

# MA 205 Complex Analysis: Exponential Function

August 27, 2020

# Cauchy-Maclaurin integral test

In the last lecture we discussed power series and their radius of convergence. I had stated in the lecture that the harmonic series  $\frac{1}{n}$  diverges. Let me begin the lecture by stating a theorem which gives a proof of this fact.

Consider a eventually non-negative, continuous function  $f$  defined on the unbounded interval  $[1, \infty)$ , on which it is eventually monotone decreasing. Then the infinite series  $\sum_1^\infty f(n)$  converges if and only the improper integral  $\int_1^\infty f(x)dx$  exists (i.e, is finite). If the integral diverges then the series diverges as well.

This test is used for instance to show the divergence of the harmonic series  $\sum_1^\infty 1/n$ . It also shows the convergence of the series  $\sum_1^\infty 1/n^\alpha$  for any  $\alpha > 1$ .

Today, we'll use our knowledge of power series to construct a few basic functions. We will begin with the all important exponential function.

Now consider the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

We've seen that its radius of convergence is  $\infty$ ; i.e., this function is well-defined for any  $z \in \mathbb{C}$ . This function will keep our company throughout this course. So let's befriend it a bit more !

# Exponential Function

If

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots,$$

then  $f(0) = 1$ . By term by term differentiation, one observes that

$$f'(z) = f(z).$$

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We'll denote this function by  $\exp(z)$ , also denoted  $e^z$ . Note that

$$(\exp(bz))' = b \exp(bz).$$

# Exponential Function

Now consider the function

$$h(z) = \exp(z) \cdot \exp(-z).$$

This is defined throughout  $\mathbb{C}$ . What's  $h'(z)$ ?

$$h'(z) = \exp(z) \cdot (-\exp(-z)) + \exp(-z) \cdot \exp(z) = -h + h = 0.$$

Therefore  $h(z) \equiv c$  (Why ?) and since  $h(0) = 1$ , it is identically equal to 1. Thus, we have proved two things:

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Therefore  $h(z) \equiv c$  (Why ?) and since  $h(0) = 1$ , it is identically equal to 1. Thus, we have proved two things:

(i)  $\exp(z)$  is never vanishing.

(ii)  $\exp(-z) = \frac{1}{\exp(z)}.$

# Exponential Function

Note that the derivative of  $f(z) = a \exp(bz)$  is  $f'(z) = bf(z)$ . Interestingly, the converse is also true. Thus,

$$f(z) = a \exp(bz) \text{ for } a, b \in \mathbb{C} \iff f'(z) = bf(z).$$

Proof: Assume  $f'(z) = bf(z)$  for  $b \in \mathbb{C}$ . Now consider

$$h(z) = f(z) \exp(-bz).$$

Then,  $h'(z) = -bh + bh = 0$ , for all  $z$  in the domain. So,  $h(z) \equiv a$  for some  $a \in \mathbb{C}$ . Therefore,

$$f(z) = \frac{a}{\exp(-bz)} = a \exp(bz),$$

by what we already know.



# Exponential Function

Corollary:  $f' = f$  and  $f(0) = 1$  characterizes the exponential function. The function

$$f(z) = e^x(\cos y + i \sin y),$$

is holomorphic throughout  $\mathbb{C}$  and  $f' = f$ . Clearly,  $f(0) = 1$  as well. Thus,

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Remark:  $e^x$  here is  $e$  to the power of  $x$ , and  $e$  is the number that you know from MA 105 (base of natural logarithm). We'll try and reconstruct everything about logarithm and exponential function from scratch.

# Exponential Function

By now we know that  $\exp$  is defined throughout  $\mathbb{C}$  and that 0 is not in the range of  $\exp(z)$ . ,  $\exp$  is a map from  $\mathbb{C} \rightarrow \mathbb{C}^\times$ . Now  $\exp$  has this wonderful property that it takes the “correct” operation in  $\mathbb{C}$  to the “correct” operation in  $\mathbb{C}^\times$ .

$$\exp(w + z) = \exp(w) \cdot \exp(z).$$

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Thus in the language of group theory  $\exp$  is a homomorphism from  $\mathbb{C}$  to  $\mathbb{C}^\times$ .

