then we can take E^*-cohomology to get a homomorphism $\mathrm{B}\alpha^* : E^*(\mathrm{B}G) \to E^*(\mathrm{B}\Lambda_r)$. If we had a homomorphism $E^*(\mathrm{B}\Lambda_r) \to R$, we would indeed get a character map.

4.2 The E^*-Cohomology of $\mathrm{B}A$ and Their Maps

We postpone the discussion of the rings, to give some properties of E^*-cohomology.

First we describe $E^*(\mathrm{B}\mathbb{Z}/m)$. Let $\psi_m : \mathbb{Z}/m \to \mathrm{U}(1)$ be the homomorphism determined by $\psi_m(1) = e^{2\pi i/m}$. This induces a map $\mathrm{B}\psi_m^* : E^*(\mathrm{BU}(1)) \to E^*(\mathrm{B}\mathbb{Z}/m)$. Denote by $x \in E^2(\mathrm{B}\mathbb{Z}/m)$ the cohomology class $\mathrm{B}\psi_m^*(x)$.

Proposition 4.2.1 ([HKR00, 5.8])**.** *The E^*-cohomology of $\mathrm{B}\mathbb{Z}/m$ is given by $E^*(\mathrm{B}\mathbb{Z}/m) = E^*[[x]]/([m](x))$. Write $m = sp^t$ with s coprime to p, then this is a free E^*-module of rank p^{nt} (where n was the height).*

Proposition 4.2.2. *Let Y be a space s.t. $E^*(Y)$ is a free E^*-module of finite rank. Then Y satisfies Künneth with respect to any X, that is, the map $E^*(X) \otimes_{E^*} E^*(Y) \xrightarrow{\sim} E^*(X \times Y)$ is an isomorphism.*

Proof. Look at the functors $X \mapsto E^*(X) \otimes_{E^*} E^*(Y)$ and $X \mapsto E^*(X \times Y)$. Both of them are manifestly homotopy invariant. Since $E^*(Y)$ is free, it is also flat, and so both functors satisfy Mayer-Vietoris. Both functors send arbitrary wedges to arbitrary products, since tensor with a free finite rank module commutes with arbitrary products. We conclude that they are both cohomology theories. Moreover, they agree on $X = *$, and therefore are isomorphic. $\square$

Using both propositions we can bootstrap to arbitrary finite abelian groups.

Proposition 4.2.3 ([HKR00, 5.8])**.** *Let A be an abelian group, and write $|A| = sp^t$ for s coprime to p. Then $E^*(\mathrm{B}A)$ is a free E^*-module of rank p^{nt}, and $\mathrm{B}A$ satisfies Künneth. Specifically, for $A = \mathbb{Z}/m_1 \oplus \cdots \oplus \mathbb{Z}/m_l$:*

$$E^*(\mathrm{B}\mathbb{Z}/m_1 \times \cdots \times \mathrm{B}\mathbb{Z}/m_l) \cong E^*[[x_1, \ldots, x_l]] / ([m_1](x_1), \ldots, [m_l](x_l)).$$

Proof. A finite abelian group is the product of finite cyclic groups. Since B commutes with products, we can induct on the number of components in the product, and the proof follows by the two previous propositions. $\square$

Recall that the formal group law on E_* was defined by taking the $E^*(\mathrm{B}-)$ of the multiplication map $\mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{U}(1)$. That is, this map induces the map $x \mapsto F(y, z) = y +_F z$ on the cohomology.

By pre-composing with the diagonal, and doing this for k-copies of $\mathrm{U}(1)$, we see that the multiplication-by-k map $\mathrm{U}(1) \xrightarrow{k} \mathrm{U}(1)$ induces the map $E^*[[x]] \to E^*[[y]]$ given by $x \mapsto [k](y)$.

Let $k : \bigoplus_{i=1}^t \mathrm{U}(1) \to \bigoplus_{j=1}^s \mathrm{U}(1)$ be the map given on the i,j-th coordinate by multiplication-by-k_{ij}. Taking $E^*(\mathrm{B}-)$ gives a map $E^*[[x_1,\dots,x_s]] \to E^*[[y_1,\dots,y_t]]$ given by $x_j \mapsto \sum_F [k_{ij}](y_i)$. From this it follows that:

$$\begin{aligned}
\sum_{j,F} [l_j](x_j) \mapsto & \sum_{j,F} [l_j] \left(\sum_{i,F} [k_{ij}](y_i) \right) \\
&= \sum_{j,F} \sum_{i,F} [k_{ij} l_j](y_i) \\
&= \sum_{i,F} \left[\sum_j k_{ij} l_j \right] (y_i)
\end{aligned}$$

Let $k : \bigoplus_{i=1}^t \mathbb{Z}/m_i \to \bigoplus_{j=1}^s \mathbb{Z}/\mu_j$ be given on the i,j-th coordinate by multiplication-by-k_{ij} (where k_{ij} is defined only modulo μ_j). Recall the maps $\psi_m : \mathbb{Z}/m \to \mathrm{U}(1)$ given by $1 \mapsto e^{2\pi i/m}$. We look at the maps $\bigoplus_{i=1}^t \psi_{m_i} : \bigoplus_{i=1}^t \mathbb{Z}/m_i \to \bigoplus_{i=1}^t \mathrm{U}(1)$, and similarly $\bigoplus_{j=1}^s \psi_{\mu_j}$. The composition $\left(\bigoplus_{j=1}^s \psi_{\mu_j}\right) \circ k$ is given on the i,j-th coordinate by $1 \mapsto k_{ij} \mapsto e^{2\pi i k_{ij}/\mu_j}$. Define a map $k : \bigoplus_{i=1}^t \mathrm{U}(1) \to \bigoplus_{j=1}^s \mathrm{U}(1)$, by letting the i,j-th coordinate being the multiplication-by-k_{ij} map (where we choose some lift of k_{ij} from $\mathbb{Z}/\mu_j$ to $\mathbb{Z}$). We then get the commutative diagram:

$$\begin{tikzcd}
\bigoplus_{i=1}^t \mathrm{U}(1) \arrow[r, "k"] & \bigoplus_{j=1}^s \mathrm{U}(1) \\
\bigoplus_{i=1}^t \mathbb{Z}/m_i \arrow[r, "k"] \arrow[u, hook, "\bigoplus_{i=1}^t \psi_{m_i}"] & \bigoplus_{j=1}^s \mathbb{Z}/\mu_j \arrow[u, hook, "\bigoplus_{j=1}^s \psi_{\mu_j}"]
\end{tikzcd}$$

By taking $E^*(\mathrm{B}-)$ we get the commutative diagram:

$$\begin{tikzcd}
E^*[[y_1,\dots,y_t]] \arrow[d] & E^*[[x_1,\dots,x_s]] \arrow[l] \arrow[d] \\
E^*[[y_1,\dots,y_t]]/([m_i](y_i)) & E^*[[x_1,\dots,x_s]]/([\mu_j](x_j)) \arrow[l]
\end{tikzcd}$$

Where the vertical maps are given by $y_i \mapsto y_i$ and $x_j \mapsto x_j$. We have computed the upper map before, and since the vertical maps are surjections, we conclude:

Proposition 4.2.4. *Let $k : \bigoplus_{i=1}^{t} \mathbb{Z}/m_i \to \bigoplus_{j=1}^{s} \mathbb{Z}/\mu_j$ be given on the i, j-th coordinate by multiplication-by-k_{ij}. After taking $E^* (\mathrm{B}-)$, it induces the map given by $x_j \mapsto \sum_F [k_{ij}] (y_i)$. Moreover, for integers $l_1, \ldots, l_s$, it gives $\sum_{j,F} [l_j] (x_j) \mapsto \sum_{i,F} \left[\sum_j k_{ij} l_j\right] (y_i)$.*

Remark 4.2.5. Note that the k_{ij}'s are defined only modulo $\mathbb{Z}/\mu_j$, but $[k_{ij}] (y_i)$ requires us to choose a lift to $\mathbb{Z}$. We see that the result is independent of the lift.

4.3 The Rings $L_r (E^*)$ and $L (E^*)$

As we have seen, to construct the character map we needed a ring R together with homomorphisms $E^* (\mathrm{B}\Lambda_r) \to R$. We will construct such a ring, $L_r = L_r (E^*)$. Moreover, we will take a colimit to construct a ring $L = L (E^*)$.

Recall that $E^* (\mathrm{BU} (1)) = E^* [[x]]$, where x is the complex orientation. For any homomorphism $\alpha : \Lambda_r \to \mathrm{U} (1)$, we can take $E^* (\mathrm{B}-)$ to get $\mathrm{B}\alpha^* : E^* (\mathrm{U} (1)) \to E^* (\mathrm{B}\Lambda_r)$. Let $S_r = \{\mathrm{B}\alpha^* (x) \mid \alpha : \Lambda_r \to \mathrm{U} (1), \alpha \neq 1\} \subseteq E^* (\mathrm{B}\Lambda_r)$.

Definition 4.3.1. We define $L_r = S_r^{-1} E^* (\mathrm{B}\Lambda_r)$. There is indeed a map $E^* (\mathrm{B}\Lambda_r) \to L_r$, namely the localization map.

We wish to describe the above construction with coordinates, to make it more explicit. Recall that $E^* (\mathrm{B}\Lambda_r) \cong E^* [[x_1, \ldots, x_n]] / ([p^r] (x_1), \ldots, [p^r] (x_n))$ by Proposition 4.2.3. Let $\Lambda_r \xrightarrow{\alpha} \mathrm{U} (1)$ be a homomorphism. Since it lands in the p^r-torsion, it factors as $\Lambda_r \xrightarrow{k} \mathbb{Z}/p^r \xrightarrow{\psi_{p^r}} \mathrm{U} (1)$, where k is given on the i-th coordinate by multiplication-by-k_i. The condition $\alpha \neq 1$ amounts to the condition $(k_1, \ldots, k_n) \neq 0 \mod p^r$. By Proposition 4.2.4, the induced map is given by $\mathrm{B}\alpha^* (x) = \sum_F [k_i] (x_i)$. Therefore $S_r = \{\sum_F [k_i] (x_i) \mid (k_1, \ldots, k_n) \neq 0 \mod p^r\}$.

