# MA 207 - Differential Equations-II

Amiya Kumar Pani

Department of Mathematics,
Indian Institute of Technology Bombay,
Powai, Mumbai 76
*akp.ma207.2020@gmail.com*

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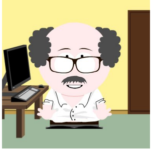
# Start with Quotation and One Cartoon

*"If God has made Nature, Man made Mathematics to study it "*

*by Anonymous*



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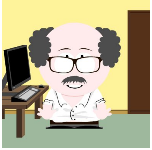

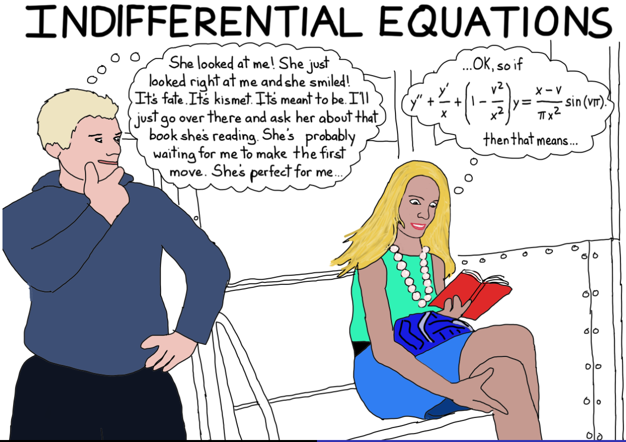

# Outline of the lecture

- Singular Sturm Liouville Problems
- Example
- Exercises

Consider the second order linear DE of the form:

$$(r(x)y')' + (q(x) + \lambda p(x))y = 0 \quad a < x < b, \tag{1}$$
$$k_1 y(a) + k_2 y'(a) = 0 \tag{2}$$
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$\boxed{\mathcal{L}(y) + \lambda p(x)y = 0,\ \mathcal{L}(y) = (r(x)y')' + q(x)y.}$

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We are interested to determine the values of the parameter $\lambda$ for which the BVP has non-trivial solutions.

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1. $r(x) > 0, \; p(x) > 0$ for $x \in [a, b]$;
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Both of these questions can be answered by studying the self-adjoint form.

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**Definition**

*$\mathcal{L}$ is called self-adjoint, if* $\boxed{(\mathcal{L}u, v) = (u, \mathcal{L}v)}$.

In order to make $\mathcal{L}$ self-adjoint, we could define $V$ as the space of twice continuously differentiable functions satisfying the boundary conditions. That is, Let $V$ be the space of twice continuously differentiable functions on $[a, b]$ with inner-product $(v, w) = \int_a^b v(x)w(x)\, dx$.

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Note that

$$\begin{aligned}
(\mathcal{L}u, v) - (u, \mathcal{L}v) &= r(x)(u'v - uv')\Big|_{x=a}^{x=b} \\
&= r(b)\big(u'(b)v(b) - v'(b)u(b)\big) \\
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If $l_2 \neq 0$ and $k_2 \neq 0$, then from the boundary condition: $u'(b) = -(l_1/l_2)u(b), \;\; v'(b) = -(l_1/l_2)v(b)$ and $u'(a) = -(k_1/k_2)u(a), \;\; v'(a) = -(k_1/k_2)v(a)$. On substitution

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Hence, $\boxed{(\mathcal{L}u, v) = (u, \mathcal{L}v).}$

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$(1 - x^2)y'' - 2xy' + p(p+1)y = 0, \;\; x \in (-1, 1)$ can be put in the self-adjoint form as

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$$\boxed{\mathcal{L}y + \lambda y := ((1 - x^2)y')' + p(p+1)y = 0}$$

On the set of all two time continuously differentiable fuctions $v$ on $(-1, 1)$ with $v, v'$ are bounded as $x \longrightarrow 1$, $\mathcal{L}$ is Self-adjoint as

$$(\mathcal{L}u, v) - (u, \mathcal{L}v) = (1 - x^2)(u'v - uv')\Big|_{x=-1}^{x=1} = 0$$

$(r(x) = 1 - x^2)$.