Proof: Fix  $w \in \mathbb{C}$ . Then the function  $f(z) = \exp(w + z)$  is holomorphic in  $\mathbb{C}$  and  $f'(z) = f(z)$ . So,  $f(z) = a \exp(z)$  for some constant  $a$ . By evaluating  $f$  at 0, see that  $a = \exp(w)$ . Thus,  $f(z) = \exp(w) \cdot \exp(z)$ .

In particular:

$$\exp(i\theta) = \cos \theta + i \sin \theta$$

and

$$\exp(nz) = \exp(z)^n.$$



Euler was a Swiss Mathematician who spent most of his professional life in Russia. He is without doubt the greatest mathematician of the 18th century. He was also extremely productive with his total works spanning 92 volumes. His areas of contribution include Algebra, Analysis, Number Theory, Graph theory, Applied Mathematics and Physics ... Euler later developed a cataract in his left eye, which was discovered in 1766. Just a few weeks after its discovery, he was rendered almost totally blind. However, his condition appeared to have little effect on his productivity, as he compensated for it with his mental calculation skills and exquisite memory. For example, Euler could repeat the Aeneid of Virgil from beginning to end without hesitation, and for every page in the edition he could indicate which line was the first and which the last. With the aid of his scribes, Euler's productivity on many areas of study actually increased. He produced on average, one mathematical paper every week in the year 1775.

"Read Euler, read Euler, he is the master of us all." (Laplace)

# Exponential Function

This property of converting addition into multiplication also characterizes  $\exp(z)$ .

*Let  $0 \in \Omega$ . Suppose  $f : \Omega \rightarrow \mathbb{C}$  is such that  $f$  is differentiable at 0 and  $f(0) \neq 0$ . Suppose  $f(w + z) = f(w)f(z)$  whenever  $w, z, w + z \in \Omega$ . Then,  $f(z) = \exp(bz)$ , where  $b = f'(0)$ .*

# Exponential Function

Now we define  $e$  as follows:

$$e = \exp(1).$$

*It is an irrational number (slightly hard exercise !) and its approx. value is 2.718. A much deeper fact that  $e$  is not the root of any polynomial with rational coefficients. Such numbers are called transcendental numbers. 'Most' numbers are transcendental. It is an open problem whether  $e + \pi$  is irrational !*

One can then define  $e^x$  for any real number; first for rational powers and then for arbitrary real powers by taking limits. One can then show that this definition of  $e^x$  coincides with the definition of  $e^x$  as defined using the power series  $1 + x + x^2/2! + \dots$





# Trigonometric Functions

Recall the Taylor expansions:

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots = \frac{\exp(\imath y) - \exp(-\imath y)}{2\imath}$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots = \frac{\exp(\imath y) + \exp(-\imath y)}{2}.$$

Motivated by this, we define complex trigonometric functions:

$$\sin z = \frac{\exp(\imath z) - \exp(-\imath z)}{2\imath} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$\cos z = \frac{\exp(\imath z) + \exp(-\imath z)}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

Define hyperbolic sine and hyperbolic cosine by:

$\cos(hz) = (e^z + e^{-z})/2$ . Its power series is given by

$$\cos(hz) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$$

$\sin(hz) = (e^z - e^{-z})/2$ . Its power series is given by

$$\sin(hz) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

We have seen that  $e^z$  is not a 1-1 function. Hence its inverse is not defined everywhere. Nevertheless we would like to construct an analytic inverse function, called logarithm on certain subsets, i.e, a analytic function  $w = \log(z)$  such that  $e^w = z$ . As remarked before,  $\log$  will be undefined at 0.

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## Definition

Let  $\Omega \subseteq \mathbb{C}$  be a domain. Let  $f(z)$  be a continuous function on  $\Omega$  such that  $\exp(f(z)) = z \ \forall z \in \Omega$ . Then  $f$  is called a branch of the logarithm.