Proposition 4.3.2. *The map $E^* (\mathrm{B}\Lambda_r) \to E^* (\mathrm{B}\Lambda_{r+1})$ induced by the projection $\Lambda_{r+1} \to \Lambda_r$, lifts to a map $L_r \to L_{r+1}$.*

Proof. The projection $\Lambda_{r+1} \to \Lambda_r$ is given by the multiplication-by-1 on each coordinate, so again by Proposition 4.2.4 they induce the maps $E^* (\mathrm{B}\Lambda_r) \to E^* (\mathrm{B}\Lambda_{r+1})$, given by $x_i \mapsto x_i$. Moreover $\sum_F [k_i] (x_i) \in S_r$ is mapped to $\sum_F [k_i] (x_i) \in S_{r+1}$. Therefore, once we invert S_{r+1} in the target, clearly S_r are sent to invertibles, so the map lifts to the localization of the source. □

Definition 4.3.3. We define $L = L(E^*) = \operatorname{colim} L_r$.

By definition, $\mathrm{Aut}(\Lambda_r)$ acts on Λ_r, so by functoriality we get that it also acts on $E^*(\mathrm{B}\Lambda_r)$.

Proposition 4.3.4. *The* $\mathrm{Aut}(\Lambda_r)$*-action lifts to an action on* L_r*.*

Proof. Let $k : \Lambda_r \to \Lambda_r$ be an automorphism given by on the i,j-th coordinate by multiplication by k_{ij}. Once again, by Proposition 4.2.4, for integers $l_1, \ldots, l_n$, the induced map sends $\sum_{j,F} [l_j](x_j)$ to $\sum_{i,F}\left[\sum_j k_{ij} l_j\right](x_i)$. Since k is an automorphism, the matrix (k_{ij}) is invertible. Therefore, if $(l_1, \ldots, l_n) \neq 0$, then also $\left(\sum_j k_{1j} l_j, \ldots, \sum_j k_{nj} l_j\right) \neq 0$, so if the source is in S_r, the result is in S_r as well. This shows that action lifts to an action on L_r. $\square$

Using the projection $\pi_r : \mathrm{Aut}(\Lambda) \to \mathrm{Aut}(\Lambda_r)$ we endow L_r with an $\mathrm{Aut}(\Lambda)$-action. By factoring the projection through $\mathrm{Aut}(\Lambda_{r+1})$, we see that the map $L_r \to L_{r+1}$ is equivariant with respect to that action. In conclusion:

Proposition 4.3.5. *The rings* L_r *have an* $\mathrm{Aut}(\Lambda)$*-action, and the maps* $L_r \to L_{r+1}$ *are equivariant with respect to this action. Therefore* L *has an* $\mathrm{Aut}(\Lambda)$*-action as well.*

One may wonder if the ring L_r is the zero ring. An argument in [HKR00] shows that this isn't the case, and even more is true.

Proposition 4.3.6 ([HKR00, 6.5, 6.6, 6.8])**.** *The element* p *is invertible in* L*, so* L *is a* $p^{-1}E^*$*-module. Furthermore,* $L^{\mathrm{Aut}(\Lambda)} = p^{-1}E^*$*, and* L *is faithfully flat over it. Moreover, this holds when* L *is replaced with* L_r*.*

4.3.1 Algebro-Geometric Interpretation

First we wish to simplify the situation a little bit. Recall from Corollary 2.7.7 that we have a formal group law $F = u\Gamma_U$ over $\mathrm{E}(k,\Gamma)_* = Wk[[u_1, \ldots, u_{n-1}]]\left[u^{\pm 1}\right]$. This came from the computation $\mathrm{E}(k,\Gamma)^*(\mathrm{BU}(1)) = \mathrm{E}(k,\Gamma)^*[[x]]$ with $|x| = -2$. It will be more convenient to work with $t = u^{-1}x$, which lives in degree 0, similarly to Example 2.2.11. On these elements, the formal group law acts the same as Γ_U. By the invertibility of u, we can do the computation of L_r with these elements, and the results will not be affected. Moreover, since everything is defined already over $E^0 = Wk\,[[u_1, \ldots, u_{n-1}]]$, we can do all the computations over it, which will give the ring L_r^0, the 0-th degree part of L_r, and add $u^{\pm 1}$ at the end to get back L_r.

The formal group law Γ_U over E^0, gives a formal group $\mathbb{G} = \operatorname{Spf} E^0\,[[t]]$ with multiplication $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$. By definition, the p^r-torsion elements in $\mathbb{G}$, is the scheme-theoretic kernel of the multiplication-by-p^r map $[p^r] : \mathbb{G} \to \mathbb{G}$, that is $\mathbb{G}\,[p^r] = \operatorname{Spec}\left(E^0\,[[t]] \,/\, ([p^r]\,(t))\right) = \operatorname{Spec} E^0\,(\mathrm{B}\mathbb{Z}/p^r)$. Since Γ_U is of height n, the leading term of the p^r-series, is $t^{p^{rn}}$, which by a variant of Weierstrass preparation (see [HKR00, 5.1]) shows that $\mathbb{G}\,[p^r]$ is of rank p^{rn}. We also see that $\operatorname{Spec} E^0\,(\mathrm{B}\Lambda_r) = (\mathbb{G}\,[p^r])^n$.

Inverting S_r is equivalent to inverting their 0-th graded analogues $S_r^0 = \left\{\sum_{\Gamma_U} [k_i]\,(t_i) \mid (k_1, \ldots, k_n) \neq 0 \mod p^r\right\}$. Algebro-geometrically, this is equivalent to restricting to the open subset where all the functions $\sum_{\Gamma_U} [k_i]\,(t_i)$ don't vanish. That is, $\operatorname{Spec} L_r^0$ is the open subset of n-tuple of points in $\mathbb{G}\,[p^r]$, i.e. n points in the p^r-torsion, which are linearly independent. In fact, the points of $\mathbb{G}\,[p^r]$, as a group, are isomorphic (non-canonically) to $(\mathbb{Z}/p^r)^n = \Lambda_r$, so this means that the n points are not only linearly independent, but also span, that is they form a basis.

Moreover, if $\sum_{\Gamma_U} [k_i]\,(t_i) = 0$ then also $\sum_{\Gamma_U} [pk_i]\,(t_i) = [p]\left(\sum_{\Gamma_U} [k_i]\,(t_i)\right) = 0$. So, if any pk_i is not 0 modulo p^r, inverting $\sum_{\Gamma_U} [pk_i]\,(t_i)$ already inverts $\sum_{\Gamma_U} [k_i]\,(t_i)$. Well, $pk_i = 0 \mod p^r$ if and only if $k_i = 0 \mod p^{r-1}$, which shows that we can invert only those where all k_i's are a multiple of p^{r-1}. Since there are n k_i's, each of them can take any of p values (numbers which are a multiple of p^{r-1}), and not all 0, we need to invert only $p^n - 1$ elements.

The description of $\operatorname{Spec} L_r^0$ as a basis for Λ_r also shows where the $\operatorname{Aut}(\Lambda_r)$ action comes from, it just changes the basis by multiplying by an invertible matrix.

Example 4.3.7 (K-Theory Saga: The Ring L_r). We continue with complex K-theory. Recall from Example 2.7.9 that $\mathrm{K}_p^\wedge \cong \mathrm{E}\,(\mathbb{F}_p, \Gamma)$, where Γ is the multiplicative formal group law, $\Gamma\,(y, z) = y + z + yz$. That is, p-complete K-theory is a Lubin-Tate spectrum at height $n = 1$, so the the construction above applies to it. It is worth noting that the computation here should be related to Atiyah-Segal, although there we considered K itself, and here $\mathrm{K}_p^\wedge$, and as we will see we will indeed get that L is a natural place to study characters in a p-complete situation.

In Example 2.6.9, we saw that $\left(\mathrm{K}_p^\wedge\right)_* = \mathbb{Z}_p\left[\beta^{\pm 1}\right]$. As in Section 4.3.1, it is easier to work with the element $t = \beta^{-1}x$ and the formal group law $u+v+uv$ over $\left(\mathrm{K}_p^\wedge\right)_0 = \mathbb{Z}_p$. The n-series then is $[n](t) = (1+t)^n - 1$.

In our case $\Lambda_r = \mathbb{Z}/p^r$, and we have $\left(\mathrm{K}_p^\wedge\right)^0(\mathrm{B}\mathbb{Z}/p^r) \cong \mathbb{Z}_p[[t]]/([p^r](t)) = \mathbb{Z}_p[[t]]/\left((1+t)^{p^r} - 1\right)$. Again, by a variant of Weierstrass preparation (see [HKR00, 5.1]) the inclusion $\mathbb{Z}_p[t] \to \mathbb{Z}_p[[t]]$ induces an isomorphism of the ring above to $\mathbb{Z}_p[t]/\left((1+t)^{p^r} - 1\right)$. To make the computation easier, we change variable $s = 1+t$, to work with the ring $\mathbb{Z}_p[s]/\left(s^{p^r} - 1\right)$.

We note that this is in accordance with the algebro-geometric point of view from Section 4.3.1, as the spectrum of this ring is isomorphic as a group scheme to the group of roots of unity of order p^r.

Under this change of variables, $S_r = \left\{s^k - 1 \mid 0 < k < p^r\right\}$. By Proposition 4.3.6, p is invertible in the localization, so we might as well invert it before inverting S_r. We then denote $R = \mathbb{Q}_p[s]/\left(s^{p^r} - 1\right)$, and our goal is to compute $S_r{}^{-1}R$. Denote by $\Phi_k(s)$ the k-th cyclotomic polynomial, and by ζ_k a primitive k-th root of unity. Recall that $s^{p^r} - 1 = \left(s^{p^{r-1}} - 1\right)\Phi_{p^r}(s)$. Therefore we have a quotient map $R \to \mathbb{Q}_p[s]/(\Phi_{p^r}) \cong \mathbb{Q}_p(\zeta_{p^r})$, and we claim that it admits the target as the S_r-localization of the source.

First, note that $s^k - 1 \in S_r$ is sent to $\zeta_{p^r}^k - 1$, and since $0 < k < p^r$, this is not zero. Since the codomain is a field, this is invertible, so by the universal property of localization, we get a map $S_r{}^{-1}R \to \mathbb{Q}_p[s]/(\Phi_{p^r})$.

Second, look at the map $\mathbb{Q}_p[s] \to S_r{}^{-1}R$ (the composition of the quotient and localization maps). We took the quotient by $s^{p^r} - 1$, and inverted $s^{p^{r-1}} - 1$, so $\Phi_{p^r}(s) = \frac{s^{p^r}-1}{s^{p^{r-1}}-1}$ is zero in $S_r{}^{-1}R$ as well. Thus, the map factors to a map $\mathbb{Q}_p[s]/(\Phi_{p^r}) \to S_r{}^{-1}R$, which is clearly an inverse to the map above.

