Integrals and Series

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Chapter 1

Gamma and Beta integrals

1.1 Beta Function

The Beta Function is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

for Re $x > 0$, Re $y > 0$. It is related to the Gamma function by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

1.1.1 Integral 1

Prove that

$$\int_0^1 x^{a-1}(1-x)^{b-1}\frac{dx}{(x+p)^{a+b}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\frac{1}{(1+p)^a p^b}$$

where $a,\ b,\ p > 0$.

Proof. Perform the change of variables $\dfrac{x}{p+x} = \dfrac{t}{p+1}$. Then

$$\frac{p}{(p+x)^2} = \frac{1}{p+1}\frac{dt}{dx}$$
$$\Rightarrow \frac{dx}{(p+x)^2} = \frac{dt}{p(p+1)}$$

After making the substitutions, the integral transforms into:

$$\begin{aligned}\int_0^1 x^{a-1}(1-x)^{b-1}\frac{dx}{(x+p)^{a+b}} &= \frac{1}{p^b(1+p)^a}\int_0^1 t^{a-1}(1-t)^{b-1}dt \\ &= \frac{1}{p^b(1+p)^a}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\end{aligned}$$

□

1.1.2 Integral 2

$$\int_0^1 \frac{1}{(2-x)\sqrt[5]{x^2(1-x)^3}}dx = \frac{2\pi\sqrt[10]{2}}{\sqrt{5+\sqrt{5}}}$$

Proof. Perform the change of variables $t = 1 - x$. Then,

$$\int_0^1 \frac{1}{(2-x)\sqrt[5]{x^2(1-x)^3}}dx = \int_0^1 \frac{1}{(1+t)\sqrt[5]{(1-t)^2 t^3}}dt$$

We now have a special case of Integral 1 with $a = \frac{2}{5}$, $b = \frac{3}{5}$ and $p = 1$. Therefore,

$$\begin{aligned}\int_0^1 \frac{1}{(2-x)\sqrt[5]{x^2(1-x)^3}}dx &= \frac{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)}{2^{\frac{2}{5}}} \\ &= \frac{\pi}{2^{\frac{2}{5}}\sin\left(\frac{2\pi}{5}\right)} \\ &= \frac{2\pi\sqrt[10]{2}}{\sqrt{5+\sqrt{5}}}\end{aligned}$$

□

Chapter 2

Residue theorem based evaluation of real integrals

2.1 Rectangular Contours

2.1.1 Integral 1

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+e^{2a}} \frac{1}{x^2+\pi^2} dx = \frac{2\pi e^{-a}}{4a^2+\pi^2} - \frac{1}{1+e^{2a}}$$

where $a \in \mathbb{R}$.

Proof. Let $f(z) = \dfrac{e^z}{(e^{2z}+e^{2a})z}$. Let Γ be the positively oriented rectangle in the complex plane with vertices $R-i\pi$, $R+i\pi$, $-R+i\pi$ and $-R-i\pi$ where $R > |a|$. There are three first order poles of $f(z)$ lying inside Γ at $z=0$, $\frac{i\pi}{2}+a$ and $-\frac{i\pi}{2}+a$. Then, by the residue theorem, we have

$$\begin{aligned}
\int_\Gamma f(z)\, dz &= 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=\frac{i\pi}{2}+a} f(z) + \operatorname*{Res}_{z=-\frac{i\pi}{2}+a} f(z) \right) \\
&= 2\pi i \left(\frac{1}{1+e^{2a}} - \frac{ie^{-a}}{2a+i\pi} + \frac{ie^{-a}}{2a-i\pi} \right) \\
&= 2\pi i \left(\frac{1}{1+e^{2a}} - \frac{2\pi e^{-a}}{4a^2+\pi^2} \right) \qquad (1)
\end{aligned}$$

By change of variables,

$$\begin{aligned}
\int_\Gamma f(z)\, dz &= \int_{-R-i\pi}^{R-i\pi} f(z)\, dz + \int_{R-i\pi}^{R+i\pi} f(z)\, dz + \int_{R+i\pi}^{-R+i\pi} f(z)\, dz + \int_{-R+i\pi}^{-R-i\pi} f(z)\, dz \\
&= \int_{-R}^{R} (f(x-i\pi) - f(x+i\pi))\, dx + \int_{R-i\pi}^{R+i\pi} f(z)\, dz + \int_{-R+i\pi}^{-R-i\pi} f(z)\, dz \\
&= -2\pi i \int_{-R}^{R} \frac{e^x}{e^{2x}+e^{2a}} \frac{1}{x^2+\pi^2} dx + \int_{R-i\pi}^{R+i\pi} f(z)\, dz + \int_{-R+i\pi}^{-R-i\pi} f(z)\, dz \qquad (2)
\end{aligned}$$

Note that

$$\begin{aligned}\left|\int_{R-i\pi}^{R+i\pi} f(z)\, dz\right| &= \left|\int_{-\pi}^{\pi} f(R+iy)\, dy\right| \\ &\leq e^R \int_{-\pi}^{\pi} \frac{dy}{|iy+R|\cdot|e^{2iy+2R}+e^{2a}|} dy \\ &\leq \frac{e^{-R}}{R}\int_{-\pi}^{\pi} \frac{dy}{|e^{2iy}+e^{2a-2R}|} \\ &\leq \frac{e^{-R}}{R}\frac{2\pi}{|1-e^{2a-2R}|} \end{aligned} \tag{3}$$

Similarly,

$$\left|\int_{-R+i\pi}^{-R-i\pi} f(z)\, dz\right| \leq \frac{e^{-R}}{R}\frac{2\pi}{|e^{2a}-e^{-2R}|} \tag{4}$$

From equations (3) and (4), we see that the vertical integrals tend to 0 as $R \to \infty$. Therefore, by equations (1) and (2):

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+e^{2a}}\frac{1}{x^2+\pi^2} dx = \frac{2\pi e^{-a}}{4a^2+\pi^2} - \frac{1}{1+e^{2a}}$$

□

2.1.2 Integral 2

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{\cosh(\pi x)} dx = \frac{1}{\cosh\left(\frac{a}{2}\right)}$$

where $a > 0$.

