Section 1C: Subspaces
Worked Solutions (a)–(d)

Problem Setup

Let F be a field and V a vector space over F. For each part below, we determine whether the given set is a subspace and justify the claim.

(a)

Claim. If $b \in F$, then

$$U = \{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\}$$

is a subspace of F^4 if and only if $b = 0$.

Proof. The zero vector $(0, 0, 0, 0)$ lies in U if and only if

$$0 = 5 \cdot 0 + b \iff b = 0.$$

If $b \neq 0$, then U does not contain the zero vector and hence is not a subspace.

If $b = 0$, then

$$U = \{x \in F^4 : x_3 = 5x_4\}$$

is defined by a homogeneous linear equation. One checks directly that U is closed under addition and scalar multiplication, hence is a subspace. $\square$

(b)

Claim. The set of continuous real-valued functions on $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

Proof. Let

$$U = \{f : [0, 1] \to \mathbb{R} : f \text{ is continuous}\}.$$

The zero function is continuous, so $0 \in U$. If $f, g \in U$, then $f + g$ is continuous, and if $a \in \mathbb{R}$ then af is continuous. Thus U is closed under addition and scalar multiplication, hence is a subspace. $\square$

(c)

Claim. The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Proof. Let

$$U = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}.$$

The zero function is differentiable, so $0 \in U$. If $f, g \in U$, then $f+g$ is differentiable with

$$(f+g)' = f' + g'.$$

If $a \in \mathbb{R}$, then af is differentiable with

$$(af)' = af'.$$

Thus U is closed under addition and scalar multiplication, and hence is a subspace. $\square$

(d)

Claim. The set of differentiable real-valued functions f on $(0,3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.

Proof. Let

$$U = \{f : (0,3) \to \mathbb{R} : f \text{ is differentiable and } f'(2) = b\}.$$

If U is a subspace, it must contain the zero function. Since $0'(2) = 0$, this forces $b = 0$.

Conversely, if $b = 0$, then

$$U = \{f : f'(2) = 0\}.$$

The zero function lies in U. If $f, g \in U$, then

$$(f+g)'(2) = f'(2) + g'(2) = 0,$$

so $f + g \in U$. If $a \in \mathbb{R}$, then

$$(af)'(2) = af'(2) = 0,$$

so $af \in U$. Thus U is a subspace. $\square$

(e)

Claim. The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}^\infty$.

Proof. Let

$$U = \{(z_n)_{n\geq 1} \in \mathbb{C}^\infty : \lim_{n\to\infty} z_n = 0\}.$$

Here $(z_n)_{n\geq 1}$ denotes an infinite sequence $(z_1, z_2, z_3, \ldots)$ of complex numbers, which we can think of as an infinite tuple. The space $\mathbb{C}^\infty$ is the set of all such sequences.

To show U is a subspace, we verify three conditions:

(i) Zero vector: The zero sequence $(0, 0, 0, \ldots)$ satisfies

$$\lim_{n\to\infty} 0 = 0,$$

so the zero vector is in U.

(ii) Closed under addition: Suppose $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \in U$. This means $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$. Their sum is the sequence

$$(x_n + y_n)_{n\geq 1} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots).$$

By limit laws,

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0 + 0 = 0,$$

so $(x_n + y_n)_{n\geq 1} \in U$.

(iii) Closed under scalar multiplication: Let $a \in \mathbb{C}$ and $(x_n)_{n\geq 1} \in U$. The scalar multiple is

$$(ax_n)_{n\geq 1} = (ax_1, ax_2, ax_3, \ldots).$$

By limit laws,

$$\lim_{n\to\infty} (ax_n) = a \cdot \lim_{n\to\infty} x_n = a \cdot 0 = 0,$$

so $(ax_n)_{n\geq 1} \in U$.

Therefore U is a subspace of $\mathbb{C}^\infty$. $\square$