We conclude that our ring is $L_r^0 = \mathbb{Q}_p(\zeta_{p^r})$. This is again in accordance with Section 4.3.1, since the points are primitive roots of unity, that is each point forms a basis for the group of roots of unity.

The whole graded ring is $L_r = \mathbb{Q}_p(\zeta_{p^r})\left[\beta^{\pm 1}\right]$. From this it is also easy to see that $L = \mathbb{Q}_p(\zeta_{p^\infty})\left[\beta^{\pm 1}\right]$.

4.4 The Generalized Class Functions Ring

As in the case of Atiyah-Segal, in order to establish Theorem C, that is, the result on the character map described in the introduction to this section, we need to formulate a more general theorem. The theorem will also depend on a G-space X, and the proof will use this freedom in a crucial way. Moreover, the proof will rely on passing to other subgroups as well, although unlike in Atiyah-Segal's proof, it will only use abelian subgroups of G. To this end, we define the generalized objects we need.

Let X be a finite G-CW complex. Recall from Section 4.1 that for $r \geq r_0$, we have $G_{n,p} = \hom_{\mathrm{Grp}}(\Lambda_r, G)$, with the G-action by conjugation. Note that for $\alpha \in G_{n,p}$, $X^{\operatorname{im}\alpha}$ is the fixed points of X at the n-tuple $(g_1, \ldots, g_n)$ which is represented by α, so the following definition is independent of $r \geq r_0$.

Definition 4.4.1. The *fixed point space* of X is $\mathrm{Fix}_{n,p}(G, X) = \coprod_{\alpha \in G_{n,p}} X^{\operatorname{im}\alpha}$. This space has a $G \times \mathrm{Aut}(\Lambda_r)$-action, described below.

$\mathrm{Fix}_{n,p}(G, X)$ admits a G-action, where $\gamma \in G$ sends $x \in X^{\operatorname{im}\alpha}$ to $\gamma x \in X^{\operatorname{im}\gamma.\alpha}$. This is well defined, since if x fixed by α, i.e. $g_i x = x$, then $\gamma g_i \gamma^{-1} \gamma x = \gamma x$, so γx is fixed by $\gamma.\alpha$. Moreover, it admits an $\mathrm{Aut}(\Lambda_r)$-action. Let $\varphi \in \mathrm{Aut}(\Lambda_r)$, for any $\alpha \in G_{n,p}$, clearly $\operatorname{im}\alpha = \operatorname{im}(\alpha \circ \varphi)$, so $x \in X^{\operatorname{im}\alpha}$ is mapped by φ to $x \in X^{\operatorname{im}(\alpha\circ\varphi)}$ (i.e. this action just permutes to coordinates labeled by the $\alpha \in G_{n,p}$). The actions commute, since $\gamma.(\alpha \circ \varphi) = (\gamma.\alpha) \circ \varphi$, because φ acts on the source and γ on the target. Therefore we have a $G \times \mathrm{Aut}(\Lambda_r)$-action.

The action on $\mathrm{Fix}_{n,p}(G, X)$ gives a $G \times \mathrm{Aut}(\Lambda_r)$-action of E^*-algebras on $E^*(\mathrm{Fix}_{n,p}(G, X))$. As we saw, L_r admits an $\mathrm{Aut}(\Lambda_r)$-action, define the trivial G-action on it, to get a $G \times \mathrm{Aut}(\Lambda_r)$-action. Take the diagonal $G \times \mathrm{Aut}(\Lambda_r)$-action on $L_r \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X))$.

Definition 4.4.2. The *class functions* are:

$$\mathrm{Cl}_{n,p}(G, X; L_r) = \left(L_r \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X))\right)^G$$

This E^*-algebra still has an $\mathrm{Aut}(\Lambda_r)$-action.

Note that for $X = *$, trivial G-space, $\mathrm{Fix}_{n,p}(G, *) = G_{n,p}$ as a G-space. Hence $E^*(\mathrm{Fix}_{n,p}(G, *)) \cong \hom_{\mathrm{Set}}(G_{n,p}, E^*)$. Taking the G fixed points gives $\hom_{G\mathrm{Set}}(G_{n,p}, E^*)$ (note that this hom is in GSet). In conclusion, we get $\mathrm{Cl}_{n,p}(G, *; L_r) = \hom_{G\mathrm{Set}}(G_{n,p}, L_r)$. This agrees with Definition 4.1.2.

We give an alternative description of the algebra before taking the G fixed points. Simply by taking the coproduct out of the cohomology as a product, and out of the tensor product, we get:

Proposition 4.4.3. $L_r \otimes_{E^*} E^* \left(\mathrm{Fix}_{n,p}(G,X)\right) \cong \prod_{\alpha \in G_{n,p}} \left(L_r \otimes_{E^*} E^* \left(X^{\mathrm{im}\,\alpha}\right)\right)$.

We wish to emphasize the combinatorial nature of the $G \times \mathrm{Aut}(\Lambda_r)$-action. To that end, we first formulate a general combinatorial statement:

Proposition 4.4.4. *Let H be a group. Let I be some indexing H-set, and let $Y_{[i]}$ be a collection of H-spaces indexed by I that depend only on the orbit. Endow $\prod_{i \in I} Y_{[i]}$ with an H-action by $h.(y_i)_{i \in I} = (h.y_{h.i})_{i \in I}$. Then $\left(\prod_{i \in I} Y_{[i]}\right)^H \cong \prod_{[i] \in I/H} Y_{[i]}^{\mathrm{St}_H(i)}$.*

Proof. First, the action is indeed well defined since $y_{h.i} \in Y_{[i]}$, because i and $h.i$ are in the same orbit. Requiring that $(y_i)_{i \in I}$ is a fixed point amounts to $y_i = h.y_{h.i}$ for all i and h. Note that this condition equates y_i only with values which are in the H-orbit of i. Moreover, the value of y_i determines the whole H-orbit $y_{h.i}$ (by $h^{-1}y_i$). Therefore a fixed point is determined by one element per orbit, y_i, that satisfies $y_i = h.y_i$ when $h.i = i$, i.e. when h is in the stabilizer $\mathrm{St}_H(i)$. So the condition is simply that $y_i \in Y_{[i]}^{\mathrm{St}_H(i)}$, and the conclusion follows. ☐

We now want to apply this to our case.

Proposition 4.4.5. *We have:*

$$\mathrm{Cl}_{n,p}(G, X; L_r)^{\mathrm{Aut}(\Lambda_r)} \cong \prod_{[\alpha] \in G_{n,p}/(G \times \mathrm{Aut}(\Lambda_r))} \left(L_r^{\mathrm{St}_{\mathrm{Aut}(\Lambda_r)}(\alpha)} \otimes_{E^*} E^* \left(X^{\mathrm{im}\,\alpha}\right)^{\mathrm{St}_G(\alpha)}\right)$$

In particular, when $X = *$, we have:

$$\mathrm{Cl}_{n,p}(G; L_r)^{\mathrm{Aut}(\Lambda_r)} \cong \prod_{[\alpha] \in G_{n,p}/(G \times \mathrm{Aut}(\Lambda_r))} L_r^{\mathrm{St}_{\mathrm{Aut}(\Lambda_r)}(\alpha)}$$

Proof. We will take $G \times \mathrm{Aut}(\Lambda_r)$ fixed points of Proposition 4.4.3 in two steps, first by G then by $\mathrm{Aut}(\Lambda_r)$. Recall that L_r is fixed by G. The action on the cohomology part comes from the action on the space, taking $x \in X^{\mathrm{im}\,\alpha}$ to $\gamma x \in X^{\mathrm{im}\,\gamma.\alpha}$. Since this is invertible, this gives a homeomorphism between $X^{\operatorname{im}\alpha}$ and $X^{\operatorname{im}\gamma.\alpha}$, which shows that $E^*\left(X^{\operatorname{im}\alpha}\right) \cong E^*\left(X^{\operatorname{im}\gamma.\alpha}\right)$. Using this isomorphism, we see that $E^*\left(X^{\operatorname{im}\alpha}\right)$ depends only on the G-orbit of α, so we can employ Proposition 4.4.4 for Proposition 4.4.3 with $H = G$ and $I = G_{n,p}$, to get:

$$\mathrm{Cl}_{n,p}\left(G, X; L_r\right) \cong \prod_{[\alpha]\in G_{n,p}/G} \left(L_r \otimes_{E^*} E^*\left(X^{\operatorname{im}\alpha}\right)^{\mathrm{St}_G(\alpha)}\right)$$

We now have the $\mathrm{Aut}\left(\Lambda_r\right)$-action. Recall that $\mathrm{Aut}\left(\Lambda_r\right)$ didn't act on the space part, since it fixes $\operatorname{im}\alpha$. We are then in the situation of Proposition 4.4.4 again, for $H = \mathrm{Aut}\left(\Lambda_r\right)$ and $I = G_{n,p}/G$, and the general case follows.

When $X = *$, all fixed points $X^{\operatorname{im}\alpha}$ are again trivial, which shows that the E^*-cohomology is simply E^*, and tensoring with it over E^* does nothing, and the specific case follows. □

4.5 The General Character Map

We now construct the character map, that also depends on r, which is omitted from the notation:

$$\chi^G_{n,p} : E^*\left(\mathrm{E}G \times_G X\right) \to \mathrm{Cl}_{n,p}\left(G, X; L_r\right)$$

This map is given by a map $E^*\left(\mathrm{E}G \times_G X\right) \to L_r \otimes_{E^*} E^*\left(\mathrm{Fix}_{n,p}\left(G, X\right)\right)$ which lands in the G fixed points. By Proposition 4.4.3, this is the data of a map $E^*\left(\mathrm{E}G \times_G X\right) \to L_r \otimes_{E^*} E^*\left(X^{\operatorname{im}\alpha}\right)$ for each $\alpha \in G_{n,p}$.

Let $\alpha \in G_{n,p}$, that is $\alpha : \Lambda_r \to G$. By functoriality of E, this induces a map $\mathrm{E}\Lambda_r \to \mathrm{E}G$. Consider the inclusion $X^{\operatorname{im}\alpha} \to X$. The multiplication of these maps gives $\mathrm{E}\Lambda_r \times X^{\operatorname{im}\alpha} \to \mathrm{E}G \times X$. Since X and $\mathrm{E}G$ have a G-action, the map $\alpha : \Lambda_r \to G$ induces a Λ_r-action on them. Moreover, by definition, this Λ_r-action restricts to a trivial action on $X^{\operatorname{im}\alpha}$. We equip both sides with the diagonal Λ_r-action, which makes the map equivariant, and we get a map between the Λ_r orbits. The Λ_r orbits of the source are $\mathrm{B}\Lambda_r \times X^{\operatorname{im}\alpha}$. Since the action on the target was pulled form the diagonal G-action, we can further take the G orbits on the target, to get a map $\mathrm{B}\Lambda_r \times X^{\operatorname{im}\alpha} \to \mathrm{E}G \times_G X$.

