Thesis

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Contents

1 Overview of Chromatic Homotopy Theory

Our goal is to motivate the introduction of Morava K-theory $\mathrm{K}(n)$ and Morava E-theory $\mathrm{E}(n)$, and other variants of Morava E-theory $\mathrm{E}(k, \Gamma)$, and their connection to formal group laws. There are different views on what chromatic homotopy theory is. **TODO write some more**

1.1 The Balmer Spectrum

We will start with an algebraic motivation. Let R be a noetherian ring. Consider the symmetric monoidal stable ∞-category $\mathrm{Ch}(R)$ of chain complexes on R. **TODO be more specific** It is then natural to ask how much information about R is encoded in the category $\mathrm{Ch}(R)$. We will try to recover $\operatorname{Spec} R$, as a topological space, from $\mathrm{Ch}(R)$.

Remark 1.1.1. Balmer's work actually recovers the structure sheaf as well. **TODO reference**

Definition 1.1.2. A *perfect complex* is a complex that is quasi-isomorphic to a bounded complex of finitely-generated projective modules. These objects are the compact objects in $\mathrm{Ch}(R)$, thus they can be defined categorically. Their full subcategory is denoted by $\mathrm{Ch}_{\mathrm{perf}}(R)$.

Definition 1.1.3. Let $\mathcal{C}$ be some symmetric monoidal stable ∞-category. A full subcategory $\mathcal{T}$ is *thick* if:

- $0 \in \mathcal{T}$,
- it is closed under cofibers,
- it is closed under retracts.

Example 1.1.4. Consider the case $\mathcal{C} = \mathrm{Ch}_{\mathrm{perf}}(R)$ (e.g. over $\mathbb{Z}$, chain complexes quasi-isomorphic to bounded chain complexes of finitely-generated free abelian groups). Let $K \in \mathrm{Ch}(R)$, and define $\mathcal{T}_K = \left\{A \in \mathrm{Ch}_{\mathrm{perf}}(R) \mid A \otimes K \cong 0\right\}$. We claim that $\mathcal{T}_K$ is thick. Clearly $0 \in \mathcal{T}_K$. Let $A \to B$ be a morphism between two complexes in $\mathcal{T}$. Since tensor is left, tensoring the cofiber with K is given by $\operatorname{cofib}(A \to B) \otimes K \cong \operatorname{cofib}(A \otimes K \to B \otimes K) \cong \operatorname{cofib}(0 \to 0) \cong 0$, therefore the cofiber is indeed in $\mathcal{T}_K$. Lastly, if $A \to B \to A$ is the identity and $B \otimes K \cong 0$, we get that $\mathrm{id}_{A \otimes K}$ factors through 0, which implies that $A \otimes K$ is 0, so that $A \in \mathcal{T}_K$.

Definition 1.1.5. A thick subcategory $\mathcal{T}$ is an *ideal* if $A \in \mathcal{T}, B \in \mathcal{C} \implies A \otimes B \in \mathcal{T}$. Furthermore, it is a *prime ideal* if it is a proper subcategory, and $A \otimes B \in \mathcal{T} \implies A \in \mathcal{T}$ or $B \in \mathcal{T}$. The *spectrum* of the category is defined similarly to the classical spectrum of a ring: As a set, $\operatorname{Spec} \mathcal{C} = \{\mathcal{P} \text{ prime ideal}\}$. For any family of objects $S \subseteq \mathcal{C}$ we define $V(S) = \{\mathcal{P} \in \operatorname{Spec} \mathcal{C} \mid S \cap \mathcal{P} = \emptyset\}$. We topologize $\operatorname{Spec} \mathcal{C}$ with the Zariski topology by declaring those to be the closed subsets. We also denote $\operatorname{Supp}(A) = V(\{A\})$.

Example 1.1.6. We continue the example of $\mathcal{T}_K$. Clearly if $A \otimes K \cong 0$ then also $A \otimes B \otimes K \cong 0$, so it is an ideal. Let $\mathfrak{p}$ be a prime ideal in R in the usual sense, and take $K = R_\mathfrak{p}$ (concentrated at degree 0), then $A \otimes K = A_\mathfrak{p}$ (levelwise localization). **TODO consider actually proving** We will omit the proof that $\mathcal{T}_K$ is a prime, but we shall prove something weaker, namely only the case where A, B are bounded complexes of finitely generate projective modules (and not merely quasi-isomorphic to such complexes). Assume then that $0 = (A \otimes B)_\mathfrak{p} = A_\mathfrak{p} \otimes_{R_\mathfrak{p}} B_\mathfrak{p}$. Assume by negation that $A_\mathfrak{p}, B_\mathfrak{p} \neq 0$, i.e. $(A_n)_\mathfrak{p}, (B_m)_\mathfrak{p} \neq 0$ but $(A_n)_\mathfrak{p} \otimes_{R_\mathfrak{p}} (B_m)_\mathfrak{p} = 0$ for some n, m. Since the localization of a projective module is again a projective module, and a projective over a local ring is free, and clearly if the tensor of two free modules vanish then one of them vanishes, it follows that $(A_n)_\mathfrak{p} = 0$ or $(B_m)_\mathfrak{p} = 0$, which is a contradiction. Therefore $\mathcal{T}_\mathfrak{p}$ is a prime ideal.

Theorem 1.1.7. *The map* $\operatorname{Spec} R \to \operatorname{Spec}\left(\mathrm{Ch}_{\mathrm{perf}}(R)\right)$*, given by* $\mathfrak{p} \mapsto \mathcal{T}_\mathfrak{p} = \{A \mid A_\mathfrak{p} = 0\}$ *is a homeomorphism.*

TODO reference

Proposition 1.1.8. *Prime ideals pullback: Let* $F : \mathcal{C} \to \mathcal{D}$ *be a reduced symmetric monoidal functor that preserves cofibers, between two symmetric monoidal stable* ∞*-categories, and let* $\mathcal{P}$ *be a prime ideal in* $\mathcal{D}$*, then* $F^*\mathcal{P} = \{A \in \mathcal{C} \mid F(A) \in \mathcal{P}\}$ *is a prime ideal.*

Proof. Clearly $F(0) = 0 \in \mathcal{P}$ since F is reduced, so $0 \in F^*\mathcal{P}$. Since F preserves cofibers, for $A, B \in F^*\mathcal{P}$, i.e. $F(A), F(B) \in \mathcal{P}$, and a map $A \to B$ we get $F(\operatorname{cofib}(A \to B)) = \operatorname{cofib}(F(A) \to F(B)) = \operatorname{cofib}(F(A) \to F(B)) \in \mathcal{P}$. Let $A \to B \to A$ be a retract, that is the composition is the identity, s.t. $B \in F^*\mathcal{P}$. We know that $F(A) \to F(B) \to F(A)$ is also a retract by functoriality, thus $F(A) \in \mathcal{P}$, that is $A \in F^*\mathcal{P}$. We conclude that $F^*\mathcal{P}$ is indeed a thick subcategory.

Let $A \in F^*\mathcal{P}, B \in \mathcal{C}$, since F is monoidal, $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{P}$, so $A \otimes B \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is an ideal.