Proof. For $k \in \{1, 2, \cdots\}$, define B_k as the positively oriented rectangle with vertices $\pm k$ and $\pm k + i\pi k$. Then, using the residue theorem:

$$\begin{aligned}\lim_{k\to\infty}\int_{B_k} \frac{e^{iaz}}{\cosh(\pi z)} dz &= 2\pi i \sum_{k=0}^{\infty} \operatorname*{Res}_{z=\frac{(2k+1)i}{2}} \frac{e^{iaz}}{\cosh(\pi z)} \\ &= 2\pi i \left(\frac{e^{-\frac{a}{2}}}{\pi i}\sum_{k=0}^{\infty}(-1)^k e^{-ak}\right) \\ &= \frac{1}{\cosh\left(\frac{a}{2}\right)}\end{aligned}$$

Since only the integral over the bottom side of the rectangle survives under the limit $k \to \infty$, we have

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{\cosh(\pi x)} dx = \lim_{k\to\infty}\int_{B_k} \frac{e^{iaz}}{\cosh(\pi z)} dz = \frac{1}{\cosh\left(\frac{a}{2}\right)}$$

□

2.2 Circular Contours

2.2.1 Integrals 1, 2

$$\int_0^\infty x^{s-1}\sin(x)\,dx = \begin{cases} \sin\left(\frac{\pi s}{2}\right)\Gamma(s) & s \in (-1,0)\cup(0,1) \\ \frac{\pi}{2} & s = 0 \end{cases}$$
$$\int_0^\infty x^{s-1}\cos(x)\,dx = \cos\left(\frac{\pi s}{2}\right)\Gamma(s) \quad 0 < s < 1$$

Proof. Let $f(z) = z^{s-1}e^{iz}$ where $0 < s < 1$. The branch of the logarithm is chosen as $-\pi < \arg z \leq \pi$. The idea is to integrate $f(z)$ around the contour shown in Fig. 2.1 and let $R \to \infty$ and $\epsilon \to 0^+$. C_R and C_ϵ are quarter circles centered at $z = 0$ and having radiuses R and ϵ, respectively. C_ϵ is used avoid the branch point at $z = 0$. Using Cauchy's theorem, we have

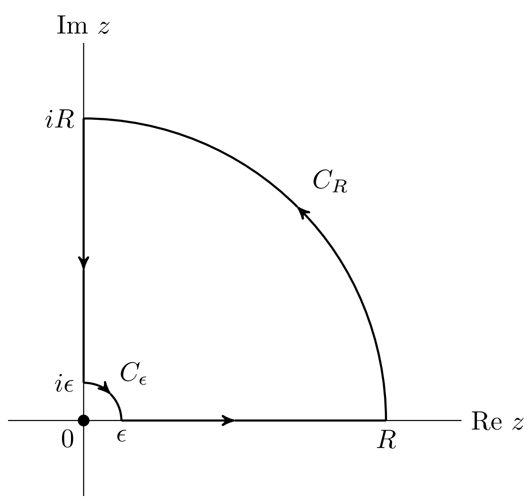


Figure 2.1: Contour for section 2.2.1

$$\int_\epsilon^R f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{iR}^{i\epsilon} f(z)\,dz + \int_{C_\epsilon} f(z)\,dz = 0 \tag{1}$$

Note that

$$\int_{iR}^{i\epsilon} f(z)\,dz = -i\int_\epsilon^R f(iz)\,dz \tag{2}$$

Using Jordan's Lemma,

$$
\begin{aligned}
\left|\int_{C_R} f(z)\, dz\right| &= R^s \left|\int_0^{\frac{\pi}{2}} e^{iRe^{i\theta}} e^{i\theta s}\, d\theta\right| \\
&\leq R^s \int_0^{\frac{\pi}{2}} \left|e^{iRe^{i\theta}}\right|\, d\theta \\
&\leq R^s \int_0^{\frac{\pi}{2}} e^{-R\frac{2\theta}{\pi}}\, d\theta \\
&= \frac{\pi}{2} R^{s-1}\left(1-e^{-R}\right)
\end{aligned}
$$

Therefore,

$$
\lim_{R\to\infty}\left|\int_{C_R} f(z)\, dz\right| = 0 \tag{3}
$$

Similarly, we have

$$
\left|\int_{C_\epsilon} f(z)\, dz\right| \leq \frac{\pi}{2}\epsilon^{s-1}\left(1-e^{-\epsilon}\right) \to 0 \quad \text{as} \quad \epsilon \to 0^+ \tag{4}
$$

Using equations (1), (2), (3) and (4), we can write

$$
\begin{aligned}
\int_0^\infty x^{s-1} e^{ix}\, dx &= i^s \int_0^\infty x^{s-1} e^{-x}\, dx \\
&= \left(\cos\left(\frac{\pi s}{2}\right) + i\sin\left(\frac{\pi s}{2}\right)\right)\Gamma(s)
\end{aligned}
$$

