MA 207 - Differential Equations-II

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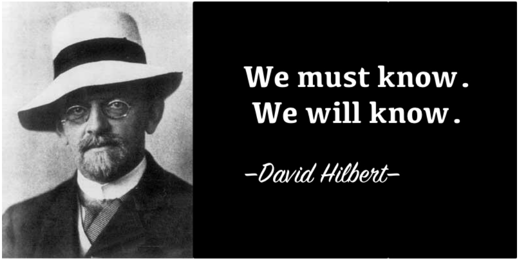
"A wide variety of economic problems lead to differential, difference, and integral equations. Ordinary differential equations appear in models of economic dynamics. Integral equations appear in dynamic programming problems and asset pricing models. Discrete-time dynamic problems lead to difference equations."

by Kenneth Judd, Numerical Methods in Economics (1998) p. 335



Outline of the lecture

- Fourier Series
- Periodic functions
- Convergence result
- Fourier series for functions with period $2L$
- Fourier series for even functions
- Fourier series for odd functions

An example

Consider the eigenvalue problem with periodic boundary conditions:

$$y'' + \lambda y = 0 \quad -\pi < x < \pi,$$
$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$$

NOT A STURM LIUOVILLE BVP!

The eigenfunctions corresponding to the eigenvalues $\lambda_n = n^2$, $n = 0, 1, 2, \cdots$ are $\boxed{\cos nx}$ and $\boxed{\sin nx}$. Given a function f, we discussed the possibility of expanding f in terms of orthonormal eigenfunctions of a Sturm-Liouville BVP.In particular, choosing the eigenvalue problem considered in the above example (which also yields orthonomal eigenfunctions, though not a SL-BVP) now, we discuss the possibility of expanding $f(x)$ in terms of a trigonometric series, say of the form

$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, and the restrictions to be imposed on $f(x)$ so that $\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$.

Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807):

***Any** periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.*

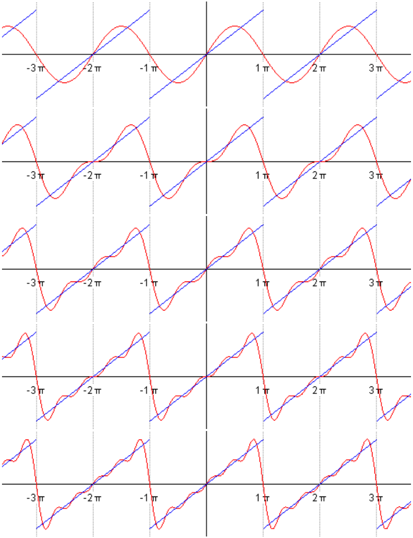
Don't believe it?

- Neither did Lagrange, Laplace, Poisson and other big wigs
- Not translated into English until 1878!

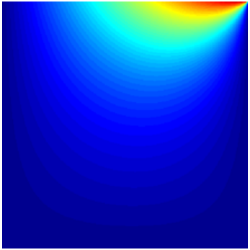
But it's true!

- called Fourier Series

Periodic Identity function (Sawtooth curve-approximations)

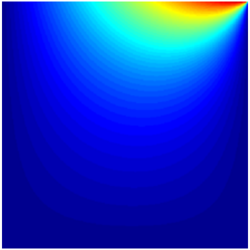


Why?



Consider a metal plate in the shape of a square whose side measures π *meters, with coordinates* $(x, y) \in [0, \pi] \times [0, \pi]$*. If there is no heat source within the plate, and if three of the four sides are held at* 0 *degrees Celsius, while the fourth side, given by* $y = \pi$*, is maintained at the temperature gradient* $T(x, \pi) = x$ *degrees Celsius, for* x *in* $(0, \pi)$ *(initial condition!), then one can show that the stationary heat distribution (or the heat distribution after a long period of time has elapsed) is given by* $T(x, y) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \frac{\sinh ny}{\sinh n\pi}$*. The initial data is used to fit the constants while solving the heat equation using the method of separation of variables.*

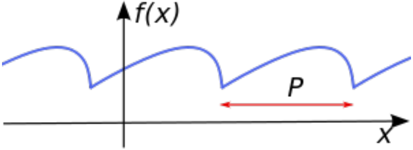
Why?



