CPHD Filtering With Unknown Clutter Rate and Detection Profile

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***Abstract*—In Bayesian multi-target filtering, we have to contend with two notable sources of uncertainty, clutter and detection. Knowledge of parameters such as clutter rate and detection profile are of critical importance in multi-target filters such as the probability hypothesis density (PHD) and cardinalized PHD (CPHD) filters. Significant mismatches in clutter and detection model parameters result in biased estimates. In practice, these model parameters are often manually tuned or estimated offline from training data. In this paper we propose PHD/CPHD filters that can accommodate model mismatch in clutter rate and detection profile. In particular we devise versions of the PHD/CPHD filters that can adaptively learn the clutter rate and detection profile while filtering. Moreover, closed-form solutions to these filtering recursions are derived using Beta and Gaussian mixtures. Simulations are presented to verify the proposed solutions.**

***Index Terms*—CPHD, Finite set statistics, multi-target tracking, parameter estimation, PHD, robust filtering.**

I. Introduction

Multi-target filtering involves the joint estimation of the number of targets and their individual states from a sequence of observations in the presence of association uncertainty, detection uncertainty and clutter [1], [2], [4]. The random finite set (RFS) approach, [4], [5] is an elegant formulation of the multi-target filtering problem in which the collection of target states, referred to as the *multi-target state*, is naturally represented as a finite set. The rationale behind this representation traces back to a fundamental consideration in estimation theory-estimation error [6].

Due to the inherent combinatorial nature of multi-target densities and the multiple integrations on the (infinite dimensional) multi-target state and observation spaces, the multi-target Bayes filter (see, e.g., [4] and [5]) is intractable in most practical applications. To alleviate this intractability, the probability hypothesis density (PHD) and subsequently cardinalized PHD (CPHD) filters have been proposed [7], [8] as moment and cardinality approximations. These filters operate on the single-target state space and avoid the combinatorial problem that arises from data association. Since their inceptions the PHD and CPHD filters have generated substantial interest from academia as well as the commercial sector with the developments of numerical solutions such as sequential Monte Carlo (SMC) or particle and Gaussian mixtures [9]–[11]. Extensions to maintain track continuity have been proposed in [12], [13], for the particle-PHD filter and [14] for the Gaussian mixture PHD filter. In [15] the Gaussian mixture PHD filter is extended to linear Jump Markov multi-target models for tracking maneuvering targets. Recently, important developments such as the auxiliary particle-PHD filter [16] and measurement-oriented particle labeling technique [17], partially solve the clustering problem in the extraction of state estimates from the particle population. Clever uses of the PHD filter with measurement-driven birth intensity were independently proposed in [18] and [17] to improve tracking performance as well as obviating exact knowledge of the birth intensity.

In PHD/CPHD filtering, no less than multi-target filtering in general, we have to contend with two notable sources of uncertainty, clutter and detection, in addition to the process and measurement noise from each target. Clutter are spurious measurements that do not belong to any target, while detection uncertainty refers to the phenomena that the sensor does not always detect the targets. Knowledge of parameters such as clutter rate and detection profile are of critical importance in Bayesian multi-target filtering, arguably, more so than measurement noise model in single-target filtering. Significant mismatches in clutter and detection model parameters inevitably result in erroneous estimates. However, except for some applications such as radar, the clutter rate and detection profile of the sensor are not available in general. Usually these parameters are either estimated from training data or manually tuned. For example, in visual tracking, measurements are extracted from images via various background or foreground modeling techniques [19]. The clutter rate and detection profile depend on the detection method. Moreover, it is not known whether these parameters are time-invariant. Thus, the ability of the PHD and CPHD filters to accommodate mismatch in clutter rate and detection profile is very important in practice.

This paper proposes various versions of the CPHD (and PHD) filter which can account for mismatches in clutter and detection model parameters. In particular we show that the CPHD (and PHD) recursion can be manipulated into forms that can adaptively learn nonuniform detection profile or/and clutter rate while filtering, provided that the detection profile and clutter background do not change too rapidly compared to the measurement-update rate. Analytic approximations to these filters

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for linear Gaussian multi-target models are also proposed and verified via simulations. Preliminary results have been reported in [20] and [21], which outline a general formulation for estimating both the detection profile and the clutter intensity function. The approach described in the paper are implementable special cases of the general theoretical approach described in [20] and [21] with additional results that enable analytic implementations.

We remark that robust Bayesian approaches to problems with model mismatch in the literature such as [22]–[27] operate with probability distributions and are not directly applicable to the CPHD and PHD filters which work with intensity functions. A related work is given in [28] which deals with the problem of calibrating time-invariant multi-target model parameters. The key idea is to find the vector of parameters that maximizes an approximate marginal likelihood of the observed data via gradient ascent. Neither the approximate likelihood function nor its gradient are computable and the authors have proposed an SMC approximation. While the approach of [28] is quite general it does not exploit analytic approximations of the CPHD and PHD filters. Moreover, it is not directly applicable to nonuniform, time-varying clutter rate and detection profile. An even more closely related work which investigates clutter estimation while filtering (with known detection profile) using the PHD filter is given in [29].

The paper is organized as follows. Section II gives the background material on the standard CPHD (and PHD) filter. Section III details the version of the CPHD (and PHD) filter for unknown clutter rate, while Section IV details the CPHD (and PHD) filter for unknown detection profile. Section V then combines these solutions in the form of CPHD (and PHD) filter that can adaptively learn a jointly unknown clutter rate and unknown detection profile. Analytic implementations are also presented based on Beta and Gaussian mixtures. Simulations are presented in Section VI.

II. Conventional CPHD Recursion

Suppose that at time k, there are $N(k)$ target states $x_{k,1},\ldots,x_{k,N(k)}$, each taking values in a state space $\mathcal{X}$ and $M(k)$ observations $z_{k,1},\ldots,z_{k,M(k)}$ each taking values in an observation space $\mathcal{Z}$. Then, the multi-target state and multi-target observations, at time k, are the finite sets [4], [5], [7]

$$X_k = \{x_{k,1},\ldots,x_{k,N(k)}\} \subset \mathcal{X}$$
$$Z_k = \{z_{k,1},\ldots,z_{k,M(k)}\} \subset \mathcal{Z}.$$

The following notation is used throughout. Denote by C_j^ℓ the *binomial coefficient* $\frac{\ell!}{j!(\ell-j)!}$, P_j^n the *permutation coefficient* $\frac{n!}{(n-j)!}$, $\langle\cdot,\cdot\rangle$ the *inner product* defined between two real valued functions α and β by $\langle\alpha,\beta\rangle = \int\alpha(x)\beta(x)dx$, (or $\sum_{\ell=0}^{\infty}\alpha(\ell)\beta(\ell)$ when α and β are real sequences) and $e_j(\cdot)$ the *elementary symmetric function* [3] of order j defined for a finite set Z of real numbers by $e_j(Z) = \sum_{S\subseteq Z,|S|=j}\left(\prod_{\zeta\in S}\zeta\right)$, with $e_0(Z)=1$ by convention and $|S|$ the cardinality of a set S.

The CPHD recursion rests on the following assumptions regarding the target dynamics and observations.

1) Each target evolves and generates measurements independently of one another.
2) The birth RFS and the surviving RFSs are independent of each other.
3) The clutter RFS is an i.i.d cluster process[1] and independent of the measurement RFSs.
4) The prior and predicted multi-target RFSs are i.i.d cluster processes.

Let $v_{k|k-1}$ and $\rho_{k|k-1}$ denote the intensity and cardinality distribution associated with the predicted multi-target state and let v_k and ρ_k denote the intensity and cardinality distribution associated with the posterior multi-target state. The intensity and cardinality distribution are summary statistics of the underlying RFS. The intensity of an RFS is analogous to the mean of a random variable. The cardinality distribution describes, probabilistically, the number of elements in an RFS. We refer the reader to [8] for the construction of an approximate RFS density from these statistics. The following propositions, which constitute the prediction and update step of the CPHD filter, show explicitly how the posterior intensity and posterior cardinality distribution are jointly propagated in time [8], [11].

Proposition 1: *If at time $k-1$, the posterior cardinality distribution ρ_{k-1} and posterior intensity v_{k-1} are given, then the predicted cardinality distribution $\rho_{k|k-1}$ and predicted intensity $v_{k|k-1}$ are given by*

$$\rho_{k|k-1}(n) = \sum_{j=0}^{n}\rho_{\Gamma,k}(n-j)\Pi_{k|k-1}[v_{k-1},\rho_{k-1}](j), \tag{1}$$

$$v_{k|k-1}(x) = \gamma_k(x) + \int p_{S,k}(\zeta)f_{k|k-1}(x|\zeta)v_{k-1}(\zeta)d\zeta \tag{2}$$

where

$$\Pi_{k|k-1}[v,\rho](j) = \sum_{\ell=j}^{\infty}C_j^\ell\rho(\ell)\frac{\langle p_{S,k},v\rangle^j\langle 1-p_{S,k},v\rangle^{\ell-j}}{\langle 1,v\rangle^\ell}$$

$\rho_{\Gamma,k}(\cdot) =$ cardinality distribution of birth RFS
$\gamma_k(\cdot) =$ intensity function of birth RFS
$p_{S,k}(\zeta) =$ probability of survival to time k given state ζ at time $k-1$
$f_{k|k-1}(x|\zeta) =$ single target Markov transition density from from $k-1$ to time k.

Proposition 2: *If at time k, the predicted cardinality distribution $\rho_{k|k-1}$ and predicted intensity $v_{k|k-1}$ are given, then for a given measurement set Z_k, the updated cardinality distribution ρ_k and updated intensity v_k are given by*

$$\rho_k(n) = \frac{\Upsilon_k^0[v_{k|k-1};Z_k](n)\rho_{k|k-1}(n)}{\langle\Upsilon_k^0[v_{k|k-1};Z_k],\rho_{k|k-1}\rangle} \tag{3}$$

$$v_k(x) = v_{k|k-1}(x)\left[q_{D,k}(x)\frac{\langle\Upsilon_k^1[v_{k|k-1};Z_k],\rho_{k|k-1}\rangle}{\langle\Upsilon_k^0[v_{k|k-1};Z_k],\rho_{k|k-1}\rangle} + \sum_{z\in Z_k}\psi_{k,z}(x)\frac{\langle\Upsilon_k^1[v_{k|k-1};Z_k-\{z\}],\rho_{k|k-1}\rangle}{\langle\Upsilon_k^0[v_{k|k-1};Z_k],\rho_{k|k-1}\rangle}\right] \tag{4}$$

[1]See [30] for more details on i.i.d. cluster process.

where

$$\Upsilon_k^u[v,Z](n) = \sum_{j=0}^{\min(|Z|,n)} (|Z|-j)!\rho_{K,k}(|Z|-j) \times P_{j+u}^n \frac{\langle 1-p_{D,k}, v\rangle^{n-(j+u)}}{\langle 1,v\rangle^n} e_j\left(\Xi_k(v,Z)\right) \quad (5)$$

$$\psi_{k,z}(x) = \frac{\langle 1,\kappa_k\rangle}{\kappa_k(z)} g_k(z|x) p_{D,k}(x) \quad (6)$$

$$\Xi_k(v,Z) = \{\langle v,\psi_{k,z}\rangle : z \in Z\} \quad (7)$$

$$p_{D,k}(x) = \text{probability of detection of state } x \text{ at time } k \quad (8)$$

$$q_{D,k}(x) = 1 - p_{D,k}(x) \quad (9)$$

$$g_k(z|x) = \text{single target measurement likelihood at time } k \quad (10)$$

$$\rho_{K,k}(\cdot) = \text{cardinality distribution of clutter RFS at time } k \quad (11)$$

$$\kappa_k(\cdot) = \text{intensity function of clutter RFS at time } k. \quad (12)$$

If the cardinalities of the RFS involved are Poisson distributed, then the above propositions reduce to the PHD recursion [7]

$$v_{k|k-1}(x) = \gamma_k(x) + \int p_{S,k}(\zeta) f_{k|k-1}(x|\zeta) v_{k-1}(\zeta) d\zeta,$$

$$v_k(x) = \left[q_{D,k}(x) + \sum_{z\in Z_k} \frac{p_{D,k}(x) g_k(z|x)}{\kappa_k(z) + \langle p_{D,k} g_k(z|\cdot), v_{k|k-1}\rangle}\right] \times v_{k|k-1}(x).$$

III. CPHD Filtering With Unknown Clutter Rate

The underlying idea in this development is to model clutter by a random finite set of "false targets" or "generator objects" (distinct from actual targets) which are specified by standard type models for births/deaths and transitions as well as misses/detections and measurements (also distinct from the respective models for actual targets). The multi-target state is then the finite set of actual targets and clutter generators, which is to be estimated from the sequence of finite sets of observations generated by them. Estimation of the hybrid state of targets/objects then yields information on the number and individual states of actual targets in addition to the unknown clutter rate. In the following, we derive the CPHD recursion, which propagates separate intensity functions for the two target types, jointly alongside the cardinality distribution of all targets/objects, i.e., both actual and clutter.

A. Hybrid State Space Model

Let $\mathcal{X}^{(1)}$ denote the state space for actual targets, $\mathcal{X}^{(0)}$ denote the state space for clutter generators and $\mathcal{Z}$ denote a common observation space. Define the hybrid state space

$$\ddot{\mathcal{X}} = \mathcal{X}^{(1)} \uplus \mathcal{X}^{(0)}$$

where $\uplus$ denotes a disjoint union. The double dot notation is used throughout to denote a function or variable defined on the hybrid state space, i.e., denote $\ddot{x} \in \ddot{\mathcal{X}}$ for a hybrid state as opposed to $x \in \mathcal{X}^{(1)}$ or $c \in \mathcal{X}^{(0)}$ for actual or clutter states. The integral of a function $\ddot{f} : \ddot{\mathcal{X}} \to \mathbb{R}$ is given by

$$\int_{\ddot{\mathcal{X}}} \ddot{f}(\ddot{x}) d\ddot{x} = \int_{\mathcal{X}^{(1)}} \ddot{f}(x)dx + \int_{\mathcal{X}^{(0)}} \ddot{f}(c)dc.$$

It is assumed throughout that actual targets and clutter objects are statistically independent. Where necessary, a superscript $^{(1)}$ is used to denote functions or variables pertaining to actual targets, while a superscript $^{(0)}$ is used to denote functions or variables on the space of clutter objects. For any space $\mathcal{X}$, let $\mathcal{F}(\mathcal{X})$ denotes the set of all finite subsets of $\mathcal{X}$.