Lastly, assume that $A \otimes B \in F^*\mathcal{P}$, again since F is monoidal, $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{P}$, so $A \in F^*\mathcal{P}$ or $B \in F^*\mathcal{P}$, that is $F^*\mathcal{P}$ is a prime ideal. ☐

Now, recall that $\mathrm{Ch}(R) \cong \mathrm{Mod}_{HR}$, therefore we can reinterpret the above theorem as $\operatorname{Spec} R \cong \operatorname{Spec}\left(\mathrm{Mod}_{HR}^{\mathrm{comp}}\right)$ (where the comp denotes the compact objects in the category). We shall turn this theorem into a definition:

Definition 1.1.9. Let R be an $\mathbb{E}_\infty$-ring. We define the *spectrum* of R to be $\operatorname{Spec} R = \operatorname{Spec}\left(\mathrm{Mod}_R^{\mathrm{comp}}\right)$.

A natural question to ask then is what is the topological space $\operatorname{Spec}\mathbb{S}$. Recall that $\mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$, the category of spectra, and that the compact objects in spectra are the finite spectra $\mathrm{Sp}^{\mathrm{fin}}$. So, unwinding the definitions, the question can rephrased as finding the prime ideals in $\mathrm{Sp}^{\mathrm{fin}}$, and their topology. Chromatic homotopy theory provides an answer to this question.

1.2 MU and Complex Orientations

Throughout this section, let E be a multiplicative cohomology theory (that is, equipped with a map $E \otimes E \to E$ and $1 \in E_0$, which is associative and unital after taking homotopy groups).

Consider the map $S^2 \to \mathrm{BU}(1)$ classifying the universal complex line bundle. Concretely, under the identifications $S^2 \cong \mathbb{C}\mathrm{P}^1$ and $\mathrm{BU}(1) \cong \mathbb{C}\mathrm{P}^\infty$, this map can be realized as the inclusion $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^\infty$. This map induces a map

$$\tilde{E}^2(\mathrm{BU}(1)) \to \tilde{E}^2\left(S^2\right) \cong \tilde{E}^0\left(S^0\right) \cong E^0(*) = E_0.$$

Since E is unital, there is a canonical generator $1 \in E_0$.

Definition 1.2.1. E is called *complex oriented* if the map $\tilde{E}^2(\mathrm{BU}(1)) \to E_0$ is surjective, equivalently, if 1 is in the image of that map. A choice of a lift $x \in \tilde{E}^2(\mathrm{BU}(1))$ of $1 \in E_0$ is called a *complex orientation*.

Example 1.2.2. Let R be some ring, and consider $\mathrm{H}R$. It is known that $\mathrm{H}R^*(\mathbb{C}\mathrm{P}^n) \cong R[x]/\left(x^{n+1}\right)$ and $\mathrm{H}R^*(\mathbb{C}\mathrm{P}^\infty) \cong R[[x]]$, where $|x| = 2$, and the maps induced by the inclusions of projective spaces maps x to x. Therefore we see that $x \in \mathrm{H}R^2(\mathrm{BU}(1))$ is mapped to $x \in \mathrm{H}R^2\left(S^2\right) = R\{x\}$, which is mapped to the generator of the reduced part of $\mathrm{H}R^0\left(S^0\right) = R \oplus R$. Hence, x is a complex orientation for $\mathrm{H}R$.

Example 1.2.3. Let K be complex K-theory, then we know that $\mathrm{K}_* = \mathbb{Z}\left[\beta^{\pm 1}\right]$ where β is the Bott element, with $|\beta| = 2$. It is also known (by the Atiyah-Hirzebruch spectral sequence) that $\mathrm{K}^*(\mathbb{C}\mathrm{P}^n) \cong \mathrm{K}_*[t]/\left(t^{n+1}\right)$ and $\mathrm{K}^*(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{K}_*[[t]]$ (here $|t| = 0$), where the maps induced by the inclusions of projective spaces maps t to t. We deduce that $\beta^{-1}t \in \mathrm{K}^2(\mathrm{BU}(1))$ is mapped to $\beta^{-1}t \in \mathrm{K}^2\left(S^2\right) = \mathbb{Z}\left\{\beta^{-1}t\right\}$, and further mapped to $t \in \mathrm{K}^0\left(S^0\right) = \mathbb{Z}\{t\}$, which is indeed the generator of the reduced part. Therefore $x = \beta^{-1}t$ is complex orientation for K. **TODO write the reduced thing more clearly**

Example 1.2.4. Recall that MU is constructed as the colimit $\mathrm{MU} = \operatorname{colim} \mathrm{MU}(n)$. Also, $\mathrm{MU}(1) \cong \Sigma^{\infty-2}\mathrm{BU}(1)$. Therefore we get a canonical map $\Sigma^{\infty-2}\mathrm{BU}(1) \to \mathrm{MU}$, which gives a cohomology class $x_{\mathrm{MU}} \in \mathrm{MU}^2(\mathrm{BU}(1))$.

Proposition 1.2.5 ([Rav86, 4.1.3])**.** x_{MU} *is a complex orientation for* MU*.*

Theorem 1.2.6 (**TODO reference**)**.** MU *is the universal complex oriented cohomology theory, in the following sense: For any multiplicative cohomology theory E, then there is a bijection between (homotopy classes of) multiplicative maps* $\mathrm{MU} \to E$ *and complex orientations on E. The bijection is given in one direction by pulling back x_{MU} along a multiplicative map.*

Assume that E is complex oriented with a complex orientation x.

Proposition 1.2.7 ([Rav86, 4.1.4])**.** *As E_*-algebras, $E^*(\mathrm{BU}(1)) \cong E^*[[x]]$ and $E^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) \cong E^*[[y,z]]$.*

TODO maybe indicate the use of AHSS

The tensor product of complex line bundles is classified by a universal map $\mathrm{BU}(1) \times \mathrm{BU}(1) \to \mathrm{BU}(1)$. Therefore we get a map $E^*(\mathrm{BU}(1)) \to E^*(\mathrm{BU}(1) \times \mathrm{BU}(1))$, which is completely determined by the image of $x \in E^*[[x]]$ in $E^*[[y,z]]$ as above. We conclude that a choice of a complex orientation on E gives rise to an element $F_E(y,z) \in E^*[[y,z]]$.

Proposition 1.2.8 ([Rav86, 4.1.4])**.** *F_E is a formal group law on E_*.*

Definition 1.2.9. The *height* of E is simply the height of F_E.

Example 1.2.10. We continue with HR from 1.2.2. It is known that the tensor of complex line bundles induces the map

$$R[[x]] = \mathrm{H}R^*(\mathrm{BU}(1)) \to \mathrm{H}R^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = R[[y,z]],$$

given by $x \mapsto y + z$. This is the additive formal group law. It is immediate that $[p] = px$. So for $R = \mathbb{Q}$ we get that the height of $\mathrm{H}\mathbb{Q}$ is 0, while for $R = \mathbb{F}_p$ we have $px = 0$ so the height of $\mathrm{H}\mathbb{F}_p$ is ∞.

Example 1.2.11. We return to the example of complex K-theory 1.2.3. It is known that the tensor of complex line bundles induces the map

$$\mathrm{K}_*[[t]] = \mathrm{K}^*(\mathrm{BU}(1)) \to \mathrm{K}^*(\mathrm{BU}(1) \times \mathrm{BU}(1)) = \mathrm{K}_*[[u,v]],$$

given by $t \mapsto u + v + uv$. Note that to comply with the definition of the formal group law, we should use the isomorphism $\mathrm{K}^*(\mathrm{BU}(1)) \cong \mathrm{K}_*[[x]]$, i.e. the element $x = \beta t$. By multiplying by β^{-1} (recall that the map is of K_*-modules) we get that

$$x = \beta^{-1} t \mapsto \beta^{-1} u + \beta^{-1} v + \beta^{-1} uv = y + z + \beta yz = F_{\mathrm{K}}(y, z).$$

By induction we prove that the n-series is $[n](x) = \beta^{-1}(1+\beta x)^n - \beta^{-1}$. This is clear for $n = 1$, and we have:

$$\begin{aligned}
[n+1](x) &= x + [n](x) + \beta x [n](x) \\
&= x + \beta^{-1}(1+\beta x)^n - \beta^{-1} + x(1+\beta x)^n - x \\
&= \beta^{-1}(1+\beta x)(1+\beta x)^n - \beta^{-1} \\
&= \beta^{-1}(1+\beta x)^{n+1} - \beta^{-1}
\end{aligned}$$

TODO consider discussing the computation of $\mathrm{BU}(1)$, maybe as part of complex K-theory example?