Consider a metal plate in the shape of a square whose side measures π meters, with coordinates $(x, y) \in [0, \pi] \times [0, \pi]$. If there is no heat source within the plate, and if three of the four sides are held at 0 degrees Celsius, while the fourth side, given by $y = \pi$, is maintained at the temperature gradient $T(x, \pi) = x$ degrees Celsius, for x in $(0, \pi)$ (initial condition!), then one can show that the stationary heat distribution (or the heat distribution after a long period of time has elapsed) is given by $T(x, y) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \frac{\sinh ny}{\sinh n\pi}$. *The initial data is used to fit the constants while solving the heat equation using the method of separation of variables.*

Definition

A function $f(x)$ is periodic if it is defined for all real x and there exists a positive number p such that $f(x+p) = f(x) \quad \forall x$.



Exs. Sine and cosine functions are periodic with period 2π.

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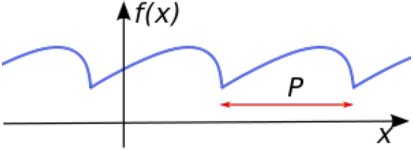


Exs. Sine and cosine functions are periodic with period 2π.

From the definition, $f(x+2p) = f((x+p)+p) = f(x+p) = f(x)$.

That is, for any integer n, $\boxed{f(x+np) = f(x) \quad \forall x}$.

If a periodic function $f(x)$ has a smallest period $p(> 0)$, this is called the fundamental period of $f(x)$.

$f(x) = constant$ has no fundamental period as it is periodic with any period p.

Fourier Series, Fourier Coefficients

CAN we express every periodic function $f(x)$ of period 2π on $[-\pi, \pi]$ as

$$f(x) \approx a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) \qquad (A)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx, \quad a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx, \;\; m = 1, 2, 3, \cdots .$$

and

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx, \;\; m = 1, \; 2, \; 3, \cdots .$$

- a_0, a_m, b_m: called Euler's formulae for the coefficients of the Fourier series.
- The right hand side of (A) is called Fourier series of f.

In addition, assume that $\int_{-\pi}^{\pi} |f| \, dx < \infty$.

How does one guess Euler's formulae

- Determine a_0 :
$$\int_{-\pi}^{\pi} f(x)\, dx = a_0(2\pi) \Longrightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx.$$
- Determine a_m : Multiply both sides of (A) by $\cos mx$ (m being a fixed positive integer) and integrate from $-\pi$ to π to obtain
$$\begin{aligned} a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\, dx \\ &= \int_{-\pi}^{\pi} (a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)) \cos mx\, dx \\ &= \pi \;\; \textit{using orthogonality property and} \\ &\qquad\qquad \textit{assuming term by term integration}. \end{aligned}$$
- Determine b_n : Multiply both sides of (A) by $\sin mx$ and integrate from $-\pi$ to π to obtain in a similar way b_m.

Some Observations:

- Any given periodic and integrable function f on $[-\pi, \pi]$ can be represented as a Fourier series and one can compute its coefficients using Euler's formulae.
- On December 21st, 1807, Joseph Fourier went one step ahead to announce that any arbitrary not necessarily periodic could be expressed as a series of Sine and Cosine functions.

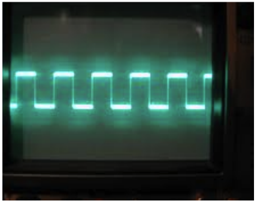
It was a revolutionary idea, which first presented in French Academy of Sciences in the context of his work on heat conduction problem.

Even Lagrange thought thought to be a purely Nonsense!!!

Let us pose a question which bothered Mathematicians for 150 years:

Qn. How can an arbitrary nonperiodic function be expressed as a series of sines and cosines?

