Weak equivalences between categories of models of type theory

Simon Cho Cory M. Knapp Clive Newstead Liang Ze Wong

Mentors: Chris Kapulkin Emily Riehl

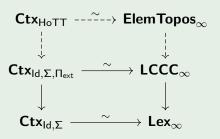
AMS Special Session on Homotopy Type Theory (a Mathematics Research Communities Session) NSF Grant No. DMS 1641020

11 Jan 2018

The Internal Language Conjectures make precise the belief that intensional type theory is the internal language of ∞ -categories:

Conjecture (Kapulkin-Lumsdaine 2016)

There are ∞ -equivalences:



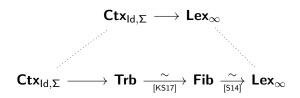
Chris Kapulkin and Peter LeFanu Lumsdaine. 'The homotopy theory of type theories'. In: arXiv preprint arXiv:1610.00037 (2016).

In this talk, we present progress towards the ∞ -equivalence:

$$\text{Ctx}_{\text{Id},\Sigma} \longrightarrow \text{Lex}_{\infty}$$

Chris Kapulkin and Karol Szumilo. 'Internal language of finitely complete (∞ , 1)-categories'. In: arXiv preprint arXiv:1709.09519 (2017).

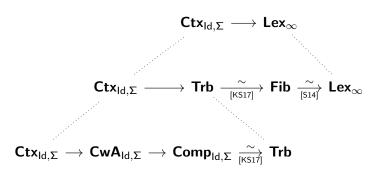
In this talk, we present progress towards the ∞ -equivalence:



Karol Szumiło. 'Two models for the homotopy theory of cocomplete homotopy theories'. In: arXiv preprint arXiv:1411.0303 (2014).

Chris Kapulkin and Karol Szumilo. 'Internal language of finitely complete (∞ , 1)-categories'. In: arXiv preprint arXiv:1709.09519 (2017).

In this talk, we present progress towards the ∞ -equivalence:



Karol Szumiło. 'Two models for the homotopy theory of cocomplete homotopy theories'. In: arXiv preprint arXiv:1411.0303 (2014).

Chris Kapulkin and Karol Szumilo. 'Internal language of finitely complete (∞ , 1)-categories'. In: arXiv preprint arXiv:1709.09519 (2017).

Theorem (CKNW)

Assume our models satisfy the Logical Framework (LF) of [LW15]. The comparison functors

$$\textbf{Ctx}_{\text{Id},\Sigma,\text{LF}} \longleftarrow \textbf{CwA}_{\text{Id},\Sigma,\text{LF}} \longleftarrow \textbf{Comp}_{\text{Id},\Sigma,\text{LF}} \longleftarrow \textbf{Trb}_{\text{LF}}$$

have homotopy inverses, hence are ∞ -equivalences.

Theorem (CKNW)

Assume our models satisfy the Logical Framework (LF) of [LW15]. The comparison functors

$$\textbf{Ctx}_{\text{Id},\Sigma,\text{LF}} \longleftarrow \textbf{CwA}_{\text{Id},\Sigma,\text{LF}} \longleftarrow \textbf{Comp}_{\text{Id},\Sigma,\text{LF}} \longleftarrow \textbf{Trb}_{\text{LF}}$$

have homotopy inverses, hence are ∞ -equivalences.

Plan:

- Compare these models of type theory
- Highlight the role that each model plays in the equivalence
- Compare weak equivalences within each model

