MA 207 - Differential Equations-II

Amiya Kumar Pani

Department of Mathematics,
Indian Institute of Technology Bombay,
Powai, Mumbai 76
akp.ma207.2020@gmail.com

October 27, 2020

" Perhaps we see equations as simple because they are easily expressed in terms of mathematical notation already invented at an earlier stage of development of the science, and thus what appears to us as elegance of description really reflects the interconnectedness of Nature's laws at different levels."
by Murray Gell-Mann

"The Infinite! No other question has ever moved so profoundly the spirit of man."
by David Hilbert

Outline of the lecture

- Recall: Matrix Eigen-Value Problem
- Sturm-Liouville Boundary Value Problems
- Properties
- Examples

Recall: Matrix EVP

Definition

Let $V = \mathbb{R}^n$ (resp. $\mathbb{C}^n$) and A be an $n \times n$ matrix with real (resp. complex) entries. A is symmetric (resp. Hermitian), if $A = A^T$ (resp. $A = \overline{A^T}$).

Consider $\boxed{AX = \lambda X}$, where A is an $n \times n$ symmetric or Hermitian matrix (coined in the last lecture as Self-adjoint), λ is a scalar parameter.

Properties :

- All eigenvalues are real.
- Eigenvectors corresponding to distinct eigenvalues are orthogonal.
- If $\lambda_1, \lambda_2, \cdots \lambda_n$ are eigenvalues and $v_1, v_2, \cdots, v_n$ are the corresponding eigenvectors, then any vector z can be written as $z = \sum_{i=1}^{n} \alpha_i v_i$, where α_i is related to z and v_i.

Consider the second order linear DE of the form:

$$(r(x)y')' + (q(x) + \lambda p(x))y = 0 \quad a \leq x \leq b, \tag{1}$$
$$k_1 y(a) + k_2 y'(a) = 0 \tag{2}$$
$$l_1 y(b) + l_2 y'(b) = 0 \tag{3}$$

Here $k_1,\ k_2$ not both are zero, $l_1,\ l_2$ not both are zeros, λ is a parameter; $k_1,\ k_2,\ l_1,\ l_2$ are given constants. (1)-(3) is referred to as STURM-LIOUVILLE BVP.

Often (1) is represented as

$$\boxed{\mathcal{L}(y) + \lambda p(x)y = 0,\ \mathcal{L}(y) = (r(x)y')' + q(x)y.}$$

$y \equiv 0$ is always a solution of (1)-(3).

We are interested to determine the values of the parameter λ for which the BVP has non-trivial solutions.

Let V be the space of twice continuously differentiable functions defined on $[a, b]$ on which the inner-product is defined

$$(u, v) = \int_a^b u(x)v(x)\, dx.$$

For $u, v \in V$,

LAGRANGE'S IDENTITY :

$$\boxed{(\mathcal{L}u, v) - (u, \mathcal{L}v) = r(x)(u'v - uv')\Big|_{x=a}^{x=b}.}$$

Regular Sturm Liouville Problems:

A Sturm-Liouville problem

$$(r(x)y')' + (q(x) + \lambda p(x))y = 0 \quad a \leq x \leq b,$$
$$k_1 y(a) + k_2 y'(a) = 0$$
$$l_1 y(b) + l_2 y'(b) = 0$$

is said to be regular if

1. $r(x) > 0$, $p(x) > 0$ for $x \in [a, b]$;
2. $p(x)$, $q(x)$, $r(x)$, $r'(x)$ are continuous on $[a, b]$.

Result

For a regular Sturm-Liouville problem

$$\mathcal{L}y + \lambda p(x)y \equiv ((r(x)y')' + q(x)y) + \lambda p(x)y = 0 \quad a \leq x \leq b, \tag{4}$$

$$k_1 y(a) + k_2 y'(a) = 0 \tag{5}$$

$$l_1 y(b) + l_2 y'(b) = 0, \tag{6}$$

all the eigenvalues are real.

Proof : If possible, let $\boxed{\lambda = \mu + i\nu}$ be a complex eigenvalue of (4)-(6), and let $\phi(x) = u(x) + iv(x)$ denote the corresponding eigenfunction.

$$\begin{aligned} & (\mathcal{L}\phi, \phi) = (\phi, \mathcal{L}\phi) \\ \implies & (-\lambda p(x)\phi,\ \phi) = (\phi,\ -\lambda p(x)\phi) \end{aligned}$$

$$\Longrightarrow \int_a^b -\lambda p(x)\phi(x)\overline{\phi(x)}\,dx = -\int_a^b \phi(x)\bar{\lambda}\overline{p(x)}\,\overline{\phi(x)}\,dx$$

$$\Longrightarrow (\lambda - \bar{\lambda})\int_a^b p(x)\phi(x)\overline{\phi(x)}\,dx = 0.$$

But $\phi(x) = u(x) + iv(x) \Longrightarrow \phi(x)\overline{\phi(x)} = u^2 + v^2$.