Find the Fourier coefficients of the periodic function



$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0; \\ k & \text{if } 0 \le x \le \pi, \end{cases}$$

$f(x + 2\pi) = f(x)$.

Note that $f(x)$ is integrable. $a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\, dx = 0,$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{0} (-k) \cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} k \cos nx \, dx \\ &= 0. \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-k)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} k\sin nx\,dx \\
&= \frac{1}{\pi}\left[\frac{k\cos nx}{n}\bigg|_{-\pi}^{0} - \frac{k\cos nx}{n}\bigg|_{0}^{\pi}\right] \\
&= \frac{k}{\pi n}\Big[1 - \cos n\pi - \cos n\pi + 1\Big] = \frac{2k}{n\pi}(1-(-1)^n).
\end{aligned}
$$

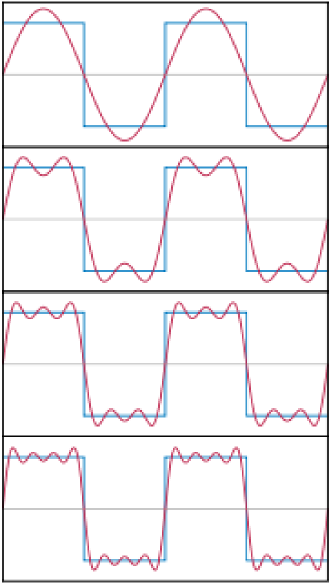
$$
b_n = \begin{cases} \dfrac{4k}{n\pi} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}
$$

$\Longrightarrow$ Fourier series of $f(x)$ is $\boxed{\frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)}$.

At $\boxed{x = 0}$, we see that the partial sums are 0. Also, $\dfrac{f(0^+)+f(0^-)}{2} = \dfrac{k+-k}{2} = 0$. At $\boxed{x = \pi/2}$,

$f(\pi/2) = \dfrac{4k}{\pi}(1 - 1/3 + 1/5 - \cdots) \Longrightarrow 1 - 1/3 + 1/5 - \cdots = \pi/4.$

Square wave - approximations by trigonometric series



Some Possible Questions...

How does one ensure the convergence of the Fourier series to $f(x)$?

More pertinent Questions:

Qn 2. Under what conditions on f, its Fourier series converges?

Qn 3. If so, Does it converge to f?

Before we proceed to provide some possible answers to $Qn2$ and $Qn3$, let us examine the Fourier coefficients.

Assume that f is absolutely integrable, that is, $\int_{-\pi}^{\pi} |f| \, dx < \infty$, then

$$
\begin{aligned}
|a_m| &= \left| \int_{-\pi}^{\pi} f(x) \cos mx \, dx \right| \leq \int_{-\pi}^{\pi} |f(x)| \, |\cos mx| \, dx \\
&\leq \int_{-\pi}^{\pi} |f| \, dx < \infty.
\end{aligned}
$$

Similarly, b_m's are bounded.

The Fourier series converges if the sequence $\{S_N(x)\}_{N=1}^{\infty}$ converges for each x, where $S_N(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$.

Fourier believed that if f is continuous, then its Fourier series would converge, which proved to be false. (counter example by du Bois Reymond in 1873)

Some History—

- 1927: M .Riesz proved that if f is square integrable on $[-\pi, \pi]$, that is, $\int_{-\pi}^{\pi} |f|^2 \, dx < \infty$, then its Fourier series converges in the sense that (called Riesz-Fisher Theorem)

$$\lim_{N \longrightarrow \infty} |f(x) - S_N(x)|^2 \, dx = 0.$$

Finally in 1966, L. Carleson settled the question of pointwise convergence for function f on $[-\pi, \pi]$: For a square integrable function, its Fourier series converges almost everywhere to f. (a part from a countable no. points).

Convergence of Fourier Series - sufficiency

- $f(x)$ is a periodic function with period 2π.
- $f(x)$ and $f'(x)$ piecewise continuous in $[-\pi, \pi]$.

