Signal Processing - 1 by One

Sibi Raj B. Pillai
Dept of Electrical Engineering
IIT Bombay

Outline

- Impulse Replacement Operation
- Generalization
- Examples
- Digital Convolution
- Analog Domain

Outline

- Impulse Replacement Operation
- Generalization
- Examples
- Digital Convolution
- Analog Domain

Outline

- Impulse Replacement Operation
- Generalization
- Examples
- Digital Convolution
- Analog Domain

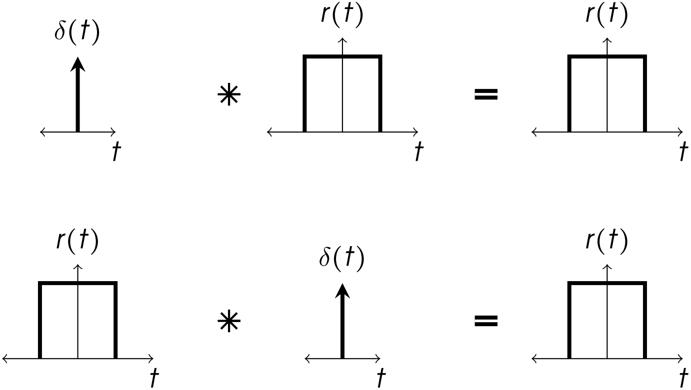
Outline

- Impulse Replacement Operation
- Generalization
- Examples
- Digital Convolution
- Analog Domain

Outline

- Impulse Replacement Operation
- Generalization
- Examples
- Digital Convolution
- Analog Domain

Dirac Formalism for Replacement

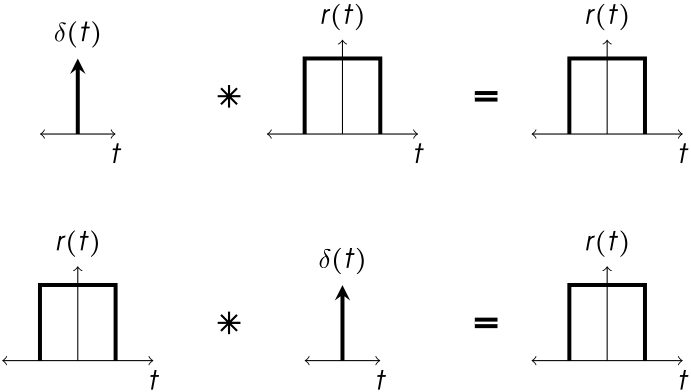


$\delta(t-\tau) * x(t) = x(t-\tau)$

$\delta(t) * x(t-\tau) = x(t-\tau)$

$f(t) * \delta(t-\tau) = ?$

Dirac Formalism for Replacement



$$\delta(t-\tau) * x(t) = x(t-\tau)$$
$$\delta(t) * x(t-\tau) = x(t-\tau)$$

$$f(t) * \delta(t-\tau) = ?$$

Desiderata on $*$

- $*$ should be commutative, i.e. $x(t) * y(t) = y(t) * x(t)$.
- $*$ with an impulse at τ should yield the function $x(t-\tau)$.
- Associativity: $\big(x(t) * y(t)\big) * z(t) = x(t) * \big(y(t) * z(t)\big)$.
- Distribution over addition:

$$x(t) * \big(y(t) + z(t)\big) = x(t) * y(t) + x(t) * z(t).$$

- Consistency when $\delta(t-\tau)$ is used in place of any signal(s).

Desiderata on $*$

- $*$ should be commutative, i.e. $x(t) * y(t) = y(t) * x(t)$.
- $*$ with an impulse at τ should yield the function $x(t - \tau)$.
- Associativity: $\big(x(t) * y(t)\big) * z(t) = x(t) * \big(y(t) * z(t)\big)$.
- Distribution over addition:

$$x(t) * \big(y(t) + z(t)\big) = x(t) * y(t) + x(t) * z(t).$$

- Consistency when $\delta(t - \tau)$ is used in place of any signal(s).

Desiderata on $*$

- $*$ should be commutative, i.e. $x(t) * y(t) = y(t) * x(t)$.
- $*$ with an impulse at τ should yield the function $x(t - \tau)$.
- Associativity: $\big(x(t) * y(t)\big) * z(t) = x(t) * \big(y(t) * z(t)\big)$.
- Distribution over addition:

$$x(t) * \big(y(t) + z(t)\big) = x(t) * y(t) + x(t) * z(t).$$

- Consistency when $\delta(t - \tau)$ is used in place of any signal(s).

