Chapter 2 Interpolation

2.1 Interpolation by Polynomials

2.1.1. Theoretical Foundation: The Interpolation Formula of Lagrange

Π_n: the set of all real or complex polynomials P whose degrees do not exceed n:

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n$$

(2.1.1.1) Theorem For $n+1$ arbitrary support points

$$(x_i, f_i), \quad i = 0, \ldots, n, \quad x_i \neq x_k \; for \; i \neq k,$$

there exists a unique polynomial $P \in \Pi_n$ with

$$P(x_i) = f_i, \quad i = 0, 1, \ldots, n.$$

Proof. Uniqueness ? Let $P_1(x_i) = P_2(x_i) = f_i$ for $i = 0, 1, \ldots, n$.
Define $P = P_1 - P_2$, then $P(x_i) = 0$ for $i = 0, 1, \ldots, n$.
Since P is a polynomial of degree n and $p(x) = c(x - x_0) \ldots (x - x_n)$, $c = 0$.

Existence ?

$$L_i(x) := \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)} : \quad \text{Lagrange polynomial}$$

$$P(x) = \sum_{i=0}^{n} f_i L_i(x) : \quad \text{Lagrange interploating polynomial}$$

Basis for $\prod_n$

$$= \{1, x, x^2, \ldots, x^n\}$$
$$= \{L_0(x), L_1(x), \ldots, L_n(x)\}$$

EXAMPLE. Given for $n = 2$:

x_i	0	1	3
f_i	1	3	2

Wanted: $P(2)$, where $P \in \Pi_2$, $P(x_i) = f_i$ for $i = 0, 1, 2$.

Solution?

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$P(x) = f_0 L_0(x) + f_1 L_1(x) + f_2 L_2(x)$$

$$P(2) = 1\frac{(2-1)(2-3)}{(0-1)(0-3)} + 3\frac{(2-0)(2-3)}{(1-0)(1-3)} + 2\frac{(2-0)(2-1)}{(3-0)(3-1)}$$

2.1.2 Neville's Algorithm

For a given set of support points $(x_i, f_i),\ i = 0, 1, \ldots, n,$ we denote by

$$P_{i_0 i_1 \ldots i_k} \in \prod_k$$

that polynomial in Π_k for which

$$P_{i_0 i_1 \ldots i_k}(x_{i_j}) = f_{i_j}, \qquad j = 0, 1, \ldots, k.$$

These polynomials are linked by the following recursion:

$$P_i(x) \equiv f_i,$$

(2.1.2.1)

$$P_{i_0 i_1 \ldots i_k}(x) \equiv \frac{(x - x_{i_0}) P_{i_1 i_2 \ldots i_k}(x) - (x - x_{i_k}) P_{i_0 i_1 \ldots i_{k-1}}(x)}{x_{i_k} - x_{i_0}}.$$

PROOF:

$$P_i(x) \equiv f_i \quad \Rightarrow \quad P_i(x_i) = f_i$$

Assume $P_{i_0 i_1 \ldots i_{k-1}}(x_{i_j}) = f(x_{i_j}), \quad 0 \le j \le k-1,$

and $P_{i_1 i_2 \ldots i_k}(x_{i_j}) = f(x_{i_j}), \quad 1 \le j \le k.$

Then

$$P_{i_0 i_1 \ldots i_k}(x_{i_0}) = f(x_{i_0}),$$
$$P_{i_0 i_1 \ldots i_k}(x_{i_k}) = f(x_{i_k}),$$
$$P_{i_0 i_1 \ldots i_k}(x_{i_j}) = f(x_{i_j}), \quad 1 \le j \le k-1.$$

	$k=0$	1	2	3
x_0	$f_0 = P_0(x)$			
		$P_{01}(x)$		
x_1	$f_1 = P_1(x)$		$P_{012}(x)$	
		$P_{12}(x)$		$P_{0123}(x)$
x_2	$f_2 = P_2(x)$		$P_{123}(x)$	
		$P_{23}(x)$		
x_3	$f_3 = P_3(x)$			

$$P_{123}(x) = \frac{(x - x_1)P_{23}(x) - (x - x_3)P_{12}(x)}{x_3 - x_1}$$

EXAMPLE. Determine $P_{012}(2)$ for the same support points as in section 2.1.1.

