Thesis

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1 Overview Of Chromatic Homotopy Theory

Our goal is to motivate the introduction of Morava K-theory $\mathrm{K}(n)$ and Morava E-theory $\mathrm{E}(n)$, and other variants of Morava E-theory $\mathrm{E}(k,\Gamma)$, and their connection to formal group laws. There are different views on what chromatic homotopy theory is. **TODO write some more**

1.1 The Balmer Spectrum

We will start with an algebraic motivation. Let R be a noetherian ring. Consider the symmetric monoidal stable ∞-category $\mathrm{Ch}(R)$ of chain complexes on R. **TODO be more specific** It is then natural to ask how much information about R is encoded in the category $\mathrm{Ch}(R)$. We will try to recover $\mathrm{Spec}\, R$, as a topological space, from $\mathrm{Ch}(R)$.

Remark. Balmer's work actually recovers the structure sheaf as well. **TODO reference**

Definition 1. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finitely-generated projective modules. These objects are the compact objects in $\mathrm{Ch}(R)$, thus they can be defined categorically. Their full subcategory is denoted by $\mathrm{Ch}_{\mathrm{perf}}(R)$.

Definition 2. Let $\mathcal{C}$ be some symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

- $0 \in \mathcal{T}$,
- it is closed under cofibers (that is if $a \to b \to c$ is a cofiber sequence in $\mathcal{C}$ and $a, b \in \mathcal{T}$, then $c \in \mathcal{T}$),
- it is closed under retracts.

Example. Consider the case $\mathcal{C} = \mathrm{Ch}_{\mathrm{perf}}(R)$ (e.g. over $\mathbb{Z}$, bounded chain complexes of finitely-generated free abelian groups). Let $K_\bullet \in \mathrm{Ch}(R)$, and define $\mathcal{T}_{K_\bullet} = \{A_\bullet \in \mathrm{Ch}_{\mathrm{perf}}(R) \mid A_\bullet \otimes K_\bullet = 0\}$. We claim that $\mathcal{T}_{K_\bullet}$ is thick. Clearly $0 \in \mathcal{T}_{K_\bullet}$. Let $A_\bullet \to B_\bullet$ be a morphism between two complexes in $\mathcal{T}$. The cofiber of $A_\bullet \to B_\bullet$ is the pushout $B_\bullet \times_{A_\bullet} 0$. Since tensor is left, tensoring the cofiber with $K_\bullet$ is given by the pushout $(B_\bullet \otimes K_\bullet) \times_{A_\bullet \otimes K_\bullet} (0 \otimes K_\bullet) = 0 \times_0 0 = 0$, therefore the cofiber is indeed in $\mathcal{T}$. Lastly, if $A_\bullet \to B_\bullet \to A_\bullet$ is the identity and $B_\bullet \otimes K_\bullet$, we get that $\mathrm{id}_{A_\bullet \otimes K_\bullet}$ factors through 0, which implies that $A_\bullet \otimes K_\bullet$ is 0, so that $A_\bullet \in \mathcal{T}$.

Definition 3. A thick subcategory $\mathcal{T}$ is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring: As a set, $\mathrm{Spec}\, \mathcal{C} = \{\mathcal{P} \text{ prime ideal}\}$. For any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \mathrm{Spec}\, \mathcal{C} \mid S \cap \mathcal{P} = \emptyset\}$. We topologize $\mathrm{Spec}\, \mathcal{C}$ with the Zariski topology by declaring those to be the closed subsets. We also denote $\mathrm{Supp}(a) = V(\{a\})$.

