Signal Processing - 1 by One

Sibi Raj B. Pillai
Dept of Electrical Engineering
IIT Bombay

Outline

- So Far: Sampling, Convolution, Interpolation
- Previous Week: Fourier Series and Fourier Transform
- Previous Class: Convolution Multiplication Theorem
- Today: Parseval's Relation

Outline

- So Far: Sampling, Convolution, Interpolation
- Previous Week: Fourier Series and Fourier Transform
- Previous Class: Convolution Multiplication Theorem
- Today: Parseval's Relation

Energy and Power

For an analog signal $x(t)$, the energy $||x||_{\ell_2}^2$ is given by

$$||x||_{\ell_2}^2 = \int_{t \in \mathbb{R}} |x(t)|^2 dt.$$

If $||x||_{\ell_2} < \infty$, then we say $x(t) \in \mathcal{L}_2$ (Class of Energy Signals) .

The only periodic signal in $\mathcal{L}_2$ is zero almost everywhere.

The **Power** of a $T-$periodic signal $x_p(t)$ is

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x_p(t)|^2 dt.$$

Express the power/energy in terms of the Fourier coefficients?

Energy and Power

For an analog signal $x(t)$, the energy $||x||_{\ell_2}^2$ is given by

$$||x||_{\ell_2}^2 = \int_{t\in\mathbb{R}} |x(t)|^2 dt.$$

If $||x||_{\ell_2} < \infty$, then we say $x(t) \in \mathcal{L}_2$ (Class of Energy Signals) .

The only periodic signal in $\mathcal{L}_2$ is zero almost everywhere.

The **Power** of a $T-$periodic signal $x_p(t)$ is

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x_p(t)|^2 dt.$$

Express the power/energy in terms of the Fourier coefficients?

Energy and Power

For an analog signal $x(t)$, the energy $||x||_{\ell_2}^2$ is given by

$$||x||_{\ell_2}^2 = \int_{t \in \mathbb{R}} |x(t)|^2 dt.$$

If $||x||_{\ell_2} < \infty$, then we say $x(t) \in \mathcal{L}_2$ (Class of Energy Signals) .

The only periodic signal in $\mathcal{L}_2$ is zero almost everywhere.

The **Power** of a $T-$periodic signal $x_p(t)$ is

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x_p(t)|^2 dt.$$

Express the power/energy in terms of the Fourier coefficients?

Energy and Power

For an analog signal $x(t)$, the energy $||x||_{\ell_2}^2$ is given by

$$||x||_{\ell_2}^2 = \int_{t\in\mathbb{R}} |x(t)|^2 dt.$$

If $||x||_{\ell_2} < \infty$, then we say $x(t) \in \mathcal{L}_2$ (Class of Energy Signals) .

The only periodic signal in $\mathcal{L}_2$ is zero almost everywhere.

The **Power** of a $T-$periodic signal $x_p(t)$ is

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x_p(t)|^2 dt.$$

Express the power/energy in terms of the Fourier coefficients?

Parseval's Relation

Notice that for a vector $\bar{x} = \left(x_1, \cdots, x_N\right)$:

$$||\bar{x}||^2 = |x_1|^2 + \cdots + |x_N|^2.$$

i.e. simply sum the *energy* in each orthogonal axis (Euclidean).

Parseval's Relation

Notice that for a vector $\bar{x} = \left(x_1, \cdots, x_N\right)$:

$$||\bar{x}||^2 = |x_1|^2 + \cdots + |x_N|^2.$$

i.e. simply sum the *energy* in each orthogonal axis (Euclidean).

Parseval's Generalization:

If $x(t) = \sum_{m \in \mathbb{Z}} \beta_m \phi_m(t)$ with $\langle \phi_m(t), \phi_n(t) \rangle = \delta[m-n]$, in $-\frac{T}{2} \leq t \leq \frac{T}{2}$,
then

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{m \in \mathbb{Z}} |\beta_m|^2.$$

Parseval's Relation

Notice that for a vector $\bar{x} = \left(x_1, \cdots, x_N\right)$:

$$||\bar{x}||^2 = |x_1|^2 + \cdots + |x_N|^2.$$

i.e. simply sum the *energy* in each orthogonal axis (Euclidean).

