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# Ax–Grothendieck Theorem

When does injectivity imply bijectivity?

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## Introduction

We all know from basic linear algebra that injective endomorphisms of a finite-dimensional vector space are also surjective. The same holds for finite sets. Over algebraically closed fields, there is an analogous result:

**Theorem 0.1** (Special case of Ax–Grothendieck Theorem). *Let  $k$  be an algebraically closed field,  $n \in \mathbb{N}$ , and  $f_1, \dots, f_n \in k[x_1, \dots, x_n]$ . If the map*

$$F : k^n \rightarrow k^n, \quad F(\underline{x}) = (f_1(\underline{x}), \dots, f_n(\underline{x}))$$

*is injective, then it is surjective.*

We will talk about an analogous and more general statement about schemes later on.

Let us comment briefly on the hypotheses. Injectivity is the correct assumption for  $F$ : the map  $z \mapsto z^2$  is surjective but not injective on  $\mathbb{C}$ . Furthermore, the result is generally false for non-algebraically closed fields; for example,  $x \mapsto x^3$  is injective but not surjective on  $\mathbb{Q}$ .

As a sanity check, let's verify the result for  $n = 1$ . Consider a polynomial  $f \in k[x]$  such that the map  $x \mapsto f(x)$  is injective; then  $f$  can have at most one root. Since  $k$  is algebraically closed, it follows that  $f = ax + b$  with  $a \neq 0$  (otherwise the map would not be injective). Hence  $f$  is surjective.

Moreover, the theorem is easy to prove when  $k = \overline{\mathbb{F}}_p$ .

For  $k = \overline{\mathbb{F}}_p$ . Let  $\underline{z} := (z_1, \dots, z_n) \in k^n$  and let  $L$  be the subfield of  $k$  generated by the coefficients of the polynomials defining  $F$  and by the  $z_i$ 's. The field  $L$  is a finite extension of  $\mathbb{F}_p$ , hence a finite set. The restriction  $F : L^n \rightarrow L^n$  is well-defined and injective; since  $L^n$  is finite it is also surjective. Therefore,  $\underline{z}$  lies in the image of  $F$ , proving the claim.  $\square$

## Proof of the theorem

We start by proving two lemmas and then proceed with the proof of Theorem 0.1.

**Lemma 0.1.** *The map  $F$  is injective if and only if there exist  $q_{i,j} \in k[x_1, \dots, x_n, y_1, \dots, y_n]$  and integers  $k_j > 0$  such that, defining*

$$g_i(x_1, \dots, x_n, y_1, \dots, y_n) := f_i(x_1, \dots, x_n) - f_i(y_1, \dots, y_n),$$

*for all  $j$  we have the identity*

$$\sum_{i=1}^n g_i \cdot q_{i,j} = (x_j - y_j)^{k_j} \quad (1)$$

*Proof.* Consider the ideal  $I = (g_1, \dots, g_n)$ . The map  $F$  is injective if and only if all the zeros of  $I$  are of the form  $(\underline{x}, \underline{x})$  for  $\underline{x} \in k^n$ . This is the case if and only if the zeros of  $I$  are a subset of the zeros of  $J = (x_1 - y_1, \dots, x_n - y_n)$ , i.e. if and only if  $V(I) \subseteq V(J)$ . By the [Hilbert Nullstellensatz](#), this holds if and only if  $J \subseteq \sqrt{I}$ ; hence  $(x_j - y_j) \in \sqrt{I}$ .  $\square$

**Lemma 0.2.** *Let  $\underline{z} \in k^n$ . Then  $\underline{z}$  is not in the image of  $F$  if and only if there exist  $r_1, \dots, r_n \in k[x_1, \dots, x_n]$  such that*

$$\sum_{i=1}^n r_i \cdot (f_i - z_i) = 1. \quad (2)$$

*Proof.* The element  $\underline{z}$  is not in the image of  $F$  if and only if the ideal  $I = (f_1 - z_1, \dots, f_n - z_n)$  has no zeros, equivalently  $1 \in I$  (again by Nullstellensatz).  $\square$

Now we are ready to prove the theorem.

*Of Theorem 0.1.* Suppose  $\underline{z} \in k^n$  is not in the image of  $F$ . Let  $A$  be the  $\mathbb{Z}$ -algebra generated by:

- the coefficients of the  $f_i$ 's defining  $F$ ;
- the  $z_i$ 's;
- the coefficients of the  $q_{i,j}$  as in Lemma 0.1;
- the coefficients of the  $r_i$  as in Lemma 0.2.