Example. We continue the example of $\mathcal{T}_{K_\bullet}$. Clearly if $A_\bullet \otimes K_\bullet = 0$ then also $A_\bullet \otimes B_\bullet \otimes K_\bullet = 0$, so it is an ideal. Let $\mathfrak{p}$ be a prime ideal in R in the usual sense, and take $K_\bullet = R_\mathfrak{p}$ (concentrated at degree 0), then $A_\bullet \otimes K_\bullet = (A_\bullet)_\mathfrak{p}$ (level-wise localization). Now, assume that $0 = (A_\bullet \otimes B_\bullet)_\mathfrak{p} = (A_\bullet)_\mathfrak{p} \otimes_{R_\mathfrak{p}} (B_\bullet)_\mathfrak{p}$ Assume by negation that $(A_\bullet)_\mathfrak{p}, (B_\bullet)_\mathfrak{p} \neq 0$, i.e. $(A_n)_\mathfrak{p}, (B_m)_\mathfrak{p} \neq 0$ but $(A_n)_\mathfrak{p} \otimes_{R_\mathfrak{p}} (B_m)_\mathfrak{p} = 0$ for some n, m. Well, localization of projective is projective, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, so $(A_n)_\mathfrak{p} = 0$ or $(B_m)_\mathfrak{p} = 0$, which is a contradiction. Therefore $\mathcal{T}_\mathfrak{p}$ is a prime ideal.

Theorem 1. *The map* $\operatorname{Spec} R \to \operatorname{Spec}\left(\mathrm{Ch}_{\mathrm{perf}}(R)\right)$*, given by* $\mathfrak{p} \mapsto \mathcal{T}_{\mathfrak{p}} = \left\{A_\bullet \mid (A_\bullet)_{\mathfrak{p}} = 0\right\}$ *is a homeomorphism.*

TODO reference

Proposition 1. *Prime ideals pullback: Let* $F : \mathcal{C} \to \mathcal{D}$ *be an exact symmetric monoidal functor between two symmetric monoidal stable* ∞*-categories, and let* $\mathcal{P}$ *be a prime ideal in* $\mathcal{D}$*, then* $F^*\mathcal{P} = \{a \in \mathcal{C} \mid F(a) \in \mathcal{P}\}$ *is a prime ideal.*

Proof. Clearly $F(0) = 0 \in \mathcal{P}$ since F is exact, so $0 \in F^*\mathcal{P}$. Since F is exact, it sends cofibers to cofibers, so for $a, b \in F^*\mathcal{P}$, i.e. $F(a), F(b) \in \mathcal{P}$, and a map $a \to b$ we get $F(\mathrm{cofib}(a \to b)) = \mathrm{cofib}(F(a) \to F(b)) = \mathrm{cofib}(F(a) \to F(b)) \in \mathcal{P}$. Let $a \to b \to a$ be a retract, that is the composition is the identity, s.t. $b \in F^*\mathcal{P}$. We know that $F(a) \to F(b) \to F(a)$ is also a retract by functoriality, thus $F(a) \in \mathcal{P}$, that is $a \in F^*\mathcal{P}$. We conclude that $F^*\mathcal{P}$ is indeed a thick subcategory.

Let $a \in F^*\mathcal{P}, b \in \mathcal{C}$, since F is monoidal, $F(a \otimes b) = F(a) \otimes F(b) \in \mathcal{P}$, so $a \otimes b \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is an ideal.

Lastly, assume that $a \otimes b \in F^*\mathcal{P}$, again since F is monoidal, $F(a \otimes b) = F(a) \otimes F(b) \in \mathcal{P}$, so $a \in F^*\mathcal{P}$ or $b \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is a prime ideal. ☐

Now, recall that $\mathrm{Ch}(R) \cong \mathrm{Mod}_{HR}$, therefore we can reinterpret the above theorem as $\operatorname{Spec} R \cong \operatorname{Spec}\left(\mathrm{Mod}_{\mathrm{HR}}^{\mathrm{comp}}\right)$ (where the comp denotes the compact objects in the category). We shall turn this theorem into a definition:

Definition 4. Let R be an E_∞ ring spectrum. We define the *spectrum* of R to be $\operatorname{Spec} R = \operatorname{Spec}\left(\mathrm{Mod}_R^{\mathrm{comp}}\right)$.

A natural question to ask then is what is $\operatorname{Spec} \mathbb{S}$. Recall that $\mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$, the category of spectra, and that the compact objects in spectra are the finite spectra $\mathrm{Sp}^{\mathrm{fin}}$. So, unwinding the definitions, the question can rephrased as finding the prime ideals in $\mathrm{Sp}^{\mathrm{fin}}$, and their topology. Chromatic homotopy theory provides an answer to this question.

