MA 207 - Differential Equations-II

Amiya Kumar Pani

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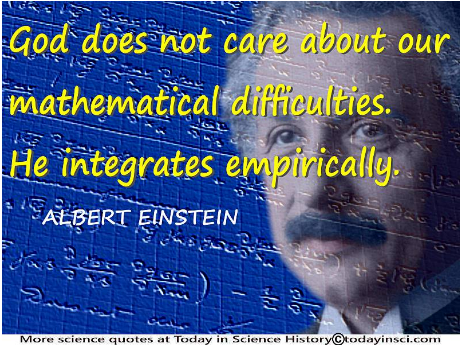
November 13, 2020

Start with ....


God does not care about our mathematical difficulties. He integrates empirically.
ALBERT EINSTEIN
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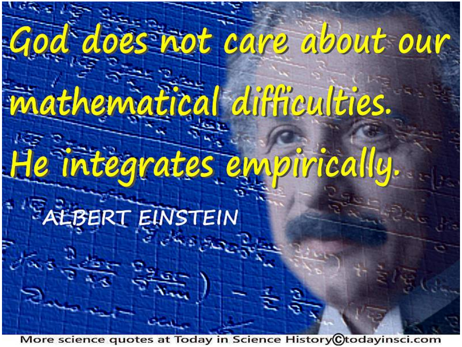
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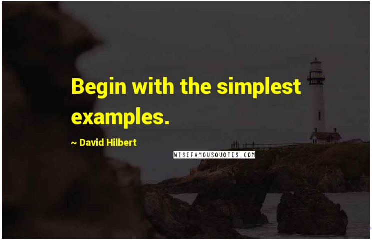
Begin with the simplest examples.
~ David Hilbert
WISEFAMOUSQUOTES.COM

Outline of the lecture

- Vibrating string-wave equation
- Method of separation of variables
- D'Alembert's Method
- Method of Fourier transforms

Wave equation (One space variable)

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Under suitable assumptions on the initial data (which will be specified later),

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ also satisfies the wave equation.

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$$\boxed{a_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})\, dx;},$$

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The formal solution of the wave equation (1D- space) with Dirichlet BC & IC is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\frac{n\pi ct}{l}) + b_n \sin(\frac{n\pi ct}{l}) \right] \sin(\frac{n\pi x}{l}),}$$ with the Fourier coefficients a_n and b_n defined as above.

Sufficient Conditions for Existence

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then, the formal solution is in fact a solution.

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Exercise :

1. Is the solution unique?

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1. $f(x),\ f'(x),\ f''(x)$ are continuous in $[0, l]$,
2. $f(0) = f(l) = 0,\ f''(0) = f''(l) = 0$
3. $g(x),\ g'(x)$ are continuous in $[0, l]$,
4. $g(0) = g(l) = 0$

then, the formal solution is in fact a solution.

Exercise :

1. Is the solution unique?

2. Tut. Sheet 5 : 4, 5, 6, 10 (ii), (iii)

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Example:

Consider $u_{tt} = 4u_{xx}, \; 0 < x < 30, \;\; t > 0$ with $u(0,t) = 0 = u(30,t)$ and initial position

$$u(x,0) = f(x) := \begin{cases} x/10, & 0 \le x \le 10, \\ (30-x)/20, & 10 < x \le 30. \end{cases}$$

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$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi t}{30}) \; \sin(\frac{n\pi x}{30}),}$$

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$$a_n \;\; = \;\; \frac{2}{30}\int_0^{10} \frac{x}{10} \sin(\frac{n\pi x}{30}) \; dx + \frac{2}{30}\int_{10}^{30} \frac{30-x}{20} \sin(\frac{n\pi x}{30}) \; dx$$

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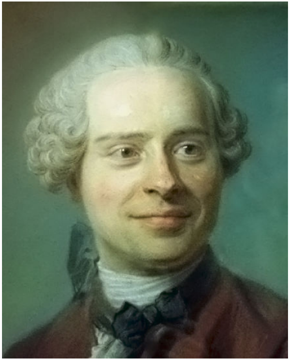
Jean-Baptiste le Rond d'Alembert (16 November 1717 - 29 October 1783)

French mathematician, mechanician, physicist, philosopher, and music theorist.
Until 1759 he was, together with Denis Diderot, a co-editor of the Encyclopédie.
D'Alembert's formula for obtaining solutions to the wave equation is named after him. The wave equation is sometimes referred to as d'Alembert's equation,
The fundamental theorem of algebra is known as the d'Alembert/Gauss theorem, as an error in d'Alembert's proof was caught by Gauss.
He also created his ratio test, a test to see if a series converges.
The D'Alembert operator, which first arose in D'Alembert's analysis of vibrating strings, plays an important role in modern theoretical physics.



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That is, $\boxed{u_{xx} = v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}.}$

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Substituting the expressions for u_{xx} and u_{yy} in the wave equation, we obtain

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$$v_{\xi\eta} = 0 \Longrightarrow \frac{\partial^2 v}{\partial\xi\partial\eta} = 0 \Longrightarrow \frac{\partial}{\partial\xi}\left(\frac{\partial v}{\partial\eta}\right) = 0 \Longrightarrow \boxed{\frac{\partial v(\xi,\eta)}{\partial\eta} = G(\eta)}.$$

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Use Plancherel's identity to find a relation between the solution and the initial data (in terms of L^2-norm) of $u_{tt} = u_{xx}$ with $u(x, 0) = u_0$, $u_t(x, 0) = u_1 = 0$.

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