Suppose at time $k-1$ that hybrid multi-target state $\ddot{X}_{k-1} \in \mathcal{F}(\ddot{\mathcal{X}})$ is given by the disjoint union of actual and clutter states respectively, i.e., $\ddot{X}_{k-1} = X_{k-1}^{(1)} \uplus X_{k-1}^{(0)}$ where $X_{k-1}^{(1)} \in \mathcal{F}(\mathcal{X}^{(1)})$ and $X_{k-1}^{(0)} \in \mathcal{F}(\mathcal{X}^{(0)})$. At time k, the multi-target state evolves to $\ddot{X}_k \in \mathcal{F}(\ddot{\mathcal{X}})$ and is given by the disjoint union of finite set states of transitioned actual targets and finite set states of clutter generators respectively at time k, i.e.,

$$\ddot{X}_k = X_k^{(1)} \uplus X_k^{(0)} \quad (13)$$

where $X_k^{(1)} \in \mathcal{F}(\mathcal{X}^{(1)})$ and $X_k^{(0)} \in \mathcal{F}(\mathcal{X}^{(0)})$. The actual multi-target state and clutter multi-target state at time k are given by the union of surviving states and new births, respectively, i.e.,

$$X_k^{(1)} = \bigcup_{x_{k-1}\in X_{k-1}^{(1)}} S_{k|k-1}^{(1)}(x_{k-1}) \cup \Gamma_k^{(1)} \quad (14)$$

$$X_k^{(0)} = \bigcup_{c_{k-1}\in X_{k-1}^{(0)}} S_{k|k-1}^{(0)}(c_{k-1}) \cup \Gamma_k^{(0)} \quad (15)$$

and $S_{k|k-1}^{(1)}(x_{k-1})$ is an RFS which takes on the empty set $\emptyset$ with probability $1 - p_{S,k}^{(1)}(x_{k-1})$ or a singleton $\{x_k\} \in \mathcal{X}^{(1)}$ with probability density $p_{S,k}^{(1)}(x_{k-1}) f_{k|k-1}^{(1)}(x_k|x_{k-1})$. Similarly, $S_{k|k-1}^{(0)}(c_{k-1})$ is an RFS which takes on the empty set $\emptyset$ with probability $1 - p_{S,k}^{(0)}(c_{k-1})$ or a singleton $\{c_k\} \in \mathcal{X}^{(0)}$ with probability density $p_{S,k}^{(0)}(c_{k-1}) f_{k|k-1}^{(0)}(c_k|c_{k-1})$. Also, $\Gamma_k^{(1)}$ and $\Gamma_k^{(0)}$ are RFSs with realizations in $\mathcal{X}^{(1)}$ and $\mathcal{X}^{(0)}$, respectively, of new births for actual and clutter targets, described by intensity functions $\gamma_k^{(1)}$ and $\gamma_k^{(0)}$ and cardinality distributions $\rho_{\Gamma,k}^{(1)}$ and $\rho_{\Gamma,k}^{(0)}$ respectively. Consequently, the RFS of all (actual and clutter) target births $\Gamma_k^{(1)} \uplus \Gamma_k^{(0)}$ has cardinality distribution $\ddot{\rho}_{\Gamma,k} = \rho_{\Gamma,k}^{(1)} * \rho_{\Gamma,k}^{(0)}$ (where $*$ denotes a convolution) and intensity function $\ddot{\gamma}_k = \gamma_k^{(1)} + \gamma_k^{(0)}$. It is assumed that conditional on $X_{k-1}^{(1)}$ and $X_{k-1}^{(0)}$ respectively, the RFSs $S_{k|k-1}^{(1)}(\cdot)$ and $S_{k|k-1}^{(0)}(\cdot)$ are statistically independent. Note that actual targets cannot become clutter objects, and vice versa. To simplify notations, it is assumed that clutter generators are identical, and hence it is possible to ignore any functional dependence on the actual state of a clutter generator.

A theoretically more general approach that does not rely on these assumptions is presented in [20] and [21].

At time k, the hybrid multi-target state $\ddot{X}_k$ produces a finite set of measurements $Z_k \in \mathcal{F}(\mathcal{Z})$ given by the union of measurements produced by actual and clutter states, i.e.,

$$Z_k = D_k^{(1)}\left(X_k^{(1)}\right) \cup D_k^{(0)}\left(X_k^{(0)}\right) \tag{16}$$

where

$$D_k^{(1)}\left(X_k^{(1)}\right) = \bigcup_{x_k \in X_k^{(1)}} \Theta_k^{(1)}(x_k) \tag{17}$$

$$D_k^{(0)}\left(X_k^{(0)}\right) = \bigcup_{i=1,\ldots,|X_k^{(0)}|} \Theta_{k,i}^{(0)} \tag{18}$$

with $\Theta_k^{(1)}(x_k)$ being an RFS which takes on the empty set $\emptyset$ with probability $q_{D,k}^{(1)}(x_k) = 1 - p_{D,k}^{(1)}(x_k)$ or a singleton $\{z_k\} \in \mathcal{Z}$ with probability density $p_{D,k}^{(1)}(x_k)g_k(z_k|x_k)$ and for each $i = 1,\ldots,|X_k^{(0)}|$, $\Theta_{k,i}^{(0)}$ is an RFS which takes on the empty set $\emptyset$ with probability $q_{D,k}^{(0)} = 1 - p_{D,k}^{(0)}$ or a singleton $\{z_k\} \in \mathcal{Z}$ with probability density $p_{D,k}^{(0)}\tilde{\varkappa}_k(z_k)$. It is implicitly assumed that conditional on X_k, the RFSs in the union of (17) and (18) are statistically independent. Since clutter generators are identical, they have the same spatial measurement distribution, can generate at most one return at each time and have the same detection or generation probability. These are reasonable modelling assumptions considering the little amount of available statistical information on clutter. Note the difference between the clutter model adopted here and the standard one: Clutter is not Poisson but is binomial (given by clutter generators).

B. Recursion

The CPHD filter with unknown clutter rate jointly propagates the posterior intensity $\ddot{v}_k(\cdot)$ and posterior cardinality distribution $\ddot{\rho}_k(\cdot)$ of the hybrid state $\ddot{X}_k$. Due to the construction of the hybrid state space as a disjoint union of actual target and clutter generator spaces, the intensity $\ddot{v}_k(\cdot)$ is decomposable into

$$\ddot{v}_k(\ddot{x}) = \begin{cases} v_k^{(1)}(x), & \ddot{x} = x \\ v_k^{(0)}(c), & \ddot{x} = c \end{cases}$$

where $v_k^{(1)}(\cdot)$ and $v_k^{(0)}(\cdot)$ are the intensities for actual and clutter targets respectively. Thus, it is sufficient to propagate the respective intensities for actual and clutter targets $v_k^{(1)}(\cdot)$ and $v_k^{(0)}(\cdot)$ alongside the hybrid cardinality distribution $\ddot{\rho}_k(\cdot)$. Moreover, the posterior intensity $v_k^{(0)}(\cdot)$ of the clutter generators is characterized by the posterior mean number of clutter generators $N_k^{(0)}$, since the detections or false alarms generated by clutter targets do not depend on the actual value of the clutter state c. The estimated posterior mean clutter rate is simply $\lambda_k = N_k^{(0)} p_{D,k}^{(0)}$ since the cardinality distribution of clutter is binomial. Consequently, the CPHD filter for unknown clutter rate recursively propagates the following quantities: the posterior intensity $v_k^{(1)}(\cdot)$ of the actual target states, $N_k^{(0)}$ the posterior mean number of clutter generators, and $\ddot{\rho}_k(\cdot)$ the posterior cardinality distribution of all targets including actual and clutter.

Remark: The posterior cardinality $\ddot{\rho}_k(\ddot{n})$ of the hybrid state gives only information on the total number of actual and clutter targets. It is important to note that as a consequence of adopting a hybrid state space, the posterior mode $\ddot{N}_k = \arg\max_{\ddot{n}} \ddot{\rho}_k(\ddot{n})$ cannot be used to estimate the number of actual targets, since this will include both actual targets and clutter generators. Instead, only the posterior mean $N_k^{(1)} = \left\langle 1, v_k^{(1)} \right\rangle$ can be used as an actual target number estimate.

Remark: The independence of the clutter returns from their clutter state values means that it is sufficient to specify the model for clutter completely in terms of: the mean number of clutter births $N_{\Gamma,k}^{(0)} = \left\langle 1, \gamma_k^{(0)} \right\rangle$ and constant probability of clutter target survival $p_{S,k}^{(0)}$, as well as the spatial likelihood $\tilde{\varkappa}_k(\cdot)$ and constant probability of clutter target detection $p_{D,k}^{(0)}$. It is not necessary to specify explicit forms for the transition density $f_{k|k-1}^{(0)}(\cdot|\cdot)$ and birth intensity $\gamma_k^{(0)}(\cdot)$.

The following results follow directly from substituting the hybrid state space model parameters into the conventional CPHD recursion, hence the proof is omitted. Notice that the use of clutter generators to model false alarms eliminates the calculation of the elementary symmetric functions and enforces the updated cardinality distribution to be zero for each argument up until $\ddot{n} = |Z_k|$. The resulting filter has a linear complexity in the number of measurements at the expense of less informative actual target cardinality.

Proposition 3: *If at time $k-1$, the posterior intensity for actual targets $v_{k-1}^{(1)}$, the posterior mean number of clutter generators $N_{k-1}^{(0)}$ and the posterior hybrid cardinality distribution $\ddot{\rho}_{k-1}$, are given, then their respective predictions to time k are given by*

$$v_{k|k-1}^{(1)}(x) = \gamma_k^{(1)}(x) + \int p_{S,k}^{(1)}(\zeta) f_{k|k-1}^{(1)}(x|\zeta) v_{k-1}^{(1)}(\zeta) d\zeta$$
$$N_{k|k-1}^{(0)} = N_{\Gamma,k}^{(0)} + p_{S,k}^{(0)} N_{k-1}^{(0)}$$
$$\ddot{\rho}_{k|k-1}(\ddot{n}) = \sum_{j=0}^{\ddot{n}} \ddot{\rho}_{\Gamma,k}(\ddot{n}-j) \sum_{\ell=j}^{\infty} C_j^{\ell} \ddot{\rho}_{k-1}(\ell)(1-\phi)^{\ell-j}\phi^j$$

where

$$\phi = \left(\frac{\left\langle v_{k-1}^{(1)}, p_{S,k}^{(1)} \right\rangle + N_{k-1}^{(0)} p_{S,k}^{(0)}}{\left\langle 1, v_{k-1}^{(1)} \right\rangle + N_{k-1}^{(0)}} \right).$$

Proposition 4: *If at time k, the predicted intensity for actual targets $v_{k|k-1}^{(1)}$, the predicted mean number of clutter generators $N_{k|k-1}^{(0)}$, the predicted hybrid cardinality distribution $\ddot{\rho}_{k|k-1}$, are given, then their respective updates for a given sensor measurement set Z_k at time k are given by*

$$v_k^{(1)}(x) = v_{k|k-1}^{(1)}(x) \left[\frac{q_{D,k}^{(1)}(x) \frac{\left\langle \ddot{\Upsilon}_k^1[\ddot{v}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1} \right\rangle}{\left\langle \ddot{\Upsilon}_k^0[\ddot{v}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1} \right\rangle}}{\left\langle 1, v_{k|k-1}^{(1)} \right\rangle + N_{k|k-1}^{(0)}} \right.$$
$$\left. + \sum_{z \in Z_k} \frac{p_{D,k}^{(1)}(x) g_k(z|x)}{p_{D,k}^{(0)} N_{k|k-1}^{(0)} \tilde{\varkappa}_k(z) + \left\langle v_{k|k-1}^{(1)}, p_{D,k}^{(1)} g_k(z|\cdot) \right\rangle} \right]$$

$$N_k^{(0)} = N_{k|k-1}^{(0)} \left[q_{D,k}^{(0)} \frac{\frac{\langle \ddot{\Upsilon}_k^1[\ddot{v}_{k|k-1}Z_k], \ddot{\rho}_{k|k-1} \rangle}{\langle \ddot{\Upsilon}_k^0[\ddot{v}_{k|k-1}Z_k], \ddot{\rho}_{k|k-1} \rangle}}{\left\langle 1, v_{k|k-1}^{(1)} \right\rangle + N_{k|k-1}^{(0)}} + \sum_{z \in Z_k} \frac{p_{D,k}^{(0)} \tilde{\varkappa}_k(z)}{p_{D,k}^{(0)} N_{k|k-1}^{(0)} \tilde{\varkappa}_k(z) + \left\langle v_{k|k-1}^{(1)}, p_{D,k}^{(1)} g_k(z|\cdot) \right\rangle} \right]$$

$$\ddot{\rho}_k(\ddot{n}) = \begin{cases} 0 & \ddot{n} < |Z_k| \\ \frac{\ddot{\rho}_{k|k-1}(\ddot{n}) \ddot{\Upsilon}_k^0[\ddot{v}_{k|k-1}Z_k](\ddot{n})}{\langle \ddot{\rho}_{k|k-1}, \ddot{\Upsilon}_k^0 \rangle} & \ddot{n} \geq |Z_k| \end{cases}$$

where

$$\ddot{\Upsilon}_k^u[\ddot{v}_{k|k-1}Z_k](\ddot{n}) = \begin{cases} 0 & \ddot{n} < |Z_k| + u \\ P_{|Z_k|+u}^{\ddot{n}} \Phi^{\ddot{n}-(|Z_k|+u)} & \ddot{n} \geq |Z_k| + u, \end{cases}$$

$$\Phi = 1 - \frac{\left\langle v_{k|k-1}^{(1)}, p_{D,k}^{(1)} \right\rangle + N_{k|k-1}^{(0)} p_{D,k}^{(0)}}{\left\langle 1, v_{k|k-1}^{(1)} \right\rangle + N_{k|k-1}^{(0)}}.$$

If the cardinalities of the RFS involved are Poisson distributed, the above propositions reduce to the following PHD recursion for unknown clutter rate:

$$v_{k|k-1}^{(1)}(x) = \gamma_k^{(1)}(x) + \int p_{S,k}^{(1)}(\zeta) f_{k|k-1}^{(1)}(x|\zeta) v_{k-1}^{(1)}(\zeta) d\zeta$$

$$N_{k|k-1}^{(0)} = \left\langle 1, \gamma_k^{(0)} \right\rangle + p_{S,k}^{(0)} N_{k-1}^{(0)}$$

$$v_k^{(1)}(x) = v_{k|k-1}^{(1)}(x) \left[q_{D,k}^{(1)}(x) + \sum_{z \in Z_k} \frac{p_{D,k}^{(1)}(x) g_k(z|x)}{p_{D,k}^{(0)} N_{k|k-1}^{(0)} \tilde{\varkappa}_k(z) + \left\langle v_{k|k-1}^{(1)}, p_{D,k}^{(1)} g_k(z|\cdot) \right\rangle} \right]$$

$$N_k^{(0)} = N_{k|k-1}^{(0)} \left[q_{D,k}^{(0)}(x) + \sum_{z \in Z_k} \frac{p_{D,k}^{(0)} \tilde{\varkappa}_k(z)}{p_{D,k}^{(0)} N_{k|k-1}^{(0)} \tilde{\varkappa}_k(z) + \left\langle v_{k|k-1}^{(1)}, p_{D,k}^{(1)} g_k(z|\cdot) \right\rangle} \right].$$

C. Analytic Implementation

A closed-form solution to the CPHD recursion with unknown clutter rate can be derived via Gaussian mixtures in a similar manner to that for the conventional CPHD recursion. In the following, $\mathcal{N}(\cdot; m, P)$ denotes a Gaussian density with mean m and covariance P. Consider the following standard linear Gaussian assumptions for the transition and observation models of individual targets, as well as certain assumptions on the birth, death and detection of targets:

- Each actual target follows a linear Gaussian dynamical model i.e.,

$$f_{k|k-1}^{(1)}(x|\zeta) = \mathcal{N}(x; F_{k-1}\zeta, Q_{k-1}) \quad (30)$$
$$g_k(z|x) = \mathcal{N}(z; H_k x, R_k) \quad (31)$$

  where F_{k-1} is the state transition matrix, Q_{k-1} is the process noise covariance, H_k is the observation matrix and R_k is the observation noise covariance.

- The survival and detection probabilities for actual targets/object are state independent, i.e.,

$$p_{S,k}^{(1)}(x) = p_{S,k}^{(1)} \quad (32)$$
$$p_{D,k}^{(1)}(x) = p_{D,k}^{(1)}. \quad (33)$$

- The intensity of the actual target birth RFS is a Gaussian mixture of the form

$$\gamma_k^{(1)}(x) = \sum_{i=1}^{J_{\gamma,k}} w_{\gamma,k}^{(i)} \mathcal{N}\left(x; m_{\gamma,k}^{(i)}, P_{\gamma,k}^{(i)}\right) \quad (34)$$

  where $J_{\gamma,k}$, $w_{\gamma,k}^{(i)}$, $m_{\gamma,k}^{(i)}$, $P_{\gamma,k}^{(i)}$, $i = 1, \ldots, J_{\gamma,k}$, are given model parameters.

