

# Thesis

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## 1 Overview of Chromatic Homotopy Theory

Our goal is to motivate the introduction of Morava K-theory  $K(n)$  and Morava E-theory  $E(n)$ , and other variants of Morava E-theory  $E(k, \Gamma)$ , and their connection to formal group laws. There are different views on what chromatic homotopy theory is. **TODO write some more**

### 1.1 The Balmer Spectrum

We will start with an algebraic motivation. Let  $R$  be a noetherian ring. Consider the symmetric monoidal stable  $\infty$ -category  $\mathrm{Ch}(R)$  of chain complexes on  $R$ . **TODO be more specific** It is then natural to ask how much information about  $R$  is encoded in the category  $\mathrm{Ch}(R)$ . We will try to recover  $\mathrm{Spec} R$ , as a topological space, from  $\mathrm{Ch}(R)$ .

*Remark 1.1.1.* Balmer's work actually recovers the structure sheaf as well. **TODO reference**

**Definition 1.1.2.** A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finitely-generated projective modules. These objects are the compact objects in  $\mathrm{Ch}(R)$ , thus they can be defined categorically. Their full subcategory is denoted by  $\mathrm{Ch}_{\mathrm{perf}}(R)$ .

**Definition 1.1.3.** Let  $\mathcal{C}$  be some symmetric monoidal stable  $\infty$ -category. A full subcategory  $\mathcal{T}$  is *thick* if:

- $0 \in \mathcal{T}$ ,
- it is closed under cofibers,
- it is closed under retracts.

*Example 1.1.4.* Consider the case  $\mathcal{C} = \mathrm{Ch}_{\mathrm{perf}}(R)$  (e.g. over  $\mathbb{Z}$ , bounded chain complexes of finitely-generated free abelian groups). Let  $K \in \mathrm{Ch}(R)$ , and define  $\mathcal{T}_K = \{A \in \mathrm{Ch}_{\mathrm{perf}}(R) \mid A \otimes K = 0\}$ . We claim that  $\mathcal{T}_K$  is thick.

Clearly  $0 \in \mathcal{T}_K$ . Let  $A \rightarrow B$  be a morphism between two complexes in  $\mathcal{T}$ . The cofiber of  $A \rightarrow B$  is the pushout  $B \times_A 0$ . Since tensor is left, tensoring the cofiber with  $K$  is given by the pushout  $(B \otimes K) \times_{A \otimes K} (0 \otimes K) = 0 \times_0 0 = 0$ , therefore the cofiber is indeed in  $\mathcal{T}$ . Lastly, if  $A \rightarrow B \rightarrow A$  is the identity and  $B \otimes K$ , we get that  $\text{id}_{A \otimes K}$  factors through 0, which implies that  $A \otimes K$  is 0, so that  $A \in \mathcal{T}$ .

**Definition 1.1.5.** A thick subcategory  $\mathcal{T}$  is an *ideal* if  $A \in \mathcal{T}, B \in \mathcal{C} \implies A \otimes B \in \mathcal{T}$ . Furthermore, it is a *prime ideal* if it is a proper subcategory, and  $A \otimes B \in \mathcal{T} \implies A \in \mathcal{T}$  or  $B \in \mathcal{T}$ . The *spectrum* of the category is defined similarly to the classical spectrum of a ring: As a set,  $\text{Spec } \mathcal{C} = \{\mathcal{P} \text{ prime ideal}\}$ . For any family of objects  $S \subseteq \mathcal{C}$  we define  $V(S) = \{\mathcal{P} \in \text{Spec } \mathcal{C} \mid S \cap \mathcal{P} = \emptyset\}$ . We topologize  $\text{Spec } \mathcal{C}$  with the Zariski topology by declaring those to be the closed subsets. We also denote  $\text{Supp}(A) = V(\{A\})$ .

*Example 1.1.6.* We continue the example of  $\mathcal{T}_K$ . Clearly if  $A \otimes K = 0$  then also  $A \otimes B \otimes K = 0$ , so it is an ideal. Let  $\mathfrak{p}$  be a prime ideal in  $R$  in the usual sense, and take  $K = R_{\mathfrak{p}}$  (concentrated at degree 0), then  $A \otimes K = A_{\mathfrak{p}}$  (level-wise localization). **TODO consider actually proving** We will omit the proof that  $\mathcal{T}_K$  is a prime, but we shall prove something weaker, namely only the case where  $A, B$  are bounded complexes of finitely generate projective modules (and not merely quasi-isomorphic to such complexes). Assume then that  $0 = (A \otimes B)_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} B_{\mathfrak{p}}$ . Assume by negation that  $A_{\mathfrak{p}}, B_{\mathfrak{p}} \neq 0$ , i.e.  $(A_n)_{\mathfrak{p}}, (B_m)_{\mathfrak{p}} \neq 0$  but  $(A_n)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (B_m)_{\mathfrak{p}} = 0$  for some  $n, m$ . Well, localization of projective is projective, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, so  $(A_n)_{\mathfrak{p}} = 0$  or  $(B_m)_{\mathfrak{p}} = 0$ , which is a contradiction. Therefore  $\mathcal{T}_{\mathfrak{p}}$  is a prime ideal.

