Signal Processing - 1 by One

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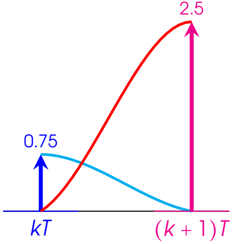
Cubic Interpolation

Let us interpolate in the interval $[0, T]$ between two samples.



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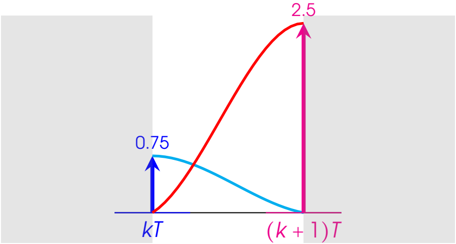

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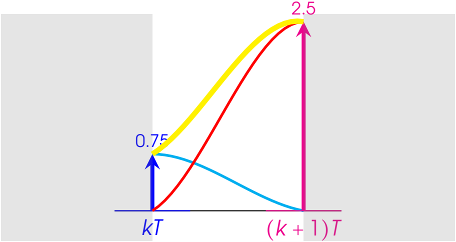

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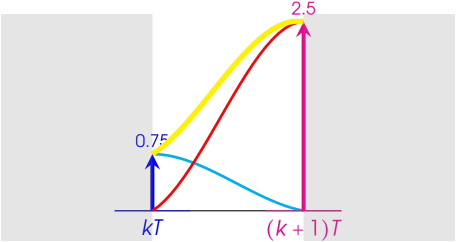

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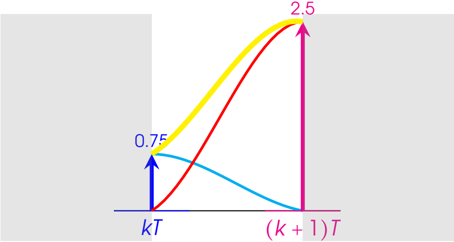

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Eg: For $\delta(t)$ as input, the output in $t \in [0, 1]$

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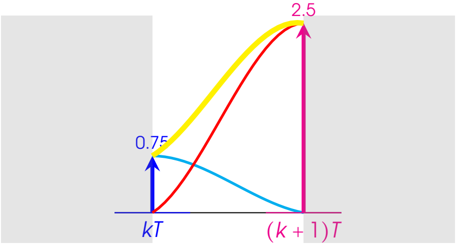

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We need to do slightly more work!

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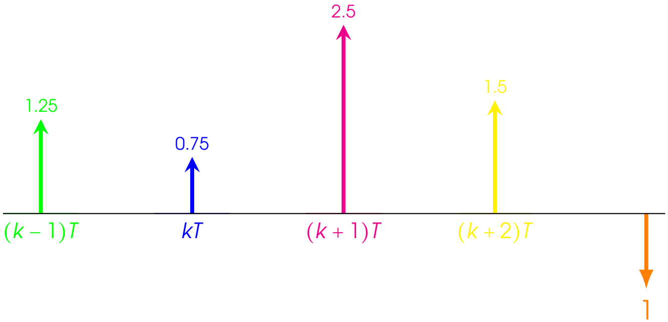

Take $T = 1$ and samples of $f(t)$ at instants $t = 0, \pm 1, \cdots$ are given.

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$f(0) = d$; $f(1) = a + b + c + d$;

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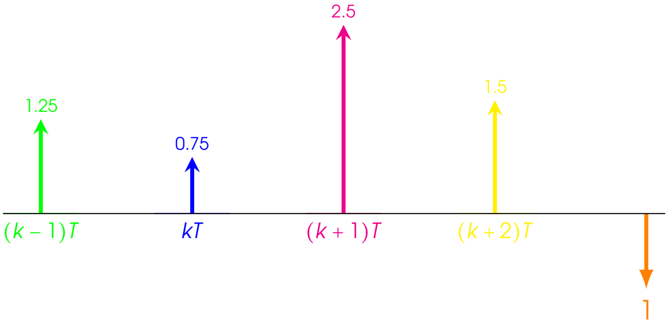