Example 1.2.12. By taking the cofiber of the multiplication-by-p map, we get a spectrum K/p, mod-p K-theory, with coefficients $(\mathrm{K}/p)_* = \mathbb{F}_p\left[\beta^{\pm 1}\right]$. It is evident that $F_{\mathrm{K}/p}(y, z) = y + z + \beta yz$ as well. From the result above, it follows that

$$[p](x) = \beta^{-1}(1+\beta x)^p - \beta^{-1} = \beta^{-1}(1^p + \beta^p x^p) - \beta^{-1} = \beta^{p-1} x^p,$$

which shows that the height is exactly 1.

A formal group law on E_* is the same data as a map from the Lazard ring L, so the complex orientation gives a map $L \to E_*$. In particular, since MU is complex oriented, there is a canonical map $L \to \mathrm{MU}_*$.

Theorem 1.2.13 (Quillen, [Rav86, 4.1.6])**.** *The canonical map* $L \to \mathrm{MU}_*$ *is an isomorphism.*

TODO consider adding something about the $X(n)$, and the obstruction to lifting to a complex orientation, with obstructions living in odd E_k, so even are automatically orientable

1.3 BP, Morava K-Theory and Morava E-Theory

A good principle in homotopy theory (and in many other areas in math) is to study it one prime at a time. This is possible in homotopy theory due to the arithmetic square **TODO reference**. So, let us fix a prime p. We can form $\mathrm{MU}_{(p)}$, the p-localization of MU.

Theorem 1.3.1 ([Ada74, II 15])**.** *There exists a unique map of ring spectra* $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ *(depending on the prime p) satisfying:*

- ε *is an idempotent, i.e.* $\varepsilon^2 = \varepsilon$*,*
- ε_* *sends* $[\mathbb{C}\mathrm{P}^n] \in \pi_* \left(\mathrm{MU}_{(p)}\right)$ *to itself if* $n = p^r - 1$ *and to* 0 *otherwise.*

The map $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ gives a cohomology operation, for every X we have $\varepsilon^* : \mathrm{MU}^*_{(p)}(X) \to \mathrm{MU}^*_{(p)}(X)$. Denote by $\mathrm{BP}^*_{(p)}(X)$ the image of ε^*.

Theorem 1.3.2 ([Ada74, II 16], [Rav86, 4.1.12])**.** BP *is a cohomology theory, represented by an associative commutative ring spectrum* BP *(depending on the prime p), which is a retract of* $\mathrm{MU}_{(p)}$*. The homotopy groups of* BP *are* $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ *where* $|v_n| = 2(p^n - 1)$*.*

For convenience we denote $v_0 = p$ (and indeed $|v_0| = 2\left(p^0 - 1\right) = 0$). Since BP is a retract of MU, it comes with a map $\mathrm{MU} \to \mathrm{BP}$, that is, a complex orientation.

Proposition 1.3.3. *[**TODO reference**] The p-series of the formal group law associated to* BP *is* $[p](x) = \sum v_n x^{p^n}$*.*

Once we have BP, we can turn to the definition of Morava K-theory and Morava E-theory

Definition 1.3.4. Let $0 < n < \infty$. *Morava K-theory* at height n and prime p, denoted by $\mathrm{K}(p,n)$ or $\mathrm{K}(n)$ when the prime is clear from the context, is the spectrum obtained by killing $p = v_0, \dots, v_{n-1}, v_{n+1}, \dots$ in BP and inverting v_n. Therefore $\mathrm{K}(n)_* = \mathbb{F}_p\left[v_n^{\pm 1}\right]$. We also define $\mathrm{K}(0) = \mathrm{H}\mathbb{Q}$ and $\mathrm{K}(\infty) = \mathrm{H}\mathbb{F}_p$. Similarly, *Morava E-theory* at height n and prime p, denoted by $\mathrm{E}(p,n)$ or $\mathrm{E}(n)$, is the spectrum obtained by killing $v_{n+1}, v_{n+2}, \dots$ in BP and inverting v_n. Therefore $\mathrm{E}(n)_* = \mathbb{Z}_{(p)}\left[v_1, \dots v_{n-1}, v_n^{\pm 1}\right]$.

Since Morava K-theory and E-theory are obtained from BP by cofibers and filtered colimits, they are equipped with a map from BP, hence also with a complex orientation. Then, from 1.3.3, we get:

Corollary 1.3.5. *The p-series associated to the formal group laws of* $\mathrm{K}(n)$ *and* $\mathrm{E}(n)$ *are* $v_n x^{p^n}$ *and* $v_0 x + \dots v_n x^{p^n}$ *respectively, and are therefore of height exactly* n *and height* $\leq n$ *respectively. (Note that by the example of* $\mathrm{H}R$*, this is also true for* $\mathrm{K}(0)$ *and* $\mathrm{K}(\infty)$*.)*

We want to describe some properties of Morava K-theory. To do so we first need some definitions.

Definition 1.3.6. Let R be an evenly graded ring. R is called a *graded field* if it satisfies one of the equivalent conditions:

- every non-zero homogenus element is invertible,
- it is a field F concentrated at degree 0, or $F\left[\beta^{\pm 1}\right]$ for β of positive even degree.

An $\mathbb{A}_\infty$-ring E is a *field* if E_* is a graded field.

Example 1.3.7. $K(n)$ is a field for $0 \leq n \leq \infty$.

Proposition 1.3.8. *A field E has Kunneth, i.e. $E_*(X \otimes Y) \cong E_*(X) \otimes_{E_*} E_*(Y)$ for any spectra X, Y.*

Proposition 1.3.9 ([Lur10, 24])**.** *Let $E \neq 0$ be a complex oriented cohomology theory, whose formal group law has height exactly n, then $E \otimes \mathrm{K}(n) \neq 0$. Let E be a field s.t. $E \otimes \mathrm{K}(n) \neq 0$, then E admits the structure of a $\mathrm{K}(n)$-module. (Here $0 \leq n \leq \infty$.)*

Example 1.3.10. As we have seen before, mod-p K-theory, K/p, has height exactly 1 and coefficients $(\mathrm{K}/p)_* = \mathbb{F}_p\left[\beta^{\pm 1}\right]$. It is also known that K and K/p, are A_∞ ring spectra, from which it follows that K/p is a field. We deduce that K/p is a $\mathrm{K}(1)$-module. Since $|\beta| = 2$ and $|v_1| = 2(p-1)$ it is free of rank $p-1$.

