Signal Processing - 1 by One

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Outline

- So Far: Sampling, Convolution, Fourier Transform
- Previous Week: Shannon Sampling Theorem
- Previous Class: Systems and Circuits
- Today: Laplace Transform

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$$\cos(2\pi ft+\theta)=\cos(2\pi ft)\cos(\theta)-\sin(2\pi ft)\sin(\theta).$$

Notice that

$$\int_{k\frac{T}{2}}^{(k+1)\frac{T}{2}}\cos(2\pi ft)\sin(2\pi ft)dt=0 \text{ for } T=\frac{1}{f}, k\in\mathbb{Z}.$$

Furthermore

$$\lim_{T_s\to\infty}\int_{-T_s}^{T_s}\cos(2\pi ft)\sin(2\pi ft)dt=0 \text{ (In a generalized sense)}$$

$$\lim_{T_s\to\infty}\int_{-T_s}^{T_s}\cos(2\pi f_1 t)\sin(2\pi f_2 t)dt=0 \text{ (as a generalized integral)} .$$

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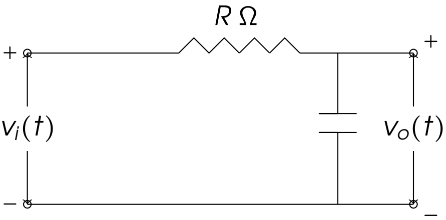
For $f > 0$, if the component $\exp(-j2\pi ft)$ corresponds to positive frequency, then $\exp(j2\pi ft)$ corresponds to negative frequency

Notice that $\exp(-j2\pi f\, t)$ and $\exp(j2\pi f\, t)$ are *orthogonal* for $f > 0$.

Complex Circuits

Complex numbers in electrical circuits suggest the presence of both $\cos(2\pi f t)$ and $\sin(2\pi f t)$ inside, even when $\cos(2\pi f t)$ is input.



Generalizing the Fourier Transform: For $s = \sigma + j2\pi f$,

$$X(s) = \int_{\mathbb{R}} x(t)\exp(-st)dt \text{ (Two-sided Laplace Transform).}$$

Region of Convergence (ROC) : $\{Real(s)\}$ s.t. Integral exists.

$$\lim_{\sigma \to 0} X(s) = X(f).$$

Kirchoff's Voltage Law:

$$v_i(t) = i(t)\,R + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau.$$

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$$V_i(s) = I(s)\,R + \frac{1}{C}\frac{I(s)}{s} \quad \text{and} \quad V_o(s) = \frac{1}{C}\,\frac{I(s)}{s}.$$

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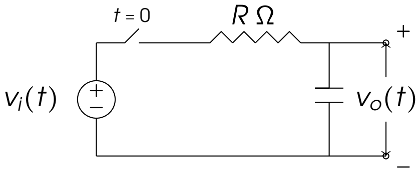
At $s = j2\pi f$,

$$V_o(f) = H(f)V_i(f) \text{ where } H(f) = \frac{1}{1+j2\pi fRC}$$

$$v_o(t) = h(t) * v_i(t) \text{ where } h(t) = \frac{1}{RC}\exp(-\frac{t}{RC}), t \geq 0.$$

Laplace with Initial Conditions

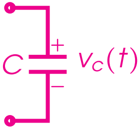


$$X(s) = \int_0^{\infty} x(t)\exp(-jst)\,dt$$

$$v_i(t) = i(t)R + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau.$$

Tool: One-sided or Unilateral Laplace Transform for $t \geq 0$.

A Charged Capacitor



$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)\,d\tau.$$

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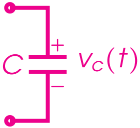

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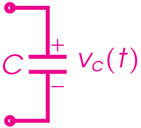

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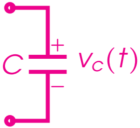

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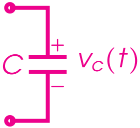

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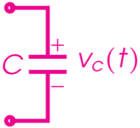

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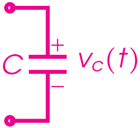

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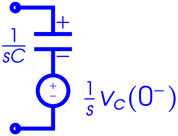

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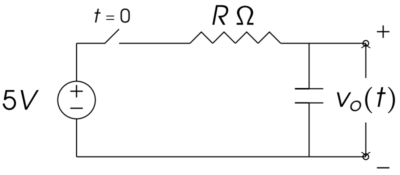
RC Circuit Example



$$X(s) = \int_0^{\infty} x(t)\exp(-st)\,dt$$

$$V_i(s) = I(s)R + \frac{v_C(0^-)}{s} + \frac{1}{sC}I(s).$$

$$I(s) = \frac{\frac{5}{s} - \frac{v_C(o^-)}{s}}{R + \frac{1}{sC}}$$

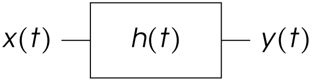
$$v_C(t) = v_C(0^-)u(t) + \left[5 - v_C(0^-)\right](1 - e^{-\frac{t}{RC}})u(t),\ t \geq 0.$$