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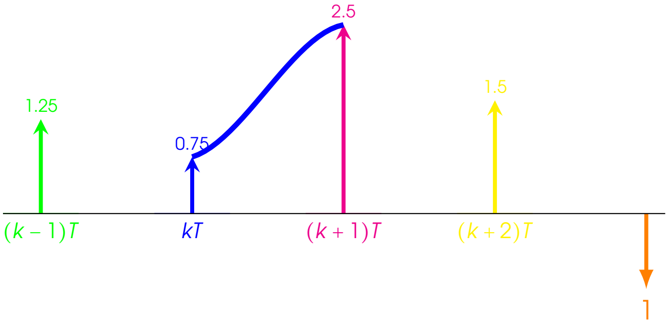

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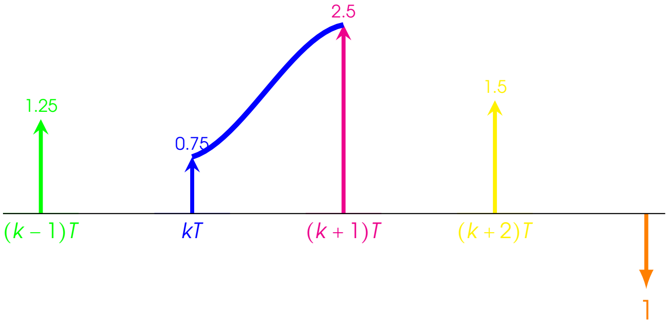

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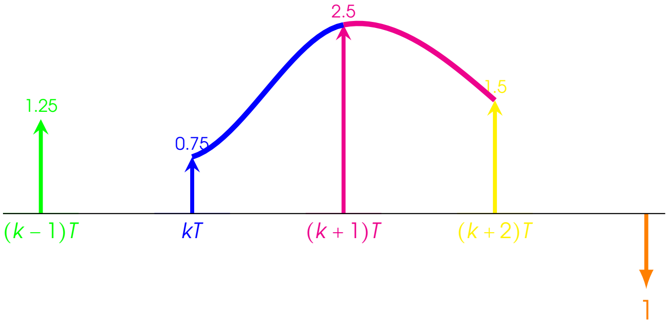

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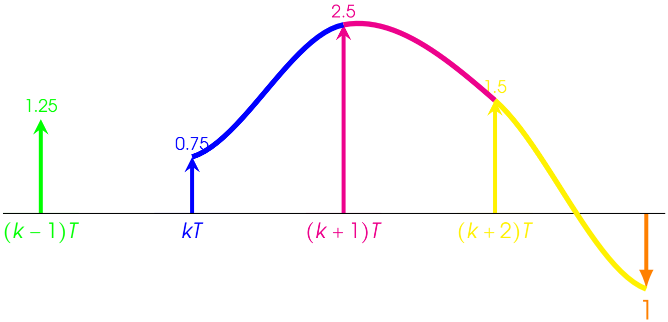

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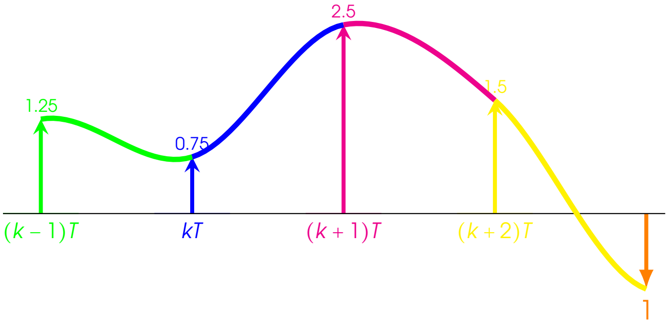

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$$p(\frac{t}{T}) = a\left(\frac{t}{T}\right)^3 + b\left(\frac{t}{T}\right)^2 + c\left(\frac{t}{T}\right) + d.$$

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Popular Choice:

$$f'(0) = \frac{f(1) - f(-1)}{2}; \; f'(1) = \frac{f(2) - f(0)}{2};$$

Superposition and Shift

$$\sum_{n\in\mathbb{Z}} x_n\delta(t-nT) \longrightarrow \boxed{\text{Cubic Interpolator}} \longrightarrow x(t)$$

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Time Shifting: Shifting the input results in a shifted output.