Type-theoretic models

 $Ctx \longrightarrow CwA \longrightarrow Comp \longleftarrow Trb$

a type theory	${\mathbb T}$	\mathcal{C}	a category
contexts	Γ, Δ, \dots	Γ, Δ, \dots	objects
substitutions	$\Delta \xrightarrow{\sigma} \Gamma$	$\Delta \xrightarrow{\sigma} \Gamma$	maps

a type theory	${\mathbb T}$	\mathcal{C}	a category
contexts	Γ, Δ, \dots	Γ, Δ, \dots	objects
substitutions	$\Delta \xrightarrow{\sigma} \Gamma$	$\Delta \xrightarrow{\sigma} \Gamma$	maps
types	$\Gamma \vdash A$ type	Г.А → Г	canonical projections
terms	Γ ⊢ <i>a</i> : <i>A</i>	$\Gamma \xrightarrow{a} \Gamma.A$	sections of $\Gamma.A o \Gamma$

	a type theory	\mathbb{T}	\mathcal{C}	a category
	contexts	Γ, Δ, \dots	Γ, Δ, \dots	objects
	substitutions	$\Delta \xrightarrow{\sigma} \Gamma$	$\Delta \xrightarrow{\sigma} \Gamma$	maps
	types	$\Gamma \vdash A$ type	Г.А → Г	canonical projections
	terms	Γ ⊢ <i>a</i> : <i>A</i>	$\Gamma \xrightarrow{a} \Gamma.A$	sections of $\Gamma.A \twoheadrightarrow \Gamma$
	substitution of A along $\Delta \xrightarrow{\sigma} \Gamma$	$A[\sigma]$	$\Delta . A[\sigma]$	sections of $\Gamma.A \rightarrow \Gamma$ pullback of $\Gamma.A \rightarrow \Gamma$ along $\Delta \xrightarrow{\sigma} \Gamma$
$\begin{array}{ccc} \Delta.A[\sigma] \stackrel{\sigma.A}{\longrightarrow} \Gamma.A \\ \downarrow & \downarrow \\ \Delta \stackrel{\sigma}{\longrightarrow} \Gamma \end{array}$				

	a type theory	\mathbb{T}	\mathcal{C}	a category
		Γ, Δ, \dots		
	substitutions	$\Delta \xrightarrow{\sigma} \Gamma$	$\Delta \xrightarrow{\sigma} \Gamma$	maps
	types	$\Gamma \vdash A$ type	Г.А → Г	canonical projections
	terms	Γ ⊢ <i>a</i> : <i>A</i>	$\Gamma \xrightarrow{a} \Gamma.A$	sections of $\Gamma.A \twoheadrightarrow \Gamma$
	substitution of A along $\Delta \xrightarrow{\sigma} \Gamma$	$A[\sigma]$	$\Delta . A[\sigma]$	pullback of $\Gamma.A woheadrightarrow \Gamma$ along $\Delta \xrightarrow{\sigma} \Gamma$
$\begin{array}{ccc} \Delta.A[\sigma] \stackrel{\sigma.A}{\longrightarrow} \Gamma.A \\ \downarrow & \downarrow \\ \Delta \stackrel{\sigma}{\longrightarrow} \Gamma \end{array}$				

Further, choice of pullbacks is functorial.

Encoding type dependencies

Two ways to keep track of type dependencies $(\Gamma.A \rightarrow \Gamma)$:

Encoding type dependencies

Two ways to keep track of type dependencies $(\Gamma.A \rightarrow \Gamma)$:

Contextual categories have an N-grading on objects

$$\mathsf{Ob}_0\mathcal{C} \longleftarrow \cdots \longleftarrow \mathsf{Ob}_n\mathcal{C} \stackrel{p_n}{\longleftarrow} \mathsf{Ob}_{n+1}\mathcal{C} \longleftarrow \cdots$$

such that $p_n(\Gamma.A) = \Gamma$ (where $\Gamma \in Ob_nC$).

Encoding type dependencies

Two ways to keep track of type dependencies $(\Gamma.A \rightarrow \Gamma)$:

Contextual categories have an N-grading on objects

$$\mathsf{Ob}_0\mathcal{C} \longleftarrow \cdots \longleftarrow \mathsf{Ob}_n\mathcal{C} \stackrel{p_n}{\longleftarrow} \mathsf{Ob}_{n+1}\mathcal{C} \longleftarrow \cdots$$

such that $p_n(\Gamma.A) = \Gamma$ (where $\Gamma \in Ob_nC$).

