1

VECTOR SPACES

1.1 BINARY OPERATION

An operation which combines two elements of a set to give another element of the same set is called a **binary operation** or a **binary composition.**

Definition: *A binary operation on a set is a rule which assigns to any two elements of the set a unique element in the set.*

If a set is closed with respect to the operation o, then we say that o is a **binary operation** in the set.

The operations of algebraic addition, subtraction, multiplication and division can now be looked upon as binary compositions over the set of numbers. Thus $aob = a \div b$ or a/b is a binary composition over the set of non-zero rational numbers.

$aob = a - b$ is a binary composition over the set of integral numbers, since $a - b$ is also an integer. But it is not a binary composition over the set of natural numbers, for $x - y$ (where x, y are natural numbers) is not always a natural number. The addition and multiplication operations, *i.e.,*

$$aob = a + b,\ aob = a \times b$$

are binary compositions over the sets of natural numbers, integral numbers, rational numbers.

Binary compositions are sometimes denoted by the symbol $*$. Thus,

$$a * b \in A,\ \forall\ a, b \in A.$$

1.2 ALGEBRAIC STRUCTURE

Definition: *A non-empty set A equipped with one or more binary operations is called an algebraic structure.*

Suppose there are binary operations $*, +, ., \circ, \oplus, \otimes, \cap, \wedge, \vee$, etc. in set A. Then $(A, *)$, $(A, \oplus, \otimes)$, $(A, *, \circ, \vee, \wedge)$ etc. are algebraic structures. Hence, if N is the set of positive integral numbers, I is the set of integers, R is the set of real numbers, then $(N, +)$, $(I, +)$, $(I, -)$, $(R, +)$, $(R, +, .)$ are all algebraic structures where $(R, +, .)$ is an algebraic structure of two binary operations "+" and ".".

1.3 GROUP

Definition: *An algebraic structure (G, o) where G is a non-empty set with a binary operation 'o' defined on it is said to be a group if the binary operation 'o' satisfies the following axioms (called group axioms):*

(G_1) **Closure axiom:** G is closed under the operation o, *i.e.,*

$$a \; o \; b \in G, \; \forall \; a, b \in G.$$

(G_2) **Associative axiom:** The binary operation 'o' is associative *i.e.,*

$$(a \; o \; b) \; o \; c = a \; o \; (b \; o \; c), \; \forall \; a, b, c \in G.$$

(G_3) **Identity axiom:** For every $a \in G$, there exists an element $e \in G$ such that

$$a \; o \; e = e \; o \; a = a, \; \forall \; a \in G.$$

The element $e \in G$ is called identity element of G w.r.t. o

(G_4) **Inverse axiom:** Each element of G possesses inverse *i.e.,* for each $a \in G$, there exists an element $b \in G$ such that

$$a \; o \; b = e = b \; o \; a.$$

The element b is then called the inverse of a with respect to 'o' and we write $b = a^{-1}$. Thus a^{-1} is an element of G such that

$$a^{-1} \; o \; a = e = a \; o \; a^{-1}.$$

Note 1. While defining group closure axiom could have been dropped as o being binary operation, it is automatically implied.

In general use we call G as the group instead of (G, o), but in doing so, no uncertainty must arise with respect to the operation under reference. We do so only for the sake of convenience in speaking.

It should be always borne into mind that a group (G, o) is comprised of two entities namely the set G and the binary operation o defined on it.

Note 2. In the above definition, the elements $a, b, c \in G$ are arbitrary and may be distinct or identical.

Note 3. Usually the identity element is expressed by 0 in additive operation. Therefore,

$$0 + a = a = a + 0.$$

Note 4. The identity element is expressed by 1 in multiplicative operation, then

$$a \cdot 1 = a = 1 \cdot a.$$

1.4 GROUPOID

Definition: *An algebraic structure (G, o) is called a groupoid (or a binary algebra or a quasi group) if binary operation 'o' satisfies only closure axiom, i.e.,*

$$a \; o \; b \in G, \; \forall \; a, b \in G.$$

1.5 SEMI GROUP

Definition: *An algebraic structure (G, o), where G is a non-empty set with binary operation 'o' defined on it is said to be a semi-group if the binary operation 'o' satisfies the following axioms:*

(G_1) **Closure axiom:** G is closed under the operation o, *i.e.,* +

$$a \; o \; b \in G, \; \forall \; a, b \in G.$$

(G_2) **Associative axiom:** The operation 'o' is associative *i.e.,*

$$(a \; o \; b) \; o \; c = a \; o \; (b \; o \; c), \; \forall \; a, b, c \in G.$$

1.6 MONOID

Definition: *An algebraic structure (G, o) where G is a non-empty set with binary operation 'o' defined on it is said to be a monoid if the binary operation satisfies the following axioms:*

(G_1) **Closure axiom:** G is closed under the operation o, *i.e.,*

$$a \; o \; b \in G, \text{ for all } a, b \in G.$$

(G_2) **Associative axiom:** The binary operation 'o' is associative *i.e.,*

$$(a \; o \; b) \; o \; c = a \; o \; (b \; o \; c), \; \forall \; a, b, c \in G.$$

(G_3) **Identity axiom:** For every $a \in G$, there exists an element $e \in G$ such that

$$a \; o \; e = a = e \; o \; a$$

The element e is called the identity element of G w.r.t. o.

1.7 FINITE AND INFINITE GROUPS

If a group contains a finite number of elements, it is called *finite* and if it contains an infinite number of elements, it is called *infinite.*

1.8 ORDER OF A GROUP

The number of elements, in a finite group is called the order of the group. An infinite group is sometimes said to be a group of infinite order. The order of a group G is usually denoted by the symbol o (G).

Thus, if G has n elements, $n \in N$, it is a *finite* group of *order n.*

The smallest group for a given composition is the set $\{e\}$ consisting of the identity element alone.

1.9 ABELIAN GROUP OR A COMMUTATIVE GROUP

Definition: *A group G is said to be abelian or commutative if in addition to the above four postulates, with operation "o" the following postulate is also satisfied.*

(G_5) **Commutative axiom:** $a \cdot b = b \cdot a$ for all $a, b \in G$.

Thus, for an *abelian group* we have, for $a, b \in G$,

$$a + b = b + a \qquad \text{(additive operation)}$$

$$a \cdot b = b \cdot a. \qquad \text{(multiplicative operation)}$$

In particular, an abelian group under addition is sometimes called **module.**

A group which is not abelian is called *non-abelian.*

Illustrations:

1. (Z, +), (Q, +), (R, +) are all infinite abelian groups (0 is the identity and $-a$ is the inverse of a in each case), + being the binary operation.
2. The set of all even integers (including zero) with addition forms an infinite abelian group.

1.10 RING

Definition: *An algebraic structure (R, +, ·) where R is a non-empty set with two binary operations "+" and "·" of addition and multiplication respectively defined on R is said to be a ring, if the following axioms are satisfied:*

R₁ (R, +) is an abelian group, *i.e.* if a, b, c be arbitrary elements (distinct or identical) of R, then

(R_{11}) **Closure property:** $a + b \in R$ (additive closure)

(R_{12}) **Associative property:** $(a + b) + c = a + (b + c)$ (additive associativity)

(R_{13}) **Existence of identity:** There exists an element 0 in R such that for every a in R, $a + 0 = a$
(existence of a zero, *i.e.*, additive identity)

(R_{14}) **Existence of inverse:** If $a \in R$, then there exists an element $-a$ in R such that $a + (-a) = 0$
(existence of additive inverse)

(R_{15}) **Commutative property:** $a + b = b + a$
(additive commutativity)

R₂ (R, ·) is a semi-group, *i.e.*,

(R_{21}) **Closure property:** $a \cdot b \in R$ (multiplicative closure)

(R_{23}) **Associative property:** Multiplication is associative, *i.e.*,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

R₃ The *left distributive law* and the *right distributive law* hold, *i.e.*,

(R_{31}) $a \cdot (b + c) = a \cdot b + a \cdot c$ (left distributive law)

and (R_{32}) $(b + c) \cdot a = b \cdot a + c \cdot a.$ (right distributive law)

Remark: Since R is an abelian group under addition +, the following properties hold.

1.11 PROPERTIES OF A RING

1. 0 is the left as well as right identity, *i.e.*,
$$0 + a = a = a + 0.$$
2. $-a$ is the left as well as right inverse of a, *i.e.*,
$$(-a) + a = 0 = a + (-a).$$
3. The additive identity 0 is unique.
4. The additive inverse $-a$ of a is unique.
5. The equation $a + x = b$, where $a, b \in R$ has a unique solution $b + (-a)$ in R. For convenience, $b + (-a)$ is generally written as $b - a$.
6. The cancellation laws hold, *i.e.*,
if $a + c = b + c$, then $a = b$
and if $c + a = c + b$, then $a = b$.

Another definition: *A module (an additive abelian group) R is a* **ring** *if R is also a semi-group under multiplication and further, satisfies right and left distributive laws under addition.*

1.12 TYPES OF RINGS

(i) Commutative Ring

Definition: *A ring R in which multiplication is commutative, i.e.,* $a \cdot b = b \cdot a$ for all, a, b in R, is called a **commutative ring.**

If multiplication is not commutative, the ring is said to be *non-commutative.*

(ii) Ring with Unity:

Definition: *A ring R with a multiplicative identity 1 such that* $1 \cdot a = a \cdot 1 = a$ *for every a in R is called a ring with unity.*

A multiplicative identity in a ring is called unity.

If R be a ring with unity, then an element $a \in$ R is called *inversible* if there exists an element b in R such that $a \cdot b = 1 = b \cdot a$.

Also, then we write $b = a^{-1}$ and call it a *multiplicative inverse* of a.

(iii) Null Ring or Zero Ring:

Definition: *The singleton set {0} with two operations '+', '·' defined by*

$$0 + 0 = 0 \text{ and } 0 \cdot 0 = 0$$

is a ring and is called the zero ring or null ring.

(iv) Boolean Ring:

Definition: *A ring R is said to be Boolean ring if each of its elements is idempotent, i.e.,* $a^2 = a, \forall\, a \in$ R.

(v) *p*-Ring:

Definition: *A ring R is called p-ring, if* $a^p = a$ *and* $p^a = 0, \forall\, a \in R$.

1.13 FIELD

Definition: *A ring (F, +, ·) with at least two elements is called a field, if it*

(i) is commutative,

(ii) has unity,

(iii) is such that each non-zero element possesses multiplicative inverse in F.

A commutative division ring is called a *field.*

Combining the properties of integral domain and the above additional property the definition of field can now be stated as given below:

A field (F, +, ·) is a set F together with two binary operations '+' and '·' of addition and multiplication respectively defined on F such that for all a, b, c in F the following axioms are satisfied:

F_1 (F, +) is an abelian group

(F_{11}) **Closure property:** $a, b \in$ F, $a + b \in$ F. [Closure axiom]

(F_{12}) **Additive associative property:** $a, b, c \in$ F, $(a + b) + c = a + (b + c)$. [Additive associative]

(F_{13}) **Existence of additive identity:** One element 0 in f such that

$$a + 0 = 0 + a = a. \qquad \text{[Additive identity]}$$

(F_{14}) **Existence of additive inverse:** For each element $a \in$ F, there is an element $-a$ in f such that

$$a + (-a) = 0. \qquad \text{[Additive inverse]}$$

(F_{15}) **Additive commutative property:**

$$a + b = b + a$$ [Additive commutative law]

F_2 (F_{21}) **Multiplicative closure property:**

$$a \cdot b \in F.$$ [Multiplicative closure]

(F_{22}) **Multiplicative associative property:** If $a \cdot b \cdot c \in F$, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$ [Multiplicative associative]

(F_{23}) **Existence of multiplicative identity:** One element 1 in F (which is called identity) is such that

$$a \cdot 1 = a.$$

(F_{24}) **Existence of multiplicative inverse:** For each non-zero element $a \in F$, there is another element a^{-1} (which is called multiplicative inverse of a) is such that

$$a \cdot a^{-1} = 1.$$ (Multiplicative inverse)

(F_{25}) **Multiplicative commutative property:** If $a, b \in F$, then

$$a \cdot b = b \cdot a.$$

F_3 (F_{31}) **Left distributive law:** If $a, b \in F$, then

$$a \cdot (b + c) = a \,.\, b + a \cdot c.$$ [Left distributive law]

(F_{32}) **Right distributive law:**

$$(b + c) \cdot a = b \,.\, a + c \cdot a.$$ [Right distributive law]

The above conditions can again be summarised as:

1. The system (F, +) is an abelian group.
2. The subset of non-zero elements of F is an abelian group with respect to multiplication.
3. All elements $\in$ F satisfy distributive laws.

Illustrations:

1. The set of all rational numbers is a *field* with respect to addition and multiplication operations.
2. The set of all real numbers is a *field* with respect to addition and multiplication operations.
3. The set of complex numbers is a *field* with respect to addition and multiplication operations.
4. The set of integers is not a *field*.

1.14 SUB-FIELD

Definition: *A subset E (containing more than one element) of a field F is called a sub-field of F, if E is a field with respect to addition and multiplication in F.*

This definition may be restated as follows:

1. $a + b \in$ E, for all $a, b \in$ E and $a \in$ E $\Rightarrow$ $-a \in$ E ; also $0 \in$ E.
2. $a \cdot b \in$ E, for all $a, b \in$ E, $a \neq 0$ and $a \in$ E $\Rightarrow$ $a^{-1} \in$ E and $1 \in$ E.

Illustrations:

1. The field of rational numbers is a sub-field of the field of complex numbers.
2. The field of real numbers is a sub-field of the field of complex numbers.

1.15 PRIME FIELD

Definition: *A field is called a prime field if it contains no proper sub-field.*

Illustrations:

1. The set **Q** of rational numbers is a prime sub-field of the field F of real numbers, if F is of characteristic zero.
2. The set of residue classes modulo a prime p is also a sub-field of F, if F is of characteristic p.

Note:

(*i*) A field is necessarily an integral domain while every finite integral domain is a field.

(*ii*) The multiplicative inverse of a non-zero element of a field is unique.

