Signal Processing - 1 by One

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Outline

- So Far: Impulse, Sampling, Replacement
- Previous Week: Convolution ($\circledast$) and Interpolation
- Previous Class: Fourier Series
- Today: Fourier Series Contd..

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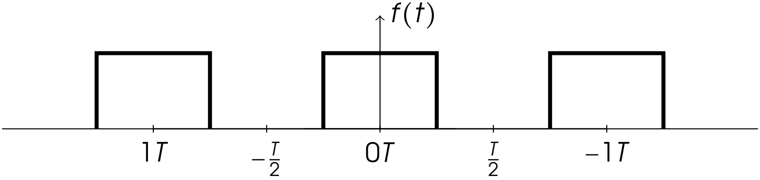

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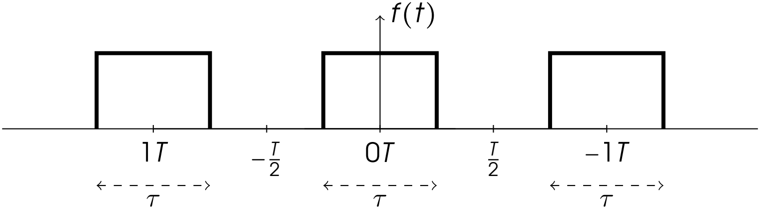

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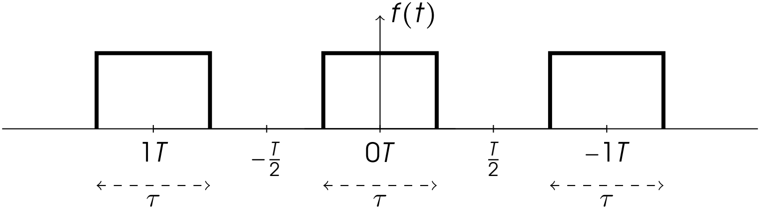

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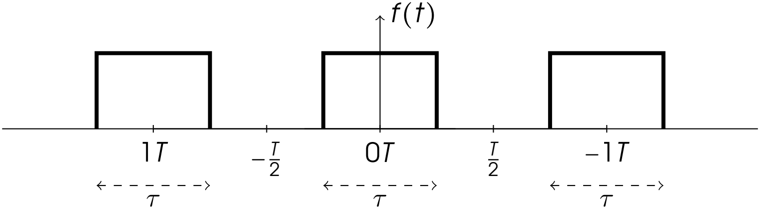

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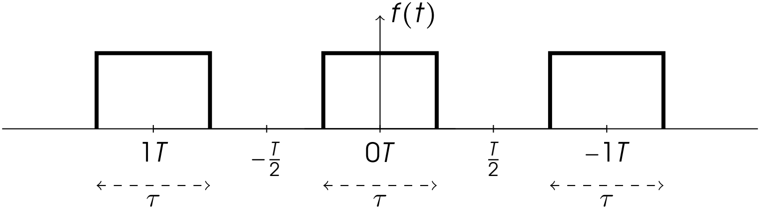

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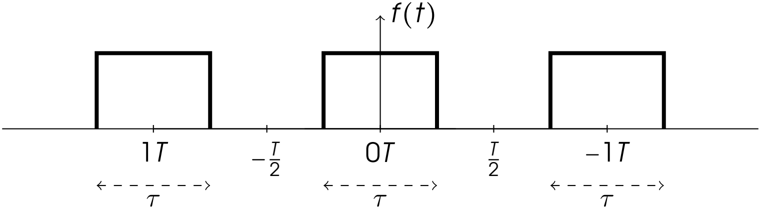

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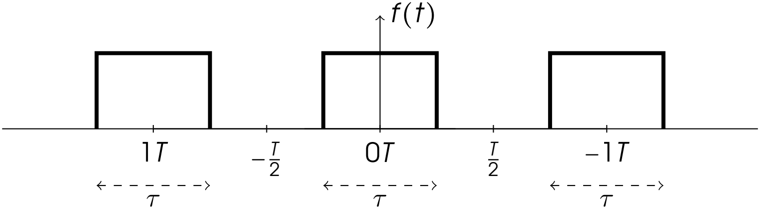

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Fourier Series

$$
\begin{aligned}
f(t) &= \alpha_0 + \sum_{n \in \mathbb{Z} \setminus \{0\}} a_n \cos\left(\frac{2\pi}{T_d} nt\right) + \sum_{n \in \mathbb{Z}} b_n \sin\left(\frac{2\pi}{T_d} nt\right) \\
&= \alpha_0 + \sum_{n \in \mathbb{Z} \setminus \{0\}} a_n \frac{\left(e^{j\frac{2\pi}{T_d} nt} + e^{-j\frac{2\pi}{T_d} nt}\right)}{2} + \sum_{n \in \mathbb{Z}} b_n \frac{\left(e^{j\frac{2\pi}{T_d} nt} - e^{-j\frac{2\pi}{T_d} nt}\right)}{2j} \\
&= \sum_{m \in \mathbb{Z}} \alpha_m e^{j\frac{2\pi}{T_d} mt}.
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Convergence

