Section 1C: Subspaces
Worked Solutions

Problem Setup

Let F be a field and V a vector space over F. For each part below, we determine whether the given set is a subspace and justify the claim.

Problem 1

For each of the following subsets of F^4, determine whether it is a subspace of F^4.

(a)

Claim.

$$U = \{(x_1 - x_2,\ x_2,\ x_1 + x_2,\ x_3 + x_4) \in F^4 :\ x_1, x_2, x_3, x_4 \in F\}$$

is a subspace of F^4. In fact,

$$U = \{(u_1, u_2, u_3, u_4) \in F^4 :\ u_3 = u_1 + 2u_2\}.$$

Proof. Define a linear map $T : F^4 \to F^4$ by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2,\ x_2,\ x_1 + x_2,\ x_3 + x_4).$$

Then $U = \mathrm{im}(T)$, hence U is a subspace.

To describe U explicitly, let $T(x_1, x_2, x_3, x_4) = (u_1, u_2, u_3, u_4)$. From $u_2 = x_2$ and $u_1 = x_1 - x_2$ we get $x_1 = u_1 + u_2$, so

$$u_3 = x_1 + x_2 = (u_1 + u_2) + u_2 = u_1 + 2u_2.$$

Conversely, given any (u_1, u_2, u_4), choosing $x_2 = u_2,\ x_1 = u_1 + u_2,\ x_3 = u_4,\ x_4 = 0$ yields $T(x_1, x_2, x_3, x_4) = (u_1, u_2, u_1 + 2u_2, u_4)$. So exactly those vectors with $u_3 = u_1 + 2u_2$ occur. □

(b)

Claim.

$$U = \{(x_1 - x_2,\ x_2,\ x_1 + x_2,\ x_3 + x_4) \in F^4 :\ x_1, x_2, x_3, x_4 \in F,\ x_3 = 0\}$$

is a subspace of F^4, and in fact it is the same subspace as in part (a).

Proof. With $x_3 = 0$ we have the form $(x_1 - x_2,\ x_2,\ x_1 + x_2,\ x_4)$. As x_4 ranges over F, the fourth coordinate is arbitrary, and the first three coordinates satisfy the same relation $u_3 = u_1 + 2u_2$ as in (a). Hence this set equals the U from (a), so it is a subspace. □

(c)

Claim.
$$U = \{(x_1 - x_2,\ x_2,\ x_1 + x_2,\ x_3 + x_4) \in F^4 :\ x_1, x_2, x_3, x_4 \in F,\ x_3 = x_4\}$$
is a subspace of F^4. Moreover,
$$U = \{(u_1, u_2, u_1 + 2u_2,\ 2t) : u_1, u_2, t \in F\}.$$
In particular, if $\mathrm{char}(F) \neq 2$ then $2t$ ranges over all of F, so U coincides with the subspace from (a); if $\mathrm{char}(F) = 2$ then $2t = 0$ and the fourth coordinate must be 0.

Proof. If $x_3 = x_4 = t$, then the fourth coordinate is $x_3 + x_4 = 2t$, and the first three coordinates are as in (a), hence satisfy $u_3 = u_1 + 2u_2$. This is the image of the linear map $F^3 \to F^4$ given by $(x_1, x_2, t) \mapsto (x_1 - x_2, x_2, x_1 + x_2, 2t)$, so it is a subspace, with the stated description. □

(d)

Claim.
$$U = \{(x_1 - x_2,\ x_2,\ x_1 + x_2,\ x_3 + x_4) \in F^4 :\ x_1 = x_2 = 5x_3\}$$
is a subspace of F^4.

Proof. Write $t = x_3$ and $s = x_4$. Then $x_1 = x_2 = 5t$ and the vector becomes
$$(5t - 5t,\ 5t,\ 5t + 5t,\ t + s) = (0,\ 5t,\ 10t,\ t + s).$$
Thus
$$U = \{(0, 5t, 10t, t + s) : t, s \in F\},$$
which is the image of the linear map $F^2 \to F^4$, $(t, s) \mapsto (0, 5t, 10t, t + s)$. Hence U is a subspace (for every field F; the dimension may depend on $\mathrm{char}(F)$). □

Problem 2 (Example 1.35)

Verify all assertions about subspaces in Example 1.35.