1.16 INTRODUCTION TO VECTOR SPACES

We have ideas of *vectors*, their operations, and two algebraic structures known as *group* and *field*. Group is an algebraic structure (G, $*$) with one binary operation $*$, while field is an algebraic structure (F, +, $*$) with two binary operations +, and $*$. We now introduce another algebraic structure known as **vector space** (or *linear vector space*) which is a generalisation arising from the ordinary notion of a vector in two or three dimensions (R^2 or R^3), in general in R^n.

1.17 PRELIMINARIES

A vector a in two dimensions (R^2) or in three dimensions (R^3) is represented as $a = (a_1, a_2)$ or $a = (a_1, a_2, a_3)$, respectively where right hand expression is ordered pair of two or three ordered triplet of real numbers called the components of the vector. An n-component vector a is an ordered n-tuple of real numbers, written as $a = (a_1, a_2, ..., a_n)$.

The addition of two vectors a and b is denoted by $a + b$, which is a vector. The identity element for the operation addition is o (null) vector, as $0 + a = a + 0 = a$.

The scalar multiplication of a vector a by a scalar α is a vector defined by αa. The addition of vectors is *commutative* and *associative* that is, $a + b = b + a$ and $(a + b) + c = a + (b + c)$.

(1) Internal composition. Let S be any non-empty set. If for any elements $a, b \in S$, $a * b \in S$ and $a * b$ is unique, then $*$ is said to be an internal composition in the set S. Here a and b both are the elements of S and the map is $S \times S \to S$.

(2) External composition. Let V and F be any two non-empty sets. If $a * x \in V$ for all $a \in F$ and for all $x \in V$ and $a * x$ be unique then $*$ is said to be the *external composition* in V over F.

Here a is an element of the set F and x is an element of the set V and the resulting element $a * x$ is an element of the set V and the map is $F \times V \to V$.

The elements of V are called *vectors* and are denoted by ***x, y, z, u, v,*** etc. and the elements of F are called *scalars* and are denoted by a, b, c, etc.

It is known earlier that the addition of vectors (internal composition) and the multiplication of vectors by scalars (external composition) satisfy the following properties:

(*i*) for any $x, y \in V$, $x + y \in V$, where V is a set of vectors

(*ii*) for any $x, y, z \in V$, $(x + y) + z = x + (y + z)$

(*iii*) for any vector x, there exists a null vector $0 \in V$ such that $x + 0 = 0 + x = x$

(*iv*) corresponding to the vector $x \in V$, there exists a vector $(-x)$, such that

$$x + (-x) = (-x) + x = 0.$$

(*v*) for any vectors $x, y \in V$, $x + y = y + x$.

(*vi*) There exists an external composition in V over the field F, of real numbers, such that

$$a \in F, x \in V \text{ implies } a\,x \in V$$

(*vii*) for any $a, b \in F$ and $x \in V$,

$$(a + b)\,x = ax + bx$$

(*viii*) for any $a \in F$, and $x, y \in V$

$$a(x + y) = ax + ay$$

(*ix*) for any $a, b \in F$ and $x \in V$

$$(ab)x = a(bx)$$

(*x*) $1 \cdot x = x$, where 1 is the identity element in F and x is any element of V.

These ideas motivate the following abstract definition of a vector space V over a field F.

1.18 VECTOR SPACE

Let (F, +, •) be the given field in which elements of F are scalars. Further let V be a non-empty set where the elements of V are vectors. Then V is said to be a vector space over the field F if it satisfies the following postulates:

A. Under vector addition

The addition of vectors (denoted by '+') is defined as internal composition in V satisfying the following:

A_1. Closure. $\forall\, \alpha, \beta \in V, \alpha + \beta \in V$

A_2. Commutativity. $\forall\, \alpha, \beta \in V, \alpha + \beta = \beta + \alpha$

A_3. Associativity. $\forall\, \alpha, \beta, \gamma \in V, \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

A_4. Existence of Identity. There exists $0 \in V$ such that

$$\alpha + 0 = \alpha = 0 + \alpha \quad \forall\, \alpha \in V$$

0 is called as zero vector in V.

A_5. Existence of inverse. There exists $-\alpha \in V\ \forall\, \alpha \in V$ such that

$$(-\alpha) + \alpha = 0 = \alpha + (-\alpha)$$

$-\alpha$ is the negative of α.

B. Under scalar multiplication

The scalar multiplication is defined as external composition in V over F satisfying the following:

B_1. Closure. $\forall\, a \in F$ and $\forall\, \alpha \in V$, $a\,\alpha \in V$

B_2. Distributivity w.r.t. vector addition

$$a(\alpha + \beta) = a\alpha + a\beta \quad \forall\, a \in F \text{ and } \forall\, \alpha, \beta \in V$$

B_3. Distributivity w.r.t. scalar addition

$$(a + b)\,\alpha = a\alpha + b\alpha \quad \forall\, a, b \in F \text{ and } \forall\, \alpha \in V$$

B_4. **Associativity**

$$(ab)\alpha = a(b\alpha) \quad \forall \quad a, b \in F \quad \text{and} \quad \forall \alpha \in V$$

B_5. $1.\alpha = \alpha \quad \forall \alpha \in V$ and $1 \in F$

The vector space of V over the field F is denoted as V(F).

Notes 1. V(F) is a real vector space if F is the field R of real numbers.

2. V(F) is a complex vector space if F is the field C of complex numbers.

3. V(F) is a rational vector space if F is the field Q of rational numbers.

ILLUSTRATIVE EXAMPLES

Example 1. *Let F be any field, and V is the set of all n-tuples $x = (x_1, x_2, ..., x_n)$ of scalars x_i in F. If $y = (y_1, y_2, ..., y_n)$ with y_i in F, the sum of x and y is defined by*

$$x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n),$$

and the product of a scalar c and a vector x is defined by

$$cx = (cx_1, cx_2, ..., cx_n),$$ *then prove that V is a vector space.*

Sol. (1) Under addition

(*i*) **Closure**

$$x_1, x_2 ..., x_n \in F$$
$$y_1, y_2, ..., y_n \in F$$

$\Rightarrow \quad x_1 + y_1, x_2 + y_2, ..., x_n + y_n \in F.$

$\therefore \quad (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \in V_n(F)$

$\therefore \quad x + y \in V_n(F) \quad \forall x, y \in V_n(F).$

(*ii*) **Commutativity**

$$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$
$$= (y_1 + x_1, y_2 + x_2, ..., y_n + x_n)$$

$\because\ x_i \in F, y_i \in F$ and Commutative law holds in F

$$= (y_1, y_2, ..., y_n) + (x_1, x_2, ..., x_n)$$

$\therefore \quad x + y = y + x \quad \forall \quad x, y \in V_n(F)$

(*iii*) **Associativity**

$$\{(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n)\} + (z_1, z_2, ..., z_n)$$
$$= (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) + (z_1, z_2, ..., z_n)$$
$$= (x_1 + y_1 + z_1, x_2 + y_2 + z_2, ..., x_n + y_n + z_n)$$
$$= (x_1, x_2, ..., x_n) + \{(y_1 + z_1, y_2 + z_2, ..., y_n + z_n)\}$$
$$= (x_1, x_2, ..., x_n) + \{(y_1, y_2, ..., y_n) + (z_1, z_2, ..., z_n)\}$$

$\therefore \quad (x + y) + z = x + (y + z) \quad \forall x, y, z \in V_n(F).$

(*iv*) **Existence of identity**

$\forall (x_1, x_2, ..., x_n) \in V_n(F), \exists (0, 0, 0, ..., 0)$ in $V_n(F)$ such that

$$(x_1, x_2, ..., x_n) + (0, 0, ..., 0) = (x_1, x_2, ..., x_n)$$
$$= (0, 0, ..., 0) + (x_1, x_2, ..., x_n)$$

$\therefore$ (0, 0, ..., 0) is the zero element of $V_n(F)$.

Thus, $x + 0 = 0 + x \quad \forall x \in V_n(F)$ and $0 \in V_n(F)$

(v) **Existence of inverse**

$$\forall\, x = (x_1, x_2, \ldots, x_n) \in V_n(F).$$
$$\exists - x = (-x_1, -x_2, \ldots, -x_n) \in V_n(F) \text{ such that}$$
$$(-x) + x = 0 = x + (-x)$$

Hence $-x$ is the inverse element of x in $V_n(F)$.

(2) **Under scalar multiplication**

$$\forall\, x, y \in V_n(F) \text{ and } a \in F$$

we have,

(i)
$$\begin{aligned} a(x+y) &= a\{(x_1, x_2 \ldots, x_n) + (y_1, y_2, \ldots, y_n)\} \\ &= a\{(x_1+y_1) + (x_2+y_2) + \ldots + (x_n+y_n)\} \\ &= a(x_1+y_1) + a(x_2+y_2) + \ldots + a(x_n+y_n) \\ &= (ax_1+ay_1) + (ax_2+ay_2) + \ldots + (ax_n+ay_n) \\ &= (ax_1, ax_2, \ldots, ax_n) + (ay_1, ay_2, \ldots, ay_n) \\ &= a(x_1, x_2, \ldots, x_n) + a(y_1, y_2, \ldots y_n) \\ &= ax + ay. \end{aligned}$$

(ii)
$$\begin{aligned} (a+b)x &= (a+b)(x_1, x_2, \ldots, x_n) \\ &= \{(a+b)x_1, (a+b)x_2, \ldots, (a+b)x_n\} \\ &= (ax_1+bx_1, ax_2+bx_2, \ldots, ax_n+bx_n) \\ &= (ax_1, ax_2, \ldots, ax_n) + (bx_1, bx_2, \ldots, bx_n) \\ &= a(x_1, x_2, \ldots, x_n) + b(x_1, x_2, \ldots, x_n) \\ &= ax + bx. \end{aligned}$$

(iii)
$$\begin{aligned} a(bx) &= a\{b(x_1, x_2, \ldots, x_n)\} = a(bx_1, bx_2, \ldots, bx_n) \\ &= (abx_1, abx_2, \ldots, abx_n) \\ &= (ab)(x_1, x_2, \ldots, x_n) = (ab)x. \end{aligned}$$

(iv)
$$\begin{aligned} 1x &= 1(x_1, x_2, \ldots, x_n) = (1x_1, 1x_2, \ldots, 1x_n) \\ &= (x_1, x_2, \ldots, x_n) = x. \end{aligned}$$

Thus, V satisfies all the conditions of vector space, hence V is a vector space.

Example 2. *Prove that $R \times R$ is a vector space over R under addition and scalar multiplication defined by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$.*

Sol. The binary operation (+) is commutative and associative and (0, 0) is the zero element.

The inverse of (x_1, x_2) is $(-x_1, -x_2)$.

Hence $(R \times R, +)$ is an abelian group.

Let $u = (x_1, x_2)$ and $v = (y_1, y_2)$ and $\alpha, \beta \in R$.

Then
$$\begin{aligned} \alpha(u+v) &= \alpha[(x_1, x_2) + (y_1, y_2)] \\ &= \alpha[(x_1+y_1, x_2+y_2)] \\ &= \alpha(x_1+y_1, x_2+y_2) \\ &= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2) \\ &= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2) \\ &= \alpha(x_1, x_2) + \alpha(y_1, y_2) = \alpha u + \alpha v \end{aligned}$$

$$\begin{aligned} (\alpha+\beta)u &= (\alpha+\beta)(x_1, x_2) = \{(\alpha+\beta)x_1, (\alpha+\beta)x_2\} \\ &= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2) \end{aligned}$$

$$= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2)$$
$$= \alpha(x_1, x_2) + \beta(x_1, x_2)$$
$$= \alpha u + \beta u.$$

Also, $$\alpha(\beta u) = \alpha\{\beta(x_1, x_2)\} = \alpha(\beta x_1, \beta x_2) = (\alpha\beta x_1, \alpha\beta x_2)$$
$$= (\alpha\beta)(x_1, x_2) = (\alpha\beta)\, u$$

and $$1u = 1(x_1, x_2) = (1x_1, 1x_2) = (x_1, x_2) = u.$$

Thus R × R satisfies all the properties of vector space.

Hence, R × R is a vector space over R.

Example 3. *Show that the set M_2 (R) of all 2 × 2 matrices is a vector space over R under matrix addition and scalar multiplication defined by*

$$\alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$$

Sol. Addition (+) is a binary operation in the set M_2(R). Associative and commutative laws hold for the binary operation (+) in M_2(R).

The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(R)$, is the identity element for the operation (+). The inverse of the matrix $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in M_2(R)$ is $\begin{bmatrix} -x_1 & -x_2 \\ -x_3 & -x_4 \end{bmatrix}$ which also belongs to M_2(R).

Hence, M_2 (R) is an abelian group under matrix addition.

Let $$M_1 = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} \in M_2(R)$$

and $$\alpha, \beta \in R.$$

Then, $$\alpha(M_1 + M_2) = \alpha\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}\right)$$

$$= \alpha\begin{bmatrix} x_1 + y_1 & x_2 + y_2 \\ x_3 + y_3 & x_4 + y_4 \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \alpha y_1 & \alpha x_2 + \alpha y_2 \\ \alpha x_3 + \alpha y_3 & \alpha x_4 + \alpha y_4 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 & \alpha x_2 \\ \alpha x_3 & \alpha x_4 \end{bmatrix} + \begin{bmatrix} \alpha y_1 & \alpha y_2 \\ \alpha y_3 & \alpha y_4 \end{bmatrix}$$

$$= \alpha\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} + \alpha\begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} = \alpha M_1 + \alpha M_2$$

$$(\alpha + \beta)\, M_1 = (\alpha + \beta)\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} (\alpha + \beta)x_1 & (\alpha + \beta)x_2 \\ (\alpha + \beta)x_3 & (\alpha + \beta)x_4 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 + \beta x_1 & \alpha x_2 + \beta x_2 \\ \alpha x_3 + \beta x_3 & \alpha x_4 + \beta x_4 \end{bmatrix} = \begin{bmatrix} \alpha x_1 & \alpha x_2 \\ \alpha x_3 & \alpha x_4 \end{bmatrix} + \begin{bmatrix} \beta x_1 & \beta x_2 \\ \beta x_3 & \beta x_4 \end{bmatrix}$$

$$= \alpha\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} + \beta\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \alpha M_1 + \beta M_1$$

$$\alpha(\beta M_1) = \alpha\left(\beta\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = \alpha\begin{bmatrix} \beta x_1 & \beta x_2 \\ \beta x_3 & \beta x_4 \end{bmatrix} = \begin{bmatrix} (\alpha\beta)x_1 & (\alpha\beta)x_2 \\ (\alpha\beta)x_3 & (\alpha\beta)x_4 \end{bmatrix}$$

$$= (\alpha\beta)\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = (\alpha\beta)M_1$$

and
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = M_1.$$

Hence, the set M_2 (R) of all 2×2 matrices is a vector space over R under the operation addition and scalar multiplication.

