Signal Processing - 1 by One

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Outline

- So Far: Sampling, Convolution, Fourier Transform
- Previous Week: Shannon Sampling Theorem
- Previous Class: Circuits, Systems and Laplace Transform
- Today: RLC Circuit, Parseval Revisited

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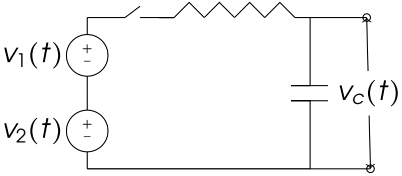
Not-So Linear Initial Conditions

Q4) Can you apply superposition theorem on the two voltage sources to find $v_c(t)$. The resistor is R ohms and capacitor is C Farads, which has an initial charge of 0.3V.



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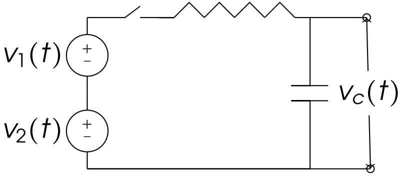


Solution: The important thing to notice that initial conditions can have an effect on the super-position principle, unless accounted properly. For example, consider the system $y(t) = ax(t) + c$, which appears *linear*, but should more aptly be called an *affine* system. Notice that

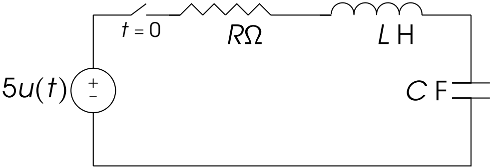
$$y_1(t) = ax_1(t)+c, y_2(t) = ax_2(t)+c \Rightarrow y_1(t)+y_2(t) = a(x_1(t)+x_2(t))+2c.$$

However, $x_1(t) + x_2(t)$ as input will produce $a(x_1(t) + x_2(t)) + c$ as output, which is different from $y_1(t) + y_2(t)$ (system *not linear*).

Similarly, in the circuit above, superposition should be applied after replacing the initial conditions by a suitable source signal.

RLC Circuit

Q5) For the serial RLC circuit find the voltage across C.



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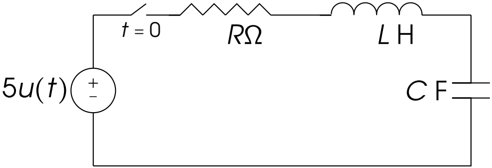

$$V_C(s) = \frac{V_i(s)}{R + sL + \frac{1}{sC}} \frac{1}{sC} = \frac{5}{LC} \frac{1}{s(s^2 + \frac{R}{L}s + \frac{1}{LC})}$$

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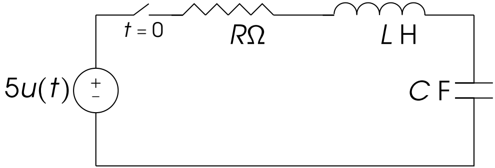

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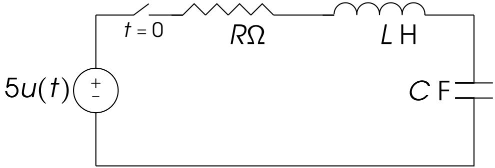

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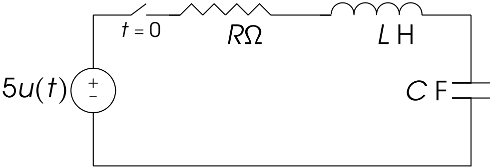

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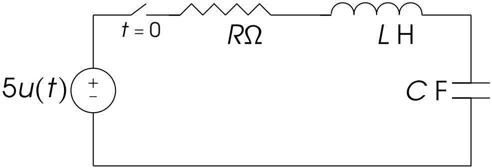

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$$v_C(t) = 5\left[1 - \exp(-\frac{R}{2L}t)\cos(\omega t) - \frac{R}{2\omega L}\exp(-\frac{R}{2L}t)\sin(\omega t)\right]u(t), \omega > 0.$$

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$$\int_{\mathbb{R}} |x(t)|^2 dt = \int_{\mathbb{R}} |X(f)|^2 df.$$

Worked Example

Question) Find the integral

$$\int_{t\in\mathbb{R}} \mathrm{sinc}(2\alpha t)\mathrm{sinc}^2(\alpha t)dt$$