

Thesis

Shay Ben Moshe

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1 Overview Of Chromatic Homotopy Theory

Our goal is to motivate the introduction of Morava K-theory $K(n)$ and Morava E-theory $E(n)$, and other variants of Morava E-theory $E(k, \Gamma)$, and their connection to formal group laws. There are different views on what chromatic homotopy theory is. **TODO write some more**

1.1 The Balmer Spectrum

We will start with an algebraic motivation. Let R be a noetherian ring. Consider the symmetric monoidal stable ∞ -category $\text{Ch}(R)$ of chain complexes on R . **TODO be more specific** It is then natural to ask how much information about R is encoded in the category $\text{Ch}(R)$. We will try to recover $\text{Spec } R$, as a topological space, from $\text{Ch}(R)$.

Remark 1.1.1. Balmer's work actually recovers the structure sheaf as well. **TODO reference**

Definition 1.1.2. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finitely-generated projective modules. These objects are the compact objects in $\text{Ch}(R)$, thus they can be defined categorically. Their full subcategory is denoted by $\text{Ch}_{\text{perf}}(R)$.

Definition 1.1.3. Let \mathcal{C} be some symmetric monoidal stable ∞ -category. A full subcategory \mathcal{T} is *thick* if:

- $0 \in \mathcal{T}$,
- it is closed under cofibers (that is if $a \rightarrow b \rightarrow c$ is a cofiber sequence in \mathcal{C} and $a, b \in \mathcal{T}$, then $c \in \mathcal{T}$),
- it is closed under retracts.

Example 1.1.4. Consider the case $\mathcal{C} = \text{Ch}_{\text{perf}}(R)$ (e.g. over \mathbb{Z} , bounded chain complexes of finitely-generated free abelian groups). Let $K_{\bullet} \in \text{Ch}(R)$, and define $\mathcal{T}_{K_{\bullet}} = \{A_{\bullet} \in \text{Ch}_{\text{perf}}(R) \mid A_{\bullet} \otimes K_{\bullet} = 0\}$. We claim that $\mathcal{T}_{K_{\bullet}}$ is thick. Clearly $0 \in \mathcal{T}_{K_{\bullet}}$. Let $A_{\bullet} \rightarrow B_{\bullet}$ be a morphism between two complexes in \mathcal{T} . The cofiber of $A_{\bullet} \rightarrow B_{\bullet}$ is the pushout $B_{\bullet} \times_{A_{\bullet}} 0$. Since tensor is left, tensoring the cofiber with K_{\bullet} is given by the pushout $(B_{\bullet} \otimes K_{\bullet}) \times_{A_{\bullet} \otimes K_{\bullet}} (0 \otimes K_{\bullet}) = 0 \times_0 0 = 0$, therefore the cofiber is indeed in \mathcal{T} . Lastly, if $A_{\bullet} \rightarrow B_{\bullet} \rightarrow A_{\bullet}$ is the identity and $B_{\bullet} \otimes K_{\bullet}$, we get that $\text{id}_{A_{\bullet} \otimes K_{\bullet}}$ factors through 0, which implies that $A_{\bullet} \otimes K_{\bullet}$ is 0, so that $A_{\bullet} \in \mathcal{T}$.

Definition 1.1.5. A thick subcategory \mathcal{T} is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring: As a set, $\text{Spec } \mathcal{C} = \{\mathcal{P} \text{ prime ideal}\}$. For any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \text{Spec } \mathcal{C} \mid S \cap \mathcal{P} = \emptyset\}$. We topologize $\text{Spec } \mathcal{C}$ with the Zariski topology by declaring those to be the closed subsets. We also denote $\text{Supp}(a) = V(\{a\})$.