Taking E^*-cohomology we get $E^*\left(\mathrm{E}G \times_G X\right) \to E^*\left(\mathrm{B}\Lambda_r \times X^{\operatorname{im}\alpha}\right)$. Since X was assumed to be a finite G-CW complex, we have Künneth for the target by Proposition 4.2.3, so the map is equivalently a map $E^*\left(\mathrm{E}G \times_G X\right) \to E^*\left(\mathrm{B}\Lambda_r\right) \otimes_{E^*} E^*\left(X^{\operatorname{im}\alpha}\right)$. Using the localization map $E^*\left(\mathrm{B}\Lambda_r\right) \to L_r$ we finally get the desired map $E^*(\mathrm{E}G \times_G X) \to L_r \otimes_{E^*} E^*\left(X^{\operatorname{im}\alpha}\right)$. This concludes the construction of the data of the character map.

Proposition 4.5.1 ([HKR00, 6.9])**.** *The map $E^*(\mathrm{E}G \times_G X) \to L_r \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X))$ constructed above lands in the $G \times \mathrm{Aut}(\Lambda_r)$ fixed points.*

Therefore, by taking the G fixed points on the target, we indeed get the desired character map $\chi^G_{n,p} : E^*(\mathrm{E}G \times_G X) \to \mathrm{Cl}_{n,p}(G, X; L_r)$, which lands in the $\mathrm{Aut}(\Lambda_r)$ fixed points.

Proposition 4.5.2. *The character maps are compatible with the maps $\mathrm{Cl}_{n,p}(G, X; L_r) \to \mathrm{Cl}_{n,p}(G, X; L_{r+1})$, coming from the maps $L_r \to L_{r+1}$. Therefore, we have a character map for L, that is $\chi^G_{n,p} : E^*(\mathrm{E}G \times_G X) \to \mathrm{Cl}_{n,p}(G, X; L)$, which lands in the $\mathrm{Aut}(\Lambda)$ fixed points.*

Proof. We constructed the character map by constructing a map for each α. It is easy to see that these maps are compatible with the maps $L_r \to L_{r+1}$, coming from the projections. $\square$

Since p is invertible in L by Proposition 4.3.6, and $\mathrm{Aut}(\Lambda)$ doesn't change p^{-1}, it also follows that after inverting p, that is, rationalizing, the map $p^{-1}\chi^G_{n,p} : p^{-1}E^*(\mathrm{E}G \times_G X) \to \mathrm{Cl}_{n,p}(G, X; L)$ still lands in the $\mathrm{Aut}(\Lambda)$ fixed points (and the same is true for L_r in place of L).

We are now in position to state the main theorem. This should remind you of Theorem 3.3.1 and Theorem 3.3.2.

Theorem 4.5.3 ([HKR00, Theorem C])**.** *First, after tensoring with L, the character map $\chi^G_{n,p} \otimes L : E^*(\mathrm{E}G \times_G X) \otimes_{E^*} L \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)$ becomes an isomorphism. Second, the map $p^{-1}\chi^G_{n,p} : p^{-1}E^*(\mathrm{E}G \times_G X) \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)^{\mathrm{Aut}(\Lambda)}$ is an isomorphism. Moreover, these statements hold when L is replaced with L_r, for $r \geq r_0$.*

Corollary 4.5.4. *Using Proposition 4.4.5, for the case $X = *$ we get isomorphisms*

$$E^*(\mathrm{B}G) \otimes_{E^*} L \cong \mathrm{Cl}_{n,p}(G; L) \cong \prod_{[\alpha] \in G_{n,p}/G} L,$$

and

$$p^{-1}E^*(\mathrm{B}G) \cong \mathrm{Cl}_{n,p}(G; L)^{\mathrm{Aut}(\Lambda)} \cong \prod_{[\alpha] \in G_{n,p}/(G \times \mathrm{Aut}(\Lambda))} L^{\mathrm{St}_{\mathrm{Aut}(\Lambda)}(\alpha)},$$

and these statements hold when L is replaced with L_r, for $r \geq r_0$.

The first part of the theorem will be proven in the remaining of the section.

Proof (of the second part). Consider the isomorphism from the first part, $E^*(\mathrm{E}G \times_G X) \otimes_{E^*} L \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)$. Endowing the source with $\mathrm{Aut}(\Lambda)$-action by acting only on L, makes it equivariant. Therefore, there is an isomorphism on the fixed points. Using $L^{\mathrm{Aut}(\Lambda)} = p^{-1}E^*$, from Proposition 4.3.6, the fixed points on the source are:

$$\begin{aligned}(E^*(\mathrm{E}G \times_G X) \otimes_{E^*} L)^{\mathrm{Aut}(\Lambda)} &= E^*(\mathrm{E}G \times_G X) \otimes_{E^*} L^{\mathrm{Aut}(\Lambda)} \\ &= E^*(\mathrm{E}G \times_G X) \otimes_{E^*} p^{-1}E^* \\ &= p^{-1}E^*(\mathrm{E}G \times_G X)\end{aligned}$$

So indeed $p^{-1}E^*(\mathrm{E}G \times_G X) \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)^{\mathrm{Aut}(\Lambda)}$ is an isomorphism. (The exact same proof works when L is replaced with L_r, for $r \geq r_0$.) $\square$

4.6 The Idea of the Proof and Complex Oriented Descent

Our next goal is to prove the first part of Theorem 4.5.3. That is, for a finite group G and a finite G-CW complex X, the character map becomes an isomorphism after tensoring with L, i.e. $\chi^G_{n,p} \otimes L : E^*(\mathrm{E}G \times_G X) \otimes_{E^*} L \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)$ is an isomorphism, and the same with L replaced with L_r, for $r \geq r_0$. One may wonder why we had to introduce the G-space X into the construction, in order to prove the case of interest, $X = *$. The reason is, that there is a trick, called *complex oriented descent*, that allows us to reduce to the case of G-spaces X with *abelian stabilizers*. Using this and some further ideas we reduce to the case where G is abelian, and $X = *$. That is, introducing the space X into the construction, allows us to reduce the statement to abelian groups.

To be more explicit, this is the strategy. We will consider the character map as a natural transformation between functors of pairs (G, X), and then we will follow these steps:

- Use complex oriented descent to reduce to X with abelian stabilizers,
- Use Mayer-Vietoris to reduce to spaces $X = D^n \times G/A$ with A abelian,
- Use homotopy invariance to reduce to $X = G/A$,
- Use induction to reduce from $(G, G/A)$ to $(A, *)$,

- Prove for $(A,*)$.

This strategy will be formulated as a theorem later, after we introduce complex oriented descent. To introduce it, we need some definitions.

Definition 4.6.1. Let ξ be a d-dimensional complex vector bundle over a space X. The *flag bundle* $F(\xi) \to X$ is the bundle of complete flags in ξ.

The fiber over a point can be described as an (ordered) d-tuple $(\ell_1, \ldots, \ell_d)$ of orthogonal lines. To define it precisely, we can take the d-fold power of the projective bundle $P(\xi)$, and restrict to the sub-bundle of orthogonal lines. We note that for a trivial bundle $X \times V \to X$, we have $F(X \times V) \cong X \times F(V)$, i.e. the flags are computed fiber-wise.

Definition 4.6.2. Let $C^* : \mathcal{S}^{\mathrm{op}} \to \mathrm{GrAb}$ be a contra-variant functor from spaces to graded abelian groups. C^* is said to satisfy *complex oriented descent*, if for every space X and bundle ξ over X, C^* sends the diagram $X \leftarrow F(\xi) \leftleftarrows F(\xi) \times_X F(\xi)$, to an equalizer diagram $C^*(X) \to C^*(F(\xi)) \rightrightarrows C^*(F(\xi) \times_X F(\xi))$.

We note that if ξ is a G-vector bundle over the G-space X, then $F(\xi) \to X$ is also a G-bundle, since G acts unitarily. We also recall that every finite group G has a faithful finite dimensional complex representation. The reason complex oriented descent is useful is the following.

Proposition 4.6.3. *Let X be a G-space, and let $\rho : G \to V$ be a faithful representations. Then the G-space $F(X \times V) \cong X \times F(V)$ has abelian stabilizers.*

Proof. Let $(x, (\ell_i))$ be a point in $X \times F(V)$. We wish to show that its stabilizer is abelian. Let $g, h \in G$ be two elements which fix the point, that is, $g.x = x = h.x$ and $\rho_g(\ell_i) = \ell_i = \rho_h(\ell_i)$. We see that the linear transformations $\rho_g : V \to V$ and $\rho_h : V \to V$ are simultaneously diagonalizable w.r.t to the decomposition (ℓ_i) of V. Therefore, by a classical result in linear algebra, they commute, $\rho_g \rho_h = \rho_h \rho_g$, i.e. $\rho_{gh} = \rho_{hg}$. Since ρ is faithful, we get that $gh = hg$. □

Definition 4.6.4. Let $H \leq G$ be a subgroup, and let Y be an H-space. We define the G-space $G \times_H Y$ as follows. Define an H-action on G by $h.g = gh^{-1}$. This gives a diagonal action on $G \times Y$, i.e. $h.(g, y) = \left(gh^{-1}, h.y\right)$. The orbits are $G \times_H Y$. This space has a G-action by $\gamma.(g, y) = (\gamma g, y)$. This is well defined, since $\gamma.\left[gh^{-1}, h.y\right] = \left[\gamma g h^{-1}, h.y\right] = [\gamma g, y] = \gamma.[g, y]$.

Definition 4.6.5. Let $\mathcal{C}$ be the category whose objects are pairs (G, X) where G is a finite group and X is a finite G-CW complex. The morphisms in $\mathcal{C}$ are generated from the following: First, the usual morphisms $(G, X) \to (G, Y)$. Second, for $H < G$ and Y, we add a morphism $(H, Y) \to (G, G \times_H Y)$.

