Computer Science 384 Tuesday, October 25, 2016
St. George Campus University of Toronto

Homework Assignment #3: Knowledge Representation
Due: Thursday, November 17, 2016 by 11:59 PM

Silent Policy: *A silent policy will take effect 24 hours before this assignment is due, i.e. no question about this assignment will be answered, whether it is asked on the discussion board, via email or in person.*

Late Policy: 10% per day after the use of 3 grace days.

Total Marks: This assignment represents 10% of the course grade.

Handing in this Assignment

What to hand in on paper: Nothing.

What to hand in electronically: You must submit your assignment electronically. Name your file `A3.pdf`.

Clarification Page: Important corrections (hopefully few or none) and clarifications to the assignment will be posted on the Assignment 3 Clarification page:

`http://www.teach.cs.toronto.edu/~csc384h/fall/Assignments/A3/a3_faq.html`.

You are responsible for monitoring the A3 Clarification page.

Questions: Questions about the assignment should be asked on Piazza:

`https://piazza.com/utoronto.ca/fall2016/csc384/home`.

If you have a question of a personal nature, please email the instructor, Steven Shapiro, at steven at cs dot toronto dot edu placing 384 and A3 in the subject line of your message.

1. **[30 pts]** For each of the following sentences, give an interpretation that makes that sentence false and the other two sentences true. Justify your answer.

   (a) $\forall x \forall y \forall z[(P(x,y) \wedge P(x,z)) \supset P(y,z)]$;

   (b) $\forall x \forall y[(P(x,y) \wedge P(y,x)) \supset (x = y)]$;

   (c) $\forall x \forall y[P(a,x) \supset P(b,y)]$.

   (a)

$$
\begin{aligned}
&\mathfrak{I} = \langle \mathscr{D}, \mathscr{I} \rangle \\
&\mathscr{D} = \{d_1, d_2, d_3\} \\
&\mathscr{I}[P] = \{\langle d_1, d_2 \rangle, \langle d_1, d_3 \rangle\} \\
&\mathscr{I}[a] = d_3 \\
&\mathscr{I}[b] = d_3
\end{aligned}
$$

Proof. i. Show $\mathfrak{I} \not\models (a)$. We need to show that there exists a variable assignment μ such that $\mathfrak{I}, \mu \models P(x,y) \wedge P(x,z) \wedge \neg P(y,z)$, but this follows if we let $\mu(x) = d_1, \mu(y) = d_2$, and $\mu(z) = d_3$, since $\langle d_1, d_2 \rangle \in \mathscr{I}[P], \langle d_1, d_3 \rangle \in \mathscr{I}[P]$, and $\langle d_2, d_3 \rangle \notin \mathscr{I}[P]$.

ii. Show $\mathfrak{I} \models (b)$. We need to show that for any variable assignment μ:

$$\mathfrak{I}, \mu \models (P(x,y) \wedge P(y,x)) \supset x = y.$$

However, it is easy to see that for any μ, if $\mathfrak{I}, \mu \models P(x,y)$ then $\mathfrak{I}, \mu \not\models P(y,x)$.

iii. Show $\mathfrak{I} \models (c)$. We need to show that for any μ,

$$\mathfrak{I}, \mu \models P(a,x) \supset P(b,y).$$

However, it is easy to see that for any μ, $\mathfrak{I}, \mu \not\models P(a,x)$.

□

The other parts are similar.

2. **[30 pts]** Consider the following scenario:

   Rosemary, Peter, and Amanda go to a restaurant for dinner. Amanda orders steak and pecan pie. Vegetarians won't order steak, but anyone with a sweet tooth will order the pecan pie. Peter is a vegetarian. Rosemary and Amanda do not order any of the same dishes. Everyone at the restaurant is either a vegetarian or has a sweet tooth (or both).

   (a) Represent this scenario as sentences in FOL, and show semantically that they logically entail that there is a vegetarian at the restaurant who does not have a sweet tooth.

   - Predicates: $\mathsf{Restaurant}(x)$ for x is at the restaurant; $\mathsf{Orders}(x,y)$ for x orders y; $\mathsf{Veg}(x)$ for x is vegetarian; and $\mathsf{Sweet}(x)$ for x has a sweet tooth.