LTI Systems

Linear System: A system such that for any $x_1(t)$ and $x_2(t)$, if

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t) \quad \text{and}$$

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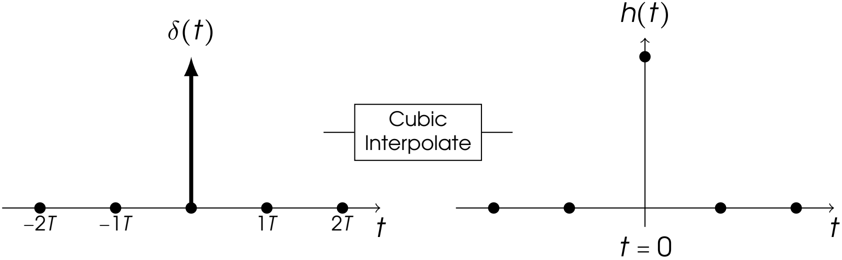
LTI system: knowing the output to an impulse is good enough.

Cubic Interpolation as LTI System



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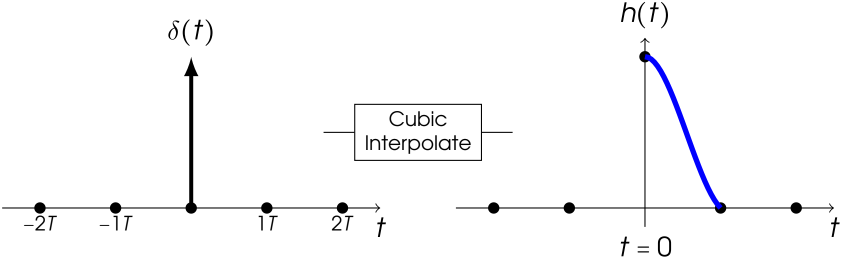

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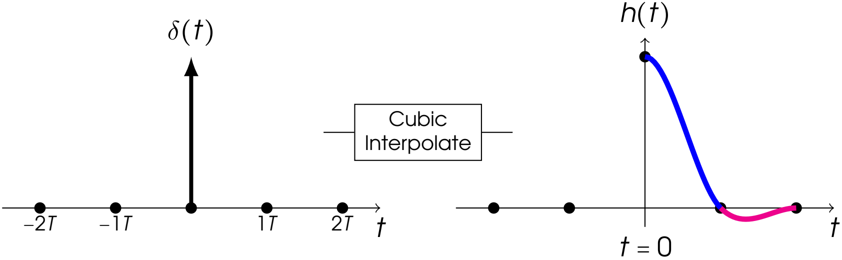

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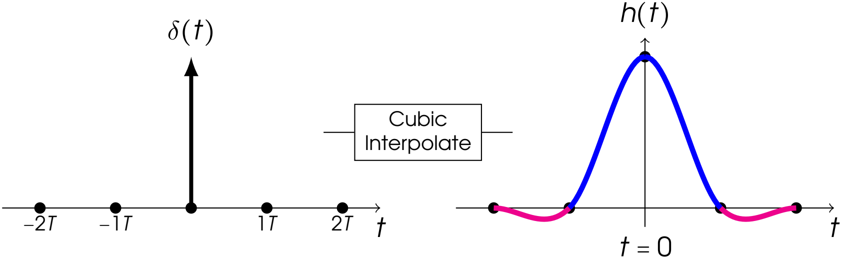