$\Longrightarrow$ The Fourier series $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ with $a_0,\ a_n,\ b_n$ determined by Euler's formula is convergent.
$\Longrightarrow$ Its sum is $f(x)$ except at the point where $f(x)$ is discontinuous, and the sum of the series is the average of the left hand and right hand limits of $f(x)$ at x_0, that is, $\dfrac{f(x_0^-) + f(x_0^+)}{2}$.
(If $f(x)$ is continuous, the convergence is uniform.)

Fourier Series for Functions with Period $2L$

If a function $f(x)$ is defined in $[-L, L]$ with period $2L$ has a Fourier series, then the series is $\boxed{a_0 + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x))}$,

with
the Fourier coefficients defined by the Euler formulas given by :

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,\\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx,\ \ n = 1,\ 2,\ 3, \cdots\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx,\ \ n = 1,\ 2,\ 3, \cdots
\end{aligned}
$$

- Let $f(x) \approx a_0 + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x))$.
- Multiply both sides of the above equation by 1, $\cos(\frac{m\pi x}{L})$, $\sin(\frac{m\pi x}{L})$ and integrate from $-L$ to L.
- Using

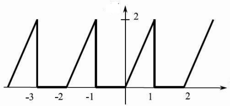
$$\int_{-L}^{L} \cos(\frac{m\pi x}{L}) \cos(\frac{n\pi x}{L})\, dx = \begin{cases} 0 & \text{if } m \neq n; \\ L & \text{if } m = n, \end{cases}$$

$$\int_{-L}^{L} \cos(\frac{m\pi x}{L}) \sin(\frac{n\pi x}{L})\, dx = 0 \text{ for all } \; m = n,$$

$$\int_{-L}^{L} \sin(\frac{m\pi x}{L}) \sin(\frac{n\pi x}{L})\, dx = \begin{cases} 0 & \text{if } m \neq n; \\ L & \text{if } m = n, \end{cases}$$

we obtain the Fourier coefficients for the case when $f(x)$ has period $2L$.

Find the Fourier series of the periodic function $f(x)$ of period $p = 2$, when $f(x) = \begin{cases} 0, & -1 < x < 0; \\ x & 0 < x < 1. \end{cases}$



$L = 1$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_0^1 x\,dx = 1/4.$$

$$a_n = \int_{-1}^{1} f(x)\,\cos n\pi x\,dx = \int_0^1 x\cos n\pi x\,dx =$$

$$x\frac{\sin n\pi x}{n\pi}\bigg|_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi}\,dx = \frac{\cos n\pi x}{n^2\pi^2}\bigg|_0^1 = \frac{(-1)^n - 1}{n^2\pi^2}.$$

$$b_n = \int_{-1}^{1} f(x)\ \sin n\pi x\ dx = \int_0^1 x \sin n\pi x\ dx =$$
$$-x \frac{\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi}\ dx = \frac{(-1)^{n-1}}{n\pi}.$$

Hence, $\forall x \in [-1, 1]$,

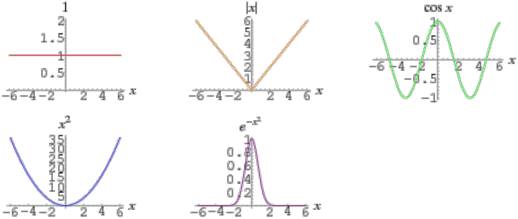
$$f(x) =$$
$$\frac{1}{4} - \frac{2}{\pi^2}(\cos \pi x + \frac{1}{9}\cos 3\pi x + \cdots) + \frac{1}{\pi}(\sin \pi x - \frac{1}{2}\sin 2\pi x + \cdots).$$