Example 4. *Let V be the set of all ordered pairs of real numbers. Addition and multiplication are defined by $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$ and $\alpha(x, y) = (x, \alpha y)$, where x, y, x_1, y_1 and α are real numbers. Show that V is not a vector space over R.*

Sol. V is an abelian group under the operation (+) as defined by the problem.

Let $\alpha, \beta \in$ R and $x, y \in$ V.

Now $\quad (\alpha + \beta)(x, y) = \{x, (\alpha + \beta)y\} = (x, \alpha y + \beta y).$

Also $\quad \alpha(x, y) + \beta(x, y) = (x, \alpha y) + (x, \beta y) = (2x, \alpha y + \beta y).$

But $\quad (x, \alpha y + \beta y) \neq (2x, \alpha y + \beta y)$

i.e., $\quad (\alpha + \beta)(x, y) \neq \alpha(x, y) + \beta(x, y)$

Hence, V is not a vector space over R.

Example 5. *Show that the set V of all real valued continuous (differentiable or integrable) functions defined in the interval (0, 1) is the vector space over the field R of real numbers with respect to addition and scalar multiplication defined as given below:*

$$(f + g)(x) = f(x) + g(x),\ \forall f, g \in V \text{ and } \forall x \in [0, 1].$$
$$(\alpha f)(x) = \alpha f(x),\ \forall f \in V, \alpha \in R \text{ and } x \in [0, 1].$$

Sol. (*i*) (V, +) is an abelian group.

(*ii*) V is closed w.r.t. scalar multiplication, since αf, for $\alpha \in$ R, $f \in$ V is also a real valued continuous functions in [0, 1].

(*iii*) If $\alpha \in$ R and $f, g \in$ V, then

$$[\alpha(f + g)](x) = \alpha[(f + g)(x)] = \alpha[f(x) + g(x)]$$
$$= \alpha f(x) + \alpha g(x) = [\alpha f + \alpha g](x)$$

$\therefore \quad \alpha(f + g) = \alpha f + \alpha g.$

(*iv*) If $\alpha, \beta \in$ R and $f \in$ V, then

$$[(\alpha + \beta)f](x) = (\alpha + \beta) f(x) = \alpha f(x) + \beta f(x)$$
$$= (\alpha f)(x) + (\beta f)(x)$$

i.e., $\quad (\alpha + \beta)f = \alpha f + \beta f.$

(*v*) If $\alpha, \beta \in$ R and $f \in$ V, then

$$[(\alpha\beta)f](x) = (\alpha\beta)f(x) = \alpha[\beta f(x)]$$
$$= \alpha[(\beta f)(x)] = [\alpha(\beta f)](x)$$

i.e., $\quad (\alpha\beta)f = \alpha(\beta f)$

(*vi*) If 1 is the unity of R and $f \in$ V, then

$$(1.f)\,(x) = 1f(x) = f(x)$$

i.e.,
$$1.f = f$$

Thus, V(R) satisfies all the properties of vector space, hence V (R) is a vector space.

Example 6. *Prove that the set of all real solutions of the differential equation* $\dfrac{d^2y}{dx^2} + w^2y = 0$ *forms a linear vector space over a real field.*

Sol. Let V be the set of real solutions of the given equation. Let y_1 and y_2 are two real solutions of the differential equation

$$\frac{d^2y}{dx^2} + w^2y = 0 \qquad \ldots(i)$$

Then, $\dfrac{d^2y_1}{dx^2} + w^2y_1 = 0$ and $\dfrac{d^2y_2}{dx^2} + w^2y_2 = 0$

Adding the above two equations, we get

$$\frac{d^2}{dx^2}(y_1 + y_2) + w^2(y_1 + y_2) = 0.$$

This shows that $(y_1 + y_2)$ satisfies the equation (*i*) that is, $y_1 + y_2$ is a real solution of (*i*).

Hence, $y_1 + y_2 \in$ V.

The function $y = 0$ is also a solution of (*i*), which is the null element. If y is a solution of (*i*) so is $-y$.

Also, for every real solution y, αy where α is a real number, is a real solution of the equation (*i*),

Since $\dfrac{d^2}{dx^2}(\alpha y) + w^2(\alpha y) = \alpha\dfrac{d^2y}{dx^2} + \alpha w^2y = \alpha\left[\dfrac{d^2y}{dx^2} + w^2y\right] = \alpha.0 = 0$

$\therefore$ $\alpha y \in$ V

$$(\alpha + \beta)\left\{\frac{d^2y}{dx} + w^2y\right\} = \frac{d^2}{dx^2}\{(\alpha + \beta)\,y\} + w^2\{(\alpha + \beta)\,y\}$$

$$= (\alpha + \beta)\frac{d^2y}{dx^2} + (\alpha + \beta)w^2y = \alpha\frac{d^2y}{dx^2} + \beta\frac{d^2y}{dx^2} + w^2\alpha y + w^2\beta y$$

$$= \alpha\left(\frac{d^2y}{dx^2} + w^2y\right) + \beta\left(\frac{d^2y}{dx^2} + w^2y\right).$$

Thus, all the axioms of vector space are satisfied, hence the set of real solution of the given equation is a vector space.

Example 7. *Given R × R with usual addition and multiplication is defined by* $\alpha(x, y) = (\alpha x, \alpha^2y)$. *Show that R × R is not a vector space over R.*

Sol. R × R with usual addition is an abelian group.

Now, $(\alpha + \beta)\,(x, y) = ((\alpha + \beta)\,x, (\alpha + \beta)^2\,y)$

$$= (\alpha x + \beta x, \alpha^2y + \beta^2y + 2\alpha\beta y)$$

and $$\alpha(x, y) + \beta(x, y) = (\alpha x, \alpha^2 y) + (\beta x, \beta^2 y)$$
$$= (\alpha x + \beta x, \alpha^2 y + \beta^2 y)$$

These show that
$$(\alpha + \beta)(x, y) \neq \alpha(x, y) + \beta(x, y).$$

Hence, R × R is not a vector space over R.

Example 8. *Let R^+ be the set of all positive real numbers. Addition and scalar multiplication in R^+ are defined as follows:*
$$a + b = ab \text{ for all } a, b \in R^+;$$
$$\alpha a = a^\alpha \text{ for all } a \in R^+ \text{ and } \alpha \in R.$$

Show that R^+ is a real vector space.

Sol. $(R^+, +)$ is an abelian group with identity 1.

Now $$\alpha(a + b) = \alpha(ab) = (ab)^\alpha \quad \text{[as defined in the problem]}$$
$$= a^\alpha b^\alpha = \alpha a + \alpha b$$
$$(\alpha + \beta)\, a = a^{\alpha+\beta} = a^\alpha a^\beta = \alpha a + \beta a$$
$$\alpha(\beta a) = \alpha a^\beta = (a^\beta)^\alpha = a^{\beta\alpha} = a^{\alpha\beta} = (\alpha\beta)a$$

Also $$1a = a1 = a.$$

Thus, all the axioms are satisfied in R^+, hence R^+ is a vector space over R.

1.19 PROPERTIES OF VECTOR SPACES

In a vector space V over a field F, if 0 is the zero vector of V, then

(*i*) $a0 = 0 \ \forall\, a \in F$

(*ii*) $0x = 0 \ \forall\, x \in V, 0 \in F$

(*iii*) $a(-x) = -(ax) \ \forall\, a \in F$ and $\forall\, x \in V$

(*iv*) $(-a)\, x = -(ax), \ \forall\, a \in F$ and $\forall\, x \in V$

(*v*) $a(x - y) = ax - ay, \ \forall\, a \in F$ and $\forall\, x, y \in V$

(*vi*) $ax = 0$ implies $a = 0$ or $x = 0$

Proof. (*i*) We have $$a0 = a(0 + 0) \qquad [\because 0 = 0 + 0]$$
$$= a0 + a0$$

Since $$a0 \in V \text{ and } 0 + a0 = a0$$

∴ $$0 + a0 = a0 + a0$$

Since V is an abelian group under addition, therefore, by right cancellation law in V, we obtain
$$0 = a0 \ \forall\, a \in F. \qquad [\because \ a0 = 0]$$

(*ii*) We have $$0x = (0 + 0)x \qquad [\because \ 0 + 0 = 0]$$
$$= 0x + 0x \ \forall\, x \in V.$$

Hence, $$0 + 0x = 0x + 0x \qquad [\because \ 0x \in V \text{ and } 0 + 0x = 0x]$$

∴ $$0 = 0x \qquad \text{[by right cancellation law]}$$

(*iii*) Since, $$a[x + (-x)] = ax + a(-x)$$

i.e., $$a0 = ax + a(-x) \qquad [\because \ x - (-x) = 0]$$

i.e., $$0 = ax + a(-x) \qquad [\because \ a0 = 0]$$

Hence, $a(-x)$ is the additive inverse of ax

i.e., $$a(-x) = -(ax)$$

(*iv*) We have $[a + (-a)]\, x = ax + (-a)x$

i.e., $0x = ax + (-a)x$

i.e., $0 = ax + (-a)x$ $[\because \ a + (-a) = 0]$

Hence, $(-a)\, x$ is the additive inverse of ax

i.e., $(-a)\, x = -(ax)$ $[0x = 0]$

(*v*) We have $a(x - y) = a\{x + (-y)\} = ax + a(-y)$

$= ax + [-(ay)]$

$= ax - ay.$ $[\because \ a(-y) = -(ay)$, by (*iii*)$]$

(*vi*) Let us consider $a \neq 0,$

Since a is a non-zero element in F, it has its inverse in F. Let a^{-1} is the inverse of a

$\because$ $ax = 0$

$\therefore$ $a^{-1}(ax) = a^{-1}(0)$

or $(a^{-1}a)\, x = 0$

or $1x = 0$

i.e., $x = 0.$ $[\because \ a^{-1}a = 1]$

Next, let $x \neq 0$. Then it requires to show that $a = 0$.

If possible, let $a \neq 0$, then a^{-1} exists,

$\therefore$ $ax = 0$

$\Rightarrow$ $a^{-1}(ax) = a^{-1}0$

$\Rightarrow$ $(a^{-1}a)\, x = 0$

$\Rightarrow$ $1\, x = 0$

$\Rightarrow$ $x = 0.$

Thus, we arrive at a contradiction that $x \neq 0$,

Hence, $a = 0.$

Thus, $x \neq 0$ and $ax = 0 \Rightarrow a = 0.$

1.19.1 Deductions

Let V be a vector space over a field F.

(*a*) If $a, b \in$ F and x be a non-zero element of V, then $ax = bx \Rightarrow a = b.$

(*b*) If $x, y \in$ V and a be a non-zero element of F, then $ax = ay \Rightarrow x = y.$

Proof. (*a*) Given $ax = bx$

i.e., $ax - bx = 0$

or $(a - b)x = 0$

$\therefore$ $a - b = 0$ $[\because \ x \neq 0]$

i.e., $a = b.$

(*b*) Given $ax = ay$

or $ax - ay = 0$

or $a(x - y) = 0$

$\because$ $a \neq 0,$

$\therefore$ $x - y = 0,$

i.e., $x = y.$

TEST YOUR KNOWLEDGE

1. Prove that R^2 is a vector space.
2. Prove that R^3 is a vector space.
3. V is the set of all ordered triplets of real numbers of the form $(0, y, z)$; addition and multiplication in V are defined as,
$$(0, y, z) \oplus (0, y', z') = (0, y + y', z + z')$$
and $$c \cdot (0, y, z) = (0, 0, cz),$$
prove that V is a vector space.
4. Verify that the set of all real continuous functions in an interval $[a, b]$ forms a linear vector space over a field (real).
5. Verify that the set of all real functions f such that $f(x + 1) = f(x)$ is a vector space over the field of real numbers.
6. Prove that the set V of all functions from R to R in which addition and multiplication are defined as $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha[f(x)]$ for all $f, g \in V$, is a vector space over R.
7. Prove that the set of all ordered pairs of real numbers (x, y) with the operations $(x, y) \oplus (x', y') = (x + x', y + y')$ and $c. (x, y) = (0, 0)$ is not a vector space.
8. Let V be the set of all ordered pairs (x, y) of real numbers and let F be the field of real numbers. Prove that V is not a vector space over the field of real numbers under addition and scalar multiplication defined as,
   (*i*) $$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$ $$c(x, y) = (|c|x, |c|y)$$
   (*ii*) $$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$ $$c(x, y) = (c^2x, c^2y).$$
9. Let V be the set of all ordered pairs (x, y) of real numbers and let F be the field of real numbers. Addition and multiplication in V are defined as,
$$(x, y) + (x_1, y_1) = (3y + 3y_1, -x - x_1),$$
$$a(x, y) = (3ay, -ax).$$
Show that V, with these operations, is not a vector space over F.
10. Let $V = \{a + b\sqrt{2} : a, b \in Q\}$. Show that V is a vector space over Q under addition and multiplication.
11. Show that the following are not vector space :
   (*i*) $R \times R$ with usual addition and scalar multiplication defined by $\alpha(a, b) = (0, \alpha b)$
   (*ii*) $R \times R$ with addition defined by $(a, b) + (c, d) = (ac, bd)$ and usual scalar multiplication.
12. Let V denote the set of all solutions of the differential equation $5\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3y = 0.$
Show that V is a vector space over R.
13. Let $V = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in R\right\}$ over R. Prove that the set V is a vector space under the usual addition and scalar multiplication in matrices.
14. Let V be the set of pairs (x, y) of real numbers and let F be the field of real numbers. Addition and scalar multiplication are defined as
$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c(x, y) = (cx, 0).$$
Is V, with these operations, a vector space?