Desiderata on $*$

- $*$ should be commutative, i.e. $x(t) * y(t) = y(t) * x(t)$.
- $*$ with an impulse at τ should yield the function $x(t-\tau)$.
- Associativity: $\big(x(t) * y(t)\big) * z(t) = x(t) * \big(y(t) * z(t)\big)$.
- Distribution over addition:

$$x(t) * \big(y(t) + z(t)\big) = x(t) * y(t) + x(t) * z(t).$$

- Consistency when $\delta(t-\tau)$ is used in place of any signal(s).

Desiderata on $*$

- $*$ should be commutative, i.e. $x(t) * y(t) = y(t) * x(t)$.
- $*$ with an impulse at τ should yield the function $x(t-\tau)$.
- Associativity: $\big(x(t) * y(t)\big) * z(t) = x(t) * \big(y(t) * z(t)\big)$.
- Distribution over addition:

$$x(t) * \big(y(t) + z(t)\big) = x(t) * y(t) + x(t) * z(t).$$

- Consistency when $\delta(t-\tau)$ is used in place of any signal(s).

Desiderata on $*$

- $*$ should be commutative, i.e. $x(t) * y(t) = y(t) * x(t)$.
- $*$ with an impulse at τ should yield the function $x(t-\tau)$.
- Associativity: $\big(x(t) * y(t)\big) * z(t) = x(t) * \big(y(t) * z(t)\big)$.
- Distribution over addition:

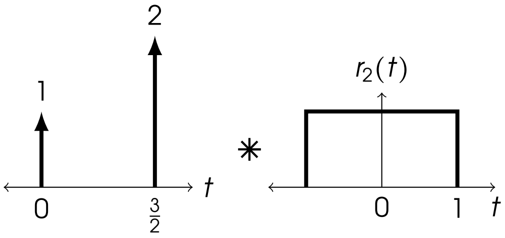
$$x(t) * \big(y(t) + z(t)\big) = x(t) * y(t) + x(t) * z(t).$$

- Consistency when $\delta(t-\tau)$ is used in place of any signal(s).

We hope that there may exist such an integral operation, which boils down to a summation for digital signals.

