Ax–Grothendieck Theorem

2025-12-07

Introduction

We all know from basic linear algebra that injective endomorphisms of a finite-dimensional vector space are also surjective. The same holds for finite sets. Over algebraically closed fields, there is an analogous result:

Theorem 0.1 (Special case of Ax–Grothendieck Theorem). *Let k be an algebraically closed field, $n \in \mathbb{N}$, and $f_1, \dots, f_n \in k[x_1, \dots, x_n]$. If the map*

$$F : k^n \to k^n, \qquad F(\underline{x}) = (f_1(\underline{x}), \dots, f_n(\underline{x}))$$

is injective, then it is surjective.

We will talk about an analogous and more general statement about schemes later on.

Let us comment briefly on the hypotheses. Injectivity is the correct assumption for F: the map $z \mapsto z^2$ is surjective but not injective on $\mathbb{C}$. Furthermore, the result is generally false for non-algebraically closed fields; for example, $x \mapsto x^3$ is injective but not surjective on $\mathbb{Q}$.

As a sanity check, let's verify the result for $n = 1$. Consider a polynomial $f \in k[x]$ such that the map $x \mapsto f(x)$ is injective; then f can have at most one root. Since k is algebraically closed, it follows that $f = ax + b$ with $a \neq 0$ (otherwise the map would not be injective). Hence f is surjective.

Moreover, the theorem is easy to prove when $k = \overline{\mathbb{F}}_p$.

For $k = \overline{\mathbb{F}}_p$. Let $\underline{z} := (z_1, \dots, z_n) \in k^n$ and let L be the subfield of k generated by the coefficients of the polynomials defining F and by the z_i's. The field L is a finite extension of $\mathbb{F}_p$, hence a finite set. The restriction $F : L^n \to L^n$ is well-defined and injective; since L^n is finite it is also surjective. Therefore, $\underline{z}$ lies in the image of F, proving the claim. □

Proof of the theorem

We start by proving two lemmas and then proceed with the proof of Theorem 0.1.

Lemma 0.1. *The map F is injective if and only if there exist $q_{i,j} \in k[x_1, \dots, x_n, y_1, \dots, y_n]$ and integers $k_j > 0$ such that, defining*

$$g_i(x_1, \dots, x_n, y_1, \dots, y_n) := f_i(x_1, \dots, x_n) - f_i(y_1, \dots, y_n),$$

for all j we have the identity

$$\sum_{i=1}^{n} g_i \cdot q_{i,j} = (x_j - y_j)^{k_j} \tag{1}$$

Proof. Consider the ideal $I = (g_1, \dots, g_n)$. The map F is injective if and only if all the zeros of I are of the form $(\underline{x}, \underline{x})$ for $\underline{x} \in k^n$. This is the case if and only if the zeros of I are a subset of the zeros of $J = (x_1 - y_1, \dots, x_n - y_n)$, i.e. if and only if $\mathrm{V}(I) \subseteq \mathrm{V}(J)$. By the Hilbert Nullstellensatz, this holds if and only if $J \subseteq \sqrt{I}$; hence $(x_j - y_j) \in \sqrt{I}$. □

Lemma 0.2. *Let $\underline{z} \in k^n$. Then $\underline{z}$ is not in the image of F if and only if there exist $r_1, \dots, r_n \in k[x_1, \dots, x_n]$ such that*

$$\sum_{i=1}^{n} r_i \cdot (f_i - z_i) = 1. \tag{2}$$

Proof. The element $\underline{z}$ is not in the image of F if and only if the ideal $I = (f_1 - z_1, \dots, f_n - z_n)$ has no zeros, equivalently $1 \in I$ (again by Nullstellensatz). □

Now we are ready to prove the theorem.

Of Theorem 0.1. Suppose $\underline{z} \in k^n$ is not in the image of F. Let A be the $\mathbb{Z}$-algebra generated by:

- the coefficients of the f_i's defining F;
- the z_i's;
- the coefficients of the $q_{i,j}$ as in Lemma 0.1;
- the coefficients of the r_i as in Lemma 0.2.

