MA 207 - Differential Equations-II

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Start with ....

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"It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity."

by Pierre Simon Laplace

Outline of the lecture

- Fourier Transforms (introduced by Joseph Fourier in 1822)
  - Basic Properties
  - Inversion and Plancherel Theorem
  - Applications

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"The integrals which we have obtained are not only general expressions which satisfy the differential equation, they represent in the most distinct manner the natural effect which is the object of the phenomenon... when this condition is fulfilled, the integral is, properly speaking, the equation of the phenomenon; it expresses clearly the character and progress of it, in the same manner as the finite equation of a line or curved surface makes known all the properties of those forms."

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"Primary causes are unknown to us; but are subject to simple and constant laws, which may be discovered by observation, the study of them being the object of natural philosophy. Heat, like gravity, penetrates every substance of the universe, its rays occupy all parts of space. The object of our work is to set forth the mathematical laws which this element obeys. The theory of heat will hereafter form one of the most important branches of general physics."

Examples

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&= \boxed{\sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}} \qquad (\mathit{CHECK!})
\end{aligned}$$

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Properties of Fourier Transforms

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$$f(x) = \begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{elsewhere.} \end{cases}$$

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$$\implies \mathfrak{F}(e^{\alpha x} f(x)) = \frac{1}{\sqrt{2\pi}} \frac{(1 - e^{-i(\omega + i\alpha)a})}{i(\omega + i\alpha)}$$

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Use induction to prove (B).

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Exercises: • Tutorial Sheet 6, Problem 6

• Kreyszig, Page 575, Qns 1-10

Application - Heat equation in unbounded domain

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We know that $\boxed{\mathfrak{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}}$

Now, $e^{-\omega^2 t} = e^{-\omega^2/(1/t))}$. Comparing, $4a = 1/t \Longrightarrow a = \dfrac{1}{4t}$

Set $\hat{g}(\omega) = e^{-\omega^2 t}$.
Then, $\hat{u}(\omega, t) = \hat{u_0}(\omega)\hat{g}(\omega)$.

Using convolution theorem, we have $\boxed{\hat{u}(\omega, t) = \frac{1}{\sqrt{2\pi}}\mathfrak{F}(f * g)}$

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That is, $\hat{g}(\omega, t) = \frac{1}{\sqrt{2t}}\mathfrak{F}(e^{-x^2/4t}) \Longrightarrow \boxed{g(x, t) = \frac{1}{\sqrt{2t}}(e^{-x^2/4t}).}$

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\end{aligned}
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Hence, $\boxed{u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} u_0(y)e^{-(x-y)^2/4t}\,dy}$

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Stability property