Hence, $(\lambda - \bar{\lambda})\displaystyle\int_a^b p(x)(u^2 + v^2)\,dx = 0$.

Since the problem is regular, $p(x) > 0$ and is continuous in $[a, b]$.

Since the integrand is positive and continuous (eigenfunctions satisfy the DE), the integral is positive.

Hence, $\lambda - \bar{\lambda} = 2i\nu = 0 \Longrightarrow \nu = 0$.

Hence, λ is real.

Exercise : Eigenfunctions corresponding to real eigenvalues are real.

Result

The eigenvalues of Sturm Liouville problems are simple; that is, corresponding to each eigen value there exists only one linearly independent eigenfunction.

Proof: Given an eigenvalue λ of SL-BVP, if possible let $\phi_1(x)$ and $\phi_2(x)$ be two linearly independent eigenfunctions.

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} \neq 0,$$

(since ϕ_1, ϕ_2 are l.i.)

$$\begin{aligned} W(\phi_1, \phi_2)(a) &= \phi_1(a)\phi_2'(a) - \phi_2(a)\phi_1'(a) \\ &= \phi_1(a)(-\frac{k_1}{k_2}\phi_2(a)) - \phi_2(a)(-\frac{k_1}{k_2}\phi_1(a)) \ \text{(using BC)} \\ &= 0; \end{aligned}$$

which is a contradiction. Hence, ϕ_1, ϕ_2 are linearly dependent.

Eigenfunctions of regular SL-BVP

Let λ denote an eigenvalue of a regular SL-BVP.
We proved that λ is real. If possible, let $\phi(x) = u(x) + iv(x)$ be an eigen function corresponding to the eigen value λ.

$$\begin{aligned} &\mathcal{L}\phi(x) + \lambda p(x)\phi(x) = 0 \quad a \leq x \leq b, \\ &k_1\phi(a) + k_2\phi'(a) = 0 \\ &l_1\phi(b) + l_2\phi'(b) = 0. \end{aligned}$$

That is, we have

$$\begin{aligned} \mathcal{L}u(x) + \lambda p(x)u(x) = 0, &\qquad \mathcal{L}v(x) + \lambda p(x)v(x) = 0, \\ k_1u(a) + k_2u'(a) = 0 &\qquad k_1v(a) + k_2v'(a) = 0 \\ l_1u(b) + l_2u'(b) = 0, &\qquad l_1v(b) + l_2v'(b) = 0. \end{aligned}$$

That is, corresponding to an eigenvalue λ, we have two eigen functions $u(x)$ and $v(x)$. But since the eigenvalues are simple, $u(x)$ and $v(x)$ are linearly dependent. Hence $\phi(x) = u(x)$, is a real eigenfunction. (of course, $c\phi(x)$, c being a complex constant is also an eigen function!)

Eigenfunctions of a Regular Sturm-Liouville Problem - Orthogonality

Result

The eigen functions of SL BVP (4)-(6) *corresponding to* **distinct** *eigenvalues are orthogonal with respect to the inner product*

$$<< f, g >>= \int_a^b p(x)f(x)g(x)\, dx.$$

That is, if $\phi_1(x)$ *and* $\phi_2(x)$ *are two eigenfunctions corresponding to the distinct eigenvalues* λ_1 *and* λ_2*, respectively, and if* $\lambda_1 \neq \lambda_2$*, then*

$$\int_a^b p(x)\phi_1(x)\phi_2(x)\, dx = 0 \tag{7}$$

(This property expresses the orthogonality property of the eigenfunctions with respect to the weight function $p(x)$.)

