Department of Mathematics
Indian Institute of Technology, Bombay
Powai, Mumbai–400 076, INDIA.

MA 207–Differential Equations-II

Autumn 2020

Instructor

Amiya Kumar Pani

Name :	
Roll No :	

Course contents of MA 207 (Differential Equations-II):

Review of power series and series solutions of ODEs; Legendre's equation and Legendre polynomials; Regular and irregular singular points and method of Frobenius; Bessel's equation and Bessel functions; Sturm-Liouville problems; Fourier series, Classification of second order linear PDEs; D'Alembert solution of second order wave equations; Laplace, heat and wave equations using separation of variables; vibration of circular membrane; Heat equations in half space.

Texts/References

E. Kreyszig, Advanced Engineering Mathematics, 8th Edition, Wiley Eastern, 1999.

W.E. Boyce and R. DiPrima, Elementary Differential Equations and Boundary Value Problems, 8th Edition, Wiley, 2005.

G. F. Simmons, Differential Equations with Applications and Historical Notes, McGraw-Hill International Publications, NY, 1991.

R. V. Churchill and J. W. Brown, Fourier Series and Boundary Value Problems (7th Edition), McGraw-Hill (2006).

Policy for Attendance

Attendance in both lecture and tutorial classes is compulsory. Students who fail to attend 80% of the lecture and tutorial classes may be awarded XX grade.

Evaluation

To be announced lateron.

Instructors

Amiya Kumar Pani - Div-S1 and Div-S2, Office: MA 206, Mathematics Department, Telephone number : 7481

Course Associates

We shall announce tutorial batches with names of the tutors at a later stage.

Teaching Plan (On line teaching: through online and then atleast last 15 to 20 minutes, question-answers problem solving, through online interaction)

[K] refers to the text book by E. Kreyszig, "Advanced Engineering Mathematics", 8th Edition, John Wiley and Sons(1999).

No.	Topic	§in [K]	No.of Lec.
1.	Power Series	14.1 - 14.2	1
2.	Series solutions: Legendre's equation and Legendre polynomials	4.3	1
3.	Frobenius method, Bessel's equation, Bessel functions	4.4–4.5	2
4.	Sturm-Liouville Problems: Eigenvalues and eigenfunctions with properties	4.7-4.8	2
5.	Fourier Series	10.1 - 10.10	1
6.	Classification of 2^{nd} Order PDEs and Different Boundary Conditions, Uniqueness of Solutions	Handout-2 1	2
7.	Separation of variables for Heat and Wave equations	11.2 - 11.5	2
8.	Rectangular membrane, Laplace in polar and Circular membrane	11.8 - 11.9 11.10	2
9.	Fourier Transforms and applications to heat equations in $\mathbb{R}$ and half space	11.6	1

Tutorial Sheet No. 1

Q.1. Find the radius of convergence of the following power series:

(i) $\sum z^n$ (ii) $\sum \frac{z^m}{m!}$ (iii) $\sum m! z^m$ (iv) $\sum_{m=k}^{\infty} m(m-1)\cdots(m-k+1)z^m$ (v) $\sum \frac{(2n)!}{2^{2n}(n!)^2} z^n$

(vi) $\sum_{1}^{\infty} \frac{z^m}{m(m+1)\cdots(m+k+1)}$ (vii) $\sum_{1}^{\infty} \frac{n^n}{n!} z^n$ (viii) $\sum_{1}^{\infty} \frac{(2n)!}{n^n} z^n$ (ix) $\sum_{1}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$

Q.2. Determine the radius of convergence of $\sum n! x^{n^2}$ and $\sum x^{n!}$.

Q.3. Show that if $\sum_{1}^{\infty} a_n x^n$ has radius of convergence R, then $\sum_{1}^{\infty} a_n x^{2n}$ has radius of convergence $\sqrt{R}$ and $\sum_{1}^{\infty} a_n^2 x^n$ has the radius of convergence R^2.

Q.4. Apply the power series method to determine the general solution of the following differential equations.

(i) $(1-x^2)y' = y$ (ii) $y' = xy, \ \ y(0) = 1$ (iii) $(1-x^2)y' = 2xy$

(iv) $y' - 2xy = 1, \quad y(0) = 0$. From (iv) deduce the Taylor series for $e^{x^2} \int_0^x e^{-t^2}\, dt$.

Q.5. Find the solution as a power series in powers of $(x-1)$.

(i) $y'' + y = 0$ (ii) $y'' - y = 0$

Q.6. Find the power series solutions for the following differential equations.

(i) Legendre's equation: $(1-x^2)y'' - 2xy' + p(p+1)y = 0$.

(ii) Tchebychev's equation: $(1-x^2)y'' - xy' + p^2 y = 0$.

(iii) Airy's equation: $y'' - xy = 0$.

(iv) Hermite's equation : $y'' - x^2 y = 0$.

Q.7. Show that the function $(\sin^{-1} x)^2$ satisfies the initial value problem: $(1-x^2)y'' - xy' = 2, \quad y(0) = 0$, $y'(0) = 0$. Hence, find the Taylor series for $(\sin^{-1} x)^2$. What is its radius of convergence ?

Q.8. Attempt a power series solution (with center at the origin) for $x^2 y'' - (1+x)y = 0$. Explain why the procedure does not give any nontrivial solutions.

Q.9. Prove the Rodrigues' formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$.

Q.10. Prove that $(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) h^n$.

Q.11. Show that if $f(x)$ is a polynomial with double roots at a and b then $f''(x)$ vanishes at least twice in (a, b). Generalize this and show (using Rodrigues' formula) that $P_n(x)$ has n distinct roots in $(-1, 1)$ and in particular, $P_n(x)$ cannot have a double root.

Q.12. Establish the following recurrence relations for $P_n(x)$.

(i) $(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0$ (ii) $P'_{n+1} - xP'_n - (n+1)P_n = 0$

(iii) $xP'_n - P'_{n-1} - nP_n = 0$ (iv) $P'_{n+1} - P'_{n-1} - (2n+1)P_n = 0$

(v) $(x^2-1)P'_n - nxP_n + nP_{n-1} = 0$

Q.13. Prove the following relations:

(i) $P_n(-x) = (-1)^n P_n(x)$ (ii) $P'_n(-x) = (-1)^{n+1} P'_n(x)$

(iii) $P_n(1) = 1; P_n(-1) = (-1)^n$ (iv) $P_{2n+1}(0) = 0; P_{2n}(0) = (-1)^n \dfrac{(2n)!}{2^{2n}(n!)^2}$

(v) $P'_n(1) = \dfrac{1}{2}n(n+1)$ (vi) $P'_n(-1) = (-1)^{n-1}\dfrac{1}{2}n(n+1)$

(vii) $P'_{2n}(0) = 0$ (viii) $P'_{2n+1}(0) = (-1)^n \dfrac{(2n+1)!}{2^{2n}(n!)^2}$.

Q.14. Prove that $\displaystyle\int_{-1}^{+1} P_m(x)P_n(x)dx = 0$ $(m \neq n)$ and $\displaystyle\int_{-1}^{1} P_n(x)^2 dx = \frac{2}{2n+1}$.

Q.15. Prove the following relation if $(n-m)$ is even $(m \leq n)$:

(i) $\displaystyle\int_{-1}^{+1} P'_m P'_n dx = m(m+1)$ (ii) $\displaystyle\int_{0}^{1} P_m P_n dx = 0$ if $(n-m)$ is even and $n \neq m$.

(iii) $\displaystyle\int_{-1}^{1} x^m P'_n(x)dx = 0$ if $m \leq n$. What is the value of the integral if instead $n-m$ is odd?

Tutorial Sheet No. 2

Q.1. Locate and classify the singular points for the following differential equations:

(i) Bessel's equation: $x^2y'' + xy' + (x^2 - p^2)y = 0$.

(ii) Laguerre's equation: $xy'' + (1 - x)y' + \lambda y = 0$.

(iii) Jacobi's equation: $x(1 - x)y'' + (\gamma - (\alpha + 1)x)y' + n(n + \alpha)y = 0$.

(iv) The hypergeometric equation: $x(1 - x)y'' + [c - (a + b + 1)x)]y' - aby = 0$.

(v) $xy'' + (\cot x)y' + xy = 0$.

Q.2. Attempt a Frobenius series solution $y(x) = x^\rho \sum_{n=0}^{\infty} a_n x^n$ for the $x^2y'' + (3x - 1)y' + y = 0$, and compute the successive coefficients and the radius of convergence of the series solution. Why does the method fail?

Q.3. Find two linearly independent solutions of the following differential equations:

(i) $x(x - 1)y'' + (4x - 2)y' + 2y = 0$. (ii) $2x(x + 2)y'' + y' - xy = 0$.

(iii) $x^2y'' + x^3y' + (x^2 - 2)y = 0$. (iv) $xy'' + 2y' + xy = 0$.

Q.4. Show that the hypergeometric equation has a regular singular point at infinity[1], but that the point of infinity is an irregular singular point for the Airy's equation.

Q.5. Using the indicated substitutions, reduce the following differential equations to Bessel's equation and find the general solution in term of the Bessel functions.

(i) $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0, \ (\lambda x = z)$. (ii) $xy'' - 5y' + xy = 0, \ (y = x^3u)$.

(iii) $y'' + k^2xy = 0, \ (y = u\sqrt{x}, \ \frac{2}{3}kx^{3/2} = z)$.

(iv) $x^2y'' + (1 - 2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0, \ (y = x^\nu u, \ x^\nu = z)$.

Q.6. (a) Prove that $[x^nJ_n]' = x^nJ_{n-1}$ and $[x^{-n}J_n]' = -x^{-n}J_{n+1}$.

(b) Use (a) to prove that (i) $J_{n-1} + J_{n+1} = \frac{2n}{x}J_n$ (ii) $J_{n-1} - J_{n+1} = 2J_n'$.

Q.7. Show that (i) $J_{1/2} = \sqrt{\frac{2}{\pi x}}\sin x$ (ii) $J_{-1/2} = \sqrt{\frac{2}{\pi x}}\cos x$ (iii) $J_{3/2} = \sqrt{\frac{2}{\pi x}}\Big(\frac{\sin x}{x} - \cos x\Big)$.

(iv) $J_{-3/2} = -\sqrt{\frac{2}{\pi x}}(\frac{\cos x}{x} + \sin x)$

Q.8. When n is an integer, show that

(i) $J_n(x)$ is an even function if n is even (ii) $J_n(x)$ is an odd function if n is odd.

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[1] The differential equation $y'' + p(x)y' + q(x)y = 0$ has a regular singular point at infinity, if after substitution of $x = 1/t$ in the ODE, the resulting ODE has a regular singular point at the origin.

Q.9. Show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$ and one zero of $J_{n-1}(x)$.

Q.10. Prove that $\exp\left(\frac{tx}{2} - \frac{x}{2t}\right) = \sum_{n=-\infty}^{\infty} J_n(x)t^n$ (This formula is due to Schlömilch). Use Schlömilch's formula to show that $J_0^2 + 2\sum_{n=1}^{\infty} J_n^2 = 1$. Deduce that $|J_0| \leq 1; |J_n| \leq \frac{1}{\sqrt{2}}$.

Q.11. Prove that

(i) $\cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} \cos 2n\theta J_{2n}(x)$ (ii) $\sin(x\sin\theta) = 2\sum_{n=1}^{\infty} \sin(2n+1)\theta J_{2n+1}(x)$.

Q.12. Show that $\frac{1}{2}\frac{d}{dx}[J_n^2 + J_{n+1}^2] = \frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2$, $\frac{d}{dx}[xJ_nJ_{n+1}] = x(J_n^2 - J_{n+1}^2)$, and deduce that

(i) $J_0^2 + 2\sum_{n=1}^{\infty} J_n^2 = 1$ (ii) $\sum_{n=0}^{\infty}(2n+1)J_nJ_{n+1} = \frac{x}{2}$.