Remark: Refer to the previous remark for the model specification of clutter returns.

The following results are straightforward adaptations of the Gaussian mixture solution to the conventional CPHD recursion [11].

Proposition 5: *If at time $k-1$, the posterior intensity $v_{k-1}^{(1)}$, posterior mean number of clutter generators $N_{k-1}^{(0)}$, posterior hybrid cardinality distribution $\ddot{\rho}_{k-1}$, are given and $v_{k-1}^{(1)}$ is a Gaussian mixture given by*

$$v_{k-1}^{(1)}(x) = \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}\left(x; m_{k-1}^{(i)}, P_{k-1}^{(i)}\right) \quad (35)$$

then, the predicted intensity $v_{k|k-1}^{(1)}$ is also a Gaussian mixture and

$$v_{k|k-1}^{(1)}(x) = p_{S,k}^{(1)} \sum_{j=1}^{J_{k-1}} w_{k-1}^{(j)} \mathcal{N}\left(x; m_{S,k|k-1}^{(j)}, P_{S,k|k-1}^{(j)}\right) + \gamma_k^{(1)}(x)$$

$$N_{k|k-1}^{(0)} = N_{\Gamma,k}^{(0)} + p_{S,k}^{(0)} N_{k-1}^{(0)}$$

$$\ddot{\rho}_{k|k-1}(\ddot{n}) = \sum_{j=0}^{\ddot{n}} \ddot{\rho}_{\Gamma,k}(\ddot{n} - j) \sum_{\ell=j}^{\infty} C_j^\ell \ddot{\rho}_{k-1}(\ell)(1-\phi)^{\ell-j}\phi^j$$

where

$$\phi = \left(\frac{p_{S,k}^{(1)} \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} + p_{S,k}^{(0)} N_{k-1}^{(0)}}{\sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} + N_{k-1}^{(0)}} \right)$$

and $\gamma_k^{(1)}(x)$ is given in (34),

$$m_{S,k|k-1}^{(j)} = F_{k-1} m_{k-1}^{(j)}$$
$$P_{S,k|k-1}^{(j)} = Q_{k-1} + F_{k-1} P_{k-1}^{(j)} F_{k-1}^T.$$

Proposition 6: *If at time k, the predicted intensity $v_{k|k-1}^{(1)}$, predicted mean number of clutter generators $N_{k|k-1}^{(0)}$, predicted* *hybrid cardinality distribution $\ddot{\rho}_{k|k-1}$, are all given and $v_{k|k-1}^{(1)}$ is a Gaussian mixture given by*

$$v_{k|k-1}^{(1)}(x) = \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}\left(x; m_{k|k-1}^{(i)}, P_{k|k-1}^{(i)}\right) \quad (41)$$

then, given a measurement set Z_k, the updated intensity $v_k^{(1)}$ is also a Gaussian mixture and

$$v_k^{(1)}(x) = q_{D,k}^{(1)} \frac{\frac{\left\langle \ddot{\Psi}_k^1[\ddot{v}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1}\right\rangle}{\left\langle \ddot{\Psi}_k^0[\ddot{v}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1}\right\rangle}}{\sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} + N_{k|k-1}^{(0)}} v_{k|k-1}^{(1)}(x) + \sum_{z\in Z_k} \sum_{j=1}^{J_{k|k-1}} w_{D,k}^{(j)}(z) \mathcal{N}\left(x; m_k^{(j)}(z), P_k^{(j)}\right)$$

$$N_k^{(0)} = N_{k|k-1}^{(0)} \left[q_{D,k}^{(0)} \frac{\frac{\left\langle \ddot{\Psi}_k^1[\ddot{v}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1}\right\rangle}{\left\langle \ddot{\Psi}_k^0[\ddot{v}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1}\right\rangle}}{\sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} + N_{k|k-1}^{(0)}} + \sum_{z\in Z_k} \frac{p_{D,k}^{(0)} \tilde{\varkappa}_k(z)}{p_{D,k}^{(0)} N_{k|k-1}^{(0)} \tilde{\varkappa}_k(z) + p_{D,k}^{(1)} \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} q_k^{(j)}(z)} \right]$$

$$\ddot{\rho}_k(\ddot{n}) = \begin{cases} 0 & \ddot{n} < |Z_k| \\ \frac{\ddot{\rho}_{k|k-1}(\ddot{n}) \ddot{\Psi}_k^0[\ddot{v}_{k|k-1} Z_k](\ddot{n})}{\left\langle \ddot{\rho}_{k|k-1}, \ddot{\Psi}_k^0 \right\rangle} & \ddot{n} \geq |Z_k| \end{cases}$$

where

$$\ddot{\Psi}_k^u[\Phi_{k|k-1} Z_k](\ddot{n}) = \begin{cases} 0 & \ddot{n} < |Z_k| + u \\ P_{|Z_k|+u}^{\ddot{n}} \Phi_{k|k-1}^{\ddot{n}-(|Z_k|+u)} & \ddot{n} \geq |Z_k| + u \end{cases}$$

$$\Phi_{k|k-1} = 1 - \frac{p_{D,k}^{(1)} \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} + p_{D,k}^{(0)} N_{k|k-1}^{(0)}}{\sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} + N_{k|k-1}^{(0)}},$$

$$w_{D,k}^{(j)}(z) = \frac{p_{D,k}^{(1)} w_{k|k-1}^{(j)} q_k^{(j)}(z)}{p_{D,k}^{(0)} N_{k|k-1}^{(0)} \tilde{\varkappa}_k(z) + p_{D,k}^{(1)} \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} q_k^{(j)}(z)}$$

$$q_k^{(j)}(z) = \mathcal{N}\left(z; H_k m_{k|k-1}^{(j)}, H_k P_{k|k-1}^{(j)} H_k^T + R_k\right)$$

$$m_k^{(j)}(z) = m_{k|k-1}^{(j)} + K_k^{(j)} \left(z - H_k m_{k|k-1}^{(j)}\right)$$

$$P_k^{(j)} = \left[I - K_k^{(j)} H_k\right] P_{k|k-1}^{(j)}$$

$$K_k^{(j)} = P_{k|k-1}^{(j)} H_k^T \left[H_k P_{k|k-1}^{(j)} H_k^T + R_k\right]^{-1}.$$

Estimates for the mean number of actual targets and mean clutter rate are $\hat{N}_k^{(1)} = \sum_{j=1}^{J_k} w_k^{(j)}$ and $\hat{\lambda}_k = N_k^{(0)} p_{D,k}^{(0)}$ respectively. In implementations, pruning and merging of mixture components is required to prevent an exponential increase in the total number of components and truncation of posterior cardinality at a sufficiently high number of terms is required to enable a tractable propagation. These procedures are exactly the same as those for the implementation of the conventional CPHD and PHD filters [10], [11].

IV. CPHD Filtering With Unknown Detection Profile

The basic idea in this development is to augment the unknown detection probability into the single target state. This approach can accommodate (spatially) nonuniform detection profiles. Estimation of the augmented state of targets then yields information on the number of targets, as well as the individual kinematic states and the unknown detection probability for the particular target. In the following, we derive the CPHD recursion for the augmented state model, which propagates the cardinality distribution of targets and the intensity function involving the augmented state.

A. Augmented State Space Model

Let $\mathcal{X}^{(\Delta)} = [0, 1]$ denote the state space for the unknown detection probability. Define the augmented state space

$$\underline{\mathcal{X}} = \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)}$$

where $\times$ denotes a Cartesian product. The underscore notation is used throughout to denote a function or variable defined on the augmented state space, i.e., denote $\underline{x} = [x, a] \in \underline{\mathcal{X}}$ for an augmented state where $x \in \mathcal{X}^{(1)}$ for the kinematic state and $a \in \mathcal{X}^{(\Delta)} = [0, 1]$ for the augmented part. The integral of a function $\underline{f} : \underline{\mathcal{X}} \rightarrow \mathbb{R}$ is given by

$$\int_{\underline{\mathcal{X}}} \underline{f}(\underline{x}) d\underline{x} = \int_{\mathcal{X}^{(\Delta)}} \int_{\mathcal{X}^{(1)}} \underline{f}(x, a) dx da.$$

The multi-target transition and measurement models are essentially the same as in the conventional case, except that the single-target transition and measurement models are extended to accommodate the augmented state. For consistency in notations, the superscript $^{(1)}$ will continue to be used to denote functions or variables on the space of actual targets. Note that the conventional model applies for clutter, hence no clutter targets and no superscript $^{(0)}$ will be encountered in this subsection.

The single target survival probability and transition density for augmented states are simply

$$\underline{p}_{S,k}(\underline{x}) = \underline{p}_{S,k}(x, a) = p_{S,k}^{(1)}(x) \quad (49)$$

$$\underline{f}_{k|k-1}\left(\underline{x}|\underline{\zeta}\right) = \underline{f}_{k|k-1}(x, a|\zeta, \alpha) = f_{k|k-1}^{(1)}(x|\zeta) f_{k|k-1}^{(\Delta)}(a|\alpha). \quad (50)$$

Target births are given by an intensity $\underline{\gamma}_k^{(1)}(\cdot)$ for augmented states as well as corresponding usual cardinality distribution $\rho_{\Gamma,k}^{(1)}(\cdot)$. The single target detection probability and measurement likelihood for augmented states are

$$\underline{p}_{D,k}(\underline{x}) = \underline{p}_{D,k}(x, a) = a \quad (51)$$

$$\underline{g}_k(z|\underline{x}) = \underline{g}_k(z|x, a) = g_k(z|x). \quad (52)$$

Clutter follows the conventional CPHD model given by Poisson false alarms with intensity function $\kappa_k(\cdot)$.

B. Recursion

The derivation of the CPHD recursion for an augmented state model featuring unknown probability of detection is straightforward. It is a matter of substituting the single target motion and observation models for the augmented state into the conventional CPHD recursion given by Propositions 1 and 2. The resultant CPHD recursion propagates the posterior cardinality distribution $\rho_k(\cdot)$ and the posterior intensity function $\underline{v}_k^{(1)}(\cdot,\cdot)$ for the augmented state which now includes the unknown probability of detection. The following results are direct consequences of substituting the augmented state space model parameters into the conventional CPHD recursion, hence the proof is omitted. The new recursion retains the cubic complexity in measurements of the conventional CPHD recursion, since the conventional model for false alarms is used and consequently the computation of elementary symmetric functions is still required.

Proposition 7: *If at time $k-1$, the posterior intensity $\underline{v}_{k-1}^{(1)}$ and posterior cardinality distribution ρ_{k-1} are given, then the predicted cardinality distribution $\rho_{k|k-1}$ and predicted intensity $\underline{v}_{k|k-1}^{(1)}$ are given by*

$$\rho_{k|k-1}(n) = \sum_{j=0}^{n} \rho_{\Gamma,k}^{(1)}(n-j)\Pi_{k|k-1}\left[\underline{v}_{k-1}^{(1)}, \rho_{k-1}\right](j)$$

$$\underline{v}_{k|k-1}^{(1)}(x,a) = \underline{\gamma}_k^{(1)}(x,a) + \int\int_0^1 p_{S,k}(\zeta) f_{k|k-1}^{(\Delta)}(a|\alpha) f_{k|k-1}^{(1)}(x|\zeta)\underline{v}_{k-1}^{(1)}(\alpha,\zeta)d\alpha d\zeta$$

where

$$\Pi_{k|k-1}[\underline{v},\rho](j) = \sum_{\ell=j}^{\infty} C_j^{\ell}\rho(\ell)\frac{\langle p_{S,k},\underline{v}\rangle^j\langle 1-p_{S,k},\underline{v}\rangle^{\ell-j}}{\langle 1,\underline{v}\rangle^{\ell}}.$$

Proposition 8: *If at time k, the predicted intensity $\underline{v}_{k|k-1}^{(1)}$ and predicted cardinality distribution $\rho_{k|k-1}$ are given, then for a given measurement set Z_k, the updated cardinality distribution ρ_k and updated intensity $\underline{v}_k^{(1)}$ are given by*

$$\rho_k(n) = \frac{\underline{\Upsilon}_k^0\left[\underline{v}_{k|k-1}^{(1)}; Z_k\right](n)\rho_{k|k-1}(n)}{\left\langle \underline{\Upsilon}_k^0\left[\underline{v}_{k|k-1}^{(1)}; Z_k\right], \rho_{k|k-1}\right\rangle}$$

$$\underline{v}_k^{(1)}(x,a) = \underline{v}_{k|k-1}^{(1)}(x,a)\left[(1-a)\frac{\left\langle \underline{\Upsilon}_k^1\left[\underline{v}_{k|k-1}^{(1)}; Z_k\right], \rho_{k|k-1}\right\rangle}{\left\langle \underline{\Upsilon}_k^0\left[\underline{v}_{k|k-1}^{(1)}; Z_k\right], \rho_{k|k-1}\right\rangle} + \sum_{z\in Z_k}\underline{\psi}_{k,z}(x,a)\frac{\left\langle \underline{\Upsilon}_k^1\left[\underline{v}_{k|k-1}^{(1)}; Z_k-\{z\}\right], \rho_{k|k-1}\right\rangle}{\left\langle \underline{\Upsilon}_k^0\left[\underline{v}_{k|k-1}^{(1)}; Z_k\right], \rho_{k|k-1}\right\rangle}\right]$$

where

$$\underline{\Upsilon}_k^u\left[\underline{v}_{k|k-1}^{(1)}, Z_k\right](n) = \sum_{j=0}^{\min(|Z_k|,n)} (|Z_k|-j)!p_{K,k}(|Z_k|-j)P_{j+u}^n \times \frac{\left\langle 1-\underline{p}_{D,k}, \underline{v}_{k|k-1}^{(1)}\right\rangle^{n-(j+u)}}{\left\langle 1, \underline{v}_{k|k-1}^{(1)}\right\rangle^n} e_j\left(\Xi_k\left(\underline{v}_{k|k-1}^{(1)}, Z_k\right)\right)$$

$$\underline{p}_{D,k}(x,a) = a$$

$$\underline{\psi}_{k,z}(x,a) = \frac{\langle 1,\kappa_k\rangle}{\kappa_k(z)} g_k(z|x)\cdot a$$

$$\Xi_k\left(\underline{v}_{k|k-1}^{(1)}, Z_k\right) = \left\{\left\langle \underline{v}_{k|k-1}^{(1)}, \underline{\psi}_{k,z}\right\rangle : z\in Z_k\right\}.$$

Unlike the unknown clutter case, in CPHD filtering with unknown detection profile, estimates of the number of targets can be obtained as usual with the mode $N_k^{(1)} = \arg\max_n \rho_k(n)$ derived from the cardinality distribution or the mean $N_k^{(1)} = \left\langle 1, \underline{v}_k^{(1)}\right\rangle$ derived from the intensity function.

