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Some interesting consequences of Chebotarev density theorem

2025-12-13

Introduction

The aim of this post is to explain some of the consequences of Cebotarev density theorem.

First of all, we define the objects we need, see chapter 13 of Neukirch (1999).

Definition: Let K be a number field and L/K be a Galois extension with Galois group G . For every $\sigma \in G$ we define $P_{L|K}(\sigma)$ as the set of all unramified prime ideals \mathfrak{p} of K such that there exists a prime ideal $\mathfrak{P}|\mathfrak{p}$ of L satisfying

$$\sigma = \left(\frac{L|K}{\mathfrak{P}} \right)$$

where $\left(\frac{L|K}{\mathfrak{P}} \right)$ is the Frobenius automorphism of \mathfrak{P} over K .

Since for all $\tau \in G$ we have

$$\left(\frac{L|K}{\tau \mathfrak{P}} \right) = \tau \left(\frac{L|K}{\mathfrak{P}} \right) \tau^{-1}$$

the set $P_{L|K}(\sigma)$ only depends on the conjugacy class

$$\langle \sigma \rangle := \{ \tau \sigma \tau^{-1} | \tau \in G \}$$

Moreover, if $\langle \sigma \rangle \neq \langle \tau \rangle$ then $P_{L|K}(\sigma) \cap P_{L|K}(\tau) = \emptyset$.

We now want to define what a density is.

Definition: Let $A \subseteq \mathbb{N}$. Set $A(n) := \{1, \dots, n\} \cap A$ and $a(n) := |A(n)|$. If the limit exists, we define the *natural density* $d(A)$ as

$$d(A) := \lim_{n \rightarrow \infty} \frac{a(n)}{n}$$

The theorem

We are now ready to state the theorem

Theorem 0.1 (Chebotarev density theorem). *Let K be a number field and L/K be a finite Galois extension with Galois group G . Then for every $\sigma \in G$, the set $P_{L|K}(\sigma)$ has density (the limit exists), and it is given by*

$$d(P_{L|K}(\sigma)) = \frac{\#\langle \sigma \rangle}{\#G}$$

We will not prove Theorem 0.1, since the argument is long and technical. For a complete proof look at Neukirch (1999).

The consequences

We finally can learn about the corollaries of Theorem 0.1.

Characterization of number fields through splitting of primes

Definition: Let L/K be a Galois extension of number fields and \mathfrak{p} be a prime of K . We say that \mathfrak{p} splits completely in L if for every prime $\mathfrak{P}|\mathfrak{p}$

$$f(\mathfrak{P}|\mathfrak{p}) = e(\mathfrak{P}|\mathfrak{p}) = 1$$

where f is the inertia index and e is the ramification index. We define $P(L|K)$ as the set of primes of K splitting completely in L .

Proposition 0.1 (M. Bauer). *Let K be a number field and let L and M be two finite Galois extensions of K . Then*

$$P(M|K) \subseteq P(L|K) \iff L \subseteq M$$

Therefore,

$$P(M|K) = P(L|K) \iff L = M$$

i.e. the primes splitting completely completely determine the number field.

Proof. If $L \subseteq M$ then $P(M|K) \subseteq P(L|K)$ because of the multiplicativity of inertia and ramification index in towers of extensions.

Conversely, define $N := LM$ as the composite field and set

$$G := \text{Gal}(N/K), \quad H_L := \text{Gal}(N/L), \quad H_M := \text{Gal}(N/M).$$

Assume for the sake of contradiction that $L \not\subseteq M$. This is equivalent to $H_M \not\subseteq H_L$. Choose $g \in H_M \setminus H_L$. Since H_M and H_L are normal,

$$\langle g \rangle \subseteq H_M, \quad \langle g \rangle \cap H_L = \emptyset$$

By Theorem 0.1 there are infinitely many primes \mathfrak{p} of K unramified in N with Frobenius conjugacy class equal to $\langle g \rangle$. For such a \mathfrak{p} , pick $\mathfrak{P}|\mathfrak{p}$ in N ; then

$$\left(\frac{N|K}{\mathfrak{P}} \right) \in \langle g \rangle \subseteq H_M$$

Claim: For every prime \mathfrak{p} of K unramified in N , pick a prime \mathfrak{P} of N above \mathfrak{p} . Then

$$\mathfrak{p} \in P(M|K) \iff \left(\frac{N|K}{\mathfrak{P}} \right) \in H_M$$

Assuming the claim we get

$$\mathfrak{p} \in P(M|K)$$

But also $\left(\frac{N|K}{\mathfrak{P}} \right) \notin H_L$, so $\mathfrak{p} \notin P(L|K)$. This contradicts $P(M|K) \subseteq P(L|K)$.

We are left to prove the claim.

of the claim. Since M/K is Galois, the restriction map

$$\text{res}_M : G = \text{Gal}(N/K) \rightarrow \text{Gal}(M/K)$$

is surjective with kernel H_M . Since \mathfrak{p} is unramified, the Frobenius automorphism is well defined and it satisfies:

$$\text{res}_M \left(\left(\frac{N|K}{\mathfrak{P}} \right) \right) = \left(\frac{M|K}{\mathfrak{P} \cap M} \right) \in \text{Gal}(M/K)$$

Now observe that \mathfrak{p} splits completely in M if and only if every prime of M above \mathfrak{p} has inertia index 1 (since we already know that \mathfrak{p} is unramified). Equivalently, the decomposition group in $\text{Gal}(E/K)$ is trivial, which is the same of saying that

$$\left(\frac{M|K}{\mathfrak{p}} \right) = 1.$$

This holds if and only if $\left(\frac{N|K}{\mathfrak{p}} \right) \in \ker(\text{res}_M) = H_M$. □

□

Neukirch, Jürgen. 1999. *Algebraic Number Theory*. 1st ed. Vol. 322. Grundlehren Der Mathematischen Wissenschaften. Berlin, Heidelberg: Springer-Verlag. <https://doi.org/10.1007/978-3-662-03983-0>.