Separating the real and imaginary parts yields,

$$
\int_0^\infty x^{s-1}\sin(x)\, dx = \sin\left(\frac{\pi s}{2}\right)\Gamma(s) \tag{5}
$$

$$
\int_0^\infty x^{s-1}\cos(x)\, dx = \cos\left(\frac{\pi s}{2}\right)\Gamma(s) \tag{6}
$$

where $0 < s < 1$. Now, let us define $I : (-1, 1) \to \mathbb{R}$ as $I(s) = \int_0^\infty x^{s-1}\sin(x)\, dx$. We have proven that $I(s) = \sin\left(\frac{\pi s}{2}\right)\Gamma(s)$ whenever $0 < s < 1$. Consider the case when $-1 < s < 0$. Applying integration by parts and using equation (6) gives

$$
\begin{aligned}
I(s) &= \left[\frac{x^s\sin(x)}{s}\right]_0^\infty - \frac{1}{s}\int_0^\infty x^s\cos(x)\, dx \\
&= -\frac{1}{s}\int_0^\infty x^s\cos(x)\, dx \\
&= -\frac{1}{s}\cos\left(\frac{\pi}{2}(s+1)\right)\Gamma(s+1) \\
&= \sin\left(\frac{\pi s}{2}\right)\Gamma(s)
\end{aligned}
$$

The case $s = 0$ corresponds to the famous Dirichlet integral whose proof is well known.

$$I(0) = \int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$

Also, $I(s)$ is continuous at $s = 0$ since

$$\lim_{s\to 0} \left[\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\right] = \frac{\pi}{2}$$

□

2.2.2 Integral 3

For $n \in \mathbb{N}$,

$$\int_0^{\frac{\pi}{2}} \frac{\cos\left((2n-1)\arcsin\left(\frac{\sin x}{\sqrt{2}}\right)\right)}{\sqrt{1-\frac{\sin^2 x}{2}}} dx = \frac{\sqrt{\pi}}{2}\sin\left(\frac{n\pi}{2}\right)\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$

Proof. Let I denote the integral. We will start off with the substitution $\sin t = \frac{\sin x}{\sqrt{2}}$. This transforms the integral into:

$$I = \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{\cos((2n-1)t)}{\sqrt{1-2\sin^2 t}} dt = \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{\cos((2n-1)t)}{\sqrt{\cos(2t)}} dt$$

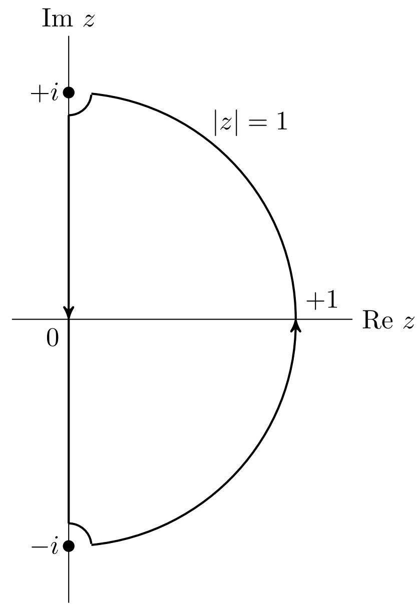
Furthermore, substituting $\theta = 2t$ and using the symmetry of the integral, we have

$$\begin{aligned} I &= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{2n-1}{2}\theta\right)}{\sqrt{\cos\theta}} d\theta \\ &= \frac{1}{2\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\left(\frac{2n-1}{2}\theta\right)}{\sqrt{\cos\theta}} d\theta \\ &= \frac{1}{2\sqrt{2}}\mathrm{Re}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\exp\left(\frac{2n-1}{2}i\theta\right)}{\sqrt{\cos\theta}} d\theta \qquad (1) \end{aligned}$$

Now integrate the principal branch of $f(z) = \frac{z^{n-1}}{\sqrt{1+z^2}}$ around the following contour:

The contour has circular indents of radius ϵ around the branch points $+i$ and $-i$. It is easily seen that integrals along these indents tend to 0 as $\epsilon \to 0^+$. Then, using Cauchy's Theorem, we have

$$\begin{aligned} \int_{\left\{z\in\mathbb{C}:\ |z|=1,\ -\frac{\pi}{2}<\arg z<\frac{\pi}{2}\right\}} \frac{z^{n-1}}{\sqrt{1+z^2}} dz &= \int_{-i}^{i} \frac{y^{n-1}}{\sqrt{1+y^2}} dy \\ &= i^n \int_{-1}^{1} \frac{y^{n-1}}{\sqrt{1-y^2}} dy \\ &= \begin{cases} 2i^n \int_0^1 \frac{y^{n-1}}{\sqrt{1-y^2}} dy & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \qquad (2) \end{aligned}$$



On the other hand, note that

$$\begin{aligned}
\int_{\left\{z\in\mathbb{C}\colon\, |z|=1,\, -\frac{\pi}{2}<\arg z<\frac{\pi}{2}\right\}} \frac{z^{n-1}}{\sqrt{1+z^2}}dz &= i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{in\theta}}{\sqrt{1+e^{2i\theta}}}d\theta \\
&= i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{\frac{2n-1}{2}i\theta}}{\sqrt{e^{i\theta}+e^{-i\theta}}}d\theta \\
&= \frac{i}{\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{\frac{2n-1}{2}i\theta}}{\sqrt{\cos\theta}}d\theta \qquad (3)
\end{aligned}$$