From this we also deduce some form of uniqueness for Morava K-theory:

Corollary 1.3.11. *Let E be a field with $E_* \cong \mathbb{F}_p\left[v_n^{\pm 1}\right]$, which is also complex oriented of height exactly n. Then $E \cong \mathrm{K}(n)$ (as spectra).*

1.4 $\operatorname{Spec} \mathbb{S}_{(p)}$ and $\operatorname{Spec} \mathbb{S}$

We are now in a position to describe the topological space $\operatorname{Spec} \mathbb{S}$. However, it will be easier to state it first for $\operatorname{Spec} \mathbb{S}_{(p)}$, and then pullback prime ideals. We know that $\mathrm{Mod}_{\mathbb{S}_{(p)}} = \mathrm{Sp}_{(p)}$, and its compact objects are $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$, the p-localizations of finite spectra.

Proposition 1.4.1. *Let $\mathcal{T}_E$ be the E-acyclics, i.e.*

$$\mathcal{T}_E = \ker E_* = \left\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid E_*(X) = 0\right\} = \left\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid X \otimes E_* = 0\right\}.$$

Then $\mathcal{T}_E$ is thick.

Proof. The proof follows the same lines of 1.1.4 for the case $\mathrm{Ch}_{\mathrm{perf}}(R)$. $\square$

Definition 1.4.2. We define $\mathcal{C}_{p,n} = \mathcal{T}_{\mathrm{K}(n)}$, the $\mathrm{K}(n)$-acyclics. By the above proposition, it is thick. Also, $\mathcal{C}_{p,\infty} = \{0\}$, which is trivially thick. When the prime is clear from the context, we write $\mathcal{C}_n$ in place of $\mathcal{C}_{p,n}$.

Proposition 1.4.3 ([Lur10, 26])**.** *For $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$, if $\mathrm{K}(n)_*(X) = 0$ then $\mathrm{K}(n-1)_*(X) = 0$.*

Definition 1.4.4. We say that a spectrum $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$ is of *type n* (possibly ∞) if its first non-zero Morava K-theory homology is $\mathrm{K}(n)$.

Corollary 1.4.5. *$\mathcal{C}_n$ is the full subcategory of finite p-local spectra of type $> n$, that is $\mathcal{C}_n = \left\{X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)} \mid \forall m \leq n : \mathrm{K}(m)_*(X) = 0\right\}$. Thus $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$.*

Proposition 1.4.6 (**TODO reference**)**.** *The inclusions $\mathcal{C}_{n+1} \subset \mathcal{C}_n$ are proper.*

Proposition 1.4.7. *If $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$ is not contractible, then is of finite type. Therefore $\bigcap_{n<\infty} \mathcal{C}_n = \{0\} = \mathcal{C}_\infty$.*

Proof. Let X be non-contractible. Then $\mathrm{H}\mathbb{Z}_*(X) \neq 0$. Let m be the first non-zero degree. Using the universal coefficient theorem and the fact that the spectrum is p-local we get that $(\mathrm{H}\mathbb{F}_p)_m(X) \neq 0$, thus $(\mathrm{H}\mathbb{F}_p)_*(X) \neq 0$. Since X is finite, $(\mathrm{H}\mathbb{F}_p)_*(X)$ is bounded. The Atiyah-Hirzebruch spectral sequence for X with cohomology $\mathrm{K}(n)$ has E^2-page given by $E^2_{p,q} = H_p\left(X; \mathrm{K}(n)_q\right)$. Since $\mathrm{K}(n)_q = \mathbb{F}_p$ for $q = 0 \mod 2(p^n - 1)$ and 0 otherwise, we see that the rows $q = 0 \mod 2(p^n-1)$ are $(\mathrm{H}\mathbb{F}_p)_*(X)$, and the others are 0. Therefore if we take n such that the period $2(p^n-1)$ is larger then the bound on $(\mathrm{H}\mathbb{F}_p)_*(X)$, then all differentials have either source or target 0. Thus, the spectral sequence collapses at the E^2-page, and since $(\mathrm{H}\mathbb{F}_p)_*(X) \neq 0$, we ge that $\mathrm{K}(n)(X) \neq 0$, i.e. X has type $< n$. $\square$

Proposition 1.4.8. *$\mathcal{C}_n$ is a prime ideal.*

Proof. For $X, Y \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$, by Kunneth we have

$$\mathrm{K}(n-1)_*(X \otimes Y) = \mathrm{K}(n-1)_*(X) \otimes \mathrm{K}(n-1)_*(Y).$$

Assume that $X \in \mathcal{C}_n$, that is $\mathrm{K}(n-1)_*(X) = 0$. It follows that $\mathrm{K}(n-1)_*(X \otimes Y) = 0$, i.e. $X \otimes Y \in \mathcal{C}_n$, so $\mathcal{C}_n$ is an ideal. Assume that $X \otimes Y \in \mathcal{C}_n$, that is $\mathrm{K}(n-1)_*(X \otimes Y)$, therefore one of the terms in the RHS of the equation must vanish (since they are graded vector spaces), so $\mathcal{C}_n$ is a prime ideal. □

Theorem 1.4.9 (Thick Subcategory Theorem [HS98])**.** *If $\mathcal{T}$ is a proper thick subcategory of $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$, then $\mathcal{T} = \mathcal{C}_n$ for some $0 \leq n \leq \infty$.*

Remark 1.4.10. The proof relies on a major theorem called the Nilpotence Theorem.

Corollary 1.4.11. $\mathrm{Spec}\,\mathbb{S}_{(p)} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_\infty\}$*, and the closed subsets in the topology are chains* $\{\mathcal{C}_k, \mathcal{C}_{k+1}, \ldots, \mathcal{C}_\infty\}$ *for some* $0 \leq k \leq \infty$*.*

Proof. Follows immediately from the previous results. □

We now want to describe $\mathrm{Spec}\,\mathbb{S}$. Note that the p-localization functor $L_{(p)}$ is a Bousfield localization. As such, it is left (its right adjoint is the inclusion), and in particular preserves cofibers. It also clearly reduced, i.e. sends 0 to 0. Now, $L_{(p)}$ is smashing, that is $L_{(p)}X = X \otimes \mathbb{S}_{(p)}$, so it is also symmetric monoidal. As we have seen in 1.1.8, under these conditions we can pullback primes. Therefore

$$\mathcal{P}_{p,n} = L_{(p)}^* \mathcal{C}_{p,n} = \left\{ X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{K}(n)_* \left(X_{(p)}\right) = 0 \right\}$$

and

$$\mathcal{P}_{p,\infty} = L_{(p)}^* \mathcal{C}_{p,\infty} = \left\{ X \in \mathrm{Sp}^{\mathrm{fin}} \mid X_{(p)} = 0 \right\}$$

are prime ideals. Note that $\mathrm{H}\mathbb{Q}_* \left(X_{(p)}\right) = 0$ if and only if $\mathrm{H}\mathbb{Q}_*(X) = 0$ (this follows from the fact that rationalization and p-localization are both smashing, and the rational sphere is obtained from the p-local by inverting p), hence $\mathcal{P}_{p,0} = \left\{ X \in \mathrm{Sp}^{\mathrm{fin}} \mid \mathrm{H}\mathbb{Q}_*(X) = 0 \right\}$. We see that $\mathcal{P}_{p,0}$ is independent of p, and we denote it by $\mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}}$.

Theorem 1.4.12 (**TODO explain/reference**)**.** $\mathrm{Spec}\,\mathbb{S} = \left\{ \mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}} \right\} \bigcup \left(\bigcup_p \{\mathcal{P}_{p,1}, \ldots, \mathcal{P}_{p,\infty}\} \right)$*, and the closed subsets in the topology are finite unions of chains* $\{\mathcal{P}_{p,k}, \mathcal{P}_{p,k+1}, \ldots, \mathcal{P}_{p,\infty}\}$ *for some* $0 \leq k \leq \infty$ *(i.e. they may include* $\mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}}$*).* ***TODO diagram***

TODO regarding the topology, maybe I should prove that the pullback is also continuous?