**Theorem 1.1.7.** *The map  $\text{Spec } R \rightarrow \text{Spec}(\text{Ch}_{\text{perf}}(R))$ , given by  $\mathfrak{p} \mapsto \mathcal{T}_{\mathfrak{p}} = \{A \mid A_{\mathfrak{p}} = 0\}$  is a homeomorphism.*

**TODO reference**

**Proposition 1.1.8.** *Prime ideals pullback: Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a reduced symmetric monoidal functor that preserves cofibers, between two symmetric monoidal stable  $\infty$ -categories, and let  $\mathcal{P}$  be a prime ideal in  $\mathcal{D}$ , then  $F^*\mathcal{P} = \{A \in \mathcal{C} \mid F(A) \in \mathcal{P}\}$  is a prime ideal.*

*Proof.* Clearly  $F(0) = 0 \in \mathcal{P}$  since  $F$  is reduced, so  $0 \in F^*\mathcal{P}$ . Since  $F$  preserves cofibers, for  $A, B \in F^*\mathcal{P}$ , i.e.  $F(A), F(B) \in \mathcal{P}$ , and a map  $A \rightarrow B$  we get  $F(\text{cofib}(A \rightarrow B)) = \text{cofib}(F(A) \rightarrow F(B)) = \text{cofib}(F(A) \rightarrow F(B)) \in \mathcal{P}$ . Let  $A \rightarrow B \rightarrow A$  be a retract, that is the composition is the identity, s.t.  $B \in F^*\mathcal{P}$ . We know that  $F(A) \rightarrow F(B) \rightarrow F(A)$  is also a retract by functoriality, thus  $F(A) \in \mathcal{P}$ , that is  $A \in F^*\mathcal{P}$ . We conclude that  $F^*\mathcal{P}$  is indeed a thick subcategory.

Let  $A \in F^*\mathcal{P}, B \in \mathcal{C}$ , since  $F$  is monoidal,  $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{P}$ , so  $A \otimes B \in F^*\mathcal{P}$ , that is  $F^*\mathcal{P}$  is an ideal.

Lastly, assume that  $A \otimes B \in F^*\mathcal{P}$ , again since  $F$  is monoidal,  $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{P}$ , so  $A \in F^*\mathcal{P}$  or  $B \in F^*\mathcal{P}$ , that is  $F^*\mathcal{P}$  is a prime ideal.  $\square$

Now, recall that  $\text{Ch}(R) \cong \text{Mod}_{HR}$ , therefore we can reinterpret the above theorem as  $\text{Spec } R \cong \text{Spec}(\text{Mod}_{HR}^{\text{comp}})$  (where the comp denotes the compact objects in the category). We shall turn this theorem into a definition:

**Definition 1.1.9.** Let  $R$  be an  $\mathbb{E}_{\infty}$ -ring. We define the *spectrum* of  $R$  to be  $\text{Spec } R = \text{Spec}(\text{Mod}_R^{\text{comp}})$ .

A natural question to ask then is what is  $\text{Spec } S$ . Recall that  $\text{Mod}_S = \text{Sp}$ , the category of spectra, and that the compact objects in spectra are the finite spectra  $\text{Sp}^{\text{fin}}$ . So, unwinding the definitions, the question can be rephrased as finding the prime ideals in  $\text{Sp}^{\text{fin}}$ , and their topology. Chromatic homotopy theory provides an answer to this question.

## 1.2 MU and Complex Orientations

Throughout this section, let  $E$  be a multiplicative cohomology theory (that is, equipped with a map  $E \otimes E \rightarrow E$  and  $1 \in E_0$ , which is associative and unital after taking homotopy groups).

Consider the map  $S^2 \rightarrow \text{BU}(1)$  classifying the universal complex line bundle. Concretely, under the identifications  $S^2 \cong \mathbb{CP}^1$  and  $\text{BU}(1) \cong \mathbb{CP}^{\infty}$ , this map can be realized as the inclusion  $\mathbb{CP}^1 \subseteq \mathbb{CP}^{\infty}$ . This map induces a map  $\tilde{E}^2(\text{BU}(1)) \rightarrow \tilde{E}^2(S^2) \cong \tilde{E}^0(S^0) \cong E^0(*) = E_0$ . Since  $E$  is unital, there is a canonical generator  $1 \in E_0$ .

**Definition 1.2.1.**  $E$  is called *complex oriented* if the map  $\tilde{E}^2(\mathrm{BU}(1)) \rightarrow E_0$  is surjective, equivalently, if 1 is in the image of that map. A choice of a lift  $x \in \tilde{E}^2(\mathrm{BU}(1))$  of  $1 \in E_0$  is called a *complex orientation*.

*Example 1.2.2.* Let  $R$  be some ring, and consider  $HR$ . It is known that  $HR^*(\mathbb{CP}^n) \cong R[x]/(x^{n+1})$  and  $HR^*(\mathbb{CP}^\infty) \cong R[[x]]$ , where  $|x| = 2$ , and the maps induced by the inclusions of projective spaces maps  $x$  to  $x$ . Therefore we see that  $x \in HR^2(\mathrm{BU}(1))$  is mapped to  $x \in HR^2(S^2) = \mathbb{Z}\{x\}$ , which is mapped to the generator of the reduced part of  $HR^0(S^0) = R \oplus R$ . Therefore  $x$  is a complex orientation for  $HR$ .

*Example 1.2.3.* Let  $K$  be complex K-theory, then we know that  $K_* = \mathbb{Z}[\beta^{\pm 1}]$  where  $\beta$  is the Bott element, with  $|\beta| = 2$ . It is also known (by Atiyah-Hirzebruch spectral sequence) that  $K^*(\mathbb{CP}^n) \cong K_*[t]/(t^{n+1})$  and  $K^*(\mathbb{CP}^\infty) \cong K_*[[t]]$  (here  $|t| = 0$ ), where the maps induced by the inclusions of projective spaces maps  $t$  to  $t$ . We deduce that  $\beta t \in K^2(\mathrm{BU}(1))$  is mapped to  $\beta t \in K^2(S^2) = \mathbb{Z}\{\beta t\}$ , which is mapped to  $t \in K^0(S^0) = \mathbb{Z}\{t\}$ , which is indeed the generator of the reduced part. Therefore  $x = \beta t$  is complex orientation for  $K$ . **TODO write the reduced thing more clearly**

*Example 1.2.4.* Recall that  $\mathrm{MU}$  is constructed as the colimit  $\mathrm{MU} = \mathrm{colim} \mathrm{MU}(n)$ . Also,  $\mathrm{MU}(1) \cong \Sigma^{\infty-2}\mathrm{BU}(1)$ . Therefore we get a canonical map  $\Sigma^{\infty-2}\mathrm{BU}(1) \rightarrow \mathrm{MU}$ , which gives a cohomology class  $x_{\mathrm{MU}} \in \mathrm{MU}^2(\mathrm{BU}(1))$ .

