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# Some basic modular arithmetic

Fermat's Christmas Theorem series

2025-12-14

ENG ITA

## Introduction

We are at the bus stop, waiting for our bus to arrive. On the table it is written that there are three buses arriving soon:

- Bus  $A$  arrives in 2 minutes;
- Bus  $B$  arrives in 5 minutes;
- Bus  $C$  arrives in 10 minutes.

Of course we are not lucky, and we have to wait for bus  $C$ . While watching buses arriving and leaving, we ask ourselves:

**Question:** Will buses  $A$ ,  $B$ , and  $C$  ever arrive simultaneously?

We read on the table that:

- Bus  $A$  passes every 3 minutes;
- Bus  $B$  passes every 24 minutes;
- Bus  $C$  passes every 23 minutes.

We have our notebook with us, so we approach the problem in the brutal way. We list all the times at which buses  $A$ ,  $B$ , and  $C$  pass (the numbers are the waiting times in minutes):

- Bus  $A$  will pass in  
 $2, 5, 8, 11, 14, 17, \dots$
- Bus  $B$  will pass in  
 $5, 29, 53, 77, 101, \dots$

- Bus  $C$  will pass in

$$10, 33, 56, 79, 102, \dots$$

We do not know whether, by continuing this list, we will find a common number in the three lists (which would mean a simultaneous arrival). This method is tedious, so we want an efficient way to solve this problem in general.

Let's translate the problem into a mathematical setting. What does it mean that bus  $A$  passes every 3 minutes and it arrives in 2 minutes? It means that the possible times (in minutes) at which bus  $A$  will pass are of the form  $3 \cdot n + 2$ , where  $n$  is a nonnegative integer (i.e.  $n = 0, 1, 2, \dots$ ; convince yourself about this). Indeed, all the numbers we listed for  $A$  are of this form.

Similarly, the waiting times for bus  $B$  are of the form  $24 \cdot k + 5$  and the waiting times for bus  $C$  are of the form  $23 \cdot \ell + 10$ . Then, finding a time  $x$  (always in minutes) when  $A$ ,  $B$ , and  $C$  arrive simultaneously means finding three nonnegative integers  $n, k, \ell$  such that

$$x = 3 \cdot n + 2, \quad x = 24 \cdot k + 5, \quad x = 23 \cdot \ell + 10.$$

In other words, the total waiting time  $x$  must be of all three forms at the same time.

## Congruences of integers

Let's make the notation a little bit better and more helpful.

**Definition:** Let  $n$ ,  $m$ , and  $q$  be integers, with  $q > 0$ . We write

$$n \equiv_q m$$

if the remainder of the division of  $n$  by  $q$  and the remainder of the division of  $m$  by  $q$  are the same.

The writing  $n \equiv_q m$  is read as “ $n$  is *congruent to  $m$  modulo  $q$* ”.

**Example 0.1.** Consider  $n = 14$  and  $q = 3$ . Then  $14 \equiv_3 2$ . Indeed,

$$14 = 3 \cdot 4 + 2$$

has remainder 2, and also 2 has remainder 2 when divided by 3.

*Remark 0.1.* For every integer  $n$  and  $q$  with  $q > 0$ , if  $r$  is the remainder of the division of  $n$  by  $q$ , then  $n \equiv_q r$ .

The following fact tells us the meaning of congruences.

**Proposition 0.1.** Let  $n$ ,  $m$ , and  $q$  be integers with  $q > 0$ . Then  $n \equiv_q m$  if and only if  $q$  divides  $n - m$ .

*Proof.* Suppose that  $n \equiv_q m$ . Then the remainder of the division of  $n$  by  $q$  is the same as the remainder of the division of  $m$  by  $q$ . Let's call this remainder  $r$ .

By Euclidean division, there exist integers  $k$  and  $\ell$  such that

$$n = q \cdot k + r, \quad m = q \cdot \ell + r,$$

with  $0 \leq r < q$ .

Subtract the second equation from the first:

$$n - m = (q \cdot k + r) - (q \cdot \ell + r).$$

Now we simplify it:

$$n - m = q \cdot k + r - q \cdot \ell - r,$$

and cancelling  $r$  on the right-hand side we get

$$n - m = q \cdot k - q \cdot \ell.$$

Factor  $q$ :

$$n - m = q \cdot (k - \ell).$$

This means exactly that  $q$  divides  $n - m$ .

