Recall

In the last class we discussed some basic notions of topology. Recall that a subset $U \subseteq \mathbb{C}$ is said to be open if given any point $z_0 \in \mathbb{C}$, there exists a real number $\delta > 0$ such that $B_\delta(z_0) \subseteq U$. Here $B_\delta(z_0)$ is the open ball of radius δ around z_0.

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Intuitively this means that if some point belongs to U, all sufficiently nearby points belong to U. The prototypical examples are $\mathbb{C}$ and the open disc minus finitely many points.
A subset $Z \subseteq \mathbb{C}$ is closed if its complement is open. Example: For any continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 0$ defines a closed subset of $\mathbb{C}$.

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A subset $Z \subseteq \mathbb{C}$ is closed if its complement is open. Example: For any continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 0$ defines a closed subset of $\mathbb{C}$.
Its a more non-trivial fact that $\mathbb{C}$ has no subsets which are both open and closed other than $\mathbb{C}$ and $\emptyset$

MA 205 Complex Analysis: CR Equations

August 20, 2020

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Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be differentiable at $z_0 \in \Omega$. Thus,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

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exists. In the last class, we have stressed the point that the existence of this complex limit means a lot; the limit exists as z approaches z_0 along any path. To derive the CR equations, we'll in particular look at the existence of this limit as $z \to z_0$ along the x-direction and the y-direction.

Cauchy-Riemann Equations

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$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h\to 0} \left[\frac{u(a + h, b) - u(a, b)}{h} + \imath \frac{v(a + h, b) - v(a, b)}{h} \right] \\ &= \lim_{h\to 0} \frac{u(a + h, b) - u(a, b)}{h} + \imath \lim_{h\to 0} \frac{v(a + h, b) - v(a, b)}{h} \\ &= u_x(a, b) + \imath v_x(a, b). \end{aligned}$$

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Similarly, in the y-direction, we get

$$f'(z_0) = \lim_{k \to 0} \frac{f(z_0 + \imath k) - f(z_0)}{\imath k} = v_y(a, b) - \imath u_y(a, b).$$

Thus, differentiability of $f = u + \imath v$ at $z_0 = a + \imath b$ implies that u_x, u_y, v_x, v_y exist at (a, b) and they satisfy

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at (a, b). These are the CR equations. If CR equations are not satisfied at a point, then f is not differentiable at that point.

Example: Consider $f(z) = |z|^2$. Here, $u(x, y) = x^2 + y^2$, $v(x, y) = 0$. Thus CR equations are satisfied only at the point $(0, 0)$.

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$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{|z|^2}{z} = 0.$$

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$$v(x,y) = \begin{cases} \frac{-3x^2y+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

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Check that CR equations are satisfied at $(0, 0)$. You'll get $u_x = v_y = 1$ and $u_y = -v_x = 0$ at $(0, 0)$.

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If $(x,y)\to(0,0)$ via either of the axes, this limit is 1. If $(x,y)\to(0,0)$ via $y=x$, this limit is -1. So limit does not exist.

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Suppose for a moment that z and $\bar{z}$ are independent variables! Formally applying chain rule:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{1}{2} + \frac{\partial f}{\partial y} \cdot \frac{1}{2\imath} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \imath \frac{\partial f}{\partial y}\right).$$

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Since $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(u_x + iv_x + iu_y - v_y)$

We can of course view $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ as a function of two real variables;

$$f(x, y) = (u(x, y), v(x, y)).$$

For such functions, in MA 105, you have seen the notion of the total derivative.

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$$\lim_{(h,k)\to(0,0)} \frac{\left\|f(a+h, b+k) - f(a, b) - Df(a, b)\begin{bmatrix} h \\ k \end{bmatrix}\right\|}{\|(h, k)\|} = 0.$$

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Existence of partial derivatives does not imply the existence of total derivative, but existence of partial derivatives which are continuous throughout the domain does imply the existence of total derivative.

Exercise: Show that if f is complex differentiable, then f is real differentiable; i.e., f has a total derivative as a function of two real variables. Show that the converse is not true.
(At the moment solve this exercise assuming the continuity of the first partial derivatives of u and v. We shall see later that this assumption can be removed (it is automatic)).

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- real and imaginary parts satisfy CR.

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Thus, complex differentiability implies:

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- real and imaginary parts satisfy CR.

