Signal Processing - 1 by One

Sibi Raj B. Pillai
Dept of Electrical Engineering
IIT Bombay

Outline

- So Far: Impulse, Sampling, Replacement
- Previous Week: Convolution ($\circledast$) and Interpolation
- Today: Drawing with Sinusoids

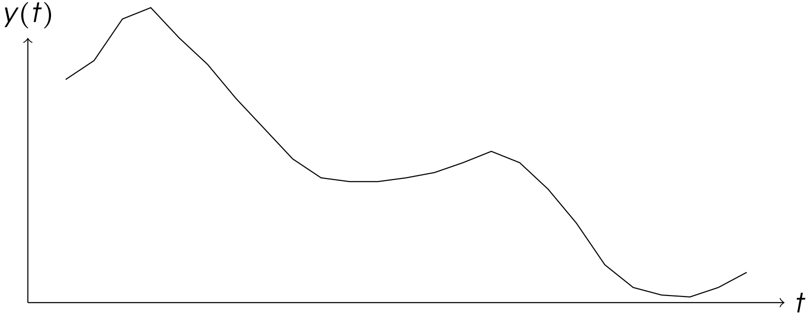
Outline

- So Far: Impulse, Sampling, Replacement
- Previous Week: Convolution ($\circledast$) and Interpolation
- Today: Drawing with Sinusoids

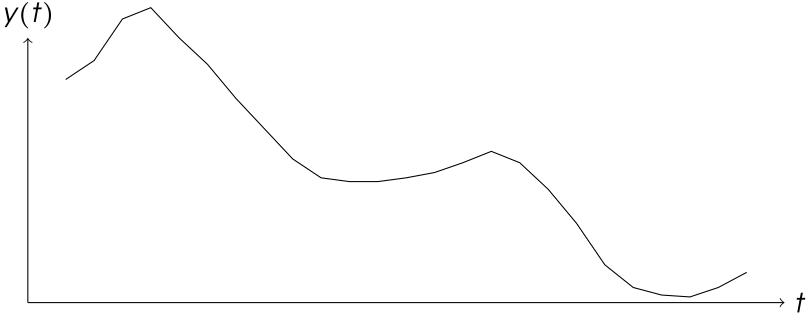
Outline

- So Far: Impulse, Sampling, Replacement
- Previous Week: Convolution ($\circledast$) and Interpolation
- Today: Drawing with Sinusoids

Sinusoidal Leopards

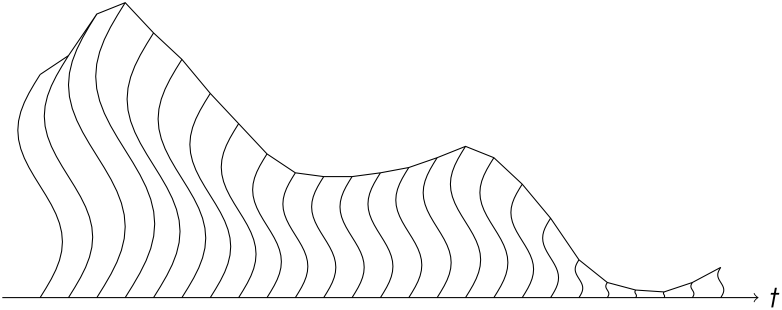


Sinusoidal Leopards



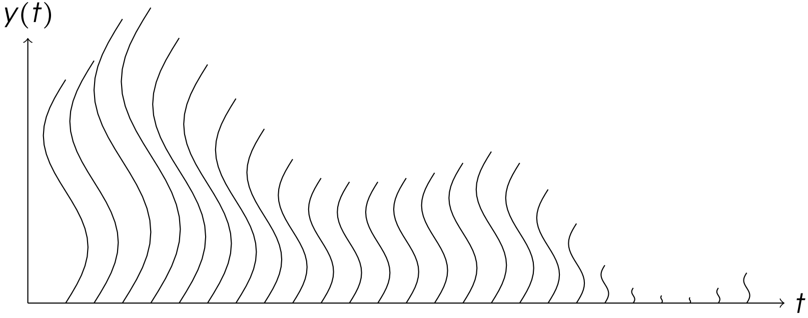
Can we plot this picture using sine waves, i.e. $\sin(2\pi f x)$.