Comparing with Lagrange interpolation formula, what is an advantage of Neville's algorithm?

2.1.3 Newton's Interpolation Formula: Divided Differences

Write the interpolating polynomial $P \in \Pi_n,\ P(x_i) = f_i,\ i = 0, 1, \ldots n,$ in the form of

$$\begin{aligned} P(x) &\equiv P_{01\ldots n}(x) \\ &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \\ &\qquad + a_n(x - x_0)\cdots(x - x_{n-1}). \end{aligned}$$

Horner Scheme: $P(\xi) = (\cdots(a_n(\xi - x_{n-1}) + a_{n-1})(\xi - x_{n-2}) + \cdots + a_1)(\xi - x_0) + a_0.$

$$\begin{aligned} f_0 &= P(x_0) = a_0 \\ f_1 &= P(x_1) = a_0 + a_1(x_1 - x_0) \\ f_2 &= P(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\ &\cdots \end{aligned}$$

$P_{i_0 i_1 \ldots i_{k-1}}(x)$ and $P_{i_0 i_1 \ldots i_k}(x)$ differ by a polynomial of degree k with k zeros $x_{i_0}, x_{i_1}, \ldots, x_{i_{k-1}}$

$$P_{i_0 i_1 \ldots i_k}(x) = P_{i_0 i_1 \ldots i_{k-1}}(x) + f_{i_0 i_1 \ldots i_k}(x - x_{i_0})(x - x_{i_1})\cdots(x - x_{i_{k-1}})$$

From the identity $P_{i_0} = f_{i_0}$,

(2.1.3.4)
$$\begin{aligned} P_{i_0 i_1 \ldots i_k}(x) = &f_{i_0} + f_{i_0 i_1}(x - x_{i_0}) + \cdots \\ &+ f_{i_0 i_1 \ldots i_k}(x - x_{i_0})(x - x_{i_1})\cdots(x - x_{i_{k-1}}) \end{aligned}$$

From (2.1.2.1)

$$f_{i_0 i_1 \ldots i_k} = \frac{f_{i_1 \ldots i_k} - f_{i_0 \ldots i_{k-1}}}{x_{i_k} - x_{i_0}}$$: k-th divided difference

The polynomial is invariant to any permutation of the indices $i_0, i_1, \ldots, i_k$, and so is its coefficient $f_{i_0 i_1 \ldots i_k}$ of x^k.

Divided-difference scheme:

$$\begin{array}{c|cccc}
 & k=0 & 1 & 2 & \cdots \\
\hline
x_0 & f_0 & & & \\
 & & f_{01} & & \\
x_1 & f_1 & & f_{012} & \\
 & & f_{12} & \vdots & \ddots \\
x_2 & f_2 & \vdots & & \\
\vdots & \vdots & & &
\end{array}$$

$$f_{01} = \frac{f_1 - f_0}{x_1 - x_0}, \quad f_{12} = \frac{f_2 - f_1}{x_2 - x_1}, \quad \ldots,$$

$$f_{012} = \frac{f_{12} - f_{01}}{x_2 - x_0}, \quad f_{123} = \frac{f_{23} - f_{12}}{x_3 - x_1}, \quad \ldots.$$

$$\begin{aligned} P(x) &\equiv P_{01\ldots n}(x) \\ &\equiv f_0 + f_{01}(x - x_0) + \cdots + f_{01\ldots n}(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \end{aligned}$$