Using equations (1), (2) and (3), we get

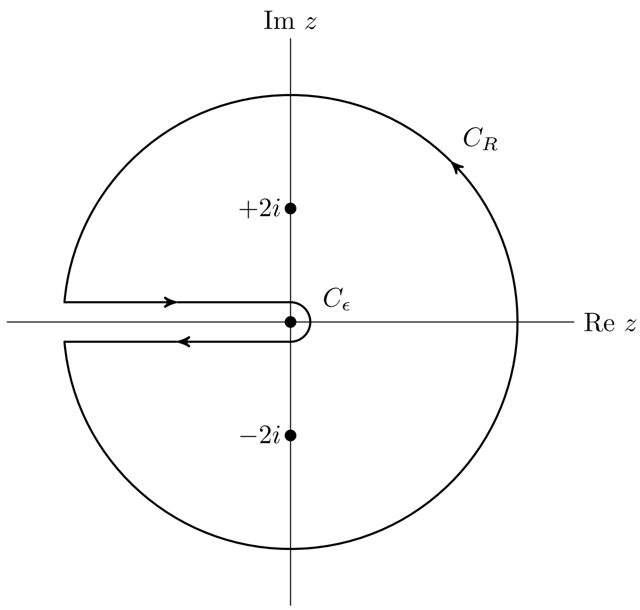
$$\begin{aligned}
I &= \sin\left(\frac{n\pi}{2}\right)\int_0^1 \frac{y^{n-1}}{\sqrt{1-y^2}}dy \\
&= \frac{1}{2}\sin\left(\frac{n\pi}{2}\right)\int_0^1 \frac{\xi^{\frac{n}{2}-1}}{\sqrt{1-\xi}}d\xi \quad (\xi = y^2) \\
&= \frac{\sqrt{\pi}}{2}\sin\left(\frac{n\pi}{2}\right)\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}
\end{aligned}$$

$\square$

2.2.3 Integral 4

$$\int_0^\infty \frac{\sin(\ln x)}{x^2+4}dx = \frac{\pi \sin(\ln 2)}{4\cosh\left(\frac{\pi}{2}\right)}$$

Proof. Consider the function $f(z) = \frac{e^{i\log z}}{4+z^2}$ where the branch of the logarithm corresponds to $-\pi < \arg z \leq \pi$. We will integrate $f(z)$ around the following "key-hole" contour:



C_R is a circle of radius R and C_ϵ is a half-circle of radius ϵ. Both of them are centered at 0. As $R \to \infty$ and $\epsilon \to 0^+$, the integrals around C_R and C_ϵ tend to 0. So, we are only left with the integrals above and below the branch cut.

Let's calculate the residues at the poles $\pm 2i$. In doing so, one must be careful about the branch of the logarithm.

$$\begin{aligned}
\operatorname*{Res}_{z=2i} f(z) &= \lim_{z\to 2i}(z-2i)\frac{e^{i\log z}}{z^2+4} \\
&= \frac{e^{i\log(2i)}}{4i} \\
&= \frac{e^{i\ln 2 - \arg(i)}}{4i} \\
&= \frac{e^{i\ln 2 - \frac{\pi}{2}}}{4i}
\end{aligned}$$

Similarly,

$$\operatorname*{Res}_{z=-2i} f(z) = -\frac{e^{i\ln 2 - \arg(-i)}}{4i} = -\frac{e^{i\ln 2 + \frac{\pi}{2}}}{4i}$$

Therefore, using the Residue Theorem,

$$
\begin{aligned}
\int_{-\infty}^{0} \frac{e^{i(\ln|x|+i\pi)}}{4+x^2}dx + \int_{0}^{-\infty} \frac{e^{i(\ln|x|-i\pi)}}{4+x^2}dx &= 2\pi i \left(\operatorname*{Res}_{z=2i} f(z) + \operatorname*{Res}_{z=-2i} f(z) \right) \\
\Rightarrow e^{-\pi} \int_{0}^{\infty} \frac{e^{i\ln x}}{4+x^2}dx - e^{\pi} \int_{0}^{\infty} \frac{e^{i\ln x}}{4+x^2}dx &= 2\pi i \left(\frac{e^{i\ln 2}}{4i} e^{-\frac{\pi}{2}} - \frac{e^{i\ln 2}}{4i} e^{\frac{\pi}{2}} \right) \\
\Rightarrow -2\sinh(\pi) \int_{0}^{\infty} \frac{e^{i\ln x}}{4+x^2}dx &= -\sinh\left(\frac{\pi}{2}\right) \pi e^{i\ln 2} \\
\Rightarrow \int_{0}^{\infty} \frac{e^{i\ln x}}{4+x^2}dx &= \frac{\pi e^{i\ln 2}}{4\cosh\left(\frac{\pi}{2}\right)}
\end{aligned}
$$

Now, separate the imaginary parts to get the answer. □

Chapter 3

Differentiation under the integral sign

3.0.1 Integral 1

$$\int_0^\infty \frac{\sin\left(x^2\right)\ln(x)}{x}\,dx = -\frac{\gamma\pi}{8}$$

where γ is the Euler's constant.