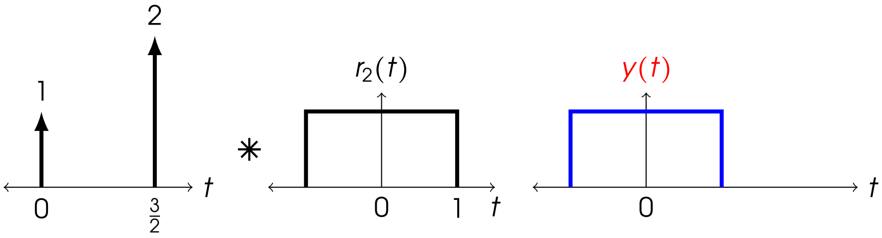
Examples

Assume the desired properties: then



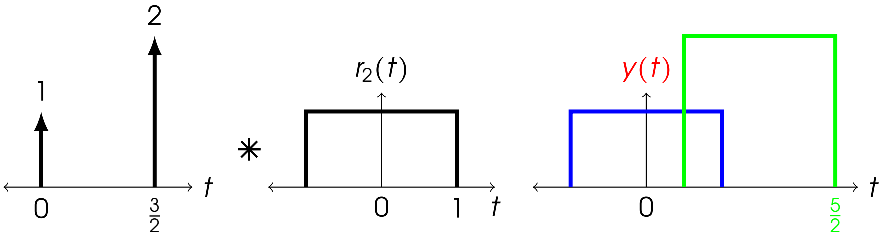
Examples

Assume the desired properties: then



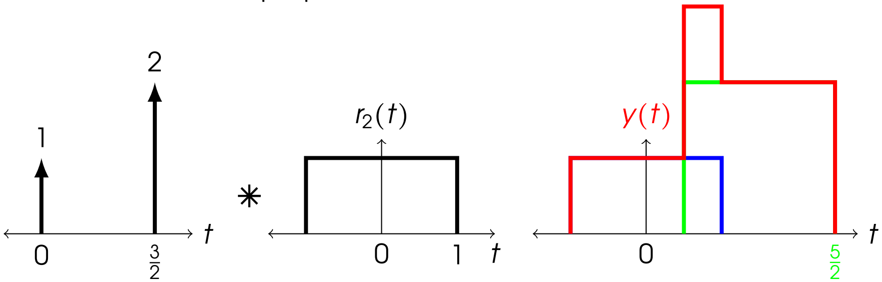
Examples

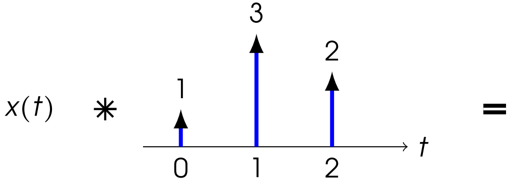
Assume the desired properties: then



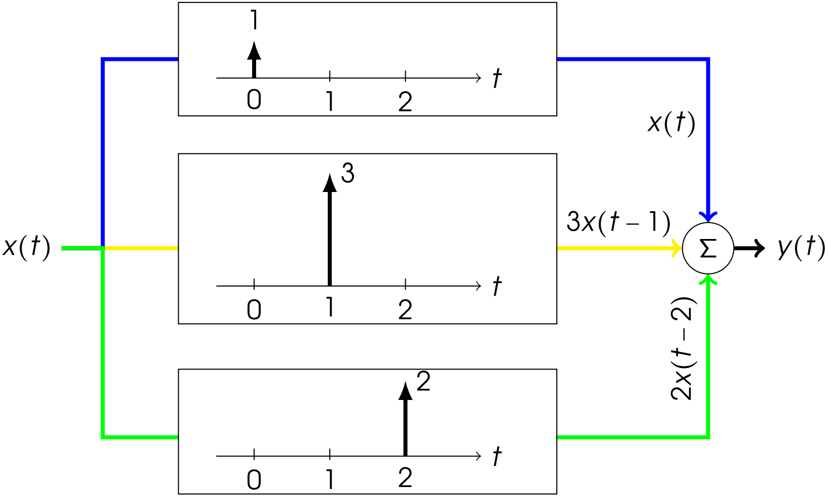
Examples

Assume the desired properties: then





Another Superposition 'Law'



Discrete Version

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + \cdots + x_m\delta(t-mT)$$
$$y(t) = \delta(t) + 3\delta(t-T) + 2\delta(t-2T)$$

Discrete Version

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + \cdots + x_m\delta(t-mT)$$
$$y(t) = \delta(t) + 3\delta(t-T) + 2\delta(t-2T)$$
$$x(t) * y(t) = x(t) + 3x(t-T) + 2x(t-2T).$$

Discrete Version

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + \cdots + x_m\delta(t-mT)$$
$$y(t) = \delta(t) + 3\delta(t-T) + 2\delta(t-2T)$$
$$x(t) * y(t) = x(t) + 3x(t-T) + 2x(t-2T).$$

Eg. let $\bar{x} = (x_0, \cdots, x_m) = (2, 3, 5, 4, 5)$ and $\bar{y} = (y_0, \cdots, y_2) = (1, 3, 2)$.

Discrete Version

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + \cdots + x_m\delta(t-mT)$$
$$y(t) = \delta(t) + 3\delta(t-T) + 2\delta(t-2T)$$
$$x(t) * y(t) = x(t) + 3x(t-T) + 2x(t-2T).$$

Eg. let $\bar{x} = (x_0, \cdots, x_m) = (2,3,5,4,5)$ and $\bar{y} = (y_0, \cdots, y_2) = (1,3,2)$.

$$\begin{array}{ccccc} 2 & 3 & 5 & 4 & 5 \\ & & 1 & 3 & 2 \\ \hline \end{array}$$

Discrete Version

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + \cdots + x_m\delta(t-mT)$$
$$y(t) = \delta(t) + 3\delta(t-T) + 2\delta(t-2T)$$
$$x(t) * y(t) = x(t) + 3x(t-T) + 2x(t-2T).$$

Eg. let $\bar{x} = (x_0, \cdots, x_m) = (2,3,5,4,5)$ and $\bar{y} = (y_0, \cdots, y_2) = (1,3,2)$.

