MA 205 Complex Analysis: Examples

September 11, 2020

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Example

Consider $f(z) = \frac{e^z}{\sin z + \cos z}$. Let's expand it as a power series centered at 0. First note that f is holomorphic at 0. The nearest pole to 0 for this function is when $sin\ z + cos\ z = 0$. On squarring, when this happens $sin(2z) = -1$. Lets understand when this happens.

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Hence radius of convergence is $\pi/4$

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Recall that if Ω is a domain in $\mathbb{C}$ and f is a holomorphic function on and inside a simply closed contour γ and z_0 is an interior point of γ, then

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Example 2:

$$\int_{|z|=2} \frac{e^z}{z^2(z-1)} dz$$
$$= \int_{|z|=\epsilon} \frac{e^z/z-1}{z^2} + \int_{|z-1|=\epsilon} \frac{e^z/z^2}{z-1} dz$$
$$= 2\pi i [\tfrac{d}{dz}(\tfrac{e^z}{z-1})]_{z=0} + 2\pi i [\tfrac{e^z}{z^2}]_{z=1}$$
$$= -4\pi i + (2\pi i)e = 2\pi i(e-2)$$

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$$= 0 \text{ - } 2\pi\imath[(3z-4)/(z-(1-\sqrt{5})/2)]_{z=(1+\sqrt{5})/2}$$

$$= (\sqrt{5}-3)\pi\imath$$

Example 4

$$I = \int_{|z|=3} \frac{z^9+1}{z^6-1}$$
$$= \int_{|z|=3} \frac{z^3(z^6-1)+z^3+1}{z^6-1}$$
$$= \int_{|z|=3} z^3 + \int_{|z|=3} \frac{z^3+1}{z^6-1}$$
$$= 0 + \int_{|z|=3} \frac{1}{z^3-1}$$
$=0$ (by an earlier exercise)

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Singularities

Many times, one has a situation where Ω is an open set and f is a holomorphic function on the complement of a certain subset. The points of this subset are called **singularities** of the function. Given the rigid nature of holomorphic functions, we can get a lot of information on the nature of the singularities; essentially by looking at the function in small punctured neighborhoods of those points. Let us see this in more detail.

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A singular point is said to be isolated if the function is holomorphic in a punctured disc around that point.
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A singularity is non-isolated if it is not isolated ! That is, in no punctured neighborhood of the singularity is the function holomorphic.
For example $f(z) = |z|$ has all points as singularities and hence no point is an isolated singularity.

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Note that if an isolated singularity at z_0 is removable, then $\lim\limits_{z \to z_0} f(z)$ exists. The converse is also true and that is the Riemann's Removable Singularity Theorem.

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$$g(z) = \begin{cases} (z - z_0)^2 f(z) & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0. \end{cases}$$

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$$g(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots.$$

Note that $c_0 = g(z_0) = 0$ and $c_1 = g'(z_0) = 0$.

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$$g(z) = c_2(z - z_0)^2 + c_3(z - z_0)^3 + \ldots.$$

If we define $f(z_0) = c_2$, then f is holomorphic throughout. i.e., z_0 is a removable singularity.

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Intuitively a pole is a point at which the function blows up from all directions. An isolated singularity z_0 is said to be a pole if $\lim_{z \to z_0} f(z)$ is ∞ (that is the function takes values outside any bounded set in any small punctured neighborhood of z_0). In this case the function $g(z) = \frac{1}{f(z)}$ is holomorphic at z_0 with $g(0) = 0$. (Why ?).

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A function $f(z)$ defined on an open set except at all the poles is called a **meromorphic function**. An isolated singularity that is neither a pole nor a removable singularity is called an **essentially singularity**. These are the most interesting to understand. Like before we have an important theorem on the values attained by a function near an essential singularity.

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Theorem: If z_0 is an isolated singularity, then it essential if and only if the values of f come arbitrarily close to every complex number in a neighborhood of z_0.

Casorati-Weierstrass Theorem

A function $f(z)$ defined on an open set except at all the poles is called a **meromorphic function**. An isolated singularity that is neither a pole nor a removable singularity is called an **essentially singularity**. These are the most interesting to understand. Like before we have an important theorem on the values attained by a function near an essential singularity.

Theorem: If z_0 is an isolated singularity, then it essential if and only if the values of f come arbitrarily close to every complex number in a neighborhood of z_0.

The if part if obvious. For the only if part, suppose f has an essential singularity. Let a be any complex number. Suppose f does not attain values arbitrarily close to a, then $\lim_{z \to z_0} (z - z_0) \dfrac{1}{(f(z) - a)} = 0$. Hence by Riemann's theorem above, it has a removable singularity at z_0.

Proof cont ..

Depending on whether the singularity can be removed by assigning the value to be zero or a non-zero value, $f(z)$ will have a pole or a removable singularity at z_0 respectively. In either case we have a contradiction.

Proof cont ..

Depending on whether the singularity can be removed by assigning the value to be zero or a non-zero value, $f(z)$ will have a pole or a removable singularity at z_0 respectively. In either case we have a contradiction. For example, the function $e^{1/z}$ has an essential

singularity at 0. (Check !)
Remark: Its important to note that we only talk about the singularity being removable, pole or essential if it is an isolated singularity. There might be small discrepencies in the convention from one reference to another.

Quote

A good mathematical joke is better, and better mathematics, than a dozen mediocre papers. - J E Littlewood.

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(Obviously to be taken in jocular vein !)

So time for a mathematical joke ...

Joke

There was a transatlantic flight and the pilot and copilot dropped dead. A desperate flight attendant asked if anyone knew how to fly a plane. An old polish man said: "Well, I used to fly planes in WW II, but nothing like this". When he brought him into the cockpit, his jaw dropped. There were so many buttons, levers, and fancy dials. "What's wrong?" the flight attendant asked.

"I'm just a simple pole in a complex plane", he responded.

Another (non-mathematical) Polish Joke

A Polish immigrant went to the DMV to apply for a driver's license. First, of course, he had to take an eyesight test. The optician showed him a card with the letters:
'C Z W I X N O S T A C Z.'
“Can you read this?” the optician asked.
“Read it?” the Polish guy replied, “I know the guy.”