Proof. The substitution $t = x^2$ transforms the integral into

$$\int_0^\infty \frac{\sin\left(x^2\right)\ln(x)}{x}dx = \frac{1}{4}\int_0^\infty \frac{\sin(t)\ln(t)}{t}dt \tag{1}$$

Using the results of section 2.2.1, we can write

$$\begin{aligned}
\int_0^\infty \frac{\sin(t)\ln(t)}{t}dt &= \lim_{s\to 0}\frac{d}{ds}\int_0^\infty x^{s-1}\sin(x)\,dx \\
&= \lim_{s\to 0}\frac{d}{ds}\left[\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\right] \\
&= \lim_{s\to 0}\left[\frac{\pi}{2}\cos\left(\frac{\pi s}{2}\right)\Gamma(s) + \sin\left(\frac{\pi s}{2}\right)\Gamma'(s)\right] \\
&= \lim_{s\to 0}\left[\frac{\pi}{2}\cos\left(\frac{\pi s}{2}\right)\left(\frac{1}{s} - \gamma + O(s)\right) + \sin\left(\frac{\pi s}{2}\right)\left(-\frac{1}{s^2} + \frac{6\gamma^2+\pi^2}{12} + O(s)\right)\right] \\
&= \lim_{s\to 0}\left[-\frac{\pi\gamma}{2} + \left(\frac{\pi}{2}\cos\left(\frac{\pi s}{2}\right)\frac{1}{s} - \sin\left(\frac{\pi s}{2}\right)\frac{1}{s^2}\right) + O(s)\right] \\
&= -\frac{\pi\gamma}{2}
\end{aligned} \tag{2}$$

Plugging this into equation (1) gives the desired result. $\square$

3.0.2 Integral 2

$$\int_0^{\frac{\pi}{2}} \frac{\arctan^2(\sin^2\theta)}{\sin^2\theta}\, d\theta = \pi \log\left(\frac{2+\sqrt{2}}{4}\right)\sqrt{\frac{\sqrt{2}+1}{2}} + \frac{\pi^2}{4}\sqrt{\frac{\sqrt{2}-1}{2}}$$

Proof. Let I denote the integral. Then, using integration by parts we can write

$$I = \int_0^{\frac{\pi}{2}} \frac{\left(\arctan(\sin^2 x)\right)^2}{\sin^2 x} dx = 4\int_0^{\frac{\pi}{2}} \frac{\cos^2 x \arctan(\sin^2 x)}{1+\sin^4 x} dx$$

The main idea of this evaluation is to use differentiation under the integral sign. Let us introduce the parameter α:

$$f(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x \arctan(\alpha \sin^2 x)}{1+\sin^4 x} dx$$

Taking derivative inside the integral,

$$\begin{aligned} f'(\alpha) &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin^4 x}\cdot\frac{\sin^2 x}{1+\alpha^2\sin^4 x} dx \\ &= \frac{1}{1-\alpha^2}\int_0^{\frac{\pi}{2}} \frac{\cos^2 x \sin^2 x}{1+\sin^4 x} dx - \frac{\alpha^2}{1-\alpha^2}\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\sin^2 x}{1+\alpha^2\sin^4 x} dx \end{aligned}$$

Let $g(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x \sin^2 x}{1+\alpha^2 \sin^4 x} dx$. Then,

$$I = 4f(1) = 4\int_0^1 \frac{g(1)-\alpha^2 g(\alpha)}{1-\alpha^2} d\alpha \tag{1}$$

$$\begin{aligned} g(\alpha) &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x\sin^2 x}{1+\alpha^2\sin^4 x} dx \\ &= \int_0^\infty \frac{t^2}{\left(t^4(1+\alpha^2)+2t^2+1\right)\left(t^2+1\right)} dt \quad (t=\tan x) \\ &= -\frac{1}{\alpha^2}\int_0^\infty \frac{1}{1+t^2} dt + \frac{1}{\alpha^2}\int_0^\infty \frac{1+(1+\alpha^2)t^2}{(1+\alpha^2)t^4+2t^2+1} dt \\ &= -\frac{\pi}{2\alpha^2} + \frac{\pi\sqrt{1+\sqrt{1+\alpha^2}}}{2\sqrt{2}\alpha^2} \end{aligned} \tag{2}$$

That last integral was evaluated using an application of the residue theorem. Substitute this into equation (1) to get

$$I = \pi\sqrt{2}\int_0^1 \frac{\sqrt{1+\sqrt{2}} - \sqrt{1+\sqrt{1+\alpha^2}}}{1-\alpha^2} d\alpha \tag{3}$$

Luckily, integral (3) has a nice elementary anti-derivative.

$$
\begin{aligned}
&\int \frac{\sqrt{1+\sqrt{2}}-\sqrt{1+\sqrt{1+\alpha^2}}}{1-\alpha^2}d\alpha \\
&= \sqrt{1+\sqrt{2}}\ \operatorname{arctanh}(\alpha) - \int \frac{\sqrt{1+\sqrt{1+\alpha^2}}}{1-\alpha^2}d\alpha \\
&= \sqrt{1+\sqrt{2}}\ \operatorname{arctanh}(\alpha) - \int \frac{t\sqrt{1+t}}{(2-t^2)\sqrt{t^2-1}}dt \quad (t=\sqrt{1+\alpha^2}) \\
&= \sqrt{1+\sqrt{2}}\ \operatorname{arctanh}(\alpha) - \int \frac{t}{(2-t^2)\sqrt{t-1}}dt \\
&= \sqrt{1+\sqrt{2}}\ \operatorname{arctanh}(\alpha) - 2\int \frac{u^2+1}{2-(u^2+1)^2}du \quad (u=\sqrt{t-1}) \\
&= \sqrt{1+\sqrt{2}}\ \operatorname{arctanh}(\alpha) - 2\int \frac{u^2+1}{(\sqrt{2}-1-u^2)(\sqrt{2}+1+u^2)}du \\
&= \sqrt{1+\sqrt{2}}\ \operatorname{arctanh}(\alpha) - \int \frac{du}{\sqrt{2}-1-u^2} + \int \frac{du}{\sqrt{2}+1+u^2} \\
&= \sqrt{1+\sqrt{2}}\ \operatorname{arctanh}(\alpha) - \sqrt{\sqrt{2}+1}\ \operatorname{arctanh}\left(u\sqrt{\sqrt{2}+1}\right) + \sqrt{\sqrt{2}-1}\operatorname{arctan}\left(u\sqrt{\sqrt{2}-1}\right) + C \\
&= \sqrt{1+\sqrt{2}}\ \operatorname{arctanh}(\alpha) - \sqrt{\sqrt{2}+1}\ \operatorname{arctanh}\left(\sqrt{\sqrt{1+\alpha^2}-1}\sqrt{\sqrt{2}+1}\right) \\
&\quad + \sqrt{\sqrt{2}-1}\operatorname{arctan}\left(\sqrt{\sqrt{1+\alpha^2}-1}\sqrt{\sqrt{2}-1}\right) + C
\end{aligned}
$$