			2	3	5	4	5
					1	3	2
2 x(t-2T)			4	6	10	8	10
3 x(t-1T)		6	9	15	12	15	
1 x(t-0T)	2	3	5	4	5		

Discrete Version

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + \cdots + x_m\delta(t-mT)$$
$$y(t) = \delta(t) + 3\delta(t-T) + 2\delta(t-2T)$$
$$x(t) * y(t) = x(t) + 3x(t-T) + 2x(t-2T).$$

Eg. let $\bar{x} = (x_0, \cdots, x_m) = (2,3,5,4,5)$ and $\bar{y} = (y_0, \cdots, y_2) = (1,3,2)$.

			2	3	5	4	5
					1	3	2
2 x(t-2T)			4	6	10	8	10
3 x(t-1T)		6	9	15	12	15	
1 x(t-0T)	2	3	5	4	5		
	2	9	18	25	27	23	10

Discrete Version

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + \cdots + x_m\delta(t-mT)$$
$$y(t) = \delta(t) + 3\delta(t-T) + 2\delta(t-2T)$$
$$x(t) * y(t) = x(t) + 3x(t-T) + 2x(t-2T).$$

Eg. let $\bar{x} = (x_0, \cdots, x_m) = (2,3,5,4,5)$ and $\bar{y} = (y_0, \cdots, y_2) = (1,3,2)$.

			2	3	5	4	5
					1	3	2
2 x(t-2T)			4	6	10	8	10
3 x(t-1T)		6	9	15	12	15	
1 x(t-0T)	2	3	5	4	5		
	2	9	18	25	27	23	10

$\boxed{\text{For } \bar{z} = \bar{x} * \bar{y}, \text{ makes sense to define } z_n := \sum_k y_k x_{n-k}}$

Primary School Days

$$
\begin{array}{ccccccccccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & \otimes \\
 & & & & & & & & & 1 & 1 & \\
\hline
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & \oplus \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & & \\
\hdashline
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} &
\end{array}
$$

Multiplication without carry addition is called **convolution**.

$\bar{x} * \bar{u} = (\bar{x}$ multiply $\bar{u})$ modulo MAXNUM.

Primary School Days

$$
\begin{array}{ccccccccccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & \otimes \\
 & & & & & & & & & 1 & 1 & \\
\hline
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & \oplus \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & & \\
\hdashline
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} &
\end{array}
$$

Multiplication without carry addition is called **convolution**.

$$\bar{x} * \bar{u} = (\bar{x} \text{ multiply } \bar{u}) \text{ modulo MAXNUM.}$$

Digital Convolution

Notation: Vector $\bar{x} := x_0, \cdots, x_{N-1}$.

Distribution of $*$ over $+$

$$\bar{x} * (\bar{u}_1 + \bar{u}_2) = (\bar{x} * \bar{u}_1) + (\bar{x} * \bar{u}_2).$$

Examples:

$$\bar{x} * [1\ \ 1] = \bar{x} * ([1\ 0] + [0\ \ 1]) = (\bar{x} * [1\ \ 0]) + (\bar{x} * [0\ \ 1])$$

$$\bar{x} * [\alpha\ \ \beta] = \bar{x} * ([\alpha\ \ 0] + [0\ \ \beta]) = (\bar{x} * [\alpha\ \ 0]) + (\bar{x} * [0\ \ \beta])$$

Makes sense to have $\bar{x} * [1\ \ 0] = \bar{x}$ "— *digital impulse*".

Digital Convolution

Notation: Vector $\bar{x} := x_0, \cdots, x_{N-1}$.

Distribution of $*$ over $+$

$$\bar{x} * (\bar{u}_1 + \bar{u}_2) = (\bar{x} * \bar{u}_1) + (\bar{x} * \bar{u}_2).$$

Examples:

$$\bar{x} * [1 \ \ 1] = \bar{x} * ([1 \ \ 0] + [0 \ \ 1]) = (\bar{x} * [1 \ \ 0]) + (\bar{x} * [0 \ \ 1])$$

$$\bar{x} * [\alpha \ \ \beta] = \bar{x} * ([\alpha \ \ 0] + [0 \ \ \beta]) = (\bar{x} * [\alpha \ \ 0]) + (\bar{x} * [0 \ \ \beta])$$

Makes sense to have $\bar{x} * [1 \ \ 0] = \bar{x}$ "— *digital impulse*".