(Hint for (ii): Look at $\frac{d}{dx}\left[x\sum_{n=0}^{\infty}(2n+1)J_nJ_{n+1}\right]$)

Q.13. Prove the following.

(i) $J_3 + 3J_0' + 4J_0''' = 0$

(ii) $J_2 - J_0 = aJ_c''$ find a and c.

(iii) $\int J_{\nu+1}dx = \int J_{\nu-1}dx - 2J_\nu$.

Q.14. Find two linearly independent series solutions of Bessel's equation with $p = 0$ (note that one of these will not be a power series).

Tutorial Sheet No. 3

Q.1. Solve the following boundary value problems.

(i) $y'' - y = 0,\ \ y(0) = 0,\ \ y(1) = 1$ (ii) $y'' - 6y' + 25y = 0,\ \ y'(0) = 1,\ \ y(\pi/4) = 0$

(iii) $x^2y'' + 7xy' + 3y = 0,\ y(1) = 1,\ y(2) = 2$

(iv) $y'' + y' + y = x,\ y(0) + 2y'(0) = 1,\ \ y(1) - y'(1) = 8$

(v) $y'' + \pi^2 y = 0,\ \ y(-1) = y(1),\ \ y'(-1) = y'(1)$.

Q.2. Find the eigenvalues and eigenfunctions of the following boundary value problems.

(i) $y'' + \lambda y = 0,\ \ \ y(0) = 0,\ \ \ y'(1) = 0.$ (ii) $y'' + \lambda y = 0,\ \ \ y(0) = 0,\ \ \ y(\ell) = 0.$

(iii) $y'' + \lambda y = 0,\ y(0) = y'(0),\ y(1) = 0.$, (iv) $y'' + \lambda y = 0,\ \ \ y(0) = y(2\pi),\ y'(0) = y'(2\pi)$.

(v) $(e^{2x}y')' + e^{2x}(\lambda + 1)y = 0,\ \ \ y(0) = 0,\ \ \ y(\pi) = 0.$

Q.3. For which values of λ, does the boundary value problem

$$y'' - 2y' + (1 + \lambda)y = 0, \quad y(0) = 0,\ \ y(1) = 0$$

have a non-trivial solution ?

Q.4. Show that the eigenvalues of the boundary value problem $y'' + \lambda y = 0,\ y(0) = 0,\ y(1) + y'(1) = 0$ are obtained as solutions of $\tan k = -k$, where $k = \sqrt{\lambda}$. Conclude from a plot that this equation has infinitely many solutions. Show that the eigenfunctions are $y_m = \sin(k_m x)$.

Q.5. Determine the normalised eigenfunctions of the Sturm-Liouville problem

$y'' + \lambda y = 0,\ \ y(0) = 0 = y(1)$.

Q.6. Expand the function $f(x) = x,\ \ x \in [0, 1]$ in terms of the normalised eigenfunctions $\phi_n(x)$ of the boundary value problem $y'' + \lambda y = 0,\ \ y(0) = 0,\ \ y(1) + y'(1) = 0$.

Q.7. Find the eigenfunctions and the eigenvalues of the following Sturm-Liouville problems.

(i) $y'' + 2y' + (\lambda + 1)y = 0;\ \ \ y(0) = y(\pi) = 0$ (ii) $x^2y'' + xy' + \lambda y = 0;\ \ \ y(1) = y(\ell) = 0.$

Q.8. Verify that $J_n(\frac{kx}{a})$ satisfies $\frac{d}{dx}\left[x\frac{d}{dx}\left\{J_n(\frac{kx}{a})\right\}\right] + \left(\frac{k^2x}{a^2} - \frac{n^2}{x}\right)J_n(\frac{kx}{a}) = 0.$

Multiply by $J_n(\frac{\ell x}{a})$ and integrate by parts from 0 to a to get

$$kJ_n'(k)J_n(l) - \frac{kl}{a^2}\int_0^a xJ_n'\left(\frac{\ell x}{a}\right)J_n'\left(\frac{kx}{a}\right)\,dx + \int_0^a (\frac{k^2}{a^2}x - \frac{n^2}{x})J_n(\frac{kx}{a})J_n(\frac{lx}{a})dx = 0,$$

where prime ($\prime$) denotes differentiation with respect to the argument of J_n. Interchange k and ℓ to obtain the relation

$$\int_0^a xJ_n(\frac{kx}{a})J_n(\frac{\ell x}{a})dx = a^2\frac{\ell J_n(k)J_n'(\ell) - kJ_n(\ell)J_n'(k)}{k^2 - \ell^2}.$$

Prove that if k and ℓ are the roots of the Bessel's equation $J_n(\lambda) = 0$ then

$$\begin{aligned}\int_0^a xJ_n(\frac{kx}{a})J_n(\frac{\ell x}{a})dx &= 0 \qquad (k \neq \ell) \\ &= \frac{1}{2}a^2[J_n'(k)]^2 \qquad (k = \ell) \\ &= \frac{1}{2}a^2 J_{n+1}^2(k) \qquad (k = \ell).\end{aligned}$$

Q.9. The function $P_n(x)$ satisfies the equation $\frac{d}{dx}[(1-x^2)P_n'] + n(n+1)P_n = 0$. Proceed as indicated in Q.8 above to prove that $\int_{-1}^{+1} P_m P_n dx = 0 \qquad (m \neq n)$.

Q.10. If $x^n = \sum_{r=0}^{n} a_r P_r(x)$ prove that $a_n = \frac{2^n(n!)^2}{(2n)!}$.

Q.11. Prove that $\int_{-1}^{+1}(1-x^2)[P_n'(x)]^2\,dx = \frac{2n(n+1)}{2n+1}$.

Q.12. Show that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\sin nx \sin^2 n\alpha &= \text{constant} \qquad (0 < x < 2\alpha) \\ &= 0 \qquad (2\alpha < x < \pi)\end{aligned}.$$

Q.13. Prove that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos nx}{n^2} = \frac{\pi^2}{12} - \frac{x^2}{4} \quad (-\pi \leq x \leq \pi)$.

Q.14. Show that $\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1)^3} = \frac{1}{8}\pi x(\pi - x) \quad (0 \leq x \leq \pi)$.

Q.15. (**Fourier Theorem**) Let $f(x)$ be a periodic function of period 2π on the real axis which is piecewise continuously differentiable. Suppose further that $\int_{-\pi}^{\pi}|f(x)|dx < \infty$. Let a_n and b_n be defined by the relations

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt, \qquad n = 0, 1, 2, \ldots$$

The series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$ converges to $f(x)$ if $f(t)$ is continuous at $t = x$ and converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ if $f(t)$ has a finite discontinuity at $t = x$. From the Fourier expansions given in Q.1 through Q.7 and the Fourier theorem stated above deduce the following results.

(i) $1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \ldots = \frac{2\pi}{3\sqrt{3}}$

(ii) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-\frac{1}{11}+\ldots=\frac{\pi}{3\sqrt{3}}$

(iii) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$ (iv) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$

(v) $1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-+\ldots=\frac{\pi^3}{32}$ (vi) $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\ldots=\frac{\pi^2}{8}$

(vii) $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\ldots=\frac{\pi}{4}-\frac{1}{2}.$

Q.16 Using Parseval's identity, prove that $1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ldots=\frac{\pi^4}{96}$.

(Hint: Use $f(x)=\begin{cases} x, & -\pi/2<x<\pi/2 \\ \pi-x, & \pi/2<x<3\pi/2. \end{cases}$)

Q.17 Find the Fourier series of the function $f(x)$ which is assumed to have the period 2π, where

(i) $f(x)=x, \quad 0<x<2\pi.$

(ii) $f(x)=\begin{cases} -x, & -\pi\leq x<0 \\ x, & 0\leq x<\pi \end{cases}$

(iii) $f(x)=x+|x|, \quad -\pi<x<\pi.$

Q.18 Find the Fourier series of the periodic function $f(x)$, of period $p=2$, when

$$f(x)=\begin{cases} 0, & -1<x<0 \\ x, & 0<x<1 \end{cases}$$

Q.19 State whether the given function is even or odd. Find its Fourier series

(i)$f(x)=\begin{cases} k, & -\pi/2<x<\pi/2 \\ 0, & \pi/2<x<3\pi/2 \end{cases}$ (ii) $f(x)=3x(\pi^2-x^2), \quad -\pi<x<\pi.$

Q.20 In each problems, find the Fourier series for the given function f on the prescribed interval.

(i)

$$f(x)=\begin{cases} -1, & -1\leq x<0 \\ 1, & 0\leq x\leq 1 \end{cases}$$

for $|x|\leq 1$.

(ii)

$$f(x)=\begin{cases} -x, & -1\leq x<0 \\ x, & 0\leq x\leq 1 \end{cases}$$

for $|x| \leq 1$.

(iii)

$$f(x) = \begin{cases} 0, & -2 \leq x < 1 \\ 3, & 1 \leq x \leq 2 \end{cases}$$

for $|x| \leq 2$.

(iv)

$$f(x) = e^x;\ |x| \leq l.$$

(v)

$$f(x) = \sin^2 x;\ |x| \leq \pi.$$

Q.21 Expand each of the following functions as a Fourier cosine series on the prescribed interval.

(i)

$$f(x) = e^{-x};\ 0 \leq x \leq 1.$$

(ii)

$$f(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ 1, & 1 \leq x \leq 2 \end{cases}$$

for $0 \leq x \leq 2$.

Q.22 Expand each of the following functions as a Fourier sine series on the prescribed interval.

(i)

$$f(x) = e^{-x};\ 0 < x < 1.$$

(ii)

$$f(x) = \begin{cases} x, & 0 < x < a \\ a, & a \leq x \leq 2a \end{cases}$$

for $0 < x < 2a$.

(iii)

$$f(x) = 2\sin x\ \cos x;\ 0 < x < \pi.$$

Tutorial Sheet No. 4

Q.1 Classify the following partial differential equations. Find the characterstic curves and transform into canonical form.

(i) $u_{xx} + 4u_{xy} + 3u_{yy} + 3u_x - u_y + 2u = 0.$

(ii) $u_{xx} + 2u_{xy} + u_{yy} + 5u_x + 3u_y + u = 0.$

(iii) $u_{xx} - 6u_{xy} + 12u_{yy} + 4u_x - u = \sin(xy).$

(iv) $u_{xx} + xu_{yy} = 0.$

Q.2 Classify the following partial differential equations:

(i) $5u_{xx} - 3u_{yy} + (\sin x)u_x + e^{xy^2}u_y + u = 0.$

(ii) $e^{xy}u_{xx} + (\cosh x)u_{yy} + u_x - u = 0.$

(iii) $[\log(1 + x^2 + y^2)]u_{xx} - [2 + \cos x]u_{yy} = 0.$

(iv) $10u_{xx} + u_{yy} - u_x + (\log(1 + x^2))u = 0.$

(v) $u_{yy} + (1 + x^2)u_x - u_y + u = 0.$

Q.3 Show that

$$3u_{x_1x_1} - 2u_{x_1x_2} + 2u_{x_2x_2} - u_{x_2x_3} + 3u_{x_3x_3} + 5u_{x_2} - u_{x_3} + 10u = 0$$

is of elliptic type by determining the matrix A (eigen values $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 4$). Determine a transformation that yields

$$u_{\xi_1\xi_1} + 3u_{\xi_2\xi_2} + 4u_{\xi_3\xi_3} + \frac{9}{\sqrt{6}}u_{\xi_1} + \frac{1}{\sqrt{2}}u_{\xi_2} - \frac{4}{\sqrt{3}}u_{\xi_3} + 10u = 0.$$

Q.4 Determine the regions where the Tricomi's equation : $u_{xx} + xu_{yy} = 0$ is of elliptic, parabolic and hyperbolic type. Obtain its characteristics and canonical form in the hyperbolic region.