Example 1.1.6. We continue the example of $\mathcal{T}_{K_{\bullet}}$. Clearly if $A_{\bullet} \otimes K_{\bullet} = 0$ then also $A_{\bullet} \otimes B_{\bullet} \otimes K_{\bullet} = 0$, so it is an ideal. Let \mathfrak{p} be a prime ideal in R in the usual sense, and take $K_{\bullet} = R_{\mathfrak{p}}$ (concentrated at degree 0), then $A_{\bullet} \otimes K_{\bullet} = (A_{\bullet})_{\mathfrak{p}}$ (level-wise localization). Now, assume that $0 = (A_{\bullet} \otimes B_{\bullet})_{\mathfrak{p}} = (A_{\bullet})_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (B_{\bullet})_{\mathfrak{p}}$. Assume by negation that $(A_{\bullet})_{\mathfrak{p}}, (B_{\bullet})_{\mathfrak{p}} \neq 0$, i.e. $(A_n)_{\mathfrak{p}}, (B_m)_{\mathfrak{p}} \neq 0$ but $(A_n)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (B_m)_{\mathfrak{p}} = 0$ for some n, m . Well, localization of projective is projective, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, so $(A_n)_{\mathfrak{p}} = 0$ or $(B_m)_{\mathfrak{p}} = 0$, which is a contradiction. Therefore $\mathcal{T}_{\mathfrak{p}}$ is a prime ideal.

Theorem 1.1.7. *The map $\text{Spec } R \rightarrow \text{Spec}(\text{Ch}_{\text{perf}}(R))$, given by $\mathfrak{p} \mapsto \mathcal{T}_{\mathfrak{p}} = \{A_{\bullet} \mid (A_{\bullet})_{\mathfrak{p}} = 0\}$ is a homeomorphism.*

TODO reference

Proposition 1.1.8. *Prime ideals pullback: Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a reduced symmetric monoidal functor that preserves cofibers, between two symmetric monoidal stable ∞ -categories, and let \mathcal{P} be a prime ideal in \mathcal{D} , then $F^*\mathcal{P} = \{a \in \mathcal{C} \mid F(a) \in \mathcal{P}\}$ is a prime ideal.*

Proof. Clearly $F(0) = 0 \in \mathcal{P}$ since F is reduced, so $0 \in F^*\mathcal{P}$. Since F preserves cofibers, for $a, b \in F^*\mathcal{P}$, i.e. $F(a), F(b) \in \mathcal{P}$, and a map $a \rightarrow b$ we get $F(\text{cofib}(a \rightarrow b)) = \text{cofib}(F(a) \rightarrow F(b)) = \text{cofib}(F(a) \rightarrow F(b)) \in \mathcal{P}$. Let $a \rightarrow b \rightarrow a$ be a retract, that is the composition is the identity, s.t. $b \in F^*\mathcal{P}$. We know that $F(a) \rightarrow F(b) \rightarrow F(a)$ is also a retract by functoriality, thus $F(a) \in \mathcal{P}$, that is $a \in F^*\mathcal{P}$. We conclude that $F^*\mathcal{P}$ is indeed a thick subcategory.

Let $a \in F^*\mathcal{P}, b \in \mathcal{C}$, since F is monoidal, $F(a \otimes b) = F(a) \otimes F(b) \in \mathcal{P}$, so $a \otimes b \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is an ideal.

Lastly, assume that $a \otimes b \in F^*\mathcal{P}$, again since F is monoidal, $F(a \otimes b) = F(a) \otimes F(b) \in \mathcal{P}$, so $a \in F^*\mathcal{P}$ or $b \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is a prime ideal. \square

Now, recall that $\text{Ch}(R) \cong \text{Mod}_{\text{HR}}$, therefore we can reinterpret the above theorem as $\text{Spec } R \cong \text{Spec}(\text{Mod}_{\text{HR}}^{\text{comp}})$ (where the comp denotes the compact objects in the category). We shall turn this theorem into a definition:

Definition 1.1.9. Let R be an \mathbb{E}_{∞} -ring. We define the *spectrum* of R to be $\text{Spec } R = \text{Spec}(\text{Mod}_R^{\text{comp}})$.