1.20 VECTOR SUBSPACES

Let V be a vector space over the field F then the non-empty subset W of V is said to be the vector subspace if W itself is a vector space over F under the operations of V.

Thus, every vector space has at least two subspaces.

(*i*) V is a subspace of V (*ii*) {0} is a subspace of V.

These two vector subspaces of V are called **trivial subspaces.** All other vector subspaces of V are called **non-trivial subspaces.**

e.g., let W be subset of R^3 consisting of all vectors of the form $(a, b, 0)$, where a, b are any real numbers of the field F, with the usual operations of vector addition and scalar multiplication.

To check if W is a subspace of R^3, we have to check whether the axioms of vector spaces hold.

Let $x = (x_1, x_2, 0)$ and $y = (y_1, y_2, 0)$ be two vectors.

Then $x + y = (x_1, x_2, 0) + (y_1, y_2, 0) = (x_1 + y_1, x_2 + y_2, 0)$

is in W, since the third component is zero. Also if c is a scalar, then

$$cx = c(x_1, x_2, 0) = (cx_1, cx_2, 0) \text{ is in W.}$$

Thus vector addition and scalar multiplication are binary operations in W, *i.e.*, W is closed under these two operations. Other properties, of vector space, can also be verified. Hence, W is a subspace of R^3.

1.21 THEOREMS

(1) *Let V be a vector space over a field F. A non-empty subset W of V is a subspace of V if and only if, W is closed with respect to vector addition and scalar multiplication in V.*

Proof. Let W be a subspace of V. Then W itself is a vector space and hence W is closed w.r.t. vector addition and scalar multiplication.

Conversely, let W be a non-empty subset of V

such that $u, v \in W \Rightarrow u + v \in W$

and $u \in W$ and $\alpha \in F \Rightarrow \alpha u \in W$.

We have to prove that W is subspace of V. Since W is non-empty, there exists an element $u \in W$

$\therefore$ $0u = 0 \in W$

Also $v \in W \Rightarrow (-1)\, v = -v \in W$.

Thus, W contains 0 and the additive inverse of each of its element.

Hence, W is an additive subgroup of V.

Also, $u \in W$ and $\alpha \in F \Rightarrow \alpha u \in W$

since the elements of W are the elements of V, the other axioms of a vector space are true in W.

Hence W is a subspace of V.

(2) *Let V be a vector space over a field F. A non-empty subset W of V is a subspace of V if and only if, $u, v \in W$ and $\alpha, \beta \in F$, implies $\alpha u + \beta v \in W$.*

Proof. Let W be a subspace of V. Let $u, v \in W$ and $\alpha, \beta \in F$.

Then by Theorem 1, αu and $\beta v \in W$ and hence $\alpha u + \beta v \in W$.

Conversely, Let $u, v \in W$ and $\alpha, \beta \in F \Rightarrow \alpha u + \beta v \in W$.

Taking $\alpha = \beta = 1$, we get $u, v \in W \Rightarrow u + v \in W$.

Taking $\beta = 0$, we get $\alpha \in F$ and $u \in W \Rightarrow \alpha u \in W$.

Hence by Theorem 1, W is a subspace of V.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the set W of ordered triad $(a_1, a_2, 0)$ where $a_1, a_2, \in F$, a field, is a sub-space of V_3 over F.*

Sol. Let $u = (a_1, a_2, 0)$ and $v = (b_1, b_2, 0)$ belong to W, where $a_1, a_2, b_1, b_2 \in F$.

If a and b be any two elements of F, we have

$$\begin{aligned} au + bv &= a(a_1, a_2, 0) + b(b_1, b_2, 0) \\ &= (aa_1, aa_2, 0) + (bb_1, bb_2, 0) \\ &= (aa_1 + bb_1, aa_2 + bb_2, 0) \in W \end{aligned}$$

Since $(aa_1 + bb_1), (aa_2 + bb_2) \in F$.

Therefore, W is a vector subspace of V_3 over F.

Example 2. *Let $V = \{(x, y, z) : x, y, z \in R\}$, where R is the field of real numbers. Show that, if $W = \{(x, y, z) : x - 3y + 4z = 0\}$, it is a subspace of V over R.*

Sol. Let $u, v \in W$, then we may write

$$u = (3y_1 - 4z_1, y_1, z_1), \qquad [\because \; x = 3y - 4z]$$

and

$$v = (3y_2 - 4z_2, y_2, z_2)$$

If $a, b \in R$, we have

$$\begin{aligned} au + bv &= a(3y_1 - 4z_1, y_1, z_1) + b(3y_2 - 4z_2, y_2, z_2) \\ &= (3ay_1 - 4az_1, ay_1, az_1) + (3by_2 - 4bz_2, by_2, bz_2) \\ &= [3(ay_1 + by_2) - 4(az_1 + bz_2), ay_1 + by_2, az_1 + bz_2] \\ &= (3l - 4m, l, m) \in W \end{aligned}$$

where $l = ay_1 + by_2 \in R$ and $m = az_1 + bz_2 \in R$.

Therefore, $a, b \in R$ and $u, v \in W \Rightarrow au + bv \in W$.

Hence, V is a vector sub-space of V in R.

Example 3. *Let $W = \{f : f \in F[x]$ and $f(a) = 0\}$ i.e., W is the set of all polynomials in $F[x]$ having 'a' as a root, where $a \in F$. Prove that W is a subspace over F.*

Sol. If a is a root of $f(x) = 0$ then $x - a \in W$ and hence W is non-empty.

Let $\quad f, g \in F[x]$

and $\quad \alpha, \beta \in F$.

To prove $\quad \alpha f + \beta g \in W$,

we have to show that a is a root of $\alpha f + \beta g = 0$.

Now $\quad (\alpha f + \beta g)(a) = \alpha f(a) + \beta g(a) = \alpha 0 + \beta 0 = 0$.

Hence, a is a root of $\alpha f + \beta g = 0$.

$\therefore \quad \alpha f + \beta g \in W$ and hence, W is a subspace of $F[x]$.

Example 4. *Prove that the intersection of two subspaces of a vector space is a subspace.*

Sol. Let A and B be two subspaces of a vector space V over a field F.

To prove, $A \cap B$ is a subspace of V.

Now, $0 \in A \cap B$ and hence $A \cap B$ is non-empty.

Let $u, v \in A \cap B$ and $\alpha, \beta \in F$.

Then $u, v \in A$

and $u, v \in B$.

$\therefore$ $\alpha u + \beta v \in A$ and $\alpha u + \beta v \in B$. (since A and B are subspaces)

$\therefore$ $\alpha u + \beta v \in A \cap B$.

Hence $A \cap B$ is a subspace of V.

Example 5. *Prove that the union of two subspaces of a vector space need not be a subspace.*

Sol. Let

$$A = \{(a, 0, 0); a \in R\}$$
$$B = \{(0, b, 0); b \in R\}$$

Let $u = (1, 0, 0)$ and $v = (a_2, 0, 0)$,

where $u, v \in A$ and $\alpha, \beta \in R$.

Then

$$\alpha u + \beta v = \alpha(a_1, 0, 0) + \beta(a_2, 0, 0)$$
$$= (\alpha a_1, 0, 0) + (\beta a_2, 0, 0)$$
$$= (\alpha a_1 + \beta a_2, 0, 0) \in A.$$

Hence, A is a subspace of R^3.

Similarly, it can be proved that B is a subspace of R^3.

But $A \cup B$ is not a subspace of R^3.

for $(1, 0, 0)$ and $(0, 1, 0) \in A \cup B$,

but $(1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin A \cup B$.

Hence, union of two subspaces of a vector space is not a subspace.

Example 6. *Let W be the set of all points in R^3 satisfying the equation $ax + by + cz = 0$. Show that W is a subspace of R^3.*

Sol. Let $u = (a_1, b_1, c_1)$ and $v = (a_2, b_2, c_2) \in W$ and $\alpha, \beta \in R$.

Then, we have

$$aa_1 + bb_1 + cc_1 = 0$$

and

$$aa_2 + bb_2 + cc_2 = 0$$

Hence,

$$\alpha(aa_1 + bb_1 + cc_1) + \beta(aa_2 + bb_2 + cc_2) = 0$$

i.e.,

$$a(\alpha a_1 + \beta a_2) + b(\alpha b_1 + \beta b_2) + c(\alpha c_1 + \beta c_2) = 0$$

i.e., $\alpha u + \beta v \in W$, so that W is a subspace of R^3.

Example 7. *Show that $S = \{(x, y, z) : x^2 + y^2 = z^2\}$ is not a subspace of R^3.*

Sol. Let $u, v \in S$, then we have

$$u = \left(\sqrt{z_1^2 - y_1^2}, y_1, z_1\right) \text{ and } v = \left(\sqrt{z_2^2 - y_2^2}, y_2, z_2\right)$$

If $\alpha, \beta \in R$

then

$$\alpha u + \beta v = \alpha\left(\sqrt{z_1^2 - y_1^2}, y_1, z_1\right) + \beta\left(\sqrt{z_2^2 - y_2^2}, y_2, z_2\right)$$
$$= \alpha\left(\sqrt{z_1^2 - y_1^2}, \alpha y_1, \alpha z_1\right) + \beta\left(\sqrt{z_2^2 - y_2^2}, \beta y_2, \beta z_2\right)$$
$$= \left(\alpha\sqrt{z_1^2 - y_1^2} + \beta\sqrt{z_2^2 - y_2^2}, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2\right) \notin S$$

$\therefore$ S is not a vector subspace of R^3.

Example 8. *Prove that a set of polynomials of degree 2 need not be a subspace.*

Sol. Let V be the set of all polynomials of degree exactly equal to 2; V is a subset of P_2 (polynomials of degree 2), but it is not a subspace of P_2, since the sum of the polynomials $-ax^2 + bx + c$ and $ax^2 + 2kx + d$ being a polynomial of degree 1, is not in V.

Example 9. *If a vector space V is the set of real valued continuous functions over R then show that the set W of solutions of the differential equation* $3\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 4y = 0$ *is a subspace of V.*

Sol. Given that the set W contains the solutions of the given equation,

i.e.,
$$W = \left\{y : 3\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 4y = 0\right\} \quad \text{where } y = f(x)$$

$\because$ $y = 0$ satisfies the given equation, hence $0 \in W$.

If y_1, y_2 be any two elements of W, then we have,

$$3\frac{d^2y_1}{dx^2} + 11\frac{dy_1}{dx} - 4y_1 = 0 \quad ...(1)$$

and
$$3\frac{d^2y_2}{dx^2} + 11\frac{dy_2}{dx} - 4y_2 = 0 \quad ...(2)$$

Now for $p, q \in R$, we obtain,

$$3\frac{d^2}{dx^2}[(py_1) + (qy_2)] + 11\frac{d}{dx}[py_1 + qy_2] - 4(py_1 + qy_2)$$
$$= 3p\frac{d^2y_1}{dx^2} + 3q\frac{d^2y_2}{dx^2} + 11p\frac{dy_1}{dx} + 11q\frac{dy_2}{dx} - 4py_1 - 4qy_2$$
$$= p\left(3\frac{d^2y}{dx^2} + 11\frac{dy_1}{dx} - 4y_1\right) + q\left(3\frac{d^2y_2}{dx^2} + 11\frac{dy_2}{dx} - 4y_2\right) = p.0 + q.0 = 0$$

[Using (1) and (2)]

Hence, $(py_1 + qy_2)$ satisfies the given equation.

$\therefore$ $(py_1 + qy_2) \in W$.

Thus the set W is a subspace of V.

Example 10. *Let A and B be subspaces of a vector space V. Then $A \cap B = \{0\}$ iff every vector $v \in A + B$ can be uniquely expressed in the form $v = a + b$, where $a \in A$ and $b \in B$.*

Sol. Let $A \cap B = \{0\}$

and $v \in A + B$.

Let $v = a_1 + b_1 = a_2 + b_2$,

where $a_1, a_2 \in A$ and $b_1, b_2, \in B$.

Then $a_1 - a_2 = b_2 - b_1$.

But $a_1 - a_2 \in A$ and $b_2 - b_1 \in B$.

Hence, $a_1 - a_2, b_2 - b_1 \in A \cap B$.

Since $A \cap B = \{0\}$,

$a_1 - a_2 = 0$

and $b_2 - b_1 = 0$

so that $a_1 = a_2$
and $b_1 = b_2$.

Hence, the expression of v in the form $a + b$, where $a \in A$ and $b \in B$ is unique.

Conversely, let us suppose that any element in A + B can be uniquely expressed in the form $a + b$, where $a \in A$ and $b \in B$.

Then we claim that $A \cap B = \{0\}$.

If $A \cap B \neq \{0\}$, let $v \in A \cap B$ and $v \neq 0$.

Then $0 = v - v = 0 + 0$.

Thus 0 has been expressed in the form $a + b$ in two different ways which is a contradiction.

Hence, $A \cap B = \{0\}$.

TEST YOUR KNOWLEDGE

1. Show that the following subsets of R^3 are subspaces:
   (i) $\{(a, 0, c) : a, c \in R\}$ (ii) $\{(a, b, c) : a = b = c\}$ (iii) $\{(a, b, c) : a = b + c\}$.
2. Show that each of the following subsets of V_3 (R) is not a subspace:
   (i) $S = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ (ii) $S = \{(x, y, z) : x + y + z = 1\}$
3. Which of the following subsets of R^3 are subspaces of R^3?
   The set of all vectors of the form
   (i) $(a, b, 3)$ (ii) (a, b, c) where $c > 0$
   (iii) (a, b, c), where $b = 2a + 1$, (iv) (a, b, c), where $a = c = 0$.
4. Which of the following subsets of P_2 (polynomials of degree 2) are subspaces? The set of all polynomials of the form
   (i) $a_2x^2 + a_1x + a_0$, where $a_0 = 0$ (ii) $a_2x^2 + a_1x + a_0$, where $a_0 = 2$
   (iii) $a_2x^2 + a_1x + a_0$, where $a_2 + a_1 = a_0$ (iv) $a_2x^2 + a_1x + a_0$, where $a_2 + a_1 + a_0 = 2$.
5. Which of the following subsets of the vector space M_{23} are subspaces? The set of all matrices of the form
   (i) $\begin{bmatrix} l & m & n \\ p & q & r \end{bmatrix}$, where $l = 2n + 1$ (ii) $\begin{bmatrix} 0 & 1 & a \\ b & c & 0 \end{bmatrix}$
   (iii) $\begin{bmatrix} l & m & n \\ p & q & r \end{bmatrix}$, where $l + n = 0$ and $m + p + r = 0$.
6. Show that for the set $S \in R_{2\times 2}$, where S is defined by $S = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_{2\times 2}; a + b = 0\right\}$, S is a subspace of $R_{2\times 2}$.
7. Show that for the subset $W \in R^3$, where $W = \{(a, b, c) \in R^3 ; b = c = 0\}$, W is a subspace of R^3.
8. Show that for the subset $W \in R^3$, where $W = \{(x, y, z) \in R^3; xy = z\}$, W is not a subspace of R^3.
9. Let $W = \{(z, 2y, 3z) : x, y, z \in R\}$, where R is the field of real numbers. Prove that W is a subspace of V_3 over R.
10. Let R^3 be the usual three dimensional vector space over the field R of real numbers. Prove that $P = \{(x, y, z) \in R^3, 2x - y + 3z = 0\}$ is a subspace of R^3.