Sinusoidal Leopards



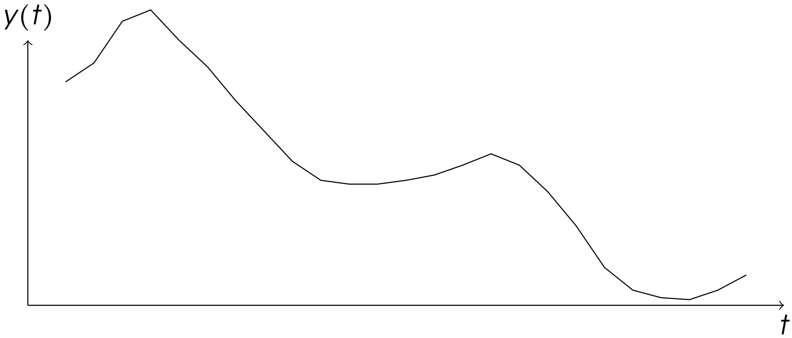
Can we plot this picture using sine waves, i.e. $\sin(2\pi f x)$.

Sinusoidal Leopards

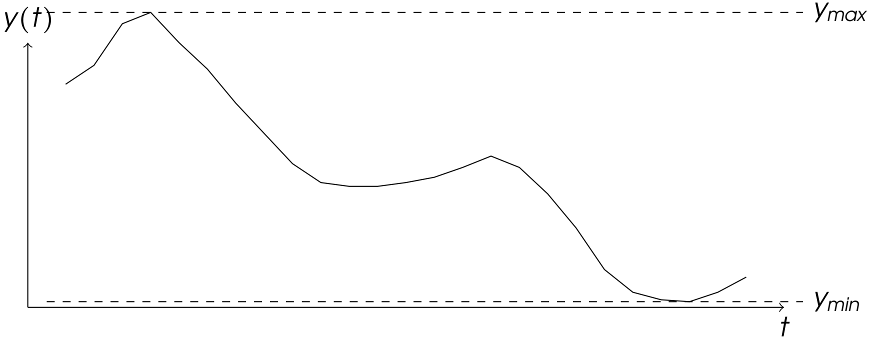


Can we plot this picture using sine waves, i.e. $\sin(2\pi f x)$.