Remark 1.4.13. Thick subcategories are interesting for another reason, unrelated to the Balmer spectrum point of view, namely they give a very powerful proof method. Say we have a property that is satisfied by 0, and is closed under cofibers and retracts. It follows that the collection of objects that satisfy it is thick. Then, for example, by the thick subcategory theorem 1.4.9, it is enough to find one object in $\mathcal{C}_n \setminus \mathcal{C}_{n-1}$ that satisfies the property, to show that all objects in $\mathcal{C}_n$ satisfy it.

1.5 The Stacky Point of View and the Relationship Between Morava K-Theory and Morava E-Theory

First we will describe, without being precise, another point of view on what chromatic homotopy theory is about.

There is a stack of formal groups with strict isomorphisms, denoted by $\mathcal{M}^s_{\mathrm{fg}}$. It can be described as the stack that sends a ring R to the groupoid of formal group laws, with strict isomorphisms between them. Quillen theorem 1.2.13 tells us that MU_* is the Lazard ring, that is the universal ring that carries the universal formal group law. It turns out that this theorem has a second part, which says that $(\mathrm{MU} \otimes \mathrm{MU})_*$ is the universal ring that carries two formal group laws and a strict isomorphism between them. Therefore, $\mathcal{M}^s_{\mathrm{fg}}$ is represented by $(\mathrm{MU}_*, (\mathrm{MU} \otimes \mathrm{MU})_*)$.

The geometric points of the stack $\mathcal{M}^s_{\mathrm{fg}}$ are describe precisely the same as $\operatorname{Spec} \mathbb{S}$, that is because for an algebraically closed field of characteristic 0 there is a unique (up to isomorphism) formal group law which is of height 0 namely the additive formal group law, and for characteristic p there is a unique (up to isomorphism) formal group law of each height $1 \leq n \leq \infty$.

For a spectrum X, $\mathrm{MU}_*(X)$ is a $(\mathrm{MU}_*, (\mathrm{MU} \otimes \mathrm{MU})_*)$-comodule, which is the same as a sheaf over $\mathcal{M}^s_{\mathrm{fg}}$. From this point of view, chromatic homotopy theory lets us study a spectrum by decomposing it over the stack $\mathcal{M}^s_{\mathrm{fg}}$.

We can restrict ourselves to the stack only over rings of characteristic p, $\mathcal{M}^s_{\mathrm{fg},p}$, which is then represented by $\left(\left(\mathrm{MU}_{(p)}\right)_*, \left(\mathrm{MU}_{(p)} \otimes \mathrm{MU}_{(p)}\right)_*\right)$. **TODO I think it's true, is it?** Similarly to MU, BP is universal ring with the universal p-typical formal group law, and $(\mathrm{BP} \otimes \mathrm{BP})_*$ is the universal ring with two p-typical formal group laws and an isomorphism between them

TODO I didn't say this before. Since every formal group law is isomorphic to a p-typical one, we know that the stack $\mathcal{M}^{s}_{\mathrm{fg},p}$ is also represented by $(\mathrm{BP}_*, (\mathrm{BP} \otimes \mathrm{BP})_*)$.

It is now reasonable that $\mathrm{K}(n)$, obtained from BP by killing the v_m's for $m \neq n$ and inverting v_n, sees the n-th level, and that $\mathrm{E}(n)$ obtained in the same way but only killing v_m for $m > n$, sees the levels $\leq n$.

Let us now claim a precise statement, formalizing this description.

Theorem 1.5.1 (**TODO reference**)**.** *$\mathrm{E}(n)$ and $\mathrm{K}(0) \oplus \cdots \oplus \mathrm{K}(n)$ are Bousfield equivalent. That is, they have the same acyclics, locals, and their localization functors are the same.*

TODO chromatic square and chromatic tower, maybe another subsection?

1.6 Landweber Exact Functor Theorem

As we have seen, a complex orientation on a cohomology theory, which is described by a map $\mathrm{MU} \to E$, has an associated formal group law, which is described by the map $L = \mathrm{MU}_* \to E_*$. Note that this formal group law is of degree -2, by virtue of the grading on $L = \mathrm{MU}_*$ **TODO say more?**. One can ask whether the converse is true, namely given a graded ring R and a formal group law F of degree -2 given by $L \to R$, is there a complex oriented cohomology theory whose coefficients are R and the associated formal group law is F.

A strategy is to define $(E_{R,F})_* (X) = \mathrm{MU}_* (X) \otimes_{\mathrm{MU}_*} R$. Unfortunately, this is not always a homology theory. However there is a condition that one can check, which guarantees that it is.

Definition 1.6.1. $L \to R$ is called *Landweber flat* if for every prime p, the image of the sequence $p = v_0, v_1, v_2, \dots$ in R, which are the coefficients of the p-series, is regular. That is, for each p and $n \geq 0$, v_n is not a zero divisor in $R/(v_0, v_1, \dots, v_{n-1})$.

Remark 1.6.2. If p is invertible in R, then p is invertible, and R/p is already 0, so we don't need to check $v_1, v_2, \dots$.

Theorem 1.6.3 (Landweber Exact Functor Theorem (LEFT), [Lur10, 15, 16])**.** *If $L \to R$ is Landweber flat, then $E_{R,F}$ defined above is a homology* *theory. Moreover, there are no phantom maps between such spectra, so $E_{R,F}$ is represented by a spectrum. This spectrum is complex oriented, has coefficients R and associated formal group law F.*

Example 1.6.4. We return to complex K-theory, from 1.2.3 and 1.2.11. We can take the completion at the element $p \in \mathrm{K}_*$, which gives the spectrum $\mathrm{K}_p^\wedge$. This spectrum has coefficients $\left(\mathrm{K}_p^\wedge\right)_* = \left(\mathrm{K}_*\right)_p^\wedge = \left(\mathbb{Z}\left[\beta^{\pm1}\right]\right)_p^\wedge = \mathbb{Z}_p^\wedge\left[\beta^{\pm1}\right]$. The formal group law, as we have seen, is given by $F_{\mathrm{K}_p^\wedge}(y,z) = y + z + \beta yz$. We claim that $F_{\mathrm{K}_p^\wedge}/\mathrm{K}_p^\wedge$ is Landweber flat. Clearly $p = v_0$ is not a zero divisor in $\mathbb{Z}_p^\wedge\left[\beta^{\pm1}\right]$. As we have seen in 1.2.12, mod-p the p-series is $\beta^{p-1}x^p$, so that $v_1 = \beta^{p-1}$ which is not a zero divisor $\mathbb{F}_p\left[\beta^{\pm1}\right]$. Modulo v_1 the ring is already 0, and we are done. For other primes, by 1.6.2 we are done. Therefore, by 1.6.8 we get that $\mathrm{K}_p^\wedge \cong E_{\mathrm{K}_p^\wedge, F_{\mathrm{K}_p^\wedge}}$.

Example 1.6.5. Morava E-theory is Landweber flat, since by 1.3.5, the p-series has coefficients $p = v_0, v_1, \ldots, v_n$. p is not a zero divisor in $\mathrm{E}(n)_* = \mathbb{Z}_{(p)}\left[v_1, \ldots v_{n-1}, v_n^{\pm1}\right]$. Then v_i is not a zero divisor in $\mathrm{E}(n)_*/(p, v_1, \ldots, v_{i-1}) \cong \mathbb{F}_p\left[v_i, \ldots v_{n-1}, v_n^{\pm1}\right]$. After v_n the ring becomes 0 and we are done. For other primes, by 1.6.2 we are done.