Definition 4.6.6. Let $C^* : \mathcal{C}^{\mathrm{op}} \to \mathrm{GrAb}$ be a contra-variant functor from $\mathcal{C}$. We define the following properties of C^*:

- *Homotopy invariance* - for every G, the functor $C^*(G, -)$ is G-homotopy invariant.
- *Mayer-Vietories* - for every G, $C^*(G, -)$ satisfies Mayer-Vietoris.
- *Complex oriented descent* - for every G, $C^*(G, -)$ satisfies complex oriented descent.
- *Induction* - for every, $H \leq G$ and H-space Y, the morphism $(H, Y) \to (G, G \times_H Y)$ induces an isomorphism $C^*(G, G \times_H Y) \xrightarrow{\sim} C^*(H, Y)$. G.

Theorem 4.6.7 ([HKR00, 6.10])**.** *Let $C^*, D^* : \mathcal{C}^{\mathrm{op}} \to \mathrm{GrAb}$ be functors satisfying the above properties. Let $\tau : C^* \to D^*$ be a natural transformation between them. Suppose that τ commutes with the connecting morphisms of Mayer-Vietoris, and that $\tau(A, *)$ is an isomorphism for all abelian groups A. Then τ is a natural isomorphism.*

Proof. We will follow the steps of the strategy outlined before (although we will describe it in the opposite order).

Let G be a group, and $A \leq G$ an abelian subgroup. The morphism $(A, *) \to (G, G \times_A *) \cong (G, G/A)$, by naturality of τ, induces a commutative square:

$$\begin{tikzcd}
C^*(G, G/A) \arrow[r] \arrow[d, "\tau(G,G/A)"] & C^*(A, *) \arrow[d, "\tau(A,*)"] \\
D^*(G, G/A) \arrow[r] & D^*(A, *)
\end{tikzcd}$$

By induction, the horizontal morphisms are isomorphisms. By assumption, $\tau(A, *)$ is an isomorphism. We conclude that $\tau(G, G/A)$ is an isomorphism.

Since C^*, D^* are homotopy invariant, for every disk D^n equipped with a trivial action, the map $(G, G/A \times D^n) \to (G, G/A)$ induces an isomorphism. Similarly to before, by naturality $\tau(G, G/A \times D^n)$ is an isomorphism.

Now, let X a finite G-CW complex, s.t. the stabilizer of every point is abelian. All the cells are then of the form $G/A \times D^n$ for some abelian subgroup $A \leq G$ and disk D^n. By an induction on the number of cells, using Mayer-Vietoris and the fact that τ commutes with the connecting morphisms, (G, X) is an isomorphism.

Lastly, let X be an arbitrary finite G-CW complex. Let V be a faithful G representation, and consider the bundle $X \times V \to V$. By naturality, the diagram $X \leftarrow X \times F(V) \leftleftarrows X \times F(V) \times F(V)$ induces a commutative diagram:

$$\begin{tikzcd}
C^*(X) \arrow[r, "f"] \arrow[d, "\alpha=\tau(G,X)"] & C^*(X \times F(V)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\beta=\tau(G,X\times F(V))"] & C^*(X \times F(V) \times F(V)) \arrow[d, "\tau(G,X\times F(V)\times F(V))"] \\
D^*(X) \arrow[r, "g"] & D^*(X \times F(V)) \arrow[r, shift left] \arrow[r, shift right] & D^*(X \times F(V) \times F(V))
\end{tikzcd}$$

By complex oriented descent, the two rows are equalizer diagrams. By Proposition 4.6.3, $X \times F(V)$ has abelian stabilizers, hence we already know that $\beta = \tau(G, X \times F(V))$ is an isomorphism. We can then construct the map $\beta^{-1} g : D^*(X) \to C^*(X \times F(V))$, which by definitions makes the diagram commute. By the universal property of the equalizer, we get a map $\alpha' : D^*(X) \to C^*(X)$ s.t. the diagram is commutative. The composition $\alpha'\alpha : C^*(X) \to C^*(X)$ makes the diagram commute, and since $C^*(X)$ is the equalizer, by uniqueness, $\alpha'\alpha = \mathrm{id}_{C^*(X)}$. Similarly $\alpha\alpha' : D^*(X) \to D^*(X)$ makes the diagram commute, so $\alpha\alpha' = \mathrm{id}_{D^*(X)}$. This shows that $\alpha = \tau(G, X)$ is invertible, which completes the proof. $\square$

4.7 Proof of the Main Theorem

We are going to use the previous results to prove the main theorem, Theorem 4.5.3. Almost all of the proof will work with L replaced by L_r, for $r \geq r_0$, without a change, so we state everything for L except for the end where there is a difference. Recall that we have already proved the second part. Therefore, what is left to prove is that $\chi^G_{n,p} \otimes L : E^*(\mathrm{E}G \times_G X) \otimes_{E^*} L \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, X; L)$ is an isomorphism. We will do this using Theorem 4.6.7.

Denote $C^*(G, X) = E^*(\mathrm{E}G \times_G X) \otimes_{E^*} L$, and $D^*(G, X) = \mathrm{Cl}_{n,p}(G, X; L) = (L_r \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X)))^G$. Their definition on morphisms $(G, X) \to (G, Y)$ is clear, simply by functoriality of all constructions when G is fixed. The definition on morphisms for induction will be given below, together with the proof that they satisfy induction. We also denote by $\tau(G, X)$ the character map $\chi_{n,p}^{G} \otimes L$ for X.

Lemma 4.7.1. *Both functors C^* and D^* are homotopy invariant.*

Proof. This is immediate since E^* is homotopy invariant, the Borel construction $X \mapsto \mathrm{E}G \times_G X$ is G-homotopy invariant, and the fixed points of a G-CW complex are also G-homotopy invariant. $\square$

Lemma 4.7.2. *Both functors C^* and D^* satisfy Mayer-Vietoris, and τ commutes with the connecting morphisms.*

Proof. The Borel construction $X \mapsto \mathrm{E}G \times_G X$ is a limit, and so are fixed points, so they commute with pushouts. Therefore, the usual pushouts that induce Mayer-Vietoris, give Mayer-Vietoris for our functors. Moreover, the definition makes it clear that the character map commutes with the connecting morphisms. $\square$

Lemma 4.7.3. *Both functors C^* and D^* satisfy complex oriented descent.*

Proof. Hopkins, Kuhn and Ravenel prove in [HKR00, 2.5] that any complex oriented cohomology theory (and not only the cohomology theories of interest to us, namely Lubin-Tate) satisfies complex oriented descent, and we will rely on this result.

Let ξ be a G-vector bundle over X.

$\mathrm{E}G \times_G \xi$ is a G-vector bundle on $\mathrm{E}G \times_G X$, and it satisfies $F(\mathrm{E}G \times_G \xi) \cong \mathrm{E}G \times_G F(\xi)$. Then the fact that E^* satisfies complex oriented descent, and that L is flat (see Proposition 4.3.6), imply that C^* satisfies complex oriented descent.

Moreover, in [HKR00, 2.6], they prove that for an abelian subgroup, $A \leq G$, the diagram $X^A \leftarrow F(\xi)^A \leftleftarrows F(\xi)^A \times_{X^A} F(\xi)^A$ gives an equalizer diagram in E^*-cohomology. In the situation of D^*, we use the result for $A = \operatorname{im} \alpha$ which is indeed abelian by the fact they are n commuting elements (equivalently, by the fact that it is the image of an abelian group). Equalizers, which are limits, commute with limits, and therefore commute with products and taking G-fixed points. Using this, and the flatness of L again, we deduce, by Proposition 4.4.3, that D^* satisfies complex oriented descent as well. $\square$

Lemma 4.7.4. *Both functors C^* and D^* satisfy induction.*

Proof. Let $H \leq G$ be a subgroup, and Y an H-space.

We have $\mathrm{E}G \times_G (G \times_H Y) \cong \mathrm{E}H \times_H Y$. Taking E^*-cohomology and tensoring with L gives an isomorphism $C^*(G, G \times_H Y) \xrightarrow{\sim} C^*(H, Y)$, which shows the functoriality for this sort of morphisms, and the fact that it is an isomorphism show that C^* satisfies induction.

We now claim that there is a homeomorphism $\varphi : G \times_H \mathrm{Fix}_{n,p}(H, Y) \xrightarrow{\sim} \mathrm{Fix}_{n,p}(G, G \times_H Y)$.

By definition:

$$G \times_H \mathrm{Fix}_{n,p}(H, Y) = G \times_H \coprod_{\alpha \in H_{n,p}} Y^{\mathrm{im}\,\alpha}$$

An element here is the data of $g \in G$, $\alpha \in H_{n,p}$ and $y \in Y^{\mathrm{im}\,\alpha}$. We will denote its H-orbit by $[g, \alpha, y]$. For an $h \in H$, the relation we get is $[g, \alpha, y] = \left[gh^{-1}, h.\alpha, h.y\right]$. An element $\gamma \in G$ acts by $\gamma.[g, \alpha, y] = [\gamma g, \alpha, y]$.

Similarly, by definition:

$$\mathrm{Fix}_{n,p}(G, G \times_H Y) = \coprod_{\alpha \in G_{n,p}} (G \times_H Y)^{\mathrm{im}\,\alpha}$$

An element here is the data of $\alpha \in G_{n,p}$, $[g, y] \in (G \times_H Y)^{\mathrm{im}\,\alpha}$. We will denote this by $(\alpha, [g, y])$. For an $h \in H$, the relation we get is $(\alpha, [g, y]) = \left(\alpha, \left[gh^{-1}, hy\right]\right)$. An element $\gamma \in G$ acts by $\gamma.(\alpha, [g, y]) = (\gamma.\alpha, [\gamma g, y])$.

Define the map $\varphi : G \times_H \mathrm{Fix}_{n,p}(H, Y) \to \mathrm{Fix}_{n,p}(G, G \times_H Y)$ by $\varphi([g, \alpha, y]) = (g.\alpha, [g, y])$.