Since finitely many coefficients are involved, A is of finite type over $\mathbb{Z}$. Moreover, the restriction $F : A^n \to A^n$ is still injective. Furthermore, we have the following identities in $A[x_1, \dots, x_n, y_1, \dots, y_n]$ given respectively by Lemma 0.1 and Lemma 0.2:

$$\sum_{i=1}^{n} g_i \cdot q_{i,j} = (x_j - y_j)^{k_j} \tag{3}$$

$$\sum_{i=1}^{n} r_i \cdot (f_i - z_i) = 1 \tag{4}$$

Now pick a maximal ideal m in A. The field A/m is finite by the following claim.

Claim: Let B be an algebra of finite type over $\mathbb{Z}$. If m is a maximal ideal of B, then B/m is a finite field.

Assuming the claim, set $L = A/m$ and note that the reduction $\overline{F} : L^n \to L^n$ of F modulo m is injective. Indeed, consider $\underline{x}, \underline{y} \in L^n$ such that $\overline{F}(\underline{x}) = \overline{F}(\underline{y})$. By evaluating Equation 3 at $(\underline{x}, \underline{y})$ we obtain $\underline{x} = \underline{y}$ in L^n, since $\overline{g}_i(\underline{x}, \underline{y}) = 0$ for all i.

Since L^n is finite and $\overline{F}$ is injective, $\overline{F}$ is surjective. However, reducing $\underline{z}$ modulo m and evaluating Equation 4 at a preimage of that reduction under $\overline{F}$ yields a contradiction. Hence, up to proving the claim, we have established Theorem 0.1.

of the Claim. Consider two cases:

- If $m \cap \mathbb{Z} = p\mathbb{Z}$, then B/m is an algebra of finite type over $\mathbb{F}_p$; by Zariski's lemma B/m is a finite extension of $\mathbb{F}_p$, hence finite.
- If $m \cap \mathbb{Z} = 0$, then in the tower $\mathbb{Z} \subset \mathbb{Q} \subset B/m$, we see that B/m is of finite type over $\mathbb{Q}$, hence is finite over $\mathbb{Q}$ by Zariski's lemma. Artin–Tate's lemma then implies that $\mathbb{Q}$ would be of finite type over $\mathbb{Z}$, a contradiction.

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Generalization to schemes

Note that the map F considered in Theorem 0.1 can be seen as the map induced on closed points by a morphism of k-schemes $\mathbb{A}^n_k \to \mathbb{A}^n_k$. This leads to the following question:

Question:

Let k be an algebraically closed field and X a scheme of finite type over k. If $f : X \to X$ is a k-endomorphism injective on k-points, is it surjective on all points?

The finite-type assumption is necessary: the map induced by

$$k[X_1, \dots, X_n, \dots] \to k[X_1, \dots, X_n, \dots]$$

sending $X_1 \mapsto 0$ and $X_i \mapsto X_{i-1}$ for $i > 1$ is injective but not surjective.

The answer is positive, and it follows (not trivially) from the following theorem.

Theorem 0.1 (Ax–Grothendieck Theorem)**.** *Let S be a scheme and let X be a scheme of finite type over S. Any injective S-endomorphism $f : X \to X$ is bijective.*

Moreover, if X is reduced and k is an algebraically closed field of characteristic 0, then an injective endomorphism on k-points is in fact an isomorphism (not merely a bijection). This is not trivial since we have bijective morphisms of schemes that are not isomorphisms (for example the one induced by $k[t^2, t^3] \to k[t]$). A reference for this is Lemma 1 of this article.

We will not give a proof of Theorem 0.1; see J. Ax's paper Injective endomorphisms of varieties and schemes, or Grothendieck's version in EGA IV_3, Proposition 10.4.11 for a complete proof.

Let us explain why Theorem 0.1 implies that injectivity on k-points gives surjectivity on all points.

The first ingredient is Chevalley's theorem on constructible sets (see Hartshorne, Ex. II.3.19).

Theorem 0.2 (Chevalley's theorem)**.** *Let $f : X \to Y$ be a finite type morphism between noetherian schemes. If $Z \subseteq X$ is a constructible subset (i.e. a finite union of locally closed subsets), then $f(Z)$ is constructible.*

The following lemma explains why Theorem 0.1 implies Theorem 0.1.