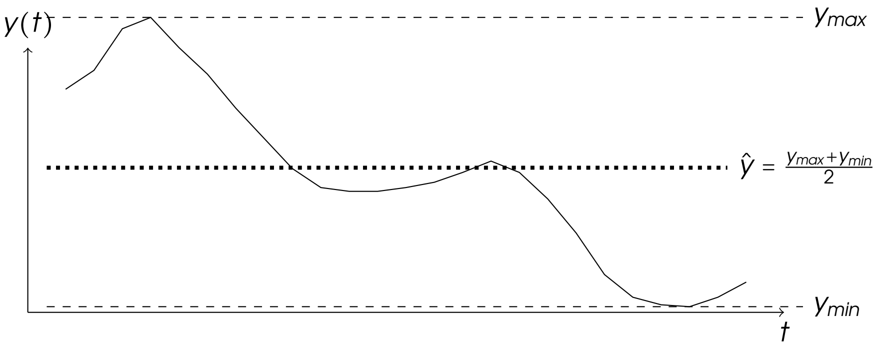
Min-Max Error



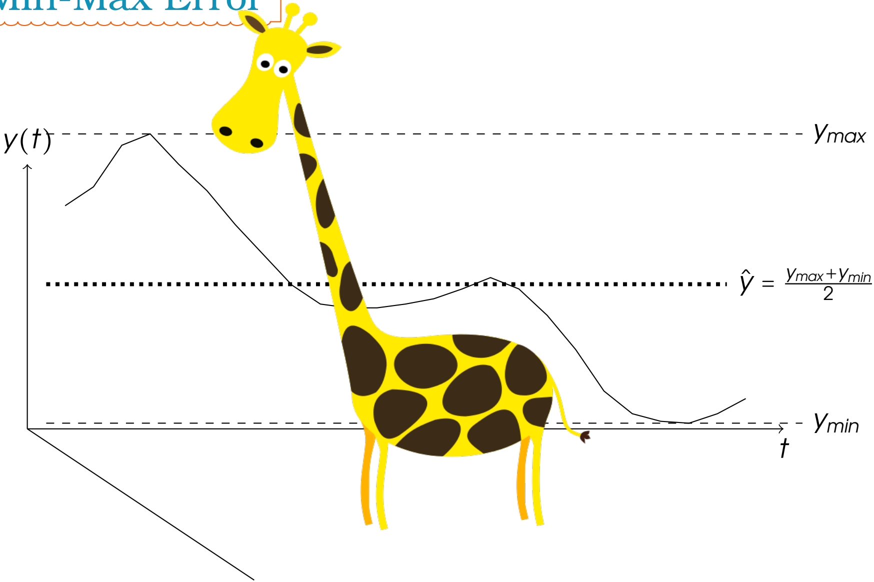
Min-Max Error



Min-Max Error



Min-Max Error



Mean Squared Error (MSE)

DC Approximation: Find the constant value a which minimizes

$$\text{MSE} := \int_{\mathbb{T}} |y(t) - a|^2 dt,$$

where we take $\mathbb{T} = [0, T_d]$ as the interval of interest.

Mean Squared Error (MSE)

DC Approximation: Find the constant value a which minimizes

$$\text{MSE} := \int_{\mathbb{T}} |y(t) - a|^2 dt,$$

where we take $\mathbb{T} = [0, T_d]$ as the interval of interest.

Notation:

$$\mu = \frac{1}{T_d} \int_{\mathbb{T}} y(t) dt \qquad \text{(Mean Value)}$$

$$\|y(t)\|_{\ell_2} = \left(\int_{\mathbb{T}} |y(t)|^2 dt \right)^{\frac{1}{2}} \qquad (\ell_2\text{- norm})$$

$$\langle y(t), x(t) \rangle = \int_{\mathbb{T}} y(t) x^*(t) dt \qquad \text{(Dot product, } \mathbb{C} \text{ valued).}$$

Mean Squared Error (MSE)

DC Approximation: Find the constant value a which minimizes

$$\text{MSE} := \int_{\mathbb{T}} |y(t) - a|^2 dt,$$

where we take $\mathbb{T} = [0, T_d]$ as the interval of interest.

Notation:

$$\mu = \frac{1}{T_d} \int_{\mathbb{T}} y(t) dt \qquad \text{(Mean Value)}$$

$$||y(t)||_{\ell_2} = \left(\int_{\mathbb{T}} |y(t)|^2 dt \right)^{\frac{1}{2}} \qquad (\ell_2\text{- norm})$$

$$\langle y(t), x(t) \rangle = \int_{\mathbb{T}} y(t) x^*(t) dt \qquad \text{(Dot product, } \mathbb{C} \text{ valued).}$$

DC Approximation : $y_{approx}(t) = \alpha_0, \forall t \in [0, T_d]$ where

$$\alpha_0 = \underset{a \in \mathbb{R}}{\text{argmin}} \int_{\mathbb{T}} |y(t) - a|^2 dt$$

Minimizing Functions

$$\int_{\mathbb{T}} |y(t) - a|^2 dt = \int_{\mathbb{T}} |y(t) - \mu + \mu - a|^2 dt$$

Minimizing Functions

$$\int_{\mathbb{T}} |y(t) - a|^2 dt = \int_{\mathbb{T}} |y(t) - \mu + \mu - a|^2 dt$$

$$= \int_{\mathbb{T}} |y(t) - \mu|^2 dt + \int_{\mathbb{T}} |a - \mu|^2 dt - 2 \int_{\mathbb{T}} (y(t) - \mu)(a - \mu) dt$$