Categories with Attributes have a functor

$$\mathsf{Ty} \colon \mathcal{C}^{\mathit{op}} \to \mathbf{Set}$$

such that $A \in Ty(\Gamma)$.

Homotopy-theoretic models

 $Ctx \longrightarrow CwA \longrightarrow Comp \longleftarrow Trb$

Tribes

Tribes are categories with a distinguished class of maps — called **fibrations**, such that pullbacks against — exist.

Tribes

Tribes are categories with a distinguished class of maps — called **fibrations**, such that pullbacks against — exist.

Fibrations determine **path objects**, which can then be used to define homotopies $f \sim g$ and homotopy equivalences $X \xrightarrow{\sim} Y$.

Tribes

Tribes are categories with a distinguished class of maps — called **fibrations**, such that pullbacks against — exist.

Fibrations determine **path objects**, which can then be used to define homotopies $f \sim g$ and homotopy equivalences $X \xrightarrow{\sim} Y$.

Tribes are categories with weak equivs. := homotopy equivs.

Contextual categories vs. Tribes

Contextual categories are tribes:

	Contextual categories	Tribes
$\rightarrow\!$	Canonical projections	Fibrations
Σ	Dependent sum	Composition
ld	Identity types	Path objects

Contextual categories vs. Tribes

Contextual categories are tribes:

	Contextual categories	Tribes
$\rightarrow\!$	Canonical projections	Fibrations
Σ	Dependent sum	Composition
ld	Identity types	Path objects

However, tribes are not contextual categories:

- no functorial choice of pullbacks
- no pullback-stable choice of Id- and Σ -types.



Comprehension categories encode pullbacks against \twoheadrightarrow via a Grothendieck fibration $P \colon \mathcal{T} \to \mathcal{C}$.

Comprehension categories encode pullbacks against \twoheadrightarrow via a Grothendieck fibration $P \colon \mathcal{T} \to \mathcal{C}$.

They provide the right setting for strictifying these pullbacks:

Theorem (Lumsdaine-Warren 2015)

Let $\mathcal C$ be a full comprehension category satisfying LF, with weakly stable Id and Σ -types.

There is an equivalent full split comprehension category C_1 with strictly stable Id and Σ -types.

Comprehension categories encode pullbacks against \twoheadrightarrow via a Grothendieck fibration $P \colon \mathcal{T} \to \mathcal{C}$.

They provide the right setting for strictifying these pullbacks:

Theorem (Lumsdaine-Warren 2015)

Let $\mathcal C$ be a full comprehension category satisfying LF, with weakly stable Id and Σ -types. \leadsto tribes

There is an equivalent full split comprehension category C_1 with strictly stable Id and Σ -types. \leadsto contextual cats/CwAs

Comprehension categories encode pullbacks against \twoheadrightarrow via a Grothendieck fibration $P \colon \mathcal{T} \to \mathcal{C}$.

They provide the right setting for strictifying these pullbacks:

Theorem (Lumsdaine-Warren 2015)

Let C be a full comprehension category satisfying LF, with weakly stable Id and Σ -types. \leadsto tribes

There is an equivalent full split comprehension category C_1 with strictly stable Id and Σ -types. \leadsto contextual cats/CwAs

Logical Framework (LF):

- Existence of categorical dependent exponentials.
- Satisfied if $\mathcal C$ has Π -types or is locally cartesian closed.

 $\mathsf{Ctx} \xrightarrow{\bot} \mathsf{CwA} \xrightarrow{\longleftarrow} \mathsf{SplComp} \xrightarrow{\longleftarrow} \mathsf{Comp} \xrightarrow{\longleftarrow} \mathsf{Trb}$

$$\mathsf{Ctx} \xrightarrow{\longleftarrow} \mathsf{CwA} \xrightarrow{\longleftarrow} \mathsf{SplComp} \xrightarrow{\stackrel{[\mathsf{LW15}]}{\longleftarrow}} \mathsf{Comp} \xleftarrow{\longleftarrow} \mathsf{Trb}$$

Need units and counits to be natural weak equivalences.