$\phi_1(x)$ and $\phi_2(x)$ satisfy

$$\begin{aligned} \mathcal{L}\phi_1(x) &= -\lambda_1 p(x)\phi_1(x) \\ \mathcal{L}\phi_2(x) &= -\lambda_2 p(x)\phi_2(x) \end{aligned}$$

Now,

$$\begin{aligned} & (\mathcal{L}\phi_1(x), \phi_2(x)) = (\phi_1(x), \mathcal{L}\phi_2(x)) \\ \Longrightarrow\ & (-\lambda_1 p(x)\phi_1(x), \phi_2(x)) = (\phi_1(x), -\lambda_2 p(x)\phi_2(x)) \\ \Longrightarrow\ & (\lambda_1 p(x)\phi_1(x), \phi_2(x)) - (\phi_1(x), \lambda_2 p(x)\phi_2(x)) = 0 \\ \Longrightarrow\ & \int_a^b (\lambda_1 p(x)\phi_1(x)\phi_2(x) - \lambda_2 p(x)\phi_1(x)\phi_2(x))\, dx = 0 \\ & \qquad\qquad (\textit{ since } p(x) \textit{ is real}, \ \lambda_2 \textit{ is real}, \ \phi_2(x) \textit{ is real}). \\ \Longrightarrow\ & \int_a^b (\lambda_1 - \lambda_2) p(x)\phi_1(x)\phi_2(x)\, dx = 0 \\ & \lambda_1 \neq \lambda_2 \Longrightarrow \int_a^b p(x)\phi_1(x)\phi_2(x)\, dx = 0 \\ \Longrightarrow\ & << \phi_1(x), \phi_2(x) >>= 0. \end{aligned}$$

Some Important Results:

• $\exists$ a countably infinite number of eigenvalues $\lambda_n,\ n = 1, 2, 3, \cdots$ with corresponding eigenfunctions $\phi_n(x)$. The eigenvalues λ_n can be arranged as a monotonically increasing sequence $\lambda_1 < \lambda_2 < \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$.

• If $f(x)$ is a continuously differentiable function on $[a, b]$, then

$\boxed{f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x)}$ where the coefficient function f_n is defined by:

$$f_n = \frac{\int_a^b p(x) f(x) \phi_n(x)\, dx}{\int_a^b p(x) \phi_n^2(x)\, dx} = \frac{<< f, \phi_n >>}{<< \phi_n, \phi_n >>}.$$

Expand the function $f(x) = x, \;\; x \in [0,1]$ in terms of the normalised eigenfunctions $\phi_n(x)$ of the boundary value problem $y'' + \lambda y = 0, \;\; y(0) = 0, \;\; y(1) + y'(1) = 0.$

Solution :

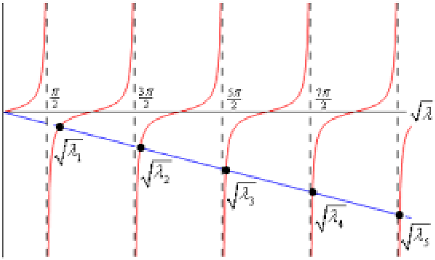
$\boxed{\text{Case 1}}$: When $\lambda \leq 0$, the BVP does not have a non-trivial solution. (Check!)

$\boxed{\text{Case 2}}$: $\lambda > 0$: The general solution is:

$\boxed{y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.}$ Subjected to the given boundary conditions, $\sqrt{\lambda}$ must satisfy $\sqrt{\lambda} = -\tan\sqrt{\lambda}.$

Setting $f = \sqrt{\lambda}$ and $g = -\tan\sqrt{\lambda}$, the graphs of f and g intersect at infinite number of points, that is the equation has infinite roots, given by $\sqrt{\lambda_n} = -\tan\sqrt{\lambda_n}$. The corresponding eigen functions are $\sin\sqrt{\lambda_n}x, \;\; n = 1, \, 2, \cdots$.

Graph of $-\sqrt{\lambda} = \tan\sqrt{\lambda}$



Let $\phi_n = \alpha_n \sin\sqrt{\lambda_n}x$ be the corresponding normalized eigenfunctions. We also have $\sin\sqrt{\lambda_n} + \sqrt{\lambda_n}\cos\sqrt{\lambda_n} = 0$. Then,

$$\int_0^1 \alpha_n^2 \sin^2\sqrt{\lambda_n}x\, dx = 1 \implies \int_0^1 \alpha_n^2 \left(\frac{1-\cos 2\sqrt{\lambda_n}x}{2}\right) dx = 1$$

$$\implies \frac{\alpha_n^2}{2}\left(x - \frac{\sin 2\sqrt{\lambda_n}x}{2\sqrt{\lambda_n}}\right)\Bigg|_{x=0}^{x=1} = 1 \implies \frac{\alpha_n^2}{2}\left(1 - \frac{\sin 2\sqrt{\lambda_n}}{2\sqrt{\lambda_n}}\right) = 1$$

$$\implies \frac{\alpha_n^2}{2}\left(1 - \frac{-\cancel{2\sqrt{\lambda_n}}\cos\sqrt{\lambda_n}\cos\sqrt{\lambda_n})}{\cancel{2\sqrt{\lambda_n}}}\right) = 1$$

$$\implies \frac{\alpha_n^2}{2}(1+\cos^2\sqrt{\lambda_n}) = 1 \implies \alpha_n = \left(\frac{2}{1+\cos^2\sqrt{\lambda_n}}\right)^{1/2}.$$