EXAMPLE in section 2.2.1

$$
\begin{array}{l|lll}
x_0 = 0 & f_0 = 1 & & \\
 & & f_{01} = 2 & \\
x_1 = 1 & f_1 = 3 & & f_{012} = -\frac{5}{6} \\
 & & f_{12} = -\frac{1}{2} & \\
x_2 = 3 & f_2 = 2 & &
\end{array}
$$

Consider multivariate functions:

$$
\begin{aligned}
f[x_0] &= f(x_0) \\
f[x_0, x_1] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\
&= \frac{f(x_0)(x_1 - x_2) + f(x_1)(x_2 - x_0) + f(x_2)(x_0 - x_1)}{(x_1 - x_0)(x_2 - x_1)(x_0 - x_2)}
\end{aligned}
$$

(2.1.3.9) Theorem. The divided difference $f[x_{i_0}, \ldots, x_{i_n}]$ is a symmetric function of their arguments.

(2.1.3.10) Theorem. If $f(x)$ is a polynomial of degree N, then

$$f[x_0, \ldots, x_k] = 0 \quad \text{for} \quad k > N.$$

PROOF.

EXAMPLE. $f(x) = x^2$

x_i	$k=0$	1	2	3	4
0	0				
		1			
1	1		1		
		3		0	
2	4		1		0
		5		0	
3	9		1		
		7			
4	16				

If $f(x)$ has a derivative at x_0, then it makes sense for certain purposes to define

$$f[x_0, x_0] := f'(x_0).$$

2.1.4 The Error in Polynomial Interpolation

Once again we consider a given function $f(x)$ and certain of its values

$$f_i = f(x_i), \qquad i = 0, 1, \dots, n.$$

Let $P(x) \equiv P_{0\dots n}(x) \in \Pi_n$ be the interpolating polynomial with

$$P(x_i) = f_i, \qquad i = 0, 1, \dots, n.$$

(2.1.4.1) **Theorem**. *If the function f has an $(n+1)$ st derivative, then for every argument $\bar{x}$, there exists a number ξ in the smallest interval $I[x_0, \dots, x_n, \bar{x}]$, which contains $\bar{x}$ and all support abscissas x_i, satisfying*

$$f(\bar{x}) - P_{01\dots n}(\bar{x}) = \frac{\omega(\bar{x}) f^{(n+1)}(\xi)}{(n+1)!},$$

where

$$\omega(x) \equiv (x - x_0)(x - x_1) \cdots (x - x_n).$$

PROOF.

Let $Q(x) = f(x) - P(x) - K\omega(x)$.
Then $Q(x_i) = 0, i = 0, \dots, n$.
Let $Q(\bar{x}) = 0$.
Then Q has $(n+2)$ zeros.

$\Longrightarrow$

By Rolle's theorem, Q' has $(n+1)$ zeros.

$\vdots$

$Q^{(n+1)}$ has 1 zero ($x = \xi$).

Hence $K = \frac{f^{(n+1)}(\xi)}{(n+1)!}$

A different error term can be derived from Newton's interpolation formula (2.1.3.4):

$$P(x) \equiv P_{01\ldots n}(x) \equiv f[x_0] + f[x_0, x_1](x - x_0) + \cdots$$
$$+ f[x_0, x_1, \ldots, x_n](x - x_0) \ldots (x - x_{n-1}).$$

to interpolate

$$(x_i, f_i): \quad f_i = f(x_i), \quad i = 0, 1, \ldots, n,$$

we introduce an $(n+2)$nd support point

$$(x_{n+1}, f_{n+1}): \quad x_{n+1} := \bar{x}, \quad f_{n+1} := f(\bar{x}), \quad \bar{x} \neq x_i, \quad i = 0, \ldots, n,$$

then by Newton's formula

(2.1.4.2) $$f(\bar{x}) = P_{0\ldots n+1}(\bar{x}) = P_{0\ldots n}(\bar{x}) + f[x_0, \ldots, x_n, \bar{x}]\omega(\bar{x}).$$