We need to show that it doesn't depend on the H-orbit representative in the source, and indeed,

$$\begin{aligned}
\varphi\left(\left[gh^{-1}, h.\alpha, h.y\right]\right) &= \left(gh^{-1}.h.\alpha, \left[gh^{-1}, h.y\right]\right) \\
&= \left(g.\alpha, \left[gh^{-1}, h.y\right]\right) \\
&= (g.\alpha, [g, y]) \\
&= \varphi([g, \alpha, y]).
\end{aligned}$$

We need to show that the element defined lands in the target, $(g.\alpha, [g, y]) \in \mathrm{Fix}_{n,p}(G, G \times_H Y)$, i.e. $[g, y] \in (G \times_H Y)^{g.\,\mathrm{im}\,\alpha}$. Since $\alpha \in H_{n,p}$, we have

$g.\operatorname{im}\alpha \leq gHg^{-1}$, so let $ghg^{-1} \in \operatorname{im}\alpha$, and we verify that $[g,y]$ is invariant under it (via the G-action). Recall that $y \in Y^{\operatorname{im}\alpha}$, so we get, $ghg^{-1}.\,[g,y] = \left[ghg^{-1}g,y\right] = [gh,y] = [g,hy] = [g,y]\,.$

We show that it is G-equivariant. So let $\gamma \in G$, and indeed,

$$\begin{aligned}\varphi\left(\gamma.\,[g,\alpha,y]\right) &= \varphi\left([\gamma g,\alpha,y]\right)\\ &= \left(\gamma g.\alpha,[\gamma g,y]\right)\\ &= \gamma.\left(g.\alpha,[g,y]\right)\\ &= \gamma.\varphi\left([g,\alpha,y]\right)\end{aligned}$$

We show that it is one-to-one. Assume $\varphi([g,\alpha,y]) = \varphi([g',\alpha',y'])$, i.e. $(g.\alpha,[g,y]) = (g'.\alpha',[g',y'])$. In particular, $[g,y] = [g',y']$. It follows that $y' = h.y$ and $g' = gh^{-1}$ for some $h \in H$. Then, $g.\alpha = g'.\alpha' = gh^{-1}.\alpha'$, so $\alpha = h^{-1}.\alpha'$, equivalently $\alpha' = h.\alpha$. We therefore conclude that they are indeed in the same H-orbit, $[g,\alpha,y] = \left[gh^{-1},h.\alpha,h.y\right] = [g',\alpha',y']$.

Lastly, we show that it is surjective, which is the only step which is not routine. Let $(\alpha,[g,y])$, i.e. $\alpha \in G_{n,p}$, $[g,y] \in (G\times_H Y)^{\operatorname{im}\alpha}$. We claim that if such a triple exists, i.e. the fixed points are not empty, then necessarily $g^{-1}.\alpha \in H_{n,p}$. Let $\gamma \in \operatorname{im} g^{-1}.\alpha$. Since $g\gamma g^{-1}$ is in $\operatorname{im}\alpha$, it fixes $[g,y]$, that is $[g,y] = g\gamma g^{-1}.\,[g,y] = \left[g\gamma g^{-1}g,y\right] = [g\gamma,y]$. Therefore, for some $h \in H$, $\left(gh^{-1},hy\right) = (g\gamma,y)$, in particular $\gamma = h^{-1}$, so $\gamma \in H$. It follows that $g^{-1}.\alpha \in H_{n,p}$. We now claim that $y \in Y^{\operatorname{im} g^{-1}.\alpha}$. So let $\eta \in \operatorname{im} g^{-1}.\alpha$, which we now know is in H as well. Since $[g,y] \in (G\times_H Y)^{\operatorname{im}\alpha}$, it is fixed by $g\eta g^{-1}$, i.e. there is some $h \in H$, s.t. $\left(gh^{-1},hy\right) = g\eta g^{-1}.\,(g,y) = \left(g\eta g^{-1}g,y\right) = (g\eta,y)$, and we get that $y = \eta y$. We shows that $y \in Y^{\operatorname{im} g^{-1}.\alpha}$, so the element $\left[g,g^{-1}\alpha,y\right]$ is well defined, and mapped to $(\alpha,[g,y])$.

It is clear that the map φ is continuous, and so is its inverse $\left[g,g^{-1}\alpha,y\right] \mapsto (\alpha,[g,y])$, which shows that indeed D^* satisfies induction. $\square$

Lemma 4.7.5. *$\tau(A,*)$ is an isomorphism for all abelian groups A.*

Proof. We need to verify that $\chi^A_{n,p} : E^*(\mathrm{B}A)\otimes_{E^*} L \to \mathrm{Cl}_{n,p}(A;L)$ is an isomorphism for abelian groups A.

By Proposition 4.2.3, we have Künneth for $E^*(\mathrm{B}A)$, so it takes direct sums in A to tensor products.

For an abelian group, the A-action on $A_{n,p}$ is trivial (since it is by conjugation). Therefore, $\mathrm{Cl}_{n,p}(A;L) = \hom_{A\mathsf{Set}}(A_{n,p},L) = \hom_{\mathsf{Set}}(A_{n,p},L) =$

$\hom_{\mathrm{Set}}\left(\hom_{\mathrm{Ab}}\left(\Lambda, A\right), L\right)$. The $\hom_{\mathrm{Ab}}\left(\Lambda, A\right)$ commutes with direct sums in the second coordinate, and the outer $\hom_{\mathrm{Set}}$ takes them to tensor product.

Both functors take direct sums to tensor products, so by the structure theorem for abelian groups, we reduce to the case $\mathbb{Z}/q^k$ for a prime q.

First we handle the case $q \neq p$. Again by Proposition 4.2.3, $E^*\left(\mathrm{B}\mathbb{Z}/q^k\right)$ is a free module of rank 1, so the source of $\chi_{n,p}^{\mathbb{Z}/q^k}$ is L. Moreover, $\hom_{\mathrm{Ab}}\left(\Lambda, \mathbb{Z}/q^k\right)$ has only the trivial homomorphism, so $\mathrm{Cl}_{n,p}\left(\mathbb{Z}/q^k; L\right) = L$. It is easy to see that this is indeed an isomorphism then.

Lastly, we handle the case $q = p$. At this point we have to prove it separately for every L_r with $r \geq r_0$, and the result follows for L as well, by taking the colimit on both sides.

In this case the group is $\mathbb{Z}/p^k$, so $r_0 = k$. Let $r \geq r_0 = k$. Now $\left(\mathbb{Z}/p^k\right)_{n,p}$, i.e. n commuting p-power-torsion elements, is just n elements, so it is canonically isomorphic to $\left(\mathbb{Z}/p^k\right)^n = \Lambda_k$. The character map $\chi_{n,p}^{\mathbb{Z}/p^k}$ is then a morphism $E^*\left(\mathrm{B}\mathbb{Z}/p^k\right) \otimes_{E^*} L_r \to \hom_{\mathrm{Set}}\left(\Lambda_k, L_r\right)$.

By Proposition 4.2.3, the source of the character map is isomorphic to $E^*\left[\left[x\right]\right]/\left(\left[p^k\right]\left(x\right)\right) \otimes_{E^*} L_r$. Using the exponential rule, and the adjunction between the forgetful from L_r-modules to E^*-modules and $- \otimes_{E^*} L_r$, we get that the character map is the same data as a map $\phi : \Lambda_k \to \hom_{L_r}\left(L_r\left[\left[x\right]\right]/\left(\left[p^k\right]\left(x\right)\right), L_r\right)$.

In [HKR00] the proof ends here, since they have an alternative definition of L_r as a ring representing some functor, which makes the above map an isomorphism almost immediately. We didn't take this route, so we make the connection here more explicit.

Recall the definition of L_r from Definition 4.3.1, and the definition of the character map. Chasing the definitions, we see that for $0 \neq (l_i) \in \Lambda_k$, $\phi\left(\left(l_i\right)\right)$ is the homomorphism which sends x to $\sum_F \left[l_i\right]\left(x_i\right) \in L_r$. Since $r \geq r_0 = k$, the last element is in S_r, and in particular invertible in L_r. We now finish the proof by applying [HKR00, 6.2] to ϕ, which shows that in this case indeed the character map is an isomorphism. We remark that in their notation, we use the result for $r = k$ and $R = L_r$. Moreover, they denote by Λ_r^* the Pontryagin dual, which is isomorphic to Λ_r, and $\phi\left(\alpha\right)$ in their notation really means the value at x (as can be seen in [HKR00, 5.5]). □

Proof (of the first part of Theorem 4.5.3). Follows immediately by combining the previous lemmas and Theorem 4.6.7. □

5 Elliptic Curves

At this point, one may wonder how we can find interesting pairs (k, Γ), of a perfect field and a formal group law over it, to obtain Lubin-Tate spectra. Two simple examples which we have already seen are the additive formal group law over $\mathbb{F}_p$, of height ∞, which gives rise to $\mathrm{H}\mathbb{F}_p$, and the multiplicative formal group law over $\mathbb{F}_p$, of height 1, which gives rise to $\mathrm{K}_p^\wedge$. Elliptic curves are another source for formal group laws.

5.1 Formal Group Laws From Elliptic Curves

Let C be an elliptic curve over a ring R, with O the point at infinity. In [Sil09, IV], there is a construction of a formal group law Γ_C over R, obtained by considering the infinitesimal neighborhood of O in C. We will also denote by $\mathbb{G}_C$ the corresponding formal group.

Now, assume that $R = k$ is a finite field of characteristic p. We denote by $C[p^r]$ the p^r-torsion, i.e. the (scheme-theoretic) kernel of the multiplication-by-p^r map. By [Sil09, IV.7.5] and [Sil09, V.3.1], we have:

Proposition 5.1.1. *The height of Γ_C is either 1 or 2. Moreover, the height is 2 if and only if the only point of $C[p^r]$ is O for all $r \geq 1$.*

In fact, there are many more equivalent conditions to the above, which can be found in the above reference, and this is turned into a definition.

Definition 5.1.2. C is called *supersingular* if Γ_C is of height 2.

From now on we assume that our elliptic curve is supersingular.

Similarly to the Lubin-Tate deformation theory of formal group laws described in Section 2.7, there is a deformation theory for elliptic curves. Recall that Γ_C over k has a universal deformation. The Serre-Tate theorem [KM85, 2.9.1] then implies the following:

Theorem 5.1.3. *There exists a deformation C_U over $Wk[[u_1]]$ of C, whose formal group law Γ_{C_U} is a universal deformation of Γ_C.*

In this case we get a Lubin-Tate spectrum $E = \mathrm{E}(k, \Gamma_C)$. We recall from Corollary 2.7.7 that the coefficients can be taken to be $E_* = Wk[[u_1]]\left[u^{\pm 1}\right]$ where $|u| = 2$, and the formal group law is $u(\Gamma_C)_U$, which by the above can be described by $u\Gamma_{C_U}$.