A natural question to ask then is what is $\text{Spec } \mathbb{S}$. Recall that $\text{Mod}_{\mathbb{S}} = \text{Sp}$, the category of spectra, and that the compact objects in spectra are the finite spectra Sp^{fin} . So, unwinding the definitions, the question can be rephrased as finding the prime ideals in Sp^{fin} , and their topology. Chromatic homotopy theory provides an answer to this question.

1.2 MU And Complex Orientations

Throughout this section, let E be a multiplicative cohomology theory (that is, equipped with a map $E \otimes E \rightarrow E$ which is associative and unital up to homotopy).

Consider the map $S^2 \rightarrow \text{BU}(1)$ classifying the universal complex line bundle. Concretely, under the identifications $S^2 \cong \mathbb{CP}^1$ and $\text{BU}(1) \cong \mathbb{CP}^{\infty}$, this map can be realized as the inclusion $\mathbb{CP}^1 \subseteq \mathbb{CP}^{\infty}$. This map induces a map $\tilde{E}^2(\text{BU}(1)) \rightarrow \tilde{E}^2(S^2) \cong \tilde{E}^0(S^0) \cong E^0(*) = E_0$. Since E is unital, there is a canonical generator $1 \in E_0$.

Definition 1.2.1. E is called *complex oriented* if the map $\tilde{E}^2(\text{BU}(1)) \rightarrow E_0$ is surjective, equivalently, if 1 is in the image of that map. A choice of a lift $x \in \tilde{E}^2(\text{BU}(1))$ of $1 \in E_0$ is called a *complex orientation*.

Example 1.2.2. Let R be some ring, and consider HR . It is known that $\text{HR}^*(\mathbb{CP}^n) \cong R[x]/(x^{n+1})$ and $\text{HR}^*(\mathbb{CP}^{\infty}) \cong R[[x]]$, where $|x| = 2$, and the maps induced by the inclusions of projective spaces maps x to x . Therefore we see that $x \in \text{HR}^2(\text{BU}(1))$ is mapped to $x \in \text{HR}^2(S^2) = \mathbb{Z}\{x\}$, which is mapped to the generator of the reduced part of $\text{HR}^0(S^0) = R \oplus R$. Therefore x is a complex orientation for HR .

Example 1.2.3. Let K be complex K -theory, then we know that $K_* = \mathbb{Z}[\beta^{\pm 1}]$ where β is the Bott element, with $|\beta| = 2$. It is also known (by Atiyah-Hirzebruch spectral sequence) that $K^*(\mathbb{CP}^n) \cong K_*[t]/(t^{n+1})$ and $K^*(\mathbb{CP}^{\infty}) \cong K_*[[t]]$ (here $|t| = 0$), where the maps induced by the inclusions of projective spaces maps t to t . We deduce that $\beta t \in K^2(\text{BU}(1))$ is mapped to $\beta t \in K^2(S^2) = \mathbb{Z}\{\beta t\}$, which is mapped to $t \in K^0(S^0) = \mathbb{Z}\{t\}$, which is indeed the generator of the reduced part. Therefore $x = \beta t$ is complex orientation for K . **TODO write the reduced thing more clearly**

Example 1.2.4. Recall that MU is constructed as the colimit $\text{MU} = \text{colim } \text{MU}(n)$. Also, $\text{MU}(1) \cong \Sigma^{\infty-2}\text{BU}(1)$. Therefore we get a canonical map $\Sigma^{\infty-2}\text{BU}(1) \rightarrow \text{MU}$, which gives a cohomology class $x_{\text{MU}} \in \text{MU}^2(\text{BU}(1))$.

Proposition 1.2.5 (TODO reference). x_{MU} is indeed a complex orientation for MU .