The difference on the left-hand side appears in Theorem (2.1.4.1), and since $\omega(\bar{x}) \neq 0$, we must have

$$f[x_0, \ldots, x_n, \bar{x}] = \frac{f^{(n+1)}(\xi)}{(n+1)!} \text{ for some } \xi \in I[x_0, \ldots, x_n, \bar{x}].$$

Also $$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!} \text{ for some } \xi \in I[x_0, \ldots, x_n]$$

EXAMPLE. $f(x) = \sin x$:

$$x_i = \frac{\pi}{10} \cdot i, \quad i = 0, 1, 2, 3, 4, 5, \quad n = 5,$$

$$\sin x - P(x) = (x - x_0)(x - x_1) \dots (x - x_5) \frac{-\sin \xi}{720}, \quad \xi = \xi(x),$$

$$\left|\sin x - P(x)\right| \leq \frac{1}{720}\left|(x - x_0)(x - x_1) \dots (x - x_5)\right| = \frac{|\omega(x)|}{720}.$$

In the exterior of the interval $I[x_0, \dots, x_n]$,the value of $|\omega(x)|$ in Theorem (2.1.4.1) grows very fast. The use of the interpolation polynomial P for approximating f at some location outside the interval $I[x_0, \dots, x_n]$ -- called *extrapolation* -- should be avoided if possible.

Within $I[x_0, \dots, x_n]$, it should not be assumed that finer and finer samplings of the function f will lead to better and better approximations through interpolation.

Consider a real function $f \in C^\infty[a, b]$. To every interval partition $\Delta = \{a = x_0 < x_1 < \dots < x_n = b\}$, there exists an interpolating polynomial $P_\Delta \in \Pi_n$ with $P_\Delta(x_i) = f_i$ for $x_i \in \Delta$. A sequence of interval partitions

$$\Delta_m = \left\{a = x_0^{(m)} < x_1^{(m)} < \dots < x_{n_m}^{(m)} = b\right\}$$

gives rise to a sequence of interpolating polynomials P_{Δ_m}.

One might expect the polynomials P_{Δ_m} to converge toward f if the fineness

$$\|\Delta_m\| := \max_i \left|x_{i+1}^{(m)} - x_i^{(m)}\right|$$

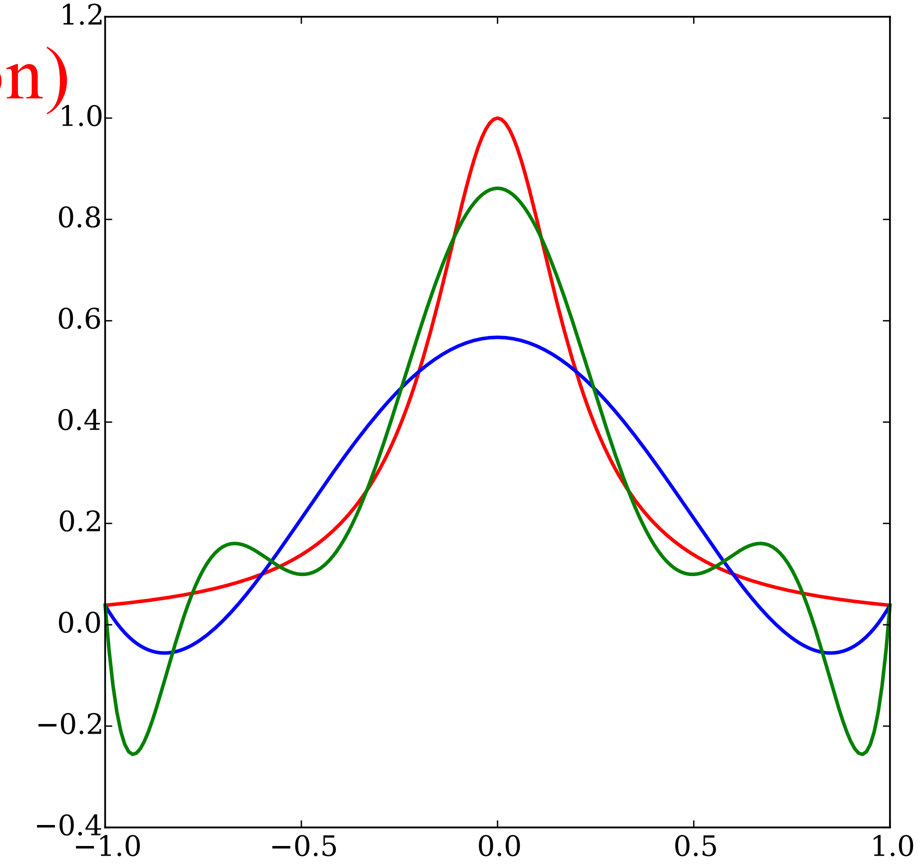
of the partitions tends to 0 as $m \to \infty$. In general this is not true. For example, it has been shown for the functions