## Lemma

*Let  $\Omega \subseteq \mathbb{C}$  be a domain and let  $f$  be a branch of the logarithm. Then any other branch of the logarithm differs from  $f$  by a constant multiple multiple of  $2\pi i$ .*

## Proof.

Let  $g(z)$  be a branch of the logarithm. Then  $f(z) - g(z)$  is a constant multiple of  $2\pi i$  for all  $z \in \Omega$ . Since  $\Omega$  is connected, and  $f(z) - g(z)$  is continuous while integral multiples of  $2\pi i$  is a discrete set, it follows that  $f(z) - g(z)$  is a constant multiple of  $2\pi i$



# Logarithm

We will usually work with a fixed branch of the logarithm called the **Principal Branch**. This is defined as follows:

Let  $\Omega \subset \mathbb{C}$  be the open subset defined by  $\mathbb{C}$  minus the negative real line. For any  $z \in \Omega$ ,  $z = |z|e^{i\theta} : -\pi < \theta < \pi = re^{i\theta}$ , define  $f(z) = \log r + i\theta = \log|z| + i\text{Arg}(z)$ .

One checks that  $f(z)$  is a branch of  $\log(z)$  on  $\Omega$ . In fact,  $f$  is analytic and  $f'(z) = \frac{1}{z}$ .

One can show that  $\log(z)$  is not defined as a analytic function on all of  $\mathbb{C}^*$ . It can only be defined on certain 'nice' open subsets of  $\mathbb{C}^*$ . We will come back to the logaithm in more detail after we discuss complex integration.

# Zero's of Analytic Functions

Let  $\Omega$  be a domain in  $\mathbb{C}$  and let  $f : \Omega \rightarrow \mathbb{C}$  be a complex analytic function defined on  $\Omega$ . This means  $f$  can be expressed by a power series expanded around any point in  $\Omega$ .

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$$\begin{aligned} f(z) &= \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(0) + \frac{z^{n+2}}{(n+2)!} f^{(n+2)}(0) + \dots \\ &= z^{n+1} \left( \frac{f^{(n+1)}(0)}{(n+1)!} + \frac{z}{(n+2)!} f^{(n+2)}(0) + \dots \right) \end{aligned}$$

# Zero's of holomorphic functions

Now as  $z \rightarrow 0$  along the sequence  $\{z_n\}$ , we see that the lhs is identically zero. Hence the rhs also vanishes identically along this sequence. Hence the term inside the bracket vanishes along  $\{z_n\}$  and hence by continuity, vanishes at the limit namely 0, thereby showing that  $f^{(n+1)}(0) = 0$ . This contradicts the maximality assumption on  $n$ , thereby completing the proof.

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One could ask if the converse is true; namely given a discrete set of points going to infinity does there exist a holomorphic function with the property that it vanishes exactly on this set ? Indeed this holds. In fact one can find a holomorphic function vanishing exactly on any discrete set with prescribed vanishing multiplicities at each of those points. This is called the **Weierstrass product theorem**.

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(Karl Weierstrass (1815-1897) was a very important mathematician of the 19th century. He was responsible for putting analysis on rigorous footing. The precise  $\epsilon - \delta$  definition of limit was formulated by Weierstrass)

# Failure of the property in the $C^\infty$ case

Note that this property fails for real differentiable functions. It however works equally well for real analytic functions as well (same proof as above). The function

$$\begin{aligned} f(x) &= e^{-1/x} \text{ for } x > 0 \\ &= 0 \text{ for } x < 0 \end{aligned}$$

is infinitely differentiable but vanishes along the entire negative real line.

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# Trigonometric Functions

## Exercises:

- (i) Define other trigonometric functions.
- (ii) Show that  $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$  is onto. Is it one-to-one ? (Show that  $e^z = e^{(z+2\pi i)}$ )
- (iii) Show that  $\sin, \cos : \mathbb{C} \rightarrow \mathbb{C}$  are surjective. In particular, note the difference with real sine and cosine which were bounded by 1.
- (iv) Show that  $\sin^2 z + \cos^2 z = 1$ ,
- (v)  $\sin(z + w) = \sin z \cos w + \cos z \sin w$ ,  
 $\cos(z + w) = \cos z \cos w - \sin z \sin w$ .