Q.5 Classify the following partial differential equations:

(i) $u_{xx} + 2u_{yz} + (\sin x)u_z - e^{y^2}u_x + u = 0.$

(ii) $u_{xx} + 2u_{xy} + u_{yy} + 2u_{zz} + (1 + xy)u = 0.$

(iii) $7u_{xx} - 10u_{xy} - 22u_{yz} + u_{yy} - 16u_{xz} - 5u_{zz} = 0.$

(iv) $u_{xx} - 2x^2u_{xz} + u_{yy} + u_{zz} = 0.$

Q.6 Using Gauss divergence theorem, prove the uniqueness result for the following boundary value problems:

(i)

$$-\Delta u = f(x),\ x \in \Omega$$

with $u = g,\ x \in \partial\Omega$.

(ii)

$$-\Delta u + u = f(x),\ x \in \Omega$$

with Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = g,\ x \in \partial\Omega$$

(iii)

$$-\Delta u = f(x),\ x \in \Omega$$

with Robin boundary condition

$$\alpha u + \frac{\partial u}{\partial \nu} = g,\ x \in \partial\Omega.$$

Here, $\partial\Omega$ is the piecewise smooth boundary, ν is the outward normal and α is a positive constant.

Q. 7 Using Gauss divergence theorem, show that each of the following problem has a unique solution:

(i)

$$-\bigtriangledown \cdot (a(x) \bigtriangledown u) = f(x),\ x \in \Omega$$

with mixed boundary conditions $u = g,\ x \in \partial\Omega_1$ and $a\frac{\partial u}{\partial \nu} = h,\ x \in \partial\Omega_2$. Here, $\partial\bar{\Omega}_1 \cup \partial\bar{\Omega}_2 = \partial\Omega$ are parts of the boundary.

(ii)

$$u_t - \triangle u = f(x,t),\ x \in \Omega\ \ t > 0$$

with Robin boundary condition

$$\alpha u + \frac{\partial u}{\partial \nu} = g,\ x \in \partial\Omega,\ \ t \geq 0$$

and initial condition: $u(x,0) = u_0(x),\ \ x \in \Omega$.

(iii)

$$u_t - \triangle u = f(x),\ x \in \Omega,\ t \in [0,T]$$

with Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = g,\ x \in \partial\Omega,\ t \in [0,T]$$

and initial condition $u(x,0) = u_0$.

(iv)

$$u_{tt} - \bigtriangledown \cdot (a(x) \bigtriangledown u) = f(x,t),\ x \in \Omega,\ t \in [0,T]$$

with Robin boundary condition

$$\alpha u + a\frac{\partial u}{\partial \nu} = g,\ x \in \partial\Omega$$

and initial conditions $u(x,0) = u_0,\ u_t(x,0) = u_1$.

Here, $\partial\Omega$ is the piecewise smooth boundary, ν is the outward normal, $a \geq \beta > 0$ and α is a positive constant.

Q. 8 If u is a solution of $-\Delta u = f,\ \ x \in \Omega$ with Neumann boundary condition :

$$\frac{\partial u}{\partial \nu} = g,\ x \in \partial\Omega,$$

then, prove that u satisfies $\displaystyle\int_{\partial\Omega} g\,ds + \int_{\Omega} f(x)\,dx = 0.$

Q.9 Show that the solution u of

$$u_{tt} - u_{xx} = 0,\ x \in \mathbb{R},\ t > 0,$$

with $u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \mathbb{R}$ and

$$\lim_{x\to-\infty} u(x,t) = \lim_{x\to+\infty} u(x,t) = 0$$

satisfies the following conservation property: $E(t) = E(0)$, where

$$E(t) = \int_{-\infty}^{\infty} |u_t(x,t)|^2\,dx + \int_{-\infty}^{\infty} |u_x(x,t)|^2\,dx.$$

Q.10 For the heat equation :

$$u_t - \bigtriangledown \cdot (a(x) \bigtriangledown u) = 0,\ x \in \Omega,\ \ t > 0$$

with homogeneous Dirichlet boundary condition : $u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0$ and initial condition: $u(x,0) = u_0(x),\ x \in \Omega$, show that the energy $E(t) = \dfrac{1}{2}\displaystyle\int_{\Omega} |u(x,t)|^2\,dx$ is monotonically decreasing and satisfies $E(t) \leq E(0)$.

Tutorial Sheet No. 5

Q.1 Using the method of separation of variables, solve : $u_t = u_{xx},\ x \in (0,1),\ t > 0$ with data

(i) $u(0,t) = u(1,t) = 0$ and $u(x,0) = x(1-x)$.

(ii) $u_x(0,t) = 0 = u_x(1,t)$ and $u(x,0) = x(1-x)$.

Q.2 For the heat equation : $u_t - c^2 u_{xx} = 0,\ 0 < x < \ell,\ t > 0$ with initial condition $u(x,0) = f(x),\ 0 < x < \ell$ and Neumann boundary conditions $u_x(0,t) = u_x(\ell,t) = 0$, show that $\displaystyle\int_0^\ell u(x,t)\,dx = C$, where C is a constant. Further, using the method of separation of variables, prove that $\displaystyle\lim_{t\to\infty} u(x,t) = \frac{1}{\ell}\int_0^\ell f(x)\,dx$. Compute the solution, when (i) $f(x) = x$ and (ii) $f(x) = \sin^2(\dfrac{\pi x}{\ell})$.

Q.3 Using the method of separation of variables, compute the solution of : $u_t - c^2 u_{xx} + a^2 u = 0,\ 0 < x < \ell, t > 0$ with initial condition $u(x,0) = f(x),\ 0 < x < \ell$ and Dirichlet boundary conditions $u(0,t) = u(\ell,t) = 0$. Find $\displaystyle\lim_{t\to\infty} u(x,t)$.

Q.4 Use the method of separation of variables to obtain the solution of following wave equation :

$$u_{tt} - c^2 u_{xx} = 0,\ 0 < x < \ell,\ t > 0$$

with initial condition $u(x,0) = f(x),\quad u_t(x,0) = g(x)\ 0 < x < \ell$ and Neumann boundary conditions $u_x(0,t) = u_x(\ell,t) = 0$. Compute the solution, when (i) $f(x) = x^2(x - \ell^2),\ g(x) = 0$, (ii) $f(x) = \sin^2(\frac{\pi x}{\ell}),\ g(x) = 0$ and (iii) $f(x) = 0,\ g(x) = 1$.

Q.5 Use the method of separation of variables to solve the following problem for the telegrapher's equation : $u_{tt} - \gamma^2 u_{xx} + 2\alpha^2 u_t = 0,\ 0 < x < \ell,\ t > 0$ with initial condition $u(x,0) = f(x),\quad u_t(x,0) = g(x)\ 0 < x < \ell$ and Dirichlet boundary conditions $u(0,t) = u(\ell,t) = 0$. Show that the solution $u(x,t)$ tends to zero as $t \to \infty$.

Q.6 Solve : $u_{tt} - c^2 u_{xx} = xe^{-t},\ 0 < x < \ell, t > 0$ with boundary conditions: $u(0,t) = \sin t,\ \ u(\ell,t) = 1$ and initial conditions: $u(x,0) = u_t(x,0) = 0$.

Q.7 Solve : $u_t - c^2 u_{xx} = 0,\ 0 < x < \ell, t > 0$ with boundary conditions: $u_x(0,t) = 0,\ \ u_x(\ell,t) = e^{-t}$ and initial condition $u(x,0) = 0$.

Q. 8 Solve the following heat equation : $u_t - c^2 u_{xx} = 0,\ 0 < x < \ell,\ t > 0$ with Dirichlet boundary conditions : $u(0,t) = 0,\ u(\ell,t) = t$ and initial condition $u(x,0) = f(x), 0 < x < \ell$. Discuss the behaviour of the solution for large t.

Q.9 Solve the following problems :

(i) $u_t - c^2 u_{xx} = 0, 0 < x < \ell,\ t > 0$ with zero initial condition and Dirichlet boundary conditions : $u(0,t) = 0,\ u(\ell,t) = e^{-t},\ t > 0$.

(ii) $u_{tt} - c^2 u_{xx} = 0,\ 0 < x < \ell,\ t > 0$ with zero initial conditions and boundary conditions : $u_x(0,t) = t,\ u_x(\ell,t) = 0$.

10. Find the solution of the following problems:

(i)
$$u_t - u_{xx} = f,\ 0 < x < 1,\ t > 0, \quad \text{with initial condition } u(x,0) = u_0,$$
and one of the following boundary conditions :
(a) Dirichlet BC : $u(0,t) = u(1,t) = 0$,
(b) Neumann BC : $u_x(0,t) = 0 = u_x(1,t)$,
(c) Mixed BC : $u(0,t) = 0 = u_x(1,t)$ and
(d) Periodic BC : $u(x+1,t) = u(x,t)$.

(ii) Repeat the same process for the second order wave equation :
$$u_{tt} - u_{xx} = f,\ 0 < x < 1,\ t > 0 \text{ with initial conditions } u(x,0) = u_0,\ u_t(x,0) = u_1,$$
with the BCs given in (i).

(iii) For problems (i) and (ii) with homogeneous Dirichlet boundary conditions and with $f = x(x-1)$ and $f = \sin \pi x\ \sin \pi t$ and zero initial conditions.

Tutorial Sheet No. 6

Q.1 Using the method of separation of variables, solve the Laplace equation: $\Delta u = 0$ in a rectangle $[0,a] \times [0,b]$ with the boundary conditions:

(i) $u(x,0) = u(x,1) = 0$ and $u(0,y) = u(1,y) = \sin \pi y$, assuming $a = 1 = b$.

(ii) $u_x(0,y) = u_x(a,y) = 0,\ 0 < y < b, \quad u_y(x,0) = f(x),\ \ u_y(x,b) = g(x),\ 0 < x < a$.

(iii) $u(0,y) = u(a,y) = 0,\ 0 < y < b, \quad u_y(x,0) = f(x),\ \ u_y(x,b) = g(x),\ 0 < x < a$.

Compute the solution for the last two cases when $f(x) = x(x-a)$ and $g(x) = 0$.

Q.2 Assuming that term-wise differentiation is permissible, show that a solution of Laplace equation: $\Delta u = 0$ in the disc of radius 1 with the boundary condition : $u(1,\theta) = f(\theta)$ is given by
$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$
where a_n, b_n are the Fourier coefficients of f.

Q.3 Solve the Neumann problem for the Laplace equation : $\Delta u = 0$ in the disc of radius 1 with the boundary condition : $\frac{\partial u}{\partial r}(1,\theta) = \sin^3 \theta$.

Q.4 Show that if $f(x)$ and $g(x)$ decay sufficiently rapidly at infinity, $\widehat{f * g}(\xi) = \widehat{f}(\xi)\ \widehat{g}(\xi)$ which is the convolution theorem for Fourier transform.

Q.5 Show that if $f(x)$ and its derivatives decay sufficiently rapidly at infinity, then

$$\widehat{\frac{df}{dx}} = i\xi\widehat{f}, \quad \widehat{xf(x)} = \frac{-1}{i}\frac{\partial\widehat{f}}{\partial\xi}.$$

Q.6 Find the Fourier transform of the following functions defined on $\mathbb{R}$:

(i) $f(x) = x^2$, if $0 < x < 1$ and 0 otherwise.

(ii) $f(x) = |x|$ if $x \in (-1, 1)$ and zero otherwise.

(iii) $f(x) = e^{-ax^2}$, $x \in \mathbb{R}$ and $a > 0$.

(iv) $f(x) = xe^{-x}$, $x > 0$ and zero otherwise.