Minimizing Functions

$$\int_{\mathbb{T}} |y(t) - a|^2 dt = \int_{\mathbb{T}} |y(t) - \mu + \mu - a|^2 dt$$

$$= \int_{\mathbb{T}} |y(t) - \mu|^2 dt + \int_{\mathbb{T}} |a - \mu|^2 dt - 2 \int_{\mathbb{T}} (y(t) - \mu)(a - \mu) dt$$

$$= \int_{\mathbb{T}} |y(t) - \mu|^2 dt + |a - \mu|^2\, T_d - 2(a - \mu) \int_{\mathbb{T}} (y(t) - \mu) dt.$$

Minimizing Functions

$$\int_{\mathbb{T}} |y(t) - a|^2 dt = \int_{\mathbb{T}} |y(t) - \mu + \mu - a|^2 dt$$

$$= \int_{\mathbb{T}} |y(t) - \mu|^2 dt + \int_{\mathbb{T}} |a - \mu|^2 dt - 2 \int_{\mathbb{T}} (y(t) - \mu)(a - \mu) dt$$

$$= \int_{\mathbb{T}} |y(t) - \mu|^2 dt + |a - \mu|^2 \, T_d - 2(a - \mu) \int_{\mathbb{T}} (y(t) - \mu) dt.$$

Notice that $\int_{\mathbb{T}} (y(t) - \mu) dt = 0$, by the definition of μ.

$$\int_{\mathbb{T}} |y(t) - a|^2 dt \geq \int_{\mathbb{T}} |y(t) - \mu|^2 dt, \;\; \forall a \in \mathbb{R}.$$

Minimizing Functions

$$\int_{\mathbb{T}} |y(t) - a|^2 dt = \int_{\mathbb{T}} |y(t) - \mu + \mu - a|^2 dt$$

$$= \int_{\mathbb{T}} |y(t) - \mu|^2 dt + \int_{\mathbb{T}} |a - \mu|^2 dt - 2 \int_{\mathbb{T}} (y(t) - \mu)(a - \mu) dt$$

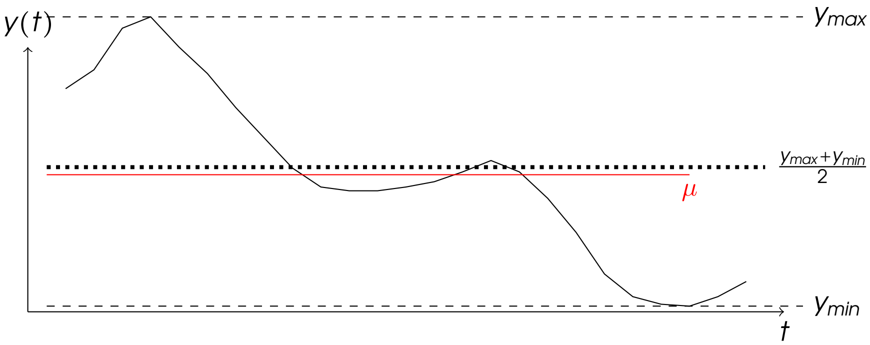
$$= \int_{\mathbb{T}} |y(t) - \mu|^2 dt + |a - \mu|^2 \, T_d - 2(a - \mu) \int_{\mathbb{T}} (y(t) - \mu) dt.$$

Notice that $\int_{\mathbb{T}} (y(t) - \mu) dt = 0$, by the definition of μ.

