SUPPLEMENTARY MATERIALS: A SEMI-ANALYTIC DIAGONALIZATION FEM FOR THE SPECTRAL FRACTIONAL LAPLACIAN

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SM1. Proof of Corollary 2.4. Let $\{\phi_j\}_{j\geq 1} \subset H_0^1(\Omega)$ and $\{\psi_k\}_{k\geq 1} \subset H^1_{\mathcal{Y}}(y^\alpha,(0,\mathcal{Y}))$ be orthonormal bases which, in addition, are orthogonal in $L^2(\Omega)$ and $L^2(y^\alpha,(0,\mathcal{Y}))$, respectively. We can, for instance, choose $\{\phi_j\}_{j\geq 1}$ to be the eigenfunctions of the Dirichlet Laplacian, as described in Subsection 2.1. An example of the family $\{\psi_k\}_{k\geq 1}$ will be given below in Theorem 3.1.

Clearly, $\phi_j\psi_k \in \mathring{H}^1_L(y^\alpha,\mathcal{C}_{\mathcal{Y}})$. Observe, in addition, that

$$
\begin{aligned}
(\phi_j\psi_k,\phi_s\psi_t)_{\mathring{H}^1_L(y^\alpha,\mathcal{C}_{\mathcal{Y}})} &= \int_0^{\mathcal{Y}}\int_\Omega \big([\nabla\phi_j(x)\cdot\nabla\phi_s(x)]\psi_k(y)\psi_t(y) \\
&\qquad + \phi_j(x)\phi_s(x)\psi_k'(y)\psi_t'(y)\big)\,dx\,dy \\
&= \delta_{js}\int_0^{\mathcal{Y}} y^\alpha\psi_k(y)\psi_t(y)\,dy + \delta_{kt}\int_\Omega\phi_j(x)\phi_s(x)\,dx \\
&= C_{jkst}\delta_{js}\delta_{kt}\,.
\end{aligned}
$$

Thus, the family $\{\phi_j\psi_k\}_{j,k\geq 1}$ is linearly independent. Using Gram-Schmidt, this linearly independent set contains an orthonormal subset which, to simplify the notation, we do not relabel.

Next we show that $\{\phi_j\psi_k\}_{j,k\geq 1}$ is complete in $\mathring{H}^1_L(y^\alpha,\mathcal{C}_{\mathcal{Y}})$. To see this we assume that $W \in \mathring{H}^1_L(y^\alpha,\mathcal{C}_{\mathcal{Y}})$ is such that

$$
\int_0^{\mathcal{Y}} y^\alpha\int_\Omega \left([\nabla_x W(x,y)\cdot\nabla\phi_j(x)]\psi_k(y) + \partial_y W(x,y)\phi_j(x)\psi_k'(y)\right)\,dx\,dy = 0\,,
$$

for all $j,k\in\mathbb{N}$. This implies that

$$
\begin{aligned}
0 = \int_0^{\mathcal{Y}} y^\alpha\psi_k(y)\left(\int_\Omega\nabla_x W(x,y)\cdot\nabla_x\phi_j(x)\,dx\right)\,dy + \\
\int_\Omega\phi_j(x)\left(\int_0^{\mathcal{Y}} y^\alpha\partial_y W(x,y)\psi_k'(y)\,dy\right)\,dx\,.
\end{aligned}
$$

But, because ϕ_j and ψ_k are linearly independent, this is only possible if

$$
\int_\Omega \nabla_x W(x,y)\cdot\nabla_x\phi_j(x)\,dx = 0\,,
$$

for all $j\in\mathbb{N}$ and a.e. $y\in(0,\mathcal{Y})$; and

$$
\int_0^{\mathcal{Y}} y^\alpha\partial_y W(x,y)\psi_k'(y)\,dy = 0\,,
$$

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for all $k \in \mathbb{N}$ and a.e. $x \in \Omega$. Let now, for each $j \in \mathbb{N}$, $S_j \subset (0, \mathcal{Y})$ be the set where the first condition is false. Since S_j is null, i.e.,

$$\int_{S_j} y^\alpha \, dy = 0 \,,$$

we have that $S = \bigcup_{j=1}^\infty S_j$ is also a null set, as it is a countable union of null sets.

Similarly, for $k \in \mathbb{N}$ we let $T_k \subset \Omega$ is the where the second condition fails. We again have that

$$\int_{T_k} dx = 0 \,,$$

and that $T = \bigcup_{k=1}^\infty T_k$ is another null set.