(v) $f(x) = e^{-|x|}$, $x \in \mathbb{R}$.

(vi) $f(x) = 1$, if $x \in (-1, 1)$ and zero otherwise.

Q.7 Consider the heat equation $u_t = u_{xx}$ in the domain $t \geq 0$, $x \in \mathbb{R}$ with initial condition $u(x, 0) = f(x)$. Taking the Fourier transform with respect to the x variable prove that $\frac{d\widehat{u}}{dt} = -\xi^2\widehat{u}$, $\widehat{u}(0) = \widehat{f}(\xi)$ where $\widehat{u}$ denotes the Fourier transform of u and hence, deduce that $\widehat{u}(\xi, t) = e^{-\xi^2 t}\widehat{f}(\xi)$. (You may interchange $\frac{d}{dt}$ and $\int_0^\infty dx$ without justification.)

Q. 8 Prove that $\widehat{u}(\xi, t)$ obtained above can be written as $\widehat{u}(\xi, t) = \frac{1}{\sqrt{4\pi t}}\widehat{G * f}$, where $G(x) = e^{-x^2/4t}$. Deduce that the solution of the initial value problem for the heat equation is given by $u(x, t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4t}\right) f(y)dy$. Note that the last formula makes sense (because of the rapid decay of the exponential) even if f doesn't decay, although the steps leading to the formula need further justification. Of course one may verify the correctness of the formula by substituting into the heat equation (pl. work it out).

Q.9 Check that $\lim_{t \to 0} u(x, t) = f(x)$ by first setting $y = x - 2\sqrt{t}u$ in the integral given in $Q.8$.

Q. 10 For the heat equation: $u_t + \beta u_x = Du_{xx}$, $x \in \mathbb{R}$, $t > 0$, with initial condition $u(x, 0) = u_0(x)$, $x \in \mathbb{R}$, apply transformations: $y = x - \beta t$, $\tau = t$, and set $v(y, \tau) = u(x, t)$. Show that v satisfies the equation: $v_\tau - Dv_{yy} = 0$, $y \in \mathbb{R}, \tau > 0$ with initial condition $v(y, 0) = u_0(x)$. Using Fourier transformation, find the solution u.

Q. 11 The mathematical model developed by Black & Scholes for an option price on a non-dividend paying stock is governed by the following PDE: $\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} - rc = 0$, where, S is the price of the underlying asset (called share of a company say for example), $c = c(S, t)$ denotes the price of the European call option[2], r denotes the short term risk free interest

[2] A call option gives the holder the right to buy an asset at a certain price (called strike or exercise price) by a certain time (called expiration time) and European option means it can be exercised at the end of the expiration time.

rate, σ denotes the volatility of the underlying stock and T is the expiration date. This is augmented by initial condition: $c(S,T) = \max(S-E,0)$ and condition at ∞ as $c(S,t) \to S$ as $S \to \infty$, where E is called exercised price. Use transformation $t = \frac{(T-\tau)}{\frac{1}{2}\sigma^2}$ (it acts as a normalizer as well as to bring T to zero), $c = Ee^{k_1 x + k_2 \tau} u(x,\tau)$, where $k_1 = -\frac{1}{2}(\frac{2r}{\sigma^2} - 1)$ and $k_2 = -\frac{1}{4}(\frac{2r}{\sigma^2} + 1)^2$ and transform the equation to a heat equation $u_\tau - u_{xx} = 0$. Apply the Fourier transformation technique to obtain an anlytical solution for the European call option.

Q.12 Solve the following initial value problems using Fourier transform techniques.

(i) $u_t = c^2 u_{xx},\ x \in \mathbb{R},\ t > 0$ with initial condition : $u(x,0) = \frac{1}{1+x^2}$.

(ii) $u_t = c^2 u_{xx},\ x \in \mathbb{R},\ t > 0$ with initial condition : $u(x,0) = 1$, if $x \in (-1,1)$ and zero otherwise.

Q.13 Use Plancherel's identity to find a relation between the solution and the initial data (in terms of L^2-norm) for the following Cauchy Problems:

(i) $u_t + u_x + bu = 0$ with given $u(x,0)$.

(ii) $u_t = u_{xxx}$ with given $u(x,0)$.

(iii) $u_t = bu_{xx} + au_x + cu$ with given $u(x,0)$.

(iv) $u_{tt} = a^2 u_{xx}$ with $u(x,0) = u_0,\ u_t(x,0) = u_1$.

Q.14 Show that if the function u is square integrable in $\mathbb{R}$ and its transform satisfies $|\hat{u}(y)| \le \frac{C}{1+y^4}$, for some constant C, then the first and second derivatives of u exist and are bounded functions.

Q. 15 Using Fourier transform, solve the following transport equation: $u_t + au_x = 0,\ x \in \mathbb{R},\ \ t > 0$ with initial condition $u(x,0) = u_0(x)$.

Q.16 Show that if u is absolutely integrable in $\mathbb{R}$, then $\hat{u}(y)$ is a continuous function on $\mathbb{R}$. Show that $\|\hat{u}\|_\infty \le (2\pi)^{-1/2} \|u\|_{L^1}$ (optional).

Handout 1

MA-207: DE-II Sturm - Liouville problems

Linear Boundary Value Problems. Besides Initial Value Problems (IVP), where the unknown function and its derivatives are prescribed at a single point x_0, we often encounter in applications the so called boundray value problems (BVP) for differential equations. In these problems, the value of the required unknown function with its derivatives is defined at both ends of the interval on which a solution is sought. The differential equation together with the boundary conditions is referred to as a boundary value problem.

Consider a general second order linear differential equation.

$$p(x)y'' + q(x)y' + r(x)y = f(x), \quad a \le x \le b \tag{1}$$

satisfying boundary conditions of the form

$$\left.\begin{aligned} b_1[y] &\equiv k_1 y(a) + k_2 y'(a) = \alpha \quad (k_1 \text{ and } k_2 \text{ not both zero}) \\ b_2[y] &\equiv l_1 y(b) + l_2 y'(b) = \beta \quad (l_1 \text{ and } l_2 \text{ not both zero}), \end{aligned}\right\} \tag{2}$$

where $k_1,\ k_2,\ l_1,\ l_2,\ \alpha$ & β are real constants. When $f \equiv 0$, the problem (1) with homogenous boundary conditions (2) that is, $\alpha = 0,\ \beta = 0$ is called a linear homogeneous boundary value problem. Otherwise it is called a linear nonhomogeneous BVP. The boundary condition given in (2) are of general type and depending on $k_1,\ k_2,\ l_1,$ & l_2, we classify these BVPs as:

(i) Dirichlet BCs:
$$y(a) = \alpha, \quad y(b) = \beta.$$

(ii) Neumann BCs:
$$y'(a) = \alpha, \quad y'(b) = \beta.$$

(iii) Mixed BCs:
$$(a) \quad y(a) = \alpha, \quad y'(b) = \beta,$$
or
$$(b) \quad y'(a) = \alpha, \quad y(b) = \beta.$$

(iv) Robin or Fourier BCs:
$$y(a) + ky'(a) = \alpha, \quad k \neq 0$$
$$y'(b) + ly'(b) = \beta, \quad l \neq 0$$
.

(v) Periodic BCs:
$$y(a) = y(b), \quad y'(a) = y'(b)$$

Depending on a, b and coefficient p, we sometimes, classify the BVPs (1) -(2) as regular or singular linear BVP.

Definition (regular linear BVP). A linear BVP (1)-(2) is called **regular** if both a and b are finite, and the coefficient $p(x) \neq 0$ for all $x \in [a, b]$. A linear BVP is called **singular** if it is not regular. For example, if either $a = -\infty$ or $b = \infty$, or $p(x) \equiv 0$ for at least one $x \in [a, b]$.

By a solution of the linear BVP (1)-(2), we mean a twice continuously differentiable function y which satisfies the DE (1) and the boundary condition (2).

The existence and uniqueness theory even for the linear BVP is more difficult than that of IVP. For example, a simple looking DE : $y'' + y = 0, \;\; 0 < x < \pi$ with $y(0) = 0$ and $y(\pi) = \beta \neq 0$ does not have a solution.

We have seen earlier in Differential Equations-I that an initial value linear problem has a unique solution on any interval about the initial point in which the coefficients $p,\ q,\ r$ & f are continuous. No such general statement can be made about the BVP (1)-(2). Although the homogeneous linear BVP has a trivial solution $y \equiv 0$ but there can be nontrivial solutions. Very often such nontrivial solutions are of paramount interest in applications. However it is possible to find a set of conditions so that the homogeneous BVP (that is, homogeneous DE with homogeneous BCs) has only one trivial solution.

Theorem 1. Let y_1 and y_2 be two linearly independent solutions of $p(x)y'' + q(x)y' + r(x)y = 0$. Then the homogeneous boundary value problem has only one trivial solution if and only if

$$W = \begin{vmatrix} b_1[y_1] & b_1[y_2] \\ b_2[y_1] & b_2[y_2] \end{vmatrix} \neq 0.$$

Hint. The general solution $y(x) = c_1y_1 + c_2y_2$. This is a solution of the homogeneous bvp if and only if

$$b_1(y) \equiv c_1b_1[y_1] + c_2b_1[y_2] = 0 \;\text{ and } b_2(y) \equiv c_1b_2[y_1] + c_2b_2[y_2] = 0.$$

This matrix equation

$$\begin{bmatrix} b_1[y_1] & b_1[y_2] \\ b_2[y_1] & b_2[y_2] \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has only trivial solution if and only if $W \neq 0$. This completes the proof.

Remark: The homogeneous bvp has infinitely many nontrivial solutions if and only if $W = 0$.

Theorem 2. The nonhomogeneous linear BVP (1)-(2) has a unique solution if and only if the corresponding homogeneous linear BVP has only a trivial solution .

Strum - Liouville Problems. The Strum - Liouville[3] problem represents a class of linear BVPs. One of the important aspect of this class of problems is that they generate a set of orthogonal functions. In what follows, we consider the second order linear differential equation of the form

$$(r(x)y')' + (q(x) + \lambda p(x))\, y = 0, \quad a \leq x \leq b \tag{3}$$

[3]Charles - Francois Sturm (1803 - 1855) and Joseph Liouville (1809 - 1882) in a series of papers in 1836 and 1837, set forth many important properties of the class of linear BVP associated with their name.

satisfying the general boundary conditions

$$\left.\begin{aligned} k_1y(a)+k_2y'(a)=0 \quad &(k_1 \text{ and } k_2 \text{ not both zero}) \\ l_1y(b)+l_2y'(b)=0 \quad &(l_1 \text{ and } l_2 \text{ not both zero}) \end{aligned}\right\}, \tag{4}$$

where λ is a parameter, and $k_1,\ k_2,\ l_1\ \&l_2$, are given constants. It is often convenient to define the linear homogeneous differential operator L as

$$L(y) = (r(x)y')' + q(x)y,$$

and then (3) can be written as

$$L(y) + \lambda p(x)y = 0, \quad 0 \le x \le b. \tag{5}$$

The BVP consisting of equation (3)-(4) or (5)-(4) is called a Sturm-Liouville boundary value problem. It is straight forward to check that $y \equiv 0$ is always a solution of (3)-(4). In applications, it is of interest to determine the values of the parameter λ for which the BVP (5) and (4) has non-trivial solutions. These special values of parameter λ are called eigenvalues and the corresponding non-trivial solutions are called eigenfunctions.