Type-theoretic vs. homotopy-theoretic equivalences

Given Id-types/path objects, we may define homotopies $f \sim g$ and homotopy equivalences $X \xrightarrow{\sim} Y$.

Type-theoretic vs. homotopy-theoretic equivalences

Given Id-types/path objects, we may define homotopies $f \sim g$ and homotopy equivalences $X \xrightarrow{\sim} Y$.

 $F \colon \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if *types* and *terms* in \mathcal{D} have preimages in \mathcal{C} , up to homotopy (*weak type* and *term lifting*).

 $F \colon \mathcal{C} \to \mathcal{D}$ is a **homotopy-theoretic equivalence** (HE) if it induces an equivalence Ho $\mathcal{C} \cong \text{Ho } \mathcal{D}$.

Type-theoretic vs. homotopy-theoretic equivalences

Given Id-types/path objects, we may define homotopies $f \sim g$ and homotopy equivalences $X \xrightarrow{\sim} Y$.

 $F: \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if *types* and *terms* in \mathcal{D} have preimages in \mathcal{C} , up to homotopy (*weak type* and *term lifting*).

 $F \colon \mathcal{C} \to \mathcal{D}$ is a **homotopy-theoretic equivalence** (HE) if it induces an equivalence Ho $\mathcal{C} \cong \text{Ho } \mathcal{D}$.

Note: LE assumes homotopies factor through chosen Id-types, and requires knowledge of 'immediate' type-dependencies.

Generalizations of logical equivalence

We may generalize LE in two ways, with a view towards HE:

- Require homotopies/homotopy equivalences to factor through some Id-type (rather than a chosen Id-type)
- Require weak context and section lifting

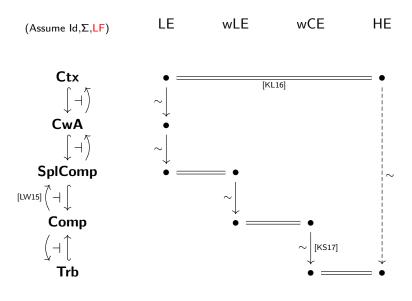
Generalizations of logical equivalence

We may generalize LE in two ways, with a view towards HE:

- Require homotopies/homotopy equivalences to factor through some Id-type (rather than a chosen Id-type)
- Require weak context and section lifting

	Type/term lifting	Context/section lifting
Chosen Id-type	LE	CE
Some Id-type	wLE	wCE

Summary



Thank you!

Questions/comments?

Categorical Semantics

- **Contexts** are finite lists $[x_1 : A_1, ..., x_n : A_n]$, up to definitional equality and renaming of variables
- Substitutions $[x_1:A_1,\ldots,x_n:A_n] \xrightarrow{f} [y_1:B_1,\ldots,y_m:B_m]$ are sequences

$$x_1 : A_1, \dots, x_n : A_n \vdash f_1 : B_1$$

$$\vdots$$

$$x_1 : A_1, \dots, x_n : A_n \vdash f_m : B_m$$

Contextual categories

Definition (Cartmell 1978)

A **contextual category** is a category C with:

- **1** A grading on objects $Ob C = \coprod_{n \in \mathbb{N}} Ob_n C$
- $oldsymbol{0}$ A terminal object 1 which is the unique object in $\mathsf{Ob}_0\mathcal{C}$
- **3** Maps pt: $Ob_{n+1}C \rightarrow Ob_nC$
- Canonical projections A → ptA
- **1** A functorial choice of pullbacks against canonical projections:

$$A[\sigma] \xrightarrow{\sigma.A} A$$

$$\downarrow \qquad \qquad \downarrow$$

$$\Delta \xrightarrow{\sigma} ptA$$

JW Cartmell. 'Generalised algebraic theories and contextual categories'. PhD thesis. University of Oxford, 1978.