**Proposition 1.2.5** ([Rav86, 4.1.3]).  $x_{\mathrm{MU}}$  is a complex orientation for  $\mathrm{MU}$ .

**Theorem 1.2.6.**  $\mathrm{MU}$  is the universal complex oriented cohomology theory: Let  $E$  be a multiplicative cohomology theory, then there is a bijection between (homotopy classes of) multiplicative maps  $\mathrm{MU} \rightarrow E$  and complex orientations on  $E$ . The bijection is given in one direction by pulling back  $x_{\mathrm{MU}}$  along a multiplicative map.

Assume that  $E$  is complex oriented with a complex orientation  $x$ .

**Theorem 1.2.7** ([Rav86, 4.1.4]). As  $E_*$ -algebras,  $E^*(\mathrm{BU}(1)) \cong E^*[[x]]$  and  $E^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) \cong E^*[[y, z]]$ .

**TODO maybe indicate the use of AHSS**

The tensor product of complex line bundles is classified by a universal map  $\mathrm{BU}(1) \times \mathrm{BU}(1) \rightarrow \mathrm{BU}(1)$ . Therefore we get a map  $E^*(\mathrm{BU}(1)) \rightarrow E^*(\mathrm{BU}(1) \times \mathrm{BU}(1))$ , which is completely determined by the image of  $x \in E^*[[x]]$  in  $E^*[[y, z]]$  as above. Therefore, a choice of a complex orientation on  $E$  gives rise to an element  $F_E(y, z) \in E^*[[y, z]]$ .

**Proposition 1.2.8** ([Rav86, 4.1.4]).  $F_E$  is a formal group law on  $E_*$ . We call the height of  $F_E$  the height of  $E$ .

*Example 1.2.9.* Consider again  $HR$ . It is known that the tensor of complex line bundles induces the map  $R[[x]] = HR^*(\mathrm{BU}(1)) \rightarrow HR^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = R[[y, z]]$  given by  $x \mapsto y + z$ . This is the additive formal group law. It is immediate that  $[p] = px$ . So for  $R = \mathbb{Q}$  we get that the height of  $H\mathbb{Q}$  is 0, while for  $R = \mathbb{F}_p$  we have  $px = 0$  so the height of  $H\mathbb{F}_p$  is  $\infty$ .

*Example 1.2.10.* We return to the example of complex K-theory 1.2.3. It is known that the tensor of complex line bundles induces the map  $K_*[[t]] = K^*(\mathrm{BU}(1)) \rightarrow K^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = K_*[[u, v]]$  given by  $t \mapsto u + v + uv$ . Note that to comply with the definition of the formal group law, we should use the isomorphism  $K^*(\mathrm{BU}(1)) \cong K_*[[x]]$ , i.e. the element  $x = \beta t$ . We get that  $x = \beta t \mapsto \beta u + \beta v + \beta uv = y + z + \beta^{-1}yz = F_K(y, z)$ . By induction we prove that the  $n$ -series is  $[n](x) = \beta(1 + \beta^{-1}x)^n - \beta$ . This is clear for  $n = 1$ , and we have:

$$\begin{aligned} [n+1](x) &= x + [n](x) + \beta^{-1}x[n](x) \\ &= x + \beta(1 + \beta^{-1}x)^n - \beta + x(1 + \beta^{-1}x)^n - x \\ &= \beta(1 + \beta^{-1}x)(1 + \beta^{-1}x)^n - \beta \\ &= \beta(1 + \beta^{-1}x)^{n+1} - \beta \end{aligned}$$

**TODO consider discussing the computation of  $\mathrm{BU}(1)$ , maybe as part of complex K-theory example?**

*Example 1.2.11.* By taking the cofiber of the multiplication-by- $p$  map, we get a spectrum  $K/p$ , mod- $p$  K-theory, with coefficients  $(K/p)_* = \mathbb{F}_p[\beta^{\pm 1}]$ . It is evident that  $F_{K/p}(y, z) = y + z + \beta^{-1}yz$  as well. From the result above, it follows that  $[p](x) = \beta(1 + \beta^{-1}x)^p - \beta = \beta(1^p + \beta^{-p}x^p) - \beta = \beta^{-(p-1)}x^p$  which shows that the height is exactly 1.

A formal group law on  $E_*$  is the same data as a map from the Lazard ring  $L$ , so the complex orientation gives a map  $L \rightarrow E_*$ . In particular, since  $\text{MU}$  is complex oriented, there is a canonical map  $L \rightarrow \text{MU}_*$ .

**Theorem 1.2.12** (Quillen, [Rav86, 4.1.6]). *The canonical map  $L \rightarrow \text{MU}_*$  is an isomorphism.*

### 1.3 BP, Morava K-Theory and Morava E-Theory

A good principle in homotopy theory (and many other areas in math) is to do study it one prime at a time. This is possible in homotopy theory due to the arithmetic square **TODO reference**. So, let us fix a prime  $p$ . We can form  $\text{MU}_{(p)}$ , the  $p$ -localization of  $\text{MU}$ .

**Theorem 1.3.1** ([Ada74, II 15]). *There exists a map of ring spectra  $\varepsilon : \text{MU}_{(p)} \rightarrow \text{MU}_{(p)}$  (which depends on the prime  $p$ ), which is an idempotent  $\varepsilon^2 = \varepsilon$  (moreover, once the action on homotopy groups is specified, it is unique).*

The map  $\varepsilon : \text{MU}_{(p)} \rightarrow \text{MU}_{(p)}$  gives a cohomology operation, for every  $X$  we have  $\varepsilon^* : \text{MU}_{(p)}^*(X) \rightarrow \text{MU}_{(p)}^*(X)$ . Denote by  $\text{BP}_{(p)}^*(X)$  the image of  $\varepsilon^*$ .