5.2 HKR From Elliptic Curves

Recall that the our main goal in HKR theory was to compute $p^{-1}E^*(\mathrm{B}G)$ for some Lubin-Tate spectrum E. We shall focus only the 0-th level, as this is 2-periodic in the usual way. The main result for us was Corollary 4.5.4 (stated here only for the 0-th level, and with L_r for $r \geq r_0$)

$$p^{-1}E^0(\mathrm{B}G) \cong \prod_{[\alpha]\in G_{n,p}/(G\times \mathrm{Aut}(\Lambda_r))} \left(L_r^0\right)^{\mathrm{St}_{\mathrm{Aut}(\Lambda_r)}(\alpha)}.$$

That is, in order to compute $p^{-1}E^0(\mathrm{B}G)$, we need to compute L_r^0, as in Definition 4.3.1, and various fixed-points sub-rings thereof.

Recall that we fixed some supersingular elliptic curve C over a finite field k of characteristic p, so that Γ_C is of height $n = 2$, and we consider $E = \mathrm{E}(k, \Gamma_C)$. As we saw in Theorem 5.1.3, we can use the deformation C_U to describe the universal deformation of the formal group. In Section 4.3.1, we have seen that L_r^0 can be described as follows. First, $\mathrm{Spec}\, E^0(\mathrm{B}\Lambda_r) = (\mathbb{G}_{C_U}[p^r])^2$. Second, $S_r^0 = \left\{[k_1](t) +_{\Gamma_{C_U}} [k_2](s) \mid (k_1, k_2) \neq 0 \mod p^r\right\}$. And $L_r^0 = \left(S_r^0\right)^{-1} E^0(\mathrm{B}\Lambda_r)$.

Now, since $\mathbb{G}_C$ is the formal neighborhood of O in C, we have a map $\mathbb{G}_C \to C$. Since the multiplication on $\mathbb{G}_C$ comes from the multiplication on C, we have the commutative square:

$$\begin{tikzcd} \mathbb{G}_C \arrow[r] \arrow[d, "{[p^r]}"] & C \arrow[d, "{[p^r]}"] \\ \mathbb{G}_C \arrow[r] & C \end{tikzcd}$$

Taking the kernels of both vertical maps, we get a map $\mathbb{G}_C[p^r] \to C[p^r]$. Since C is supersingular, by Proposition 5.1.1, the only point of $C[p^r]$ is O, i.e. it is a nilpotent thickening of the point O, which means that the map $\mathbb{G}_C[p^r] \to C[p^r]$ is an isomorphism of schemes, since $\mathbb{G}_C$ is the formal neighborhood of O. Since the group structure on $\mathbb{G}_C$ is inherited from that of C, this is also an isomorphism of group schemes.

In the same way as above, we have a map between the p^r-torsion of the deformations, $\mathbb{G}_{C_U}[p^r] \to C_U[p^r]$. Reducing modulo the maximal ideal, i.e. the map $Wk[[u_1]] \to k$, gives the map above $\mathbb{G}_C[p^r] \to C[p^r]$, which is an isomorphism. By Nakayama's lemma we see that the map $\mathbb{G}_{C_U}[p^r] \to$

$C_U[p^r]$ is also an isomorphism of schemes, and again this is actually an isomorphism of group schemes.

This means that in our computations of L_r^0, we can use the elliptic curve C_U rather then its formal group law, as the p^r-torsion groups are isomorphic. This has the advantage that the operations on the elliptic curve are given by polynomials, rather then formal power series. More explicitly, we have that $\operatorname{Spec} E^0(\mathrm{B}\Lambda_r) \cong (C_U[p^r])^2$, i.e. the scheme-theoretic kernel of the multiplication-by-p^r map on the elliptic curve, squared. We then need to localize away from the zeros of the functions $[k_1](-) +_{C_U} [k_2](-)$, for $(k_1, k_2) \neq 0 \mod p^r$. As we have seen in Section 4.3.1, we can actually consider only k_i's which are a multiple of p^{r-1}, which means that we need to consider only $p^2 - 1$ pairs.

5.3 Specific Elliptic Curve

We now restrict ourselves to a special case. Take $p = 2$ and $k = \mathbb{F}_4$. We take C to be the elliptic curve given by the Weierstrass equation $y^2 + y = x^3$. It is supersingular as follows from [Sil09, exercise V.5.7 combined with proposition A.1.1.c]. Another way to see that is by a direct computation of the terms of the formal group law, which shows that the coefficient of x in the 2-series is $2 = 0$, see [Bea17, 6.1.4].

Furthermore, denote by C_U the elliptic curve given by $y^2 + u_1xy + y = x^3$ over $Wk[[u_1]] = \mathbb{Z}_2[\zeta_3][[u_1]]$. The maximal ideal is $(1 - \zeta_3, u_1)$ with residue field $\mathbb{F}_4$, and modulo this ideal C_U reduces to C. Furthermore, in [LT66, 3.5], it is proven that the formal group law of C_U is indeed a universal deformation of that of C. Specifically, there the ring $\mathbb{Z}_2[\zeta_3]$ is denoted by R, and u_1 by t. It is claimed that the formal group law (up to order 2) is given by $x + y + u_1xy$. Then, by [LT66, 1.1], it is the universal deformation, because $C_2 = \frac{1}{2}\left((x+y)^2 - x^2 - y^2\right) = xy$.

Our next goal is to compute the ring L_r^0 corresponding to $E = \mathrm{E}(\mathbb{F}_4, \Gamma_C)$. To that end, we first need to compute $E^0(\mathrm{B}\Lambda_r)$, which as we saw is given by $\mathcal{O}\left((C_U[2^r])^2\right) = (\mathcal{O}(C_U[2^r]))^{\otimes 2}$. We then need to localize away from the zeros of $[k_1](-) +_{C_U} [k_2](-)$. We have only $2^2 - 1 = 3$ pairs, which are $\left(2^{r-1}, 0\right), \left(0, 2^{r-1}\right), \left(2^{r-1}, 2^{r-1}\right)$. Note that the first two are symmetric, and can be computed even before taking the tensor product. That is, L_r^0 is given by computing $\mathcal{O}(C_U[2^r])$ (to get the 2^r-torsion), localizing away from $[2^{r-1}](-)$ (to get the points of order exactly 2^r), tensoring with itself (to get pairs of such points), and localizing away from $[2^{r-1}](-) +_{C_U} [2^{r-1}](-)$ (to get such pairs that span).

Furthermore, we recall that by Proposition 4.3.6, 2 is invertible in L_r^0, so the whole computation can be carried with C_U base changed to $\mathbb{Q}_2[\zeta_3][[u_1]]$, and we will obtain the same result. Moreover, we note that the elliptic curve, and all the operations described above, are defined already over $R = \mathbb{Q}[u_1]$, so we can carry the whole computation over R, and tensor in the end with $\mathbb{Q}_2[\zeta_3][[u_1]]$ to get L_r^0.

5.4 Concrete Computations

We describe a way to do the calculation described above, namely over $R = \mathbb{Q}[u_1]$. We will display this along with the Macaulay2 code, that carries out the computation. Note that the code will be at times more general then this specific example, but not at all times.

The basic operations on the elliptic curve are developed in projective coordinates. Therefore, we homogenize the elliptic curve above to $X^3 = Y^2Z + u_1XYZ + YZ^2$. When we need to compute the steps described above, we will work with the affine patch where $Y = 1$, which contains the origin $O = [0;1;0]$. We note that in this patch we remove exactly one point from the curve, for if $Y = 0$, then $X^3 = 0 + 0 + 0 = 0$, i.e. $X = 0$, so we remove only the point $[0;0;1]$. This will be useful to give a computation of the addition map, and more specifically for the multiplication-by-2^r map.

The code is given below, and what follows is an explanation of the code.

5.4.1 General Remarks

We will use in the code matrices (rather then other data types which store a list of values), as Macaulay2 has the best support for matrices.

We note that we know a priori the ranks of all the constructions over R.

- The points of $C_U[p^r]$ are isomorphic to $\Lambda_r = (\mathbb{Z}/p^r)^2$, so this is of rank p^{2r}.
- The points of order exactly p^r are then of rank $p^{2r} - p^{2(r-1)}$.
- The points of L_r^0 are of rank $p^{4(r-1)}\left(p^2-1\right)\left(p^2-p\right)$. To see this, we note that they have a transitive free action of $\mathrm{Aut}(\Lambda_r) = \mathrm{GL}(2, \mathbb{Z}/p^r)$

(since we pick two elements that generate Λ_r). Being invertible is equivalent to having an invertible determinant, and an element is invertible in $\mathbb{Z}/p^r$ if and only if it is non-zero modulo p. There are p^{r-1} ways to lift a non-zero number in $\mathbb{Z}/p$ to $\mathbb{Z}/p^r$, so there are $p^{4(r-1)}$ lifts to an invertible matrix. Lastly, $\mathrm{GL}(2, \mathbb{Z}/p)$ has $\left(p^2-1\right)\left(p^2-p\right)$ elements, since the first column can be any non-zero vector, and the second must be linearly independent of it.

Knowing the ranks will be useful later on. First and foremost, as some of our constructions will involve arbitrary choices, which might lead to a result which is too large, but having the correct rank guarantees that our choice is valid. Second, this is a useful sanity-check to verify our results.

5.4.2 Basic Objects

We define the basic objects concerning the computation. Note that `r = 2` can be replaced in principal by any value. Moreover, we can change `R = QQ[u1]` to `QQ` or even `GF(p)` if we want to work over them, and then we also need to remove the `u1 * X * Y * Z` term from `F`.

In the code `RemovedP` is the point described above which is not in the affine patch we will be using ($Y = 1$).

5.4.3 Util Functions

The function `DivideGcd` has a matrix as its input, which will always be a list of polynomials, and outputs the matrix divided by the GCD of all of the elements. The function `comp` receives two matrices of polynomials, and computes the composition by substituting the variables. It also divides by the GCD, which does not affect the function in projective coordinates, but is essential in some instances to get the correct affine functions.

5.4.4 Functions on the Elliptic Curve

The first function calculated is what we call `Star`. This operation is used to define the addition (as explained below), and it satisfies $P_1 \star P_2 = -(P_1 + P_2)$, in other words, $(P_1 \star P_2) + P_1 + P_2 = O$. Geometrically, $P_1 \star P_2$ is the third intersection point of the line through P_1 and P_2 and the elliptic curve. Equivalently, projectively, the line is given by $tP_1 + sP_2$, and we are looking for the places where $F(tP_1 + sP_2) = 0$. The two trivial solutions are where $t = 0$ or $s = 0$. We think of $F(tP_1 + sP_2)$ as a polynomial in t, s. Since F is homogeneous of degree 3, all terms will have total degree 3. The cubic terms t^3 and s^3 are then precisely those that are in $F(tP_1)$ and $F(sP_2)$ respectively. We assumed that P_1, P_2 are on the curve, so we can subtract these terms, and look for the non-trivial zero of $F(tP_1 + sP_2) - F(tP_1) - F(sP_2)$. This is now a homogeneous polynomial of degree 3 without t^3 and s^3, that is, it is of the form $ts(c_t t + c_s s)$. Therefore the third solution is for $t = -c_s$ and $s = c_t$, i.e. $P_1 \star P_2 = -c_s P_1 + c_t P_2$.

