Solutions To Tutorial Sheet 4

Qns. 1-5, Classification based problems.

Q.1 (i) Hyperbolic, ($b^2 - ac = 1 > 0$), canonical form is $2u_{\xi\eta} + 2u_\xi + 5u_\eta - 2u = 0$

(ii) Parabolic, ($b^2 - ac = 0$)

(iii) elliptic, ($b^2 - ac = -3 < 0$), Characteristic curves are $\xi = (y + 3x) + i(\sqrt{3}x) = \alpha + i\beta$, $\eta = (y + 3x) - i(\sqrt{3}x) = \alpha - i\beta$.

$u_x = 3u_\alpha + \sqrt{3}u_\beta$, $u_y = u_\alpha$,

$u_{xx} = 9u_{\alpha\alpha} + 6\sqrt{3}u_{\alpha\beta} + 3u_{\beta\beta}$, $u_{xy} = 3u_{\alpha\alpha} + \sqrt{3}u_{\alpha\beta}$, $u_{yy} = u_{\alpha\alpha}$.

Substituting in the PDE, we obtain $3u_{\alpha\alpha} + 3u_{\beta\beta} + 12u_\alpha + 4\sqrt{3}u_\beta - u = \sin(\frac{\beta}{\sqrt{3}}(\alpha - \sqrt{3}\beta))$.

Q.2 (i) $b^2 - ac = 15 > 0$, hyperbolic.

(ii) $b^2 - ac = -e^{xy} \cosh x < 0$, elliptic.

(iii) $b^2 - ac = \log(x^2 + y^2 + 1)(2 + \cos x) > 0$, elliptic.

(iv) $b^2 - ac = 10 > 0$, hyperbolic.

(v) $b^2 - ac = 0$, parabolic.

Q.3 Hints given,

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1/2 \\ 0 & -1/2 & 3 \end{pmatrix},$$

Use the transformation $\xi_i = r_i^T x \quad i = 1, 2, \cdots, n$, r_i denoting the orthonormal eigenvector corresponding to the eigenvalue λ_i to obtain the canonical form.

Q.4 Discussed in class.

$x = 0$ - Parabolic, $x < 0$-hyperbolic, $x > 0$-elliptic;

For hyperbolic case, characteristic curves are $\xi = y - (2/3)(-x)^{3/2}$, $\eta = y + (2/3)(-x)^{3/2}$.

Canonical form for hyberbolic case: $u_{\xi\eta} = 0$.

5. (i)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

The eigenvalues of A are 1, 1, -1. One eigenvalue is negative, remaining are positive. Hence, hyperbolic.

(ii)

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

The eigenvalues of A are 2, 2, 0. It has a zero eigenvalue. Hence, parabolic.

(iii)

$$A = \begin{pmatrix} 7 & -5 & -8 \\ -5 & 1 & -11 \\ -8 & -11 & -5 \end{pmatrix},$$

Find eigenvalues.

(iv)

$$A = \begin{pmatrix} 1 & 0 & -x^2 \\ 0 & 1 & 0 \\ -x^2 & 0 & 1 \end{pmatrix},$$

$$det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 0 & -x^2 \\ 0 & 1-\lambda & 0 \\ -x^2 & 0 & 1-\lambda \end{pmatrix},$$

$det(A - \lambda I) = 0 \Longrightarrow (1-\lambda)^3 - x^2(x^2(1-\lambda)) = 0$

$\Longrightarrow (1-\lambda)((1-\lambda)^2 - x^4) = 0 \Longrightarrow \lambda = 1,\ 1+x^2,\ 1-x^2.$

For $|x| = 1$, parabolic; $|x| > 1$, hyperbolic; $|x| < 1$, elliptic.

Qns. 6-10, Uniqueness of solution

6. (i) If possible, let u_1 and u_2 be two distinct solutions of the Dirichlet problem, that is u_i, $(i = 1,\ 2)$ satisfies

$$\begin{aligned} -\Delta u_i &= f \ in\ \Omega \\ u_i &= g \ \ on\ \partial\Omega \end{aligned}$$

Let $w = u_1 - u_2$. Then, w satisfies

$$\begin{aligned} -\Delta\, w &= 0 \text{ in } \Omega \\ w &= 0 \ \text{ on } \partial\Omega. \end{aligned}$$

Claim : $w \equiv 0$. Multiply the first equation above by w and integrate over Ω to obtain

$$\begin{aligned} 0 &= -\int_\Omega w\, \Delta w\, d\Omega \\ &= -\int_{\partial\Omega} w \frac{\partial w}{\partial \nu}\, ds + \int_\Omega |\nabla w|^2\, dx \end{aligned}$$

Now $w = 0$ on $\partial\Omega$.

Hence $\int_\Omega |\nabla w|^2\, dx = 0 \Longrightarrow w$ is a constant in Ω.
$w = 0$ on $\partial\Omega$; w being a continuous function, we deduce $w \equiv 0$ in Ω.

$\Longrightarrow u_1 \equiv u_2$ in Ω.

$\Longrightarrow$ The Dirichlet problem has a unique solution.

(ii) Proceed as in the previous case, here w satisfies

$$\begin{aligned} -\Delta\, w + w = 0 \text{ in } \Omega \\ \partial w/\partial \nu = 0 \ \text{ on } \partial\Omega. \end{aligned}$$

Claim : $\;w \equiv 0$. Multiply the first equation above by w and integrate over Ω to obtain

$$\begin{aligned} 0 &= -\int_\Omega (w\,\Delta w + w^2)\, d\Omega \\ &= -\int_{\partial\Omega} w\frac{\partial w}{\partial \nu}\, ds + \int_\Omega (|w|^2 + |\nabla w|^2)\, dx \end{aligned}$$

Use BC to conclude $w = 0 \Longrightarrow$ The Neumann problem has a unique solution.