**Theorem 1.3.2** ([Ada74, II 16], [Rav86, 4.1.12]). *BP is a cohomology theory, represented by an associative commutative ring spectrum BP (which depends on the prime  $p$ ), which is a retract of  $\text{MU}_{(p)}$ . The homotopy groups of BP are  $\text{BP}_* = \mathbb{Z}_{(p)}[v_1, \dots]$  where  $|v_n| = 2(p^n - 1)$ .*

For convenience we denote  $v_0 = p$  (and indeed  $|v_0| = 2(p^0 - 1) = 0$ ). Since BP is a retract of MU it comes with a map  $\text{MU} \rightarrow \text{BP}$ , that is a complex orientation.

**Proposition 1.3.3** (**TODO reference**). *The  $p$ -series of the formal group law associated to BP is  $[p](x) = \sum v_n x^{p^n}$ .*

Once we have BP, we can turn to the definition of Morava K-theory and Morava E-theory

**Definition 1.3.4.** Let  $0 < n < \infty$ . *Morava K-theory at height  $n$  and prime  $p$ , denoted by  $K(p, n)$  or  $K(n)$  when the prime is clear, is the spectrum obtained by killing  $p = v_0, \dots, v_{n-1}, v_{n+1}, \dots$  in BP and inverting  $v_n$ . Therefore  $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ . We also define  $K(0) = \text{H}\mathbb{Q}$  and  $K(\infty) = \text{H}\mathbb{F}_p$ . Similarly, *Morava E-theory at height  $n$  and prime  $p$ , denoted by  $E(p, n)$  or  $E(n)$ , is the spectrum obtained by killing  $v_{n+1}, v_{n+2}, \dots$  in BP and inverting  $v_n$ . Therefore  $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ .**

Since Morava K-theory and E-theory are obtained from BP by cofibers and filtered colimits, they are equipped with a map from BP, so they are also equipped with a complex orientation. It is then evident that the  $p$ -series associated to the formal group laws of  $K(n)$  and  $E(n)$  are  $v_n x^{p^n}$  and  $v_0 x + \dots v_n x^{p^n}$  respectively, and are therefore of height exactly  $n$  and height  $\leq n$  respectively. (Note that by the example of  $\text{HR}$ , this is also true for  $K(0)$  and  $K(\infty)$ .)

We want to describe some properties of Morava K-theory. To do so we first need some definitions.

**Definition 1.3.5.** Let  $R$  be an evenly graded ring.  $R$  is called a *graded field* if every non-zero homogenous element is invertible, equivalently it is a field  $F$  concentrated at degree 0, or  $F[\beta^{\pm 1}]$  for  $\beta$  of positive even degree. An  $\mathbb{A}_\infty$ -ring  $E$  is a *field* if  $E_*$  is a graded field.

*Example 1.3.6.* Clearly  $K(n)$  for  $0 \leq n \leq \infty$  is a field.

**Proposition 1.3.7.** *A field  $E$  has Kunneth, i.e.  $E_*(X \otimes Y) \cong E_*(X) \otimes_{E_*} E_*(Y)$  for any spectra  $X, Y$ .*

**Proposition 1.3.8** ([Lur10, 24]). *Let  $E \neq 0$  be a complex oriented cohomology theory, whose formal group law has height exactly  $n$ , then  $E \otimes K(n) \neq 0$ . Let  $E$  be a field s.t.  $E \otimes K(n) \neq 0$ , then  $E$  admits the structure of a  $K(n)$ -module. (Here  $0 \leq n \leq \infty$ .)*

*Example 1.3.9.* As we have seen before mod- $p$  K-theory,  $K/p$ , has height exactly 1 and coefficients  $(K/p)_* = \mathbb{F}_p[\beta^{\pm 1}]$ . It is also known that  $K$ , and  $K/p$ , are  $A_\infty$  ring spectra, from which it follows that  $K/p$  is a field. Therefore, we deduce that  $K/p$  is a  $K(1)$ -module. Since  $|\beta| = 2$  and  $|v_1| = 2(p - 1)$  it is free of rank  $p - 1$ .

From this we also deduce some form of uniqueness for Morava K-theory:

**Corollary 1.3.10.** *Let  $E$  be a field with  $E_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ , which is also complex oriented with height exactly  $n$ , then  $E \cong K(n)$  (as spectra).*

## 1.4 $\text{Spec } \mathbb{S}_{(p)}$ and $\text{Spec } \mathbb{S}$

We are now in a position to state the answer for  $\text{Spec } \mathbb{S}$ . However, it will be easier to state it first for  $\text{Spec } \mathbb{S}_{(p)}$ , and then pullback prime ideals. We know that  $\text{Mod}_{\mathbb{S}_{(p)}} = \text{Sp}_{(p)}$ , and the compact objects there are  $\text{Sp}_{(p)}^{\text{fin}}$ , the  $p$ -localizations of finite spectra.

**Proposition 1.4.1.** *Let  $\mathcal{T}_E = \ker E_* = \left\{ X \in \text{Sp}_{(p)}^{\text{fin}} \mid E_*(X) = 0 \right\}$  (equivalently  $X \otimes E = 0$ ) i.e. the  $E$ -acyclics, then  $\mathcal{T}_E$  is thick.*

*Proof.* The exact same proof from  $\text{Ch}_{\text{perf}}(R)$  works.  $\square$

**Definition 1.4.2.** We define  $\mathcal{C}_{p,n} = \mathcal{T}_{K(n)}$ , the  $K(n)$ -acyclics. By the above it is thick. Also,  $\mathcal{C}_{p,\infty} = \{0\}$ , which are trivially thick. When the prime is clear, we will denote by  $\mathcal{C}_n$ .

**Proposition 1.4.3** ([Lur10, 26]). *For  $X \in \text{Sp}_{(p)}^{\text{fin}}$ , if  $K(n)_*(X) = 0$  then  $K(n-1)_*(X) = 0$ .*

**Definition 1.4.4.** We say that a spectrum  $X \in \text{Sp}_{(p)}^{\text{fin}}$  is of *type  $n$*  (possibly  $\infty$ ), if its first non-zero Morava  $K$ -theory homology is  $K(n)$ .