$$f(x) := \frac{1}{1+25x^2}, \quad [a,b] = [-1,1]$$ (Runge function)

$$f(x) := \sqrt{x}, \quad [a,b] = [0,1]$$

that the polynomials P_{Δ_m} do not converge pointwise to f for arbitrarily fine uniform partitions Δ_m,

$$x_i^{(m)} = a + \frac{i(b-a)}{m}, i = 0, \dots, m.$$



The red curve is the Runge function. The blue curve is a 5th-order interpolating polynomial (using six equally spaced interpolating points). The green curve is a 9th-order interpolating polynomial (using ten equally spaced interpolating points).

2.1.5 Hermite-Interpolation

Consider the real numbers $x_i, f_i^{(k)}$, $i = 0,1,\dots,m$, $k = 0,1,\dots,n_i - 1$, with

$$x_0 < x_1 < \cdots < x_m.$$

The *Hermite interpolation problem* for these data consists of determining a polynomial P whose degree does not exceed n, $P \in \Pi_n$, where

$$n+1 := \sum_{i=0}^{m} n_i ,$$

and which satisfies the following interpolation conditions:

Example:
$x_0 : f(x_0), f'(x_0)$
$x_1 : f(x_1), f'(x_1), f''(x_1)$
$x_2 : f(x_2)$

(2.1.5.1) $$P^{(k)}(x_i) = f_i^{(k)}, \quad i = 0, 1, \dots, m \quad k = 0, 1, \dots, n_i - 1.$$

(2.1.5.2) **Theorem**. *For arbitrary numbers* $x_0 < x_1 < \cdots < x_m$ *and* $f_i^{(k)}$, $i = 0,1,\dots,m$, $k = 0,1,\dots,n_i - 1$, *there exists precisely one polynomial*

$$P \in \Pi_n, \quad n+1 := \sum_{i=0}^{m} n_i ,$$

which satisfies (2.1.5.1).

PROOF. Uniqueness?

Existence?

Uniqueness implies existence

Hermite interpolating polynomials can be given explicitly in a form analogous to the interpolation formula of Lagrange. The polynomial $P \in \Pi_n$ given by

$$P(x) = \sum_{i=0}^{m} \sum_{k=0}^{n_i - 1} f_i^{(k)} L_{ik}(x) \tag{2.1.5.3}$$

satisfies (2.1.5.1). The polynomials $L_{ik} \in \Pi_n$ are *generalized Lagrange polynomials.* They are defined as follows: Starting with the auxiliary polynomials

$$l_{ik}(x) := \frac{(x - x_i)^k}{k!} \prod_{\substack{j=0 \\ j \neq i}}^{m} \left(\frac{x - x_j}{x_i - x_j} \right)^{n_j}, \quad 0 \le i \le m, \quad 0 \le k \le n_i,$$

