MA 205 Complex Analysis: Some More Theorems

October 2, 2020

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I. For instance suppose there exists a constant C such that $|f(z)| \leq \frac{C}{|z^r|}$ for sufficiently large $|z|$ and for some $r > 1$ (here $f(z)$ is an extension of $f(x)$ to a function of the complex variable). Note that this happens for instance in the case when $f(x) = P(x)/Q(x)$ where $\deg Q(x) \geq \deg P(x) + 2$. Then close up the interval with a semicircle into the upper half plane and integrate along the contour and take limit as the radius of semicircle goes to infinity. Use ML inequality to show that the integral along the semicircle goes to zero as radius goes to ∞

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Otherwise, try a rectangular contour and show that the integral over the extra sides goes to zero in the limit.

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II. If the integrand is of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} sin(x)dx$, where $P(x)$ and $Q(x)$ are polynomials with $deg(Q(x)$ atleast one more than that of $P(x)$, close up the interval by the semicircular region in the upper half plane and use Jordan's lemma to show that the integral over the semicircle goes to zero. (We have seen this when we integrated $sin(x)/x$).

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III. If the integrand is of the type $\int_0^{2\pi} P(cos(t), sin(t))dt$, set $z = e^{it}$ and use $cos(t) = \frac{z+z^{-1}}{2}$ and $sin(t) = \frac{z-z^{-1}}{2i}$. dt becomes $\frac{dz}{iz}$ and then the integral assumes the form $\int_{|z|=1} P(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i})\frac{dz}{iz}$ which can then be computed by using residue theorem.

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IV. If the integrand has infinitely many poles going to infinity, you are usually better off using a rectangular contour which emcompasses only finitely many poles.

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As before one tries to show that in the limit, the integral over the extra added vertical sides goes to zero in the limit and the intergals over the two horizontal sides are related ; usually proportional to each other. Thus taking limit as the length of the rectangular sides goes to infinity, one gets the desired answer.

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The following rather obvious lemma is often useful for contour integration:

Fractional Residue Theorem : Suppose z_0 is a simple pole of $f(z)$ and C_δ is an arc of the circle $|z - z_0| = \delta$ of angle α, then

$$\lim_{\delta \to 0} \int_{C_\delta} f(z)dz = \alpha i \mathrm{Res}(f(z), z_0)$$

Argument principle

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Argument Principle : Let γ be a simple closed contour contained in $\mathbb{C}$ and let $f(z)$ be a meromorphic function on an open set containing γ and its interior such that γ does not pass through any of the zeros and poles of $f(z)$. Then

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = N - P$$

where N and P denote the number of zero's and poles enclosed by γ with each zero and pole counted as many times as its order.

Argument principle

Proof: Let $z_1, \ldots, z_r$ be the set of zeros of $f(z)$ inside γ with multiplicities m_i at z_i and let $p_1, \ldots, p_s$ be the set of poles of $f(z)$ inside γ with multiplicites n_j at p_j. As in the proof of the residue theorem, we break up the integral into sum of integrals along each of the zeros and poles. Let γ_i's be the loops around the zero's and Γ_j's be the loops around the poles. Thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\sum_{i=1}^{r}\int_{\gamma_i}\frac{f'(z)dz}{f(z)} + \frac{1}{2\pi i}\sum_{j=1}^{s}\int_{\Gamma_j}\frac{f'(z)dz}{f(z)}$$

Consider the zero z_i with multiplicity m_i. Then locally around z_i, $f(z) = (z-z_i)^{m_i}g(z)$ for some holomorphic function $g(z)$ with $g(z) \neq 0$ at any point inside γ_i. Then $\frac{f'(z)}{f(z)} = \frac{m_i}{z-z_0} + \frac{g'(z)}{g(z)}$. Since $g(z)$ has no zero's in γ_i, it follows that $\int_{\gamma_i}\frac{g'(z)dz}{g(z)} = 0$ by Cauchy's theorem. Hence $\int_{\gamma_i}\frac{f'(z)}{f(z)}dz = m_i$. Similar argument shows that every pole of order n_j contributes $-n_j$ to the integral. Hence we get $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N - P$ as desired.

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<u>**Rouche's Theorem:**</u> Let γ be a simple closed contour and let $f(z)$ and $g(z)$ be two functions holomorphic on an open set containing γ and its interior. Suppose $|f(z) - g(z)| < |f(z)|$ at all points on γ. Then γ encloses the same number of zero's of $f(z)$ and $g(z)$.

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The proof of Rouche's theorem follows fairly easily from the argument principle I'll leave the proof as an exercise. You can also find a proof in most of the standard books in Complex Analysis.

Remark: The hypothesis $|f(z) - g(z)| < |f(z)|$ on γ implies that f and g have no zeros on γ.

Example

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Take $g(z) = 11z^4$. Then $|g(z) - f(z)| = |z^6 + z^3 + 2z + 4| < 8$ on the unit circle. Hence $|g(z) - f(z)| < |g(z)|$ on the unit circle.
Hence $g(z)$ has the same number of roots as $f(z)$ inside the unit circle. But the number of roots of $g(z)$ inside unit circle is 4 (counting mutiplicity) which therefore equals number of roots of $f(z)$.