Parseval's Generalization:

If $x(t) = \sum_{m \in \mathbb{Z}} \beta_m \phi_m(t)$ with $\langle \phi_m(t), \phi_n(t) \rangle = \delta[m-n]$, in $-\frac{T}{2} \leq t \leq \frac{T}{2}$,

then

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{m \in \mathbb{Z}} |\beta_m|^2.$$

Proof:

$$LHS = \int_{-\frac{T}{2}}^{\frac{T}{2}} |\sum_m \beta_m \phi_m(t)|^2 dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{m,n} \beta_m \beta_n^* \phi_m(t) \phi_n^*(t) dt$$

$$= \sum_{m,n} \beta_m \beta_n^* \langle \phi_m(t), \phi_n(t) \rangle = \sum_{m,n} \beta_m \beta_n^* \, \delta[m-n].$$

Power in Fourier Series

Theorem

For a T-periodic signal $x(t) = \sum_{m\in\mathbb{Z}} \alpha_m \exp(-j\frac{2\pi}{T}mt)$, *we have*

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{m\in\mathbb{Z}} |\alpha_m|^2.$$

Eg: Rectifier and Ripple

$$V_{rms}^2 = \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos^2(\frac{2\pi}{T}t)dt = |\alpha_0|^2 + \sum_{m\neq 0}|\alpha_m|^2$$

Power in Fourier Series

Theorem

For a T-periodic signal $x(t) = \sum_{m \in \mathbb{Z}} \alpha_m \exp(-j\frac{2\pi}{T}mt)$*, we have*

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{m \in \mathbb{Z}} |\alpha_m|^2.$$

Eg: Rectifier and Ripple

$$V_{rms}^2 = \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos^2(\frac{2\pi}{T}t)dt = |\alpha_0|^2 + \sum_{m \neq 0} |\alpha_m|^2$$

Power in Fourier Series

Theorem

For a T-periodic signal $x(t) = \sum_{m\in\mathbb{Z}} \alpha_m \exp(-j\frac{2\pi}{T}mt)$, *we have*

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{m\in\mathbb{Z}} |\alpha_m|^2.$$

Eg: Rectifier and Ripple

$$\begin{aligned} V_{rms}^2 = \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos^2(\frac{2\pi}{T}t)dt &= |\alpha_0|^2 + \sum_{m\neq 0} |\alpha_m|^2 \\ &= V_{dc}^2 + V_{r-rms}^2. \end{aligned}$$

Power in Fourier Series

Theorem

For a T-periodic signal $x(t) = \sum_{m \in \mathbb{Z}} \alpha_m \exp(-j\frac{2\pi}{T}mt)$*, we have*

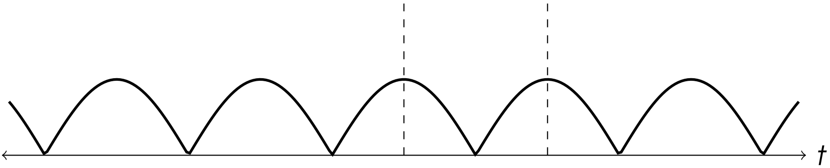
$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{m \in \mathbb{Z}} |\alpha_m|^2.$$

Eg: Rectifier and Ripple

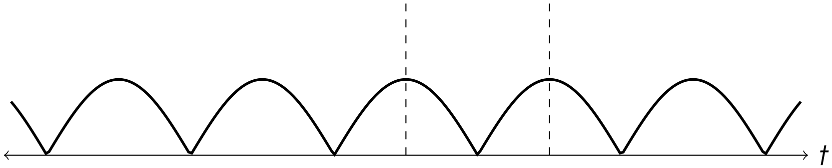
$$V_{rms}^2 = \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos^2(\frac{2\pi}{T}t)\,dt = |\alpha_0|^2 + \sum_{m \neq 0} |\alpha_m|^2$$
$$= V_{dc}^2 + V_{r-rms}^2.$$

$$\alpha_m = \frac{(-1)^{m+1}}{2\pi(m^2 - \frac{1}{4})}.$$

Circuit Analysis

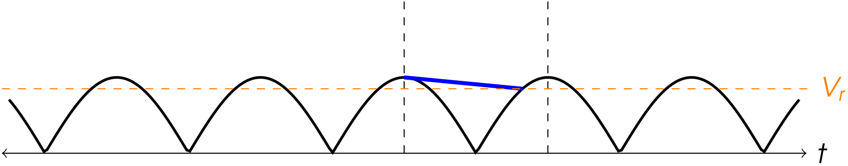


$$\left[\cos(\frac{2\pi}{T}t) \times \mathrm{rect}_{\frac{T}{2}}(t)\right] * \sum_{n} \delta(t - \frac{nT}{2})$$