Theorem 1.2.6. *MU is the universal complex oriented cohomology theory: Let E be a multiplicative cohomology theory, then there is a bijection between (homotopy classes of) multiplicative maps $\text{MU} \rightarrow E$ and complex orientations on E . The bijection is given in one direction by pulling back x_{MU} along a multiplicative map.*

Assume that E is complex oriented with a complex orientation x .

Theorem 1.2.7 ([Rav86, 4.1.4]). *As E_* -algebras, $E^*(\text{BU}(1)) \cong E^*[[x]]$ and $E^*(\text{BU}(1) \times \text{BU}(1)) \cong E^*[[y, z]]$.*

TODO maybe indicate the use of AHSS

The tensor product of complex line bundles is classified by a universal map $\text{BU}(1) \times \text{BU}(1) \rightarrow \text{BU}(1)$. Therefore we get a map $E^*(\text{BU}(1)) \rightarrow E^*(\text{BU}(1) \times \text{BU}(1))$, which is completely determined by the image of $x \in E^*[[x]]$ in $E^*[[y, z]]$ as above. Therefore, a choice of a complex orientation on E gives rise to an element $F_E(y, z) \in E^*[[y, z]]$.

Proposition 1.2.8 ([Rav86, 4.1.4]). *F_E is a formal group law on E_* . We call the height of F_E the height of E .*

Example 1.2.9. Consider again HR . It is known that the tensor of complex line bundles induces the map $R[[x]] = \text{HR}^*(\text{BU}(1)) \rightarrow \text{HR}^*(\text{BU}(1) \times \text{BU}(1)) = R[[y, z]]$ given by $x \mapsto y + z$. This is the additive formal group law. It is immediate that $[p] = px$. So for $R = \mathbb{Q}$ we get that the height of $\text{H}\mathbb{Q}$ is 0, while for $R = \mathbb{F}_p$ we have $px = 0$ so the height of $\text{H}\mathbb{F}_p$ is ∞ .

Example 1.2.10. We return to the example of complex K-theory. It is known that the tensor of complex line bundles induces the map $K_*[[t]] = K^*(\text{BU}(1)) \rightarrow K^*(\text{BU}(1) \times \text{BU}(1)) = K_*[[u, v]]$ given by $t \mapsto u + v + uv$. Note that to comply with the definition of the formal group law, we should use the isomorphism $K^*(\text{BU}(1)) \cong K_*[[x]]$, i.e. the element $x = \beta t$. We get that $x = \beta t \mapsto \beta u + \beta v + \beta uv = y + z + \beta^{-1}yz = F_K(y, z)$. By induction we prove that the n -series is $[n](x) = \beta(1 + \beta^{-1}x)^n - \beta$. This is clear for $n = 1$, and we have:

$$\begin{aligned} [n+1](x) &= x + [n](x) + \beta^{-1}x[n](x) \\ &= x + \beta(1 + \beta^{-1}x)^n - \beta + x(1 + \beta^{-1}x)^n - x \\ &= \beta(1 + \beta^{-1}x)(1 + \beta^{-1}x)^n - \beta \\ &= \beta(1 + \beta^{-1}x)^{n+1} - \beta \end{aligned}$$

Example 1.2.11. By taking the cofiber of the multiplication-by- p map, we get a spectrum K/p , mod- p K-theory, with coefficients $(K/p)_* = \mathbb{F}_p[\beta^{\pm 1}]$. It is evident that $F_{K/p}(y, z) = y + z + \beta^{-1}yz$ as well. From the result above, it follows that $[p](x) = \beta(1 + \beta^{-1}x)^p - \beta = \beta(1^p + \beta^{-p}x^p) - \beta = \beta^{-p}x^p$ which shows that the height is exactly 1.

A formal group law on E_* is the same data as a map from the Lazard ring L , so the complex orientation gives a map $L \rightarrow E_*$. In particular, since MU is complex oriented, there is a canonical map $L \rightarrow \text{MU}_*$.