Proposition

No two continuous functions in $[-\frac{T_d}{2}, \frac{T_d}{2}]$ *will have all the Fourier Series coefficients the same.*

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If the FS coefficients $\alpha_m = 0$, identically $\forall m \in \mathbb{Z}$, then $f(t) = 0$,

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Proof [Stein-Schakarchi]

From our assumption of FS coefficients being zero, i.e.

$$\frac{1}{T}\int_{\mathbb{T}} f(t)\exp(-j\frac{2\pi}{T}mt)\,dt = 0,\ \forall m \in \mathbb{Z}$$

Furthermore

$$\int_{\mathbb{T}} f(t)\big(P(t)\big)^k dt = 0,\ \ \forall k = 0, 1, 2, \cdots$$

Take for some parameter $\epsilon > 0$,

$$P(t) = \epsilon + \cos\left(\frac{2\pi}{T}t\right).$$

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for any trigonometric polynomial $P(t)$ of the form

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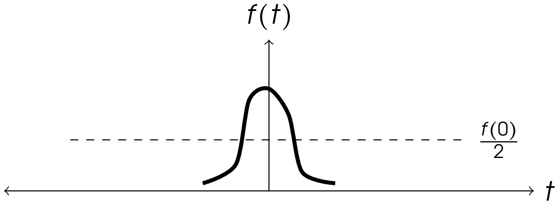

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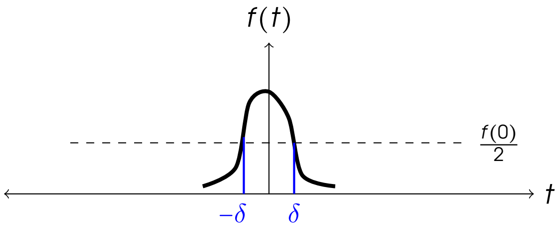

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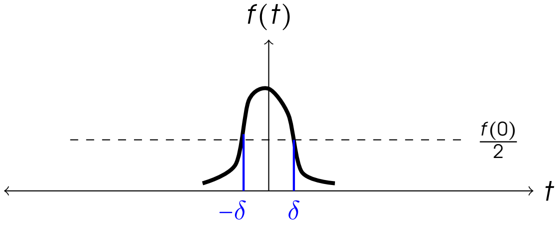

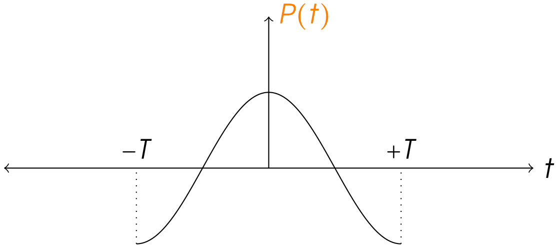

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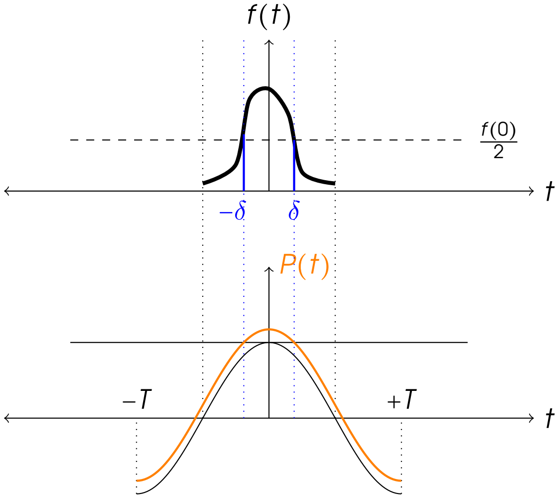

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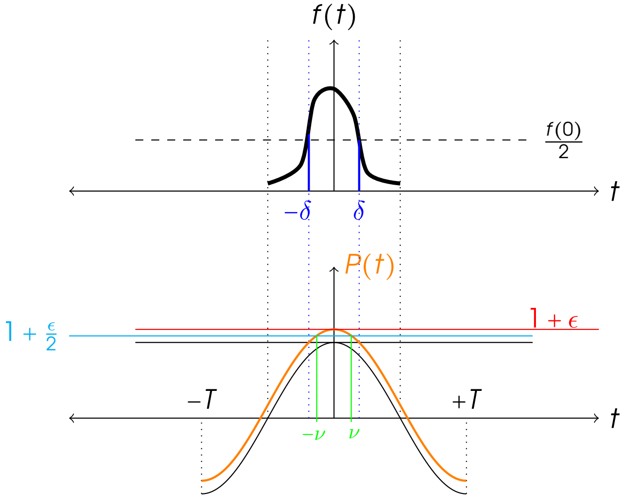