The normalized eigenfunctions are

$$\boxed{\phi_n = \frac{\sqrt{2}\sin\sqrt{\lambda_n}x}{\sqrt{1+\cos^2\sqrt{\lambda_n}}},\ n = 1, 2, \cdots}.$$

Now $f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$, where

$$\boxed{a_n = \int_0^1 f(x)\phi_n(x) = \alpha_n \int_0^1 f(x) \sin\sqrt{\lambda_n}x\,dx = \alpha_n \int_0^1 x \sin\sqrt{\lambda_n}x\,dx.}$$

$$\int_0^1 x\sin\sqrt{\lambda_n}x\,dx = -x\frac{\cos\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}\Bigg|_{x=0}^{x=1} + \int_0^1 \frac{\cos\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}\,dx$$

$$= -\frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}} + \frac{1}{\sqrt{\lambda_n}}\frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}\Bigg|_{x=0}^{x=1}$$

$$= \frac{\sin\sqrt{\lambda_n}}{\lambda_n} + \frac{1}{\sqrt{\lambda_n}}\frac{\sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}} \quad (\textit{since}\ \sin\sqrt{\lambda_n} = -\sqrt{\lambda_n}\cos\sqrt{\lambda_n})$$

$$= 2\frac{\sin\sqrt{\lambda_n}}{\lambda_n}.$$

Since $\alpha_n = \left(\frac{2}{1+\cos^2\sqrt{\lambda_n}}\right)^{1/2}$, $a_n = \frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n\sqrt{1+\cos^2\sqrt{\lambda_n}}}$. Also,

$\phi_n = \frac{\sqrt{2}\sin\sqrt{\lambda_n}x}{\sqrt{1+\cos^2\sqrt{\lambda_n}}}$, $n = 1, 2, \cdots$ Hence,

An example

Consider the eigenvalue problem with periodic boundary conditions:

$$y'' + \lambda y = 0 \quad -\pi < x < \pi,$$
$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$$

NOT A STURM LIUOVILLE BVP!

For the case $\lambda < 0$, verify that the BVP has ONLY TRIVIAL SOLUTION.

For the case $\lambda = 0$, we obtain $y = \mathit{constant}$ as eigen function.

For $\lambda > 0$, the general solution is $\boxed{y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x}$.

$$\begin{aligned} y(\pi) = y(-\pi) \quad &\Longrightarrow \quad \cancel{A\cos\sqrt{\lambda}\pi} + B\sin\sqrt{\lambda}\pi = \cancel{A\cos\sqrt{\lambda}\pi} - B\sin\sqrt{\lambda}\pi \\ &\Longrightarrow \quad B\sin\sqrt{\lambda}\pi = 0 \Longrightarrow \sqrt{\lambda}\pi = n\pi \quad (B \neq 0) \\ &\Longrightarrow \quad \lambda_n = n^2, \quad n = 1,\ 2,\ 3, \cdots . \end{aligned}$$

The eigenfunctions corresponding to $\lambda_n = n^2$ are $\sin nx, \ n = 1, 2, \cdots$.

$$\boxed{y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x}$$

$$\boxed{y'(x) = -\sqrt{\lambda}A\sin\sqrt{\lambda}x + \sqrt{\lambda}B\cos\sqrt{\lambda}x}$$

$$\begin{aligned}
& y'(\pi) = y'(-\pi) \\
\Longrightarrow \quad & -\sqrt{\lambda}A\sin\sqrt{\lambda}\pi + \cancel{\sqrt{\lambda}B\cos\sqrt{\lambda}\pi} = \sqrt{\lambda}A\sin\sqrt{\lambda}\pi + \cancel{\sqrt{\lambda}B\cos\sqrt{\lambda}\pi} \\
\Longrightarrow \quad & \sin\sqrt{\lambda}\pi = 0 \quad (A \neq 0, \ \lambda \neq 0) \\
\Longrightarrow \quad & \sqrt{\lambda}\pi = n\pi \\
\Longrightarrow \quad & \lambda_n = n^2, \quad n = 1, \ 2, \ 3, \cdots .
\end{aligned}$$

