

Section 1C: Subspaces

Worked Solutions (a)–(d)

Problem Setup

Let F be a field and V a vector space over F . For each part below, we determine whether the given set is a subspace and justify the claim.

(a)

Claim. If $b \in F$, then

$$U = \{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\}$$

is a subspace of F^4 if and only if $b = 0$.

Proof. The zero vector $(0, 0, 0, 0)$ lies in U if and only if

$$0 = 5 \cdot 0 + b \iff b = 0.$$

If $b \neq 0$, then U does not contain the zero vector and hence is not a subspace.

If $b = 0$, then

$$U = \{x \in F^4 : x_3 = 5x_4\}$$

is defined by a homogeneous linear equation. One checks directly that U is closed under addition and scalar multiplication, hence is a subspace. \square

(b)

Claim. The set of continuous real-valued functions on $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

Proof. Let

$$U = \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is continuous}\}.$$

The zero function is continuous, so $0 \in U$. If $f, g \in U$, then $f + g$ is continuous, and if $a \in \mathbb{R}$ then af is continuous. Thus U is closed under addition and scalar multiplication, hence is a subspace. \square

(c)

Claim. The set of differentiable real-valued functions on \mathbb{R} is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Proof. Let

$$U = \{f : \mathbb{R} \rightarrow \mathbb{R} : f \text{ is differentiable}\}.$$

The zero function is differentiable, so $0 \in U$. If $f, g \in U$, then $f + g$ is differentiable with

$$(f + g)' = f' + g'.$$

If $a \in \mathbb{R}$, then af is differentiable with

$$(af)' = af'.$$

Thus U is closed under addition and scalar multiplication, and hence is a subspace. \square

(d)

Claim. The set of differentiable real-valued functions f on $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.

Proof. Let

$$U = \{f : (0, 3) \rightarrow \mathbb{R} : f \text{ is differentiable and } f'(2) = b\}.$$

If U is a subspace, it must contain the zero function. Since $0'(2) = 0$, this forces $b = 0$.

Conversely, if $b = 0$, then

$$U = \{f : f'(2) = 0\}.$$

The zero function lies in U . If $f, g \in U$, then

$$(f + g)'(2) = f'(2) + g'(2) = 0,$$

so $f + g \in U$. If $a \in \mathbb{R}$, then

$$(af)'(2) = af'(2) = 0,$$

so $af \in U$. Thus U is a subspace. \square