Since finitely many coefficients are involved,  $A$  is of finite type over  $\mathbb{Z}$ . Moreover, the restriction  $F : A^n \rightarrow A^n$  is still injective. Furthermore, we have the following identities in  $A[x_1, \dots, x_n, y_1, \dots, y_n]$  given respectively by Lemma 0.1 and Lemma 0.2:

$$\sum_{i=1}^n g_i \cdot q_{i,j} = (x_j - y_j)^{k_j} \quad (3)$$

$$\sum_{i=1}^n r_i \cdot (f_i - z_i) = 1 \quad (4)$$

Now pick a maximal ideal  $m$  in  $A$ . The field  $A/m$  is finite by the following claim.

**Claim:** Let  $B$  be an algebra of finite type over  $\mathbb{Z}$ . If  $m$  is a maximal ideal of  $B$ , then  $B/m$  is a finite field.

Assuming the claim, set  $L = A/m$  and note that the reduction  $\bar{F}: L^n \rightarrow L^n$  of  $F$  modulo  $m$  is injective. Indeed, consider  $\underline{x}, \underline{y} \in L^n$  such that  $\bar{F}(\underline{x}) = \bar{F}(\underline{y})$ . By evaluating Equation 3 at  $(\underline{x}, \underline{y})$  we obtain  $\underline{x} = \underline{y}$  in  $L^n$ , since  $\bar{g}_i(\underline{x}, \underline{y}) = 0$  for all  $i$ .

Since  $L^n$  is finite and  $\bar{F}$  is injective,  $\bar{F}$  is surjective. However, reducing  $z$  modulo  $m$  and evaluating Equation 4 at a preimage of that reduction under  $\bar{F}$  yields a contradiction. Hence, up to proving the claim, we have established Theorem 0.1.

*of the Claim.* Consider two cases:

- If  $m \cap \mathbb{Z} = p\mathbb{Z}$ , then  $B/m$  is an algebra of finite type over  $\mathbb{F}_p$ ; by Zariski's lemma  $B/m$  is a finite extension of  $\mathbb{F}_p$ , hence finite.
- If  $m \cap \mathbb{Z} = 0$ , then in the tower  $\mathbb{Z} \subset \mathbb{Q} \subset B/m$ , we see that  $B/m$  is of finite type over  $\mathbb{Q}$ , hence is finite over  $\mathbb{Q}$  by Zariski's lemma. Artin–Tate's lemma then implies that  $\mathbb{Q}$  would be of finite type over  $\mathbb{Z}$ , a contradiction.

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## Generalization to schemes

Note that the map  $F$  considered in Theorem 0.1 can be seen as the map induced on closed points by a morphism of  $k$ -schemes  $\mathbb{A}_k^n \rightarrow \mathbb{A}_k^n$ . This leads to the following question:

**Question:**

Let  $k$  be an algebraically closed field and  $X$  a scheme of finite type over  $k$ . If  $f: X \rightarrow X$  is a  $k$ -endomorphism injective on  $k$ -points, is it surjective on all points?

The finite-type assumption is necessary: the map induced by

$$k[X_1, \dots, X_n, \dots] \rightarrow k[X_1, \dots, X_n, \dots]$$

sending  $X_1 \mapsto 0$  and  $X_i \mapsto X_{i-1}$  for  $i > 1$  is injective but not surjective.

The answer is positive, and it follows (not trivially) from the following theorem.

**Theorem 0.1** (Ax–Grothendieck Theorem). *Let  $S$  be a scheme and let  $X$  be a scheme of finite type over  $S$ . Any injective  $S$ -endomorphism  $f : X \rightarrow X$  is bijective.*

Moreover, if  $X$  is reduced and  $k$  is an algebraically closed field of characteristic 0, then an injective endomorphism on  $k$ -points is in fact an isomorphism (not merely a bijection). This is not trivial since we have bijective morphisms of schemes that are not isomorphisms (for example the one induced by  $k[t^2, t^3] \rightarrow k[t]$ ). A reference for this is Lemma 1 of [this article](#).

We will not give a proof of Theorem 0.1; see J. Ax’s paper [Injective endomorphisms of varieties and schemes](#), or Grothendieck’s version in [EGA IV<sub>3</sub>, Proposition 10.4.11](#) for a complete proof.

Let us explain why Theorem 0.1 implies that injectivity on  $k$ -points gives surjectivity on all points.

The first ingredient is Chevalley’s theorem on constructible sets (see Hartshorne, Ex. II.3.19).

**Theorem 0.2** (Chevalley’s theorem). *Let  $f : X \rightarrow Y$  be a finite type morphism between noetherian schemes. If  $Z \subseteq X$  is a constructible subset (i.e. a finite union of locally closed subsets), then  $f(Z)$  is constructible.*

The following lemma explains why Theorem 0.1 implies Theorem 0.1.