Categories with Attributes

Definition (Cartmell 1978)

A category with attributes is a category C with:

- A terminal object 1
- **2** A functor Ty: $C^{op} \rightarrow \mathbf{Set}$
- **3** For each $A \in Ty(\Gamma)$, an object $\Gamma.A \in \mathcal{C}$ and map $\Gamma.A \rightarrow \Gamma$
- **4** A functorial choice of a map $\sigma.A$, for each $A \in \mathsf{Ty}(\Gamma)$ and $\Delta \xrightarrow{\sigma} \Gamma$, fitting into a pullback square:

$$\begin{array}{ccc} \Delta.(\mathsf{Ty}f)(A) & \xrightarrow{\sigma.A} & \Gamma.A \\ \downarrow & & \downarrow \\ & \downarrow & & \downarrow \\ & \Delta & \xrightarrow{\sigma} & \Gamma \end{array}$$

Tribes

Definition (Joyal 2014, Shulman 2015)

A **tribe** is a category C with a subcategory F of *fibrations* (\twoheadrightarrow) containing all isomorphisms, such that

- $oldsymbol{0}$ C has a terminal object, and all objects are fibrant
- Pullbacks along fibrations exist, and fibrations are stable under pullback
- $\textbf{ anodyne maps } (\stackrel{\sim}{\rightarrowtail}) \text{ are stable under pullback along fibrations}$
- Every map factors as $\cdot \stackrel{\sim}{\longrightarrow} \cdot \longrightarrow \cdot$

Paths and homotopies within tribes

Definition

A **path object** for $Y \in \mathcal{C}$ is a factorization of the diagonal:

$$\Delta_Y = (Y \not\stackrel{\sim}{\longrightarrow} PY \xrightarrow{} Y \times Y)$$

Two maps $f, g: X \to Y$ are **homotopic** $(f \sim g)$ if there is a factorization of (f, g) through PY:

$$(f,g) = (X \xrightarrow{H} PY \longrightarrow Y \times Y)$$

A map $f: X \to Y$ is a **homotopy equivalence** $(f: X \xrightarrow{\sim} Y)$ if there is a map $g: Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$.

Paths and homotopies within tribes

Definition

A **path object** for $Y \in \mathcal{C}$ is a factorization of the diagonal:

$$\Delta_Y = (Y \not\stackrel{\sim}{\longrightarrow} PY \xrightarrow{} Y \times Y)$$

Two maps $f,g:X\to Y$ are **homotopic** $(f\sim g)$ if there is a factorization of (f,g) through PY:

$$(f,g) = (X \xrightarrow{H} PY \longrightarrow Y \times Y)$$

A map $f: X \to Y$ is a **homotopy equivalence** $(f: X \xrightarrow{\sim} Y)$ if there is a map $g: Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$.

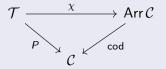
Every tribe is a category with weak equivalences, where

Weak equivalences := Homotopy equivalences

Comprehension categories

Definition (Jacobs 1993)

A **comprehension category** consists of a Grothendieck fibration $\mathcal{T} \xrightarrow{P} \mathcal{C}$ and a *comprehension functor* $\chi \colon \mathcal{T} \to \operatorname{Arr} \mathcal{C}$ such that



commutes, and χ sends cartesian maps to pullback squares. A comprehension category is **split** if P is a split fibration, and **full** if χ is fully faithful.

Categories with attributes are full split comprehension categories.

Bart Jacobs. 'Comprehension categories and the semantics of type dependency'. In: Theoretical Computer Science 107.2 (1993), pp. 169–207.

Homotopy equivalence

Let $\mathcal C$ be a model of type theory, with Id-types which give rise to homotopies $f\sim g$.