(a)

Claim. If $b \in F$, then
$$U = \{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\}$$
is a subspace of F^4 iff $b = 0$.

Proof. $(0, 0, 0, 0) \in U$ iff $0 = 5 \cdot 0 + b$, i.e. iff $b = 0$. If $b \neq 0$, U cannot be a subspace.

If $b = 0$, then $U = \{x \in F^4 : x_3 = 5x_4\}$ is defined by a homogeneous linear equation, hence is closed under addition and scalar multiplication, so it is a subspace. □

(b)

Claim. The set of continuous real-valued functions on $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

Proof. Let $U = \{f : [0, 1] \to \mathbb{R} : f \text{ is continuous}\}$. The zero function is continuous, so $0 \in U$. If $f, g \in U$, then $f + g$ is continuous; if $a \in \mathbb{R}$, then af is continuous. Thus U is closed under addition and scalar multiplication, hence is a subspace. □

(c)

Claim. The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Proof. Let $U = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}$. The zero function is differentiable, so $0 \in U$. If $f, g \in U$, then $f + g$ is differentiable and $(f + g)' = f' + g'$. If $a \in \mathbb{R}$, then af is differentiable and $(af)' = af'$. Hence U is closed under addition and scalar multiplication and is a subspace. $\square$

(d)

Claim. The set of differentiable real-valued functions f on $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ iff $b = 0$.

Proof. Let $U = \{f : (0, 3) \to \mathbb{R} : f \text{ differentiable and } f'(2) = b\}$. If U is a subspace then it contains the zero function, so $0'(2) = 0 = b$, hence $b = 0$.

Conversely, if $b = 0$, then $U = \{f : f'(2) = 0\}$. If $f, g \in U$, then $(f + g)'(2) = f'(2) + g'(2) = 0$, so $f + g \in U$. If $a \in \mathbb{R}$, then $(af)'(2) = af'(2) = 0$, so $af \in U$. Thus U is a subspace. $\square$

(e)

Claim. The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}^\infty$.

Proof. Let

$$U = \{(z_n)_{n\geq 1} \in \mathbb{C}^\infty : \lim_{n\to\infty} z_n = 0\}.$$

The zero sequence has limit 0, so $0 \in U$. If $(x_n), (y_n) \in U$, then by limit laws,

$$\lim_{n\to\infty}(x_n + y_n) = \lim x_n + \lim y_n = 0 + 0 = 0,$$

so $(x_n + y_n) \in U$. If $a \in \mathbb{C}$ and $(x_n) \in U$, then

$$\lim_{n\to\infty}(ax_n) = a \lim x_n = a \cdot 0 = 0,$$

so $(ax_n) \in U$. Hence U is a subspace. $\square$

Problem 3

Claim. The set of differentiable functions f on $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

Proof. Let $U = \{f : (-4, 4) \to \mathbb{R} : f \text{ differentiable and } f'(-1) = 3f(2)\}$. The zero function satisfies the condition. If $f, g \in U$, then $(f+g)'(-1) = f'(-1)+g'(-1) = 3f(2)+3g(2) = 3(f+g)(2)$, so $f + g \in U$. If $a \in \mathbb{R}$, then $(af)'(-1) = af'(-1) = 3af(2) = 3(af)(2)$, so $af \in U$. Thus U is a subspace. $\square$

Problem 4

Claim. Fix $b \in \mathbb{R}$. The set of continuous f on $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ iff $b = 0$.