The previous observations show that

$$A(x,y) = |\nabla_x W(x,y)| + |\partial_y W(x,y)| = 0, \quad \forall (x,y) \in \mathcal{C}_\mathcal{Y} \setminus (T \times S).$$

However, $T \times S$ is a null set and therefore, $A(x,y) = 0$ almost everywhere in $\mathcal{C}_\mathcal{Y}$. As a consequence, $W = 0$ a.e. in $\mathcal{C}_\mathcal{Y}$.

In conclusion, we have shown that $\{\phi_j \psi_k\}_{j,k \geq 1}$ is a complete orthonormal basis in $\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$.

This shows that the mapping defined by

$$\begin{aligned} \mathcal{W} : H^1_0(\Omega) \otimes H^1_\mathcal{Y}(y^\alpha, (0, \mathcal{Y})) &\to \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y}) \\ \phi_j \otimes \psi_k &\mapsto \phi_j \psi_k \,, \end{aligned}$$

and extended by linearity, is the requisite isomorphism.

SM2. Convergence Figures. The convergence results from Subsection 5.1 shown in Table 1 is presented visually in Figure SM1 and from Table 2 in Figure SM2.

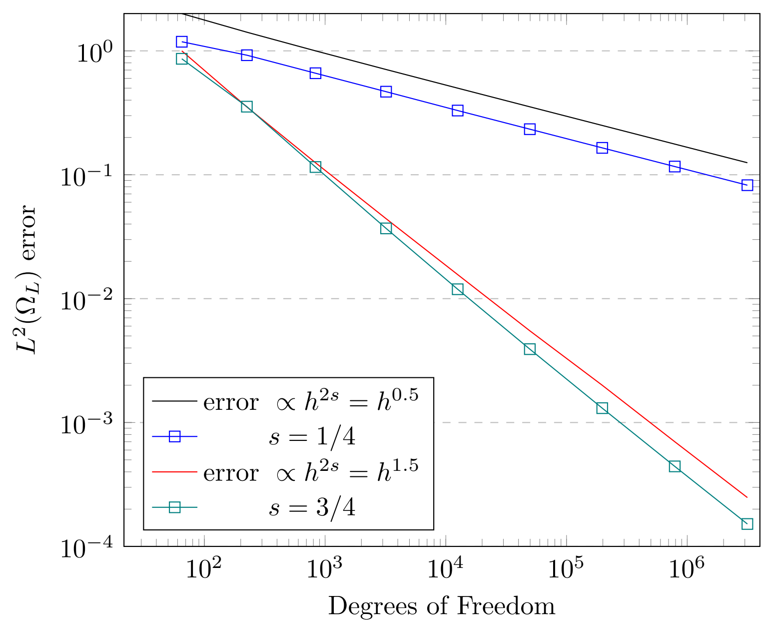

FIG. SM1. *Error in the $L^2(\Omega_L)$ norm versus the number of degrees of freedom using $\mathbb{Q}_1$ finite elements for $s = 1/4$ and $s = 3/4$ on uniformly refined meshes of Ω_L.*

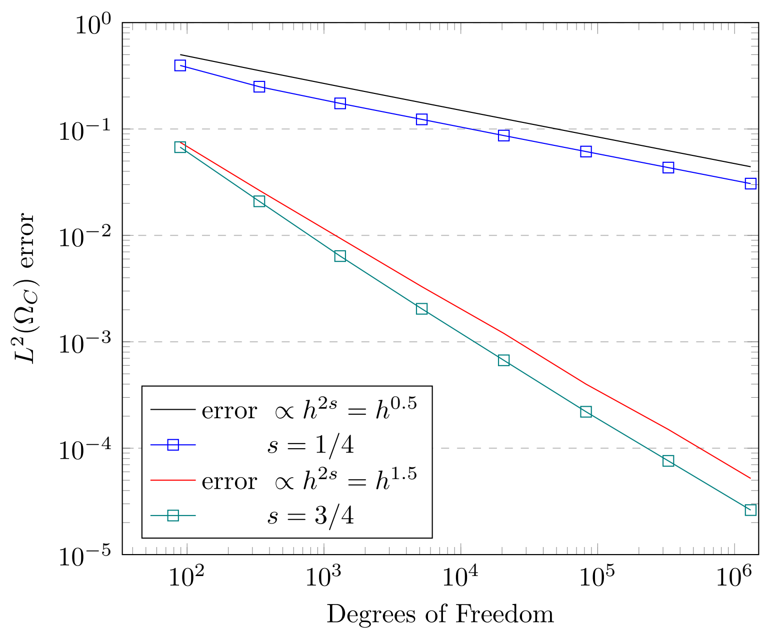

FIG. SM2. *Error in the $L^2(\Omega_C)$ norm versus the number of degrees of freedom using $\mathbb{Q}_1$ finite elements for $s = 1/4$ and $s = 3/4$ on uniformly refined meshes of Ω_C.*