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by choosing $\mu(x) = \dfrac{1}{a(x)} e^{\int \frac{b(x)}{a(x)}\, dx}$.

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Hence, the self-adjoint form for Chebyshev's equation is

$$\boxed{(\sqrt{1-x^2}y')' + \frac{\lambda}{\sqrt{(1-x^2)}} y = 0}.$$

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On the space of twice continuously differential functions on $(-1,1)$ with function and its first derivative as bounded as $x \longrightarrow \pm 1$

$$\begin{aligned} \mu(x) &= \frac{1}{1-x^2} e^{-\int \frac{x}{1-x^2}\,dx} \quad (put\ 1-x^2 = t, \;\; -2x\,dx = dt) \\ &= \frac{1}{1-x^2} \times \sqrt{(1-x^2)} = \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

Hence, the self-adjoint form for Chebyshev's equation is

$$\boxed{(\sqrt{1-x^2}y')' + \frac{\lambda}{\sqrt{(1-x^2)}} y = 0}.$$

(Check: $\mathcal{L}$ is self-adjoint?).

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Let us try to write the Hermite equation $\boxed{y'' - xy' + \lambda y = 0 \quad -\infty < x < \infty}$ in the self-adjoint form.

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To make $\mathcal{L}$ self-adjoint, impose BC's on $u$ and $v$ as

$$\boxed{\lim_{x\to\pm\infty} e^{-x^2/2}u(x) = 0}, \quad \boxed{\lim_{x\to\pm\infty} e^{-x^2/2}v(x) = 0}.$$

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Example 4 : Express the Laguerre equation
$xy'' + (1-x)y' + \lambda y = 0,\ 0 < x < \infty$ in the self-adjoint form. ( try it!)

# Eigenvalues and eigenfunctions of Singular SL Problem

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  $\boxed{\mathcal{L} \text{ is self adjoint in } \mathcal{L}y + \lambda p(x)y = 0,\ \mathcal{L}(y) = (r(x)y')' + q(x)y,}$

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- Most striking difference between regular and singular SL problems: In a singular SL problem, the eigenvalues may not be discrete, that is, the problem may have nontrivial solution corresponding to every value of $\lambda$ in some interval. In that case, we call that the problem has continuous spectrum.

  However, all four examples discussed in earlier slides have countable number of eigenvalues and eigenfunctions.

# Orthogonality of Eigenfunctions

For all four examples ( Legendre , Chebyshev's, Hermite and Laguerre equations), if $\{\phi_n\}_{n=0}^{\infty}$ be the set of eigenfunctions corresponding to eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$, then eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $p(x)$ (appearing in their respective self-adjoint form).

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Example 1 : Legendre equation
$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \;\; x \in (-1,1)$ in its self adjoint form: $((1-x^2)y')' - 2xy' + \lambda y = 0, \;\; x \in (-1,1)$.
Eigenvalues: $\lambda_n = n(n+1)$ with corresponding eigenfunction $P_n(x)$ for $n = 0, 1, \cdots\cdots$
Here, $p(x) = 1$ and orthogonality property:

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0, \;\; (m \neq n).$$

Note in addition, if we assume $y(0) = 0$, then eigenvalues $\lambda_n = n(n+1)$, when $n$ is odd nonnegative integers.

# Example 2: Chebyshev's equation

In its standard form: $(1-x^2)y'' - xy' + \lambda y = 0, \;\; x \in (-1, 1)$, with $y, y'$ are bounded as $x \longrightarrow \pm 1$.