Answers

3. (*iii*) 4. (*i*), (*iii*) 5. (*iii*).

1.22 DIRECT SUM

Let A and B be two subspaces of a vector space V. Then V is called the *direct sum* of A and B if

(*i*) $A + B = V$ (*ii*) $A \cap B = \{0\}$.

If V is the direct sum of A and B then it is written as $V = A \oplus B$.

Remark. $V = A \oplus B$ iff, every element of V can be uniquely expressed in the form $a + b$, where $a \in A$ and $b \in B$.

Example 1. In $V_3(R)$, let $A = \{(a, b, 0) : a, b \in R\}$ and $B = \{(0, 0, c) : c \in R\}$.

Then A and B are subspaces of V and $A \cap B = \{0\}$.

Let $v = (a, b, c) \in V_3(R)$.

Then $v = (a, b, 0) + (0, 0, c)$.

so that $A + B = V_3(R)$.

Hence, $V_3(R) = A \oplus B$.

Example 2. In $M_3(R)$, let A be the set of all matrices of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, and B be the set of all matrices of the form $\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$, then A and B are subspaces of $M_2(R)$ and

$A \cap B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A + B = M_2(R)$.

Hence, $M_2(R) = A \oplus B$.

Note. If $W_1, W_2, W_3, \ldots, W_n$ be subspaces of the vector space V over a field F then their sum is called *linear sum*, written as $W_1 + W_2 + W_3 + \ldots + W_n$, and is defined by

$W_1 + W_2 + W_3 + \ldots + W_n = \{x_1 + x_2 + x_3 + \ldots x_n : x_i \in W_i, i = 1, 2, \ldots, n\}$.

We state and prove a theorem on linear sum as follows.

1.23 THEOREM

The linear sum $W_1 + W_2 + W_3 + \ldots + W_n$ of subspaces W_i $(i = 1, 2, \ldots, n)$ of a vector space V is also a subspace of V.

Proof. Since each W_i is a subspace, so each W_i contains the zero vector.

$$\therefore \quad 0 + 0 + 0 + \ldots. + 0 \in W_1 + W_2 + W_3 + \ldots + W_n.$$

Hence, the sum $W_1 + W_2 + \ldots + W_n$ is not empty.

Let x and y be any two elements of the sum

so that $x = x_1 + x_2 + x_3 + \ldots + x_n$

and $y = y_1 + y_2 + y_3 + \ldots y_n$,

where $x_i, y_i \in W_i$ $(i = 1, 2, 3, \ldots, n)$.

Then for any scalars λ and $\mu \in F$,

$$\lambda x + \mu y = \lambda(x_1 + x_2 + \ldots + x_n) + \mu(y_1 + y_2 + \ldots + y_n)$$

Since W_i is a subspace of V, therefore,

$$(\lambda x_i + \mu y_i) \in W_i \text{ for each } i = 1, 2, 3, \ldots, n.$$

$$\therefore \quad (\lambda x + \mu y) \in (W_1 + W_2 + W_3 + \ldots + W_n) \text{ and}$$

hence $\quad W_1 + W_2 + \ldots + W_n \subseteq V.$

Thus, the sum of the subspaces forms a subspace of V.

Note. From above theorem, it follows that if S is any collection of vectors in V, then there exists a smallest subspace of V which contains S, that is, a subspace which contains S and which is contained in every other subspace containing S.

This leads to the following definition:

Definition. Let S be a set of vectors in a vector space V. The *subspace spanned* by S is defined to be the intersection W of all subspaces of V which contain S. When S is a finite set of vectors, $S = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$, we shall simply call W the *subspace spanned by the vectors* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$.

Earlier we constructed subspace W of a vector space V in R^2 over a field of scalars F by considering two vectors $\boldsymbol{u}$ and v of V and two scalars $c_1, c_2 \in F$ such that $w_1 = c_1\boldsymbol{u} + c_2\boldsymbol{v}$, where $w_1 \in W$, which is known as linear combination of the vectors u and v. The construction that carried one for two vectors can now easily be performed for more than two vectors. We now give a formal definition.

1.24 LINEAR COMBINATION OF VECTORS

Let V be a vector space over a field F. Let $v_1, v_2, \ldots v_k$ be the vectors of V. A vector v in V is called a linear combination of $v_1, v_2, \ldots, v_k$ if

$$v = c_1v_1 + c_2v_2 + \ldots, + c_kv_k$$

for some real scalars $c_1, c_2, \ldots, c_k$.

In figure 1, the linear combination of two vectors v_1 and v_2 in R^2 is shown geometrically.

Figure 2 shows v as a linear combination of v_1, v_2 and v_3

such that $\quad v = v_1 + 2v_2 - v_3$

when, $\quad v_1 = (1, 2, 1), v_2 = (1, 0, 2), v_3 = (1, 1, 0)$

and $\quad v = (2, 1, 5)$

i.e., $\quad (2, 1, 5) = (1, 2, 1) + 2(1, 0, 2) - (1, 1, 0).$

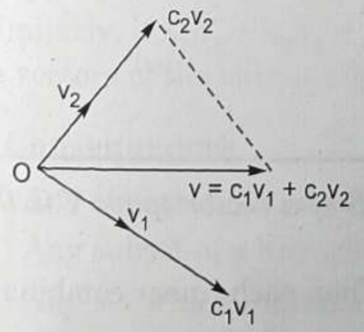

Figure 1

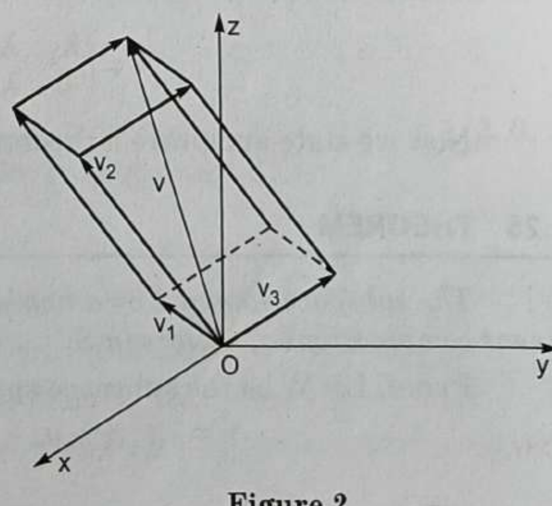

Figure 2

ILLUSTRATIVE EXAMPLES

Example 1. *Express $v = (5, 9)$ as a linear combination of the vectors $v_1 = (1, 2)$ and $v_2 = (2, 3)$.*

Sol. Let $v = c_1 v_1 + c_2 v_2$

i.e., $(5, 9) = c_1(1, 2) + c_2(2, 3)$, where c_1 and c_2 are scalars.

$$= (c_1 + 2c_2, 2c_1 + 3c_2).$$

Equating the components from both sides, we have

$$c_1 + 2c_2 = 5 \quad \text{and} \quad 2c_1 + 3c_2 = 9.$$

Solving these equations, we get $c_1 = 3$, $c_2 = 1$

Hence, $(5, 9) = 3(1, 2) + 1(2, 3)$.

Example 2. *Express $v = (3, 4, 5)$ as a linear combination of $v_1 = (1, 2, 3)$, $v_2 = (2, 3, 4)$ and $v_3 = (4, 3, 2)$.*

Sol. $v = c_1 v_1 + c_2 v_2 + c_3 v_3$.

i.e., $(3, 4, 5) = c_1(1, 2, 3) + c_2(2, 3, 4) + c_3(4, 3, 2)$

$$= (c_1 + 2c_2 + 4c_3, 2c_1 + 3c_2 + 3c_3, 3c_1 + 4c_2 + 2c_3)$$

Equating the components from both sides, we get

$$c_1 + 2c_2 + 4c_3 = 3, \quad 2c_1 + 3c_2 + 3c_3 = 4 \quad \text{and} \quad 3c_1 + 4c_2 + 2c_3 = 5.$$

Solving these equations, we get $c_1 = \frac{1}{2}, c_2 = \frac{3}{4}$ and $c_3 = \frac{1}{4}$

Hence, $(3, 4, 5) = \frac{1}{2}(1, 2, 3) + \frac{3}{4}(2, 3, 4) + \frac{1}{4}(4, 3, 2)$.

Note. If $S = \{v_1, v_2, ..., v_k\}$ is a set of vectors in a vector space V, then the set of all vectors in V that are linear combinations of the vectors in S is denoted by *Span S or Span* $\{v_1, v_2, ..., v_n\}$.

As an example, let the set S of 2×3 matrices be given by

$$S = \left\{\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}\right\}.$$

Then span S is the set in M_{23} consisting of all vectors of the form

$$\lambda_1\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix} + \lambda_2\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix} + \lambda_3\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix} + \lambda_4\begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}\lambda_1&\lambda_2&0\\0&\lambda_3&\lambda_4\end{pmatrix}, \text{ where } \lambda_1, \lambda_2, \lambda_3, \lambda_4 \text{ are real scalars.}$$

Now we state and prove a theorem.

1.25 THEOREM

The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

Proof. Let W be the subspace spanned by S. Then each linear combination

$$v = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_m v_m$$

of vectors $v_1, v_2, ..., v_m$ in S is also in W. Thus W contains the set L of all linear combinations of vectors in S. On the other hand, the set L contains S and hence the set L is non-empty.

If u, v belong to L then u and v can be expressed as s linear combination of the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, ... \boldsymbol{u}_n$ and $v_1, v_2, v_3, ..., v_m$ respectively.

i.e., $$\boldsymbol{u} = \mu_1\boldsymbol{u}_1 + \mu_2\boldsymbol{u}_2 + ... + \mu_n\boldsymbol{u}_n$$

and $$v = \lambda_1 v_1 + \lambda_2 u_2 + ... \lambda_m v_m.$$

Now for each scalar k,

$$kv + u = \sum_{i=1}^{m} (k\lambda_i) v_i + \sum_{i=1}^{n} \mu_j u_j.$$

Hence $kv + u$ belongs to L. Thus L is a subspace of V. Thus, we have shown that L is a subspace of V which contains S, and also that any subspace which contains S contains L.

Hence, it follows that L is the intersection of all subspaces containing S, that is, L is the subspace spanned by the set S.

1.26 LINEAR SPAN

Let V be a vector space over the field F and S be any non-empty subset of V. Then the *linear span* of S is defined as the set of all linear combinations of finite sets of elements of S. It is denoted by L(S).

Thus, $\quad L(S) = \{\lambda_1\boldsymbol{u}_1 + \lambda_2\boldsymbol{u}_2 + ... + \lambda_n\boldsymbol{u}_n, \boldsymbol{u}_i \in S, \lambda_i \in F (i = 1, 2, ..., n)\}$

L(S) is said to be *generated* or *spanned* by the set S and S is said to be the set of *generators* of L(S).

For example, the set $L(S) = \left\{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right\}$ of all 2×2 diagonal matrices is the linear span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

1.27 LINEAR DEPENDENCE AND INDEPENDENCE

Let V be a vector space over F. A subset S of V is said to be **linearly dependent** if ∃ distinct vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, ..., \boldsymbol{a}_n$ in S and scalars $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ in F, not all of them zero, such that

$$\lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2 + \lambda_3\,\boldsymbol{a}_3 + ... + \lambda_n\,\boldsymbol{a}_n = \mathbf{0}$$

Similarly, if $\lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2 + \lambda_3\,\boldsymbol{a}_3 + ... + \lambda_n\,\boldsymbol{a}_n = \mathbf{0}$ provided all $\lambda_i = 0, i = 1, 2, 3, ..., n$ then the vectors of the subset S are called **linearly independent.**

1.27.1 Consequences

(*i*) Any superset of a linearly dependent set is linearly dependent.

(*ii*) Any subset of a linearly independent set is linearly independent.

(*iii*) Any set which contains the **0** vector is linearly dependent.

(*iv*) If a vector can be expressed as a scalar multiple of another vector then the vectors are linearly dependent.

Example 3. *Prove that, in $V_3(R)$, the vectors (1, 2, 1), (2, 1, 0) and (1, – 1, 2) are linearly independent.*

Sol. Let $\lambda_1 (1, 2, 1) + \lambda_2 (2, 1, 0), + \lambda_3 (1, -1, 2) = (0, 0, 0)$

$$\Rightarrow \quad \lambda_1 + 2\lambda_2 + \lambda_3 = 0 \quad ...(1)$$
$$2\lambda_1 + \lambda_2 - \lambda_3 = 0 \quad ...(2)$$
$$\lambda_1 + 2\lambda_3 = 0 \quad ...(3)$$

Solving (1), (2) and (3) we get $\lambda_1 = \lambda_2 = \lambda_3 = 0$

Hence the given set of vectors is linearly independent.