Now, taking $P_1 = P, P_2 = O$ we get $(P \star O) + P + O = O$, so $P \star O = -P$. This explains the introduction of `Neg`.

Now that we have `Star` and `Neg` we can define the addition by $-(P_1 \star P_2)$. We call this `AddCalc`, as this is not quiet the addition function we will be using. This function has a problem, it vanishes on the diagonal $P_1 = P_2$, which is precisely what we need to multiply by 2. The reason is that already the function we denoted by `Star` vanishes on the diagonal, because when $P_1 = P_2$, the line through the points $tP_1 + sP_2$ is singular (it is just the point). However, this function is defined everywhere else. Luckily we can use the following trick, $P_1 + P_2 = (P_1 - Q) + (P_2 + Q)$. Specifically, we take Q to be the removed point $[0; 0; 1]$.

At this point we can introduce `Mul2` simply by $P + P$. We further define the function `MulNTwoDiv` which computes $[n]$ by expanding n in binary form (if it is divisible by 2, then $[n] = [2]\left[\frac{n}{2}\right]$, otherwise $[n] = [n-1] + \mathrm{id}$). This gives us $\left[p^{r-1}\right], [p^r]$ denoted by `Mulprm1` and `Mulpr` respectively.

5.4.5 Order Exactly p^r

The next step is to compute $\mathcal{O}(C_U[p^r])$, and its localization away from $[p^{r-1}](-)$. Then $\mathcal{O}(C_U[p^r])$ is described as the quotient of $R[x, z]$ (where $R = \mathbb{Q}[u_1]$) by some ideal. First, we want to restrict to the curve, which gives $F(x, 1, z)$, this explains the first element appended to `quoPr`. Second, we want only p^r-torsion points, that is $[p^r](x, 1, z) = O$. As $O = [0; 1; 0]$, this is equivalent to requiring that the first and last coordinates of $[p^r](x, 1, z)$ are 0, this explains the second and third element appended to `quoPr`.

Next, we actually wish our points to be of order exactly p^r (and not just p^r-torsion). To require this, we want that $\left[p^{r-1}\right](x, 1, z) \neq O$. This means that we to ensure that two values (the first and last coordinates) are not

0 together, so we can't simply invert both of them (as this requires that both are non-zero, we only need one of them to be non-zero). Note that if they vanish together, then any linear combination of them vanishes as well, but the other direction is not immediate. However, since there are only finitely many points, the number of different values attained is finite, and there are infinitely many different linear combinations over R, so there are linear combinations that vanish if and only if the two vanish together. We can then choose an arbitrary linear combination, and the result might be too large, but if the rank is correct then our choice was valid. Since we a priori know the ranks, as described in Section 5.4.1, we can verify this. It turns out that here taking the sum of the first and last coordinates works. Now, inverting an element $x \in S$ is equivalent to adding a formal element d and requiring it to be its inverse, i.e. $x^{-1}S = S\left[d\right]/(1 - dx)$. This explains the last element appended to `quoPr`.

We then define `IPr`, which is the ideal spanned by those elements, i.e. the quotient by it gives the points of order exactly p^r. Furthermore, `gbIPr` is a Gröbner basis for it. The `MonomialOrder` used in the definition of `BasePr` was found experimentally to yield faster computations (the heuristic is that x appears with the highest power, and it determines z so z will be a polynomial in x, and d will be a polynomial in x, z).

5.4.6 Spanning Pairs

Here we compute L_r^0. First we just take the tensor product of the previous with itself, which is given simply by duplicating all the variables to `x1,z1,x2,z2,d1,d2`, and all the relations (we use the Gröbner basis for faster computation). Then, we need to localize away from the kernel of $\left[p^{r-1}\right](x_1, 1, z_1) +_{C_U} \left[p^{r-1}\right](x_2, 1, z_2) \neq O$. We use the same trick, it turns out that here the sum is a again a valid linear combination, and we add another variable `d3`, and require it to be its inverse. The `MonomialOrder` used in the definition of `BaseLr0` was found experimentally to yield faster computations (the heuristic is similar to the previous).

5.4.7 The Code

```
-------- Basic Objects --------

p = 2;
```

```
r = 2;
R = QQ[u1];

A3 = R[X,Y,Z];
A33 = R[X1,Y1,Z1,X2,Y2,Z2];
A332 = A33[t,s];

F = X^3 - (Y^2 * Z + u1 * X * Y * Z + Y * Z^2);

Mt = matrix{{t}};
Ms = matrix{{s}};

O = matrix{{0, 1, 0}};
P = matrix{{X, Y, Z}};
P1 = matrix{{X1, Y1, Z1}};
P2 = matrix{{X2, Y2, Z2}};
RemovedP = matrix{{0, 0, 1}};



-------- Util Functions --------

DivideGcd = mat -> (
    g := gcd flatten entries mat;
    matrix {(flatten entries mat) // g}
);

comp = (a, b) -> (
    DivideGcd(sub(a, b))
);



-------- Functions on the Elliptic Curve --------

StarCalc = sub(F, Mt * P1 + Ms * P2) - sub(F, Mt * P1)
    ↪  - sub(F, Ms * P2);
StarCt = StarCalc_(t^2 * s);
StarCs = StarCalc_(t * s^2);
Star = -StarCs * P1 + StarCt * P2;
```

```
Neg = comp(Star, P|0);

AddCalc = comp(Neg, sub(Star, P1|P2));
MovedP1 = comp(AddCalc, P1|sub(Neg, RemovedP));
MovedP2 = comp(AddCalc, P2|RemovedP);
Add = comp(AddCalc, MovedP1|MovedP2);

Mul2 = comp(Add, P|P);

MulNTwoDiv = n -> (
    if n == 1 then P
    else if n % 2 == 0 then comp(Mul2, MulNTwoDiv(n /
      ↪ 2))
    else comp(Add, P|MulNTwoDiv(n-1))
);

Mulprm1 = MulNTwoDiv(p^(r-1));

Mulpr = MulNTwoDiv(p^r);



-------- Order Exactly p^r --------

BasePr = R[x,z,d, MonomialOrder=>{Weights => {1, 1000
   ↪ ,1000000}, Lex}];
p0 = matrix{{x, 1, z}};

quoPr = {};

quoPr = append(quoPr, sub(F, p0));

mulpr = sub(Mulpr, p0);
quoPr = append(quoPr, mulpr_(0,0));
quoPr = append(quoPr, mulpr_(0,2));

mulprm1 = sub(Mulprm1, p0);
```

```
quoPr = append(quoPr, 1 - d * (mulprm1_(0,0) +
   ↪ mulprm1_(0,2)));

IPr = ideal quoPr;

gbIPr = flatten entries gens gb IPr;



-------- Spanning Pairs --------

BaseLr0 = R[x1,z1,x2,z2,d1,d2,d3, MonomialOrder=>{
   ↪ Weights =>
   ↪ {1,1000,1,1000,1000000,1000000,3000000}, Lex}];
p01d = matrix{{x1, z1, d1}};
p02d = matrix{{x2, z2, d2}};
p1 = matrix{{x1, 1, z1}};
p2 = matrix{{x2, 1, z2}};

quoLr0 = {};

quoLr0 = quoLr0 | (flatten entries sub(matrix{gbIPr},
   ↪ p01d));
quoLr0 = quoLr0 | (flatten entries sub(matrix{gbIPr},
   ↪ p02d));

Mulprm1 = MulNTwoDiv(p^(r-1));

mulprm11 = sub(Mulprm1, p1);
mulprm12 = sub(Mulprm1, p2);
addMulprm11and2 = sub(Add, mulprm11|mulprm12)
quoLr0 = append(quoLr0, 1 - d3 * (addMulprm11and2_
   ↪ (0,0) + addMulprm11and2_(0,2)));

ILr0 = ideal quoLr0;

gbILr0 = flatten entries gens gb ILr0;
```

References

[Ada74] J. F. Adams. *Stable homotopy and generalised homology.* The University of Chicago press, 1974.

[AS69] M. F. Atiyah and G. B. Segal. Equivariant K-theory and completion. *J. Differential Geometry*, 3(1-18):9, 1969.

[Bal05] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *Journal für die reine und angewandte Mathematik*, 2005(588):149–168, 2005.

[Bal10] P. Balmer. Spectra, spectra, spectra–tensor triangular spectra versus zariski spectra of endomorphism rings. *Algebraic & Geometric Topology*, 10(3):1521–1563, 2010.

[Bea17] A Beaudry. Towards the homotopy of the $K(2)$-local moore spectrum at $p = 2$. *Advances in Mathematics*, 306:722–788, 2017.

[Hat17] A. Hatcher. *Vector Bundles and K-Theory.* November 2017. Version 2.2.

[HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Generalized group characters and complex oriented cohomology theories. *Journal of the American Mathematical Society*, 13(3):553–594, 2000.

[HS98] M. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory II. *Annals of mathematics*, 148(1):1–49, 1998.

[KM85] N. Katz and B. Mazur. *Arithmetic Moduli of Elliptic Curves.* Number 108 in Annals of Mathematics Studies. Princeton University Press, 1985.

[LT66] J. Lubin and J. Tate. Formal moduli for one-parameter formal lie groups. *Bulletin de la Société Mathématique de France*, 94:49–59, 1966.

[Lur10] J. Lurie. Chromatic homotopy theory. *252x course notes, http://www.math.harvard.edu/~lurie/252x.html*, 2010.

[Rav86] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres.* Academic Press, New York, 1986.

[Rav92] D. C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128. Princeton University Press, 1992.

[Rez98] C. Rezk. Notes on the Hopkins-Miller theorem. *Contemporary Mathematics*, 220:313–366, 1998.

[Seg68] G. B. Segal. Equivariant K-theory. *Publications mathématiques de l'IHÉS*, 34(1):129–151, 1968.

[Ser77] J.-P. Serre. *Linear representations of finite groups*, volume 42. Springer-Verlag, New York, 1977.

[Sil09] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*, volume 106. Springer-Verlag, New York, 2nd edition edition, 2009.