MA 207 - Differential Equations-II

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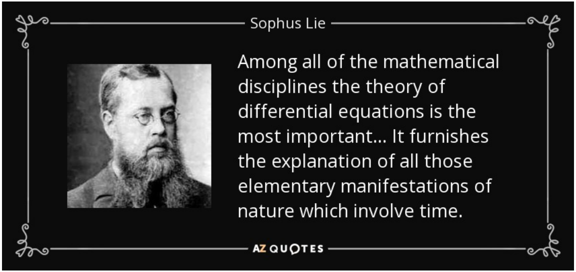
Start with Two Quotations

"Newton has shown us that a law is only a necessary relation between the present state of the world and its immediately subsequent state. All the other laws since discovered are nothing else; they are in sum, differential equations. "

by Henri Poincare



Outline of the lecture

- Recall- Legendre polynomials
- Exercises, Rodrigue's formula.
- Ordinary Point / Singular point
- Regular singular point
- Frobenius method: An outline

RECALL - Legendre Polynomials

Consider the Legendre DE: $\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0,}$ where n is a non-negative integer.

Recall that

$$
\begin{aligned}
y_1(x) &= 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \cdots \\
&\quad \textit{and} \\
y_2(x) &= x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \cdots
\end{aligned}
$$

are linearly independent solutions of the Legendre DE.

If n is a non-negative even integer (resp. an odd integer), the solution $y_1(x)$ (resp. $y_2(x)$) terminates.

That is, we obtain polynomial solutions to Legendre equation.

The polynomial solutions $P_n(x)$ which satisfy $P_n(1) = 1$, for non-negative integers n are called **LEGENDRE POLYNOMIALS**.

Exercise

Assuming that $a_n = 1$ for the case $n = 0$ and

$$a_n = \frac{(2n)!}{2^n (n!)^2}, \tag{1}$$

prove that the polynomial solutions

$$P_n(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots \tag{2}$$

(the sum terminating with a_0 if n is even and $a_1 x$ if n is odd) of the Legendre equation $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ can be expressed as $\boxed{P_n(x) = \sum_{k=0}^{M} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}}$,

where $M = n/2$ or $(n-1)/2$, whichever is an integer.

Proceed to derive the Rodrigue's formula (Q9, Tut. Sheet 1) :

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n}.$$

$$P_n(x) = a_n x^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} + \cdots \tag{3}$$

(the sum terminating with a_0 if n is even and $a_1 x$ if n is odd)
Now from recurrence relation, we have

$$a_{k+2} = \frac{(k-n)(k+n+1)}{(k+2)(k+1)} a_k$$

$$\boxed{a_k = \frac{(k+2)(k+1)}{(k-n)(k+n+1)} a_{k+2}}$$

That is,

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n,$$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = (-1)^2 \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} a_n, \vdots$$

$$\boxed{a_{n-2k} = (-1)^k \frac{n(n-1)(n-2)\cdots(n-2k+1)}{2 \cdot 4 \cdots (2k)(2n-1)(2n-3)\cdots(2n-2k+1)} a_n \ .}$$

$$\boxed{a_{n-2k} = (-1)^k \frac{n(n-1)(n-2)\cdots(n-2k+1)}{2\cdot 4\cdots(2k)(2n-1)(2n-3)\cdots(2n-2k+1)} a_n}\ .$$

$$\text{Numerator} = n(n-1)(n-2)\cdots(n-2k+1) = \frac{n!}{(n-2k)!}$$

$$\begin{aligned}
Dr &= 2\cdot 4\cdots(2k)(2n-1)(2n-3)\cdots(2n-2k+1) \\
&= 2^k\cdot k!\frac{(2n)(2n-1)(2n-2)\cdots(2n-2k+1)(2n-2k)!}{(2n)(2n-2)\cdots(2n-2k+2)\times(2n-2k)!} \\
&= 2^k\cdot k!\frac{(2n)!}{2^k(n)(n-1)\cdots(n-k+1)\times(2n-2k)!} \\
&= k!\frac{(2n)!(n-k)!}{(n!)(2n-2k)!}
\end{aligned}$$

$$\begin{aligned}
a_{n-2k} &= (-1)^k\frac{n!}{(n-2k)!}\times\frac{(n!)(2n-2k)!}{k!\,(2n)!(n-k)!}\times\frac{(2n)!}{2^n(n!)^2} \\
&= (-1)^k\frac{(2n-2k)!}{k!2^n(n-k)!(n-2k)!}
\end{aligned}$$

$$\boxed{a_{n-2k} = (-1)^k \frac{(2n-2k)!}{k!2^n(n-k)!(n-2k)!}}$$

Substituting in

$P_n(x) = a_n x^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} + \cdots$

$$\boxed{P_n(x) = \sum_{k=0}^{M} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}}$$, Is $P_n(x)$ even ??