Therefore, the integral is equal to

$$
\begin{aligned}
I &= \pi\sqrt{2}\lim_{\alpha\to 1}\left\{\sqrt{1+\sqrt{2}}\ \operatorname{arctanh}(\alpha) - \sqrt{\sqrt{2}+1}\ \operatorname{arctanh}\left(\sqrt{\sqrt{1+\alpha^2}-1}\sqrt{\sqrt{2}+1}\right)\right. \\
&\quad \left. + \sqrt{\sqrt{2}-1}\operatorname{arctan}\left(\sqrt{\sqrt{1+\alpha^2}-1}\sqrt{\sqrt{2}-1}\right)\right\} \\
&= \pi\sqrt{2}\left\{\frac{\sqrt{\sqrt{2}+1}}{2}\log\left(\frac{\sqrt{2}+2}{4}\right) + \sqrt{\sqrt{2}-1}\operatorname{arctan}\left(\sqrt{2}-1\right)\right\} \\
&= \pi\sqrt{2}\left\{\frac{\sqrt{\sqrt{2}+1}}{2}\log\left(\frac{\sqrt{2}+2}{4}\right) + \frac{\pi}{8}\sqrt{\sqrt{2}-1}\right\} \\
&= \pi\log\left(\frac{2+\sqrt{2}}{4}\right)\sqrt{\frac{\sqrt{2}+1}{2}} + \frac{\pi^2}{4}\sqrt{\frac{\sqrt{2}-1}{2}}
\end{aligned}
$$

□

Chapter 4

Integral evaluations using series expansions

4.0.1 Integral 1

$$\int_0^{2-\sqrt{3}} \frac{\arctan(x)}{x} dx = \frac{\pi}{12} \ln\left(2-\sqrt{3}\right) + \frac{2}{3}G$$

where $G = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)^2}$ is the Catalan's constant.

Proof. Let I denote the integral. Using Integration by parts, we can write

$$\begin{aligned} I &= [\ln(x)\arctan(x)]_0^{2-\sqrt{3}} - \int_0^{2-\sqrt{3}} \frac{\ln(x)}{1+x^2} dx \\ &= \frac{\pi}{12} \ln\left(2-\sqrt{3}\right) - \int_0^{\frac{\pi}{12}} \ln\left(\tan\theta\right)\, d\theta \qquad (x=\tan\theta) \end{aligned} \tag{1}$$

For $\theta \in \left(0, \frac{\pi}{2}\right)$, the following series expansions hold:

$$\begin{aligned} \log(2\sin\theta) &= -\sum_{k=1}^{\infty} \frac{\cos(2k\theta)}{k} \\ \log(2\cos\theta) &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos(2k\theta)}{k} \end{aligned}$$

Therefore, we have

$$\begin{aligned} I &= \frac{\pi}{12} \ln\left(2-\sqrt{3}\right) + 2\int_0^{\frac{\pi}{12}} \left(\sum_{k=0}^{\infty} \frac{\cos(2(2k+1)\theta)}{2k+1} \right) d\theta \\ &= \frac{\pi}{12} \ln\left(2-\sqrt{3}\right) + 2\sum_{k=0}^{\infty} \frac{1}{2k+1} \int_0^{\frac{\pi}{12}} \cos(2(2k+1)\theta)\, d\theta \end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi}{12}\ln\left(2-\sqrt{3}\right) + \sum_{k=0}^{\infty}\frac{\sin\left((2k+1)\frac{\pi}{6}\right)}{(2k+1)^2} \\
&= \frac{\pi}{12}\ln\left(2-\sqrt{3}\right) + \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} + \sum_{k=0}^{\infty}\frac{(-1)^k}{(6k+3)^2}\right) + \sum_{k=0}^{\infty}\frac{(-1)^k}{(6k+3)^2} \\
&= \frac{\pi}{12}\ln\left(2-\sqrt{3}\right) + \frac{1}{2}\left(G+\frac{G}{9}\right) + \frac{G}{9} \\
&= \frac{\pi}{12}\ln\left(2-\sqrt{3}\right) + \frac{2}{3}G
\end{aligned}
$$

□

4.0.2 Integral 2

$$
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \arctan\left(\sqrt{3}\tan\theta\right)\, d\theta - \int_{0}^{\frac{\pi}{6}} \arctan\left(\sqrt{3}\tan\theta\right)\, d\theta = \frac{\pi^2}{12}
$$

Proof. Let I denote the expression on the left hand side of the equation. The substutution $\sqrt{3}\tan\theta = \tan\varphi$ transforms I into

$$
I = \frac{\sqrt{3}}{2}\left(\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\varphi}{1+\frac{1}{2}\cos(2\varphi)}d\varphi - \int_{0}^{\frac{\pi}{4}} \frac{\varphi}{1+\frac{1}{2}\cos(2\varphi)}d\varphi\right)
$$