1.2 MU And Complex Orientations

Throughout this section, let E be a multiplicative cohomology theory (that is, equipped with a map $E \otimes E \to E$ which is associative and unital up to homotopy).

Consider the map $S^2 \to \mathrm{BU}(1)$ classifying the universal complex line bundle. Concretely, under the identifications $S^2 \cong \mathbb{C}\mathrm{P}^1$ and $\mathrm{BU}(1) \cong \mathbb{C}\mathrm{P}^\infty$, this map can be realized as the inclusion $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^\infty$. This map induces a map $\tilde{E}^2(\mathrm{BU}(1)) \to \tilde{E}^2\left(S^2\right) \cong \tilde{E}^0\left(S^0\right) \cong E^0(*) = E_0$. Since E is unital, there is a canonical generator $1 \in E_0$.

Definition 5. E is called *complex oriented* if the map $\tilde{E}^2(\mathrm{BU}(1)) \to E_0$ is surjective, equivalently, if 1 is in the image of that map. A choice of a lift $x \in \tilde{E}^2(\mathrm{BU}(1))$ of $1 \in E_0$ is called a *complex orientation*.

Example. Let R be some ring, and consider $\mathrm{H}R$. It is known that $\mathrm{H}R^*(\mathbb{C}\mathrm{P}^n) \cong R[x]/\left(x^{n+1}\right)$ and $\mathrm{H}R^*(\mathbb{C}\mathrm{P}^\infty) \cong R[[x]]$, where $|x| = 2$, and the maps induced by the inclusions of projective spaces maps x to x. Therefore we see that $x \in \mathrm{H}R^2(\mathrm{BU}(1))$ is mapped to $x \in \mathrm{H}R^2\left(S^2\right) = \mathbb{Z}\{x\}$, which is mapped to the generator of the reduced part of $\mathrm{H}R^0\left(S^0\right) = R \oplus R$. Therefore x is a complex orientation for $\mathrm{H}R$.

Example. Let K be complex K-theory, then we know that $\mathrm{K}_* = \mathbb{Z}\left[\beta^{\pm 1}\right]$ where β is the Bott element, with $|\beta| = 2$. It is also known (by Atiyah-Hirzebruch spectral sequence) that $\mathrm{K}^*(\mathbb{C}\mathrm{P}^n) \cong \mathrm{K}_*[t]/\left(t^{n+1}\right)$ and $\mathrm{K}^*(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{K}_*[[t]]$ (here $|t| = 0$), where the maps induced by the inclusions of projective spaces maps t to t. We deduce that $\beta t \in \mathrm{K}^2(\mathrm{BU}(1))$ is mapped to $\beta t \in \mathrm{K}^2\left(S^2\right) = \mathbb{Z}\{\beta t\}$, which is mapped to $t \in \mathrm{K}^0\left(S^0\right) = \mathbb{Z}\{t\}$, which is indeed the generator of the reduced part. Therefore $x = \beta t$ is complex orientation for K. **TODO write the reduced thing more clearly**

Example. Recall that MU is constructed as the colimit $\mathrm{MU} = \operatorname{colim} \mathrm{MU}(n)$. Also, $\mathrm{MU}(1) \cong \Sigma^{\infty-2}\mathrm{BU}(1)$. Therefore we get a canonical map $\Sigma^{\infty-2}\mathrm{BU}(1) \to \mathrm{MU}$, which gives a cohomology class $x_{\mathrm{MU}} \in \mathrm{MU}^2(\mathrm{BU}(1))$.

Proposition 2 (**TODO reference**)**.** x_{MU} *is indeed a complex orientation for* MU*.*

Theorem 2. MU *is the universal complex oriented cohomology theory: Let* E *be a multiplicative cohomology theory, then there is a bijection between (homotopy classes of) multiplicative maps* $\mathrm{MU} \to E$ *and complex orientations on* E*. The bijection is given in one direction by pulling back* x_{MU} *along a multiplicative map.*

Assume that E is complex oriented with a complex orientation x.