What if we assume both these? Can we then say f is complex differentiable? And the answer is Yes.
Proof: Since $f = u + \imath v$ is real differentiable,

$$\lim_{(x,y)\to(a,b)} \frac{\left\| \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} - \begin{bmatrix} u(a,b) \\ v(a,b) \end{bmatrix} - \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} \right\|}{\|(x-a, y-b)\|} = 0.$$

Cauchy-Riemann Equations

Note that the numerator is nothing but

$$|f(z) - f(z_0) - \alpha(x - a) - \beta(y - b)|,$$

where $\alpha = u_x + \imath v_x$, $\beta = u_y + \imath v_y$.

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$$\eta(z) = \frac{f(z) - f(z_0) - \alpha(x-a) - \beta(y-b)}{z - z_0}.$$

Observe that

$$\lim_{z \to z_0} \eta(z) = 0.$$

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with $\eta(z) \to 0$ as $z \to z_0$. Write this as

$$f(z) - f(z_0) = \frac{\alpha - \imath\beta}{2}(z - z_0) + \frac{\alpha + \imath\beta}{2}\overline{z - z_0} + \eta(z)(z - z_0).$$

Cauchy-Riemann Equations

Thus,

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)\frac{\overline{z-z_0}}{z-z_0} + \eta(z).$$

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does not exist (why?) and $\lim_{z\to z_0} \eta(z)$ exists, this happens if and only if

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i.e., CR equations are satisfied at z_0. Also, if this is the case,

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0),$$

since $\lim_{z\to z_0} \eta(z) = 0$.

Corollary: Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be such that it has continuous partial derivatives throughout Ω. Then if they satisfy the CR equations at a point, f is differentiable at that point. (Proof?)

Cauchy-Riemann Equations

Corollary: Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be such that it has continuous partial derivatives throughout Ω. Then if they satisfy the CR equations at a point, f is differentiable at that point. (Proof?)

Exercise: Show that $f(z) = e^x(\cos y + \imath \sin y)$ is holomorphic throughout $\mathbb{C}$. Note that $f'(z) = f(z)$. This is the complex exponential function.

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The assumptions in the statement of the corollary can be weakened. In fact, the following is true:

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Theorem

Let f be continuous on Ω. Suppose the partial derivatives exist and satisfy the Cauchy-Riemann equations at every point in Ω. Then f is holomorphic in Ω.

We shall not prove this theorem.

Cauchy-Riemann Equations

Exercise: Show that the CR equations take the form

$$u_r = \frac{1}{r} v_\theta \;\&\; v_r = -\frac{1}{r} u_\theta$$

in polar coordinates.

Harmonic Functions

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$$u_x = v_y \ \& \ u_y = -v_x.$$

Thus,

$$u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0.$$

Similarly for v.

Harmonic Functions

Suppose u and v are harmonic functions on Ω. We say that v is a harmonic conjugate of u if $f = u + \imath v$ is holomorphic in Ω.

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Example: $v(x, y) = 2xy$ is a harmonic conjugate of $x^2 - y^2$ in any domain. Indeed, $f(z) = z^2$ is holomorphic everywhere.
Note that v is a harmonic conjugate of u does not mean that u is a harmonic conjugate of v! In fact:

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Example: $v(x, y) = 2xy$ is a harmonic conjugate of $x^2 - y^2$ in any domain. Indeed, $f(z) = z^2$ is holomorphic everywhere.

Note that v is a harmonic conjugate of u does not mean that u is a harmonic conjugate of v! In fact:

Exercise: If u and v are harmonic conjugates of each other, show that they are constant functions.

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Here's a general method to find a harmonic conjugate: given a harmonic u, find u_x. Equate $v_y = u_x$ and integrate wrt y. You'll get $v(x, y) = \ldots + \phi(x)$. Now $v_x = \ldots + \phi'(x)$. Equate this to $-u_y$ to find $\phi(x)$. That gives you v.

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Unfortunately this method fails in general. Try and think of the reason !

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But if Ω is "nice", then every harmonic u on Ω has a harmonic conjugate. Conversely, if every harmonic u on Ω has a harmonic conjugate, then Ω has to be "nice". Thus, the question in analysis: "does every harmonic function has a harmonic conjugate?" is answered by geometry: "answer depends on the shape of the domain". It's relevant at this point to recall from MA 105 that curl of grad is always zero but curl free is certainly a grad of something only when the domain is "nice" (for example $\mathbb{C}$ or a disc in $\mathbb{C}$). Remember this !