Example 4. *In $V_3 (R)$, prove that the vectors (1, 4, – 2), (– 2, 1, 3) and (– 4, 11, 5) are linearly dependent.*

Sol. Let $\lambda_1 (1, 4, -2) + \lambda_2 (-2, 1, 3) + \lambda_3 (-4, 11, 5) = (0, 0, 0)$

$$\therefore \quad \lambda_1 - 2\lambda_2 - 4\lambda_3 = 0 \quad ...(1)$$
$$4\lambda_1 + \lambda_2 + 11\lambda_3 = 0 \quad ...(2)$$
$$-2\lambda_1 + 3\lambda_2 + 5\lambda_3 = 0 \quad ...(3)$$

From (1) and (2),

$$\frac{\lambda_1}{-18} = \frac{\lambda_2}{-27} = \frac{\lambda_3}{9}$$

$$\Rightarrow \quad \frac{\lambda_1}{-2} = \frac{\lambda_2}{-3} = \frac{\lambda_3}{1} = k \text{ (say)}$$

$$\therefore \quad \lambda_1 = -2k, \lambda_2 = -3k, \lambda_3 = k$$

Taking $k = 1$, $\lambda_1 = -2$, $\lambda_2 = -3$ and $\lambda_3 = 1$

which satisfy eqn. (3) also. Since λ_1, λ_2 and λ_3 non-zero scalars hence the given set of vectors is linearly dependent.

1.28 BASES AND DIMENSION

Let V be a vector space. A **basis** for V is a linearly independent set of vectors in V which spans the space V. The number of elements (vectors) present in the basis of a vector V is known as the **dimension** of V. The dimension of V is denoted by dim V.

The space V is *finite dimensional* if it has a finite basis. The vector space V is called n-dimensional, if in its basis contains n elements (vectors).

As an example, if S = {(1, 0, 0), (0, 1, 0), (0, 0, 1)} is the basis of the vector space R^3, then dimension of R^3 is 3, since its basis contains three linearly independent vectors.

Note. 1. A vector space which is not finitely generated is called an infinite dimensional space.

2. The vector space {0} is of zero dimension.

3. A vector space may have more than one basis, that is, basis of a vector space is not unique.

4. The set $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n\}$, of unit vectors, is a basis for E^n which is known as '*standard basis*' of E^n.

1.29 THEOREMS

(1) *There exists a basis for each finite dimensional vector space.*

Proof. Let V be a finite dimensional vector space over the field F. Then by definition V is linear span of the set S having finite number of vectors belonging to V.

Let $\quad S = \{\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_n\} \subseteq V$ and $V = L(S)$.

If we assume that no member of S is zero, that is $0 \notin S$, then it will not affect the generality of the theme, because the contribution of 0 in the linear combination of the elements of S is zero.

Thus $S \subseteq V \Rightarrow S$ is linearly independent or S is linearly dependent.

Let S be linearly independent, then S will be a basis of V, implies basis of a finite dimensional vector space exists.

Next, let S be linearly dependent.

We know that a set S of non-zero vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\}$ forming a vector space V over F is linearly dependent, if and only if, some vectors $a_k (k > 1)$ belonging to S is a linear combination of the preceding vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{k-1}$, in S.

Let $\boldsymbol{a}_k = \lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2 + \dots + \lambda_{k-1} \boldsymbol{a}_{k-1}$ ...(1)

Let $S_1 = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{k-1}, \boldsymbol{a}_{k+1}, \dots, \boldsymbol{a}_n\}$.

But $S = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{k-1}, \boldsymbol{a}_k, \boldsymbol{a}_{k+1}, \dots, \boldsymbol{a}_n\}$.

Hence, $S_1 \subseteq S \Rightarrow L(S_1) \subseteq L(S)$

But $L(S) = V$,

$\therefore$ $L(S_1) \subseteq V$ ...(2)

Now, we prove $V \subseteq L(S_1)$.

Let $\boldsymbol{a} \in V$ and

$\boldsymbol{a}$ = a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_k, \dots, \boldsymbol{a}_n$

= linear combination of $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{k-1}, \boldsymbol{a}_{k+1}, \dots, \boldsymbol{a}_n)$

= linear combination of the elements of $S_1 + \lambda_k$(*l.c.* of $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{k-1}$)

= *l.c.* of the elements of S_1

and hence $\boldsymbol{a} \in L(S_1)$

$\therefore$ $\boldsymbol{a} \in V \Rightarrow \boldsymbol{a} \in L(S_1)$, hence $V \subseteq L(S_1)$.

Thus, $L(S_1) \subseteq V$ and $V \subseteq L(S_1) \Rightarrow V = L(S_1)$.

Let S_1 be linearly independent, then S_1 will be basis of V, implies basis of a finite dimensional vector space exists.

If S_1 be linearly dependent, then as before we get a set of $(n - 2)$ vectors which generate the vector space V(F).

Continuing this process, we get a linearly independent subset of S, which generates V(F) and is therefore a basis of V(F).

(2) *Any vector of a given vector space can be expressed as a linear combination of the vectors belonging to a given basis in one and only one way.*

Proof. Let us consider a set of vectors say, $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$, which form a basis of the vector space V_n. Let υ be a vector of V_n and υ can be expressed as a linear combination with the set of vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$ that is

$$\upsilon = \lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2 + \dots + \lambda_n \boldsymbol{a}_n \quad \text{...(1)}$$

where λ_i's are scalars. Since the sum of any two vectors, and the scalar multiplication of any vector of V_n is a vector of V_n, so υ is a vector of V_n.

If possible, let us now assume another representation of υ as,

$$\upsilon = \mu_1 \boldsymbol{a}_1 + \mu_2 \boldsymbol{a}_2 + \dots + \mu_n \boldsymbol{a}_n \quad \text{...(2)}$$

where μ_i's are scalars.

Therefore, from (1) and (2) we have

$$\lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2 + \ldots + \lambda_n \boldsymbol{a}_n = \mu_1 \boldsymbol{a}_1 + \mu_2 \boldsymbol{a}_2 + \ldots + \mu_n \boldsymbol{a}_n$$

or $$(\lambda_1 - \mu_1)\boldsymbol{a}_1 + (\lambda_2 - \mu_2)\boldsymbol{a}_2 + \ldots + (\lambda_n - \mu_n)\boldsymbol{a}_n = 0.$$

Since $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ form the basis of V_n, so $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are linearly independent.

$\therefore$ $$\lambda_1 - \mu_1 = 0, \lambda_2 - \mu_2 = 0, \ldots, \lambda_n - \mu_n = 0$$

i.e., $$\lambda_1 = \mu_1, \lambda_2 = \mu_2, \ldots, \lambda_n = \mu_n.$$

Hence, the representation of υ is unique.

(3) *Let V be a vector space which is spanned by a finite set of vectors $a_1, a_2, \ldots, a_m$. Then any independent set of vectors in V is finite and contains no more than m vectors.*

Proof. If we can show that every subset S of V, which contains more than m vectors is linearly dependent then it suffices to prove the Theorem.

Let S be such a set. In S, let us consider there are n distinct vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $n > m$. Since, $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ span V, there exist scalars λ_{ij} in F such that $\alpha_i = \sum_{i=1}^{m} \lambda_{ij} \boldsymbol{a}_i$

For any n scalars $c_1, c_2, \ldots, c_n$, we have

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \sum_{j=1}^{n} c_j \alpha_j = \sum_{j=1}^{n} c_j \sum_{i=1}^{m} \lambda_{ij} \boldsymbol{a}_i$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} (\lambda_{ij} c_j)\boldsymbol{a}_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \lambda_{ij} c_j \right) \boldsymbol{a}_i$$

Since $n > m$, there exist scalars $c_1, c_2, \ldots, c_n$, not all 0, such that

$$\sum_{j=1}^{n} \lambda_{ij} c_j = 0, 1 \le i \le m.$$

Hence, $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0.$

This proves that S is a linearly dependent set.

Corollary 1. *If V is a finite-dimensional vector space, then any two bases of V have the same (finite) number of elements.*

Proof. Since V is finite-dimensional, it has a finite basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\}$. By Theorem 3 above, every basis of V is finite and contains, no more than m elements. Thus if $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis, then $n \le m$. By the same argument, $m \le n$.

Hence, $$m = n.$$

In the above lines, we have shown that the number of elements in a basis is the same. But basis of a vector space may not be unique. We prove it by the following theorem.

(4) *The basis of a vector space V is not unique.*

Proof. Let $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\}$ be a basis of V.

Let $$\alpha = c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \ldots + c_m\boldsymbol{a}_m, \text{ and } c_m \ne 0,$$ (c_i'*s* are scalars).

Let us take another set of vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{m-1}, \alpha\}$.

Now, $$d_1\boldsymbol{a}_1 + d_2\boldsymbol{a}_2 + \ldots + d_{m-1}\boldsymbol{a}_{m-1} + d_m\alpha$$
$$= (d_1\boldsymbol{a}_1 + d_2\boldsymbol{a}_2 + \ldots + d_{m-1}\boldsymbol{a}_{m-1}) + d_m(c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \ldots + c_m\boldsymbol{a}_m)$$

(where d_i's are scalars)

$$= (d_1 + d_m c_1)\boldsymbol{a}_1 + (d_2 + d_m c_2)\boldsymbol{a}_2 + \dots + (d_{m-1} + d_m c_{m-1})\boldsymbol{a}_{m-1} + d_m c_m \boldsymbol{a}_m$$

since $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{m-1}, \boldsymbol{a}_m$ are linearly independent,

therefore, $\quad d_1\boldsymbol{a}_1 + d_2\boldsymbol{a}_2 + \dots + d_{m-1}\boldsymbol{a}_{m-1} + d_m\alpha = 0$

which is possible, when

$$d_1 + d_m c_1 = 0,\ d_2 + d_m c_2 = 0,\ \dots,\ d_m c_m = 0$$

i.e., when $\quad d_m = 0,\ d_{m-1} = 0,\ \dots,\ d_2 = 0,\ d_1 = 0, \qquad (\because\ c_m \neq 0).$

Hence, $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{m-1}, \alpha$ are linearly independent.

Next, let the vector space generated by the above mentioned vectors be denoted by V′.

Let us take a vector υ of V′.

Then
$$\begin{aligned}\upsilon &= \lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2 + \dots + \lambda_{m-1}\boldsymbol{a}_{m-1} + \lambda_m\alpha \\ &= \lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2 + \dots + \lambda_{m-1}\boldsymbol{a}_{m-1} + \lambda_m(c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \dots + c_m\boldsymbol{a}_m) \\ &= \mu_1\boldsymbol{a}_1 + \mu_2\boldsymbol{a}_2 + \dots + \mu_{m-1}\boldsymbol{a}_{m-1} + \lambda_m\boldsymbol{a}_m.\end{aligned}$$

where $\mu_1 = \lambda_1 + c_1\lambda_m,\ \mu_2 = \lambda_2 + c_2\lambda_m,\ \dots,$

$\mu_{m-1} = \lambda_{m-1} + c_{m-1}\lambda_m$ and $\mu_m = c_m\lambda_m$.

Hence, $\quad \upsilon \in V.$

Thus every vector of V′ is also a vector V.

Similarly, it can be shown that every vector of V is also a vector of V′.

Hence, it follows that V = V′.

Thus, $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{m-1}, \boldsymbol{a}_m$ and $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{m-1}, \alpha$ form two bases of the same vector space V. This proves the theorem.

We shall state now some useful theorems without proof.

(5) *If W is a subspace of a finite-dimensional vector space V, every linearly independent subset of W is finite and is part of a (finite) basis for W.*

(6) *If W is a proper subspace of a finite-dimensional vector space V, then W is finite-dimensional and dim W < dim V.*

(7) *If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$ is finite-dimensional and*

$$\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2).$$

Corollary 2. Let S be a linearly independent subset of a vector space V. If υ is a vector in V which is not in the subspace spanned by S, then the set obtained by joining υ to S is linearly independent.

Corollary 3. In a finite-dimension vector space V every non-empty linearly independent set of vectors is a part of a basis.

Example 5. *Show that the vectors (3, 0, 2), (7, 0, 9) and (4, 1, 2) form a basis for E^3.*

Sol. We have easily proved that the given vectors are linearly independent.

Now we have to show that the vectors span (E^3). To show it we require to prove that for any vector (a_1, a_2, a_3) in E^3, it is possible to find real scalars $\lambda_1, \lambda_2, \lambda_3$, such that

$$(a_1, a_2, a_3) = \lambda_1(3, 0, 2) + \lambda_2(7, 0, 9) + \lambda_3(4, 1, 2)$$

Now,
$$(a_1, a_2, a_3) = (3\lambda_1 + 7\lambda_2 + 4\lambda_3,\ \lambda_2,\ 2\lambda_1 + 9\lambda_2 + 2\lambda_3)$$

Equating the like components from both sides, we get

$$3\lambda_1 + 7\lambda_2 + 4\lambda_3 = a_1, \quad \lambda_2 = a_2 \quad \text{and} \quad 2\lambda_1 + 9\lambda_2 + 2\lambda_3 = a_3.$$

From these equations, real values of $\lambda_1, \lambda_2, \lambda_3$ can be found. Hence the given vectors form a basis for E^3.

Example 6. *Find the basis of all real solutions of the differential equation*

$$\frac{d^2y}{dx^2} + w^2y = 0.$$

Sol. All real solutions of the differential equation can be written as

$$y = a \sin wx + b \cos wx,$$

where a and b are real numbers. The functions $\sin wx$ and $\cos wx$ are two linearly independent solutions of the equation.

Hence $\{\sin wx, \cos wx\}$ is the basis.

Example 7. *Find a basis for E^3 that contains the vectors (1, 2, 0) and (2, 0, 1).*

Sol. Let the given vectors be

$$\boldsymbol{a}_1 = (1, 2, 0) \text{ and } \boldsymbol{a}_2 = (2, 0, 1).$$

The standard basis for E^3 is $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$,

where
$$\boldsymbol{e}_1 = (1, 0, 0), \boldsymbol{e}_2 = (0, 1, 0) \text{ and } \boldsymbol{e}_3 = (0, 0, 1)$$

$$\boldsymbol{a}_1 = (1, 2, 0) = 1\boldsymbol{e}_1 + 2\boldsymbol{e}_1 + 0\boldsymbol{e}_3$$

$\therefore$ $\boldsymbol{a}_1$ can replace any one vector from $\{e_1, e_2, e_3\}$ to form a new basis. Let us consider a new basis $\{\boldsymbol{a}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$.