Digital Convolution

Notation: Vector $\bar{x} := x_0, \cdots, x_{N-1}$.

Distribution of $*$ over $+$

$$\bar{x} * (\bar{u}_1 + \bar{u}_2) = (\bar{x} * \bar{u}_1) + (\bar{x} * \bar{u}_2).$$

Examples:

$$\bar{x} * [1 \;\; 1] = \bar{x} * ([1 \;\; 0] + [0 \;\; 1]) = (\bar{x} * [1 \;\; 0]) + (\bar{x} * [0 \;\; 1])$$

$$\bar{x} * [\alpha \;\; \beta] = \bar{x} * ([\alpha \;\; 0] + [0 \;\; \beta]) = (\bar{x} * [\alpha \;\; 0]) + (\bar{x} * [0 \;\; \beta])$$

Makes sense to have $\bar{x} * [1 \;\; 0] = \bar{x}$ "— *digital impulse*".

Digital Convolution

Notation: Vector $\bar{x} := x_0, \cdots, x_{N-1}$.

Distribution of $*$ over $+$

$$\bar{x} * (\bar{u}_1 + \bar{u}_2) = (\bar{x} * \bar{u}_1) + (\bar{x} * \bar{u}_2).$$

Examples:

$$\bar{x} * [1 \;\; 1] = \bar{x} * ([1 \;\; 0] + [0 \;\; 1]) = (\bar{x} * [1 \;\; 0]) + (\bar{x} * [0 \;\; 1])$$

$$\bar{x} * [\alpha \;\; \beta] = \bar{x} * ([\alpha \;\; 0] + [0 \;\; \beta]) = (\bar{x} * [\alpha \;\; 0]) + (\bar{x} * [0 \;\; \beta])$$

Makes sense to have $\bar{x} * [1 \;\; 0] = \bar{x}$ "— *digital impulse*".