Impulse Response

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Q1) With unit step as the input $x(t)$, it is given that $Y(s) = \frac{1}{s(s+\alpha)}$, where $\alpha \in \mathbb{R}^{+}$. Find the impulse response $h(t)$.

Impulse Response



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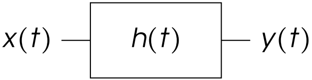

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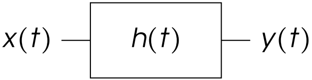

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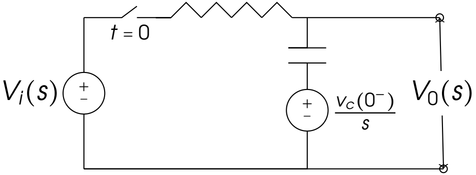
GNURADIO: Showing the step response and identify the RC ckt.

Superposition

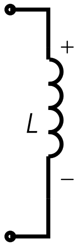
Q2) Apply super-position theorem and Laplace Transform to find the output voltage $v_0(t)$, if the input is $5u(t)$. The resistor is R ohms and capacitor is C Farads.



Output to Bounded Inputs

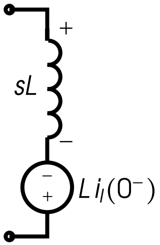
Q3) Consider an LTI system with impulse response $h(t)$, which is integrable, i.e. $\int_{\mathbb{R}} |h(t)| dt < \infty$. If the input is bounded, i.e. $\max_{t \in \mathbb{R}} |x(t)| = v_m < \infty$, show that the output is also bounded (this is called **BIBO** stability).

Inertial Inductor



$$v_l(t) = L\frac{d}{dt}i_l(t)$$

$$V_l(s) = s\,L\,I_l(s) - L\,i_l(0^-).$$



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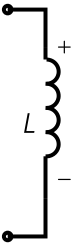

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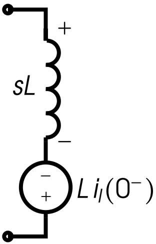
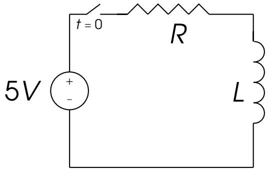




$$I_l(s) = \frac{V_i(s) + L\,i_l(0^-)}{sL + R} = \frac{\frac{V_m}{s} + L\,i_l(0^-)}{sL + R}$$

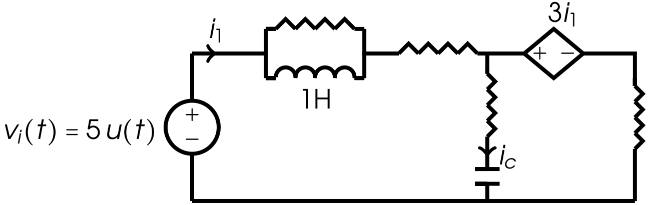
$$= \frac{V_m}{L\,s(s + \frac{R}{L})} + \frac{i_l(0^-)}{s + \frac{R}{L}}$$

$$= \frac{V_m}{R}\left(\frac{1}{s} - \frac{1}{s + \frac{R}{L}}\right) + \frac{i_l(0^-)}{s + \frac{R}{L}}.$$

$$V_i(s) = RI_l(s) + sLI_l(s) - Li_l(0^-)$$

$$i_l(t) = i_{dc} - \left[i_{dc} - i_l(0^-)\right]e^{-\frac{R}{L}t},\ t \geq 0.$$

Mesh Analysis



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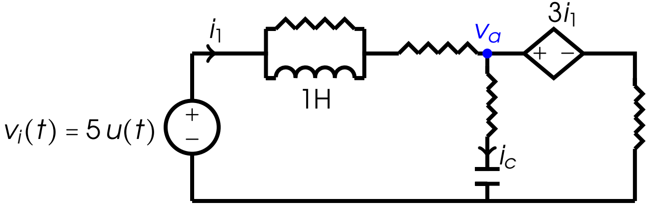