Theorem 1.2.12 (Quillen, [Rav86, 4.1.6]). *The canonical map $L \rightarrow \text{MU}_*$ is an isomorphism.*

1.3 BP, Morava K-Theory And Morava E-Theory

A good principle in homotopy theory (and many other areas in math) is to do study it one prime at a time. This is possible in homotopy theory due to the arithmetic square **TODO reference**. So, let us fix a prime p . It turns out that once we p -localize MU to $\text{MU}_{(p)}$, it splits:

Theorem 1.3.1 ([Rav86, 4.1.12]). *There exists an associative commutative ring spectrum BP (which depends on the prime p), which is a retract of $\text{MU}_{(p)}$. The homotopy groups of BP are $\text{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ where $|v_n| = 2(p^n - 1)$.*

For convenience we denote $v_0 = p$ (and indeed $|v_0| = 2(p^0 - 1) = 0$). Since BP is a retract of MU it comes with a map $\text{MU} \rightarrow \text{BP}$, that is a complex orientation.

Proposition 1.3.2 (TODO reference). *The p -series of the formal group law associated to BP is $[p](x) = \sum v_n x^{p^n}$.*

Once we have BP, we can turn to the definition of Morava K-theory and Morava E-theory

Definition 1.3.3. Let $0 < n < \infty$. *Morava K-theory* at height n and prime p , denoted by $K(p, n)$ or $K(n)$ when the prime is clear, is the spectrum obtained by killing $p = v_0, \dots, v_{n-1}, v_{n+1}, \dots$ in BP and inverting v_n . Therefore $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$. We also define $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$. Similarly, *Morava E-theory* at height n and prime p , denoted by $E(p, n)$ or $E(n)$, is the spectrum obtained by killing v_{n+1}, v_{n+2}, \dots in BP and inverting v_n . Therefore $E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_{n-1}, v_n^{\pm 1}]$.

Since Morava K-theory and E-theory are obtained from BP by cofibers and filtered colimits, they are equipped with a map from BP, so they are also equipped with a complex orientation. It is then evident that the p -series associated to the formal group laws of $K(n)$ and $E(n)$ are $v_n x^{p^n}$ and $v_0 x + \dots v_n x^{p^n}$ respectively, and are therefore of height exactly n and height $\leq n$ respectively. (Note that by the example of HR , this is also true for $K(0)$ and $K(\infty)$.)

We want to describe some properties of Morava K-theory. To do so we first need some definitions.

Definition 1.3.4. Let R be an evenly graded ring. R is called a *graded field* if every non-zero homogenous element is invertible, equivalently it is a field F concentrated at degree 0, or $F[\beta^{\pm 1}]$ for β of positive even degree. An \mathbb{A}_∞ -ring E is a *field* if E_* is a graded field.

Example 1.3.5. Clearly $K(n)$ for $0 \leq n \leq \infty$ is a field.

Proposition 1.3.6. A field E has *Kunneth*, i.e. $E_*(X \otimes Y) \cong E_*(X) \otimes_{E_*} E_*(Y)$ for any spectra X, Y .

Proposition 1.3.7 ([Lur10, 24]). Let $E \neq 0$ be a complex oriented cohomology theory, whose formal group law has height exactly n , then $E \otimes K(n) \neq 0$. Let E be a field s.t. $E \otimes K(n) \neq 0$, then E admits the structure of a $K(n)$ -module. (Here $0 \leq n \leq \infty$.)

Example 1.3.8. As we have seen before mod- p K-theory, K/p , has height exactly 1 and coefficients $(K/p)_* = \mathbb{F}_p[\beta^{\pm 1}]$. It is also known that K , and K/p , are \mathbb{A}_∞ ring spectra, from which it follows that K/p is a field. Therefore, we deduce that K/p is a $K(1)$ -module. Since $|\beta| = 2$ and $|v_1| = 2(p-1)$ it is free of rank $p-1$.