The Sturm - Liouville problems can be viewed as belonging to a much more extensive class of problems that have many common properties. Say for example, there are similarities between Sturm - Liouville problems and the algebric eigenvalue problems:

$$A\mathbf{x} = \lambda\mathbf{x}, \tag{6}$$

where A is $n\times n$ real symmetric or Hermitian matrix (sometimes, we call self-adjoint matrix)[4]. Note that every symmetric and Hermitian matrix has real eigenvalues and the corresponding eigenfunctions or eigenvectors form an orthogonal basis. Observe that a matrix has only a finite number of eigenvalues and eigenvectors. We would like to see "*How far the results for self adjoint matrix* (real symmetric or Hermitian matrix) *can be extended to the Sturm-Liouville problem (5) and (4)?"* We can gain a lot of insight into this problem by studying it from the point of view of linear algebra. To imitate the results, let V be a space of all twice continuously differentiable real valued functions on $[a, b]$. V is indeed a real vector space, which is infinite dimensional. On V, define a relation

$$(f,\ g) = \int_a^b f(x)g(x)\ dx.$$

[4]Let us recall the definitions from our linear algebra course. Denoting the innerproduct on $V = \mathbb{R}^n$or C^n by $<\cdot\,,\ \cdot>$, i.e., $<\mathbf{x},\ \mathbf{z}>=\sum_{i=1}^n x_i\overline{z_i} = \mathbf{z}^T\mathbf{x}$, we define adjoint of an $n\times n$ matrix A as an $n\times n$ matrix A^* satisfying

$$<A\mathbf{x},\ \mathbf{z}>=<\mathbf{x},\ A^*\mathbf{z}>,$$

for all vectors $\mathbf{x}$ and $\mathbf{z}$. Note that if $A = [a_{ij}]$, the elements a^*_{ij} of matrix A^* becomes $a^*_{ij} = \overline{a}_{ji}$. When $A = A^*$,then the matrix A is called Hermitian or self-adjoint. Note that we define selfadjoint or Hermitian matrix via innerproduct as

$$<A\mathbf{x},\ \mathbf{z}>=<\mathbf{x},\ A\mathbf{z}>, \quad \text{for all } \mathbf{x} \text{ and } \mathbf{z}.$$

If A is real, $A^* = A^T$ and A is real and symmetric.

It is easy to check that $(\cdot,\ \cdot)$ is an innerproduct on V and V equipped with $(\cdot,\ \cdot)$ is an innerproduct space. A linear operator L on V is called selfadjoint if

$$(Lu,\ v) = (u,\ Lv)\,, \tag{7}$$

for all u and $v \in V$. Note that if L is defined as in (5), then using integration by parts

$$\begin{aligned}(Lu,\ v) \equiv \int_a^b L(u)v\,dx &= r(x)u'v]_{x=a}^{x=b} - \int_a^b r(x)u'v'\,dx + \int_a^b q(x)uv\,dx \\ &= r(x)u'v]_{x=a}^{x=b} - r(x)uv']_{x=a}^{x=b} + \int_a^b u\left((rv')' + qv\right)\,dx \\ &= r(x)\left(u'v - uv'\right)|_{x=a}^{x=b} + (u,\ Lv)\,.\end{aligned}$$

Thus

$$(Lu,\ v) - (u,\ Lv) = r(x)\left(u'v - uv'\right)|_{x=a}^{x=b}. \tag{8}$$

The equation (8) is known as **Lagrange's identity** .

In order to make L selfadjoint, define the space V as the space of twice continuously differentiable function on $[a,\ b]$, that satisfies the boundary conditions (4). Note that as u and v satisfy the boundary condition (4),

$$\begin{aligned}r(x)\left(u'(x)v(x) - u(x)v'(x)\right)|_{x=a}^{x=b} &= r(b)\left(u'(b)v(b) - u(b)v'(b)\right) - r(a)\left(u'(a)v(a) - u(a)v'(a)\right) \\ &= r(b)\left(-\frac{l_1}{l_2}u(b)v(b) + u(b)\frac{l_1}{l_2}v(b)\right) - r(a)\left(-\frac{k_1}{k_2}u(a)v(a) + u(a)\frac{k_1}{k_2}v(a)\right) = 0.\end{aligned}$$

Hence, $(Lu,\ v) = (u,\ Lv)$ and, therefore, L is selfadjoint on V.

Remark. We have derived the selfadjointness of L when u and v are both real. However, it is easy to prove selfadjointness if u and v are complex valued, provided we use innerproduct

$$(u, v) = \int_a^b u(x)\overline{v(x)}\,dx.$$

Example 1. Given a 2^{nd} order linear DE:

$$a(x)y'' + b(x)y' + c(x)y = 0$$

we seek an integrating factor $\mu(x)$ such that the resulting equation can be written in the form

$$(\mu a y')' + \mu c y = 0.$$

By equating the coefficient of y', we find that μ must be a solution of

$$a\mu' = (b - a')\mu.$$

On solving for μ, we obtain

$$\mu(x) = \frac{1}{a} exp^{\left(\int_{x_0}^x \frac{b(s)}{a(s)}\,ds\right)}.$$

Definition. A Sturm-Liouville BVP (5) and (4) is said to be *regular* if

(i) $r(x) > 0$ and $p(x) > 0$ for $x \in [a, b]$.

(ii) $p,\ q,\ r$ & r' are continuous on $[a,\ b]$.

(iii) (a, b) is finite .

Definition. A Sturm-Liouville Problem is called *singular*, if it is not regular.

Theorem 3. For a regular Sturm-Liouville boundary value problem (5) and (4), the following results hold true :

(i) All the eigenvalues (and consequently, all the eigenvectors) are real.

(ii) Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the inner product

$$(f, g) = \int_a^b p(x)f(x)g(x)\ dx. \tag{9}$$

(iii) To each eigenvalue, there corresponds one and only one eigenfunction, that is, all the eigenvalues are simple.

(iv) There exists countably infinite number of eigenvalues $\lambda_n;\quad n = 1, 2, \cdots$ with corresponding eigenfunction $y_n(x);\quad n = 1, 2, \cdots$. The eigenvalues λ_n can be arranged as a monotonically increasing sequence $\lambda_1 < \lambda_2 < \quad \cdots$ such that $\lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

(v) Let f be any continuously differentiable function on $[a, b]$. Then f can be expanded in a convergent series of eigenfunctions of L, i.e,

$$f(x) = \sum_{n=1}^{\infty} \alpha_n y_n(x) \tag{10}$$

where

$$\alpha_n = \frac{\int_a^b p(x)f(x)y_n(x)\ dx}{\int_a^b p(x)y_n^2(x)\ dx} = \frac{(f,\ y_n)}{(y_n,\ y_n)}. \tag{11}$$

Proof. (i) Let $\lambda = \alpha + i\beta$ be an eigenvalue of (5) and let $y(x) = u(x) + iv(x)$ be the corresponding eigenvectors. On substituting in (5), we find that

$$(r(u + iv)')' + q(u + iv) + (\alpha + i\beta)p(u + iv) = 0.$$

Separating real and imaginary parts, we obtain

$$\begin{aligned} (ru')' + (q + \alpha p)u - \beta pv &= 0 \\ (rv')' + (q + \alpha p)v + \beta pu &= 0. \end{aligned}$$

Multiply the first equation by v and second one by $-u$. Then, adding together, it becomes

$$\begin{aligned}-\beta(u^2+v^2)p &= u(rv')' - v(ru')' \\ &= \left[(rv')u - (ru')v\right]'.\end{aligned}$$

Integrating from a to b, we obtain

$$-\beta \int_a^b p(u^2+v^2)\,dx = r(uv' - u'v)]_a^b.$$

Using boundary condition, the right hand side becomes zero and thus, we obtain

$$\beta \int_a^b p(u^2+v^2)\,dx = 0.$$

Since $\displaystyle\int_a^b p(u^2+v^2)\,dx > 0$, we have $\beta = 0$. Hence $\lambda = \alpha$ is real. This proves (i).

(ii) Let λ_n and λ_m be two distinct eigenvalues with corresponding eigenvectors as y_n and y_m respectively. Thus, y_n satisfies

$$(ry_n')' + (q + \lambda_n p)y_n = 0,$$

and y_m satisfies

$$(ry_m')' + (q + \lambda_m p)y_m = 0.$$

Multiply first equation by $-y_m$ and second by y_n and add to find that

$$\begin{aligned}(\lambda_m - \lambda_n)\,p\,y_m\,y_n &= y_m(ry_n')' - y_n(ry_m')' \\ &= \left[(ry_n')y_m - y_n(ry_m')\right]'.\end{aligned}$$

Integrating from a to b, and using the boundary condition, we obtain

$$\begin{aligned}(\lambda_m - \lambda_n)\int_a^b p\,y_m\,y_n\,dx &= \left[(ry_n')y_m - y_n(ry_m')\right]_a^b \\ &= 0.\end{aligned}$$

Since $\lambda_m \neq \lambda_n$, we have

$$\int_a^b p(x)\,y_m(x)\,y_n(x)\,dx = 0$$

and this completes the proof of (ii).

The proof of (iii)-(iv) is beyond the scope of this course and the statements (iii)-(iv) have been included without proof for the sake of comparison with algebraic eigenvalue problems.

Singular Sturm-Liouville Problems. There are equations of physical interest which are called singular Sturm - Liouville problems. For examples:

(i) Hermite equation:

$$y'' - xy' + \lambda y = 0, \quad -\infty < x < \infty. \tag{12}$$

(ii) Legendre equation:

$$(1-x^2)y'' - 2xy' + \lambda(\lambda+1)y = 0, \quad -1 < x < 1. \tag{13}$$

(iii) Tchebychev equation:

$$(1-x^2)y'' - xy' + \lambda^2 y = 0, \quad -1 < x < 1. \tag{14}$$

(iv) Laguerre equation:

$$xy'' + (1-x)y' + \lambda y = 0, \quad 0 < x < \infty. \tag{15}$$

Using appropriate integrating factor (See Example 1), we write the above equations in Sturm-Liouville form as

$$(e^{-x^2/2}y')' + \lambda e^{-x^2/2}y = 0, \quad -\infty < x < \infty. \tag{12$'$}$$

$$((1-x^2)y')' + \lambda(\lambda+1)y = 0, \quad -1 < x < 1. \tag{13$'$}$$

$$((1-x^2)^{1/2}y')' + \frac{\lambda^2}{(1-x^2)^{1/2}} = 0, \quad -1 < x < 1. \tag{14$'$}$$

$$(xe^{-x}y')' + \lambda e^{-x}y = 0, \quad 0 < x < \infty. \tag{15$'$}$$

For Legendre and Tchebychev equations, (13$'$)−(14$'$) the operator L is selfadjoint, if $y'(x)$ is bounded as $x \to \pm 1$. Note that

$$(Lu,\ v) - (u,\ Lv) = r(x)\left(u'(x)v(x) - u(x)v'(x)\right)\Big|_{x=-1}^{x=1}$$

and both u' and v' are bounded as $x \to \pm 1$, since, $r(x) \to 0$ as $x \to \pm 1$ we have

$$(Lu,\ v) = (u,\ Lv)\,,$$

and hence, L is selfadjoint. In case of Hermite equation, we need to impose the boundary conditions

$$\lim_{x\to\pm\infty} e^{-x^2/2}y(x) = 0.$$

If both u and v satisfy the above condition, then

$$\begin{aligned}(Lu,\ v) - (u,\ Lv) &= \lim_{x\to\pm\infty} e^{-x^2/2}\left(u'(x)v(x) - u(x)v'(x)\right)\\ &= 0,\end{aligned}$$

and thus, L is selfadjoint.

For Laguerre equation, we again impose boundary condition

$$\lim_{x\to\infty} xe^{-x}y(x) = 0.$$

With this boundary condition, L becomes selfadjoint (pl. check this).

The selfadjointness of L will guarantee that parts (i)-(ii) of Theorem 3 hold even for singular Sturm-Liouville problems. However, parts (iii)-(iv) of Theorem 3 may sometimes fail, although they are true for the Hermite, Legendre, Tchebychev and Laguerre equations. Even the consequences Theorem 3 hold good for the Bessel's equation.