Example 1.6.6. Morava K-theory $\mathrm{K}(n)$ for $n > 0$ is not Landweber flat since $p = v_0$ is not invertible in $\mathrm{K}(n)_* = \mathbb{F}_p\left[v_n^{\pm1}\right]$.

Example 1.6.7. $\mathrm{H}\mathbb{Z}$ is not Landweber flat since although $p = v_0$ is invertible, as we have seen in 1.2.10 the p-series is px, so v_1 is 0 in $\mathbb{Z}/p = \mathbb{F}_p$, which is a zero divisor.

We can also ask the following question: given complex oriented cohomology theory $\mathrm{MU} \to E$, such that $L \to E_*$ is Landweber flat, is $E_{R,F}$ equivalent to E? The answer is yes, at least in some cases.

Theorem 1.6.8. *Let E be as above, which is also evenly graded (i.e. E_* is an evenly graded ring), then there is an equivalence $E_{R,F} \to E$.*

Proof. This is a slight variation on [Lur10, 18, proposition 11]. First note that for every spectrum X we have $\mathrm{MU} \otimes X \to E \otimes X$, which induces $\mathrm{MU}_*(X) \to E_*(X)$, a map of MU_*-modules. Moreover, since $E_* \to E_*(X)$ is a map of E_*-module, the map $\mathrm{MU}_* \to E_*$ makes it a map of MU_*-modules. Together this gives a map $\left(E_{R,F}\right)_*(X) = \mathrm{MU}_*(X) \otimes_{\mathrm{MU}_*} E_* \to E_*(X)$. This map is a map of homology theories. **TODO should I explain why?** By [Lur10, 17, theorem 6] **TODO does it follow?**, this map lifts to a map of spectra $E_{R,F} \to E$. Since by construction when $X = \mathbb{S}$ the map above is

$E_* \to E_*$ which is an isomorphism, we see that the map $E_{R,F} \to E$ is an equivalence. $\square$

1.7 Lubin-Tate Deformation Theory

The Morava E-theory we have considered until now $\mathrm{E}(n)$, also called Johnson-Wilson spectrum was constructed from BP. As we noted, it is Landweber flat, which indicates that there is another approach to constructing it. Indeed there is a way to construct a related spectrum, which will be called the Lubin-Tate spectrum.

To that end, we first define the category CompRing as the category of complete local rings. The objects are complete local rings $(R, \mathfrak{m})$, we also denote by $\pi : R \to R/\mathfrak{m}$ the projection. Morphisms $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ are local homomorphisms, i.e. a homomorphism $\varphi : R \to S$ s.t. $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. In particular it induces a homomorphism $\varphi/\mathfrak{m} : R/\mathfrak{m} \to S/\mathfrak{n}$, which satisfies $\varphi/\mathfrak{m} \circ \pi_R = \pi_S \circ \varphi$.

We fix k be a perfect field of characteristic p (i.e. the Frobenius is an isomorphism), and Γ a formal group law over k of height $n < \infty$. Lubin and Tate [LT66] considered a moduli problem associated to Γ/k, described by a functor $\mathrm{Def} : \mathrm{CompRing} \to \mathrm{Grpds}$.

Definition 1.7.1. Let $(R, \mathfrak{m})$ be a complete local ring and denote by $\pi : R \to R/\mathfrak{m}$ the quotient. A *deformation* of Γ/k to $(R, \mathfrak{m})$, is (G, i), where G is a formal group law over R, $i : k \to R/\mathfrak{m}$ is a homomorphism of fields, such that $i^*\Gamma = \pi^* G$. A $\star$*-isomorphism* between two deformations to $(R, \mathfrak{m})$, $f : (G_1, i_1) \to (G_2, i_2)$, is defined only when $i_1 = i_2$, and consists of an isomorphism $f : G_1 \to G_2$, such that $\pi^* f : i^*\Gamma = \pi^* G_1 \to \pi^* G_2 \to i^*\Gamma$ is the identity, i.e. $f(x) = x \mod \mathfrak{m}$. These assemble to a groupoid $\mathrm{Def}(R, \mathfrak{m})$, whose objects are deformations to $(R, \mathfrak{m})$, and morphisms are $\star$-isomorphisms.

Remark 1.7.2. $\mathrm{Def}(R, \mathfrak{m})$ can be seen as the pullback of the groupoids $\mathrm{FGL}(R)$ and $\coprod_{i:k \to R/\mathfrak{m}} \{\Gamma\}$ over $\mathrm{FGL}(R/\mathfrak{m})$ (where the maps are $G \mapsto q^* G$ and $i \mapsto i^*\Gamma$ respectively).

Proposition 1.7.3 (/definition)**.** *The construction* $\mathrm{Def}(R, \mathfrak{m})$ *is functorial.*

Proof. Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism.

For a deformation (G, i) to $(R, \mathfrak{m})$, we define $\mathrm{Def}(\varphi)(G, i) = (\varphi^* G, \varphi/\mathfrak{m} \circ i)$. Note that $\varphi^* G$ is a formal group law over S, and $\varphi/\mathfrak{m} \circ i : k \to R/\mathfrak{m} \to S/\mathfrak{n}$ is a homomorphism. Moreover, $(\varphi/\mathfrak{m} \circ i)^* \Gamma = (\varphi/\mathfrak{m})^* i^* \Gamma = (\varphi/\mathfrak{m})^* \pi_R^* G = (\varphi/\mathfrak{m} \circ \pi_R)^* G = (\pi_S \circ \varphi)^* G = \pi_S^* \varphi^* G$, which shows that $\mathrm{Def}(\varphi)(G, i)$ is a deformation to $(S, \mathfrak{n})$.

For a $\star$-isomorphism $f : (G_1, i) \to (G_2, i)$, which is the data of an isomorphism $f : G_1 \to G_2$ such that $\pi_R^* f = \mathrm{id}_{i^*\Gamma}$ is the identity, we need to define a $\star$-isomorphism $\mathrm{Def}(\varphi)(G, i_1) \to \mathrm{Def}(\varphi)(G, i_2)$. Take it to be the isomorphism $\varphi^* f : \varphi^* G_1 \to \varphi^* G_2$, which satisfies $\pi_S^* \varphi^* f = (\varphi/\mathfrak{m})^* \pi_R^* f = (\varphi/\mathfrak{m})^* \mathrm{id}_{i^*\Gamma} = \mathrm{id}_{(\varphi/\mathfrak{m})^* i^* \Gamma} = \mathrm{id}_{(\varphi/\mathfrak{m} \circ i)^* \Gamma}$. The identity $\mathrm{id}_G : (G, i) \to (G, i)$ is clearly sent to $\mathrm{id}_{\varphi^* G}$, and compositions are sent to compositions.

This shows that $\mathrm{Def}(\varphi) : \mathrm{Def}(R, \mathfrak{m}) \to \mathrm{Def}(S, \mathfrak{n})$ is indeed a functor. Moreover, it is clear that $\mathrm{Def}(\mathrm{id}_R)$ is the identity and compositions are sent to compositions, which shows that $\mathrm{Def} : \mathrm{CompRing} \to \mathrm{Grpds}$ is indeed a functor. $\square$

Remark 1.7.4. We recall quickly that the Witt vectors Wk is a ring of characteristic 0, with maximal ideal (p), and residue field $Wk/p \cong k$. For example, $W\mathbb{F}_p = \mathbb{Z}_p^\wedge$.