**Corollary 1.4.5.**  $\mathcal{C}_n$  is the full subcategory of finite  $p$ -local spectra of type  $> n$  (i.e.  $\left\{ X \in \text{Sp}_{(p)}^{\text{fin}} \mid \forall m \leq n : K(m)_*(X) = 0 \right\}$ ). Thus clearly  $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$ .

**Proposition 1.4.6 (TODO reference).** *The inclusions are proper  $\mathcal{C}_{n+1} \subsetneq \mathcal{C}_n$ .*

**Proposition 1.4.7.** *If  $X \in \text{Sp}_{(p)}^{\text{fin}}$  is not contractible, then  $X$  has a finite type. Therefore  $\bigcap_{n < \infty} \mathcal{C}_n = \{0\} = \mathcal{C}_\infty$ .*

*Proof.* Let  $X$  be non-contractible. Then  $\text{HZ}_*(X) \neq 0$ . Let  $m$  be the first non-zero degree. Using the universal coefficient theorem and the fact that the spectrum is  $p$ -local we get that  $(\text{HF}_p)_m(X) \neq 0$ , thus  $(\text{HF}_p)_*(X) \neq 0$ . Since  $X$  is finite,  $(\text{HF}_p)_*(X)$  is bounded. Atiyah-Hirzebruch spectral sequence for  $X$  with cohomology  $K(n)$  has  $E^2$ -page given by  $E_{p,q}^2 = H_p(X; K(n)_q)$ . Since  $K(n)_q = \mathbb{F}_p$  for  $q = 0 \pmod{2(p^n - 1)}$  and 0 otherwise, we see that the rows  $q = 0 \pmod{2(p^n - 1)}$  are  $(\text{HF}_p)_*(X)$ , and the others are 0. Therefore if we take  $n$  such that the period  $2(p^n - 1)$  is larger then the bound on  $(\text{HF}_p)_*(X)$ , then all differentials have either source or target 0. Thus, the spectral sequence collapses at the  $E^2$ -page, and since  $(\text{HF}_p)_*(X) \neq 0$ , we get that  $K(n)_*(X) \neq 0$ , i.e.  $X$  has type  $< n$ .  $\square$

**Proposition 1.4.8.**  $\mathcal{C}_n$  is a prime ideal.

*Proof.* For  $X, Y$  by Kunneth we have  $K(n-1)_*(X \otimes Y) = K(n-1)_*(X) \otimes K(n-1)_*(Y)$ . Therefore, if  $X \in \mathcal{C}_n$ , i.e. the homology vanishes, then so does the homology of  $X \otimes Y$ , i.e.  $X \otimes Y \in \mathcal{C}_n$ , so  $\mathcal{C}_n$  is an ideal. If  $X \otimes Y \in \mathcal{C}_n$  then the homology of the product vanishes, therefore one in the right side must vanish (they are graded vector spaces), so  $\mathcal{C}_n$  is a prime ideal.  $\square$

**Theorem 1.4.9** (Thick Subcategory Theorem [HS98]). *If  $\mathcal{T}$  is a proper thick subcategory of  $\text{Sp}_{(p)}^{\text{fin}}$ , then  $\mathcal{T} = \mathcal{C}_n$  for some  $0 \leq n \leq \infty$ .*

*Remark 1.4.10.* The proof relies on a major theorem called the Nilpotence Theorem.

**Corollary 1.4.11.**  $\text{Spec } \mathbb{S}_{(p)} = \{\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_\infty\}$ , and the topology is such that the closed subsets are chains  $\{\mathcal{C}_k, \mathcal{C}_{k+1}, \dots, \mathcal{C}_\infty\}$  for some  $0 \leq k \leq \infty$ .

*Proof.* Follows immediately from the previous results.  $\square$

We now want to move to finding  $\text{Spec } \mathbb{S}$ . Note that the  $p$ -localization functor  $L_{(p)}$  is a Bousfield localization. As such it is left (its right adjoint is the inclusion), and in particular preserves cofibers. It also clearly sends 0 to 0, i.e. reduced. Now, since  $L_{(p)}$  is smashing, i.e.  $L_{(p)}X = X \otimes \mathbb{S}_{(p)}$ , we also get that it is symmetric monoidal. As we have seen in 1.1.8, under these conditions we can pullback primes. Therefore  $\mathcal{P}_{p,n} = L_{(p)}^* \mathcal{C}_{p,n} = \left\{ X \in \text{Sp}^{\text{fin}} \mid K(n)_*(X_{(p)}) = 0 \right\}$  and  $\mathcal{P}_{p,\infty} = L_{(p)}^* \mathcal{C}_{p,\infty} = \left\{ X \in \text{Sp}^{\text{fin}} \mid X_{(p)} = 0 \right\}$  are prime ideals. Note that

$\mathcal{P}_{p,0} = \left\{ X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_*(X_{(p)}) = 0 \right\} = \left\{ X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_*(X) = 0 \right\}$  **TODO explain** so it is independent of  $p$ , and we will denote it by  $\mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}}$ .

**Theorem 1.4.12 (TODO explain/reference).**  $\mathrm{Spec} \mathbb{S} = \left\{ \mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}} \right\} \cup \bigcup_p \{ \mathcal{P}_{p,1}, \dots, \mathcal{P}_{p,\infty} \}$ , and the topology is such that the closed subsets finite unions of chains  $\{ \mathcal{P}_{p,k}, \mathcal{P}_{p,k+1}, \dots, \mathcal{P}_{p,\infty} \}$  for some  $0 \leq k \leq \infty$  (i.e. they may include  $\mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}}$ ). **TODO diagram**

**TODO** regarding the topology, maybe I should prove that the pullback is also continuous?

## 1.5 The Stacky Point of View and the Relationship Between Morava K-Theory and Morava E-Theory

First we will describe, without being precise, another point of view on what chromatic homotopy theory is about.