From this we also deduce some form of uniqueness for Morava K-theory:

Corollary 1.3.9. Let E be a field with $E_* \cong \mathbb{F}_p[v_n^{\pm 1}]$, which is also complex oriented with height exactly n , then $E \cong K(n)$ (as spectra).

1.4 The Balmer Spectrum $\text{Spec } \mathbb{S}_{(p)}$ And $\text{Spec } \mathbb{S}$

We are now in a position to state the answer for $\text{Spec } \mathbb{S}$. However, it will be easier to state it first for $\text{Spec } \mathbb{S}_{(p)}$, and then pullback prime ideals. We know that $\text{Mod}_{\mathbb{S}_{(p)}} = \text{Sp}_{(p)}$, and the compact objects there are $\text{Sp}_{(p)}^{\text{fin}}$, the p -localizations of finite spectra.

Proposition 1.4.1. Let $\mathcal{T}_E = \ker E_* = \{X \in \text{Sp}_{(p)}^{\text{fin}} \mid E_*(X) = 0\}$ (equivalently $X \otimes E = 0$) i.e. the E -acyclics, then \mathcal{T}_E is thick.

Proof. The exact same proof from $\text{Ch}_{\text{perf}}(R)$ works. □

Definition 1.4.2. We define $\mathcal{C}_{p, \geq n} = \mathcal{T}_{K(n-1)}$, the $K(n-1)$ -acyclics. By the above it is thick. Also, $\mathcal{C}_{p, \geq 0} = \text{Sp}_{(p)}^{\text{fin}}$ and $\mathcal{C}_{p, \geq \infty} = \{0\}$, which are trivially thick. When the prime is clear, we will denote by $\mathcal{C}_{\geq n}$.

Proposition 1.4.3 ([Lur10, 26]). For $X \in \text{Sp}_{(p)}^{\text{fin}}$, if $K(n)_*(X) = 0$ then $K(n-1)_*(X) = 0$.

Definition 1.4.4. We say that a spectrum $X \in \text{Sp}_{(p)}^{\text{fin}}$ is of *type* n (possibly ∞), if its first non-zero Morava K-theory homology is $K(n)$.

Corollary 1.4.5. $\mathcal{C}_{\geq n}$ is the full subcategory of finite p -local spectra of type $\geq n$ (i.e. $\{X \in \text{Sp}_{(p)}^{\text{fin}} \mid \forall m < n : K(m)_*(X) = 0\}$). Thus clearly $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$.

Proposition 1.4.6 (TODO reference). The inclusions are proper $\mathcal{C}_{\geq n+1} \subsetneq \mathcal{C}_{\geq n}$.

Proposition 1.4.7. *If $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ is not contractible, then X has a finite type. Therefore $\bigcap_{n < \infty} \mathcal{C}_{\geq n} = \{0\} = \mathcal{C}_{\geq \infty}$.*

Proof. Let X be non-contractible. Then $\mathrm{HZ}_*(X) \neq 0$. Let m be the first non-zero degree. Using the universal coefficient theorem and the fact that the spectrum is p -local we get that $(\mathrm{H}\mathbb{F}_p)_m(X) \neq 0$, thus $(\mathrm{H}\mathbb{F}_p)_*(X) \neq 0$. Since X is finite, $(\mathrm{H}\mathbb{F}_p)_*(X)$ is bounded. Atiyah-Hirzebruch spectral sequence for X with cohomology $K(n)$ has E^2 -page given by $E_{p,q}^2 = H_p(X; K(n)_q(*))$. Since $K(n)_q = \mathbb{F}_p$ for $q = 0 \bmod 2(p^n - 1)$ and 0 otherwise, we see that the rows $q = 0 \bmod 2(p^n - 1)$ are $(\mathrm{H}\mathbb{F}_p)_*(X)$, and the others are 0. Therefore if we take n such that the period $2(p^n - 1)$ is larger than the bound on $(\mathrm{H}\mathbb{F}_p)_*(X)$, then all differentials have either source or target 0. Thus, the spectral sequence collapses at the E^2 -page, and since $(\mathrm{H}\mathbb{F}_p)_*(X) \neq 0$, we get that $K(n)(X) \neq 0$, i.e. X has type $< n$. \square

Proposition 1.4.8. $\mathcal{C}_{\geq n}$ is a prime ideal (note that $\mathcal{C}_{\geq 0}$ is not a proper subcategory, thus only for $n = 1, 2, \dots, \infty$).