put

$$L_{i0}(x) = l_{i0}(x) + \sum_{k=1}^{n_i-1} c_{0k} l_{ik}(x),$$

$$L_{i1}(x) = \sum_{k=1}^{n_i-1} c_{1k} l_{ik}(x),$$

$$L_{i2}(x) = \sum_{k=2}^{n_i-1} c_{2k} l_{ik}(x),$$

$$\vdots$$

$$L_{i,n_i-1}(x) = c_{n_i-1,n_i-1} l_{i,n_i-1}(x)$$

$$L_{ik}^{(\sigma)}(x_j) = \begin{cases} 1, & \text{if } i = j \text{ and } k = \sigma, \\ 0, & \text{otherwise.} \end{cases}$$ implies

$$L_{i,n_i-1}(x) := l_{i,n_i-1}(x), \quad i = 0, 1, \ldots, m.$$

For $k = n_i - 2, n_i - 3, \ldots, 0$, $$L_{ik}(x) := l_{ik}(x) - \sum_{\nu=k+1}^{n_i-1} l_{ik}^{(\nu)}(x_i) L_{i\nu}(x).$$

Thus P in (2.1.5.3) is indeed the desired Hermite interpolating polynomial.

An alternative way to describe Hermite interpolation is to generalize divided differences $f[x_0, x_1, \dots, x_k]$ so as to accommodate repeated abscissae. To this end, we expand the sequence of abscissae $x_0 < x_1 < \cdots < x_m$ by replacing each x_i by n_i copies of itself:

$$\underbrace{x_0 = \cdots = x_0}_{n_0} < \underbrace{x_1 = \cdots = x_1}_{n_1} < \cdots < \underbrace{x_m = \cdots = x_m}_{n_m}.$$

The $n + 1 = \sum_{i=0}^{m} n_i$ elements in this sequence are then renamed

$$t_0 = x_0 \le t_1 \le \cdots \le t_n = x_m,$$

and will be referred to as *virtual abscissae.*

We now reformulate the Hermite interpolation problem (2.1.5.1) in terms of the virtual abscissae. In order to stress the dependence of the Hermite interpolant $P(.)$ on $t_0, t_1, \dots, t_n$ we write $P_{01\dots n}(.)$ for $P(.)$. Also it will be convenient to write $f^{(r)}(x_i)$ instead of $f_i^{(r)}$ in (2.1.5.1). An essential observation is that the interpolation conditions in (2.1.5.1) belonging to the following linear ordering of the index pairs (i, k),

$$(0,0),\ (0,1),\ \dots,\ (0, n_0 - 1),\ (1,0),\ \dots,\ (1, n_1 - 1),\ \dots,\ (m, n_m - 1),$$

have the form

$$P_{01\dots n}^{(s_j - 1)}(t_j) = f^{(s_j - 1)}(t_j), \qquad j = 0, 1, \dots, n, \tag{2.1.5.4}$$

if we define $s_j, j=0,1,\ldots,n$, *as* the number of times the value of t_j occurs in the subsequence

$$t_0 \le t_1 \le \cdots \le t_j.$$

The equivalence of (2.1.5.1) and (2.1.5.4) follows directly from

$$x_0 = t_0 = t_1 = \cdots = t_{n_0-1} < x_1 = t_{n_0} = \cdots = t_{n_0+n_1-1} < \cdots, \ .$$

We now define the *generalized divided difference*

$$f[t_i, t_{i+1}, \ldots, t_{i+k}]$$

as the coefficient of x^k in the polynomial $P_{i,i+1,\ldots,i+k}(x) \in \Pi_k$ By comparing the coefficients of x^k, we find, if, $t_i = \ldots = t_{i+k} = x_l$,

(2.1.5.7a) $$f[t_i, t_{i+1}, \ldots, t_{i+k}] = \frac{1}{k!} f_l^{(k)},$$

and if $t_i < t_{i+k}$

(2.1.5.7b) $$f[t_i, t_{i+1}, \ldots, t_{i+k}] = \frac{f[t_{i+1}, \ldots, t_{i+k}] - f[t_i, t_{i+1}, \ldots, t_{i+k-1}]}{t_{i+k} - t_i}.$$