**Lemma 0.1.** *Let  $X, Y$  be schemes of finite type over an algebraically closed field  $k$ . Let  $f : X \rightarrow Y$  be a morphism whose restriction to  $k$ -points*

$$f : X(k) \rightarrow Y(k)$$

*is injective. Then  $f$  is universally injective.*

Assuming this lemma, consider a morphism  $f : \mathbb{A}_k^n \rightarrow \mathbb{A}_k^n$  over  $k$  which is injective on closed points (which are exactly the  $k$ -points, since  $k$  is algebraically closed). Then in particular  $f$  is injective on all points. From Theorem 0.1 we get the surjectivity of  $f$  on all points (answering the question raised before) and in particular also surjectivity on closed points, since only closed points are sent to closed points. In particular, we obtain Theorem 0.1.

*Of Lemma 0.1.* Observe that the diagonal  $\Delta_f : X \rightarrow X \times_Y X$  is of finite type. Indeed, it is locally of finite type because  $X \rightarrow \operatorname{Spec}(k)$  is locally of finite type (and  $\Delta_f$  can be seen as the first factor of this map), and it is quasi-compact because  $X$  is noetherian (being of finite type over  $k$ ).

Moreover, the diagonal  $\Delta_f$  is surjective on  $k$ -points. Indeed, for any  $k$ -point of  $X \times_Y X$  given by a morphism

$$\psi : \operatorname{Spec}(k) \rightarrow X \times_Y X$$

we can consider the diagram

and by injectivity of the morphism  $X \rightarrow Y$  we obtain  $\phi_1 = \phi_2$  as maps of topological spaces. They are also the same on the structure sheaf, because  $k$  is algebraically closed. Therefore,  $\phi_1 = \phi_2$ , which means that  $\psi$  factorizes through the diagonal  $\Delta_f$ .

Furthermore,  $X \times_Y X$  is a [Jacobson](#) scheme (i.e. its underlying topological space is Jacobson) because it is of finite type over  $k$ , and  $k$  is Jacobson. By [Theorem 0.2](#) the set  $\Delta_f(X)$  is constructible. Consider  $Z := (X \times_Y X) \setminus \Delta_f(X)$ . This is constructible and every closed point of  $X \times_Y X$  lies inside  $\Delta_f(X)$  (by surjectivity of  $\Delta_f$  on closed points). Since every nonempty constructible subset in a Jacobson scheme contains a closed point, it follows that  $Z = \emptyset$ . Therefore,  $\Delta_f$  is surjective on all points.

Now we prove that the surjectivity of  $\Delta_f$  implies the universal injectivity of  $f$ . Indeed, consider any  $g : Z \rightarrow Y$  and the diagonal

$$\Delta : X \times_Y Z \rightarrow (X \times_Y Z) \times_Z (X \times_Y Z)$$

of the base change. This diagonal map  $\Delta$  is surjective because it is the base change of  $\Delta_f$ , which is surjective. Thus, we are reduced to proving that surjectivity of  $\Delta_f$  implies injectivity of  $f$  (the same will then hold for any base change).

Consider  $x_1, x_2 \in X$  with  $f(x_1) = f(x_2) =: y$ . We can find a field  $L$  such that the following diagram commutes:

We get the following commutative diagram:

By surjectivity of  $\Delta_f$  we can find  $x \in X$  such that  $\Delta_f(x)$  is the image of  $\epsilon$ , hence we get  $x_1 = x_2$ . This shows that  $f$  is injective, and therefore universally injective.  $\square$

Note that the complex-geometric picture does not match the algebraic one. For example, there are holomorphic maps  $\mathbb{C}^n \rightarrow \mathbb{C}^n$  that are injective but not surjective. However, this cannot happen for  $n = 1$ , as one can show that all holomorphic injective maps  $f : \mathbb{C} \rightarrow \mathbb{C}$  are of the form  $f = ax + b$ ; see the [Casorati–Weierstrass theorem](#).

## Conclusion

To conclude, we point your attention to the fact that this theorem is linked to a famous conjecture, namely the [Jacobian conjecture](#).

**Conjecture 0.1** (Jacobian conjecture). *Let  $k$  be a field of characteristic 0 and let  $F : k^n \rightarrow k^n$  be a polynomial mapping. If the Jacobian determinant  $J_F$  is invertible, then  $F$  is an isomorphism.*

Over an algebraically closed field of characteristic 0, the fact that the bijection in Theorem [0.1](#) becomes an isomorphism shows that, to prove the Jacobian conjecture for  $\mathbb{A}_k^n$ , it suffices to show that a map

$$F : \mathbb{A}_k^n \rightarrow \mathbb{A}_k^n$$

with invertible Jacobian determinant is injective.

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I would like to thank my friend who suggested this topic to me and helped me write this post.

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## References

This post was mainly inspired by [this post](#) from Terence Tao's blog. Other references are cited in the text when used.