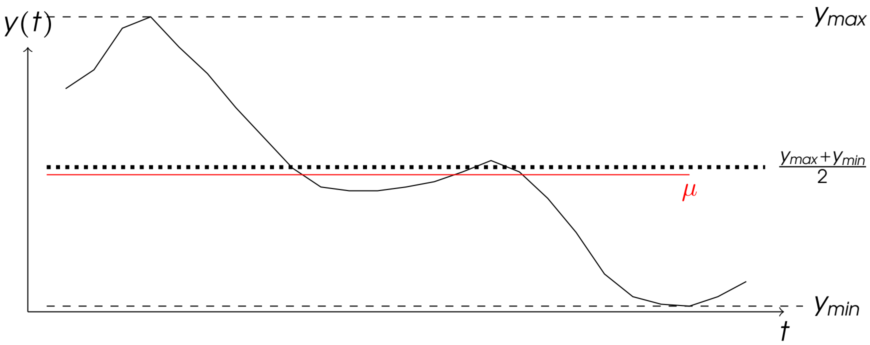
$$\int_{\mathbb{T}} |y(t) - a|^2 dt \geq \int_{\mathbb{T}} |y(t) - \mu|^2 dt, \;\; \forall a \in \mathbb{R}.$$

Mean Value is the best DC approximation for MSE.

Mean Animal



Mean Animal

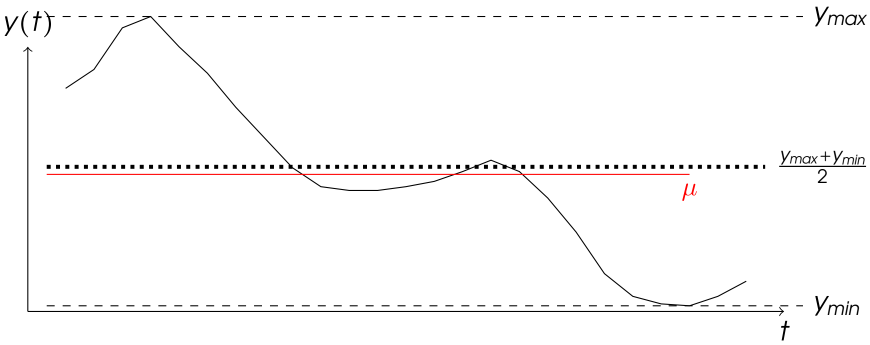


Why not fit the sinusoids we started with?

Fitting a Sine Wave

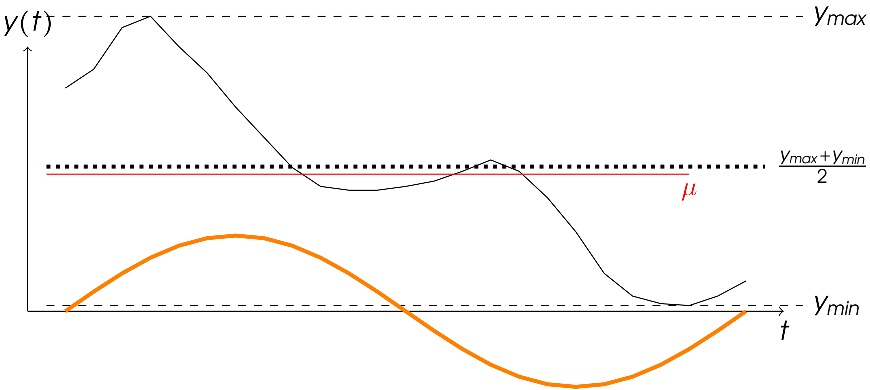
$$b_1 = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \int_{\mathbb{T}} \left| y(t) - a \sin\left(\frac{2\pi}{T_d} t\right) \right|^2 dt.$$