Lemma 0.1. *Let X, Y be schemes of finite type over an algebraically closed field k. Let $f : X \to Y$ be a morphism whose restriction to k-points*

$$f : X(k) \to Y(k)$$

is injective. Then f is universally injective.

Assuming this lemma, consider a morphism $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ over k which is injective on closed points (which are exactly the k-points, since k is algebraically closed). Then in particular f is injective on all points. From Theorem 0.1 we get the surjectivity of f on all points (answering the question raised before) and in particular also surjectivity on closed points, since only closed points are sent to closed points. In particular, we obtain Theorem 0.1.

Of Lemma 0.1. Observe that the diagonal $\Delta_f : X \to X \times_Y X$ is of finite type. Indeed, it is locally of finite type because $X \to \mathrm{Spec}(k)$ is locally of finite type (and Δ_f can be seen as the first factor of this map), and it is quasi-compact because X is noetherian (being of finite type over k).

Moreover, the diagonal Δ_f is surjective on k-points. Indeed, for any k-point of $X \times_Y X$ given by a morphism

$$\psi : \mathrm{Spec}(k) \to X \times_Y X$$

we can consider the diagram

and by injectivity of the morphism $X \to Y$ we obtain $\phi_1 = \phi_2$ as maps of topological spaces. They are also the same on the structure sheaf, because k is algebraically closed. Therefore, $\phi_1 = \phi_2$, which means that ψ factorizes through the diagonal Δ_f.

Furthermore, $X \times_Y X$ is a Jacobson scheme (i.e. its underlying topological space is Jacobson) because it is of finite type over k, and k is Jacobson. By Theorem 0.2 the set $\Delta_f(X)$ is constructible. Consider $Z := (X \times_Y X) \setminus \Delta_f(X)$. This is constructible and every closed point of $X \times_Y X$ lies inside $\Delta_f(X)$ (by surjectivity of Δ_f on closed points). Since every nonempty constructible subset in a Jacobson scheme contains a closed point, it follows that $Z = \emptyset$. Therefore, Δ_f is surjective on all points.

Now we prove that the surjectivity of Δ_f implies the universal injectivity of f. Indeed, consider any $g : Z \to Y$ and the diagonal

$$\Delta : X \times_Y Z \to (X \times_Y Z) \times_Z (X \times_Y Z)$$

of the base change. This diagonal map Δ is surjective because it is the base change of Δ_f, which is surjective. Thus, we are reduced to proving that surjectivity of Δ_f implies injectivity of f (the same will then hold for any base change).

Consider $x_1, x_2 \in X$ with $f(x_1) = f(x_2) =: y$. We can find a field L such that the following diagram commutes:

We get the following commutative diagram:

By surjectivity of Δ_f we can find $x \in X$ such that $\Delta_f(x)$ is the image of ϵ, hence we get $x_1 = x_2$. This shows that f is injective, and therefore universally injective. $\square$

Note that the complex-geometric picture does not match the algebraic one. For example, there are holomorphic maps $\mathbb{C}^n \to \mathbb{C}^n$ that are injective but not surjective. However, this cannot happen for $n = 1$, as one can show that all holomorphic injective maps $f : \mathbb{C} \to \mathbb{C}$ are of the form $f = ax + b$; see the Casorati–Weierstrass theorem.

Conclusion

To conclude, we point your attention to the fact that this theorem is linked to a famous conjecture, namely the Jacobian conjecture.

Conjecture 0.1 (Jacobian conjecture)**.** *Let k be a field of characteristic 0 and let $F : k^n \to k^n$ be a polynomial mapping. If the Jacobian determinant J_F is invertible, then F is an isomorphism.*

Over an algebraically closed field of characteristic 0, the fact that the bijection in Theorem 0.1 becomes an isomorphism shows that, to prove the Jacobian conjecture for $\mathbb{A}^n_k$, it suffices to show that a map

$$F : \mathbb{A}^n_k \to \mathbb{A}^n_k$$

with invertible Jacobian determinant is injective.

I would like to thank my friend who suggested this topic to me and helped me write this post.

References

This post was mainly inspired by this post from Terence Tao's blog. Other references are cited in the text when used.