Conversely, suppose that  $q$  divides  $n - m$ . This means that there exists an integer  $t$  such that

$$n - m = q \cdot t.$$

Rearrange:

$$n = m + q \cdot t.$$

Now divide  $m$  by  $q$ : by Euclidean division there exist integers  $\ell$  and  $r$  with  $0 \leq r < q$  such that

$$m = q \cdot \ell + r.$$

Plug this into the previous equation:

$$n = (q \cdot \ell + r) + q \cdot t = q \cdot (\ell + t) + r.$$

So when you divide  $m$  by  $q$  you get remainder  $r$ , and when you divide  $n$  by  $q$  you also get remainder  $r$ . Therefore  $n \equiv_q m$ .  $\square$

Let us now make some examples. Observe that  $14 \equiv_3 2$  and that  $19 \equiv_3 1$ . We ask what is

$$14 + 19 \equiv_3 ?$$

Maybe you already guessed it: we have

$$14 + 19 \equiv_3 2 + 1.$$

Therefore we are saying that

$$33 \equiv_3 3.$$

But we also know that  $3 \equiv_3 0$  (indeed 3 has remainder 0 when divided by 3), therefore

$$33 \equiv_3 0.$$

What we are saying is that 33 is divisible by 3.

We ask the same for the product:

$$14 \cdot 19 \equiv_3 ?$$

We are lucky again: we get

$$14 \cdot 19 \equiv_3 2 \cdot 1,$$

which means that

$$266 \equiv_3 2,$$

i.e. 266 has remainder 2 when divided by 3.

We state this precisely in the following proposition.

**Proposition 0.2.** *Let  $n, m, n', m'$ , and  $q$  be integers with  $q > 0$ . If  $n \equiv_q m$  and  $n' \equiv_q m'$ , then*

$$n + n' \equiv_q m + m'$$

and

$$n \cdot n' \equiv_q m \cdot m'.$$

### Easy part of Fermat's Christmas Theorem

**Proposition 0.3.** *Let  $n$  be an odd integer that can be written as the sum of two squares. Then  $n \equiv_4 1$ , or equivalently  $n$  has remainder 1 when divided by 4.*

*Remark 0.2.* In particular, the theorem also tells us that if  $p$  is an odd prime that can be written as the sum of two squares, then it has remainder 1 when divided by 4, which is part of what Fermat's Christmas Theorem says.

*Proof.* If  $n$  can be written as the sum of two squares, it means that there exist integers  $a, b$  such that

$$n = a^2 + b^2.$$

Observe that  $n$  is odd, therefore it has remainder 1 when divided by 2, i.e.

$$n \equiv_2 1.$$

Therefore

$$a^2 + b^2 \equiv_2 1.$$

If both  $a$  and  $b$  were odd, then  $a \equiv_2 1$  and  $b \equiv_2 1$ , therefore

$$a^2 \equiv_2 1, \quad b^2 \equiv_2 1,$$

which would mean

$$a^2 + b^2 \equiv_2 1 + 1 \equiv_2 2 \equiv_2 0,$$

and therefore  $n \equiv_2 0$ , which is impossible since  $n$  is odd.

Similarly, if  $a$  and  $b$  are both even, then  $a^2 + b^2 \equiv_2 0$ , which is again impossible.

Therefore, one is odd and one is even. Call  $a$  the odd one and  $b$  the even one. We have

$$a \equiv_2 1, \quad b \equiv_2 0.$$

Since  $b$  is even, when we divide  $b$  by 4 we get either remainder 0 or remainder 2 (otherwise it would be odd). So

$$b \equiv_4 0 \quad \text{or} \quad b \equiv_4 2.$$

In both cases we get

$$b^2 \equiv_4 0.$$

Similarly, since  $a$  is odd, we have

$$a \equiv_4 1 \quad \text{or} \quad a \equiv_4 3.$$

In both cases (check!) we get

$$a^2 \equiv_4 1.$$

Therefore

$$a^2 + b^2 \equiv_4 1 + 0 \equiv_4 1,$$

which means

$$n \equiv_4 1.$$

□

## Invertible numbers modulo $q$

Recall what the greatest common divisor is:

**Definition:** Let  $n$  and  $m$  be two integers different from 0. The greatest common divisor  $\gcd(n, m)$  is the product of the common primes appearing in the factorizations of  $n$  and  $m$ , taken with the least power.

**Example 0.2.** We compute the greatest common divisor between 72 and 540. We have

$$72 = 2^3 \cdot 3^2, \quad 540 = 2^2 \cdot 3^3 \cdot 5,$$

therefore

$$\gcd(72, 540) = 2^2 \cdot 3^2.$$

Suppose we have two integers  $n$  and  $q$  with  $q > 0$ . We ask when there exists another integer  $m$  such that

$$n \cdot m \equiv_q 1.$$

Let's try a few examples:

- If  $n = 4$  and  $q = 7$ , then

$$4 \cdot 2 \equiv_7 1.$$

- If  $n = 4$  and  $q = 6$ , then we cannot find any such  $m$ .
- If  $n = 3$  and  $q = 9$ , we cannot find any such  $m$ .
- If  $n = 3$  and  $q = 5$ , then

$$3 \cdot 2 \equiv_5 1.$$

Try some examples yourself.