The following shows the reduction to the PHD recursion when the cardinalities of the RFS involved are Poisson distributed:

$$\underline{v}_{k|k-1}^{(1)}(x,a) = \underline{\gamma}_k(x,a) + \int\int_0^1 p_{S,k}(\zeta) f_{k|k-1}^{(\Delta)}(a|\alpha) f_{k|k-1}^{(1)}(x|\zeta)\underline{v}_{k-1}^{(1)}(\alpha,\zeta)d\alpha d\zeta \tag{61}$$

$$\underline{v}_k^{(1)}(x,a) = [1-a]\underline{v}_{k|k-1}^{(1)}(x,a) + \sum_{z\in Z_k}\frac{a\cdot g_k(z|x)\underline{v}_{k|k-1}^{(1)}(x,a)}{\kappa_k(z) + \left\langle \underline{p}_{D,k} g_k(z|\cdot), \underline{v}_{k|k-1}^{(1)}\right\rangle}. \tag{62}$$

C. Analytic Implementation

A closed-form implementation for the CPHD recursion with unknown detection profile is derived based on Beta–Gaussian mixtures. The Gaussian distribution is used to model the kinematic part of the state while the Beta distribution is used to model the augmented part of the state. Note that the measurement-update equation for the CPHD filter involves factors of both "a" and "$1-a$," which after multiple recursions, result in products of the form $a^s(1-a)^t$. Beta distributions are introduced naturally as an implementation strategy. The term Beta–Gaussian distribution is used to refer to a product of a Beta and a Gaussian distribution and the term Beta–Gaussian mixture to refer to a weighted sum of the former.

In the following, $\beta(\cdot; s,t)$ denotes a Beta distribution with parameters $s > 1$ and $t > 1$, with mean $\mu_\beta = \frac{s}{s+t}$ and variance $\sigma_\beta^2 = \frac{st}{(s+t)^2(s+t+1)}$. Also denote by $B(s,t) = \int_0^1 a^{s-1}(1-a)^{t-1}da$ the Beta function evaluated at s, t. As before, $\mathcal{N}(\cdot; m, P)$ denotes a Gaussian density with mean m and covariance P. Consider the following Beta and Gaussian assumptions:

- Target kinematics are linear Gaussian as given by (30) and (31).
- The survival probability for targets are state independent (32) but the detection probability is the augmented part of the state (51).
- The intensity of the target birth RFS is a Beta–Gaussian mixture of the form

$$\underline{\gamma}_k^{(1)}(x,a) = \sum_{i=1}^{J_{\gamma,k}} w_{\gamma,k}^{(i)}\beta\left(a; s_{\gamma,k}^{(i)}, t_{\gamma,k}^{(i)}\right)\mathcal{N}\left(x; m_{\gamma,k}^{(i)}, P_{\gamma,k}^{(i)}\right)$$

where $J_{\gamma,k}$, $w_{\gamma,k}^{(i)}$, $s_{\gamma,k}^{(i)}$, $t_{\gamma,k}^{(i)}$, $m_{\gamma,k}^{(i)}$, $P_{\gamma,k}^{(i)}$, $i=1,\ldots,J_{\gamma,k}$, are given model parameters.

- The time-prediction for the augmented variable a is completely governed by Beta densities (independent of the kinematic variable x)

$$\beta(a; s_{k-1}, t_{k-1}) \longrightarrow \beta(a; s_{k|k-1}, t_{k|k-1})$$

and preserves the mean $\mu_{\beta,k|k-1} = \mu_{\beta,k-1}$ but dilates the variance $\sigma^2_{\beta,k|k-1} = k_\beta \sigma^2_{\beta,k-1}$ by a prescribed factor $k_\beta > 0$ (a typical choice is $k_\beta > 1$ which enlarges the predicted variance).

Remark: For reasons of consistency, the parameters must be chosen such that $\sigma^2_{\beta,k|k-1} < \mu_{\beta,k-1}(1-\mu_{\beta,k-1})$. This choice of parameters is analogous to a random walk model for the augmented part of the state. In essence, in the prediction, we effectively maintain the mean of the augmented part of the state while increasing its uncertainty. This is consistent with physical intuition. It can be easily verified that the updated parameters which satisfy the mean and variance constraints are given by

$$s_{k|k-1} = \left(\frac{\mu_{\beta,k|k-1}(1-\mu_{\beta,k|k-1})}{\sigma^2_{\beta,k|k-1}} - 1 \right) \mu_{\beta,k|k-1}$$
$$t_{k|k-1} = \left(\frac{\mu_{\beta,k|k-1}(1-\mu_{\beta,k|k-1})}{\sigma^2_{\beta,k|k-1}} - 1 \right) \left(1-\mu_{\beta,k|k-1}\right).$$

Other choices for the prediction model are possible but will not be considered further here.

Under these assumptions, the following propositions present an analytic solution to the CPHD filter with unknown detection profile.

Proposition 9: *If at time $k-1$, the posterior intensity $\underline{v}^{(1)}_{k-1}$ and posterior cardinality distribution ρ_{k-1} are given and $\underline{v}^{(1)}_{k-1}$ is a Beta–Gaussian mixture of the form*

$$\underline{v}^{(1)}_{k-1}(x,a) = \sum_{i=1}^{J_{k-1}} w^{(i)}_{k-1} \beta\left(a; s^{(i)}_{k-1}, t^{(i)}_{k-1}\right) \mathcal{N}\left(x; m^{(i)}_{k-1}, P^{(i)}_{k-1}\right)$$

then, the predicted intensity $\underline{v}_{k|k-1}$ is also a Beta–Gaussian mixture and the prediction is given by

$$\rho_{k|k-1}(n) = \sum_{j=0}^{n} \rho_{\Gamma,k}(n-j) \sum_{\ell=j}^{\infty} C^\ell_j \rho_{k-1}(\ell) \left\langle p^{(1)}_{S,k}, \underline{v}^{(1)}_{k-1} \right\rangle^j \left(1-p^{(1)}_{S,k}\right)^{\ell-j}$$

$$\underline{v}^{(1)}_{k|k-1}(x,a) = p^{(1)}_{S,k} \sum_{j=1}^{J_{k-1}} w^{(j)}_{k-1} \beta\left(a; s^{(j)}_{S,k|k-1}, t^{(j)}_{S,k|k-1}\right) \times \mathcal{N}\left(x; m^{(j)}_{S,k|k-1}, P^{(j)}_{S,k|k-1}\right) + \underline{\gamma}^{(1)}_k(x,a)$$

where $\underline{\gamma}^{(1)}_k(x,a)$ is given in (34),

$$s^{(j)}_{S,k|k-1} = \left(\frac{\mu^{(j)}_{\beta,k|k-1}\left(1-\mu^{(j)}_{\beta,k|k-1}\right)}{\left[\sigma^{(j)}_{\beta,k|k-1}\right]^2} - 1 \right) \mu^{(j)}_{\beta,k|k-1}$$

$$t^{(j)}_{S,k|k-1} = \left(\frac{\mu^{(j)}_{\beta,k|k-1}\left(1-\mu^{(j)}_{\beta,k|k-1}\right)}{\left[\sigma^{(j)}_{\beta,k|k-1}\right]^2} - 1 \right) \left(1-\mu^{(j)}_{\beta,k|k-1}\right)$$

$$m^{(j)}_{S,k|k-1} = F_{k-1} m^{(j)}_{k-1}$$
$$P^{(j)}_{S,k|k-1} = Q_{k-1} + F_{k-1} P^{(j)}_{k-1} F^T_{k-1}.$$

and the parameters $\mu^{(j)}_{\beta,k|k-1} = \mu^{(j)}_{\beta,k-1} = \frac{s^{(j)}_{k-1}}{s^{(j)}_{k-1}+t^{(j)}_{k-1}}$ and $[\sigma^{(j)}_{\beta,k|k-1}]^2 = |k_\beta|[\sigma^{(j)}_{\beta,k-1}]^2 = |k_\beta| \frac{s^{(j)}_{k-1} t^{(j)}_{k-1}}{(s^{(j)}_{k-1}+t^{(j)}_{k-1})^2 (s^{(j)}_{k-1}+t^{(j)}_{k-1}+1)}$ are the predicted (preserved) mean and (enlarged) variance for the $j\mathrm{th}$ component of the augmented part of the state.

A formal proof is straightforward and only an outline is given here. The previous intensity is a Beta-Gaussian mixture, where each component of the mixure is a product of a Beta density (on the augmented part of the state) and a Gaussian density (on the kinematic part of the state). The prediction of each Beta-Gaussian component is then given by the prediction of the Beta part multiplied by the prediction of the Gaussian part. The prediction of the Gaussian part follows from the Gaussian Mixture solution to the conventional CPHD recursion [11]. The prediction for the Beta part follows directly from the last assumption in the above single target model.

Proposition 10: *If at time k, the predicted intensity $\underline{v}^{(1)}_{k|k-1}$ and predicted cardinality distribution $\rho_{k|k-1}$ are given and $\underline{v}^{(1)}_{k|k-1}$ is a Beta–Gaussian mixture of the form*

$$\underline{v}^{(1)}_{k|k-1}(x,a) = \sum_{i=1}^{J_{k|k-1}} w^{(i)}_{k|k-1} \beta\left(a; s^{(i)}_{k|k-1}, t^{(i)}_{k|k-1}\right) \mathcal{N}\left(x; m^{(i)}_{k|k-1}, P^{(i)}_{k|k-1}\right)$$

then, given a measurement set Z_k, the updated intensity $\underline{v}^{(1)}_k$ is also a Beta–Gaussian mixture and

$$\rho_k(n) = \frac{\underline{\Psi}^0_k[d_{k|k-1}, w_{k|k-1}, Z_k](n) \rho_{k|k-1}(n)}{\left\langle \underline{\Psi}^0_k[d_{k|k-1}, w_{k|k-1}, Z_k], \rho_{k|k-1} \right\rangle}$$

$$\underline{v}^{(1)}_k(x,a) = \sum_{j=1}^{J_{k|k-1}} w^{(j)}_{M,k} \beta\left(a; s^{(j)}_{k|k-1}, t^{(j)}_{k|k-1}+1\right) \times \mathcal{N}\left(x; m^{(j)}_{k|k-1}, P^{(j)}_{k|k-1}\right) + \sum_{z\in Z_k} \sum_{j=1}^{J_{k|k-1}} w^{(j)}_{D,k}(z) \times \beta\left(a; s^{(j)}_{k|k-1}+1, t^{(j)}_{k|k-1}\right) \mathcal{N}\left(x; m^{(j)}_k(z), P^{(j)}_k\right)$$

where

$$\underline{\Psi}^u_k[d_{k|k-1}, w_{k|k-1}, Z_k](n) = \sum_{j=0}^{\min(|Z_k|,n)} (|Z_k|-j)! \times \rho_{K,k}(|Z_k|-j) P^n_{j+u} \frac{\langle 1-d_{k|k-1}, w_{k|k-1}\rangle^{n-(j+u)}}{\langle 1, w_{k|k-1}\rangle^n} \times e_j\left(\Lambda_k(d_{k|k-1}, w_{k|k-1}, Z_k)\right)$$

$$\Lambda_k(d_{k|k-1}, w_{k|k-1}, Z_k) = \left\{ \frac{\langle 1, \kappa_k\rangle}{\kappa_k(z)} \sum_{j=1}^{J_{k|k-1}} d^{(j)}_{k|k-1} w^{(j)}_{k|k-1} q^{(j)}_k(z) : z \in Z \right\}$$

$$w_{k|k-1} = \left[w^{(1)}_{k|k-1}, \ldots, w^{(J_{k|k-1})}_{k|k-1}\right]^T$$

$$d_{k|k-1} = \left[d^{(1)}_{k|k-1}, \ldots, d^{(J_{k|k-1})}_{k|k-1}\right]^T$$

$$d^{(j)}_{k|k-1} = \frac{s^{(j)}_{k|k-1}}{s^{(j)}_{k|k-1} + t^{(1)}_{k|k-1}}$$

$$q_k(z) = \left[q^{(1)}_k(z), \ldots, q^{(J_{k|k-1})}_k(z)\right]^T$$

$$q_k^{(j)}(z) = \mathcal{N}\left(z; H_k m_{k|k-1}^{(j)}, H_k P_{k|k-1}^{(j)} H_k^T + R_k\right)$$

$$w_{M,k}^{(j)} = w_{k|k-1}^{(j)} \frac{B\left(s_{k|k-1}^{(j)}, t_{k|k-1}^{(j)}+1\right)}{B\left(s_{k|k-1}^{(j)}, t_{k|k-1}^{(j)}\right)} \times \frac{\left\langle \Psi_k^1\left[d_{k|k-1}, w_{k|k-1}, Z_k\right], \rho_{k|k-1}\right\rangle}{\left\langle \Psi_k^0\left[d_{k|k-1}, w_{k|k-1}, Z_k\right], \rho_{k|k-1}\right\rangle}$$

$$w_{D,k}^{(j)}(z) = w_{k|k-1}^{(j)} \frac{q_k^{(j)}(z)}{\frac{\kappa_k(z)}{\langle 1,\kappa_k\rangle}} \frac{B\left(s_{k|k-1}^{(j)}+1, t_{k|k-1}^{(j)}\right)}{B\left(s_{k|k-1}^{(j)}, t_{k|k-1}^{(j)}\right)} \times \frac{\left\langle \Psi_k^1\left[d_{k|k-1}, w_{k|k-1}, Z_k-\{z\}\right], \rho_{k|k-1}\right\rangle}{\left\langle \Psi_k^0\left[d_{k|k-1}, w_{k|k-1}, Z_k\right], \rho_{k|k-1}\right\rangle}$$

$$m_k^{(j)}(z) = m_{k|k-1}^{(j)} + K_k^{(j)}\left(z - H_k m_{k|k-1}^{(j)}\right)$$

$$P_k^{(j)} = \left[I - K_k^{(j)} H_k\right] P_{k|k-1}^{(j)}$$

$$K_k^{(j)} = P_{k|k-1}^{(j)} H_k^T \left[H_k P_{k|k-1}^{(j)} H_k^T + R_k\right]^{-1}.$$

A formal proof is straightforward but cumbersome and will not be shown here. Instead we outline the core idea behind the result. The predicted intensity is a Beta–Gaussian mixture, where each component of the mixure is a product of a Beta density (on the augmented part of the state) and a Gaussian density (on the kinematic part of the state). The update of each Beta–Gaussian componenent then involves computing particular product forms which are described as follows. A product of Gaussians $\mathcal{N}(x;m,P)\mathcal{N}(z;Hx,R)$ results in a single weighted Gaussian $q(z)\mathcal{N}\left(x;\tilde{m},\tilde{P}\right)$ as per the Gaussian mixture solution to the conventional CPHD recursion [11]. The other products encountered are given by the identities, $(1-a)\beta(a;s,t) = \frac{B(s,t+1)}{B(s,t)}\beta(a;s,t+1)$ and $a\beta(a;s,t) = \frac{B(s+1,t)}{B(s,t)}\beta(a;s+1,t)$, which result in weighted Beta densities in a. The update also involves the computation of integrals of the form $\int\int a\beta(a;s,t)\mathcal{N}(x;m,P)dadx$ which reduces to $\frac{s}{s+t}$. Thus, substitution of the Beta–Gaussian mixture form for the predicted intensity into the expression for the CPHD update with unknown detection profile and subsequent algebraic simplification using the previously stated identities yields the required result.