Let
$$\boldsymbol{a}_2 = (2, 0, 1) = \lambda_1(1, 2, 0) + \lambda_2(0, 1, 0) + \lambda_3(0, 0, 1).$$

Equating the components from both sides, we get

$$\lambda_1 = 2,\ 2\lambda_1 + \lambda_2 = 0 \text{ and } \lambda_3 = 1$$

i.e.,
$$\lambda_1 = 2,\ \lambda_2 = -4,\ \lambda_3 = 1$$

$\therefore$
$$(2, 0, 1) = 2(1, 2, 0) - 4(0, 1, 0) + 1(0, 0, 1) = 2\boldsymbol{a}_1 - 4\boldsymbol{e}_2 + \boldsymbol{e}_3$$

$\therefore$ $\boldsymbol{a}_2$ can replace $\boldsymbol{e}_2$ or $\boldsymbol{e}_3$ from $\{\boldsymbol{a}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ to form a new basis.

Hence, the new basis is $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{e}_3\}$.

Example 8. *Prove that the vectors (1, 1, 1), (1, 1, 0) and (1, 0, 0) form a basis in E^3. Also prove that the vector (1, 3, 1) can replace any one of the three vectors of the basis to form a new basis.*

Sol. Let A be the matrix represented by the given vectors,

i.e.,
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \text{ Then } |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1.$$

Hence, the rank of A is 3 (since the matrix is non-singular).

Therefore, the vectors forming A are linearly independent. Now, we have to show that the vectors span E^3.

Let (a_1, a_2, a_3) be a vector of E^3 and let

$$(a_1, a_2, a_3) = \lambda_1(1, 1, 1) + \lambda_2(1, 1, 0) + \lambda_3(1, 0, 0) = (\lambda_1 + \lambda_2 + \lambda_3, \lambda_1 + \lambda_2, \lambda_1).$$

Equating the corresponding components from both sides, we get

$$\lambda_1 + \lambda_2 + \lambda_3 = a_1,\quad \lambda_1 + \lambda_2 = a_2 \quad \text{and} \quad \lambda_1 = a_3.$$

From these equations real values of $\lambda_1, \lambda_2, \lambda_3$ can be found.

Hence, the vectors span E^3.

$\therefore$ The given vectors form a basis for E^3.

Again, let $$(1, 3, 1) = \mu_1(1, 1, 1) + \mu_2(1, 1, 0) + \mu_3(1, 0, 0)$$
where μ_1, μ_2, μ_3 are scalars.

Equating like components from both sides, we have
$$\mu_1 + \mu_2 + \mu_3 = 1, \quad \mu_1 + \mu_2 = 3 \quad \text{and} \quad \mu_1 = 1$$
Solving these equations, we obtain
$$\mu_1 = 1, \mu_2 = 2, \mu_3 = -2.$$
Since μ_1, μ_2 and μ_3 are non-zero scalars and (1, 3, 1) is a non-null vector, hence (1, 3, 1) together with any two vectors of the given basis can form a new basis.

Example 9. *Find the dimension of the subspace spanned by the vectors (1, 0, 2), (2, 0, 1), (1, 0, 1) in $V_3(R)$.*

Sol. Let A be the matrix represented by the given vectors,

i.e., $$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Then $$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$\therefore$ A is singular. Hence the given three vectors forming A are linearly dependent and so the given three vectors do not form the basis of the subspace in $V_3(R)$.

Now, let $\lambda_1(1, 0, 2) + \lambda_2(2, 0, 1) = 0$

i.e., $$(\lambda_1 + 2\lambda_2, 0, 2\lambda_1 + \lambda_2) = 0 = (0, 0, 0).$$
Equating like components from both sides, we get
$$\lambda_1 + 2\lambda_2 = 0 \quad \text{and} \quad 2\lambda_1 + \lambda_2 = 0$$
These two equations have no other solutions except $\lambda_1 = 0$ and $\lambda_2 = 0$.

Since λ_1 and λ_2 are zero, so the vectors (1, 0, 2) and (2, 0, 1) are linearly independent.

So (1, 0, 2) and (2, 0, 1) form the basis of the subspace of V_3 (R).

Hence, the dimension of the basis is 2.

Example 10. *Show that the vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) and (1, 2, 3) generate the same space as generated by the vectors (1, 0, 0), (0, 1, 0) and (0, 0, 1).*

Sol. Let us consider the linear combination of the set of vectors (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 2, 3).
$$\begin{aligned} &\lambda_1(1, 0, 0) + \lambda_2(0, 1, 0) + \lambda_3(0, 0, 1) + \lambda_4(1, 2, 3) \\ &= (\lambda_1 + \lambda_4, \lambda_2 + 2\lambda_4, \lambda_3 + 3\lambda_4) \\ &= (\lambda_1 + \lambda_4)(1, 0, 0) + (\lambda_2 + 2\lambda_4)(0, 1, 0) + (\lambda_3 + 3\lambda_4)(0, 0, 1). \end{aligned}$$
Thus the two sets of vectors generate the same space.

Example 11. *If **u**, **v**, **w** be linearly independent vectors of V (F), where F is the field of real numbers, show that (**u** + **v**), (**v** + **w**), (**w** + **u**) are also linearly independent.*

Sol. Since $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent, therefore, their linear combination
$$\lambda \boldsymbol{u} + \mu \boldsymbol{v} + \upsilon \boldsymbol{w} = 0 \text{ implies, } \lambda = \mu = \upsilon = 0.$$

Now, $\lambda_1(u + v) + \lambda_2(v + w) + \lambda_3(w + u) = (\lambda_1 u + \lambda_2 v + \lambda_3 w) + (\lambda_3 u + \lambda_1 v + \lambda_2 w)$

Since u, v, w are linearly independent, therefore

$$\lambda_1 u + \lambda_2 v + \lambda_3 w = 0 \quad \text{and} \quad \lambda_3 u + \lambda_1 v + \lambda_2 w = 0$$

implies,
$$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Hence, $\lambda_1(u + v) + \lambda_2(v + w) + \lambda_3(w + u) = 0$

where $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

$\therefore$ The vectors $u + v$, $v + w$ and $w + u$ are linearly independent.

Example 12. *Show that* $S = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ *spans the subspace of* M_{22} *consisting of all symmetric matrices.*

Sol. An arbitrary symmetric matrix has the form $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, where a, b, c are any real numbers.

Let
$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{bmatrix}.$$

The solution is $\lambda_1 = a$, $\lambda_2 = b$, $\lambda_3 = c$

Since we have obtained a solution for every choice of a, b and c, we conclude that S spans the given subspace.

TEST YOUR KNOWLEDGE

1. Express $(3, -5)$ as a linear combination of $\boldsymbol{a} = (1, 2)$, $\boldsymbol{b} = (-3, 1)$ in V_2 (R).
2. Express the vector $(1, 7, -4)$ as a linear combination of the vectors $(1, -3, 2)$, $(2, -1, 1)$ in V_3 (R).
3. Express $(1, 1, 2)$ as a linear combination of $(0, 2, 1)$, $(2, 2, 4)$ in V_3 (R).
4. Show that the vector $(3, -4, 2)$ cannot be expressed as a linear combination of the vectors, $(2, 1, 5)$ and $(-1, 3, 2)$.
5. Show that the vector $(0, 2, 1)$ cannot be expressed as a linear combination of the vectors $(1, 1, 2)$ and $(2, 2, 4)$.
6. Can the matrix $M = \begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix}$ be expressed as the linear combination of the matrices
$$P = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}?$$
7. Show that the following vectors are linearly dependent.

   (*i*) $(4, 3), (8, 6)$ (*ii*) $\left(\frac{3}{2}, \frac{1}{3}, 1\right), (9, 2, 6)$

   (*iii*) $(1, 1, 2), (2, 1, 0), (0, 4, 1)$ (*iv*) $(1, 2, 3), (4, 1, 5), (-4, 6, 2)$.
8. Show that the set of matrices $\left\{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}\right\}$ in M_2 (R) is linearly independent.

9. Are the following sets of vectors linearly independent or dependent?
(i) $\{(2, -4), (1, 9), (3, 5)\}$ (ii) $\{(-1, 5, 0), (16, 8, -3), (-64, 56, 9)\}$
(iii) $\{(1, -1, 1), (1, 1, -1), (-1, 1, 1)\ (0, 1, 0)\}$ (iv) $\left\{\left(\frac{1}{4}, 0, -\frac{1}{4}\right), \left(0, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{3}, -\frac{1}{3}, 0\right)\right\}$
(v) $\{(1, 9, 9, 8), (2, 0, 0, 3), (2, 0, 0, 8)\}$.

10. Show that the vectors $(1, 2, 1), (2, 1, 0), (1, -1, 2)$ form a basis of the vector space E^3 over the field of real numbers.

11. Is the vector $(2, -5, 3)$ in the subspace of R^3 spanned by the vectors $(1, -3, 2), (2, -4, -1)$ and $(1, -5, 7)$?

12. Show that the following sets of vectors form a basis of the vector space R^3.
(i) $\{(2, -1, 0), (3, 5, 1), (1, 1, 2)\}$ (ii) $\{(1, -3, 2), (2, 4, 1), (1, 1, 1)\}$
(iii) $\{(1, 2, -3), (2, 5, 1), (-1, 1, 4)\}$ (iv) $\{(2, -3, 1), (0, 1, 2), (1, 1, 2)\}$
(v) $\{(1, 2, 2), (1, -1, 2), (1, 0, 1)\}$.

13. Show that the following sets of vectors do not form a basis of the vector space R^3.
(i) $\{(3, 2, 1), (3, 1, 5), (3, 4, -7)\}$ (ii) $\{(1, 2, 3), (2, 3, 4), (3, 4, 5)\}$
(iii) $\{(1, 2, 1), (1, 3, 5), (-1, 0, 1), (1, -1, 2)\}$ (iv) $\{(1, 2, -3), (2, -3, 1)\ (-3, 1, 1)\}$.

14. If α, β, γ be linearly independent vectors of V(F), where F is the field of real numbers, show that $(\alpha + \beta), (\alpha - \beta), (\alpha - 2\beta + \gamma)$ are also linearly independent.

15. If $\{\alpha, \beta, \gamma\}$ be a basis of a vector space R^3, then show that $\{\alpha + \beta, \beta + \gamma, \gamma + \alpha\}$ is also a basis of R^3.

16. In the vector space V^3 of real numbers, $\alpha = (1, 2, 1), \beta = (3, 1, 5), \gamma = (3, -4, 7)$, show that the subspaces spanned by $S = \{\alpha, \beta\}$ and $T = \{\alpha, \beta, \gamma\}$ are the same.

17. In the vector space R^3, $\alpha = (1, 2, 1), \beta = (3, 1, 5), \gamma = (3, -4, 7)$. Show that there exist more than one basis for the subspace spanned by the set $S = \{\alpha, \beta, \gamma\}$.

18. Find the dimension of the subspace spanned by the following vectors in $V_3(R)$:
(i) $(1, 1, 1), (-1, -1, -1)$ (ii) $(1, 2, -3), (0, 0, 1), (-1, 2, 1)$.

19. Show that the set of matrices $\left\{\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}\right\}$ forms a basis for the vector space M_{22}.

20. If the vectors $(0, 1, c), (1, c, 1)$, and $(c, 1, 0)$ of $V_3(R)$ are linearly dependent, find the value of c.

Answers

1. $(3, -5) = -\frac{12}{7}(1, 2) - \frac{11}{7}(-3, 1)$
2. $(1, 7, -4) = -3(1, -3, 2) + 2(2, -1, 1)$
3. $(1, 1, 2) = 0(0, 2, 1) + \frac{1}{2}(2, 2, 4)$
6. No.
9. (i) linearly dependent (ii) linearly dependent (iii) linearly independent
(iv) linearly dependent (v) linearly independent
11. No
18. (i) 1 (ii) 3
20. $0, \pm\sqrt{2}$.

1.30 CO-ORDINATE VECTORS

1.30.1 Co-ordinates

The n vectors $e_1 = (1, 0, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, 0, 0, ..., 0, 1)$ are called *unit* or *elementary* vectors over the field F. The unit vector e_j whose jth component is 1, is known as the jth *unit vector*.

$\{e_1, e_2, ..., e_n\}$ is a standard basis for $V_n(F)$. Every vector $\alpha \in V_n(F)$ can be expressed uniquely as

$$\alpha = a_1e_1 + a_2e_2 + ... + a_ne_n \ \forall \ a_j \in F$$

The coefficients $a_1, a_2, ..., a_n$ are called the *components* or *co-ordinates of* α *with respect to the basis* $\{e_1, e_2, ..., e_n\}$.

1.30.2 Co-ordinate Vectors

Let $e_1 = (1, 0, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, 0, 0, ..., 0, 1)$ be *elementary* vectors over the field F *i.e.*, let $\{e_1, e_2, ..., e_n\}$ be a standard basis for $V_n(F)$ and every vector $\alpha \in V_n(F)$ can be expressed uniquely as

$$\alpha = a_1e_1 + a_2e_2 + ... + a_ne_n, \ \forall \ a_i \in F.$$

Then, the vector $(a_1, a_2, ..., a_n)$ is called the *co-ordinate vector of* α *with respect to the basis* $\{e_1, e_2, ..., e_n\}$. Sometimes, we denote

$$[\alpha]_e = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \text{ co-ordinate vector as a column vector.}$$

or $$[\alpha]_e = (a_1, a_2, ..., a_n)'.$$

1.30.3 Ordered Basis

There is no standard basis for the other vector spaces. [*i.e.*, other than $V_n(F)$ or R^n, C^n, Q^n]. That is, why we introduce the concept of ordered basis.

Let V be any vector space of finite dimension $k \geq 1$ over the field F.

Then, for any k-tuple $B = \{\alpha_1, \alpha_2, ..., \alpha_k\}$, the set of vectors of V is called an *ordered basis* of V if the set B is a basis for V.

In particular, $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is the standard ordered basis for R^3, but $\{(0, 1, 0), (0, 1, 0), (0, 0, 1)\}$ is not a standard ordered basis for R^3. However, we can say that $\{(0, 1, 0), (0,1, 0), (0, 0, 1)\}$ is an ordered basis for some vector space of dimension three.