There is a stack of formal groups, denoted by  $\mathcal{M}_{\mathrm{fg}}$ . It can be described as the stack that sends a ring  $R$  to the groupoid of formal group laws, with isomorphisms between them. Quillen theorem 1.2.12 tells us that  $\mathrm{MU}_*$  is the Lazard ring, that is the universal ring that carries the universal formal group law. It turns out that this theorem has a second part, which says that  $(\mathrm{MU} \otimes \mathrm{MU})_*$  is the universal ring that carries two formal group laws and an isomorphism between them. Therefore,  $\mathcal{M}_{\mathrm{fg}}$  is represented by  $(\mathrm{MU}_*, (\mathrm{MU} \otimes \mathrm{MU})_*)$ .

The geometric points of the stack  $\mathcal{M}_{\mathrm{fg}}$  describe precisely the same as  $\mathrm{Spec} \mathbb{S}$ , that is because for an algebraically closed field of characteristic 0 there is a unique (up to isomorphism) formal group law which is of height 0 namely the additive formal group law, and for characteristic  $p$  there is a unique (up to isomorphism) formal group law of each height  $1 \leq n \leq \infty$ .

For a spectrum  $X$ ,  $\mathrm{MU}_*(X)$  is a  $(\mathrm{MU}_*, (\mathrm{MU} \otimes \mathrm{MU})_*)$ -comodule, which is the same as a sheaf over  $\mathcal{M}_{\mathrm{fg}}$ . From this point of view, chromatic homotopy theory lets us study a spectrum by decomposing it over the stack  $\mathcal{M}_{\mathrm{fg}}$ .

We can restrict ourselves to the stack only over rings of characteristic  $p$ ,  $\mathcal{M}_{\mathrm{fg},p}$ , which is then represented by  $((\mathrm{MU}_{(p)})_*, (\mathrm{MU}_{(p)} \otimes \mathrm{MU}_{(p)})_*)$ . **TODO I think it's true, is it?** Similarly to  $\mathrm{MU}$ ,  $\mathrm{BP}$  is universal ring with the universal  $p$ -typical formal group law, and  $(\mathrm{BP} \otimes \mathrm{BP})_*$  is the universal ring with two  $p$ -typical formal group laws and an isomorphism between them **TODO I didn't say this before**. Since every formal group law is isomorphic to a  $p$ -typical one, we know that the stack  $\mathcal{M}_{\mathrm{fg},p}$  is also represented by  $(\mathrm{BP}_*, (\mathrm{BP} \otimes \mathrm{BP})_*)$ .

It is now reasonable that  $\mathrm{K}(n)$ , obtained from  $\mathrm{BP}$  by killing the  $v_m$ 's for  $m \neq n$  and inverting  $v_n$ , sees the  $n$ -th level, and that  $\mathrm{E}(n)$  obtained in the same way but only killing  $v_m$  for  $m > n$ , sees the levels  $\leq n$ . **TODO I didn't interpret the  $v_n$ 's before as coefficients in the  $p$ -series which determines the height**

Let us now claim a precise statement, formalizing this description.

**Theorem 1.5.1 (TODO reference).**  $\mathrm{E}(n)$  and  $\mathrm{K}(0) \oplus \dots \oplus \mathrm{K}(n)$  are Bousfield equivalent. That is, they have the same acyclics, locals, and their localization functors are the same.

**TODO chromatic square and chromatic tower, maybe another subsection?**

## 1.6 Landweber Exact Functor Theorem

As we have seen, a complex orientation on a cohomology theory, which is described by a map  $\mathrm{MU} \rightarrow E$ , has an associated formal group law, which is described by the map  $L = \mathrm{MU}_* \rightarrow E_*$ . Note that this formal group law is of degree  $-2$ , by virtue of the grading on  $L = \mathrm{MU}_*$  **TODO say more?**. One can ask whether the converse is true, namely given a graded ring  $R$  and a formal group law  $F$  of degree  $-2$  given by  $L \rightarrow R$ , is there a complex oriented cohomology theory whose coefficients are  $R$  and the associated formal group law is  $F$ .

A strategy is to define  $(E_{R,F})_*(X) = \mathrm{MU}_*(X) \otimes_{\mathrm{MU}_*} R$ . Unfortunately, this is not always a homology theory. However there is a condition which one can check which guarantees that it is.

**Definition 1.6.1.**  $L \rightarrow R$  is called *Landweber flat* if for every prime  $p$ , the sequence  $p = v_0, v_1, v_2, \dots$  is regular, that is, for each  $n$ ,  $v_n$  is not a zero divisor in  $R/(v_0, v_1, \dots, v_{n-1})$ .

**Theorem 1.6.2** (Landweber Exact Functor Theorem (LEFT), [Lur10, 15, 16]). *If  $L \rightarrow R$  is Landweber flat, then  $E_{R,F}$  defined above is a homology theory. Moreover, there are no phantom maps between such spectra, so  $E_{R,F}$  is represented by a spectrum. This spectrum is complex oriented, has coefficients  $R$  and associated formal group law  $F$ .*

*Example 1.6.3.* We return to complex K-theory, from 1.2.3 and 1.2.10. We can take the completion at the element  $p \in K_*$ , which gives the spectrum  $K_p^\wedge$ . This spectrum has coefficients  $(K_p^\wedge)_* = (K_*)_p^\wedge = (\mathbb{Z}[\beta^{\pm 1}])_p^\wedge = \mathbb{Z}_p^\wedge[\beta^{\pm 1}]$ . The formal group law, as we have seen, is given by  $F_{K_p^\wedge}(y, z) = y + z + \beta^{-1}yz$ . We claim that  $F_{K_p^\wedge}/K_p^\wedge$  is Landweber flat. Clearly  $p = v_0$  is invertible in  $\mathbb{Z}_p^\wedge[\beta^{\pm 1}]$ , thus not a zero divisor. As we have seen in 1.2.11, mod- $p$  the  $p$ -series is  $\beta^{-(p-1)}x^p$ , so that  $v_1 = \beta^{-(p-1)}$  which is not a zero divisor. Modulo  $v_1$  the ring is already 0, and we are done. Therefore, by 1.6.5 we get that  $K_p^\wedge \cong E_{K_p^\wedge, F_{K_p^\wedge}}$ .