$$\left[\cos(\frac{2\pi}{T}t) \times \mathrm{rect}_{\frac{T}{2}}(t)\right] * \sum_n \delta(t - \frac{nT}{2}) \overset{F.S.}{\rightleftharpoons} \left[\sum_{k=1,-1} \delta(f - \frac{k}{T}) * \frac{T}{2}\mathrm{sinc}\frac{fT}{2}\right] \sum_m \delta(f - \frac{2m}{T})$$

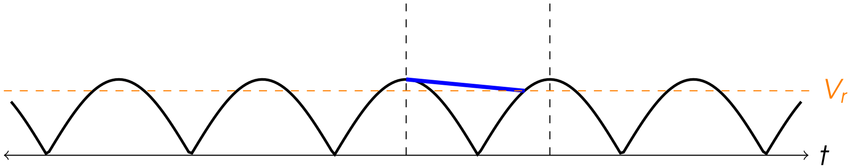
Circuit Analysis



$$\left[\cos(\frac{2\pi}{T}t) \times \mathrm{rect}_{\frac{T}{2}}(t)\right] * \sum_n \delta(t - \frac{nT}{2}) \xrightleftharpoons{F.S.} \left[\sum_{k=1,-1} \delta(f - \frac{k}{T}) * \frac{T}{2}\mathrm{sinc}\frac{fT}{2}\right] \sum_m \delta(f - \frac{2m}{T})$$

$$V_m \exp(-\frac{t}{R_L C}) = V_r \text{ at } t \approx \frac{T}{2}.$$

Circuit Analysis

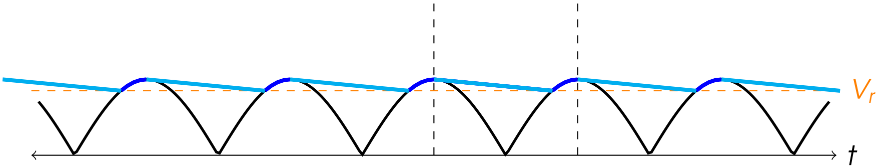


$$\left[\cos\left(\frac{2\pi}{T}t\right) \times \mathrm{rect}_{\frac{T}{2}}(t)\right] * \sum_n \delta\left(t - \frac{nT}{2}\right) \xrightleftharpoons{F.S.} \left[\sum_{k=1,-1} \delta\left(f - \frac{k}{T}\right) * \frac{T}{2}\mathrm{sinc}\frac{fT}{2}\right] \sum_m \delta\left(f - \frac{2m}{T}\right)$$

$$V_m \exp\left(-\frac{t}{R_L C}\right) = V_r \text{ at } t \approx \frac{T}{2}.$$

$$V_m\left(1 - \frac{T}{2R_L C}\right) \approx V_r.$$

Circuit Analysis



$$\left[\cos\left(\frac{2\pi}{T}t\right) \times \mathrm{rect}_{\frac{T}{2}}(t)\right] * \sum_n \delta\left(t - \frac{nT}{2}\right) \xrightleftharpoons{F.S.} \left[\sum_{k=1,-1} \delta\left(f - \frac{k}{T}\right) * \frac{T}{2}\mathrm{sinc}\frac{fT}{2}\right] \sum_m \delta\left(f - \frac{2m}{T}\right)$$

$$V_m \exp\left(-\frac{t}{R_L C}\right) = V_r \text{ at } t \approx \frac{T}{2}.$$

$$V_m\left(1 - \frac{T}{2R_L C}\right) \approx V_r.$$

Linear Vs Non-linear

If the waveform $v_I(t)$ is directly supplied, then at $R_L = \infty$:

$$\beta_m = \alpha_m \frac{1}{1 + j2\pi\frac{2m}{T}RC} = \frac{(-1)^{m+1}}{2\pi(m^2 - \frac{1}{4})} \times \frac{1}{1 + j2\pi\frac{2m}{T}RC}.$$

$$V_{dc} = \beta_0.$$