Fitting a Sine Wave



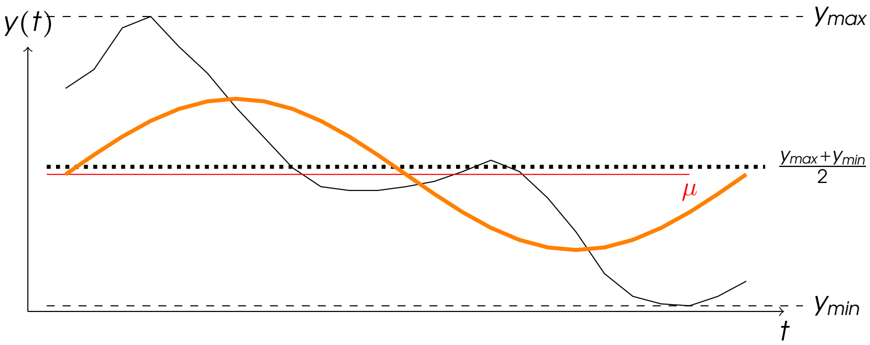
$$b_1 = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \int_{\mathbb{T}} \left| y(t) - a \sin\left(\frac{2\pi}{T_d} t\right) \right|^2 dt.$$

Fitting a Sine Wave



$$b_1 = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \int_{\mathbb{T}} \left| y(t) - a \sin\left(\frac{2\pi}{T_d} t\right) \right|^2 dt.$$

Fitting a Sine Wave



$$b_1 = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \int_{\mathbb{T}} \left| y(t) - a \sin\left(\frac{2\pi}{T_d} t\right) \right|^2 dt.$$

'Tailored' Approximation

$$\gamma = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \int_{\mathbb{T}} \left| y(t) - a\,f(t) \right|^2 dt.$$

The derivative of the MSE term should be zero at $a = \gamma$, i.e.

$$\frac{d}{da} \int_{\mathbb{T}} \left| y(t) - \gamma f(t) \right|^2 dt = 0$$

'Tailored' Approximation

$$\gamma = \operatorname*{argmin}_{a\in\mathbb{R}} \int_{\mathbb{T}} \left| y(t) - a\, f(t) \right|^2 dt.$$

The derivative of the MSE term should be zero at $a = \gamma$, i.e.

$$\frac{d}{da} \int_{\mathbb{T}} \left| y(t) - \gamma f(t) \right|^2 dt = 0$$

'Tailored' Approximation

$$\gamma = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \int_{\mathbb{T}} \left| y(t) - a\, f(t) \right|^2 dt.$$

The derivative of the MSE term should be zero at $a = \gamma$, i.e.

$$\frac{d}{da} \int_{\mathbb{T}} \left| y(t) - \gamma f(t) \right|^2 dt = 0$$

$$\Rightarrow \langle y(t) - af(t), f(t) \rangle = 0 \text{ at } a = \gamma$$

'Tailored' Approximation

$$\gamma = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \int_{\mathbb{T}} \left| y(t) - a\, f(t) \right|^2 dt.$$

The derivative of the MSE term should be zero at $a = \gamma$, i.e.

$$\frac{d}{da} \int_{\mathbb{T}} \left| y(t) - \gamma f(t) \right|^2 dt = 0$$

$$\Rightarrow \langle y(t) - af(t), f(t) \rangle = 0 \text{ at } a = \gamma$$

$$\Rightarrow \gamma = \frac{\langle y(t), f(t) \rangle}{\langle f(t), f(t) \rangle}$$

'Tailored' Approximation

$$\gamma = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \int_{\mathbb{T}} \left| y(t) - a\, f(t) \right|^2 dt.$$

The derivative of the MSE term should be zero at $a = \gamma$, i.e.

$$\frac{d}{da} \int_{\mathbb{T}} \left| y(t) - \gamma f(t) \right|^2 dt = 0$$

$$\Rightarrow \langle y(t) - af(t), f(t) \rangle = 0 \text{ at } a = \gamma$$

$$\Rightarrow \gamma = \frac{\langle y(t), f(t) \rangle}{\langle f(t), f(t) \rangle} = \frac{\langle y(t), f(t) \rangle}{||f||_{\ell_2}^2}$$