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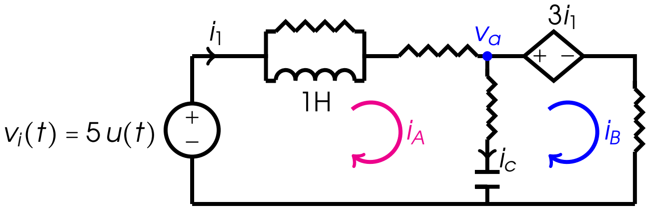

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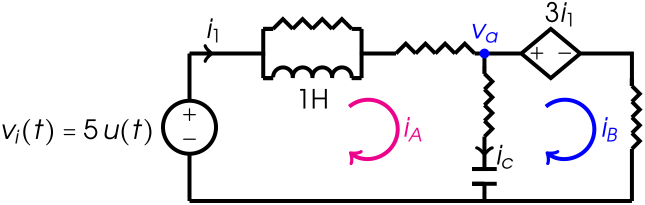

$$V_i(s) = \frac{s}{1+s} I_A(s) + I_A(s) + \left[I_A(s) - I_B(s)\right]\left(1 + \frac{1}{s}\right)$$

$$3I_A(s) = \left[I_A(s) - I_B(s)\right]\left(1 + \frac{1}{s}\right) - I_B(s)$$

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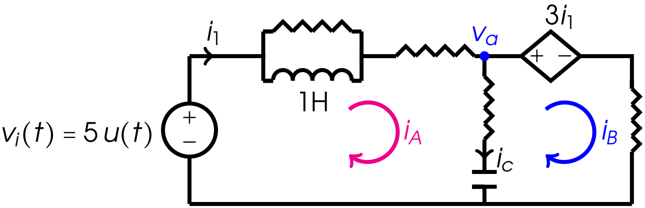

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$$\begin{bmatrix} \frac{s}{s+1} + 2 + \frac{1}{s} & 1 + \frac{1}{s} \\ -2 + \frac{1}{s} & 2 + \frac{1}{s} \end{bmatrix} \begin{bmatrix} I_A \\ -I_B \end{bmatrix} = \begin{bmatrix} V_i(s) \\ 0 \end{bmatrix} \qquad Det = \frac{8s^2 + 12s + 5}{s(s+1)}$$

Matrix Inversion

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^{-1} \frac{1}{ad-bc}.$$

$$A = \begin{bmatrix} \frac{s}{s+1} + 2 + \frac{1}{s} & 1 + \frac{1}{s} \\ -2 + \frac{1}{s} & 2 + \frac{1}{s} \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{Det} \begin{bmatrix} 2 + \frac{1}{s} & -1 - \frac{1}{s} \\ 2 - \frac{1}{s} & \frac{s}{s+1} + 2 + \frac{1}{s} \end{bmatrix}.$$

$$A^{-1} \begin{bmatrix} V_i(s) \\ 0 \end{bmatrix} = \frac{1}{Det} \begin{bmatrix} 2 + \frac{1}{s} \\ 2 - \frac{1}{s} \end{bmatrix}$$

$$Det = \frac{8s^2 + 12s + 5}{s(s+1)}$$

Mesh Solution

$$\begin{bmatrix} I_A \\ -I_B \end{bmatrix} = A^{-1} \begin{bmatrix} V_i(s) \\ 0 \end{bmatrix} = \frac{1}{Det} \begin{bmatrix} 2 + \frac{1}{s} \\ 2 - \frac{1}{s} \end{bmatrix} V_i(s) = \frac{1}{Det} \begin{bmatrix} 2 + \frac{1}{s} \\ 2 - \frac{1}{s} \end{bmatrix} \frac{5}{s}.$$

Mesh Solution

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$$I_A - I_B = [1 \;\; 1]\begin{bmatrix} I_A \\ I_B \end{bmatrix} = \frac{4}{Det}\times\frac{5}{s} \Rightarrow V_C(s) = \frac{1}{s}(I_A - I_B) = \frac{20}{8}\,\frac{s+1}{s(s^2+\frac{3}{2}s+\frac{5}{8})}$$

Mesh Solution

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Find the inverse Laplace Transform of this rational function

Mesh Solution

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Find the inverse Laplace Transform of this rational function

$$\frac{s+1}{s(s^2+\frac{3}{2}s+\frac{5}{8})} = \frac{1+\frac{1}{s}}{s^2+\frac{3}{2}s+\frac{5}{8}} = \frac{1}{s^2+\frac{3}{2}s+\frac{5}{8}} + \frac{1}{s(s^2+\frac{3}{2}s+\frac{5}{8})}$$

$$= \frac{1}{s^2+\frac{3}{2}s+\frac{5}{8}} + \left[\frac{1}{s} - \frac{s+\frac{3}{2}}{(s^2+\frac{3}{2}s+\frac{5}{8})}\right]\frac{8}{5}.$$

