Thesis

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1 Overview Chromatic Homotopy Theory

Chromatic homotopy theory is an organizing principle for stable homotopy theory. Our goal is to motivate the introduction of Morava K-Theory $\mathrm{K}(n)$ and Morava E-Theory $\mathrm{E}(n)$, and other variants of Morava E-Theory $\mathrm{E}(k, \Gamma)$, and their connection to formal group laws. There are different views on what chromatic homotopy theory is.

1.1 The Balmer Spectrum

We will start with an algebraic motivation. Let R be a noetherian ring. Consider the symmetric monoidal stable ∞-category $\mathrm{Ch}(R)$ of chain complexes on R. **TODO be more specific** It is then natural to ask how much information about R is encoded in the category $\mathrm{Ch}(R)$. We will try to recover $\operatorname{Spec} R$, as a topological space, from $\mathrm{Ch}(R)$.

Remark. Balmer's work actually recovers the structure sheaf as well. **TODO reference**

Definition 1. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finitely-generated projective modules. These objects are the compact objects in $\mathrm{Ch}(R)$, thus they can be defined categorically. Their full subcategory is denoted by $\mathrm{Ch}_{\mathrm{perf}}(R)$.

Definition 2. Let $\mathcal{C}$ be some symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

- $0 \in \mathcal{T}$,
- it is closed under cofibers (that is if $a \to b \to c$ is a cofiber sequence in $\mathcal{C}$ and $a, b \in \mathcal{T}$, then $c \in \mathcal{T}$),
- it is closed under retracts.

Example. Consider the case $\mathcal{C} = \mathrm{Ch}_{\mathrm{perf}}(R)$ (e.g. over $\mathbb{Z}$, bounded chain complexes of finitely-generated free abelian groups). Let $K_\bullet \in \mathrm{Ch}(R)$, and define $\mathcal{T}_{K_\bullet} = \{A_\bullet \in \mathrm{Ch}_{\mathrm{perf}}(R) \mid A_\bullet \otimes K_\bullet = 0\}$. We claim that $\mathcal{T}_{K_\bullet}$ is thick. Clearly $0 \in \mathcal{T}_{K_\bullet}$. Let $A_\bullet \to B_\bullet$ be a morphism between two complexes in $\mathcal{T}$. The cofiber of $A_\bullet \to B_\bullet$ is the pushout $B_\bullet \times_{A_\bullet} 0$. Since tensor is left, tensoring the cofiber with $K_\bullet$ is given by the pushout $(B_\bullet \otimes K_\bullet) \times_{A_\bullet \otimes K_\bullet} (0 \otimes K_\bullet) = 0 \times_0 0 = 0$, therefore the cofiber is indeed in $\mathcal{T}$. Lastly, if $A_\bullet \to B_\bullet \to A_\bullet$ is the identity and $B_\bullet \otimes K_\bullet$, we get that $\mathrm{id}_{A_\bullet \otimes K_\bullet}$ factors through 0, which implies that $A_\bullet \otimes K_\bullet$ is 0, so that $A_\bullet \in \mathcal{T}$.

Definition 3. A thick subcategory $\mathcal{T}$ is an *ideal* if $a \in \mathcal{T}, b \in \mathcal{C} \implies a \otimes b \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $a \otimes b \in \mathcal{T} \implies a \in \mathcal{T}$ or $b \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring: As a set, $\operatorname{Spec} \mathcal{C} = \{\mathcal{P} \text{ prime ideal}\}$. For any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \operatorname{Spec} \mathcal{C} \mid S \cap \mathcal{P} = \emptyset\}$. We topologize $\operatorname{Spec} \mathcal{C}$ with the Zariski topology by declaring those to be the closed subsets. We also denote $\operatorname{Supp}(a) = V(\{a\})$.

Theorem 1. *There is a homeomorphism* $\operatorname{Spec} R \to \operatorname{Spec}(\mathrm{Ch}_{\mathrm{perf}}(R))$, *given by* $\mathfrak{p} \mapsto \mathcal{T}_{\mathfrak{p}} = \left\{A_\bullet \mid (A_\bullet)_{\mathfrak{p}} = 0\right\}$.

TODO reference

Now, recall that $\mathrm{Ch}(R) \cong \mathrm{Mod}_{\mathrm{H}R}$, therefore we can reinterpret the above theorem as $\operatorname{Spec} R \cong \operatorname{Spec}(\mathrm{Mod}_{\mathrm{H}R}^{\mathrm{comp}})$ (where the comp denotes the compact objects in the category). We shall turn this theorem into a definition:

Definition 4. Let R be an $\mathbb{E}_\infty$ ring spectrum. We define the *spectrum* of R to be $\operatorname{Spec} R = \operatorname{Spec}(\mathrm{Mod}_R^{\mathrm{comp}})$.

A natural question to ask then is what is $\operatorname{Spec} \mathbb{S}$. Recall that $\mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$, the category of spectra, and that the compact objects in spectra are the finite spectra $\mathrm{Sp}^{\mathrm{fin}}$. So, unwinding the definitions, the question can rephrased as finding the prime ideals in $\mathrm{Sp}^{\mathrm{fin}}$, and their topology. Chromatic homotopy theory provides an answer to this question.