Handout 2

MA-207: DE-II First Order PDE : Conservation Laws

Mathematical equations very often provide a language to formulate physical phenomena in terms of some tractable forms so that it is possible to study them, predict their behaviour etc. Say for example, Newton invented Calculus to describe accurately the motion of bodies, Maxwell equations were written down to describe electrodynamics. Similarly, Schrödinger's equation was developed to describe aspects of quantum mechanics, Navier-Stokes equations[5] were formulated to model the motion of an incompressible viscous flow and Black-Scholes equations helped to understand option derivatives in stock market.

In fact, a mathematical model is an equation or a set of equations whose solution describes the physical phenomenon in an approximate manner. It is indeed a simplified description of a physical reality which is expressed in the language of mathematics. It involves observation or experiments, and picking up important factors influencing the system, and then a description in terms of mathematical equations. Once a mathematical formulation is achieved, analysis and simulation do provide results which are again to be validated against the observations or experiments. This will complete the cycle

$$\mathbf{I} - - - - - - \mathbf{F} - - - - - - \mathbf{S} - - - - - - \mathbf{I}$$

of *Identification* (to extract mathematical essence out of the physical phenomenon at hand), *Formulation* (to put the problem in a familiar setting about which we already know something (sometimes called modelling activity)). Usually in your respective branches you shall be dealing with various such models), *Solution process* (which involves mathematical, statistical and computational tools- a major part of it we shall be providing in courses given by Mathematics Department), and finally, *Interpretation* (to provide an answer to the physical problem that we have started with). We shall be talking about the models described by PDEs in this handout.

Many PDE models come naturally from basic conservation principles or laws. In fact, we can simply desrcibe the conservation principle as:

> *the rate at which a quantity of interest changes in a given domain must be equal to the rate at which the quantity flows across the boundary of the domain plus the rate at which the quantity is created or destroyed within the domain.*

For example: in population dynamics, the rate of change of population of certain animal species in a fixed geographical region must be equal to the rate at which animals migrate into the region *minus* the rate at which they migrate out *plus* the birth rate *minus* the death rate. Here

[5] Although this system of equations is used for the last two centuries by Engineers and Scientists to model the motion of an incompressible fluid flow with moderate velocity, the global existence of a unique smooth solution in 3-dimensions remains an unsolved problem. In 2D, it is completely resolved by Olfa Ladyzhenskaya from Russia and J. L. Lions with Prodi from France separately at the sametime. In fact it is one of the seven (now is called six, as Poincaré conjecture has been fully solved by Grusa Perelman) millenium prize problem proposed by the Clay Mathematical Institute (see the URL site: http://www.claymath.org/millennium/ for more detailed information).

population means population density. Similar situations can occur in many of the procesess which you may learn from your own stream.

To quantify such verbal statements, we require to put them in some mathematical form. Say for example, assume that the state variable $u = u(x,t)$ denotes the density[6] of a given quantity of interest. For simplicity, we assume that any variation of quantity is restricted to one spatial domain, say a tube with cross sectional area say A, where the cross section is labeled by the spatial variable x and the quantities of interest vary only in the x direction and also in time. The amount of the quantity in a small section of width dx is given by $u(x,t)A\,dx$, where A is the cross sectional area of the tube. Further, let $q = q(x,t)$ denote the flux of the quantity at x at time t. The flux is defined as the amount of the quantity crossing the section x at time t and its unit is given by amount per unit area per unit time. Therefore, the amount of the quantity that is crossing the section at x and at time t is $Aq(x,t)$. By convention, the flux is positive if the flow is from left to right and negative if it is from right to left. Finally, let $f = f(x,t)$ be the given rate at which the quantity is created or destroyed within the section at x and at time t. It is called a source term if it is positive and called a sink if it is negative. Hence, the amount of quantity that is created (or destroyed) in a small width dx per unit time is $f(x,t)A\,dx$. We can formulate the conservation law by considering a fixed, but arbitrary section say $a \le x \le b$ of the tube by requiring that the rate of change of the total amount of the quantity in this section must be equal to the rate at which it flows at $x = a$ minus the rate at which it flows out at $x = b$ plus the rate at which it is created within the section $a \le x \le b$. In the language of Mathematics, we write this conservation law as:

$$\frac{d}{dt}\int_a^b u(x,t)A\,dx = Aq(a,t) - Aq(b,t) + \int_a^b f(x,t)A\,dx. \qquad (*)$$

This is a fundamental conservation principle,which is indeed written in an integral form. However, if u and q are sufficiently smooth (u and q have continuous first derivatives : will be sufficient for the time being), then we write:

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b u_t(x,t)\,dx,$$

and

$$q(a,t) - q(b,t) = -\int_a^b q_x(x,t)\,dx.$$

On substituting in $(*)$, we now obtain

$$\int_a^b \left(u_t(x,t) + q_x(x,t) - f(x,t)\right)\,dx = 0.$$

Since the section $a \le x \le b$ is arbitrary and the integrand is continuous, it therefore, follows that the integrand must vanish identically leading to :

$$u_t(x,t) + q_x(x,t) = f(x,t).$$

This is a differential form of the conservation principle which is valid in the domain of interest. Unfortunately, this is one equation with two unknowns: u and q. Thus to close the system, we need

[6]Density measured as mass per unit volume or length.

to have a relation between the flux variable q and the density u, which is known as the constitutive relation or equation of state representing a particular physical situation.

For example, a model where the flux is proportional to the density, that is,

$$q = \alpha u,$$

where α is a constant is called an advection model. The situation like spread of pollutants from an industrial chimney mainly due to a strong wind (with negligible diffusion) with velocity say α and no other external input, that is, with $f = 0$ can be described by

$$u_t + \alpha u_x = 0, \quad x \in \mathbb{R}, \;\; t > 0$$

and initial concentration $u(x, 0) = u_0(x), \; x \in \mathbb{R}$, where u denotes the concentration of the pollutants and the wind velocity $\alpha > 0$ means wind is blowing from left to right. An exactly similar model is also valid for water pollution if the velocity of the flow of water is given by α. This model is also known in literature as *transport model.* By a solution u of the above first order PDE (as the highest derivative is of first order), we mean that u is continuously differentiable and it satisfies the PDE and the initial condition.

Our next question is *'How do we solve it ?'* that is, how to get an analytical or exact expression for u ?

Let us use a change of variable, that is : $\xi = x - \alpha t$ and $\tau = t$. Let us denote u in the new variables by $U(\xi, \tau)$, that is, $U(\xi, \tau) = u(\xi + \alpha\tau, \tau)$ or $u(x, t) = U(x - \alpha t, t)$. Then, using chain rule, we write :

$$u_t = U_\xi \xi_t + U_\tau \tau_t = -\alpha U_\xi + U_\tau,$$

and

$$u_x = U_\xi \xi_x + U_\tau \tau_x = U_\xi.$$

On substituting in the main PDE, we obtain

$$U_\tau = 0.$$

It is an ordinary differential equation in the new transformed dependent variable U. Hence we write the solution as

$$U(\xi, \tau) = C(\xi),$$

where C is a function of ξ and independent of τ. At $\tau = t = 0$, we have $U(\xi, 0) = u_0(\xi)$. Hence, $U(\xi, \tau) = u_0(\xi)$ and in the original variable $u(x, t) = u_0(x - \alpha t)$. It can be shown easily that u given by the above expression is a solution of the 1^{st} order PDE provided u_0 is continuously differentiable. If $\alpha > 0$, the solution is a right travelling wave which preserves the initial profile. In the context of air pollution model that too if the wind velocity is strong (with negligible diffusion), one can feel the same effect in near by areas which are in the direction of the wind even after a later time. This model can be modified to include the effect of diffusion so that a distance far away from the chimney one does not feel the strong impact of the pollutant-a case of Bhopal gas tragedy is an example. Why should we bother about such a model? In case such a model with a simulator is available, then one can take a decision to evacuate people so that the damaged can be minimal and hence, it helps in level of planning.

Exercise. Solve the following first order PDE using the transformation technique:

$$u_t + \alpha u_x + au = f(x,t), \quad x \in \mathbb{R}, \;\; t > 0,$$

with initial condition : $u(x,0) = u_0(x), \quad x \in \mathbb{R}$, where α and a are constants and f is a given function. Using the transformation technique we now arrive at:

$$U_\tau + aU = f(\xi + \alpha\tau, \tau)$$

with $U(\xi) = u_0(\xi)$. Its solution can be written as:

$$U(\xi,\tau) := u_0(\xi)e^{-a\tau} + \int_0^\tau e^{-a(\tau-s)} f(\xi + \alpha\tau, \tau)\, d\xi.$$

In the original variable, we obtain the solution as

$$u(x,t) := u_0(x - \alpha t)e^{-at} + \int_0^t e^{-a(t-s)} f(x,t)\, dx.$$

One may be curious to know *'How do we choose the transformations?'* Note that the straight lines $\xi = x - \alpha t \equiv$ constant in space-time domain are solutions of

$$\frac{dx}{dt} = \alpha,$$

which is called characteristic equation and the straight lines $\xi = C$, where C is an arbitrary constant are called characteristics. Thus it is possible to get the transformation by solving the characteristic equation.

Now for a problem like:

$$u_t + a(x,t)u_x = f(x,t,u),$$

we again form the characteristic equation :

$$\frac{dx}{dt} = a(x,t),$$

let $\phi(x,t) = C$ be its general solution. Thus we obtain the characteristic coordinates as :

$$\xi = \phi(x,t), \qquad \tau = t.$$

Again using chain rule, we obtain the following transformed equation in these new co-ordinates :

$$U_\tau = F(\xi, \tau, U),$$

where $U = U(\xi,\tau)$.

Another Example. Solve : $u_t + 2tu_x = 0, \quad x \in \mathbb{R}, \;\; t > 0$. As $a = 2t$, the characteristic equation is given by : $\dfrac{dx}{dt} = 2t$. On solving: we obtain $x - t^2 = C$. Thus set $\xi = x - t^2$ and $\tau = t$. Using chain rule we obtain

$$u_t = U_\xi(-2t) + U_\tau, \quad u_x = U_\xi$$

and therefore, we obtain $U_\tau = 0$. Its general solution is now given by $U = F(\xi)$, where F is an arbitrary function. In the original variable we obtain : $u(x,t) = F(x - t^2)$.

Few more exercises for practice: Solve:

1. $u_t + \alpha u_x = 0, \ \ x \in \mathbb{R}, \ t > 0$ with $u(x,0) = e^{-x^2}, \ \ x \in \mathbb{R}.$
2. $u_t + xtu_x = 0, \ \ x \in \mathbb{R}, \ t > 0$ with $u(x,0) = u_0(x), \ x \in \mathbb{R}.$
3. $u_t + xu_x = e^t, \ \ x \in \mathbb{R}, \ t > 0$ with $u(x,0) = u_0(x), \ x \in \mathbb{R}.$
4. $u_t + u_x - 3u = t, \ \ x \in \mathbb{R}, \ t > 0$ with $u(x,0) = x^2, \ x \in \mathbb{R}.$

In some traffic flow problem, the flux can be $q = g(u)$, where g is a nonlinear function. The form of g can very often be found out through experimental data by fitting an appropriate curve.

In order to model a simple diffusion process, like diffusion of a solute in a solvent, diffusion of the pollutant in air which is coming from a chimney in a very calm morning, and heat conduction through materials etc., the flux q is given by

$$q = -\alpha u_x,$$

where the constant α is known as conductivity in heat transfer problem, or called diffusivity in diffusion of a solute in a solvent (in that case u is known as concentration). Thus, we write the equation as diffusion equation :

$$u_t - \alpha u_{xx} = f.$$

We shall be discussing extensively the above equation in this course.