Estimates for the number of targets are derived either from the mean $\hat{N}_k^{(1)} = \sum_{j=1}^{J_k} w_k^{(j)}$ or the mode $\hat{N}_k^{(1)} = \arg\max_n \rho_k(n)$. Pruning and merging of mixture components is required to prevent an unbounded growth. A simple procedure adapted for the case of Beta–Gaussian mixtures is described here. Component pruning is performed by removing mixture components whose weights fall below a predetermined threshold T'. Component merging is performed by using the Hellinger distance $0 < d_{ij} < 1$ between two Beta–Gaussian components $\beta\left(a; s_k^{(i)}, t_k^{(i)}\right)\mathcal{N}\left(x; m_k^{(i)}, P_k^{(i)}\right)$ and $\beta\left(a; s_k^{(j)}, t_k^{(j)}\right)\mathcal{N}\left(x; m_k^{(j)}, P_k^{(j)}\right)$ [see the equation at the bottom of the page]. This similarity measure is used to identify groups of mixture components within a certain threshold S' of a reference component. Consequently, the entire group indexed by the set $I = \{i : d_{ij} < S'\}$ with reference component j, is replaced by a single Beta–Gaussian component $\tilde{w}_k^{(\tilde{j})}\beta\left(a; \tilde{s}_k^{(\tilde{j})}, \tilde{t}_k^{(\tilde{j})}\right)\mathcal{N}\left(x; \tilde{m}_k^{(\tilde{j})}, \tilde{P}_k^{(\tilde{j})}\right)$ with matching mean $\left(\tilde{\mu}_{\beta,k}^{(\tilde{j})}, \tilde{m}_k^{(\tilde{j})}\right)$ and approximate covariance $\left(\left[\tilde{\sigma}_{\beta,k}^{(\tilde{j})}\right]^2, \tilde{P}_k^{(\tilde{j})}\right)$, i.e.,

$$\sum_{i\in I} w_k^{(i)}\beta\left(a; s_k^{(i)}, t_k^{(i)}\right)\mathcal{N}\left(x; m_k^{(i)}, P_k^{(i)}\right) \approx \tilde{w}_k^{(\tilde{j})}\beta\left(a; \tilde{s}_k^{(\tilde{j})}, \tilde{t}_k^{(\tilde{j})}\right)\mathcal{N}\left(x; \tilde{m}_k^{(\tilde{j})}, \tilde{P}_k^{(\tilde{j})}\right)$$

where

$$\tilde{w}_k^{(\tilde{j})} = \sum_{i\in I} w_k^{(i)}$$

$$\tilde{m}_k^{(\tilde{j})} = \frac{1}{\tilde{w}_k^{(\tilde{j})}}\sum_{i\in I} w_k^{(i)} m_k^{(i)}$$

$$\tilde{P}_k^{(\tilde{j})} = \frac{1}{\tilde{w}_k^{(\tilde{j})}}\sum_{i\in I} w_k^{(i)} P_k^{(i)}$$

$$\tilde{s}_k^{(\tilde{j})} = \left(\frac{\tilde{\mu}_{\beta,k}^{(\tilde{j})}\left(1-\tilde{\mu}_{\beta,k}^{(\tilde{j})}\right)}{\left[\tilde{\sigma}_{\beta,k}^{(\tilde{j})}\right]^2} - 1\right)\tilde{\mu}_{\beta,k}^{(\tilde{j})}$$

$$\tilde{t}_k^{(\tilde{j})} = \left(\frac{\tilde{\mu}_{\beta,k}^{(\tilde{j})}\left(1-\tilde{\mu}_{\beta,k}^{(\tilde{j})}\right)}{\left[\tilde{\sigma}_{\beta,k}^{(\tilde{j})}\right]^2} - 1\right)\left(1-\tilde{\mu}_{\beta,k}^{(\tilde{j})}\right)$$

$$\tilde{\mu}_{\beta,k}^{(\tilde{j})} = \frac{1}{\tilde{w}_k^{(\tilde{j})}}\sum_{i\in I} w_k^{(i)}\mu_{\beta,k}^{(i)}$$

$$\left[\tilde{\sigma}_{\beta,k}^{(\tilde{j})}\right]^2 = \frac{1}{\tilde{w}_k^{(\tilde{j})}}\sum_{i\in I} w_k^{(i)}\left[\sigma_{\beta,k}^{(i)}\right]^2$$

$$\mu_{\beta,k}^{(i)} = \frac{s_k^{(i)}}{s_k^{(i)}+t_k^{(i)}}$$

$$\left[\sigma_{\beta,k}^{(i)}\right]^2 = \frac{s_k^{(i)} t_k^{(i)}}{\left(s_k^{(i)}+t_k^{(i)}\right)^2\left(s_k^{(i)}+t_k^{(i)}+1\right)}.$$

Starting with the component with the highest weight, identify all other components which fall within a certain similarity

$$d_{ij} = \sqrt{1 - \frac{B\left(\frac{s_k^{(i)}+s_k^{(j)}}{2}, \frac{t_k^{(i)}+t_k^{(j)}}{2}\right)}{\sqrt{B\left(s_k^{(i)}, t_k^{(i)}\right)B\left(s_k^{(j)}, t_k^{(j)}\right)}} \frac{\sqrt{\mathcal{N}\left(0; m_k^{(i)}-m_k^{(j)}, P_k^{(i)}+P_k^{(j)}\right)}}{\left(\det 8\pi\left(\left[P_k^{(j)}\right]^{-1}+\left[P_k^{(j)}\right]^{-1}\right)\right)^{\frac{1}{4}}}}.$$

threshold to the center of the group, then delete and replace the group with a single component. The process is repeated until all components have been accounted for. The number of components is capped to a predetermined value $J_{\max}$, retaining only those with the highest weights and re-normalizing the remaining weights to preserve the total mass. In addition, truncation of the cardinality distribution to $N_{\max}$ terms is required to ensure a tractable propagation.

V. CPHD Filtering With Jointly Unknown Clutter Rate and Detection Profile

To accommodate jointly unknown clutter rate and detection profile, we simply combine the previous two techniques outlined in Sections III and IV. Consequently, the single target state space is both hybridized and augmented. Clutter or false alarms are modelled by an unknown and time varying number of clutter generators. The cardinality distribution of clutter is again binomial. Both actual and clutter targets have an augmented state variable, in addition to their kinematic states, to describe their unknown and possibly time varying probability of detection. The state space model, single target models and resultant recursions are stated formally as follows.

A. Hybrid and Augmented State Space Model

Define the hybrid and augmented state space

$$\underline{\ddot{\mathcal{X}}} = \left(\mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)}\right) \uplus \left(\mathcal{X}^{(0)} \times \mathcal{X}^{(\Delta)}\right).$$

Consistent with previous notations, the double dot is used throughout to denote a function or variable defined on the hybrid and augmented state space and the underscore notation is used throughout to denote a function or variable defined on the augmented state space. Where a hybrid and augmented function or variable is encountered, a joint double dot and underscore notation is used, i.e., denote $\underline{\ddot{x}} \in \underline{\ddot{\mathcal{X}}}$ for a hybrid and augmented state as opposed to $\underline{x} = (x,a) \in \mathcal{X}^{(1)}\times\mathcal{X}^{(\Delta)}$ or $\underline{c} = (c,b) \in \mathcal{X}^{(0)}\times\mathcal{X}^{(\Delta)}$ for (augmented) actual or clutter states comprising the kinematic and augmented components respectively. The integral of a function $\underline{\ddot{f}} : \underline{\ddot{\mathcal{X}}} \to \mathbb{R}$ is given by

$$\int_{\underline{\ddot{\mathcal{X}}}} \underline{\ddot{f}}(\ddot{x})d\ddot{x} = + \int_{\mathcal{X}^{(\Delta)}}\int_{\mathcal{X}^{(1)}} \underline{\ddot{f}}(x,a)dxda \int_{\mathcal{X}^{(\Delta)}}\int_{\mathcal{X}^{(0)}} \underline{\ddot{f}}(c,b)dcdb.$$

As per previous developments, it is assumed throughout that actual targets and clutter objects are statistically independent. It is similarly assumed that all clutter generators are identical as far as the kinematic state is concerned, but not the augmented state and hence it is possible to ignore any functional dependence on the kinematic part of the state of a clutter generator. The dynamical and measurement models are essentially the same as in the development of the filter for the hybrid state space, except that these models are now amended to account for the additional incorporation of an augmented state space.

The joint probability of survival is defined piecewise:

$$\underline{\ddot{p}}_{S,k}(\underline{\ddot{x}}) = \begin{cases} p_{S,k}^{(1)}(x), & \underline{\ddot{x}} = (x,a) \in \mathcal{X}^{(1)}\times\mathcal{X}^{(\Delta)} \\ p_{S,k}^{(0)}, & \underline{\ddot{x}} = (c,b) \in \mathcal{X}^{(0)}\times\mathcal{X}^{(\Delta)}. \end{cases}$$

The joint transition density is defined piecewise:

$$\underline{\ddot{f}}_{k|k-1}(\ddot{x}|\ddot{\zeta}) = \begin{cases} f_{k|k-1}^{(1)}(x|\zeta)f_{k|k-1}^{(\Delta)}(a|\alpha), & \underline{\ddot{x}} = (x,a), \\ & \ddot{\zeta} = (\zeta,\alpha) \in \mathcal{X}^{(1)}\times\mathcal{X}^{(\Delta)} \\ f_{k|k-1}^{(0)}(c|\tau), & \underline{\ddot{x}} = (c,b), \\ & \ddot{\zeta} = (\tau,\upsilon) \in \mathcal{X}^{(0)}\times\mathcal{X}^{(\Delta)} \\ 0, & \text{otherwise.} \end{cases}$$

The joint birth intensity is defined piecewise and the joint birth cardinality is given by a convolution

$$\underline{\ddot{\gamma}}_k(\underline{\ddot{x}}) = \begin{cases} \underline{\gamma}_k^{(1)}(x,a), & \underline{\ddot{x}} = (x,a) \in \mathcal{X}^{(1)}\times\mathcal{X}^{(\Delta)} \\ \underline{\gamma}_k^{(0)}(c,b), & \underline{\ddot{x}} = (c,b) \in \mathcal{X}^{(0)}\times\mathcal{X}^{(\Delta)} \end{cases}$$

$$\ddot{\rho}_{\Gamma,k}(\ddot{n}) = \left(\rho_{\Gamma,k}^{(1)} * \rho_{\Gamma,k}^{(0)}\right)(\ddot{n}).$$

The joint probability of detection is defined piecewise:

$$\underline{\ddot{p}}_{D,k}(\underline{\ddot{x}}) = \begin{cases} a, & \underline{\ddot{x}} = (x,a) \in \mathcal{X}^{(1)}\times\mathcal{X}^{(\Delta)} \\ b, & \underline{\ddot{x}} = (c,b) \in \mathcal{X}^{(1)}\times\mathcal{X}^{(\Delta)}. \end{cases}$$

The joint likelihood is defined piecewise:

$$\underline{\ddot{g}}_k(z|\underline{\ddot{x}}) = \begin{cases} g_k(z|x), & \underline{\ddot{x}} = (x,a) \in \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)} \\ \tilde{\varkappa}_k(z), & \underline{\ddot{x}} = (c,b) \in \mathcal{X}^{(1)}\times\mathcal{X}^{(\Delta)}. \end{cases}$$

B. Recursion

The CPHD filter with jointly unknown clutter rate and detection probability propagates the posterior intensity $\underline{\ddot{v}}_k$ and posterior cardinality distribution $\ddot{\rho}_k(\cdot)$ of the hybrid and augmented state $\underline{\ddot{X}}_k$. Like the CPHD filter for unknown clutter rate, the intensity $\underline{\ddot{v}}_k(\cdot)$ of the hybridized and augmented state is decomposable into an intensity function of actual targets $\underline{v}_k^{(1)}(\cdot,\cdot)$ and clutter generators $\underline{v}_k^{(0)}(\cdot,\cdot)$. Note that the posterior intensity $\underline{v}_k^{(0)}(\cdot,\cdot)$ of the clutter generators is characterized by a single dependent variable $\underline{v}_k^{(0)}(\cdot)$, since the detections or false alarms generated by clutter targets do not depend on the actual value of the clutter state c. Again, the following propositions follow as direct result of combining the techniques presented in Sections III-B and IV-B and hence the proof is omitted.

Proposition 11: *If at time $k-1$, the posterior intensity for actual targets $\underline{v}_{k-1}^{(1)}$, the posterior intensity for clutter generators $\underline{v}_{k-1}^{(0)}$ and the posterior hybrid cardinality distribution $\ddot{\rho}_{k-1}$, are all given, then their respective predictions to time k are given by*

$$\underline{v}_{k|k-1}^{(1)}(x,a) = \underline{\gamma}_k^{(1)}(x,a) + \int\int_0^1 p_{S,k}^{(1)}(\zeta)f_{k|k-1}^{(\Delta)}(a|\alpha)f_{k|k-1}^{(1)}(x|\zeta)\underline{v}_{k-1}^{(1)}(\alpha,\zeta)d\alpha d\zeta$$

$$\underline{v}_{k-1}^{(0)}(b) = \underline{\gamma}_k^{(0)}(b) + p_{S,k}^{(0)}\underline{v}_{k-1}^{(0)}(b)$$

$$\ddot{\rho}_{k|k-1}(\ddot{n}) = \sum_{j=0}^{\ddot{n}} \ddot{\rho}_{\Gamma,k}(\ddot{n}-j)\sum_{\ell=j}^{\infty} C_j^\ell \ddot{\rho}_{k-1}(\ell)(1-\phi)^{\ell-j}\phi^j$$

where

$$\phi = \left(\frac{\left\langle \underline{v}_{k-1}^{(1)}, p_{S,k}^{(1)}\right\rangle + \left\langle \underline{v}_{k-1}^{(0)}, p_{S,k}^{(0)}\right\rangle}{\left\langle 1, \underline{v}_{k-1}^{(1)}\right\rangle + \left\langle 1, \underline{v}_{k-1}^{(0)}\right\rangle}\right).$$

Proposition 12: *If at time k, the predicted intensity for actual targets $\underline{v}_{k|k-1}^{(1)}$, the predicted intensity for clutter generators $\underline{v}_{k|k-1}^{(0)}$, the predicted hybrid cardinality distribution $\ddot{\rho}_{k|k-1}$, are all given, then their respective updates for a given sensor measurement set Z_k at time k are given by*

$$\underline{v}_k^{(1)}(x,a) = \underline{v}_{k|k-1}^{(1)}(x,a)\left[\frac{(1-a)\frac{\left\langle \ddot{\Upsilon}_k^1[\ddot{v}_{k|k-1}Z_k],\ddot{\rho}_{k|k-1}\right\rangle}{\left\langle \ddot{\Upsilon}_k^0[\ddot{v}_{k|k-1}Z_k],\ddot{\rho}_{k|k-1}\right\rangle}}{\left\langle 1,\underline{v}_{k|k-1}^{(1)}\right\rangle + \left\langle 1,\underline{v}_{k|k-1}^{(0)}\right\rangle} + \sum_{z\in Z_k}\frac{a\cdot g_k(z|x)}{\left\langle \underline{v}_{k|k-1}^{(0)},p_{D,k}^{(0)}\tilde{\varkappa}_k\right\rangle + \left\langle \underline{v}_{k|k-1}^{(1)},p_{D,k}^{(1)}g_k(z|\cdot)\right\rangle}\right]$$

$$\underline{v}_k^{(0)}(b) = \underline{v}_{k|k-1}^{(0)}(b)\left[\frac{(1-b)\frac{\left\langle \ddot{\Upsilon}_k^1[\ddot{v}_{k|k-1}Z_k],\ddot{\rho}_{k|k-1}\right\rangle}{\left\langle \ddot{\Upsilon}_k^0[\ddot{v}_{k|k-1}Z_k],\ddot{\rho}_{k|k-1}\right\rangle}}{\left\langle 1,\underline{v}_{k|k-1}^{(1)}\right\rangle + \left\langle 1,\underline{v}_{k|k-1}^{(0)}\right\rangle} + \sum_{z\in Z_k}\frac{b\cdot \tilde{\varkappa}_k(z)}{\left\langle \underline{v}_{k|k-1}^{(0)},p_{D,k}^{(0)}\tilde{\varkappa}_k\right\rangle + \left\langle \underline{v}_{k|k-1}^{(1)},p_{D,k}^{(1)}g_k(z|\cdot)\right\rangle}\right]$$

$$\ddot{\rho}_k(\ddot{n}) = \begin{cases} 0 & \ddot{n} < |Z_k| \\ \frac{\ddot{\rho}_{k|k-1}(\ddot{n})\ddot{\Upsilon}_k^0[\underline{v}_{k|k-1}Z_k](\ddot{n})}{\left\langle \ddot{\rho}_{k|k-1},\ddot{\Upsilon}_k^0\right\rangle} & \ddot{n} \geq |Z_k| \end{cases}$$

where

$$\ddot{\Upsilon}_k^u[\ddot{v}_{k|k-1}Z_k](\ddot{n}) = \begin{cases} 0 & \ddot{n} < |Z_k|+u \\ P_{|Z_k|+u}^{\ddot{n}}\Phi_{k|k-1}^{\ddot{n}-(|Z_k|+u)} & \ddot{n} \geq |Z_k|+u \end{cases}$$

$$\Phi = 1 - \frac{\left\langle \underline{v}_{k|k-1}^{(1)},\underline{p}_{D,k}^{(1)}\right\rangle + \left\langle \underline{v}_{k|k-1}^{(0)},\underline{p}_{D,k}^{(0)}\right\rangle}{\left\langle 1,\underline{v}_{k|k-1}^{(1)}\right\rangle + \left\langle 1,\underline{v}_{k|k-1}^{(0)}\right\rangle}$$

$$p_{D,k}^{(1)}(x,a) = a$$

$$p_{D,k}^{(0)}(b) = b.$$

Remark: Estimates of the posterior cardinality for actual targets and clutter targets must be calculated as an expected a posterior estimate $\hat{N}_k^{(1)} = \left\langle v_k^{(1)},1\right\rangle$. The expected *a posteriori* estimate of the mean clutter rate is $\hat{\lambda}_k = \left\langle \underline{v}_k^{(0)},\underline{p}_{D,k}^{(0)}\right\rangle$.