Integration

$$I = \int_2^7 |x - \sin(x)|\, dx$$

Figure: Riemann Sum for $f(x) = x - \sin(x)$

Integration

$$I = \int_2^7 |x - \sin(x)|\, dx$$

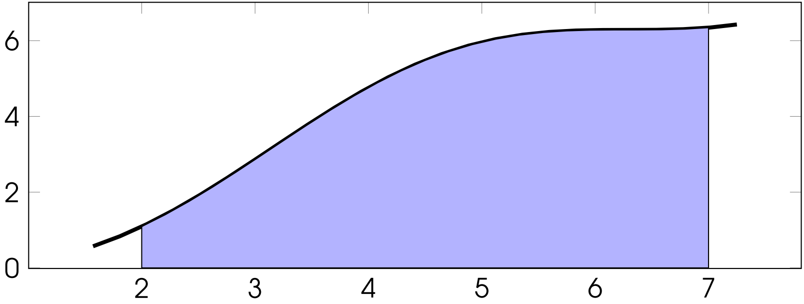


Figure: Riemann Sum for $f(x) = x - \sin(x)$

Integration

$$I = \int_2^7 |x - \sin(x)|\, dx$$

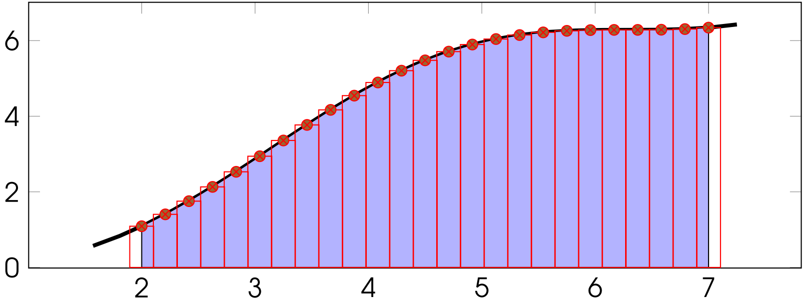


Figure: Riemann Sum for $f(x) = x - \sin(x)$

Riemann Approximation

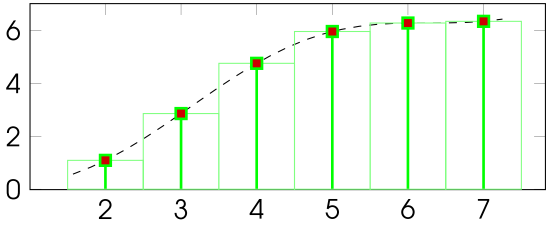


Figure: Piecewise approximation of $f(x) = x - \sin(x)$

Riemann Approximation

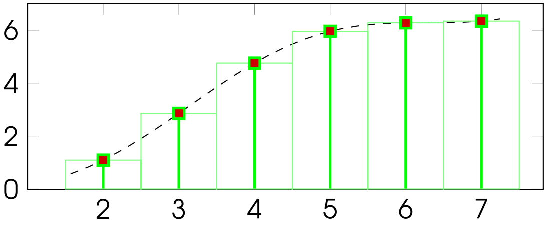


Figure: Piecewise approximation of $f(x) = x - \sin(x)$

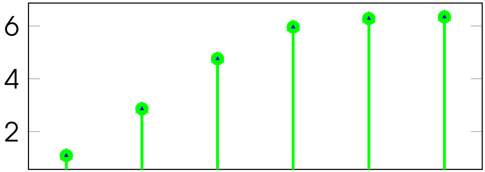


Figure: $f(x)$ as a *Composition*

Riemann Approximation

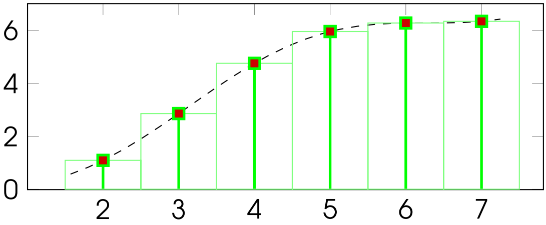
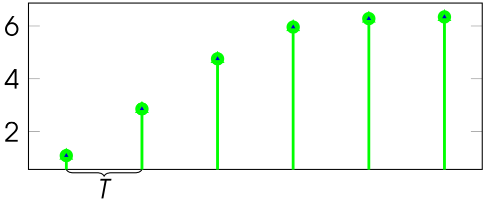


Figure: Piecewise approximation of $f(x) = x - \sin(x)$



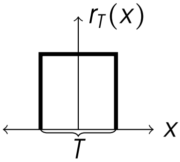


Figure: $f(x)$ as a *Composition*

Riemann Approximation

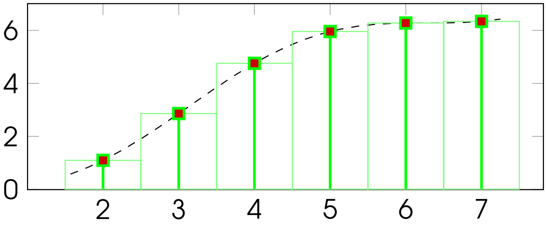


Figure: Piecewise approximation of $f(x) = x - \sin(x)$

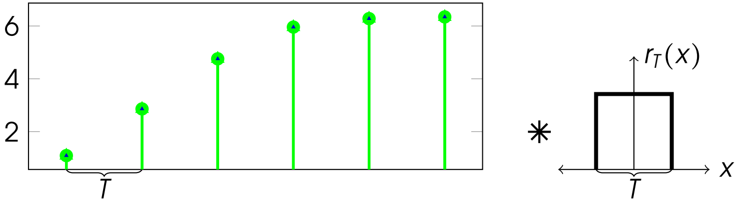


Figure: $f(x)$ as a *Composition*

Riemann Approximation

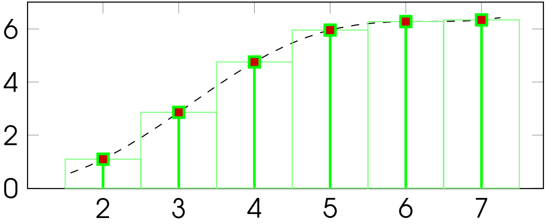


Figure: Piecewise approximation of $f(x) = x - \sin(x)$

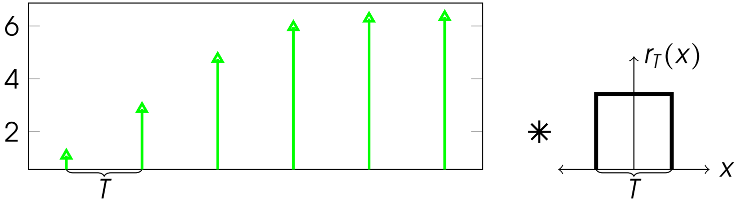
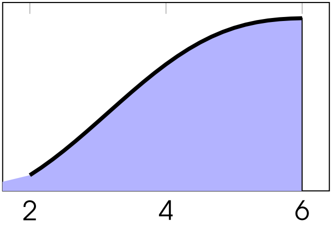
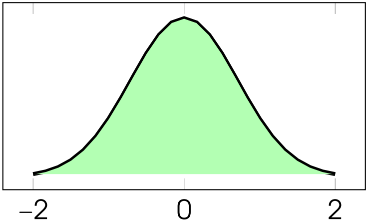