Exercise : Find the Fourier series of the periodic function $f(x)$ of period $p = 4$, when

$$f(x) = \begin{cases} -x, & -2 < x < 0; \\ x & 0 < x < 2. \end{cases}$$

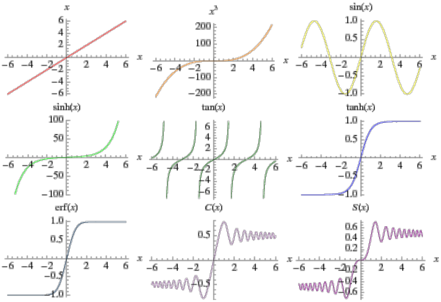
Solution (Check!): $f(x) = 1 - \dfrac{8}{\pi^2} \displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi x/2}{(2n-1)^2}.$

Fourier Series of Even Functions



- $f(x)$ is even if $\boxed{f(-x) = f(x)}$, product of even functions is even, product of even & odd function is odd.
- $a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx = \dfrac{1}{L}\displaystyle\int_{0}^{L} f(x)\,dx$
- $a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos(\dfrac{n\pi x}{L})\,dx = \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\cos(\dfrac{n\pi x}{L})\,dx$
- $b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin(\dfrac{n\pi x}{L})\,dx = 0.$
- Fouries series is $\boxed{f(x) \approx a_0 + \displaystyle\sum_{n=1}^{\infty} a_n \cos(\dfrac{n\pi x}{L})}$.

Fourier Series of Odd Functions

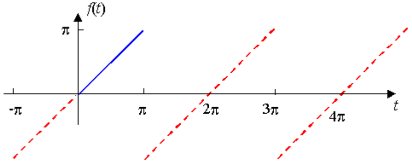


- $f(x)$ is even if $f(-x) = -f(x)$, product of odd functions is even, product of even & odd function is odd.
- $a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx = 0,\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos(\dfrac{n\pi x}{L})\,dx = 0,$
- $b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin(\dfrac{n\pi x}{L})\,dx = \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\sin(\dfrac{n\pi x}{L})\,dx.$
- Fourier series is $f(x) \approx \displaystyle\sum_{n=1}^{\infty} b_n \sin(\dfrac{n\pi x}{L})$.

Half Range sine series

- $f(x)$ is defined only in $[0, L]$, say. Consider the following extensions of $f(x)$.

Case 1 : $f(x)$ defined in $[0, L]$, an odd extension to $[-L, 0]$, and a periodic extension to $\mathbb{R}$.



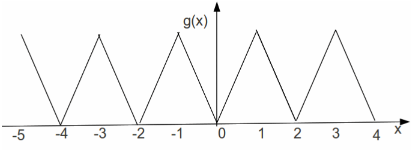
The extended function is an odd function.

The Fourier series is a sine series given by : $\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$;

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L}).$$

The series represents f on $[0, L]$.

Case 2 : $f(x)$ defined in $[0, L]$, an even extension to $[-L, 0]$, and a periodic extension to $\mathbb{R}$.



Here, $g(x)$ is an even function.
The Fourier series is a cosine series.

$$g(x) \approx a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L});$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx; \; a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L}).$$

The series represents $g(x)$ on $[0, L]$.

Under suitable assumptions on $f(x)$ (sufficiency results in the theorem), we have, the Parseval's identity :

$$\boxed{2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\, dx.}$$

Example (Tut. Sheet 15 (iv)): Show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ using the Fourier series for $f(x) = x,\ -\pi < x < \pi,\ f(-\pi) = f(\pi) = 0.$

Hints: Derive the Fourier series for $f(x)$ as $\displaystyle 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$.

Using Parseval's identity,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = 4\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Million Dollar Open Problem

Under Million Dollar Prize Problem by Clay Mathematical Institute, one such problem is Riemann Hypothesis and it states:

"All nontrivial zeros of the Riemann Zeta function:

$$\boxed{\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) \neq 1}$$

have real part $\frac{1}{2}$."

trivial zeros at even negative integers.

Although the modern computer calculations have shown that the first 10 trillion non trivial zeros have real part $1/2$, but it still remains unsolved, that is, it needs a mathematical proof to settle it.

Exercises

Tut. Sheet 3 - Qns. 12-22.
Read Page 30, Hand out 3.
Tut. Sheet 4 - Qns. 1 &2, do the classification.