*Example 1.6.4.* Morava E-theory is Landweber flat. Morava K-theory and  $H\mathbb{Z}$  are not. **TODO elaborate? reference?**

We can also ask the following question: given complex oriented cohomology theory  $MU \rightarrow E$ , such that  $L \rightarrow E_*$  is Landweber flat, is  $E_{R,F}$  equivalent to  $E$ ? The answer is yes, at least in some cases.

**Theorem 1.6.5.** *Let  $E$  be as above, which is also evenly graded (i.e.  $E_*$  is an evenly graded ring), then there is an equivalence  $E_{R,F} \rightarrow E$ .*

*Proof.* This is a slight variation on [Lur10, 18, proposition 11]. First note that for every spectrum  $X$  we have  $MU \otimes X \rightarrow E \otimes X$ , which induces  $MU_*(X) \rightarrow E_*(X)$ , a map of  $MU_*$ -modules. Moreover, since  $E_* \rightarrow E_*(X)$  is a map of  $E_*$ -module, the map  $MU_* \rightarrow E_*$  makes it a map of  $MU_*$ -modules. Together this gives a map  $(E_{R,F})_*(X) = MU_*(X) \otimes_{MU_*} E_* \rightarrow E_*(X)$ . This map is a map of homology theories. **TODO should I explain why?** By [Lur10, 17, theorem 6] **TODO does it follow?**, this map lifts to a map of spectra  $E_{R,F} \rightarrow E$ . Since by construction when  $X = \mathbb{S}$  the map above is  $E_* \rightarrow E_*$  which is an isomorphism, we see that the map  $E_{R,F} \rightarrow E$  is an equivalence.  $\square$

## 1.7 Lubin-Tate Deformation Theory

The Morava E-theory we have considered until now  $E(n)$ , also called Johnson-Wilson spectrum was constructed from BP. As we noted, it is Landweber flat, which indicates that there is another approach to constructing it. Indeed there is a way to construct a related spectrum, which will be called the Lubin-Tate spectrum.

To that end, we first define the category  $\text{CompRing}$  as the category of complete local rings. The objects are complete local rings  $(R, \mathfrak{m})$ , we also denote by  $\pi : R \rightarrow R/\mathfrak{m}$  the projection. Morphisms  $\varphi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  are local homomorphisms, i.e. a homomorphism  $\varphi : R \rightarrow S$  s.t.  $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ . In particular it induces a homomorphism  $\varphi/\mathfrak{m} : R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ , which satisfies  $\varphi/\mathfrak{m} \circ \pi_R = \pi_S \circ \varphi$ .

We fix  $k$  be a perfect field of characteristic  $p$  (i.e. the Frobenius is an isomorphism), and  $\Gamma$  a formal group law over  $k$  of height  $n < \infty$ . Lubin and Tate [LT] considered a moduli problem associated to  $\Gamma/k$ , described by a functor  $\text{Def} : \text{CompRing} \rightarrow \text{Grpds}$ .

**Definition 1.7.1.** Let  $(R, \mathfrak{m})$  be a complete local ring and denote by  $\pi : R \rightarrow R/\mathfrak{m}$  the quotient. A *deformation* of  $\Gamma/k$  to  $(R, \mathfrak{m})$ , is  $(G, i)$ , where  $G$  is a formal group law over  $R$ ,  $i : k \rightarrow R/\mathfrak{m}$  is a homomorphism of fields, such that  $i^*\Gamma = \pi^*G$ . A  $\star$ -*isomorphism* between two deformations to  $(R, \mathfrak{m})$ ,  $f : (G_1, i_1) \rightarrow (G_2, i_2)$ , is defined only when  $i_1 = i_2$ , and consists of an isomorphism  $f : G_1 \rightarrow G_2$ , such that  $\pi^*f : i^*\Gamma = \pi^*G_1 \rightarrow \pi^*G_2 \rightarrow i^*\Gamma$  is the identity, i.e.  $f(x) = x \pmod{\mathfrak{m}}$ . These assemble to a groupoid  $\text{Def}(R, \mathfrak{m})$ , whose objects are deformations to  $(R, \mathfrak{m})$ , and morphisms are  $\star$ -isomorphisms.

*Remark 1.7.2.*  $\text{Def}(R, \mathfrak{m})$  can be seen as the pullback of the groupoids  $\text{FGL}(R)$  and  $\coprod_{i:k \rightarrow R/\mathfrak{m}} \{\Gamma\}$  over  $\text{FGL}(R/\mathfrak{m})$  (where the maps are  $G \mapsto \pi^*G$  and  $i \mapsto i^*\Gamma$  respectively).

**Proposition 1.7.3** (/definition). *The construction  $\text{Def}(R, \mathfrak{m})$  is functorial.*

*Proof.* Let  $\varphi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  be a local homomorphism.

For a deformation  $(G, i)$  to  $(R, \mathfrak{m})$ , we define  $\text{Def}(\varphi)(G, i) = (\varphi^*G, \varphi/\mathfrak{m} \circ i)$ . Note that  $\varphi^*G$  is a formal group law over  $S$ , and  $\varphi/\mathfrak{m} \circ i : k \rightarrow R/\mathfrak{m} \rightarrow S/\mathfrak{n}$  is a homomorphism. Moreover,  $(\varphi/\mathfrak{m} \circ i)^*\Gamma = (\varphi/\mathfrak{m})^*i^*\Gamma =$

$(\varphi/\mathfrak{m})^* \pi_R^* G = (\varphi/\mathfrak{m} \circ \pi_R)^* G = (\pi_S \circ \varphi)^* G = \pi_S^* \varphi^* G$ , which shows that  $\text{Def}(\varphi)(G, i)$  is a deformation to  $(S, \mathfrak{n})$ .

