Section 1.

(1) $\sum_{n=0}^{\infty} x^n = \frac{1}{x}$ converges for $|x| < 1$

(2) $\sum_{n=0}^{\infty} (f(x))^n = \frac{1}{1-f(x)}$ for any x such that $|f(x)| < 1$.

1.

$$\sum_{n=0}^{\infty} (-1)^n x^n =$$

$$\sum_{n=0}^{\infty} \left(f(x)\right)^n \text{ with}$$

$f(x) = -x$. So by (2)

$$\sum_{n=0}^{\infty} \left(f(x)\right)^n = \frac{1}{1-f(x)}$$

$$= \frac{1}{1-(-x)}$$

$$= \frac{1}{1+x}$$

2. $$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} =$$

$\frac{1}{1-f(x)}$ where $f(x) = -x^2$

So by (2) $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n .$$

3. $$\frac{1}{1+x^2} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= (-x^2)^0 + (-x^2)^1 + (-x^2)^2 + (-x^3)^3 + \cdots$$

(by probem 2.)

$$= 1 + (-1)x^2 + (1)x^4 + (-1)x^6 + \cdots$$

$$= 1 + (0)x + (-1)x^2 + (0)x^3 + (1)x^4$$
$$+ (0)x^5 + (-1)x^6 + \cdots$$

Let $s(x) = 1 - x^2 + x^4 - x^6 + \cdots$

then

$$(1+x^2)\frac{1}{(1+x^2)} = 1$$

and $(1+x^2)\, s(x) =$

$$(1+x^2)(1 - x^2 + x^4 - x^6 + \cdots) =$$

$$(1 - x^2 + x^4 - x^6 + \cdots) +$$
$$(x^2 - x^4 + x^6 + \cdots) = 1$$

showing $s(x) = \frac{1}{1+x^2}$ (things that are equal to the same thing are themselves equal).

Lectio 2.

1. We have

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \{\text{def}^{\underline{n}}\}.$$

$$= \frac{n\cdot(n-1)\cdots(n-k+1)(n-k)!}{k!\,(n-k)!}$$

$$\cancel{= n\cdot(n-1)\cdots(n-(k}$$

$$= \frac{n\cdot(n-1)\cdots(n-k+1)}{k\cdot(k-1)\cdots(2)(1)}$$

providing the alternative $\text{def}^{\underline{n}}$.

2. If $n\in\mathbb{Z}^+$ and $k>n$ then from 1. we see the numerator must contain the factor $(n-n)=0$ so $\binom{n}{k}=0$.

We know that $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$

but the above fact means the we can say $(a+b)^n = \sum_{k=0}^{\infty}\binom{n}{k}a^{n-k}b^k$

since all terms where $k > n$ will be zero.

3. If $n \in \mathbb{Z}^+$ and $k > n$ then from

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k(k-1) \cdots (2)(1)}$$

we see the numerator must contain a factor $(n-i)$ where $i = n$. This implies $\binom{n}{k} = 0$.

If n is fractional ~~and $k > 0$~~ then there is no whole number i such that $(n-i) = 0$ so $\binom{n}{k} \neq 0$.

If n is negative then ~~again~~ since there is no positive i such that $(n-i) = 0$ we have $\binom{n}{k} \neq 0$.

~~4. If a^x~~

4. The only possible value for a is $a = 1$.

5.

a) $\sqrt{1-x} = (1-(-x))^{1/2}$

so $a = 1/2$.

(b). OK. This is not explained well in the text so let me clarify. We use this defn. (*)

$$(*) \quad \binom{\alpha}{k} = \frac{\alpha \cdot (\alpha-1)(\alpha-2) \cdots (\alpha-k+1)}{k \cdot (k-1) \cdots 1}$$

$$= \prod_{j=1}^{k} \frac{\alpha - j + 1}{j} \quad \text{if } k \geq 0$$

$\binom{\alpha}{k} = 0$ if $k < 0$ and if $k = 0$, (*) is the empty product and we define it as 1.