The corresponding PHD recursion follows as a special case:

$$\underline{v}_{k|k-1}^{(1)}(x,a) = \underline{\gamma}_k^{(1)}(x,a) + \int\int_0^1 p_{S,k}^{(1)}(\zeta)f_{k|k-1}^{(\Delta)}(a|\alpha)f_{k|k-1}^{(1)}(x|\zeta)\underline{v}_{k-1}^{(1)}(\alpha,\zeta)d\alpha d\zeta$$

$$\underline{v}_{k-1}^{(0)}(b) = \underline{\gamma}_k^{(0)}(b) + p_{S,k}^{(0)}\underline{v}_{k-1}^{(0)}(b)$$

$$\underline{v}_k^{(1)}(x,a) = \underline{v}_{k|k-1}^{(1)}(x,a)\left[1-a + \sum_{z\in Z_k}\frac{a\cdot g_k(z|x)}{\left\langle \underline{v}_{k|k-1}^{(0)},p_{D,k}^{(0)}\tilde{\varkappa}_k\right\rangle + \left\langle \underline{v}_{k|k-1}^{(1)},p_{D,k}^{(1)}g_k(z|\cdot)\right\rangle}\right]$$

$$\underline{v}_k^{(1)}(b) = \underline{v}_{k|k-1}^{(1)}(b)\left[1-b + \sum_{z\in Z_k}\frac{b\cdot \tilde{\varkappa}_k(z)}{\left\langle \underline{v}_{k|k-1}^{(0)},p_{D,k}^{(0)}\tilde{\varkappa}_k\right\rangle + \left\langle \underline{v}_{k|k-1}^{(1)},p_{D,k}^{(1)}g_k(z|\cdot)\right\rangle}\right].$$

C. Analytic Implementations

A closed-form implementation for the CPHD recursion with jointly unknown clutter rate and detection probability can be derived via Beta and Gaussian mixtures simply by combining the ideas previously adopted to derive closed-form implementations for the separate filters. The model assumptions for the Markov transition, measurement likelihood, survival probability, detection probability and clutter distribution are the same. The following are slightly different:

- The intensity of the actual target birth RFS is a Beta–Gaussian mixture of the form

$$\underline{\gamma}_k^{(1)}(x,a) = \sum_{i=1}^{J_{\gamma,k}^{(1)}} w_{\gamma,k}^{(i,1)}\beta\left(a;s_{\gamma,k}^{(i,1)},t_{\gamma,k}^{(i,1)}\right)\mathcal{N}\left(x;m_{\gamma,k}^{(i,1)},P_{\gamma,k}^{(i,1)}\right).$$

- The intensity of the birth RFS for clutter generators is a Beta mixture

$$\underline{\gamma}_k^{(0)}(b) = \sum_{i=1}^{J_{\gamma,k}^{(0)}} w_{\gamma,k}^{(i,0)}\beta\left(b;s_{\gamma,k}^{(i,0)},t_{\gamma,k}^{(i,0)}\right).$$

Again, the following propositions follow as direct result of combining the techniques presented in Sections III-C and IV-C and hence the proof is omitted.

Proposition 13: *If at time $k-1$, the posterior intensities $\underline{v}_{k-1}^{(1)}$ and $\underline{v}_{k-1}^{(0)}$ and posterior hybrid cardinality distribution $\ddot{\rho}_{k-1}$ are given, with $\underline{v}_{k-1}^{(1)}$ and $\underline{v}_{k-1}^{(0)}$ respectively given by the Beta–Gaussian and Beta mixtures*

$$\underline{v}_{k-1}^{(1)}(x,a) = \sum_{i=1}^{J_{k-1}^{(1)}} w_{k-1}^{(i,1)}\beta\left(a;s_{k-1}^{(i,1)},t_{k-1}^{(i,1)}\right)\mathcal{N}\left(x;m_{k-1}^{(i,1)},P_{k-1}^{(i,1)}\right)$$

$$\underline{v}_{k-1}^{(0)}(b) = \sum_{i=1}^{J_{k-1}^{(0)}} w_{k-1}^{(i,0)}\beta\left(b;s_{k-1}^{(i,0)},t_{k-1}^{(i,0)}\right)$$

then, the predicted intensities $\underline{v}_{k|k-1}^{(1)}$ and $\underline{v}_{k|k-1}^{(0)}$ are respectively also Beta–Gaussian and Beta mixtures and the prediction is given by

$$\ddot{\rho}_{k|k-1}(\ddot{n}) = \sum_{j=0}^{\ddot{n}}\ddot{\rho}_{\Gamma,k}(\ddot{n}-j)\sum_{\ell=j}^{\infty}C_j^{\ell}\ddot{\rho}_{k-1}(\ell)(1-\phi)^{\ell-j}\phi^j$$

$$\underline{v}_{k|k-1}^{(1)}(x,a) = p_{S,k}^{(1)}\sum_{j=1}^{J_{k-1}^{(1)}} w_{k-1}^{(j,1)}\beta\left(a;s_{S,k|k-1}^{(j,1)},t_{S,k|k-1}^{(j,1)}\right) \times\mathcal{N}\left(x;m_{S,k|k-1}^{(j,1)},P_{S,k|k-1}^{(j,1)}\right) + \underline{\gamma}_k^{(1)}(x,a)$$

$$\underline{v}_{k|k-1}^{(0)}(b) = p_{S,k}^{(0)}\underline{v}_{k-1}^{(0)}(b) + \underline{\gamma}_k^{(0)}(b)$$

where

$$\phi = \left(\frac{p_{S,k}^{(1)} \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i,1)} + p_{S,k}^{(0)} \sum_{i=1}^{J_{k-1}^{(0)}} w_{k-1}^{(i,0)}}{\sum_{i=1}^{J_{k-1}^{(1)}} w_{k-1}^{(i,1)} + \sum_{i=1}^{J_{k-1}^{(0)}} w_{k-1}^{(i,0)}} \right)$$

and $\underline{\gamma}_k^{(1)}(x,a)$ *is given in (34),*

$$s_{S,k|k-1}^{(j,1)} = \left(\frac{\mu_{\beta,k|k-1}^{(j,1)} \left(1 - \mu_{\beta,k|k-1}^{(j,1)}\right)}{\left[\sigma_{\beta,k|k-1}^{(j,1)}\right]^2} - 1 \right) \mu_{\beta,k|k-1}^{(j,1)}$$

$$t_{S,k|k-1}^{(j,1)} = \left(\frac{\mu_{\beta,k|k-1}^{(j,1)} \left(1 - \mu_{\beta,k|k-1}^{(j,1)}\right)}{\left[\sigma_{\beta,k|k-1}^{(j,1)}\right]^2} - 1 \right) \left(1 - \mu_{\beta,k|k-1}^{(j,1)}\right)$$

$$m_{S,k|k-1}^{(j,1)} = F_{k-1} m_{k-1}^{(j,1)}$$

$$P_{S,k|k-1}^{(j,1)} = Q_{k-1} + F_{k-1} P_{k-1}^{(j,1)} F_{k-1}^T$$

with $\mu_{\beta,k|k-1}^{(j,1)} = \mu_{\beta,k-1}^{(j,1)} = \frac{s_{k-1}^{(j,1)}}{s_{k-1}^{(j,1)} + t_{k-1}^{(j,1)}}$ *and* $\left[\sigma_{\beta,k|k-1}^{(j,1)}\right]^2 = |k_\beta| \left[\sigma_{\beta,k-1}^{(j,1)}\right]^2 = |k_\beta| \frac{s_{k-1}^{(j,1)} t_{k-1}^{(j,1)}}{\left(s_{k-1}^{(j,1)} + t_{k-1}^{(j,1)}\right)^2 \left(s_{k-1}^{(j,1)} + t_{k-1}^{(j,1)} + 1\right)}$.

Proposition 14: *If at time k, the predicted intensities* $\underline{v}_{k|k-1}^{(1)}$ *and* $\underline{v}_{k|k-1}^{(0)}$ *and predicted hybrid cardinality distribution* $\ddot{\rho}_{k|k-1}$*, are all given, with* $\underline{v}_{k|k-1}^{(1)}$ *and* $\underline{v}_{k|k-1}^{(0)}$ *respectively given by the Beta–Gaussian and Gaussian mixtures*

$$\underline{v}_{k|k-1}^{(1)}(x,a) = \sum_{i=1}^{J_{k|k-1}^{(1)}} w_{k|k-1}^{(i,1)} \beta\left(a; s_{k|k-1}^{(i,1)}, t_{k|k-1}^{(i,1)}\right) \times \mathcal{N}\left(x; m_{k|k-1}^{(i,1)}, P_{k|k-1}^{(i,1)}\right)$$

$$\underline{v}_{k|k-1}^{(0)}(b) = \sum_{i=1}^{J_{k|k-1}^{(0)}} w_{k|k-1}^{(i,0)} \beta\left(b; s_{k|k-1}^{(i,0)}, t_{k|k-1}^{(i,0)}\right)$$

then, given a measurement set Z_k*, the updated intensities* $\underline{v}_k^{(1)}$ *and* $\underline{v}_k^{(0)}$ *are respectively also Beta–Gaussian and Gaussian mixtures and the update simplifies to*

$$\begin{aligned} \underline{v}_k^{(1)}(x,a) = & \sum_{j=1}^{J_{k|k-1}^{(1)}} w_{M,k}^{(j,1)} \beta\left(a; s_{k|k-1}^{(j,1)}, t_{k|k-1}^{(j,1)} + 1\right) \\ & \times \mathcal{N}\left(x; m_{k|k-1}^{(j,1)}, P_{k|k-1}^{(j,1)}\right) \\ & + \sum_{z \in Z_k} \sum_{j=1}^{J_{k|k-1}^{(1)}} w_{D,k}^{(j,1)}(z) \beta\left(a; s_{k|k-1}^{(j,1)} + 1, t_{k|k-1}^{(j,1)}\right) \\ & \times \mathcal{N}\left(x; m_k^{(j,1)}(z), P_k^{(j,1)}\right) \end{aligned}$$

$$\begin{aligned} \underline{v}_k^{(0)}(b) = & \sum_{j=1}^{J_{k|k-1}^{(0)}} w_{M,k}^{(j,0)} \beta\left(a; s_{k|k-1}^{(j,0)}, t_{k|k-1}^{(j,0)} + 1\right) \\ & + \sum_{z \in Z_k} \sum_{j=1}^{J_{k|k-1}} w_{D,k}^{(j,0)}(z) \beta\left(a; s_{k|k-1}^{(j,0)} + 1, t_{k|k-1}^{(j,0)}\right) \end{aligned}$$

$$\ddot{\rho}_k(\ddot{n}) = \begin{cases} 0 & \ddot{n} < |Z_k| \\ \frac{\ddot{\rho}_{k|k-1}(\ddot{n}) \underline{\ddot{\Psi}}_k^0 [\ddot{v}_{k|k-1} Z_k](\ddot{n})}{\left\langle \ddot{\rho}_{k|k-1}, \underline{\ddot{\Psi}}_k^0 \right\rangle} & \ddot{n} \geq |Z_k|. \end{cases}$$

where

$$\underline{\ddot{\Psi}}_k^u [\Phi_{k|k-1}, Z_k](\ddot{n}) = \begin{cases} 0 & \ddot{n} < |Z_k| + u \\ P_{|Z_k|+u}^{\ddot{n}} \Phi_{k|k-1}^{\ddot{n} - (|Z_k|+u)} & \ddot{n} \geq |Z_k| + u \end{cases}$$

$$\Phi_{k|k-1} = 1 - \frac{\sum_{i=1}^{J_{k|k-1}^{(1)}} w_{k|k-1}^{(i,1)} d_{k|k-1}^{(j,1)} + \sum_{i=1}^{J_{k|k-1}^{(0)}} w_{k|k-1}^{(i,0)} d_{k|k-1}^{(j,0)}}{\sum_{i=1}^{J_{k|k-1}^{(1)}} w_{k|k-1}^{(i,1)} + \sum_{i=1}^{J_{k|k-1}^{(0)}} w_{k|k-1}^{(i,0)}},$$

$$d_{k|k-1}^{(j,u)} = \frac{s_{k|k-1}^{(j,u)}}{s_{k|k-1}^{(j,u)} + t_{k|k-1}^{(j,u)}}, u = 0, 1$$

$$w_{M,k}^{(j,u)}(z) = w_{k|k-1}^{(j,u)} \frac{\frac{B\left(s_{k|k-1}^{(j,u)}, t_{k|k-1}^{(j,u)} + 1\right)}{B\left(s_{k|k-1}^{(j,u)}, t_{k|k-1}^{(j,u)}\right)} \frac{\left\langle \underline{\ddot{\Psi}}_k^1 [\underline{\ddot{v}}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1} \right\rangle}{\left\langle \underline{\ddot{\Psi}}_k^0 [\underline{\ddot{v}}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1} \right\rangle}}{\sum_{i=1}^{J_{k|k-1}^{(1)}} w_{k|k-1}^{(i,1)} + \sum_{i=1}^{J_{k|k-1}^{(0)}} w_{k|k-1}^{(i,0)}}, \quad u = 0, 1$$

$$w_{D,k}^{(j,0)}(z) = \frac{w_{k|k-1}^{(j,0)} \frac{B\left(s_{k|k-1}^{(j,0)} + 1, t_{k|k-1}^{(j,0)}\right)}{B\left(s_{k|k-1}^{(j,0)}, t_{k|k-1}^{(j,0)}\right)} \tilde{\varkappa}_k(z)}{\sum_{i=1}^{J_{k|k-1}^{(0)}} d_{k|k-1}^{(j,0)} w_{k|k-1}^{(i,0)} \tilde{\varkappa}_k(z) + \sum_{i=1}^{J_{k|k-1}^{(1)}} d_{k|k-1}^{(j,1)} w_{k|k-1}^{(i,1)} q_k^{(j,1)}(z)}$$

$$w_{D,k}^{(j,1)}(z) = \frac{w_{k|k-1}^{(j,1)} \frac{B\left(s_{k|k-1}^{(j,1)} + 1, t_{k|k-1}^{(j,1)}\right)}{B\left(s_{k|k-1}^{(j,1)}, t_{k|k-1}^{(j,1)}\right)} q_k^{(j,1)}(z)}{\sum_{i=1}^{J_{k|k-1}^{(0)}} d_{k|k-1}^{(j,0)} w_{k|k-1}^{(i,0)} \tilde{\varkappa}_k(z) + \sum_{i=1}^{J_{k|k-1}^{(1)}} d_{k|k-1}^{(j,1)} w_{k|k-1}^{(i,1)} q_k^{(j,1)}(z)}$$

$$q_k^{(j,1)}(z) = \mathcal{N}\left(z; H_k m_{k|k-1}^{(j,1)}, H_k P_{k|k-1}^{(j,1)} H_k^T + R_k\right)$$

$$m_k^{(j,1)}(z) = m_{k|k-1}^{(j,1)} + K_k^{(j,1)} \left(z - H_k m_{k|k-1}^{(j,1)}\right)$$

$$P_k^{(j,1)} = \left[I - K_k^{(j,1)} H_k\right] P_{k|k-1}^{(j,1)}$$

$$K_k^{(j,1)} = P_{k|k-1}^{(j,1)} H_k^T \left[H_k P_{k|k-1}^{(j,1)} H_k^T + R_k\right]^{-1}.$$

The estimated mean number of actual targets is $\hat{N}_k^{(1)} = \sum_{j=1}^{J_k^{(1)}} w_k^{(i,1)}$, while the estimated mean number of clutter generators is $\hat{N}_k^{(0)} = \sum_{j=1}^{J_k^{(0)}} w_k^{(i,0)}$ and the estimated mean clutter rate is $\hat{\lambda}_k = \sum_{j=1}^{J_k^{(0)}} w_k^{(i,0)} d_k^{(j,0)}$. Pruning and merging of mixture components is still required and truncation of the cardinality distribution is needed, to ensure a tractable propagation. The procedure is essentially the same as previously outlined.