Sines and Cosines

$$f(t) = \sin(\frac{2\pi}{T_d}t)$$

Sines and Cosines

$$f(t) = \sin(\frac{2\pi}{T_d} t)$$

Since $||f(t)||_{\ell_2}^2 = \frac{T_d}{2}$, we get

$$b_1 = \frac{2}{T_d} \int_{\mathbb{T}} y(t) \sin(\frac{2\pi}{T_d} t) dt.$$

Sines and Cosines

$$f(t) = \sin(\frac{2\pi}{T_d}t)$$

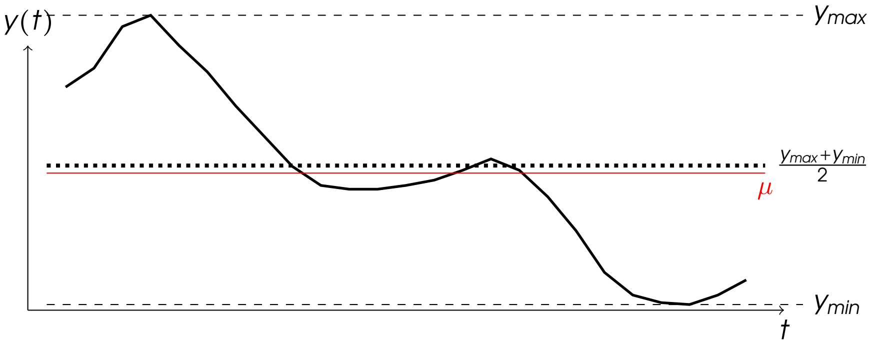
Since $||f(t)||_{\ell_2}^2 = \frac{T_d}{2}$, we get

$$b_1 = \frac{2}{T_d}\int_{\mathbb{T}} y(t)\sin(\frac{2\pi}{T_d}t)dt.$$

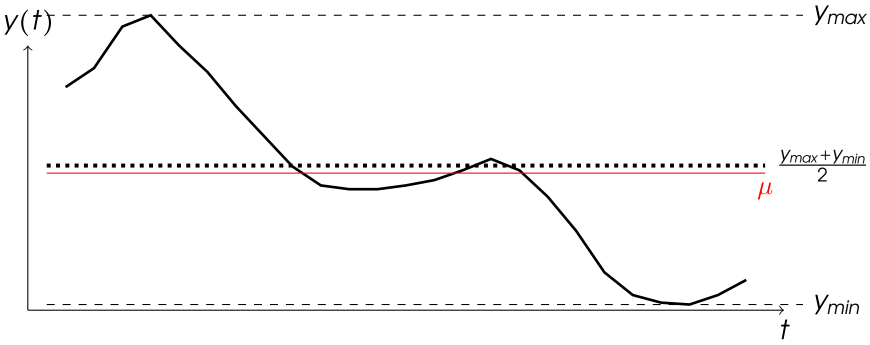
To be fair, we should fit $\cos(\frac{2\pi}{T_d}t)$ as well.

$$a_1 = \frac{2}{T_d}\int_{\mathbb{T}} y(t)\cos(\frac{2\pi}{T_d}t)dt.$$

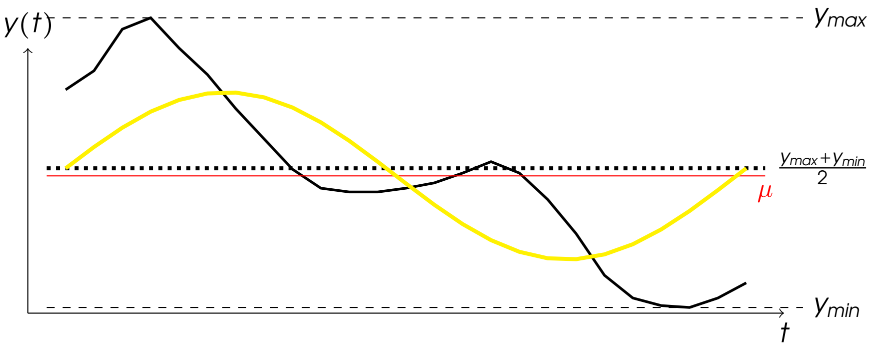
DC+Co+Sine



DC+Co+Sine



DC+Co+Sine



DC+Co+Sine