For $0 \le \phi \le \frac{\pi}{2}$, consider the following generalized integral:

$$
I(\phi) = \frac{\sqrt{3}}{2}\int_0^{\phi} \frac{\varphi}{1+\frac{1}{2}\cos(2\varphi)}d\varphi
$$

Using the series identity,

$$
1 + 2\sum_{k=1}^{\infty} x^k\cos(k\theta) = \frac{1-x^2}{1-2x\cos(\theta)+x^2} \qquad |x|<1
$$

we easily deduce that

$$
\frac{1}{1+\frac{1}{2}\cos(2\varphi)} = \frac{2}{\sqrt{3}}\left(1+2\sum_{k=1}^{\infty}(-1)^k\rho^k\cos(2k\varphi)\right)
$$

where $\rho = 2-\sqrt{3}$. Therefore,

$$
\begin{aligned}
I(\phi) &= \int_0^{\phi} \varphi\left(1+2\sum_{k=1}^{\infty}(-1)^k\rho^k\cos(2k\varphi)\right)d\varphi \\
&= \frac{\phi^2}{2} + 2\sum_{k=1}^{\infty}(-1)^k\rho^k\int_0^{\phi}\varphi\cos(2k\varphi)\, d\varphi \\
&= \frac{\phi^2}{2} + 2\sum_{k=1}^{\infty}(-1)^k\rho^k\left[\frac{\phi\sin(2k\phi)}{2k} - \frac{1-\cos(2k\phi)}{4k^2}\right]
\end{aligned}
$$

$$= \frac{\phi^2}{2} - \phi \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\rho^k \sin(2k\phi)}{k} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1-\cos(2k\phi)}{k^2} \rho^k$$

$$= \frac{\phi^2}{2} - \phi \arctan\left(\frac{\rho \sin(2\phi)}{1+\rho\cos(2\phi)}\right) + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1-\cos(2k\phi)}{k^2} \rho^k \tag{1}$$

We have used the following identity in the last step:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \sin(k\theta) = \arctan\left(\frac{x\sin(\theta)}{1+x\cos(\theta)}\right) \quad |x| \leq 1$$

Using equation (1), we get

$$I\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8} + \chi_2(\rho) \tag{2}$$

$$I\left(\frac{\pi}{3}\right) = \frac{\pi^2}{36} + \frac{1}{12}\mathrm{Li}_2\left(-\rho^3\right) - \frac{3}{4}\mathrm{Li}_2\left(-\rho\right) \tag{3}$$

$$I\left(\frac{\pi}{4}\right) = \frac{\pi^2}{96} + \frac{1}{2}\chi_2(\rho) - \frac{1}{4}\chi_2\left(\rho^2\right) \tag{4}$$

Now, we have

$$\begin{aligned} I &= I\left(\frac{\pi}{2}\right) - I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{4}\right) \\ &= \frac{25\pi^2}{288} + \left[-\frac{1}{4}\mathrm{Li}_2(\rho) + \frac{1}{2}\mathrm{Li}_2(\rho^2) + \frac{1}{12}\mathrm{Li}_2(\rho^3) - \frac{1}{16}\mathrm{Li}_2(\rho^4) - \frac{1}{24}\mathrm{Li}_2(\rho^6)\right] \end{aligned} \tag{5}$$

To complete the proof, we must prove that

$$-\frac{1}{4}\mathrm{Li}_2(\rho) + \frac{1}{2}\mathrm{Li}_2(\rho^2) + \frac{1}{12}\mathrm{Li}_2(\rho^3) - \frac{1}{16}\mathrm{Li}_2(\rho^4) - \frac{1}{24}\mathrm{Li}_2(\rho^6) = -\frac{\pi^2}{288} \tag{6}$$

This can be done by using the following dilogarithm identites:

$$\mathrm{Li}_2\left(\tan a, \frac{\pi}{2} - 2a\right) = a^2 + \frac{3}{4}\mathrm{Li}_2(\tan^2 a) - \frac{1}{8}\mathrm{Li}_2(\tan^4 a)$$

$$\mathrm{Li}_2\left(x, \frac{\pi}{3}\right) = \frac{1}{6}\mathrm{Li}_2(-x^3) - \frac{1}{2}\mathrm{Li}_2(-x)$$

where the notation $\mathrm{Li}_2(x,\theta)$ is used for the real part of $\mathrm{Li}_2(xe^{i\theta})$. Substituting $a = \frac{\pi}{12}$ and $x = \tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$ gives:

$$\begin{aligned} \frac{1}{6}\mathrm{Li}_2(-\rho^3) - \frac{1}{2}\mathrm{Li}_2(-\rho) &= \frac{\pi^2}{144} + \frac{3}{4}\mathrm{Li}_2(\rho^2) - \frac{1}{8}\mathrm{Li}_2(\rho^4) \\ \implies \frac{1}{12}\mathrm{Li}_2(\rho^6) - \frac{1}{6}\mathrm{Li}_2(\rho^3) - \frac{1}{4}\mathrm{Li}_2(\rho^2) + \frac{1}{2}\mathrm{Li}_2(\rho) &= \frac{\pi^2}{144} + \frac{3}{4}\mathrm{Li}_2(\rho^2) - \frac{1}{8}\mathrm{Li}_2(\rho^4) \end{aligned} \tag{7}$$