For nonlinear models, extended and unscented Kalman approximations can be applied to the analytic implementations of three CPHD filters proposed in Sections III–V. Note that the

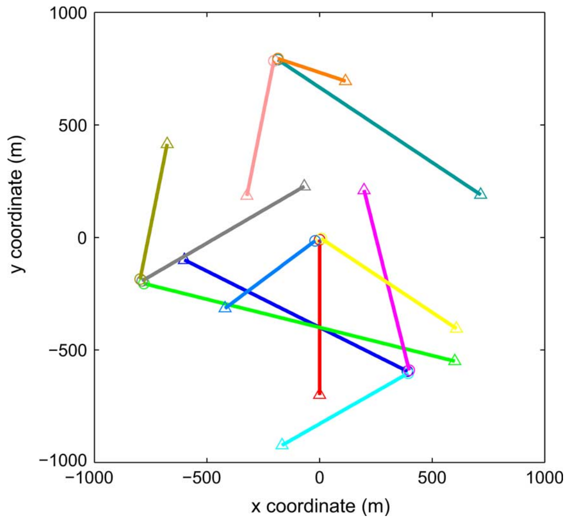

Fig. 1. Trajectories in the xy plane. Start/Stop positions for each track are shown with $\circ/\triangle$.

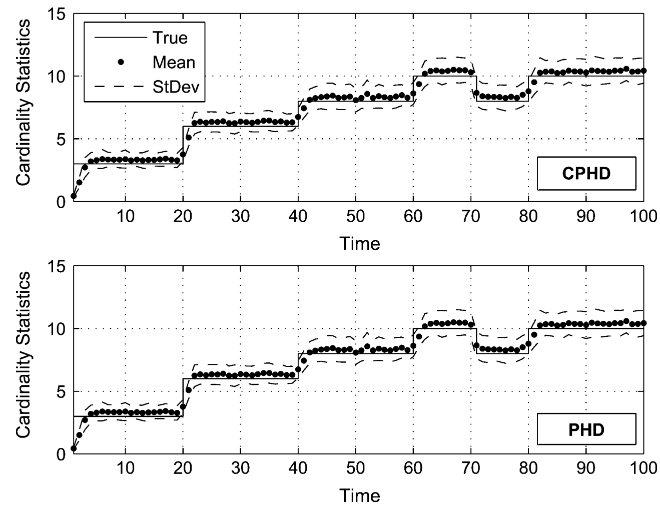

Fig. 2. 100 Monte Carlo run average results for CPHD and PHD filters for unknown clutter rate. True and estimated target numbers for CPHD (top) and PHD (bottom).

nonlinear approximations apply only to the kinematic part of the state and for actual targets which are represented by Gaussian distributions. The augmented part represented by Beta distributions remains unchanged.

VI. Numerical Studies

This section presents numerical studies for the new CPHD filters. A linear Gaussian example is used to illustrate and examine the performance of the CPHD filter for unknown clutter rate as well as the CPHD for unknown detection probability. A nonlinear example is also used to illustrate the performance of the CPHD filter for jointly unknown clutter rate and detection probability, via modification of the Beta–Gaussian solution with the unscented Kalman approximations to cope with the nonlinearity. The optimal subpattern assignment (OSPA) metric [31] is used for performance assessments.

A. Linear Gaussian Constant Velocity Example

Consider a 10 target scenario on the region $[-1000, 1000]\text{m}\times[-1000, 1000]\text{m}$. Targets move with constant velocity as shown in Fig. 1. The kinematic target state is a vector of planar position and velocity $x_k = [p_{x,k}, p_{y,k}, \dot{p}_{x,k}, \dot{p}_{y,k}]^T$. Measurements are noisy vectors of planar position only $z_k = [z_{x,k}, z_{y,k}]^T$. The single-target state space model is linear Gaussian with parameters

$$F_k = \begin{bmatrix} I_2 & \Delta I_2 \\ 0_2 & I_2 \end{bmatrix} \quad Q_k = \sigma_\nu^2 \begin{bmatrix} \frac{\Delta^4}{4} I_2 & \frac{\Delta^3}{2} I_2 \\ \frac{\Delta^3}{2} I_2 & \Delta^2 I_2 \end{bmatrix}$$
$$H_k = \begin{bmatrix} I_2 & 0_2 \end{bmatrix} \quad R_k = \sigma_\varepsilon^2 I_2$$

where I_n and 0_n denote the $n\times n$ identity and zero matrices respectively, $\Delta = 1\text{s}$ is the sampling period, $\sigma_\nu = 5\text{ms}^{-2}$ and $\sigma_\varepsilon = 10\text{m}$ are the standard deviations of the process noise and measurement noise.

1) Unknown Clutter Rate Only: For the filter with unknown clutter rate, the following scenario parameters are used. The target state consists of the kinematic components only. The survival probability for actual targets is $p_{S,k}^{(1)} = 0.99$. The birth process for actual targets is a Poisson RFS with intensity $\gamma_k^{(1)}(x) = \sum_{i=1}^{4} w_\gamma \mathcal{N}\left(x; m_\gamma^{(i)}, P_\gamma\right)$, where $w_\gamma = 0.03$, $m_\gamma^{(1)} = [0,0,0,0]^T$, $m_\gamma^{(2)} = [400, -600, 0, 0]^T$, $m_\gamma^{(3)} = [-800, -200, 0, 0]^T$, $m_\gamma^{(4)} = [-200, 800, 0, 0]^T$ and $P_\gamma = \text{diag}([50, 50, 50, 50]^T)^2$. The detection probability for measurements is a constant $p_{D,k}^{(1)} = 0.98$. Clutter returns are generated according to a binomial cardinality with parameters $N_k^{(0)} = 100$ and $p_{D,k}^{(0)} = 0.5$ and uniform spatial probability density $1/V$ over the surveillance region where $V = 4\times 10^6\ \text{m}^2$ is the 'volume' of the surveillance region. The mean clutter rate is hence 50 points per scan and the intensity of clutter is $\lambda_k^{(0)} = N_k^{(0)} p_{D,k}^{(0)}/V = 1.25\times 10^{-5}\text{m}^{-2}$. This information however is not known to the filter. The model for clutter generators given to the filter is that of births given by a mean rate of $N_{\Gamma,k}^{(0)} = 10$ while deaths are given by the survival probability of $p_{S,k}^{(0)} = 0.9$ and returns are given by detection probability $p_{D,k}^{(0)} = 0.5$. The density of clutter returns $\tilde{\varkappa}_k$ is presumed to be uniform on the measurement space. The filter is initialized with a zero intensity for actual targets and hence zero number of actual targets, and with a clutter rate equal to the total number of measurements received at the first time step minus the average birth and detection rate for actual targets.

Pruning and merging of Gaussian components is performed at each time step using a weight threshold of $T' = 10^{-5}$, a merging threshold of $U' = 4\ \text{m}$ and a maximum of $J_{\max} = 100$ Gaussian components. The number of actual targets is estimated as the expected *a posteriori* cardinality and state estimates are extracted as the means of corresponding highest Gaussian components of the posterior intensity. The joint actual target and clutter generator cardinality distribution is capped at $N_{\max} = 300$ terms.

Over 100 Monte Carlo runs, it can be seen from the averaged results in Figs. 2 and 3 that both the CPHD and PHD filters converge to the correct number of targets and that both produce

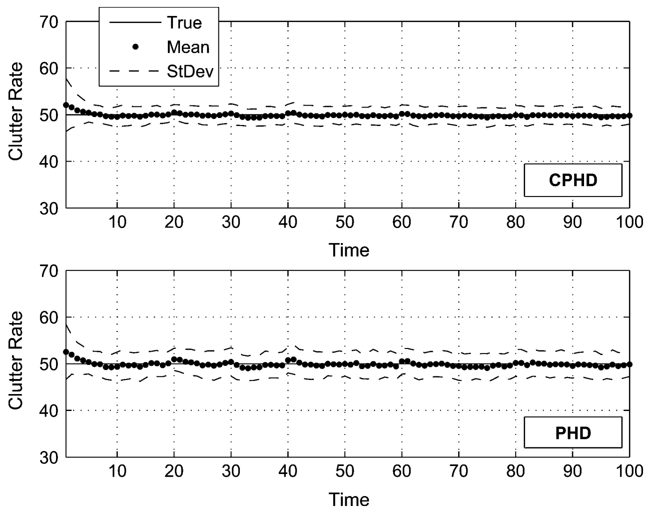

Fig. 3. 100 Monte Carlo run average results for CPHD and PHD filters for unknown clutter rate. True and estimated mean clutter rate for CPHD (top) and PHD (bottom).

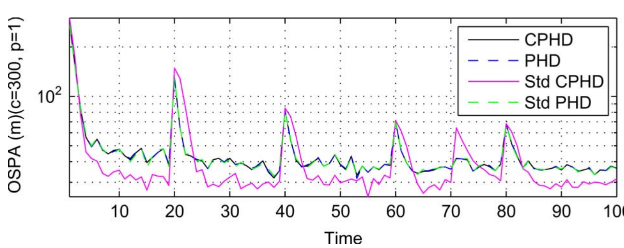

Fig. 4. OSPA miss distance versus time for the CPHD and PHD filters for unknown clutter rate.

accurate estimates of the mean clutter rate. In terms of miss distance performance as shown in Fig. 4, using the OSPA metric with parameters $p = 1$ and $c = 300\mathrm{m}$ on the estimated positions only, the CPHD and PHD filters both yield reasonable performance. It can be seen that there is an initial settling in period, but after this the miss distance achieves a value consistent with the measurement noise profile and exhibits peaks with changes in target numbers. The CPHD appears to have a slight advantage over the PHD. Comparisons are also shown with the standard CPHD and PHD filters which are supplied with the correct clutter rate. It can be seen that the new filters exhibit similar performance and converge to the same error value as that for the standard PHD filter, but cannot outperform the standard CPHD filter because the latter has better cardinality estimation and higher computational complexity.

2) Unknown Detection Profile Only: For the filter with unknown detection probability, the following scenario parameters are used. The target state is now augmented $\underline{x}_k = [a_k, x_k]^T$ to additionally include the unknown detection probability. The prediction for the augmented part of the state preserves the mean but increases the variance by 10%. The survival probability for actual targets is $p_{S,k}^{(1)} = 0.99$. The birth process for actual targets is a Poisson RFS with intensity $\underline{\gamma}_k^{(1)}(a, x) = \sum_{i=1}^{4} w_\gamma \beta(a; u_\gamma, v_\gamma)\mathcal{N}\left(x; m_\gamma^{(i)}, P_\gamma\right)$, where $w_\gamma = 0.03$, $u_\gamma = v_\gamma = 1$, $m_\gamma^{(1)} = [0, 0, 0, 0]^T$,

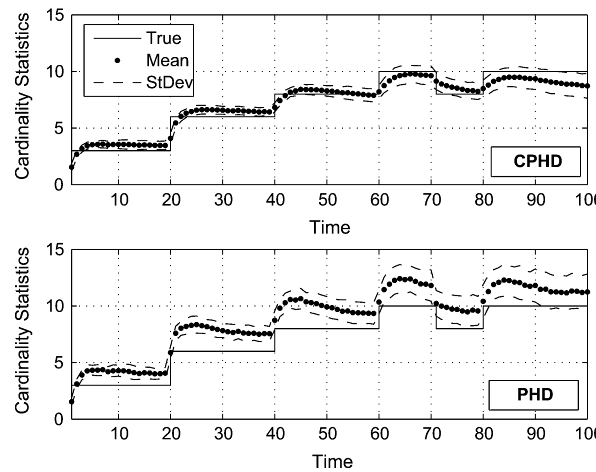

Fig. 5. 100 Monte Carlo run average results for CPHD and PHD filters for unknown detection probability. True and estimated target numbers for CPHD (top) and PHD (bottom).

$m_\gamma^{(2)} = [400, -600, 0, 0]^T$, $m_\gamma^{(3)} = [-800, -200, 0, 0]^T$, $m_\gamma^{(4)} = [-200, 800, 0, 0]^T$ and $P_\gamma = \mathrm{diag}([10, 10, 10, 10]^T)^2$. Measurements are generated according to a constant detection probability $p_{D,k}^{(1)} = 0.98$ but this is not known to the filter and must be implicitly estimated at the location of each of the tracks. Clutter returns follow a Poisson RFS with a mean rate uniform spatial probability density $1/V$ over the surveillance region where $V = 4 \times 10^6 \mathrm{m}^2$ is the "volume" of the surveillance region. The mean clutter rate is 20 points per scan and the intensity of clutter is $\lambda_k^{(0)} = 5.0 \times 10^{-6}\mathrm{m}^{-2}$. The filter is initialized to a zero state.

Pruning and merging of Beta–Gaussians is performed at each time step using a weight threshold of $T' = 10^{-3}$, a merging threshold of $S' = 1\%$ and a maximum of $J_{\max} = 1000$ Beta–Gaussians. The number of actual targets is estimated as the expected *a posteriori* number and state estimates are extracted as the means of corresponding highest components of the posterior intensity, ignoring the augmented part of the state. The cardinality distribution is calculated to a maximum of $N_{\max} = 300$ terms.

Over 100 Monte Carlo runs, it can be seen from the averaged results in Fig. 5 that the CPHD filter converges to the correct number of targets in general whereas the PHD filter has some bias which appears to slowly self-correct. In terms of miss distance performance as shown in Fig. 6, using the OSPA metric with parameters $p = 1$ and $c = 300\mathrm{m}$ on the estimated positions only, both filters perform reasonably, but the CPHD performs better than the PHD filter. The miss distance increases in the latter half of the simulation due to the fact that both filters have difficulty resolving closely spaced targets. Comparisons are also shown with the standard CPHD and PHD filters which are supplied with the correct detection probability. As expected, it can be seen that the new filters cannot achieve the same performance as their counterparts which are given the correct detection model.