$$\binom{1/2}{0} = \prod_{j=1}^{1} \frac{\cancel{1/2} - j + 1}{j} = 1. \quad \text{(see last remark)}$$

$$\binom{1/2}{1} = \prod_{j=1}^{1} \frac{1/2 - j + 1}{j} = \frac{1}{2}.$$

$$\binom{1/2}{2} = \prod_{j=1}^{2} \frac{1/2 - j + 1}{j}$$

$$= \left(\frac{\frac{1}{2} - 1 + 1}{1}\right)\left(\frac{\frac{1}{2} - 2 + 1}{2}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{4}\right) = -\frac{1}{8}$$

$$\binom{1/2}{3} = \prod_{j=1}^{3} \frac{1/2 - j + 1}{j}$$

$$= -\frac{1}{8}\left(\frac{\frac{1}{2} - 3 + 1}{3}\right)$$

$$= \left(-\frac{1}{8}\right)\left(-\frac{1}{2}\right) = \frac{1}{16}.$$

(c) $\sqrt{1-x} = (1 + (-x))^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k}(-x)^k$

$$= \binom{1/2}{0} + \binom{1/2}{1}(-x) + \binom{1/2}{2}(-x)^2 + \binom{1/2}{3}(-x^3)$$

$$+ \ldots$$

$$= \binom{1/2}{0} + \left[-\binom{1/2}{1}\right]x + \frac{1/2}{2}x^2 + \left[-\binom{1/2}{3}\right]x^3 + \ldots$$

So

$a_0 = 1$

$a_1 = -\binom{1/2}{1} = -\frac{1}{2}$

$a_2 = \binom{1/2}{2} = -\frac{1}{8}$

$a_3 = -\binom{1/2}{3} = -\frac{1}{16}.$

(d) $a_0^2 = (1)^2 = 1$

$2a_0a_1 = 2\cdot 1\cdot\left(\frac{-1}{2}\right) = -1.$

$2a_0a_2 + a_1^2 = -1 + 1 = 0$

$2a_0a_3 + 2a_1a_2 = \frac{-2}{16} + \frac{2}{16} = 0.$

6. (a)

$$\sqrt{15} = \sqrt{16-1} = \sqrt{16\left(1-\frac{1}{16}\right)}$$

$$= \left(\sqrt{16}\right)\left(\sqrt{1-\frac{1}{16}}\right) = 4\sqrt{1-\frac{1}{16}}$$

(b) $\sqrt{1-x} \doteqdot 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

$$\sqrt{1-1/16} \doteqdot 1 - \frac{1}{2}\left(\frac{1}{16}\right) - \frac{1}{8}\left(\frac{1}{16}\right)^2 - \frac{1}{16}\left(\frac{1}{16}\right)^3$$

$$= 1 - \frac{1}{32} - \frac{1}{8}\left(\frac{1}{256}\right) - \frac{1}{16}\left(\frac{1}{4096}\right)$$

$$= 1 - \frac{1}{32} - \frac{1}{2048} - \frac{1}{65,536}$$

$$= 0.96824\ldots \Rightarrow$$

(c) $\sqrt{15} \doteqdot 4 \cdot (0.96824) = 3.87298$

(d) It's correct to 5 decimal places. To get better accuracy increase the number of terms in the approximation of $\sqrt{1-x}$.

Lectia 3.

1. $\frac{1}{1-x} = [1+(-x)]^{-1}$

$X = -x$
$a = -1$.

2.

$$\sum_{k=0}^{\infty} \binom{-1}{k}(-X)^k = (1-X)^{-1}$$

$$= \frac{1}{1-X} = \sum_{k=0}^{\infty} X^k$$

That is, after a change of variables

$$\sum_{k=0}^{\infty} \binom{-1}{k}(-x)^k = \sum_{k=0}^{\infty} x^k.$$

3. Write out the LHS:

$$(-1)^0\binom{-1}{0} + (-1)^1\binom{-1}{k}x + (-1)^2\binom{-1}{2}x^2 + \dots$$

Write out the RHS.

$$1 + x + x^2 + \dots$$

Compare coefficients

$$(-1)^0\binom{-1}{0} = 1 \ ; \ (-1)^1\binom{-1}{1} = 1 \ ; \ (-1)^2\binom{-1}{2} = 1$$

Generalize:

$$\binom{-1}{k}(-1)^k = 1$$

[a formal proof would proceed by induction].