(iii) Here w satisfies

$$\begin{aligned} -\Delta\, w = 0 \text{ in } \Omega \\ \alpha w + \partial w/\partial \nu = 0 \ \text{ on } \partial\Omega. \end{aligned}$$

Claim : $\;w \equiv 0$. Multiply the first equation above by w and integrate over Ω to obtain

$$\begin{aligned} 0 &= -\int_\Omega w\,\Delta w\, d\Omega \\ &= -\int_{\partial\Omega} w\frac{\partial w}{\partial \nu}\, ds + \int_\Omega |\nabla w|^2\, dx \\ &\Longrightarrow \int_{\partial\Omega} \alpha |w|^2\, dx + \int_\Omega |\nabla w|^2\, dx = 0 \end{aligned}$$

Conclude $w = 0$, using the condition on α.

7. (i) If possible, let u_1 and u_2 be two distinct solutions of the problem, that is u_i, $(i = 1,\ 2)$ satisfies

$$\begin{aligned} -\nabla \cdot (a(x)\nabla u_i) = f \; in \; \Omega \\ u_i = g \;\; on \; \partial\Omega_1, \; a\partial u_i/\partial\nu = h \quad on \; \partial\Omega_2 \end{aligned}$$

Let $w = u_1 - u_2$. Then, w satisfies

$$\begin{aligned} -\nabla \cdot (a(x)\nabla w) = 0 \text{ in } \Omega \\ w = 0 \ \text{ on } \partial\Omega_1, \; a\partial w/\partial\nu = 0 \quad on \; \partial\Omega_2. \end{aligned}$$

Claim : $\;w \equiv 0$. Multiply the first equation above by w, integrate over Ω and apply divergence theorem to obtain

$$0 = -\int_{\partial\Omega_1} a(x) w \frac{\partial w}{\partial \nu}\, ds - \int_{\partial\Omega_2} a(x) w \frac{\partial w}{\partial \nu}\, ds + \int_\Omega a(x)|\nabla w|^2\, dx$$

Using BC, $\int_\Omega |\nabla w|^2 \, dx = 0 \Longrightarrow w$ is a constant in Ω.
$w = 0$ on $\partial\Omega_1$; w being a continuous function, we deduce $w \equiv 0$ in Ω.
$\Longrightarrow u_1 \equiv u_2$ in Ω.
(ii) Work out seeing hints from lecture notes and the previous question.

(iii) Done in lecture class, see lecture notes.

(iv) $a(x) \geq \alpha_0 > 0$ ASSUMPTION TO BE MADE. If possible, let u_1 and u_2 be two distinct solutions of the problem. Let $v = u_1 - u_2$. Then v satisfies

$$\begin{aligned} & v_{tt} - \nabla \cdot (a\nabla v) = 0 \text{ in } \Omega,\ t \in (0,T] \\ BC: \quad & a\frac{\partial v}{\partial \nu} + \alpha v = 0 \ \text{ on } \partial\Omega,\ t \in [0,T] \\ IC: \quad & v(x,0) = 0,\ v_t(x,0) = 0, \ \ x \in \Omega \end{aligned}$$

Claim : $\ v \equiv 0$.

Multiply the first equation above by v_t, integrate over Ω and apply divergence theorem to obtain

$$\begin{aligned} 0 &= \int_\Omega v_{tt} v_t \, dx - \int_\Omega \nabla \cdot (a\nabla v) v_t \, dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega |v_t|^2 \, dx - \int_{\partial\Omega} a\frac{\partial v}{\partial \nu} v_t \, ds + \int_\Omega a\nabla v \cdot \nabla v_t \, dx \end{aligned}$$

Using BC's,

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{dt}\int_\Omega |v_t|^2 \, dx + \int_{\partial\Omega} \alpha v v_t \, dx + \int_\Omega a\nabla v \cdot \nabla v_t \, dx \\ &= \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |v_t|^2 \, dx + \int_\Omega a|\nabla v|^2 \, dx + \alpha \int_{\partial\Omega} |v|^2 \, ds \Big) \end{aligned}$$

On integrating from 0 to t, we obtain

$$\Big(\int_\Omega |v_t|^2 \, dx + \int_\Omega a|\nabla v|^2 \, dx + \alpha \int_{\partial\Omega} |v|^2 \, ds \Big) = \int_\Omega |v_t(0)|^2 \, dx + \int_\Omega a|\nabla v(0)|^2 \, dx + \alpha \int_{\partial\Omega} |v(0)|^2 \, ds$$

Using IC's, obtain $\int_{\partial\Omega} |v|^2 \, ds = 0,\ \alpha > 0 \Longrightarrow v = 0$ on $\partial\Omega$.

8. Integrate the PDE over Ω, use Gauss's divergence theorem and the BC.

9. Multiply the PDE by u_t and integrate over R, use integration by parts and BC's to obtain $\partial_t(E(t)) = 0$. Integrate from 0 to t to obtain the required result.

10. $a(x) \geq \alpha_0 > 0$ - ASSUMPTION TO BE MADE.

Multiply the PDE by u, integrate over Ω, use Gauss's divergence theorem and BC to obtain

$$\begin{aligned} & \int_\Omega u u_t \, dx + \int_\Omega a(x)|\nabla u|^2 \, dx = 0 \\ & \frac{d}{dt}E(t) + 2\int_\Omega a(x)|\nabla u|^2 \, dx = 0 \end{aligned}$$

Integrating from 0 to t, we obtain the required results, $E(t) \leq E(0)$ and $E'(t) \leq 0$.