**Definition:** Let  $n$  and  $q$  be integers with  $q > 0$ . We say that  $n$  is *invertible modulo  $q$*  if there exists an integer  $m$  such that

$$n \cdot m \equiv_q 1.$$

The answer to the question is given by the following proposition.

**Proposition 0.4.** *Let  $n$  and  $q$  be integers with  $q > 0$ . Then  $n$  is invertible modulo  $q$  if and only if  $\gcd(n, q) = 1$ .*

## Chinese Remainder Theorem

We now state the theorem that will let us solve our bus problem efficiently.

**Theorem 0.1.** *Let  $s, t > 1$  be integers with  $\gcd(s, t) = 1$ . Then for all integers  $a$  and  $b$  there exists an integer  $x$  such that*

$$x \equiv_s a \quad \text{and} \quad x \equiv_t b.$$

*Proof.* We want

$$x \equiv_s a \quad \text{and} \quad x \equiv_t b.$$

The first congruence  $x \equiv_s a$  means that  $x - a$  is divisible by  $s$ , hence  $x$  is of the form

$$x = a + s \cdot n$$

for some integer  $n$ .

Now we impose the second congruence. We want  $x \equiv_t b$ , i.e.

$$a + s \cdot n \equiv_t b.$$

Subtract  $a$  from both sides:

$$s \cdot n \equiv_t b - a.$$

So the whole problem becomes: can we solve

$$s \cdot n \equiv_t (b - a)?$$

Since  $\gcd(s, t) = 1$ , by Proposition 0.4 the number  $s$  is invertible modulo  $t$ . This means that there exists an integer  $u$  such that

$$s \cdot u \equiv_t 1.$$

Now multiply the congruence  $s \cdot n \equiv_t (b - a)$  by  $u$ :

$$(s \cdot u) \cdot n \equiv_t u \cdot (b - a).$$

But  $s \cdot u \equiv_t 1$ , so the left-hand side becomes

$$1 \cdot n \equiv_t u(b - a),$$

that is,

$$n \equiv_t u(b - a).$$

Therefore, if we choose  $n := u(b - a)$ , we get our solution, which is

$$x = a + s \cdot u(b - a).$$

□

*Remark 0.3.* Observe that Theorem 0.1 tells us about the *existence* of a solution. However, in general the solution is not unique. Indeed, suppose that we have a solution  $x$ . Then also  $x + st$  is a solution, because

$$x + st \equiv_s x, \quad x + st \equiv_t x.$$

In particular, after having found a solution  $x$ , all the other solutions are of the form  $x + st \cdot k$  with  $k$  an integer.

## Solution to the bus problem

Let  $x$  be the waiting time (in minutes) until a simultaneous arrival.

From the data, we get:

- Bus  $A$ : passes every 3 minutes and arrives in 2 minutes, so  $x = 3 \cdot n + 2$ , i.e.

$$x \equiv_3 2.$$

- Bus  $B$ : passes every 24 minutes and arrives in 5 minutes, so

$$x \equiv_{24} 5.$$

- Bus  $C$ : passes every 23 minutes and arrives in 10 minutes, so

$$x \equiv_{23} 10.$$

So we want to understand whether there exists an  $x$  such that

$$x \equiv_3 2, \quad x \equiv_{24} 5, \quad x \equiv_{23} 10.$$

Since 24 is a multiple of 3, from  $x \equiv_{24} 5$  we get that 24 divides  $x - 5$ . In particular, 3 divides  $x - 5$ , so  $x \equiv_3 5$ .

But  $5 \equiv_3 2$ , therefore  $x \equiv_3 2$  automatically.

So the condition for bus  $A$  is already forced by the condition for bus  $B$ , and the problem reduces to solving only

$$x \equiv_{24} 5, \quad x \equiv_{23} 10.$$

Once we find such an  $x$ , bus  $A$  will also arrive at time  $x$ . Theorem 0.1 tells us that such a solution exists, since  $\gcd(24, 23) = 1$ . Therefore, the buses will arrive simultaneously some time in the future (you can check that  $x = 125$  works).

This article is part of the series *Fermat's Christmas Theorem*.

**Previous:** [How many primes are there?](#)