Figure: $f(x)$ as a *Composition*

Generalizing Our Operator

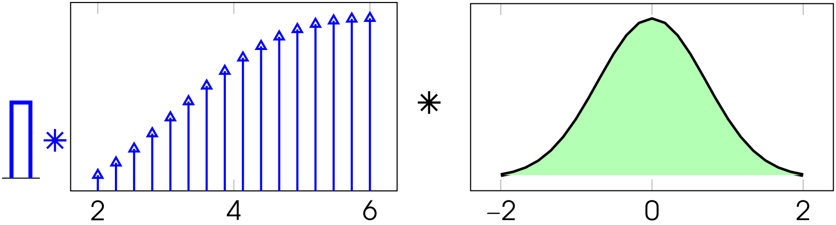


$*$



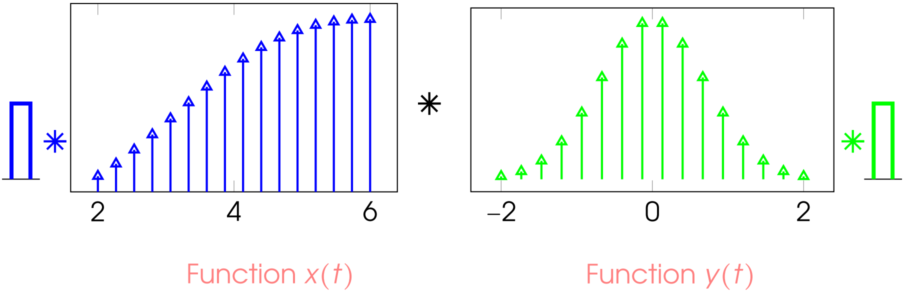
Function $x(t)$ Function $y(t)$

Generalizing Our Operator

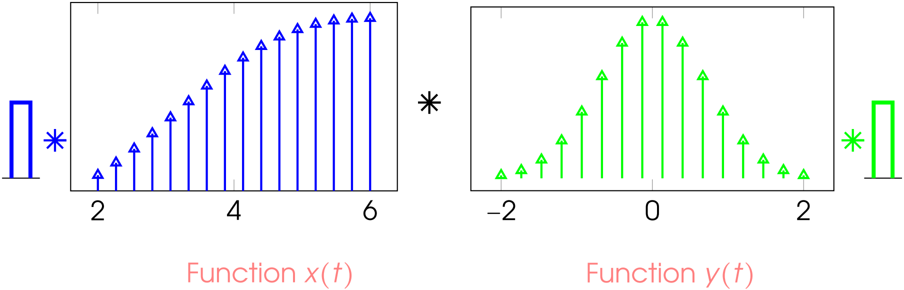


Function $x(t)$ Function $y(t)$

Generalizing Our Operator

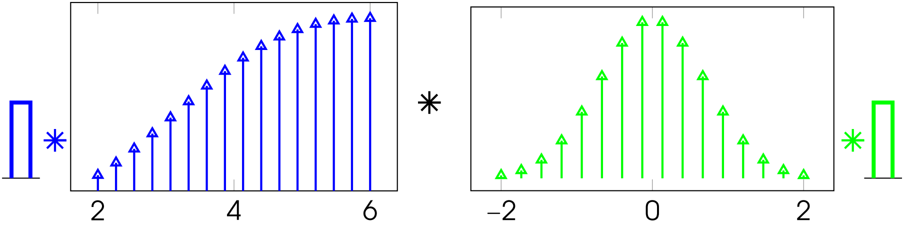


Generalizing Our Operator



$$\bar{x}_T = \sum_{m \in I} x(mT)\delta(t - mT) \quad , \quad \bar{y}_T = \sum_{n \in J} y(nT)\delta(t - nT).$$