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Eigenvalues: $\{\lambda_n = n^2\}_{n=0}^{\infty}$ and the corresponding eigenfunctions are the Chebyshev polynomials $\{T_n(x)\}_{n=0}^{\infty}$ with
$T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 1 - 2x^2, \cdots\cdots$
Here, $p(x) = 1/sqrt(1-x^2)$ and orthogonality property:

$$\int_{-1}^{1} (1-x^2)^{-1/2} \; T_m(x) T_n(x) \, dx = 0, \;\; (m \neq n).$$

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In its standard form:

$$\boxed{y'' - xy' + \lambda y = 0 \quad -\infty < x < \infty}$$

with $y$ and $y'$ polynomially bounded at $\infty$.

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Eigenvalues: $\{\lambda_n = n^2\}_{n=0}^{\infty}$ and the corresponding eigenfunctions are the Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ with
$H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2, \cdots\cdots$
Here, $p(x) = e^{-x^2/2}$ and orthogonality property:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, H_m(x)H_n(x)\, dx = 0, \ (m \neq n).$$

In its standard form:

$$\boxed{xy'' + (1-x)y' + \lambda y = 0 \quad 0 < x < \infty}$$

with $y$ and its derivative bounded at $\infty$.

# Example 4: Laguerre equation

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In the self-adjoint form:

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with $y$ and its derivative bounded at $\infty$.

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$$\boxed{(xe^{-x}y')' + \lambda e^{-x} y = 0, \quad 0 < x < \infty.}$$

Eigenvalues: $\{\lambda_n = n\}_{n=0}^{\infty}$ and the corresponding eigenfunctions are the Laguerre polynomials $\{L_n(x)\}_{n=0}^{\infty}$ with

$L_0(x) = 1, L_1(x) = 1 - x, L_2(x) = \dfrac{1}{2}(x^2 - 4x + 2), \cdots\cdots$

Here, $p(x) = e^{-x)}$ and orthogonality property:

$$\int_0^{\infty} e^{-x} \, L_m(x)L_n(x) \, dx = 0, \;\; (m \neq n).$$

Many PDE models come naturally from:

**Conservation Principle**

*The rate at which a quantity of interest changes in a given domain $=$ the rate at which the quantity flows across the boundary of the domain $+$ the rate at which the quantity is created or destroyed within the domain.*

For Air polution model: Consider a tube of cross sectional area $A$

- $u = u(x, t)$ denotes the concentration of the pollutants. Then, the amount of the concentration in a small section of width $dx$ is given by $u(x, t)Adx$.

- $q = q(x, t)$ denotes the flux

**Definition**

*Flux is defined as the amount of concentration crossing the section $x$ at $t$ and its unit is given by amount per unit area per unit time.*

The amount of the concentration that is crossing at $x$ and at $t$ is $Aq(x, t)$. By convention, the flus is positive if the flow is from left to right and negative if it is from right to left.

- $f = f(x, t)$ denotes the rate at the quantity is created or destroyed within the section $x$ and at time $t$. Then the amount is created or destroyed in a width $dx$ per unit time is $f(x, t)Adx$.

Formulate the conservation law by considering a fixed, but arbitrary section $a \leq x \leq b$ of the tube. By the Conservation principle:

**Conservation Principle**

$$\frac{d}{dt}\int_a^b u(x,t)A\,dx = Aq(a,t) - Aq(b,t) + \int_a^b f(x,t)A\,dx$$

Assuming $u$ and $q$ have continuous first derivatives, we rewrite as

$$\int_a^b \Big(u_t(x,t) + q_x(x,t) - f(x,t)\Big)\,dx = 0.$$

Since the section $[a,b]$ is arbitrary and the integrand is continuous, therefore,

$$u_t(x,t) + q_x(x,t) = f(x,t).$$

Here, there are two unknowns and one equation, therefore, a relation between the flux and the concentration, which should be given by a physical law. In air pollution model $q = \alpha u$.
Hence, we obtain

$$u_t + \alpha u_x = f(x,t).$$

Here, $\alpha$ is the wind velocity.
This is the mathematical model of air pollution ( in the absence of diffusion). How does one solve it?