- Constants: rosemary for Rosemary; peter for Peter; amanda for Amanda; steak for steak; and pie for pecan pie. The KB is:

$$\mathsf{Restaurant}(\mathsf{rosemary}) \tag{1}$$
$$\mathsf{Restaurant}(\mathsf{peter}) \tag{2}$$
$$\mathsf{Restaurant}(\mathsf{amanda}) \tag{3}$$
$$\mathsf{Orders}(\mathsf{amanda}, \mathsf{steak}) \tag{4}$$
$$\mathsf{Orders}(\mathsf{amanda}, \mathsf{pie}) \tag{5}$$
$$\forall x[\mathsf{Veg}(x) \supset \neg\mathsf{Orders}(x, \mathsf{steak})] \tag{6}$$
$$\forall x[\mathsf{Sweet}(x) \supset \mathsf{Orders}(x, \mathsf{pie})] \tag{7}$$
$$\mathsf{Veg}(\mathsf{peter}) \tag{8}$$
$$\forall x[\mathsf{Orders}(\mathsf{rosemary}, x) \equiv \neg\mathsf{Orders}(\mathsf{amanda}, x)] \tag{9}$$
$$\forall x[\mathsf{Restaurant}(x) \supset (\mathsf{Veg}(x) \vee \mathsf{Sweet}(x))] \tag{10}$$

- Show KB $\models \exists x[\mathsf{Restaurant}(x) \wedge \mathsf{Veg}(x) \wedge \neg\mathsf{Sweet}(x)]$.

  Proof. Let $\mathfrak{I} \models$ KB. We need to show that there exists a variable assignment μ such that:

$$\mathfrak{I}, \mu \models \mathsf{Restaurant}(x) \wedge \mathsf{Veg}(x) \wedge \neg\mathsf{Sweet}(x).$$

  Let $\mu(x) = \mathscr{I}[\mathsf{rosemary}]$. Then, it suffices to show that: $\mathscr{I}[\mathsf{rosemary}] \in \mathscr{I}[\mathsf{Restaurant}]$, $\mathscr{I}[\mathsf{rosemary}] \in \mathscr{I}[\mathsf{Veg}]$, and $\mathscr{I}[\mathsf{rosemary}] \notin \mathscr{I}[\mathsf{Sweet}]$. Let $\mu'(x) = \mathscr{I}[\mathsf{pie}]$. Since $\mathfrak{I} \models (9)$,

$$\mathfrak{I}, \mu' \models \mathsf{Orders}(\mathsf{rosemary}, x) \equiv \neg\mathsf{Orders}(\mathsf{amanda}, x), \tag{11}$$

  therefore $\langle \mathscr{I}[\mathsf{rosemary}], \mathscr{I}[\mathsf{pie}] \rangle \in \mathscr{I}[\mathsf{Orders}]$ iff $\langle \mathscr{I}[\mathsf{amanda}], \mathscr{I}[\mathsf{pie}] \rangle \notin \mathscr{I}[\mathsf{Orders}]$. But since $\mathfrak{I} \models (5)$, it follows that:

$$\langle \mathscr{I}[\mathsf{rosemary}], \mathscr{I}[\mathsf{pie}] \rangle \notin \mathscr{I}[\mathsf{Orders}]. \tag{12}$$

  Let $\mu''(x) = \mathscr{I}[\mathsf{rosemary}]$. Since $\mathfrak{I} \models (7)$, it follows that:

$$\mathfrak{I}, \mu'' \models \mathsf{Sweet}(x) \supset \mathsf{Orders}(x, \mathsf{pie}]),$$

  therefore $\mathscr{I}[\mathsf{rosemary}] \notin \mathscr{I}[\mathsf{Sweet}]$ or $\langle \mathscr{I}[\mathsf{rosemary}], \mathscr{I}[\mathsf{pie}] \rangle \in \mathscr{I}[\mathsf{Orders}]$. From this and (12), we can infer:

$$\mathscr{I}[\mathsf{rosemary}] \notin \mathscr{I}[\mathsf{Sweet}]. \tag{13}$$

  Since $\mathfrak{I} \models (10)$, it follows that:

$$\mathfrak{I}, \mu'' \models \mathsf{Restaurant}(x) \supset (\mathsf{Veg}(x) \vee \mathsf{Sweet}(x)),$$

  therefore $\mathscr{I}[\mathsf{rosemary}] \notin \mathscr{I}[\mathsf{Restaurant}]$ or $\mathscr{I}[\mathsf{rosemary}] \in \mathscr{I}[\mathsf{Veg}]$ or $\mathscr{I}[\mathsf{rosemary}] \in \mathscr{I}[\mathsf{Sweet}]$. However, since $\mathscr{I} \models (1)$, it follows that $\mathscr{I}[\mathsf{rosemary}] \in \mathscr{I}[\mathsf{Veg}]$ or $\mathscr{I}[\mathsf{rosemary}] \in \mathscr{I}[\mathsf{Sweet}]$. It follows from this and (13) that $\mathscr{I}[\mathsf{rosemary}] \in \mathscr{I}[\mathsf{Veg}]$. But now the proposition follows from this, (13), and the fact that $\mathfrak{I} \models (1)$. $\square$

(b) Use Resolution with answer extraction to find the vegetarian at the restaurant who does not have a sweet tooth.