For any $\alpha \in V$, there exists a unique k-tuple $\{a_1, a_2, ..., a_k\}$ of scalars such that $\alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_k\alpha_k$ for all $a_i \in F$, then $k \times 1$ matrix $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}_{k\times 1}$ is called the co-ordinate vector of α relative to the ordered basis B and is denoted by $[\alpha]_B$ as a column vector, that is, $[\alpha]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}_{k\times 1}$.

or $$[\alpha]_B = (a_1, a_2, ..., a_k) \text{ as a row vector.}$$

The coefficients $a_1, a_2, \ldots\ldots\ldots, a_k \in F$ are called the *components or co-ordinates of* α *relative to the ordered basis* $B = \{\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_k\}$.

Each choice of basis results in different co-ordinates and to identify a particular co-ordinate we use an ordered basis, where an ordered basis means a basis with fixed ordering of elements of that basis.

The basis set B of a vector space can be ordered in several ways and the co-ordinate of α may change with the change in the ordering of the basis B. Similarly, we can define the co-ordinate vector of a matrix $A_{m\times n}$ in $M_{m\times n}$ vector space of matrices of order $(m \times n)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Let* $S = \{e_1, e_2, e_3\}$ *be the natural (standard) basis for* R^3 *and let* $v = (3, -2, 4)$. *Compute the coordinate vector of* v *with respect to the basis S.*

Sol. Since S is the natural basis, the linear combination of the vector v is

$$v = 3e_1 - 2e_2 + 4e_3$$

Hence the coordinate vector of v w.r.t. the basis S is $[v]_S = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$.

Example 2. *Compute the coordinate vector of V with respect to the given basis S for V where*

(*i*) *V is* R^2, $\quad S = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}, v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

(*ii*) *V is* R^3, $\quad S = \{(1, -1, 0)\}, (0, 1, 0), (1, 0, 2)\}, v = (2, -1, -2)$.

Sol. (*i*)

$$S = \{e_1, e_2\}$$

$\therefore$

$$v = 3e_1 - 2e_2$$

Hence,

$$[v]_S = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

(*ii*) Let

$$v = c_1(1, -1, 0) + c_2(0, 1, 0) + c_3(1, 0, 2)$$

or

$$(2, -1, -2) = (c_1 + c_3, -c_1 + c_2, 2c_3).$$

$\therefore$

$$c_1 + c_3 = 2, -c_1 + c_2 = -1, 2c_3 = -2.$$

Solving these equations, we get $c_3 = -1$, $c_1 = 3$ and $c_2 = 2$.

Hence,

$$[v]_S = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}.$$

Example 3. *Compute the vector v if the coordinate vector* $[v]_S$ *is given with respect to the basis S for V*

(*i*) *V is* R^2, $\quad S = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}, [v]_s = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(ii) V is M_{22}, $S = \left\{\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}\right\}$

$$[v]_S = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}.$$

Sol. (i) $v = c_1(2, 1) + c_2(-1, 1) = 1(2, 1) + 2(-1, 1)$

or $v = (2 - 2, 1 + 2) = (0, 3)$ $(\because \quad c_1 = 1, c_2 = 2)$

(ii) $$v = 2\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -2+2-1+0 & 0+2-2+0 \\ 2+0+1+6 & 0+1-3+9 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 9 & 7 \end{bmatrix}.$$

Example 4. *Let $S_1 = \{\alpha_1 = (1, -2), \alpha_2 = (3, -4)\}$ be a basis of R^2. Find the co-ordinates of an arbitrary vector (a, b) of R^2 with respect to S_1.*

Sol. $$(a, b) = a_1\alpha_1 + a_2\alpha_2$$

$$= a_1(1, -2) + a_2(3, -4)$$

$$= [(a_1 + 3a_2), (-2a_1 - 4a_2)]$$

$\Rightarrow$ $$a_1 + 3a_2 = a$$

$$-2a_1 - 4a_2 = b$$

$\Rightarrow$ $$a_1 = -2a - \frac{3}{2}b, \; a_2 = \frac{2a+b}{2} = a + \frac{1}{2}b$$

$\therefore$ Co-ordinates of (a, b) are $a_1 = -2a - \frac{3}{2}b$, $a_2 = a + \frac{1}{2}b$ and co-ordinate vector of (a, b) relative to S_1 is

$$[a, b]_{S_1} = \left[-2a - \frac{3}{2}b, a + \frac{1}{2}b\right] \text{ as row vector}$$

$$= \begin{bmatrix} -2a - \frac{3}{2}b \\ a + \frac{1}{2}b \end{bmatrix} \text{ as column vector.}$$

Example 5. *Find the co-ordinate vector of α relative to the set $S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ of $V_3(R)$ where $\alpha = (a, b, c)$.*

Sol. For $a_1, a_2, a_3 \in R$, we have

$$a_1(1, 1, 1) + a_2(1, 1, 0) + a_3(1, 0, 0) = (0, 0, 0)$$

$\Rightarrow$ $$\{(a_1 + a_2 + a_3), (a_1 + a_2), a_1\} = (0, 0, 0)$$

$\Rightarrow$ $$a_1 + a_2 + a_3 = 0$$

$$a_1 + a_2 = 0$$

$$a_1 = 0$$

$\Rightarrow$ $$a_1 = 0,\ a_2 = 0,\ a_3 = 0$$

$\Rightarrow$ S is linearly independent.

Also, S generates $V_3(R)$, hence S is a basis of $V_3(R)$.

Now, let

$$\alpha = (a, b, c) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0)$$

$$(a, b, c) = \{(c_1 + c_2 + c_3), (c_1 + c_2), c_1\}$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_3 = a \\ c_1 + c_2 = b \\ c_1 = c \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = c \\ c_2 = b - c_1 = b - c \\ c_3 = (a - c_1 - c_2) \\ \quad = a - c - (b - c) \\ \quad = a - b \end{cases}$$

Hence, the components of α relative to the basis S are c, $(b - c)$, $(a - b)$ and co-ordinate vector of α relative to the basis S is $[\alpha]_S = (c, b - c, a - b)$.

Example 6. *If $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be an ordered basis for R^3, where $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (1, 0, 0)$, obtain the co-ordinate vector of (a, b, c) in the ordered basis B.*

Sol. Represent (a, b, c) as a linear function of $\alpha_1, \alpha_2, \alpha_3$:

$$\begin{aligned}(a, b, c) &= a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 \\ &= a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(1, 0, 0) \\ &= (a_1 + a_2 + a_3, a_2, -a_1 + a_2)\end{aligned}$$

$$\Rightarrow \begin{cases} a_1 + a_2 + a_3 = a \\ a_2 = b \\ -a_1 + a_2 = c \end{cases}$$

$$\Rightarrow \quad a_2 = b,\ a_1 = b - c,\ a_3 = a - (b - c) - b = a - 2b + c.$$

Hence, required co-ordinate vector is (a_1, a_2, a_3) *i.e.*, $\{(b - c), b, (a - 2b + c)\}$.

Example 7. *Let V be the vector space of 2×2 matrices over R. Find the co-ordinate vector of matrix $A \in V$ relative to the basis*

$$S = \left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\} \text{ where } A = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}$$

Sol. Let the unknown scalars be $a_1, a_2, a_3, a_4 \in R$

$\therefore$ $$A = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4$$

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} = a_1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + a_2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + a_3\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + a_4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & a_1 \\ a_1 & a_1 \end{bmatrix} + \begin{bmatrix} 0 & -a_2 \\ a_2 & 0 \end{bmatrix} + \begin{bmatrix} a_3 & -a_3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + a_3 + a_4 & a_1 - a_2 - a_3 \\ a_1 + a_2 & a_1 \end{bmatrix}$$

Comparing the entries on both sides, we have

$$a_1 + a_3 + a_4 = 4 \quad \ldots(i)$$
$$a_1 - a_2 - a_3 = 3 \quad \ldots(ii)$$
$$a_1 + a_2 = 4 \quad \ldots(iii)$$
$$a_1 = -7 \quad \ldots(iv)$$

Solving equations (*i*), (*ii*), (*iii*) and (*iv*), we get

$$a_1 = -7,\ a_2 = 4 - a_1 = 4 + 7 = 11$$
$$a_3 = a_1 - a_2 - 3 = -7 - 11 - 3 = -21$$
$$a_4 = 2 - a_1 - a_3 = 2 + 7 + 21 = 30$$

Hence, $[A]_S = (-7, 11, -21, 30)$.

1.30.4 Change of Basis

One of the problems under change of basis is the determination of components (or co-ordiantes) of a vector in a space V with respect to the given basis when we know the components of the vector in some other basis.

ILLUSTRATIVE EXAMPLES

Example 1. *Let the components of a vector be (1, 2) relative to the basis B = {(3, 1), (– 1, 2)} of R^2. Find the components of the vector in the basis S = {(5, 3), (– 2, 4)}*

Sol. Let $\alpha \in R^2$ with components $a_1 = 1$, $a_2 = 2$. Then,

$$\alpha_1 = (3, 1),\ \alpha_2 = (-1, 2) \in R^2$$

Basis, B = $\{\alpha_1, \alpha_2\}$ and each vector in R^2 can be written in precisely one way as a linear combination of the basis vectors.

$$\alpha = a_1\alpha_1 + a_2\alpha_2$$
$$= 1 \cdot (3, 1) + 2 \cdot (-1, 2)$$
$$= (3, 1) + (-2 + 4) = (3 - 2), (1 + 4) = (1, 5) \quad \ldots(i)$$

Let $\alpha = b_1\beta_1 + b_2\beta_2$

New basis S = {(5, 3), (– 2, 4)} = $\{\beta_1, \beta_2\}$, say, where b_1 and b_2 are the components of α in the basis S.

$$\Rightarrow \quad (1, 5) = b_1(5, 3) + b_2(-2, 4)$$
$$= \{(5b_1 - 2b_2), (3b_1 - 4b_2)\}$$
$$\Rightarrow \quad 5b_1 - 2b_2 = 1 \quad \ldots(ii)$$
$$\Rightarrow \quad 3b_1 - 4b_2 = 5 \quad \ldots(iii)$$

Solving (*ii*) and (*iii*), we get

$$b_1 = \frac{7}{13}, b_2 = \frac{24}{13}$$

Thus, the components of α relative to the new basis S are $\frac{7}{13}, \frac{24}{13}$ and co-ordinate of α with respect to S is $[\alpha]_S = \left\{\frac{7}{13}, \frac{24}{13}\right\}$ as a row vector.

Example 2. *Let $V = R^3$. Let $B = \{e_1, e_2, e_3\}$ with $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ and $S = \{\alpha_1, \alpha_2, \alpha_3\}$ with $\alpha_1 = (1, 1, 0), \alpha_2 = (1, -1, 0), \alpha_3 = (0, 1, 1)$. Consider $\alpha = (1, 2, 3) \in R^3$. Find the co-ordinate vector α relative to S.*

Sol. For $[\alpha]_B = (1, 2, 3)$, components of a relative to the basis B, we have

$$\begin{aligned} \alpha &= 1 \cdot e_1 + 2 \cdot e_2 + 3 \cdot e_3 \\ &= 1 \cdot (1, 0, 0) + 2 \cdot (0, 1, 0) + 3 \cdot (0, 0, 1) \\ &= (1, 0, 0) + (0, 2, 0) + (1, 0, 3) \\ &= (1 + 0 + 0, 0 + 2 + 0, 0 + 0 + 3) \\ &= (1, 2, 3) \end{aligned} \quad \ldots(i)$$

Let a_1, a_2, a_3 be the components of α relative to the basis S. Then,

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3, \text{ for } [\alpha]_S = (a_1, a_2, a_3)$$

$$\begin{aligned} \Rightarrow \quad (1, 2, 3) &= a_1(1, 1, 0) + a_2(1, -1, 0) + a_3(0, 1, 1) \\ &= (a_1, a_1, 0) + (a_2 - a_2, 0) + (0, a_3, a_3) \\ &= (a_1 + a_2 + 0, a_1 - a_2 + a_3, 0 + 0 + a_3) \\ &= (a_1 + a_2, a_1 - a_2 + a_3, a_3) \end{aligned}$$

$$\Rightarrow \quad \begin{cases} a_1 + a_2 = 1 \\ a_1 - a_2 + a_3 = 2 \\ a_3 = 3 \end{cases}$$

$$\Rightarrow \quad a_3 = 3, a_2 = 1, a_1 = 0$$

Hence, the components of α relative to the basis S are

$$[\alpha]_S = (0, 1, 3) \text{ as a row vector and} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \text{ as a column vector.}$$

TEST YOUR KNOWLEDGE

1. Define co-ordinate and co-ordinate vector of a vector α in a vector space V over the field F.
2. Find the co-ordinate vector of α relative to the set $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ of $V_3(R)$ when $\alpha = (4, -3, 2)$.
3. Find the co-ordinate vector of $v = \{(3, 5, -2)$, relative to the basis $e_1 = (1, 1, 1), e_2 = (0, 2, 3), e_3 = (0, 2, -1)$.
4. Let $V_3(R)$ be a finite dimensional vector space. Find the co-ordinate vector of $\alpha = (3, 1, -4)$ relative to the basis $S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$.
5. Find the co-ordinates of the vector of $(2, 1, 3, 4)$ of R^4 relative to the basis $\alpha_1 = (1, 1, 0, 0)$, $\alpha_2 = (1, 0, 1, 1)$, $\alpha_3 = (2, 0, 0, 2)$, $\alpha_4 = (0, 0, 2, 2)$.
6. State whether the following statement is True or False?

   Let $\alpha = a_1 l_1 + a_2 l_2 + \ldots\ldots\ldots + a_p l_p$. Then $(a_1, a_2, \ldots\ldots\ldots, a_p)$ is called co-ordinate vector of a with respect to (or related to) the basis $\alpha = (l_1, l_2, \ldots\ldots\ldots, l_p)$.

Answers

2. $(2, 5, -7)$
3. $(3, -1, 2)$
4. $(3, -2, -5)$
5. $\left(1, 0, \frac{1}{2}, \frac{3}{2}\right)$
6. True