For a  $\star$ -isomorphism  $f : (G_1, i) \rightarrow (G_2, i)$ , which is the data of an isomorphism  $f : G_1 \rightarrow G_2$  such that  $\pi_R^* f = \text{id}_{i^* \Gamma}$  is the identity, we need to define a  $\star$ -isomorphism  $\text{Def}(\varphi)(G, i_1) \rightarrow \text{Def}(\varphi)(G, i_2)$ . Take it to be the isomorphism  $\varphi^* f : \varphi^* G_1 \rightarrow \varphi^* G_2$ , which satisfies  $\pi_S^* \varphi^* f = (\varphi/\mathfrak{m})^* \pi_R^* f = (\varphi/\mathfrak{m})^* \text{id}_{i^* \Gamma} = \text{id}_{(\varphi/\mathfrak{m})^* i^* \Gamma} = \text{id}_{(\varphi/\mathfrak{m} \circ i)^* \Gamma}$ . The identity  $\text{id}_G : (G, i) \rightarrow (G, i)$  is clearly sent to  $\text{id}_{\varphi^* G}$ , and compositions are sent to compositions.

This shows that  $\text{Def}(\varphi) : \text{Def}(R, \mathfrak{m}) \rightarrow \text{Def}(S, \mathfrak{n})$  is indeed a functor. Moreover, it is clear that  $\text{Def}(\text{id}_R)$  is the identity and compositions are sent to compositions, which shows that  $\text{Def} : \text{CompRing} \rightarrow \text{Grpds}$  is indeed a functor.  $\square$

**Theorem 1.7.4** ([Rez, 4.4, 5.10], originally due to [LT]). *The functor  $\text{Def}$  lands in discrete groupoids (i.e.  $\text{Def}(R, \mathfrak{m})$  has 0 or 1 morphisms between objects). Furthermore the functor  $\text{Def}$  is co-represented, that is there exists a universal deformation, and the complete local ring can be chosen (non-canonically) to be  $Wk[[u_1, \dots, u_{n-1}]]$ .*

Let us unravel what that means. First note that the quotient of  $Wk[[u_1, \dots, u_{n-1}]]$  by the maximal ideal  $(p, u_1, \dots, u_{n-1})$  is  $k$ . The universal deformation can be chosen such that the formal group law over it  $\Gamma_U$  over  $Wk[[u_1, \dots, u_{n-1}]]$  satisfies  $\pi^* \Gamma_U$  is  $\Gamma$ . The universality means that for  $(R, \mathfrak{m})$ , the assignment

$$\text{hom}_{\text{CompRing}}(Wk[[u_1, \dots, u_{n-1}]], R) \rightarrow \text{Def}(R, \mathfrak{m}), \quad \varphi \mapsto \varphi^* \Gamma_U$$

is an equivalence.

#### TODO more about the witt vectors?

Now, we can form the graded ring  $Wk[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$  where  $|u| = 2$ . We can define the formal group law  $(u\Gamma_U)(x, y) = u^{-1} \Gamma_U(uy, uz)$ , which is of degree  $-2$ , thus we get a map  $L \rightarrow Wk[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$ .

**Proposition 1.7.5** ([Rez, 6.9]).  *$L \rightarrow Wk[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$  is Landweber flat.*

Using LEFT 1.6.2, we immediately get:

**Corollary 1.7.6.** *There is a complex oriented cohomology theory  $E(k, \Gamma) = E_{Wk[[u_1, \dots, u_{n-1}]] [u^{\pm 1}], u\Gamma_U}$ , called Lubin-Tate spectrum, with coefficients  $E(k, \Gamma)_* = Wk[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$  and associated formal group law  $u\Gamma_U$ .*

**Example 1.7.7.** We continue the complex K-theory saga from 1.6.3. Take the field  $k = \mathbb{F}_p$  and the formal group law  $\Gamma(y, z) = y + z + yz$ , of height  $n = 1$ . By the above construction, the ring of the universal deformation is  $W\mathbb{F}_p = \mathbb{Z}_p^\wedge$ . The universal formal group law of the universal deformation is again  $\Gamma_U(y, z) = y + z + yz$ . **TODO show** We look at the ring  $\mathbb{Z}_p^\wedge [u^{\pm 1}]$ , and at the formal group law over it  $(u\Gamma_U)(y, z) = u^{-1} (uy + uz + u^2 yz) = y + z + uyz$ . It is clear that the isomorphism  $\mathbb{Z}_p^\wedge [u^{\pm 1}] \rightarrow \mathbb{Z}_p^\wedge [\beta^{\pm 1}]$ , sends  $u\Gamma_U$  to  $F_{K_p^\wedge}$ . It follows that  $K_p^\wedge \cong E_{K_p^\wedge, F_{K_p^\wedge}} \cong E(\mathbb{F}_p, \Gamma)$ , i.e.  $p$ -complete K-theory  $K_p^\wedge$  is a Lubin-Tate spectrum at height 1.

This concludes the construction of the Lubin-Tate variant of Morava E-theory. To compare this version with the previous, we define another variant of Morava E-theory.

**Definition 1.7.8.** Let  $I \leq E(n)_* = \mathbb{Z}_{(p)} [v_1, \dots, v_{n-1}, v_n^{\pm 1}]$  be the ideal  $I = (p = v_0, v_1, \dots, v_{n-1})$ . We define the spectrum  $\widehat{E(n)} = E(n)_I^\wedge$ , called completed Johnson-Wilson spectrum.

**Theorem 1.7.9.** *The following three forms of Morava E-theory are Bousfield equivalent:*

1. Johnson-Wilson  $E(n)$ ,
2. completed Johnson-Wilson  $\widehat{E(n)}$ ,
3. Lubin-Tate  $E(k, \Gamma)$ .

**TODO reference Hovey Strickland Morava K-Theories and Localization 5.3**



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