Proof. If $U = \{f \text{ continuous} : \int_0^1 f = b\}$ is a subspace, it contains the zero function, hence $0 = \int_0^1 0 = b$, so $b = 0$. Conversely, if $b = 0$, then for $f, g \in U$, $\int_0^1 (f + g) = \int_0^1 f + \int_0^1 g = 0$, and for $a \in \mathbb{R}$, $\int_0^1 (af) = a \int_0^1 f = 0$, so U is closed under addition and scalar multiplication and hence a subspace. $\square$

Problem 5

Claim. $\mathbb{R}^2$ is not a subspace of the complex vector space $\mathbb{C}^2$.

Proof. As a subset, $\mathbb{R}^2 \subseteq \mathbb{C}^2$, but it is not closed under scalar multiplication by complex scalars. For example, $(1,0) \in \mathbb{R}^2$ but $i(1,0) = (i,0) \notin \mathbb{R}^2$. $\square$

Problem 6

(a)

Claim.

$$U = \{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$$

is a subspace of $\mathbb{R}^3$. In fact $U = \{(t,t,s) : t,s \in \mathbb{R}\}$.

Proof. Over $\mathbb{R}$, the map $x \mapsto x^3$ is injective, so $a^3 = b^3$ implies $a = b$. Hence $U = \{(t,t,s) : t,s \in \mathbb{R}\}$. This set contains $(0,0,0)$, is closed under addition and scalar multiplication, and is therefore a subspace. $\square$

(b)

Claim.

$$U = \{(a,b,c) \in \mathbb{C}^3 : a^3 = b^3\}$$

is *not* a subspace of $\mathbb{C}^3$.

Proof. Let $\omega = e^{2\pi i/3}$, so $\omega^3 = 1$ and $\omega \neq 1$. Then $(1,1,0) \in U$ and $(1,\omega,0) \in U$ (since $1^3 = \omega^3 = 1$). If U were a subspace it would be closed under addition, so their sum $(2, 1+\omega, 0)$ would lie in U. But $1 + \omega = -\omega^2$ (because $1 + \omega + \omega^2 = 0$), hence

$$(1+\omega)^3 = (-\omega^2)^3 = -\omega^6 = -1 \neq 8 = 2^3,$$

so $(2, 1+\omega, 0) \notin U$. Therefore U is not closed under addition and is not a subspace. $\square$

Problem 7

Claim. If $U \subseteq \mathbb{R}^2$ is nonempty, closed under addition, and closed under additive inverses, then U is a subspace of $\mathbb{R}^2$.

Counterexample. Let $U = \mathbb{Z}^2 = \{(m,n) : m,n \in \mathbb{Z}\}$. It is nonempty, closed under addition and under additive inverses, but not closed under scalar multiplication: $\frac{1}{2}(1,0) = (\frac{1}{2}, 0) \notin U$. So the claim is false. $\square$

Problem 8

Claim. There exists a nonempty subset $U \subseteq \mathbb{R}^2$ closed under scalar multiplication that is not a subspace.

Example. Let U be the union of the two coordinate axes:

$$U = \{(x,0) : x \in \mathbb{R}\} \;\cup\; \{(0,y) : y \in \mathbb{R}\}.$$

Then U is closed under scalar multiplication, but not under addition since $(1,0),(0,1) \in U$ while $(1,1) \notin U$. $\square$

Problem 9

Claim. The set of periodic functions $\mathbb{R} \to \mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Proof. Let $U = \{f : \mathbb{R} \to \mathbb{R} : \exists p > 0\ \forall x,\ f(x+p) = f(x)\}$. The zero function is periodic. If $f, g \in U$ with periods p_f, p_g, then $p = p_f p_g$ is a common period, so $f + g$ is periodic. If $a \in \mathbb{R}$, then af has the same period as f. Hence U is a subspace. $\square$

Problem 10

Claim. If V_1, V_2 are subspaces of V, then $V_1 \cap V_2$ is a subspace of V.

Proof. $0 \in V_1 \cap V_2$. If $u, v \in V_1 \cap V_2$, then $u + v \in V_1$ and $u + v \in V_2$, hence $u + v \in V_1 \cap V_2$. Similarly $au \in V_1 \cap V_2$ for any scalar a. Thus $V_1 \cap V_2$ is a subspace. $\square$

Problem 12

Claim. The union of two subspaces is a subspace iff one is contained in the other.