By means of the generalized divided differences

$$a_k := f[t_0, t_1, \ldots, t_k], \qquad k = 0, 1, \ldots, n,$$

the solution $P(x) = P_{01\ldots n}(x) \in \Pi_n$ of the Hermite interpolation problem (2.1.5.1) can be represented explicitly in its Newton form

(2.1.5.8)
$$\begin{aligned} P_{01\cdots n}(x) &= a_0 + a_1(x - t_0) + a_2(x - t_0)(x - t_1) + \cdots \\ &\quad + a_n(x - t_0)(x - t_1)\cdots(x - t_{n-1}). \end{aligned}$$

Why?

EXAMPLE 3. We illustrate the calculation of the generalized divided differences with the data of Example 2 $(m = 1, n_0 = 2, n_1 = 3)$. Find the Hermite interpolating polynomial.

$x_0 = 0 : f_0^{(0)} = -1, f_0^{(1)} = -2,$
$x_1 = 1 : f_1^{(0)} = 0, f_1^{(1)} = 10, f_1^{(2)} = 40$

(2.1.5.9) **Theorem**. *Let the real function* f *be* $(n+1)$ *times differentiable* on *the interval* $[a,b]$, and *consider* $(m+1)$ *support abscissae* $x_i \in [a,b]$,

$$x_0 < x_1 < \cdots < x_m.$$

If the polynomial $P(x)$ *is of degree at most* n,

$$\sum_{i=0}^{m} n_i = n+1,$$

and satisfies the interpolation conditions

$$P^{(k)}(x_i) = f^{(k)}(x_i), \qquad i = 0, 1, \ldots, m, \quad k = 0, 1, \ldots, n_i - 1,$$

then for every $\bar{x} \in [a,b]$ *there exists* $\bar{\xi} \in I[x_0, \ldots, x_m, \bar{x}]$ *such that*

$$f(\bar{x}) - P(\bar{x}) = \frac{\omega(\bar{x}) f^{(n+1)}(\bar{\xi})}{(n+1)!},$$

where

$$\omega(x) := (x - x_0)^{n_0}(x - x_1)^{n_1} \ldots (x - x_m)^{n_m}.$$

Hermite interpolation is frequently used to approximate a given real function f by a piecewise polynomial function φ. Given a partition

$$\Delta : a = x_0 < x_1 < \cdots < x_m = b$$

of an interval $[a, b]$, the corresponding *Hermite function space* $H_\Delta^{(\nu)}$ is defined as consisting of all functions $\varphi: [a, b] \to \mathbb{R}$ with the following properties:

(2.1.5.10)

(a) $\varphi \in C^{\nu-1}[a,b]$: *The ($\nu$-1)st derivative of φ exists and is continuous on* $[a,b]$.

(b) $\varphi|_{I_i} \in \prod_{2\nu-1}$: *On each subinterval* $I_i := [x_i, x_{i+1}], i = 0,1, \dots, m-1$, *$\varphi$ agrees with a polynomial of degree at most* $2\nu - 1$.

Thus the function φ consists of polynomial pieces of degree $2\nu - 1$ or less which are $\nu - 1$ times differentiable at the “knots” x_i. In order to approximate a given real function $f \in C^{\nu-1}[a, b]$ by a function $\varphi \in H_\Delta^{(\nu)}$, we choose the component polynomials $P_i = \varphi|_{I_i}$ of φ so that $P_i \in \Pi_{2\nu-1}$ and so that the Hermite interpolation conditions

$$P_i^{(k)}(x_i) = f^{(k)}(x_i), \quad P_i^{(k)}(x_{i+1}) = f^{(k)}(x_{i+1}), \quad k = 0, 1, \dots, \nu - 1,$$

are satisfied.