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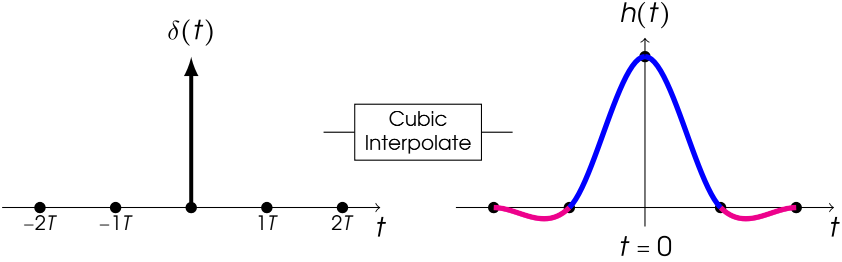

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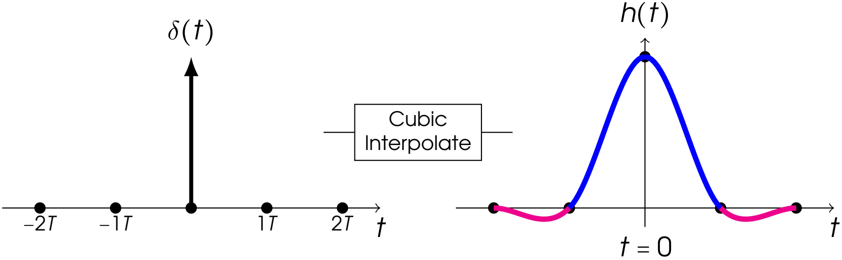

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$$y(t) = \sum_{n \in \mathbb{Z}} x_n \, h(t - nT)$$

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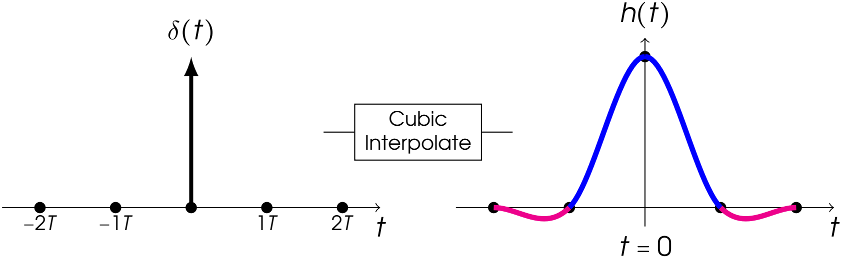

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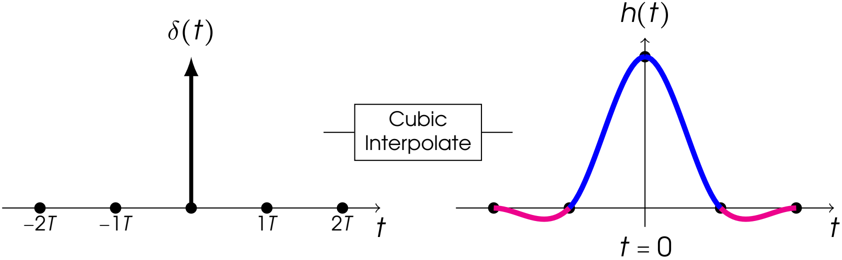

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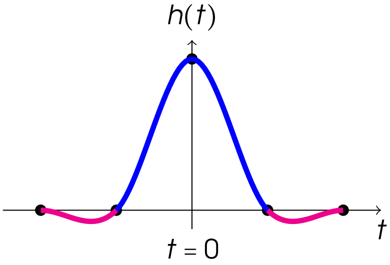
Causality



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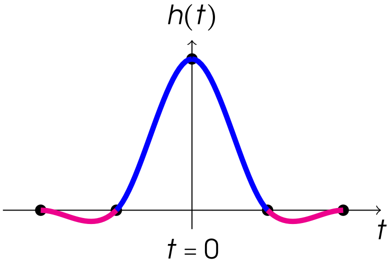

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Practical Implementation:

Use the causal version $h(t - 2T)$ instead $\Rightarrow$ **delay at the output!**.