Now, it is easy to check that equation (6) and (7) are equivalent. □

Another interesting identity, though not related to the problem above, is obtained using Hill's two-variable relation for the dilogarithm:

$$\mathrm{Li}_2(xy) = \mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(-x\left(\frac{1-y}{1-x}\right)\right) + \mathrm{Li}_2\left(-y\left(\frac{1-x}{1-y}\right)\right) + \frac{1}{2}\log^2\left(\frac{1-x}{1-y}\right)$$

Substitute $x = -y = e^{-i\frac{\pi}{4}}\frac{\sqrt{3}-1}{\sqrt{2}}$. Then $\frac{1-x}{1+x} = ix = e^{i\frac{\pi}{4}}\frac{\sqrt{3}-1}{\sqrt{2}}$ and $ix^2 = \rho$.

$$\mathrm{Li}_2(-x^2) = \mathrm{Li}_2(x) + \mathrm{Li}_2(-x) + \mathrm{Li}_2(i) + \mathrm{Li}_2(ix^2) + \frac{1}{2}\log^2(ix)$$

$$\implies \mathrm{Li}_2(i\rho) = \frac{1}{2}\mathrm{Li}_2(-i\rho) + \mathrm{Li}_2(i) + \mathrm{Li}_2(\rho) + \frac{1}{2}\log^2\left(e^{i\frac{\pi}{4}}\frac{\sqrt{3}-1}{\sqrt{2}}\right)$$

Extracting the real part from the above equation gives:

$$\frac{\mathrm{Li}_2(\rho^4)}{16} - \frac{\mathrm{Li}_2(\rho^2)}{8} - \mathrm{Li}_2(\rho) = -\frac{5\pi^2}{96} + \frac{1}{8}\log^2(\rho)$$

4.0.3 Integral 3

$$\int_0^\infty \ln\left(\frac{1+a\sin^2(bx)}{1-a\sin^2(bx)}\right)\frac{1}{x^2}\,dx = \pi b\left(\sqrt{1+a} - \sqrt{1-a}\right)$$

where $|a| < 1,\ b > 0$.

Proof. We will make use of the following well known series identity:

$$\sum_{n=-\infty}^{\infty}\frac{1}{(x+n\pi)^2} = \frac{1}{\sin^2(x)}$$

Let I denote the integral. Then,

$$\begin{aligned}
I &= \frac{1}{2}\int_{-\infty}^{\infty}\ln\left(\frac{1+a\sin^2(bx)}{1-a\sin^2(bx)}\right)\frac{1}{x^2}dx \\
&= \frac{b}{2}\int_{-\infty}^{\infty}\ln\left(\frac{1+a\sin^2(t)}{1-a\sin^2(t)}\right)\frac{1}{t^2}dt \qquad (t = bx) \\
&= \frac{b}{2}\sum_{n=-\infty}^{\infty}\int_{n\pi-\frac{\pi}{2}}^{n\pi+\frac{\pi}{2}}\ln\left(\frac{1+a\sin^2(t)}{1-a\sin^2(t)}\right)\frac{1}{t^2}dt \\
&= \frac{b}{2}\sum_{n=-\infty}^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\ln\left(\frac{1+a\sin^2(t)}{1-a\sin^2(t)}\right)\frac{1}{(t+n\pi)^2}dt \\
&= \frac{b}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\ln\left(\frac{1+a\sin^2(t)}{1-a\sin^2(t)}\right)\left(\sum_{n=-\infty}^{\infty}\frac{1}{(t+n\pi)^2}\right)dt \\
&= \frac{b}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\ln\left(\frac{1+a\sin^2(t)}{1-a\sin^2(t)}\right)\frac{1}{\sin^2(t)}dt \\
&= b\int_0^{\frac{\pi}{2}}\ln\left(\frac{1+a\sin^2(t)}{1-a\sin^2(t)}\right)\frac{1}{\sin^2(t)}dt
\end{aligned}$$

$$
\begin{aligned}
&= 2b \int_0^{\frac{\pi}{2}} \left(\sum_{k=0}^{\infty} \frac{a^{2k+1} \sin^{4k+2}(t)}{2k+1} \right) \frac{1}{\sin^2(t)} dt \\
&= 2b \sum_{k=0}^{\infty} \frac{a^{2k+1}}{2k+1} \int_0^{\frac{\pi}{2}} \sin^{4k}(t)\ dt \\
&= \pi b \sum_{k=0}^{\infty} \frac{a^{2k+1}}{(2k+1)2^{4k}} \binom{4k}{2k} \\
&= \pi b \left(\sqrt{1+a} - \sqrt{1-a}\right)
\end{aligned}
$$

$\square$

A similar calculation shows that

$$\int_0^{\infty} \log\left(\frac{1 + a\sin bx}{1 - a\sin bx}\right) \frac{dx}{x} = \pi \arcsin(a)$$

where $|a| < 1$ and $b > 0$. Dividing the above by a and integrating both sides yields the following identity:

$$\int_0^{\infty} \frac{\mathrm{Li}_2\left(\frac{\sin x}{\sqrt{2}}\right) - \mathrm{Li}_2\left(-\frac{\sin x}{\sqrt{2}}\right)}{x} dx = \frac{\pi G}{2} + \frac{\pi^2 \ln(2)}{8}$$

where G is Catalan's constant. Another integral obtained using this technique is (refer section 3.0.2):

$$\int_0^{\infty} \frac{\arctan^2(\sin^2 x)}{x^2} dx = \pi \log\left(\frac{2+\sqrt{2}}{4}\right) \sqrt{\frac{\sqrt{2}+1}{2}} + \frac{\pi^2}{4}\sqrt{\frac{\sqrt{2}-1}{2}}$$