First we convert the KB to clausal form.

$$[\mathsf{Restaurant}(\mathsf{rosemary})] \tag{14}$$
$$[\mathsf{Restaurant}(\mathsf{peter})] \tag{15}$$
$$[\mathsf{Restaurant}(\mathsf{amanda})] \tag{16}$$
$$[\mathsf{Orders}(\mathsf{amanda},\mathsf{steak})] \tag{17}$$
$$[\mathsf{Orders}(\mathsf{amanda},\mathsf{pie})] \tag{18}$$
$$[\neg\mathsf{Veg}(x),\neg\mathsf{Orders}(x,\mathsf{steak})] \tag{19}$$
$$[\neg\mathsf{Sweet}(x),\mathsf{Orders}(x,\mathsf{pie})] \tag{20}$$
$$[\mathsf{Veg}(\mathsf{peter})] \tag{21}$$
$$[\neg\mathsf{Orders}(\mathsf{rosemary},x),\neg\mathsf{Orders}(\mathsf{amanda},x)] \tag{22}$$
$$[\mathsf{Orders}(\mathsf{amanda},x),\mathsf{Orders}(\mathsf{rosemary},x)] \tag{23}$$
$$[\neg\mathsf{Restaurant}(x),\mathsf{Veg}(x),\mathsf{Sweet}(x))] \tag{24}$$

And the query is: $\exists x[\mathsf{Restaurant}(x)\wedge\mathsf{Veg}(x)\wedge\neg\mathsf{Sweet}(x)\wedge\neg\mathsf{Ans}(x)]$. Negating the query and converting to clausal form yields:

$$[\neg\mathsf{Restaurant}(x),\neg\mathsf{Veg}(x),\mathsf{Sweet}(x),\mathsf{Ans}(x)] \tag{25}$$

The Resolution proof is as follows. It roughly follows the semantic proof.

$$R[18,22b]\{x=\mathsf{pie}\}[\neg\mathsf{Orders}(\mathsf{rosemary},\mathsf{pie})] \tag{26}$$
$$R[26,20b]\{x=\mathsf{rosemary}\}[\neg\mathsf{Sweet}(\mathsf{rosemary})] \tag{27}$$
$$R[27,25c]\{x=\mathsf{rosemary}\}[\neg\ \mathsf{Restaurant}(\mathsf{rosemary}),\neg\mathsf{Veg}(\mathsf{rosemary}),\mathsf{Ans}(\mathsf{rosemary})] \tag{28}$$
$$R[28,14]\{\}[\neg\mathsf{Veg}(\mathsf{rosemary}),\mathsf{Ans}(\mathsf{rosemary})] \tag{29}$$
$$R[29a,24b]\{x=\mathsf{rosemary}\}[\neg\mathsf{Restaurant}(\mathsf{rosemary}),\mathsf{Sweet}(\mathsf{rosemary}),\mathsf{Ans}(\mathsf{rosemary})] \tag{30}$$
$$R[30a,14]\{\}[\mathsf{Sweet}(\mathsf{rosemary}),\mathsf{Ans}(\mathsf{rosemary})] \tag{31}$$
$$R[31a,27]\{\}[\mathsf{Ans}(\mathsf{rosemary})] \tag{32}$$

Rosemary is the vegetarian who does not have a sweet tooth.

3. **[40 pts]** This question involves formalizing the properties of mathematical *groups* in FOL. Recall that a set is considered to be a group relative to a binary function f and an object e if and only if (1) f is associative; (2) e is an identity element for f, that is, for any x, $f(e,x) = f(x,e) = x$; and (3) every element has an inverse, that is, for any x, there is an i such that $f(x,i) = f(i,x) = e$. Formalize these as sentences of FOL with two nonlogical symbols, a function symbol f, and a constant symbol e, and show semantically that the sentences logically entail the following property of groups:

For every x and y, there is a z such that $f(x,z) = y$.

Explain how your answer shows the value of z as a function of x and y.

$$\forall x\forall y\forall z[f(f(x,y),z) = f(x,f(y,z))] \tag{33}$$
$$\forall x[f(e,x) = f(x,e)\wedge f(x,e) = x] \tag{34}$$
$$\forall x\exists i[f(x,i) = f(i,x)\wedge f(i,x) = e] \tag{35}$$

Show $\{(33),(34),(35)\} \models \forall x \forall y \exists z [f(x,z) = y]$.