Theorem 3 ([Rav86, 4.1.4])**.** *As E_*-algebras, $E^*(\mathrm{BU}(1)) \cong E^*[[x]]$ and $E^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) \cong E^*[[y,z]]$.*

TODO maybe indicate the use of AHSS

The tensor product of complex line bundles is classified by a universal map $\mathrm{BU}(1) \times \mathrm{BU}(1) \to \mathrm{BU}(1)$. Therefore we get a map $E^*(\mathrm{BU}(1)) \to E^*(\mathrm{BU}(1) \times \mathrm{BU}(1))$, which is completely determined by the image of $x \in E^*[[x]]$ in $E^*[[y,z]]$ as above. Therefore, a choice of a complex orientation on E gives rise to an element $F_E(y,z) \in E^*[[y,z]]$.

Proposition 3 ([Rav86, 4.1.4])**.** *F_E is a formal group law on E_*. We call the height of F_E the* height *of E.*

Example. Consider again $\mathrm{H}R$. It is known that the tensor of complex line bundles induces the map $R[[x]] = \mathrm{H}R^*(\mathrm{BU}(1)) \to \mathrm{H}R^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = R[[y,z]]$ given by $x \mapsto y+z$. This is the additive formal group law. It is immediate that $[p] = px$. So for $R = \mathbb{Q}$ we get that the height of $\mathrm{H}\mathbb{Q}$ is 0, while for $R = \mathbb{F}_p$ we have $px = 0$ so the height of $\mathrm{H}\mathbb{F}_p$ is ∞.

Example. We return to the example of complex K-theory. It is known that the tensor of complex line bundles induces the map $\mathrm{K}_*[[t]] = \mathrm{K}^*(\mathrm{BU}(1)) \to \mathrm{K}^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = \mathrm{K}_*[[u,v]]$ given by $t \mapsto u+v+uv$. Note that to comply with the definition of the formal group law, we should use the isomorphism $\mathrm{K}^*(\mathrm{BU}(1)) \cong \mathrm{K}_*[[x]]$, i.e. the element $x = \beta t$. We get that $x = \beta t \mapsto \beta u + \beta v + \beta uv = y + z + \beta^{-1} yz = F_{\mathrm{K}}(y,z)$. By induction we prove that the n-series is $[n](x) = \beta\left(1+\beta^{-1}x\right)^n - \beta$. This is clear for $n=1$, and we have:

$$\begin{aligned}
[n+1](x) &= x + [n](x) + \beta^{-1} x [n](x) \\
&= x + \beta\left(1+\beta^{-1}x\right)^n - \beta + x\left(1+\beta^{-1}x\right)^n - x \\
&= \beta\left(1+\beta^{-1}x\right)\left(1+\beta^{-1}x\right)^n - \beta \\
&= \beta\left(1+\beta^{-1}x\right)^{n+1} - \beta
\end{aligned}$$

Example. By taking the cofiber of the multiplication-by-p map, we get a spectrum K/p, mod-p K-theory, with coefficients $(\mathrm{K}/p)_* = \mathbb{F}_p\left[\beta^{\pm 1}\right]$. It is evident that $F_{\mathrm{K}/p}(y,z) = y + z + \beta^{-1}yz$ as well. From the result above, it follows that $[p](x) = \beta\left(1+\beta^{-1}x\right)^p - \beta = \beta\left(1^p + \beta^{-p}x^p\right) - \beta = \beta^{-p}x^p$ which shows that the height is exactly 1.

A formal group law on E_* is the same data as a map from the Lazard ring L, so the complex orientation gives a map $L \to E_*$. In particular, since MU is complex oriented, there is a canonical map $L \to \mathrm{MU}_*$.

Theorem 4 (Quillen, [Rav86, 4.1.6])**.** *The canonical map $L \to \mathrm{MU}_*$ is an isomorphism.*

1.3 BP, Morava K-Theory And Morava E-Theory

A good principle in homotopy theory (and many other areas in math) is to do study it one prime at a time. This is possible in homotopy theory due to the arithmetic square **TODO reference**. So, let us fix a prime p. It turns out that once we p-localize MU to $\mathrm{MU}_{(p)}$, it splits:

Theorem 5 ([Rav86, 4.1.12])**.** *There exists an associative commutative ring spectrum* BP *(which depends on the prime p), which is a retract of $\mathrm{MU}_{(p)}$. The homotopy groups of* BP *are $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ where $|v_n| = 2(p^n - 1)$.*

For convenience we denote $v_0 = p$ (and indeed $|v_0| = 2\left(p^0 - 1\right) = 0$). Since BP is a retract of MU it comes with a map $\mathrm{MU} \to \mathrm{BP}$, that is a complex orientation.