Proof. If $U \subseteq W$ then $U \cup W = W$ is a subspace (and similarly if $W \subseteq U$).

Conversely, suppose $U \cup W$ is a subspace and neither contains the other. Pick $u \in U \setminus W$ and $w \in W \setminus U$. Then $u + w \in U \cup W$. If $u + w \in U$, then $w = (u + w) - u \in U$, contradiction. If $u + w \in W$, then $u = (u + w) - w \in W$, contradiction. Hence one must contain the other. $\square$

Problem 13

Claim. The union of three subspaces of V is a subspace iff one of them contains the other two.

Proof. If (say) $V_1 \supseteq V_2$ and $V_1 \supseteq V_3$, then $V_1 \cup V_2 \cup V_3 = V_1$ is a subspace.

Conversely, suppose $U = V_1 \cup V_2 \cup V_3$ is a subspace. If V_1 contains $V_2 \cup V_3$ we are done, so assume not. Then there exists $v_2 \in V_2 \setminus V_1$ or $v_3 \in V_3 \setminus V_1$; WLOG pick $v_2 \in V_2 \setminus V_1$. Similarly, if V_2 contains $V_1 \cup V_3$ we are done; otherwise pick $v_1 \in V_1 \setminus V_2$. Then $v_1 + v_2 \in U$. It cannot lie in V_1 (else $v_2 = (v_1 + v_2) - v_1 \in V_1$) and cannot lie in V_2 (else $v_1 \in V_2$). Hence $v_1 + v_2 \in V_3$. Now for any $x \in V_1$, we have $x + v_2 \in U$. The same subtraction argument shows $x + v_2$ cannot lie in V_1 or V_2, so $x + v_2 \in V_3$, hence $x = (x + v_2) - v_2 \in V_3$. Thus $V_1 \subseteq V_3$. Similarly, for any $y \in V_2$, $v_1 + y \in U$ forces $v_1 + y \in V_3$, hence $y \in V_3$. So $V_2 \subseteq V_3$. Therefore V_3 contains V_1 and V_2. $\square$

Problem 14

Suppose $U = \{(x, -x, 2x) \in F^3 : x \in F\}$ and $W = \{(x, x, 2x) \in F^3 : x \in F\}$. Describe $U + W$.

Solution.

$$U + W = \{(a, -a, 2a) + (b, b, 2b) : a, b \in F\} = \{(a + b,\ -a + b,\ 2a + 2b) : a, b \in F\}.$$

Let $s = a + b$ and $t = -a + b$. Then $2s = 2a + 2b$ and every pair (s, t) arises from some a, b. Hence

$$U + W = \{(s, t, 2s) : s, t \in F\}.$$

In words: $U + W$ is the set of all vectors in F^3 whose third coordinate is twice the first. $\square$

Problem 15

Claim. If U is a subspace of V, then $U + U = U$.

Proof. If $u_1, u_2 \in U$, then $u_1 + u_2 \in U$, so $U + U \subseteq U$. Also $u = u + 0$ shows $U \subseteq U + U$. Thus $U + U = U$. $\square$

Problem 16

Claim. Subspace addition is commutative: $U + W = W + U$.

Proof. $U + W = \{u + w\}$ and $u + w = w + u$, so $U + W = W + U$. $\square$

Problem 17

Claim. Subspace addition is associative: $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$.

Proof. Both sides equal $\{v_1 + v_2 + v_3 : \ v_i \in V_i\}$ by associativity of vector addition. $\square$

Problem 18

Claim. $\{0\}$ is the additive identity for subspace addition. A subspace has an additive inverse iff it is $\{0\}$.