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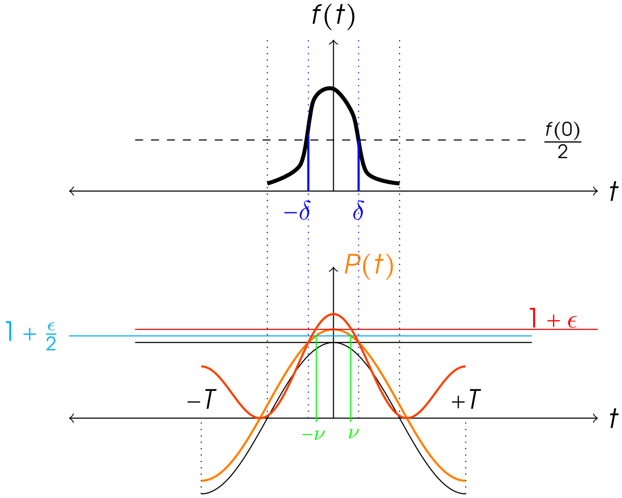

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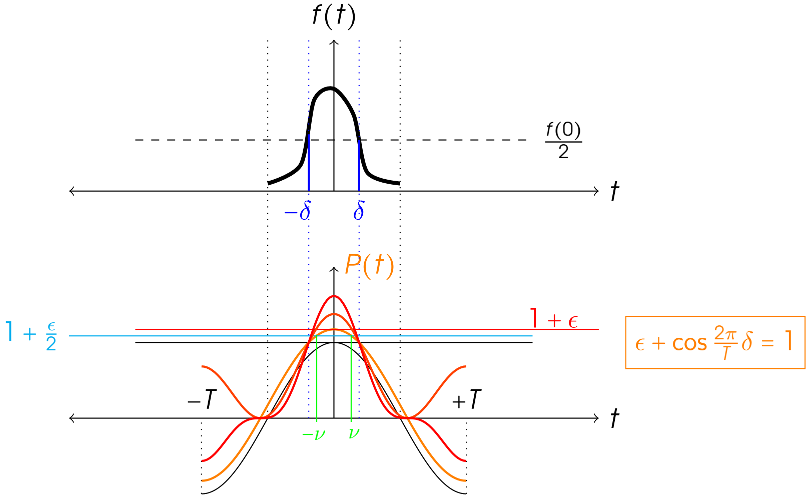

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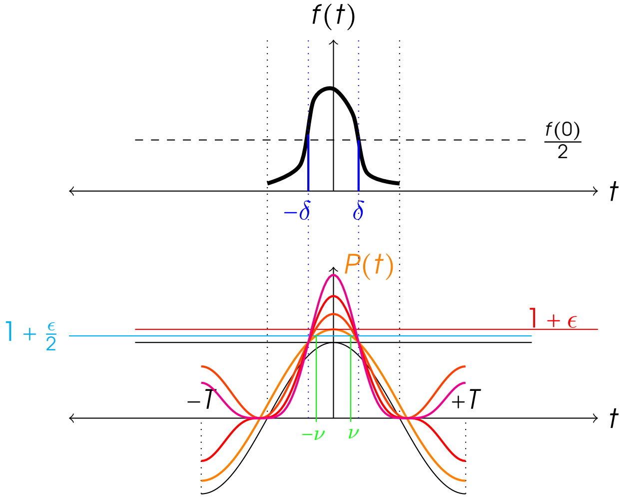

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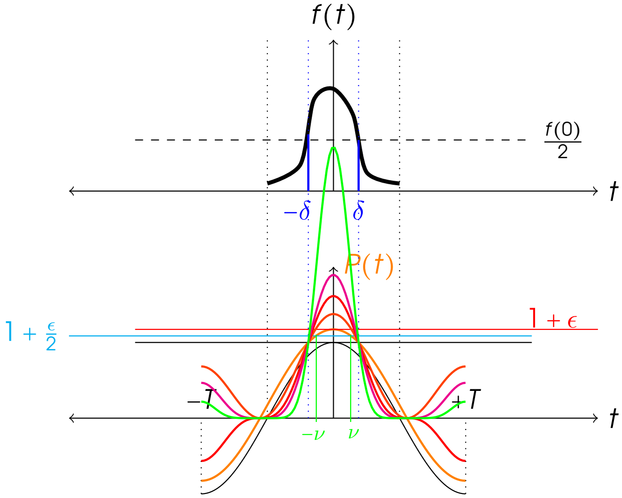

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Proof Details

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1. $I_1 > -\infty$ as $f(t)$ is locally integrable.
2. $I_2 > 0$.
3. $I_3 \geq \frac{f(0)}{2}(1 + \frac{\epsilon}{2})^k \to \infty$ when $k \uparrow \infty$, as $\epsilon > 0$.

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We have our contradiction, thus $f(0)$ has to be zero.

What this was about!

Assume a function $f(t)$ integrable in $[-\frac{T}{2}, \frac{T}{2}]$, and

$$\alpha_m = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\exp(-j\frac{2\pi}{T}mt)dt,\ m \in \mathbb{Z}.$$

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Now consider the function

$$g(t) = \sum_{n\in\mathbb{Z}} \alpha_n \exp(j\frac{2\pi}{T}nt)dt.$$

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Our lemma says $g(t) = f(t)$, if $f(t)$ is continuous at t. Therefore,

$$\sum_{n\in\mathbb{Z}} \alpha_n \exp\left(j\frac{2\pi}{T}nt\right)dt = f(t)$$

at instants where $f(t)$ is continuous.