Proposition 4 (**TODO reference**)**.** *The p-series of the formal group law associated to* BP *is $[p](x) = \sum v_n x^{p^n}$.*

Once we have BP, we can turn to the definition of Morava K-theory and Morava E-theory

Definition 6. Let $0 < n < \infty$. *Morava K-theory* at height n (and prime p, which is omitted in the notation), denoted by $\mathrm{K}(n)$, is the spectrum obtained by killing $p = v_0, \dots, v_{n-1}, v_{n+1}, \dots$ in BP and inverting v_n. Therefore $\mathrm{K}(n)_* = \mathbb{F}_p\left[v_n^{\pm 1}\right]$. We also define $\mathrm{K}0 = \mathrm{H}\mathbb{Q}$ and $\mathrm{K}\infty = \mathrm{H}\mathbb{F}_p$. Similarly, *Morava E-theory* at height n (and prime p), denoted by $\mathrm{E}(n)$, is the spectrum obtained by killing $v_{n+1}, v_{n+2}, \dots$ in BP and inverting v_n. Therefore $\mathrm{E}(n)_* = \mathbb{Z}_{(p)}\left[v_1, v_2, \dots v_{n-1}, v_n^{\pm 1}\right]$.

Since Morava K-theory and E-theory are obtained from BP by cofibers and filtered colimits, they are equipped with a map from BP, so they are also equipped with a complex orientation. It is then evident that the p-series associated to the formal group laws of $\mathrm{K}(n)$ and $\mathrm{E}(n)$ are $v_n x^{p^n}$ and $v_0 x + \dots v_n x^{p^n}$ respectively, and are therefore of height exactly n and height $\leq n$ respectively. (Note that by the example of $\mathrm{H}R$, this is also true for $\mathrm{K}(0)$ and $\mathrm{K}(\infty)$.)

We want to describe some properties of Morava K-theory. To do so we first need some definitions.

Definition 7. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenus element is invertible, equivalently it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree. An A_∞ ring spectrum E is a *field* if E_* is a graded field.

Example. Clearly $K(n)$ for $0 \leq n \leq \infty$ is a field.

Proposition 5. *A field E is has Kunneth, i.e. $E_*(X \otimes Y) \cong E_*(X) \otimes_{E_*} E_*(Y)$ for any spectra X, Y.*

Proposition 6 ([Lur10, 24])**.** *Let $E \neq 0$ be a complex oriented cohomology theory, whose formal group law has height exactly n, then $E \otimes \mathrm{K}(n) \neq 0$. Let E be a field s.t. $E \otimes \mathrm{K}(n) \neq 0$, then E admits the structure of a $\mathrm{K}(n)$-module. (Here $0 \leq n \leq \infty$.)*

Example. As we have seen before mod-p K-theory, K/p, has height exactly 1 and coefficients $(\mathrm{K}/p)_* = \mathbb{F}_p\left[\beta^{\pm 1}\right]$. It is also known that K, and K/p, are A_∞ ring spectra, from which it follows that K/p is a field. Therefore, we deduce that K/p is a $\mathrm{K}(1)$-module. Since $|\beta| = 2$ and $|v_1| = 2(p-1)$ it is free of rank $p-1$.

From this we also deduce some form of uniqueness for Morava K-theory:

Corollary. Let E be a field with $E_* \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$, which is also complex oriented with height exactly n, then $E \cong \mathrm{K}(n)$ (as spectra).

1.4 The Balmer Spectrum $\operatorname{Spec} \mathbb{S}_{(p)}$ And $\operatorname{Spec} \mathbb{S}$

References

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[Rav86] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheresa.* Academic Press, New York, 1986.