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Let us consider $g(z) = -2z$. Then

$$|g(z) - f(z)| = |e^z - 1| = |\sum_1^\infty \frac{z^n}{n!}| \leq \sum_1^\infty \frac{|z^n|}{n!} = e - 1 < |g(z)|$$

on the unit circle. Hence by Rouche's theorem $f(z)$ and $g(z)$ have equal number of roots in the unit circle, namely 1.

Example

Let us determine the number of zeros of $2z^5 - 6z^2 + z + 1$ in the annulus $1 < |z| < 2$. Consider the open disc $|z| < 1$ bounded by the unit circle $C_1 : |z| = 1$. Taking $f(z) = 2z^5 - 6z^2 + z + 1$ and $g(z) = -6z^2$, we find that
$|f(z) - g(z)| = |2z^5 + z + 1| < |-6z^2| = 6$ on C_1. Hence $f(z)$ has the same number of roots in the open unit disc as $g(z)$ which is 2.
Now consider the region $|z| < 2$ bounded by $C_2 : |z| = 2$. Then on C_2, $|f(z) - 2z^5| = |-6z^2 + z + 1| < |27| < |2z^5| = 64$ on C_2.
Hence $f(z)$ has the same number of roots as $2z^5$ inside C_2. Since $2z^5$ has 5 roots counting multiplicity in the region $|z| < 2$, it follows that $f(z)$ has 5 roots inside the disc of radius 2.
Therefore $f(z)$ has 3 roots in the region $1 \leq |z| < 2$.
By the remark following Rouche's theorem, f has no zeros on $|z| = 1$ and hence $f(z)$ has 3 roots in the open annulus $1 < |z| < 2$ as well.

FTA

Here's another quick and pretty proof of FTA using Rouche's theorem.
Let $f(z) = a_0 + a_1 z + \cdots + z^n$ be a non-constant polynomial. Take $g(z) = z^n$. Then on a sufficiently large circle around 0 of radius R, $|f(z) - g(z)| < |f(z)|$. Hence $f(z)$ and $g(z)$ have same number of zero's in the disc of radius R. Since $g(z)$ has n zero's, so does $f(z)$!

Harmonic Functions

We now revisit a topic we studied at the beginning of the course, namely harmonic functions. Recall that a function $u(x, y)$ of real variables is said to be harmonic if it is twice differentiable and $u_{xx} + u_{yy} = 0$. It turns out that harmonic functions share many properties similar to holomorphic functions. We'll see some of them.

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Recall that if u is a harmonic function, then a harmonic conjugate of u is another harmonic function v such that $u + iv$ is holomorphic. We saw some examples of computing harmonic conjugates and that time I commented that if the domain is "nice", then a harmonic conjugate always exists. The mathematical notion that replaces "nice" is simply connectedness.

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Theorem: Let U be a simply-connected domain in $\mathbb{C}$ and let u be a harmonic fuction on U. Then u admits exactly one harmonic conjugate up to a constant.

Proof: Let's dismiss the uniqueness first. Suppose u has a harmonic conjugate v. Let $f(z) = u + iv$. By CR equations, v_x and v_y are determined and hence v is determined upto a constant. To prove existence, let $g(z) = u_x - iu_y$. Then by CR equations, $g(z)$ is holomorphic. Now fix $z_0 \in U$, and define f to be the anti-derivative of g:

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$f(z) = u(z_0) + \int_{z_0}^{z} g(z)dz$ with the integral being along a path in U connecting z_0 and z. As U is simply-connected, this function is well-defined. By construction, f is holomorphic and
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Corollary: Harmonic functions are infinitely differentiable.

This follows since on a small open disc around each point, every harmonic function admits a harmonic conjugate by simply-connectedness of open discs. Since holomorphic functions are infinitely differentiable, the corollary follows.

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Theorem: (Mean-Value Property): Let u be a harmonic function on a disc of radius R. Then for any $r < R$, we have,

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Proof: Can assume $w = 0$ without loss of generality. Since u is harmonic and the domain is simply-connected, there exists a holomorphic function $f(z)$ such that $Re(f) = u$. By Cauchy's integral formula,

$$f(0) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z} dz$$

Hence

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

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Identity Principle

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Proof: Set $f = u_x - iu_y$. Then as before, f is holomorphic on Ω. Since $u = 0$ on U then so is f. Hence, by the Identity Principle for the holomorphic functions $f = 0$ on Ω, and consequently, $u_x = u_y = 0$ on Ω. Therefore u is constant on Ω, and as it is zero on U, it must be zero on Ω.

Identity Theorem

Remark: Recall that the identity theorem for holomorphic functions is stronger; namely if a holomorphic function vanishes on a set of points having a limit point, then it is identically zero. This is not true for harmonic functions. The function $Re(z)$ vanishes identically on imaginary axis but is non-zero elsewhere.