The eigenfunctions corresponding to $\lambda_n = n^2$ are $\cos nx, \ n = 1, 2, \cdots$.

$$\boxed{y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x}$$

$$\boxed{y'(x) = -\sqrt{\lambda}A\sin\sqrt{\lambda}x + \sqrt{\lambda}B\cos\sqrt{\lambda}x}$$

$$\begin{aligned}
& y'(\pi) = y'(-\pi) \\
\implies & -\sqrt{\lambda}A\sin\sqrt{\lambda}\pi + \cancel{\sqrt{\lambda}B\cos\sqrt{\lambda}\pi} = \sqrt{\lambda}A\sin\sqrt{\lambda}\pi + \cancel{\sqrt{\lambda}B\cos\sqrt{\lambda}\pi} \\
\implies & \sin\sqrt{\lambda}\pi = 0 \quad (\ A \neq 0,\ \lambda \neq 0) \\
\implies & \sqrt{\lambda}\pi = n\pi \\
\implies & \lambda_n = n^2, \ \ n = 1,\ 2,\ 3, \cdots .
\end{aligned}$$

The eigenfunctions corresponding to $\lambda_n = n^2$ are
$\cos nx, \ n = 1, 2, \cdots$.

- For each λ_n, there exists eigenfunctions $\cos nx$ and $\sin nx$.
- Orthogonality Properties :

$$\boxed{y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x}$$

$$\boxed{y'(x) = -\sqrt{\lambda}A\sin\sqrt{\lambda}x + \sqrt{\lambda}B\cos\sqrt{\lambda}x}$$

$$\begin{aligned}
& y'(\pi) = y'(-\pi) \\
\Longrightarrow\quad & -\sqrt{\lambda}A\sin\sqrt{\lambda}\pi + \cancel{\sqrt{\lambda}B\cos\sqrt{\lambda}\pi} = \sqrt{\lambda}A\sin\sqrt{\lambda}\pi + \cancel{\sqrt{\lambda}B\cos\sqrt{\lambda}\pi} \\
\Longrightarrow\quad & \sin\sqrt{\lambda}\pi = 0 \quad (\, A \neq 0,\ \lambda \neq 0) \\
\Longrightarrow\quad & \sqrt{\lambda}\pi = n\pi \\
\Longrightarrow\quad & \lambda_n = n^2, \quad n = 1,\ 2,\ 3, \cdots .
\end{aligned}$$

The eigenfunctions corresponding to $\lambda_n = n^2$ are
$\cos nx, \ n = 1, 2, \cdots .$

- For each λ_n, there exists eigenfunctions $\cos nx$ and $\sin nx$.
- Orthogonality Properties :

$$\int_{-\pi}^{\pi} \cos mx \sin mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin 2mx \, dx = 0$$

For $m \neq n$, $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0.$

Further,
$$\int_{-\pi}^{\pi} \sin^2 mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2mx)\, dx = \pi.$$
Similarly, $\displaystyle\int_{-\pi}^{\pi} \cos^2 mx \, dx = \pi.$

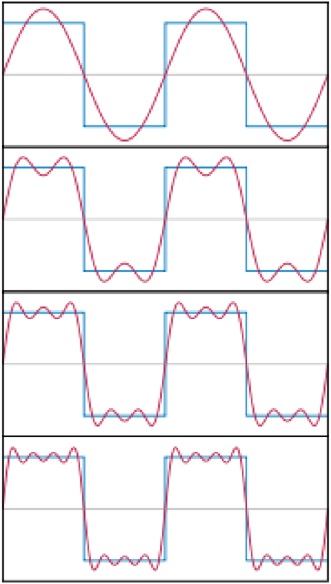
Thus, the orthonormal eigenfunctions corresponding to the eigenvalues $\boxed{\lambda_n = n^2}$ are $\boxed{\left\{\frac{\sin nx}{\sqrt{\pi}}\right\}}$ and $\boxed{\left\{\frac{\cos nx}{\sqrt{\pi}}\right\}}$.

Given a function f, we discussed the possibility of expanding f in terms of orthonormal eigenfunctions of a Sturm-Liouville BVP.

In particular, choosing the eigenvalue problem considered in the above example (which also yields orthonomal eigenfunctions!) can we discuss the possibility of expanding $f(x)$ in terms of a trigonometric series, say of the form

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)?$$

Square wave - approximations by trigonometric series



Periodic Identity function (Sawtooth curve-approximations)