Proof. For X, Y by Kunnetth we have $K(n-1)_*(X \otimes Y) = K(n-1)_*(X) \otimes K(n-1)_*(Y)$. Therefore, if $X \in \mathcal{C}_{\geq n}$, i.e. the homology vanishes, then so does the homology of $X \otimes Y$, i.e. $X \otimes Y \in \mathcal{C}_{\geq n}$, so $\mathcal{C}_{\geq n}$ is an ideal. If $X \otimes Y \in \mathcal{C}_{\geq n}$ then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces), so $\mathcal{C}_{\geq n}$ is a prime ideal. \square

Theorem 1.4.9 (Thick Subcategory Theorem [HS98]). *If \mathcal{T} is a thick subcategory of $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$, then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $n = 0, 1, 2, \dots, \infty$.*

Remark 1.4.10. The proof relies on a major theorem called the Nilpotence Theorem.

Corollary 1.4.11. $\mathrm{Spec} \mathbb{S}_{(p)} = \{\mathcal{C}_{\geq 1}, \mathcal{C}_{\geq 2}, \dots, \mathcal{C}_{\geq \infty}\}$, and the topology is such that the closed subsets are chains $\{\mathcal{C}_{\geq k}, \mathcal{C}_{\geq k+1}, \dots, \mathcal{C}_{\geq \infty}\}$ for some $k \geq 1$.

Proof. Follows immediately from the previous results. \square

We now want to move to finding $\mathrm{Spec} \mathbb{S}$. Note that the p -localization functor $L_{(p)}$ is a Bousfield localization. As such it is left (its right adjoint is the inclusion), and in particular preserves cofibers. It also clearly sends 0 to 0, i.e. reduced. Now, since $L_{(p)}$ is smashing, i.e. $L_{(p)}X = X \otimes \mathbb{S}_{(p)}$, we also get that it is symmetric monoidal. As we have seen in 1.1.8, under these conditions we can pullback primes. Therefore $\mathcal{P}_{p, \geq n} = L_{(p)}^* \mathcal{C}_{p, \geq n} = \{X \in \mathrm{Sp}^{\mathrm{fin}} \mid K(n-1)_*(X_{(p)}) = 0\}$ and $\mathcal{P}_{p, \geq \infty} = L_{(p)}^* \mathcal{C}_{p, \geq \infty} = \{X \in \mathrm{Sp}^{\mathrm{fin}} \mid X_{(p)} = 0\}$ are prime ideals. Note that $\mathcal{P}_{p, \geq 1} = \{X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_*(X_{(p)}) = 0\} = \{X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_*(X) = 0\}$ **TODO explain** so it is independent of p , and we will denote it by $\mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}}$.

Theorem 1.4.12 (TODO explain/reference). $\mathrm{Spec} \mathbb{S} = \{\mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}}\} \cup \bigcup_p \{\mathcal{P}_{p, \geq 2}, \dots, \mathcal{P}_{p, \geq \infty}\}$, and the topology is such that the closed subsets finite unions of chains $\{\mathcal{P}_{\geq k}, \mathcal{P}_{\geq k+1}, \dots, \mathcal{P}_{\geq \infty}\}$ for some $k \geq 1$ (i.e. they may include $\mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}}$). **TODO diagram**

TODO regarding the topology, maybe I should prove that the pullback is also continuous?

References

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