Generalizing Our Operator

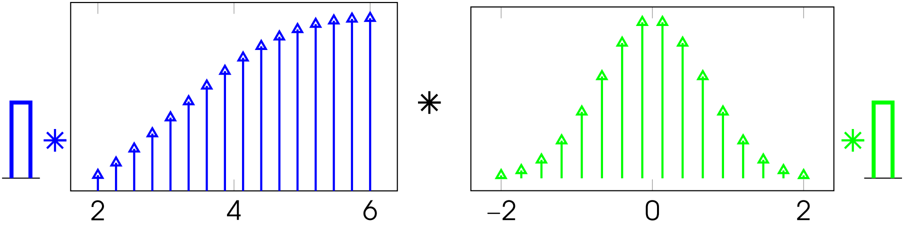


Function $x(t)$ Function $y(t)$

$$\bar{x}_T = \sum_{m \in I} x(mT)\delta(t - mT) \quad , \quad \bar{y}_T = \sum_{n \in J} y(nT)\delta(t - nT).$$

$$x(t) * y(t) \approx r_T(t) * \bar{x}_T * \bar{y}_T * r_T(t)$$

Generalizing Our Operator

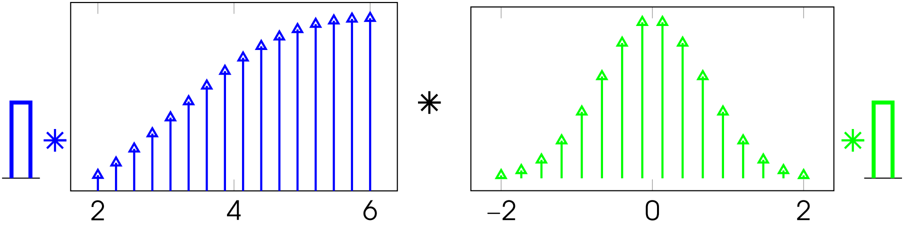


Function $x(t)$ Function $y(t)$

$$\bar{x}_T = \sum_{m \in I} x(mT)\delta(t - mT) \quad , \quad \bar{y}_T = \sum_{n \in J} y(nT)\delta(t - nT).$$

$$\begin{aligned} x(t) * y(t) &\approx r_T(t) * \bar{x}_T * \bar{y}_T * r_T(t) \\ &= \bar{x}_T * \bar{y}_T * r_T(t) * r_T(t). \end{aligned}$$

Generalizing Our Operator



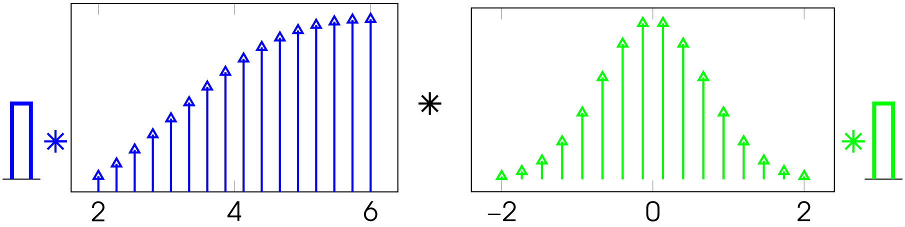
Function $x(t)$ Function $y(t)$

$$\bar{x}_T = \sum_{m \in I} x(mT)\delta(t - mT) \quad , \quad \bar{y}_T = \sum_{n \in J} y(nT)\delta(t - nT).$$

Discrete Convolution

$$\begin{aligned} x(t) * y(t) &\approx r_T(t) * \bar{x}_T * \bar{y}_T * r_T(t) \\ &= \boxed{\bar{x}_T * \bar{y}_T} * r_T(t) * r_T(t). \end{aligned}$$

Generalizing Our Operator



Function $x(t)$ Function $y(t)$

$$\bar{x}_T = \sum_{m \in I} x(mT)\delta(t - mT) \quad , \quad \bar{y}_T = \sum_{n \in J} y(nT)\delta(t - nT).$$

Discrete Convolution

Analog Convoltn

$$\begin{aligned} x(t) * y(t) &\approx r_T(t) * \bar{x}_T * \bar{y}_T * r_T(t) \\ &= \boxed{\bar{x}_T * \bar{y}_T} * \boxed{r_T(t) * r_T(t).} \end{aligned}$$

Leaky Bucket

Courtesy: Open Clipart