Homotopy equivalence

Let $\mathcal C$ be a model of type theory, with Id-types which give rise to homotopies $f\sim g$.

Definition

The **homotopy category** of $\mathcal C$ is the category Ho $\mathcal C$ with the same objects, and homotopy classes of maps.

Definition

A functor of models $F \colon \mathcal{C} \to \mathcal{D}$ is a **homotopy equivalence** (HE) if it induces an equivalences of categories $\operatorname{Ho} \mathcal{C} \cong \operatorname{Ho} \mathcal{D}$.

Definition

A functor $F: \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

• weak type lifting: For all $\Gamma \in \mathcal{C}, A \in \mathsf{Ty}(F\Gamma)$, there exists $\tilde{A} \in \mathsf{Ty}(\Gamma)$ and $F\tilde{A} \xrightarrow{\sim} A$ over $F\Gamma$.





Definition

A functor $F: \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

• weak type lifting: For all $\Gamma \in \mathcal{C}, A \in \mathsf{Ty}(F\Gamma)$, there exists $\tilde{A} \in \mathsf{Ty}(\Gamma)$ and $F\tilde{A} \xrightarrow{\sim} A$ over $F\Gamma$.





Definition

A functor $F: \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

• weak type lifting: For all $\Gamma \in \mathcal{C}, A \in \mathsf{Ty}(F\Gamma)$, there exists $\tilde{A} \in \mathsf{Ty}(\Gamma)$ and $F\tilde{A} \xrightarrow{\sim} A$ over $F\Gamma$.

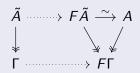




Definition

A functor $F: \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

• weak type lifting: For all $\Gamma \in \mathcal{C}, A \in \mathsf{Ty}(F\Gamma)$, there exists $\tilde{A} \in \mathsf{Ty}(\Gamma)$ and $F\tilde{A} \xrightarrow{\sim} A$ over $F\Gamma$.

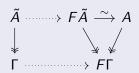




Definition

A functor $F: \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

• weak type lifting: For all $\Gamma \in \mathcal{C}, A \in \mathsf{Ty}(F\Gamma)$, there exists $\tilde{A} \in \mathsf{Ty}(\Gamma)$ and $F\tilde{A} \xrightarrow{\sim} A$ over $F\Gamma$.



Definition

A functor $F: \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

• weak type lifting: For all $\Gamma \in \mathcal{C}, A \in \mathsf{Ty}(F\Gamma)$, there exists $\tilde{A} \in \mathsf{Ty}(\Gamma)$ and $F\tilde{A} \xrightarrow{\sim} A$ over $F\Gamma$.

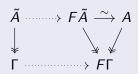


$$A \longrightarrow FA$$
 $\tilde{a} \qquad \tilde{b} \qquad \tilde{b}$

Definition

A functor $F: \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

• weak type lifting: For all $\Gamma \in \mathcal{C}, A \in \mathsf{Ty}(F\Gamma)$, there exists $\tilde{A} \in \mathsf{Ty}(\Gamma)$ and $F\tilde{A} \xrightarrow{\sim} A$ over $F\Gamma$.

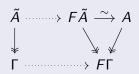


$$\begin{array}{ccc}
A & \cdots & FA \\
\downarrow & \uparrow & F \downarrow & \uparrow \\
\Gamma & \cdots & \downarrow & F\Gamma
\end{array}$$

Definition

A functor $F: \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

• weak type lifting: For all $\Gamma \in \mathcal{C}, A \in \mathsf{Ty}(F\Gamma)$, there exists $\tilde{A} \in \mathsf{Ty}(\Gamma)$ and $F\tilde{A} \xrightarrow{\sim} A$ over $F\Gamma$.



$$\begin{array}{ccc}
A & \cdots & FA \\
\tilde{a} & & F\tilde{a} \left(\sim \right) a \\
\Gamma & \cdots & & F\Gamma
\end{array}$$