Remark: In this example, the initialization in the birth model for the unknown detection probability is uniform on $[0, 1]$ which is the most uninformative. It is remarkable that the filter is still

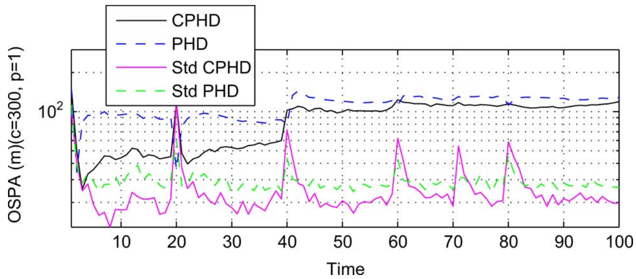

Fig. 6. OSPA miss distance versus time for the CPHD and PHD filters for unknown detection probability.

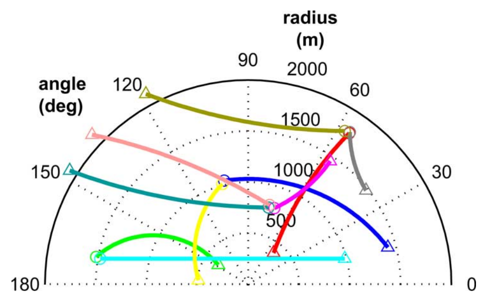

Fig. 7. Trajectories in the $r\theta$ plane. Start/Stop positions for each track are shown with $\circ/\triangle$.

able to confirm target tracks and reject false alarms, but as expected there is a noticeable delay before the filter is able to produce accurate estimates. When the initialization for the detection probability is more reasonable, the filter responds faster to cardinality changes and produces a smaller error.

B. Nonlinear Bearings and Range Example

Consider now a nonlinear bearings and range scenario with a total of 10 targets. Target tracks are shown in Fig. 7 on the half disc of radius 2000 m with the start and stop positions of each track. The augmented target state $x_k = [\tilde{x}_k^T, \omega_k, a_k]^T$ comprises the planar position and velocity $\tilde{x}_k^T = [\, p_{x,k}, \dot{p}_{x,k}, p_{y,k}, \dot{p}_{y,k}\,]^T$, the turn rate ω_k and the unknown detection probability a_k. Target generated measurements are noisy bearings and range vectors $z_k = [\theta_k, r_k]^T$. The single-target transition model is given by

$$\begin{aligned}\tilde{x}_k &= F(\omega_{k-1})\tilde{x}_{k-1} + G w_{k-1}\\ \omega_k &= \omega_{k-1} + \Delta u_{k-1}\end{aligned}$$

where

$$F(\omega) = \begin{bmatrix} 1 & \frac{\sin\omega\Delta}{\omega} & 0 & -\frac{1-\cos\omega\Delta}{\omega} \\ 0 & \cos\omega\Delta & 0 & -\sin\omega\Delta \\ 0 & \frac{1-\cos\omega\Delta}{\omega} & 1 & \frac{\sin\omega\Delta}{\omega} \\ 0 & \sin\omega\Delta & 0 & \cos\omega\Delta \end{bmatrix}$$

$$G = \begin{bmatrix} \frac{\Delta^2}{2} & 0 \\ \Delta & 0 \\ 0 & \frac{\Delta^2}{2} \\ 0 & \Delta \end{bmatrix}$$

$w_{k-1} \sim \mathcal{N}(\cdot; 0, \sigma_w^2 I)$, $u_{k-1} \sim \mathcal{N}(\cdot; 0, \sigma_u^2 I)$, $\Delta = 1$ s, $\sigma_w = 15\ \mathrm{m/s^2}$ and $\sigma_u = \pi/180$ rad/s. The prediction for the augmented part of the state is set according to $k_\beta = 1.1$. The survival probability is $p_{S,k}^{(1)}(x) = 0.99$. The single-target measurement model is given by

$$z_k = \begin{bmatrix} \arctan\left(\frac{p_{x,k}}{p_{y,k}}\right) \\ \sqrt{p_{x,k}^2 + p_{y,k}^2} \end{bmatrix} + \varepsilon_k$$

(the angle is measured from the vertical axis) where $\varepsilon_k \sim \mathcal{N}(\cdot; 0, R_k)$, with $R_k = \mathrm{diag}([\sigma_\theta^2, \sigma_r^2]^T)$, $\sigma_\theta = (\pi/180)$ rad and $\sigma_r = 5\mathrm{m}$. The birth process follows a Poisson RFS with intensity $\underline{\gamma}_k^{(1)}(a,x) = \sum_{i=1}^4 w_\gamma^{(i)} \beta(a; u_\gamma, v_\gamma) \mathcal{N}\left(x; m_\gamma^{(i)}, P_\gamma\right)$ where $w_\gamma^{(1)} = w_\gamma^{(2)} = 0.02$ and $w_\gamma^{(3)} = w_\gamma^{(4)} = 0.03$, $u_\gamma = 98$, $v_\gamma = 2$, $m_\gamma^{(1)} = [-1500, 0, 250, 0\, 0]^T$, $m_\gamma^{(2)} = [-250, 0, 1000, 0\, 0]^T$, $m_\gamma^{(3)} = [250, 0, 750, 0\ 0]^T$, $m_\gamma^{(4)} = [1000, 0, 1500, 0\ 0]^T$ and $P_\gamma = \mathrm{diag}\left([70, 70, 70, 70, 7\,(\pi/180)]^T\right)^2$.

Measurements are generated with detection probability $p_{D,k}^{(1)} = 0.98$ (this is not known to the filter). Clutter returns are generated according to a binomial cardinality with parameters $N_k^{(0)} = 20$ and $p_{D,k}^{(0)} = 0.5$ and with uniform spatial density $1/V$ over the surveillance region where $V = 2000\pi\mathrm{radm}$ is the 'volume' of the surveillance region. The mean clutter rate is hence 10 points per scan and the intensity of clutter is $\lambda_k^{(0)} = N_k^{(0)} p_{D,k}^{(0)}/V = 1.5 \times 10^{-3}\ (\mathrm{radm})^{-1}$ (this is similarly not known and must be estimated by the filter). The model for clutter generators given to the filter is that of births given by a Poisson RFS with intensity $\underline{\gamma}_k^{(0)}(b) = 0.2\beta(b; 1, 1)$ while deaths are given by the survival probability of $p_{S,k}^{(0)} = 0.9$. Pruning, merging and capping of mixture components is also performed. The estimated number of targets is the posterior mean while the state estimates are the corresponding means of the posterior Gaussian components only.

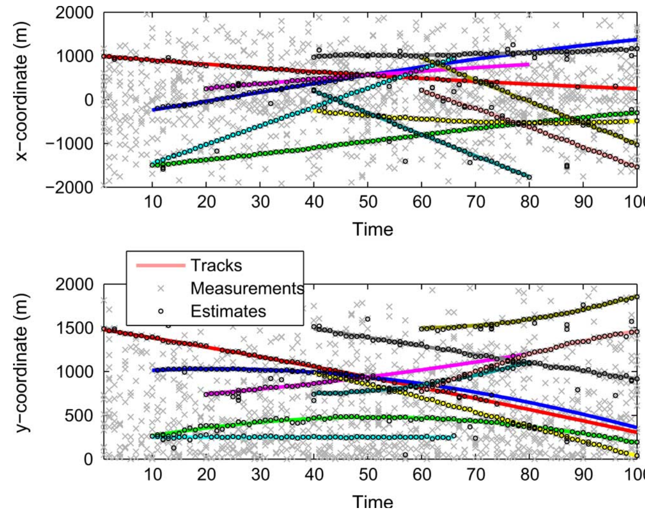

Fig. 8. Unscented Kalman CPHD filter for jointly unknown clutter rate and detection probability: filter estimates and true tracks in x and y coordinates versus time.

The performance of the unscented Kalman CPHD filter for jointly unknown clutter rate and detection profile is now shown. In Fig. 8, the target trajectories in x and y coordinates versus time are plotted against those estimated from the unscented

Kalman approximation to the CPHD filters. Results for the extended Kalman approximation are similar. The plots indicate that the CPHD filter is able to identify all object births and deaths, but still has some difficulty resolving closely spaced targets. It also appears that a jointly unknown clutter rate and detection probability is a difficult tracking scenario to contend with and consequently the proposed CPHD filter can only be expected to perform well when it is given a reasonable indication of the possible ranges of the clutter rate and detection probability. The degree of difficulty is compounded by the nonlinearity of the model, which requires further approximation in the UK implementation. Nonetheless, it is expected that a higher data or sampling rate should result in increased performance, since the predictions are performed on shortened time scales and updates are performed with increased data rates (all resulting in smaller uncertainty).

VII. Conclusion

Knowledge of parameters such as clutter rate and detection profile are of critical importance in Bayesian multi-target filtering. Estimating clutter rate and detection profile are difficult problems in practice and the capability of multi-target filters to adaptively learn these model parameters is very important. It has been shown that mismatches in clutter and detection model parameters can be accommodated within the PHD/CPHD multi-target filtering framework. Specifically, we have proposed certain versions of the CPHD (and PHD) filters that adaptively learn nonuniform detection profile and clutter rate while filtering. While the tracking performance cannot rival the ideal scenario where the clutter rate and detection profile are known *a priori,* numerical studies show that the proposed technique can correct for discrepancies in these parameters and produces promising results. The PHD filter is biased in its estimation of the unknown detection profile, while the CPHD filter appears not to have this limitation. Both filters however are unbiased in their estimate of the unknown clutter rate. Future works will continue the development of CPHD/PHD filters to additionally accommodate other unknown model parameters. In addition, nonlinearities in the model may be better handled by SMC implementations but another source of error arises in the extraction of the state estimate from the particle population and the multi-Bernoulli approach of [32] may be preferable.

References

[1] Y. Bar-Shalom and T. E. Fortmann, *Tracking and Data Association*. San Diego, CA: Academic, 1988.
[2] S. Blackman, *Multiple Target Tracking With Radar Applications*. Norwood, MA: Artech House, 1986.
[3] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*. New York: Springer-Verlag, 1995.
[4] R. Mahler, *Statistical Multisource-Multitarget Information Fusion*. Norwood, MA: Artech House, 2007.
[5] I. Goodman, R. Mahler, and H. Nguyen, *Mathematics of Data Fusion*. Norwell, MA: Kluwer, 1997.
[6] B.-N. Vo, B.-T. Vo, N.-T. Pham, and D. Suter, "Joint detection and estimation of multiple objects from image observations," *IEEE Trans. Signal Process.*, vol. 58, no. 10, pp. 5129–5241, Oct. 2010.
[7] R. Mahler, "Multi-target Bayes filtering via first-order multi-target moments," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 39, no. 4, pp. 1152–1178, 2003.
[8] R. Mahler, "PHD filters of higher order in target number," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 43, no. 3, pp. 1523–1543, 2007.
[9] B.-N. Vo, S. Singh, and A. Doucet, "Sequential Monte Carlo methods for multi-target filtering with random finite sets," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 41, no. 4, pp. 1224–1245, 2005.
[10] B.-N. Vo and W.-K. Ma, "The Gaussian mixture probability hypothesis density filter," *IEEE Trans. Signal Process.*, vol. 54, no. 11, pp. 4091–4104, Nov. 2006.
[11] B.-T. Vo, B.-N. Vo, and A. Cantoni, "Analytic implementations of the cardinalized probability hypothesis density filter," *IEEE Trans. Signal Process.*, vol. 55, no. 7, pp. 3553–3567, Jul. 2007.
[12] L. Lin, Y. Bar-Shalom, and T. Kirubarajan, "Track labeling and PHD filter for multitarget tracking," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 42, no. 3, pp. 778–795, Jul. 2006.
[13] K. Panta, B.-N. Vo, and S. Singh, "Novel data association schemes for the probability hypothesis density filter," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 43, no. 2, pp. 556–570, 2007.
[14] K. Panta, D. Clark, and B.-N. Vo, "Data association and track management for the Gaussian mixture probability hypothesis density filter," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 45, no. 3, pp. 1003–1016, 2009.
[15] A. Pasha, B.-N. Vo, H. D. Tuan, and W. K. Ma, "A Gaussian mixture PHD filter for jump Markov system model," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 45, no. 3, pp. 919–936, 2009.
[16] N. Whiteley, S. Singh, and S. Godsill, "Auxiliary particle implementation of probability hypothesis density filter," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 46, no. 3, pp. 1437–1454, 2010.
[17] B. Ristic, D. Clark, and B.-N. Vo, "Improved SMC implementation of the PHD filter," presented at the 13th Annu. Conf. Inf. Fusion, Edinburgh, U.K., 2010.
[18] J. Houssineau and D. Laneuville, "PHD filter with diffuse spatial prior on the birth process with applications to GM-PHD filter," presented at the Proc. 13th Annu. Conf. Inf. Fusion, Edinburgh, U.K., 2010.
[19] A. Elgammal, R. Duraiswami, D. Harwood, and L. S. Davis, "Background and foreground modeling using nonparametric kernel density estimation for visual surveillance," *Proc. IEEE*, vol. 90, no. 7, pp. 1151–1162, 2002.
[20] R. Mahler and A. El-Fallah, I. Kadar, Ed., "CPHD filtering with unknown probability of detection," in *Signal Process., Sens. Fusion, Target Recognit. XIX, SPIE Proc.*, 2010, vol. 7697.
[21] R. Mahler and A. El-Fallah, O. Drummond, Ed., "CPHD and PHD filters for unknown backgrounds, III: Tractable multitarget filtering in dynamic clutter," in *Signal Data Process. Small Targets 2010, SPIE Proc.*, 2010, vol. 7698.
[22] F. G. Cozman, "A brief introduction to the theory of sets of probability measures," Robotics Insititute, Carnegie Mellon Universities, Pittsburgh, PA, Tech Rep. CMU-RI-TR 97-24, 1999.
[23] B. Noack, V. Klumpp, D. Brunn, and U. Hanebeck, "Nonlinear Bayesian estimation with convex sets of probability densities," in *FUSION 2008*, Cologne.
[24] P. Walley, "Statistical reasonning with imprecise probabilities," in *Monographs on Statistics and Applied Probability*. London, U.K.: Chapman & Hall, 1991, vol. 42.
[25] S. Basu, "Ranges of posterior probabilities over a distribution band," *J. Statist. Planning Inference*, vol. 44, pp. 149–166, 1995.
[26] J. Berger, D. Insua, and F. Ruggeri, "Robust Bayesian analysis," in *Lecture Notes in Statistics*. New York: Springer, 2000, vol. 152, pp. 1–32.
[27] M. Berliner, "Hierachical Bayesian time series models," in *Maximum Entropy and Bayesian Methods*, K. Hauson and R. Silver, Eds. Norwell, MA: Kluwer, 1996, pp. 15–22.
[28] S. Singh, N. Whiteley, and S. Godsil, An "Approximate likelihood method for estimating the static parameters in multi-target tracking models," Univ. of Cambridge, Cambridge, U.K., Tech. Rep. CUED/F-INFENG/TR-606, 2009.
[29] X. Chen, R. Tharmarasa, T. Kirubarajan, and M. Pelletier, O. Drummond, Ed., "Integrated clutter estimation and target tracking using Poisson point process," in *Proc. SPIE, Signals Data Process. Small Targets*, 2009, vol. 7445.
[30] D. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes*. New York: Springer-Verlag, 1988.
[31] D. Schuhmacher, B.-T. Vo, and B.-N. Vo, "A consistent metric for performance evaluation of multi-object filters," *IEEE Trans. Signal Process.*, vol. 56, no. 8, pp. 3447–3457, Aug. 2008.
[32] B.-T. Vo, B.-N. Vo, and A. Cantoni, "The cardinality balanced multi-target multi-Bernoulli filter and its implementations," *IEEE Trans. Signal Process.*, vol. 57, no. 2, pp. 409–423, Feb. 2009.

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