Handout 3

MA-207: DE-II Classification of Second Order PDEs

Let us recall a general second order algebraic equation in two variables with real coefficients:

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0. \tag{1}$$

The nature of the curves will be decided by the principal part

$$P(x, y) = ax^2 + 2bxy + cy^2,$$

that is, the terms in quadratic equation (1) containing highest degree. Depending on the sign of the discriminant, that is, $b^2 - ac$, we classify the curve as

$$b^2 - ac > 0 \quad \text{hyperbola}$$

$$b^2 - ac = 0 \quad \text{parabola}$$

and

$$b^2 - ac < 0 \quad \text{ellipse.}$$

With suitable co-ordinate transformations: $x \mapsto X$ and $y \mapsto Y$, which depend on the roots of the polynomial equation $P(x, y) = 0$, we transfrom (1) into the normal form :

$$\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1 \quad \text{hyperbola}$$

$$Y^2 = 4AX \qquad \text{parabola}$$

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1 \quad \text{ellipse.}$$

Let us now consider the following general linear second order partial differential equation with constant cofficients:

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_x + fu + g = 0, \tag{2}$$

where a, b, c, d, e, f and g are real constants and $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, and $u_{xy} = \dfrac{\partial u}{\partial x \partial y}$ etc. The nature of the equation (2) is again determined by its principal part, that is, the part containing highest order partial derivatives :

$$P(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})u := au_{xx} + 2bu_{xy} + cu_{yy}.$$

Here,

$$P(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) := a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2}. \tag{3}$$

For classification of the second order PDE (2), it is essential to attach a symbol to (3) by replacing $\frac{\partial}{\partial x}$ by x and $\frac{\partial}{\partial y}$ by y in (3). Thus, we write $P(x, y)$ as

$$P(x, y) = ax^2 + 2bxy + cy^2.$$

Now depending whether the quadratic equation $P(x, y)$ represents an ellipse, a parabola and a hyperbola, we call (2) as elliptic, parabolic and hyperbolic. More precisely, if

$$\begin{aligned} b^2 - ac &< 0, \quad \text{then (2) is called elliptic,} \\ &= 0, \quad \text{then (2) is called parabolic,} \\ &> 0 \quad \text{then (2) is called hyperbolic.} \end{aligned}$$

As in the case of the quadratic equation in two variables, we transform the equation in its normal form by appealing to co-ordinate transformation. Now we pose a similar question in case of second order linear PDE in two variables as: "*How do we transform (2) into its normal from?* "

More precisely, *how do we obtain transformations to put (2) in its normal form ?*

1. Equation of Hyperbolic type (the case when $b^2 - ac > 0$).

Form a family of curves

$$\frac{dy}{dx} = -\omega^{\pm}, \tag{4}$$

where $\omega^{\pm}$ are the two roots of the polynomial equation $P(x, y) = 0$, i.e,

$$\omega^{\pm} = \frac{-b \pm \sqrt{b^2 - ac}}{a}.$$

After integrating (4), we obtain the following two families of curves: $y + \omega^+ x = c_1$ and $y + \omega^- x = c_2$, where c_1 and c_2 are arbitrary constants. Note that these two families of curves determined by the equation (4)

$$\xi(x, y) = c_1 \quad \text{and} \quad \eta(x, y) = c_2$$

are called characterstic curves of (2), where $\xi(x, y) := y + \omega^+ x = c_1$ and $\eta(x, y) := y + \omega^- x = c_2$.

With the help the transformations (ξ, η), we write $u = u(\xi(x, y), \eta(x, y))$. Using chain rule, we replace first partial derivatives of u as:

$$\frac{\partial u}{\partial x} = u_\xi \xi_x + u_\eta \eta_x, \qquad \frac{\partial u}{\partial y} = u_\xi \xi_y + u_\eta \eta_y.$$

Similarly, using again chain rule, we compute the second derivatives as:

$$\frac{\partial^2 u}{\partial x^2} = (u_\xi \xi_x + u_\eta \eta_x)_x = u_{\xi\xi} \xi_x^2 + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2,$$

$$\frac{\partial^2 u}{\partial x \partial y} = (u_\xi \xi_x + u_\eta \eta_x)_y = u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} \xi_x \eta_y + u_{\xi\eta} \xi_y \eta_x + u_{\eta\eta} \eta_x \eta_y,$$

and

$$\frac{\partial^2 u}{\partial y^2} = (u_\xi \xi_y + u_\eta \eta_y)_y = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2.$$

On substitution in (2) and collecting the similar terms, we obtain:

$$\begin{aligned} &au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_x + fu + g = (a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)u_{\xi\xi} \\ &+2(\xi_x\eta_x + \xi_x\eta_y + \xi_y\eta_x + \xi_y\eta_y)u_{\xi\eta} + (a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2)u_{\eta\eta} \\ &+(d\xi_x + e\xi_y)u_\xi + (d\eta_x + e\eta_y)u_\eta + fu + g. \end{aligned} \tag{5}$$

Note that the coefficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ in (5) become zero. Now using the form of ξ and η, we find that $\xi_x = \omega^+, \eta_x = \omega^-$ and $\xi_y = 1, \eta_y = 1$. On substitution in (5), we obtain

$$u_{\xi\eta} + \bar{a}u_\xi + \bar{b}u_\eta + \bar{c}u = \bar{d},$$

where $\bar{a} = \dfrac{(d\omega^+ + e)}{\phi}$, $\bar{b} = \dfrac{(d\omega^- + e)}{\phi}$, $\bar{c} = \dfrac{f}{\phi}$, $\bar{d} = \dfrac{-g}{\phi}$ and $\phi = \dfrac{2}{a}(c - 2b + a)$.

Alternate form: Take $\xi = \alpha + \beta$ and $\eta = \alpha - \beta$, where $\alpha = y - (\frac{b}{a})x$ and $\beta = \frac{\sqrt{(b^2-ac)}}{a}x$. Thus, we obtain,

$$u_{\alpha\alpha} - u_{\beta\beta} + \bar{a}u_\alpha + \bar{b}u_\beta + \bar{c}u = \bar{d}.$$

Example 1. Note that the following equation

$$u_{tt} - u_{xx} + au = f$$

is a second order hyperbolic and is already in normal form.

2. Equation of Parabolic type (when $(b^2 - ac = 0)$).

In this case, $w^+ = w^- = w$ and $\frac{dy}{dx} = -w = b/a$. Thus, solving the ODE, we obtain one characteristic curve as $\xi = y + wx$. We need to find one characteristic curve η so that both ξ and η are linearly independent. Now we choose η as $\eta = wy + \hat{c}x$, where $\hat{c} \neq w^2$ is chosen arbitrarily.

As in hyperbolic case, we transform the equation using the new coordinates ξ, η as

$$u_{\eta\eta} + \bar{a}u_\xi + \bar{b}u_\eta + \bar{c}u = \bar{d}.$$

Please work out in detail using the chain rule and the form of ξ and η.

Example 2. Note that the following equation

$$u_t - u_{xx} + du_x + du = f$$

is parabolic and is already in its normal form.

3. Equation of Elliptic type (when $(b^2 - ac < 0)$).

In this case, w^+ and w^- are complex and hence, the characteristic curves are complex-valued. Here, we solve the characteristic equation

$$\frac{dy}{dx} = -w^{\pm},$$

to obtain the characteristic curves as :

$$\xi = y - (b/a)x + i(\frac{\sqrt{(ac - b^2)}}{a})x = \alpha + i\beta$$

and

$$\eta = y - (b/a)x - i(\frac{\sqrt{(ac - b^2)}}{a})x = \alpha - i\beta,$$

where, $\alpha = y - (b/a)x$ and $\beta = \frac{\sqrt{(ac-b^2)}}{a}x$. Then using chain rule, the equation (2) is transformed into its canonical form as:

$$u_{\alpha\alpha} + u_{\beta\beta} + \bar{a}u_{\alpha} + \bar{b}u_{\beta} + \bar{c}u = \bar{d}.$$

Following the analysis of hyperbolic case, derive the above equation in detail.

Remark. The classification is still valid if the cofficients a, b, c, d, e, f, g depends on (x, y). In this case, the condition $b^2 - ac \leq 0$ or $b^2 - ac > 0$ should be satisfied at each point (x, y) in the region (that is, for $b^2(x, y) - a(x, y)c(x, y) < 0$ at each point (x, y) is a region of interest where we want to describe its nature).

The characterstic curves are the solutions of $\frac{dy}{dx} = -w^{\pm}(x, y)$ and the transformations do not pose any special problem in principle.

Exampe 3. (equation of mixed type: Tricomy Problem).

Classify:

$$u_{xx} + yu_{yy} = 0.$$

Since the discriminant $b^2 - ac = -y$, we classify the above equation depending on the sign of y. Note that for $y < 0$, the equation is hyperbolic, while for $y = 0$, it is parabolic and for $y > 0$, it is elliptic.

Now when $y = 0$, we have the normal form as $u_{xx} = 0$. For nonzero y, we discuss below the normal form. In case $y < 0$, we have $w^{\pm} = \pm\sqrt{-y}$ and hence, the characteristic equations become

$$y'(x) = -w^{\pm} = \mp\sqrt{-y}.$$

After solving the above ODE, we obtain the families of characteristic curves as:

$$\xi \equiv x + 2\sqrt{-y} = c_1, \quad \eta \equiv x - 2\sqrt{-y} = c_2.$$

Using the chain rule, the normal form can be easily derived as:

$$u_{\xi\eta} + \frac{(u_\xi - u_\eta)}{2(\xi - \eta)} = 0.$$

When $y > 0$, we have $w^{\pm} = \pm i\sqrt{y}$ and hence, we obtain the transformations: $\xi = x + 2i\sqrt{y}$ and $\eta = x - 2i\sqrt{y}$. With $\alpha = x$ and $\beta = 2\sqrt{y}$ we find that $u_{xx} = u_{\alpha\alpha}$, $u_y = \dfrac{u_\beta}{\sqrt{y}}$, $u_{yy} = \dfrac{u_{\beta\beta}}{y} - \dfrac{u_\beta}{2y^{3/2}}$. Therefore, the canonical form (or normal form) becomes

$$u_{\alpha\alpha} + u_{\beta\beta} - \frac{u_\beta}{\beta} = 0.$$

Classification of second order PDE with more than two variables.

Consider the following general second order linear PDE in n variables:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0, \tag{6}$$

where u, a_{ij}, b_i, c, d are functions of $x = (x_1, x_2, \ldots, x_n)$. Its principal part becomes

$$P(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{7}$$

Since the classification of the second order PDE in two variables depends on the sign of the discriminant, it is difficult to extend it to more than two dimensional situation. However, in case of two variables, we write the principal part as $X^T AX$, where

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Since A is symmetric, it is diagonalizable and its eigenvalues λ_1 and λ_2 are real. Note that $\lambda_1\lambda_2 = -(b^2 - ac)$. Now, the discriminant $(b^2 - ac) > 0 \Longrightarrow \lambda_1\lambda_2 < 0$, that is, λ_1 and λ_2 are of different signs. Thus, the curve traces a hyperbola. The corresponding PDE (2) is called hyperbolic provided the eigenvalues are of different signs. Similarly, if $b^2 - ac = 0$, then one of the eigenvalues is zero and hence, the PDE (2) called parabolic. Finally, with $(b^2 - ac) < 0$ the eigenvalues are of different signs and therefore, the corresponding PDE (2) is called elliptic. Note that this procedure is capable of generalization to more than two variables.