Theorem 1.7.5 ([Rez98, 4.4, 5.10], originally due to [LT66])**.** *The functor* Def *lands in discrete groupoids (i.e.* $\mathrm{Def}(R, \mathfrak{m})$ *has* 0 *or* 1 *morphisms between objects). Furthermore the functor* Def *is co-represented, that is there exists a* universal deformation*, and the complete local ring can be chosen (non-canonically) to be* $Wk[[u_1, \dots, u_{n-1}]]$.

Let us unravel what that means. First note that the quotient of $Wk[[u_1, \dots, u_{n-1}]]$ by the maximal ideal $(p, u_1, \dots, u_{n-1})$ is k. The universal deformation can be chosen such that the formal group law over it Γ_U over $Wk[[u_1, \dots, u_{n-1}]]$ satisfies $\pi^* \Gamma_U$ is Γ. The universality means that for $(R, \mathfrak{m})$, the assignment

$$\hom_{\mathrm{CompRing}}(Wk[[u_1, \dots, u_{n-1}]], R) \to \mathrm{Def}(R, \mathfrak{m}), \quad \varphi \mapsto \varphi^* \Gamma_U$$

is an equivalence.

Now, we can form the graded ring $Wk[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$ where $|u| = 2$. We can define the formal group law $(u\Gamma_U)(x, y) = u^{-1} \Gamma_U(uy, uz)$, which is of degree -2, thus we get a map $L \to Wk[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$.

Proposition 1.7.6 ([Rez98, 6.9])**.** $L \to Wk[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$ *is Landweber flat.*

Using LEFT 1.6.3, we immediately get:

Corollary 1.7.7. *There is a complex oriented cohomology theory* $\mathrm{E}(k,\Gamma) = E_{Wk[[u_1,\ldots,u_{n-1}]][u^{\pm 1}],u\Gamma_U}$*, called Lubin-Tate spectrum, with coefficients* $\mathrm{E}(k,\Gamma)_* = Wk[[u_1,\ldots,u_{n-1}]]\left[u^{\pm 1}\right]$ *and associated formal group law* $u\Gamma_U$*.*

Example 1.7.8. We continue the complex K-theory saga from 1.6.4. Take the field $k = \mathbb{F}_p$ and the formal group law $\Gamma(y,z) = y + z + yz$, of height $n = 1$. By the above construction, the ring of the universal deformation is $W\mathbb{F}_p = \mathbb{Z}_p^{\wedge}$. The universal formal group law of the universal deformation can be taken to be $\Gamma_U(y,z) = y + z + yz$ (this follows from the proof at [Rez98, 5.10], since here $n = 1$ so there are no u_i's). We look at the ring $\mathbb{Z}_p^{\wedge}\left[u^{\pm 1}\right]$, and at the formal group law over it $(u\Gamma_U)(y,z) = u^{-1}\left(uy + uz + u^2yz\right) = y + z + uyz$. It is then clear that the isomorphism $\mathbb{Z}_p^{\wedge}\left[u^{\pm 1}\right] \to \mathbb{Z}_p^{\wedge}\left[\beta^{\pm 1}\right]$, sends $u\Gamma_U$ to $F_{\mathrm{K}_p^{\wedge}}$. It follows that $\mathrm{K}_p^{\wedge} \cong E_{\mathrm{K}_p^{\wedge},F_{\mathrm{K}_p^{\wedge}}} \cong \mathrm{E}(\mathbb{F}_p,\Gamma)$, i.e. p-complete K-theory $\mathrm{K}_p^{\wedge}$ is a Lubin-Tate spectrum at height 1.

This concludes the construction of the Lubin-Tate variant of Morava E-theory. To compare this version with the previous, we define another variant of Morava E-theory.

Definition 1.7.9. Let $I \leq \mathrm{E}(n)_* = \mathbb{Z}_{(p)}\left[v_1,\ldots v_{n-1},v_n^{\pm 1}\right]$ be the ideal $I = (p = v_0, v_1, \ldots v_{n-1})$. We define the spectrum $\widehat{\mathrm{E}(n)} = \mathrm{E}(n)_I^{\wedge}$, called completed Johnson-Wilson spectrum.

Theorem 1.7.10. *The following three forms of Morava E-theory are Bousfield equivalent:*

1. *Johnson-Wilson* $\mathrm{E}(n)$*,*
2. *completed Johnson-Wilson* $\widehat{\mathrm{E}(n)}$*,*
3. *Lubin-Tate* $\mathrm{E}(k,\Gamma)$*.*

TODO reference Hovey Strickland Morava K-Theories and Localization 5.3

2 Atiyah-Segal

We now leave the realm of chromatic homotopy theory. One aspect of algebraic topology is to try to capture properties of spaces using algebraic invariants. One of the most fruitful such invariants is complex K-theory K, and one of the most important spaces in homotopy theory is $\mathrm{B}G$, so it is natural to ask for a description of $\mathrm{K}(\mathrm{B}G)$ (by Bott periodicity, we will consider only $\mathrm{K} = \mathrm{K}^0$). Atiyah and Segal [AS69] gave a description of this, and more, in the case that G is a compact Lie group, in terms of representations.

From now we fix a compact Lie group G. Also, a representation means a unitary representation. We should also note that beyond this part, we will be mostly interested in finite groups.

2.1 The Atiyah-Segal Theorem

We denote by $\mathrm{R}(G)$ the *representation ring* of G, that is the collection of virtual representations of G (which can be written is as a formal difference $V - U$) where the addition is given by direct sum and the product is given by tensor product. This is an augmented ring $\varepsilon : \mathrm{R}(G) \to \mathbb{Z}$ by the virtual dimension (i.e. $\varepsilon(V - U) = \dim(V - U) = \dim V - \dim U$). The *augmentation ideal* is $I = \ker \varepsilon = \{V - U \in \mathrm{R}(G) \mid 0 = \dim(V - U)\}$.

Theorem 2.1.1 ([AS69])**.** $\mathrm{K}(\mathrm{B}G) \cong \mathrm{R}(G)^{\wedge}_{I}$.

We will not prove the theorem, but we will indicate some of the key ingredients.

First of all, to show that objects are isomorphic, we need a map. Before giving the map actually used in the proof, we describe an easier way to see where this map comes from. Recall that $\mathrm{K}(X) \cong [X, \mathrm{BU} \times \mathbb{Z}]$. The data of a representation of G is the same thing as a homomorphism $G \to \mathrm{U}(n)$. Since B is a functor, we get a map $\mathrm{B}G \to \mathrm{BU}(n)$, and by composing with the injection $\mathrm{BU}(n) \cong \mathrm{BU}(n) \times \{n\} \to \mathrm{BU} \times \mathbb{Z}$, we indeed get a map $\mathrm{B}G \to \mathrm{BU} \times \mathbb{Z}$, that is, and element of $\mathrm{K}(\mathrm{B}G)$. Therefore we get a map $\mathrm{R}(G) \to \mathrm{K}(\mathrm{B}G)$. The theorem is that it is a ring homomorphism which exhibits $\mathrm{K}(\mathrm{B}G)$ as the completion of $\mathrm{R}(G)$ at I.