Proof. Let $\mathfrak{I} = \langle \mathscr{D}, \mathscr{I} \rangle$ be such that $\mathfrak{I} \models \{(33),(34),(35)\}$. We need to show that for any variable assignment μ, there exists a variable assignment μ^* that differs from μ on at most z such that:

$$\mathfrak{I}, \mu^* \models f(x,z) = y.$$

Since $\mathfrak{I} \models (35)$, it follows that for any variable assignment μ_1, there exists a μ_1' that differs from μ_1 on at most i such that:

$$\mathfrak{I}, \mu_1' \models f(x,i) = f(i,x) \wedge f(i,x) = e. \tag{36}$$

Let $\mu_1(x) = \mu(x)$. Then it follows that $\mathscr{I}[f](\mu(x), \mu_1'(i))$ is the same element as $\mathscr{I}[f](\mu_1'(i), \mu(x))$ and $\mathscr{I}[f](\mu_1'(i), \mu(x))$ is the same element as $\mathscr{I}[e]$. Therefore:

$$\mathscr{I}[f](\mu(x), \mu_1'(i)) \text{ is the same element as } \mathscr{I}[e]. \tag{37}$$

Let $\mu_2(x) = \mu(x)$, $\mu_2(y) = \mu_1'(i)$, and $\mu_2(z) = \mu(y)$. Then, since $\mathfrak{I} \models (33)$, it follows that:

$$\mathfrak{I}, \mu_2 \models f(f(x,y),z) = f(x, f(y,z)).$$

Therefore, it follows that:

$$\mathscr{I}[f](\mathscr{I}[f](\mu(x), \mu_1'(i)), \mu(y)) \text{ is the same element as } \mathscr{I}[f](\mu(x), \mathscr{I}[f](\mu_1'(i), \mu(y))).$$

This and (37) imply that

$$\mathscr{I}[f](\mu(x), \mathscr{I}[f](\mu_1'(i), \mu(y))) \text{ is the same element as } \mathscr{I}[f](\mathscr{I}[e], \mu(y)). \tag{38}$$

Let $\mu_3(x) = \mu(y)$. Then, since $\mathfrak{I} \models (34)$, it follows that:

$$\mathfrak{I}, \mu_3 \models f(e,x) = f(x,e) \wedge f(x,e) = x,$$

which implies that $\mathscr{I}[f](\mathscr{I}[e], \mu(y))$ is the same element as $\mathscr{I}[f](\mu(y), \mathscr{I}[e])$ and $\mathscr{I}[f](\mu(y), \mathscr{I}[e])$ is the same element as $\mu(y)$. Therefore $\mathscr{I}[f](\mathscr{I}[e], \mu(y))$ is the same element as $\mu(y)$. This and (38) imply that:

$$\mathscr{I}[f](\mu(x), \mathscr{I}[f](\mu_1'(i), \mu(y))) \text{ is the same element as } \mu(y) \tag{39}$$

Recall that we were trying to show that there exists a variable assignment μ^* that differs from μ on at most z such that:

$$\mathfrak{I}, \mu^* \models f(x,z) = y$$

However, if we let $\mu^*[z] = \mathscr{I}[f](\mu_1'(i), \mu(y))$, then this is the same as showing:

$$\mathscr{I}[f](\mu^*(x), \mathscr{I}[f](\mu_1'(i), \mu(y))) = \mu^*(y). \tag{40}$$

But since μ^* differs from μ on at most y, $\mu^*(x) = \mu(x)$ and $\mu^*(y) = \mu(y)$. Therefore (40) follows from (39). $\square$

To see that the value of z is a function of x and y, we look at the variable assignment, μ^* that assigned values to the variables in the sentence we proved. Note that $\mu^*[z] = \mathscr{I}[f](\mu'_1(i), \mu(y))$, and $\mu^*(y) = \mu(y)$, therefore the value assigned to z is a function of the value assigned to y and $\mu'_1(i)$. But from (36), it can be seen that the choice of $\mu'_1(i)$ depends on $\mu_1(x)$ and $\mu_1(x) = \mu(x) = \mu^*(x)$. This is because (36) is of the form: for any variable assignment μ_1, there exists a μ'_1 such that $P(\mu_1, \mu'_1)$,[1] where μ_1 assigned a value for x and μ'_1 assigned a value for i. Since we know that at least one such μ'_1 exists, we can always choose one such μ'_1. This means that we can define a function, which, given a μ_1, chooses a μ'_1 such that $P(\mu_1, \mu'_1)$. This shows that value of $\mu'_1(i)$ is a function of $\mu(x)$, therefore the value assigned to z is also a function of the value assigned to x. Note that this reasoning can be used as a justification for using Skolem functions in Resolution, where we replace a formula of the form $\forall x \exists y[P(x,y)]$ with one of the form $\forall x.P(x, f(x))$. Here the value assigned to y is seen as a function of the value assigned to x.

[1] Here, the relation $P(\alpha, \beta)$ can be defined as: β differs from α on at most i and $\mathscr{I}, \beta \models f(x,i) = f(i,x) \wedge f(i,x) = e$.