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$$= \frac{1}{s^2+\frac{3}{2}s+\frac{5}{8}} + \left[\frac{1}{s} - \frac{s+\frac{3}{2}}{(s^2+\frac{3}{2}s+\frac{5}{8})}\right]\frac{8}{5}.$$

Let us invert the first term

Inverse Laplace Transform

$$\exp(-at)u(t) \overset{L.T.}{\rightleftharpoons} \frac{1}{s+a}, \ \text{Real}(s+a) \geq 0. \qquad (\ast)$$

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$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_n s^n}{a_0 + a_1 s_1 + \cdots + s^m}$$

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Thus, $(\ast)$ under suitable conditions sufice to find the inverse.

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$$h(t) = \sum_{i=1}^{m} A_i \exp(\alpha_i t) u(t), \ \text{if Real}(s - \alpha_i) \geq 0, \forall i.$$

Inverse Laplace Transform

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Clearly if Real$(\alpha_i) \leq 0, \forall i, \Rightarrow h(t)$ stays bounded & integrable.

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$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_n s^n}{a_0 + a_1 s_1 + \cdots + s^m} = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \cdots + \frac{A_m}{s-\alpha_m}.$$

Thus, (✷) under suitable conditions sufice to find the inverse.

$$h(t) = \sum_{i=1}^{m} A_i \exp(\alpha_i t)u(t), \text{ if Real}(s-\alpha_i) \geq 0, \forall i.$$

Clearly if Real$(\alpha_i) \leq 0, \forall i, \Rightarrow h(t)$ stays bounded & integrable.

Thus, $\alpha_i < 0, \forall i$ will imply BIBO stability.

Oscillatory Responses

Second order circuits may have poles α_i complex

$$\int_0^\infty \exp(-at)\cos(\omega t)dt = \frac{s+a}{(s+a)^2+\omega^2}, \; \text{Real}(s+a) > 0$$

$$\int_0^\infty \exp(-at)\sin(\omega t)dt = \frac{\omega}{(s+a)^2+\omega^2}, \; \text{Real}(s+a) > 0$$

RLC Solution

$$V_C(s) = \frac{5}{2}\frac{1}{s^2 + \frac{3}{2}s + \frac{5}{8}} + \frac{5}{2}\frac{1}{s(s^2 + \frac{3}{2}s + \frac{5}{8})}$$

RLC Solution

$$V_C(s) = \frac{5}{2}\frac{1}{s^2 + \frac{3}{2}s + \frac{5}{8}} + \frac{5}{2}\frac{1}{s(s^2 + \frac{3}{2}s + \frac{5}{8})}$$

$$V_C(s) = \frac{20}{2}\frac{\frac{1}{4}}{(s + \frac{3}{4})^2 + \frac{1}{4^2}} + \frac{5}{2}\left(\frac{1}{s} - \frac{s + \frac{3}{2}}{(s + \frac{3}{4})^2 + \frac{1}{4^2}}\right)\frac{8}{5}$$

RLC Solution

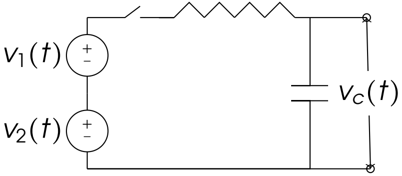
$$V_C(s) = \frac{5}{2}\frac{1}{s^2 + \frac{3}{2}s + \frac{5}{8}} + \frac{5}{2}\frac{1}{s(s^2 + \frac{3}{2}s + \frac{5}{8})}$$

$$V_C(s) = \frac{20}{2}\frac{\frac{1}{4}}{(s + \frac{3}{4})^2 + \frac{1}{4^2}} + \frac{5}{2}\left(\frac{1}{s} - \frac{s + \frac{3}{2}}{(s + \frac{3}{4})^2 + \frac{1}{4^2}}\right)\frac{8}{5}$$

$$v_C(t) = 4u(t) - 4e^{-\frac{3}{4}t}\cos(\frac{t}{4}) - 2e^{-\frac{3}{4}t}\sin(\frac{t}{4}),\ t \geq 0.$$

Not-So Linear Initial Conditions

Q4) Can you apply superposition theorem on the two voltage sources to find $v_c(t)$. The resistor is R ohms and capacitor is C Farads, which has an initial charge of 0.3V.



RLC Circuit

Q5) For the serial RLC circuit find the voltage across C.