There is an alternative description of this map. In [Seg68], Segal described equivariant K-theory K_G. This is a variant of K-theory, which assigns to a G-space the ring of virtual G-bundles, that is bundles equipped with an action of G, compatible with the action on the base G-space. Note that this is no longer a homotopy invariant, since it also takes into account the action of G. First we note the following:

Proposition 2.1.2. $\mathrm{K}_G(*) = \mathrm{R}(G)$ *(where $*$ denotes the trivial G-space)*

Proof. This is by definition, since a vector bundle over a point is just a vector space, and it is equipped with a G action. $\square$

For any G-space X, the projection map $\mathrm{pr} : X \to X/G$ allows us to pullback vector bundles on X/G, to G-bundles on X, that is it induces a map $\mathrm{pr}^* : \mathrm{K}(X/G) \to \mathrm{K}_G(X)$.

Proposition 2.1.3 ([Seg68, 2.1])**.** *If G acts* freely *on X, then there is an inverse to* pr^**, so* $\mathrm{K}(X/G) \cong \mathrm{K}_G(X)$*. The inverse is given by taking a bundle $E \to X$ to $E/G \to X/G$.*

Now, we have a map of G-spaces given by $\mathrm{E}G \to *$. By the above we get:

$$\mathrm{R}(G) \cong \mathrm{K}_G(*) \to \mathrm{K}_G(\mathrm{E}G) \cong \mathrm{K}(\mathrm{E}G/G) = \mathrm{K}(\mathrm{B}G)$$

This is again the map we need, which exhibits $\mathrm{K}(\mathrm{B}G)$ as the I-completion of $\mathrm{R}(G)$. Atiyah and Segal use this map and variants to show the theorem.

The way Atiyah and Segal's proof works is as follows. First of all we note that they prove it for the whole K^* rather than for $\mathrm{K} = \mathrm{K}^0$. Also note that $\mathrm{R}^*(G) = \mathrm{K}^*_G(*)$ is a 2-periodic version of the representation ring (because K^*_G also satisfies Bott periodicity). Within it there is the 2-periodic version of the augmentation ideal I. **TODO is this what they really use?** They use the Milnor join construction $\mathrm{E}G_n = \underbrace{G * \cdots * G}_{n \text{ times}}$ and $\mathrm{B}G_n = \mathrm{E}G_n/G$, which has the property that $\operatorname{colim} \mathrm{E}G_n \to \operatorname{colim} \mathrm{B}G_n$ is a model for $\mathrm{E}G \to \mathrm{B}G$. Then, for any compact G-space X there is a similar map to the map above, using $X \times \mathrm{E}G_n \to X$ we get a map $\mathrm{K}^*_G(X) \to \mathrm{K}^*_G(X \times \mathrm{E}G_n)$. All of these are $\mathrm{R}^*(G)$-modules, and they show that this map factors through the quotient I^n, to give a map $\mathrm{K}^*_G(X)/I^n \to \mathrm{K}^*_G(X \times \mathrm{E}G_n)$. The two sides assemble into pro-rings, and the maps assemble to a map between the pro-rings. What they actually prove is the strong form:

Theorem 2.1.4 ([AS69])**.** *If $\mathrm{K}^*_G(X)$ is finite over $\mathrm{R}^*(G)$, then the above map of pro-rings is an isomorphism.*

Their proof is interesting in another respect, it uses different groups to deduce the result for general G. In particular, to prove the result for example for a finite group, they have to deal with more general compact Lie groups. The proof has 4 steps:

- Prove for $G = \mathrm{U}(1)$ (circle group),

- Prove for $G = \mathrm{U}(1)^n$ (torus group),
- Prove for $G = \mathrm{U}(n)$,
- Prove for general compact Lie group G by embedding in $\mathrm{U}(n)$.

2.2 Recollections from Character Theory

We restrict ourselves to the case of finite groups G. We recall that representations of groups can be studied by their characters. Specifically the trace map $\chi : \mathrm{R}(G) \to \mathbb{Z}[\chi_{\rho_i}]$, defined by $\chi_\rho = \operatorname{tr}\rho$, is an isomorphism, where the ring on the right is the ring of functions generated by the irreducible characters (the multiplication of two characters is a character so it is indeed closed under multiplication).

We also recall that characters are class functions, that is, they are constant on conjugacy classes. Let L be some field containing all the values of all characters. Then a natural place to study characters is in the ring of class functions with values in L, denote by $\mathrm{Cl}(G; L)$. We can of course extend the range of the trace map to get an injection $\chi : \mathrm{R}(G) \to \mathrm{Cl}(G; L)$. The first classical theorem regarding the relationship between characters and class functions is:

Theorem 2.2.1. *After tensoring with L, the character map $\chi \otimes L : \mathrm{R}(G) \otimes L \to \mathrm{Cl}(G; L)$ becomes an isomorphism.*

Proof. Similarly to the proof in [Ser77, 9.1] for $L = \mathbb{C}$, we can view $\mathrm{Cl}(G; L)$ as a vector space over L, and the characters are linearly independent, so by counting them we see that the image of $\chi \otimes L$ has the dimension of the whole vector space and we are done. $\square$

By definition the value of a character is the trace of a linear transformation $\chi_\rho(g) = \operatorname{tr}\rho(g) = \sum \lambda_i$ where λ_i are the eigenvalues (which exist since the representation is unitary). Since $g^{|G|} = e$, we get $\rho\left(g^{|G|}\right) = \rho(e) = \mathrm{id}$, but then we get that the eigenvalues of $\rho\left(g^{|G|}\right)$ are on the one hand $\lambda_i^{|G|}$ and on the other hand they are all 1, which means that all the eigenvalues are roots of unity. Therefore $L = \mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\zeta_\infty)$ is always a valid choice for L (regardless of G). To be concrete, we will take this choice.

The Galois group of $\mathbb{Q}^{\mathrm{ab}}$ is $\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right) \cong \hat{\mathbb{Z}}^\times$. For every $m \in \hat{\mathbb{Z}}^\times$ we also denote by $m \in \mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)$ the corresponding element, which can be described as the homomorphism which raises a root of unity to the power of m. Similarly it acts on G by taking g to g^m. Then, for every such m and g we have that $\chi_\rho(g^m) = \operatorname{tr}\rho(g) = \sum\lambda_i^m = m.(\sum\lambda_i) = m.(\chi_\rho(g))$. We replace g with $g^{m^{-1}}$ (m is invertible), and rewrite this as $\chi_\rho(g) = m.\left(\chi_\rho\left(g^{m^{-1}}\right)\right)$. Similarly to this equality, we can define an action of $\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)$ on $\mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)$, by taking a class function f to $m.f$ defined by $(m.f)(g) = m.\left(f\left(g^{m^{-1}}\right)\right)$. As we just saw, the characters are in the fixed points $\mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$. Also, since the rationals are fixed by the action of the Galois group, rational linear combinations of characters are in the fixed points. We therefore conclude that the character map after tensoring with $\mathbb{Q}$ lands in the fixed points, i.e. $\chi\otimes\mathbb{Q} : \mathrm{R}(G)\otimes\mathbb{Q} \to \mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$. Moreover, the second classical theorem is:

Theorem 2.2.2. *The map* $\chi\otimes\mathbb{Q} : \mathrm{R}(G)\otimes\mathbb{Q} \to \mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$ *is an isomorphism.*

The proof is essentially the same as in [Ser77, 13.1, theorem 29a].

To conclude, 2.2.1 tells us that $\mathrm{R}(G)\otimes\mathbb{Q}^{\mathrm{ab}} \cong \mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)$, and 2.2.2 tells us that $\mathrm{R}(G)\otimes\mathbb{Q} \cong \mathrm{Cl}\left(G;\mathbb{Q}^{\mathrm{ab}}\right)^{\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right)}$.

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