Proof. $U + \{0\} = U$ since $u + 0 = u$. If $U + W = \{0\}$, then for any $u \in U$ we have $u = u + 0 \in U + W = \{0\}$, so $U = \{0\}$. Conversely $\{0\} + \{0\} = \{0\}$. $\square$

Problem 19

Claim. If V_1, V_2, U are subspaces with $V_1 + U = V_2 + U$, it need not follow that $V_1 = V_2$.

Counterexample. Take $V = \mathbb{R}^2$, $U = \mathbb{R}^2$, $V_1 = \{(x, 0)\}$ and $V_2 = \{(x, x)\}$. Then $V_1 + U = U = V_2 + U$ but $V_1 \neq V_2$. $\square$

Problem 20

Suppose $U = \{(x, x, y, y) \in F^4 : x, y \in F\}$. Find W such that $F^4 = U \oplus W$.

Solution. Let

$$W = \{(a, 0, b, 0) \in F^4 : a, b \in F\}.$$

If $(x, x, y, y) \in U \cap W$, then $(x, x, y, y) = (a, 0, b, 0)$ forces $x = 0$ and $y = 0$, so $U \cap W = \{0\}$. Also every $(p, q, r, s) \in F^4$ decomposes as

$$(p, q, r, s) = (q, q, s, s) + (p - q, 0, r - s, 0),$$

with the first term in U and the second in W. Hence $F^4 = U \oplus W$. $\square$

Problem 21

Suppose $U = \{(x, y, x + y, x - y, 2x) \in F^5 : x, y \in F\}$. Find W such that $F^5 = U \oplus W$.

Solution. Take

$$W = \{(0, 0, a, b, c) \in F^5 : a, b, c \in F\}.$$

If $(x, y, x+y, x-y, 2x) \in W$, then $x = y = 0$, hence $U \cap W = \{0\}$. Also every $(p, q, r, s, t) \in F^5$ decomposes as

$$(p, q, r, s, t) = (p, q, p+q, p-q, 2p) + (0, 0, r-(p+q),\ s-(p-q),\ t-2p),$$

with the first term in U and the second in W. Thus $F^5 = U \oplus W$. $\square$

Problem 22

With U as in Problem 21, find nonzero subspaces W_1, W_2, W_3 such that $F^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution. Let

$$W_1 = \operatorname{span}\{(0,0,1,0,0)\}, \quad W_2 = \operatorname{span}\{(0,0,0,1,0)\}, \quad W_3 = \operatorname{span}\{(0,0,0,0,1)\}.$$

Then $W = W_1 \oplus W_2 \oplus W_3 = \{(0, 0, a, b, c)\}$ as in Problem 21, so $F^5 = U \oplus W_1 \oplus W_2 \oplus W_3$. $\square$

Problem 23

Claim. If $V = V_1 \oplus U$ and $V = V_2 \oplus U$, it need not follow that $V_1 = V_2$.

Counterexample. Let $V = F^2$, $U = \{(0, y) : y \in F\}$, $V_1 = \{(x, 0) : x \in F\}$, and $V_2 = \{(x, x) : x \in F\}$. Then $F^2 = V_1 \oplus U$ since $(a, b) = (a, 0) + (0, b)$ and $V_1 \cap U = \{0\}$. Also $F^2 = V_2 \oplus U$ since $(a, b) = (a, a) + (0, b-a)$ and $V_2 \cap U = \{0\}$. But $V_1 \neq V_2$. $\square$

Problem 24

Claim. Let V_e be the even functions $\mathbb{R} \to \mathbb{R}$ and V_o the odd functions. Then $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

Proof. V_e and V_o are subspaces. If f is both even and odd, then $f(x) = -f(x)$ for all x, so $f = 0$, hence $V_e \cap V_o = \{0\}$. For any $f : \mathbb{R} \to \mathbb{R}$ define

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \qquad f_o(x) = \frac{f(x) - f(-x)}{2}.$$

Then f_e is even, f_o is odd, and $f = f_e + f_o$, giving $V_e + V_o = \mathbb{R}^{\mathbb{R}}$. Uniqueness follows from $V_e \cap V_o = \{0\}$. $\square$