It is alright to assume that $A = [a_{ij}]$ is symmetric. If not, set $\bar{a}_{ij} = \dfrac{(a_{ij} + a_{ji})}{2}$ and rewrite

$$P(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{a}_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

We can do this as $\dfrac{\partial^2}{\partial x_i \partial x_j} = \dfrac{\partial^2}{\partial x_j \partial x_i}$, assuming that u is twice continuously differentiable. As in two variable case, let us attach a quadratic form P with (6)

$$P\left(x_1, x_2, \ldots, x_n\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j, \tag{8}$$

that is replacing $\frac{\partial}{\partial x_i}$ by x_i. Since the matrix A is symmetric, that is, $(a_{ij} = a_{ji})$, it is diagonalizable with real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ counted with multiplicity. In other words, there exists a corresponding set of orthonormal n eigenvectors say $r_1, r_2, \cdots, r_n$ with $R = (r_1, r_2, \cdots, r_n)$ as columns such that

$$R^T A R = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & . & & \\ & & & . & \\ 0 & & & & \lambda_n \end{bmatrix} = D.$$

Now the classification of (6) depends on the sign of eigenvalues of A :

$$\begin{cases} \lambda_i > 0 \quad \forall \ i \\ \text{or} \\ \lambda_i < 0 \quad \forall \ i \end{cases} \qquad \textbf{elliptic,}$$

$$\begin{cases} \text{one of } \ \lambda_i < 0 \ \text{ or } \ \lambda_i > 0 \\ \\ \text{all other remaining have opposite signs} \end{cases} \qquad \textbf{hyperbolic type,}$$

One or more of $\lambda_i = 0$, then **parabolic**.

Remarks. It works well for $n \leq 3$. When $n \geq 4$, it may happen that two or more of λ_i have one sign, where as two or more of remaining λ_i have the opposite signs. Such equations are called ultra hyperbolic type.

Example 4.

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0 \qquad \text{elliptic}$$

$$u_t - \Delta u = 0 \qquad ; \qquad \text{parabolic}$$

$$u_{tt} - \Delta u = 0 \qquad \text{hyperbolic}$$

A natural question is *How do we to transform (5) into canonical form ?*
Introduce the linear transformation

$$\xi_i = r_i^T x, \qquad i = 1, 2, 3......n$$

which transforms the principal part into

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} u_{x_i x_j} = \sum_{i=1}^{n} \lambda_i u_{\xi_i \xi_i}.$$

Note that $\partial_{\xi_i} = r_i \partial_{x_i}$, that is, $\partial_\xi = R\partial_x$.

Example 5. Classify

$$u_{x_1 x_1} + 2(1 + cx_2) u_{x_2 x_3} = 0.$$

It order to symmetrize the above equation, we write it as

$$u_{x_1 x_1} + (1 + cx_2) u_{x_2 x_3} + (1 + cx_2) u_{x_3 x_2} = 0$$

and then put in the form: $\partial_x^T A \partial_x - c\partial_{x_3} = 0$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & (1+cx_2) \\ 0 & (1+cx_2) & 0 \end{bmatrix} \quad \text{and} \quad \partial_x = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{bmatrix}$$

For the eigenvalue $\lambda_1 = 1$, the normalized eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

For $\lambda_2 = 1 + cx_2$, the normalized eigenvector is $r_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

For $\lambda_3 = -(1+cx_2)$, the normalized eigenvector is $r_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

Thus,

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

and $R = R^T = R^{-1}$. Note that

$$R^T A R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1+cx_2) & 0 \\ 0 & 0 & -(1+cx_2) \end{bmatrix} = D$$

Now,

$$\partial_\xi = R\partial_x = \begin{bmatrix} \partial_{x_1} \\ \frac{(\partial_{x_2} + \partial_{x_3})}{\sqrt{2}} \\ \frac{(\partial_{x_2} - \partial_{x_3})}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \partial_{\xi_1} \\ \partial_{\xi_2} \\ \partial_{\xi_3} \end{bmatrix}$$

Thus,

$\partial_x^T A \partial_x = (R\partial_\xi)^T A (R\partial_\xi) = \lambda_1 \partial_{\xi_1}^2 + \lambda_2 \partial_{\xi_2}^2 + \lambda_3 \partial_{\xi_3}^2 + \frac{c}{\sqrt{2}}\partial_{\xi_2} - \frac{c}{\sqrt{2}}\partial_{\xi_3}$.

The equation is called parabolic, if $x_2 = -1/c \quad (c \neq 0)$, and it is called hyperbolic, provided $x_2 > -1/c$, and $x_2 < -1/c$.

For $c = 0, \lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$. Thus, the equation becomes hyperbolic.

Handout 4

MA-207: DE-II Fourier Transforms and Applications

Like Laplace transform, Fourier transform is very useful as a computational tool in solving PDEs defined on whole of $\mathbb{R}$. It converts derivatives into multiplication operations in the transform domain. There by, time dependent linear heat or wave equations with constant coefficients are converted to linear ODEs in the transform domain (sometimes called Fourier or frequency domain) and then after solving the linear ODEs, one can using inverse Fourier transform to obtain a solution in the original domain (called physical or state-space domain).

The Fourier transform of a function $u = u(x),\ x \in \mathbb{R}$ is defined by

$$(\mathcal{F}u)(\xi) \equiv \hat{u}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u(x)\, dx. \tag{9}$$

Information on 'From Fourier series to the Fourier integral' can be found in your text book pp. 558-559.

Now one can ask : Under what condition on u, its Fourier transform is meaningful ?'

Note that

$$|\hat{u}(\xi)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-i\xi x}||u(x)|\, dx \quad \leq \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u(x)|\, dx.$$

Thus, if u is absolutely integrable, that is, $\int_{-\infty}^{\infty} |u(x)|\, dx < \infty$, or more precisely $u \in L^1(\mathbb{R})$, then the Fourier transform $\hat{u}(\xi)$ of u is meaningful. In fact, the following hold (since the proofs are nontrivial and are beyond the scope of this course, we skip those) : If $u \in L^1(\mathbb{R})$, then

(i) the Fourier transform $\hat{u}(\xi),\ \xi \in \mathbb{R}$ is uniformly continuous in $\mathbb{R}$.

(ii) (Riemann-Lebesgue Lemma): $\lim_{\xi \to \pm\infty} \hat{u}(\xi) = 0$.

For more properties on linearity of the Fourier transform, convolution theorem, see pages 572-575 of your text book. More precisely, assuming that the Fourier transform of u and v exist, the following properties hold true :

(iii) $\mathcal{F}(u + \alpha v) = \mathcal{F}(u) + \alpha \mathcal{F}(v)$, for constant α.

(iv) the Fourier transform of the convolution of u and v is equal to $\sqrt{2\pi}$ times the product of their Fourier transforms, that is, $\mathcal{F}(u * v) = \sqrt{2\pi}\mathcal{F}(u)\mathcal{F}(v)$.

From application point of view, it is important to know the inverse Fourier transform. The inverse Fourier transform is defined as:

$$(\mathcal{F}^{-1}\hat{u})(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \hat{u}(\xi)\, d\xi. \tag{10}$$

Another related question:

"*When can we ensure the existence of an inverse of a Fourier transform ?*"
To answer this question, we introduce the space of square integrable functions. We define $L^2(\mathbb{R}) = \{v : \int_{\mathbb{R}} |v(x)|^2\,dx < \infty\}$. This is an important space with norm given by $\|v\|_{L^2(\mathbb{R})} \equiv \Big(\displaystyle\int_{\mathbb{R}} |v(x)|^2\,dx\Big)^{1/2}$. This is related to energy norm in some applications.

Plancherel's Theorem. If $u \in L^2(\mathbb{R})$, then $\hat{u} \in L^2(\mathbb{R})$. Moreover, the following holds true:

$$\|u\|_{L^2(\mathbb{R})} \equiv \Big(\int_{\mathbb{R}} |u(x)|^2\,dx\Big)^{1/2} = \Big(\int_{\mathbb{R}} |\hat{u}(\xi)|^2\,d\xi\Big)^{1/2} \equiv \|\hat{u}\|_{L^2(\mathbb{R})}.$$

The above important theorem states that the energy in state space or physical space is same as energy in frequency or Fourier space. This can be used to discuss the stability of solution to time dependent problems with constant coefficients like heat, wave or Schrodinger wave equations on initial data with respect to energy or L^2 norm.

Since in PDEs we shall be dealing with derivatives, it is, therefore, common to work with a smaller set of functions. As the integrals involved are improper integrals (because of infinite integrals), the functions must decay rapidly at $\pm\infty$ for integrals to exist. Thus, we define $\mathcal{S}(\mathbb{R})$ as the set of rapidly decreasing functions on $\mathbb{R}$ that have continuous derivatives of all orders. By rapidly decreasing functions, we mean those functions which along with all their derivatives decay to zero as $x \mapsto \pm\infty$ faster than any power functions. An example of such a function is $\exp(-x^2)$. Mathematically, we define $\mathcal{S}(\mathbb{R})$ as:

$$\mathcal{S}(\mathbb{R}) := \{\phi \in C^\infty(\mathbb{R}) : |\frac{d^k\phi}{dx^k}| \leq M\frac{1}{|x|^N} \ \text{ as } \ |x| \longrightarrow \infty, \ \ k = 0, 1, \ldots; \ \text{for all integers } \ N\}.$$

In literature, this set is called Schwartz space which is named after L. Schwartz. It can be shown that if $u \in \mathcal{S}$, then $\hat{u} \in \mathcal{S}$, and conversely. Therefore, it is a good set to workwith. Below we discuss an important property : under Fourier transform, the derivatives are turned into multiplication operation in the Fourier domain. If $u \in \mathcal{S}$, then

$$(\mathcal{F}u^k)(\xi) = (i\xi)^k\mathcal{F}(u),$$

where u^k is the k-th derivative of u. The proof uses induction combined with integration by parts and the rapidly decaying property at $\pm\infty$.

The Fourier transform interchanges translation and phase factor in the following sense:

- The Fourier transform of $u(x-a)$ is equal to $e^{-i\xi a}$ times Fourier transform of u, that is, $\mathcal{F}(u(x-a)) = e^{-i\xi a}\mathcal{F}(u(x))$.
- The Fourier transform of $e^{iax}u(x)$ is equal to $\mathcal{F}(u)(\xi - a)$.

The proof of the above two properties is a direct consequence of the definition of Fourier transform and hence, it is left to the reader. (Try !)

Although, we shall be working out the example of heat equation in the class, but below, we present an example of solving Laplace equation defined on the upper half plane using Fourier transform.

Example. Consider the Laplace equation on the upper half of the plane as:

$$u_{xx} + u_{yy} = 0, \;\; x \in \mathbb{R}, \; y > 0; \quad u(x,0) = f(x), \;\; x \in \mathbb{R}.$$

Taking Fourier transform in the direction of x, keeping y as a parameter, we obtain:

$$\hat{u}_{yy} - \xi^2 \hat{u} = 0, \;\; \text{with} \;\; \hat{u}(\xi, 0) = \hat{f}(\xi).$$

Its general solution can be written as :

$$\hat{u}(\xi, y) = a(\xi) e^{-\xi y} + b(\xi) e^{\xi y}.$$

The boundedness of u enforces that $b(\xi) = 0$, if $\xi > 0$ and $a(\xi) = 0$, if $\xi < 0$. Thus we have :

$$\hat{u}(\xi, y) = c(\xi) e^{-|\xi| y}.$$

Using boundary condition we now obtain :

$$\hat{u}(\xi, y) = e^{-|\xi| y} \hat{f}(\xi).$$

Thus, using convolution theorem and inverse Fourier transform of $e^{-y|\xi|}, \; y > 0$, that is $\dfrac{y}{\pi} \dfrac{1}{x^2 + y^2}$, we obtain

$$u(x, y) := \frac{y}{\pi} \frac{1}{x^2 + y^2} * f = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{(x - \tau)^2 + y^2} \, d\tau,$$

which is the required solution of the Laplace equation defined on the upper half plane.