where $M = n/2$ or $(n-1)/2$, whichever is an integer.

$$\begin{aligned} P_n(x) &= \frac{1}{2^n} \sum_{k=0}^{M} \frac{(-1)^k}{k!(n-k)!} \frac{(2n-2k)!}{(n-2k)!} x^{n-2k} \\ &= \frac{1}{2^n n!} \sum_{k=0}^{M} \frac{(-1)^k}{k!} \frac{n!}{(n-k)!} \frac{d^n}{dx^n}(x^{2n-2k}) \\ &= \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{n!}{(n-k)!} x^{2n-2k} \right) \\ &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n. \ \ (Rodrigue's\ formula) \end{aligned}$$

To prove that $\boxed{P_n(1) = 1}$ (Tut. Sheet 1, 13(i)).

(From Rodrigue's formula, $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}(x^2 - 1)^n$.)

That is, to prove that $\boxed{\phi(1) = n!2^n}$, where $\phi(x) = \dfrac{d^n}{dx^n}(x^2 - 1)^n$.

Note that $\phi(x) = \dfrac{d^n}{dx^n}(x^2 - 1)^n = \dfrac{d^n}{dx^n}\left((x-1)^n (x+1)^n\right)$

Use LEIBNITZ FORMULA given by

$\boxed{(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}}$ to differentiate (n) times to obtain

$\phi(x) = \dfrac{d^n}{dx^n}\left((x-1)^n\right)(x+1)^n +$ *terms involving* $(x-1)$ *as a factor*

(second term onwards go to 0 at $x = 1$).

Now, $\dfrac{d}{dx}(x-1)^n = n(x-1)^{n-1}$, $\dfrac{d^2}{dx^2}(x-1)^n = n(n-1)(x-1)^{n-2}$

$\vdots$

$$\frac{d^n}{dx^n}(x-1)^n = n(n-1)\cdots(n-(n-1))(x-1)^{n-n}$$

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Now, $\dfrac{d}{dx}(x-1)^n = n(x-1)^{n-1}$, $\dfrac{d^2}{dx^2}(x-1)^n = n(n-1)(x-1)^{n-2}$

$\vdots$

$$\frac{d^n}{dx^n}(x-1)^n = n(n-1)\cdots(n-(n-1))(x-1)^{n-n} = n!$$

Hence, $\phi(1) = n!(1+1)^n$

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Now, $\dfrac{d}{dx}(x-1)^n = n(x-1)^{n-1}$, $\dfrac{d^2}{dx^2}(x-1)^n = n(n-1)(x-1)^{n-2}$

$\vdots$

$\dfrac{d^n}{dx^n}(x-1)^n = n(n-1)\cdots(n-(n-1))(x-1)^{n-n} = n!$

Hence, $\phi(1) = n!(1+1)^n = n!2^n$. (Prove $P_n(-1) = (-1)^n$).

Power Series Method doesn't work always! (Tut. sheet 1, Problem 8)

Let us try to obtain power series solution for

$$\boxed{x^2y'' - (1+x)y = 0}$$

Seek a solution $\boxed{y = \sum_{n=0}^{\infty} a_n x^n}$.

$$x^2y'' = \sum_{n=2}^{\infty} (n)(n-1)a_n x^n,$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1}$$

On substituting y, xy, x^2y'' in the DE, we obtain,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \tag{4}$$

Shifting indices so that x in each of the summation has exponent

∞ ∞ ∞

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

The common ranges of the summation is from 1 to ∞; write out the terms in each summation which do not belong to this common range.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - a_0 - a_1 x - \sum_{n=2}^{\infty} a_n x^n - a_0 x - \sum_{n=2}^{\infty} a_{n-1}x^n = 0$$

That is,

$$\sum_{n=2}^{\infty} \Big(n(n-1)a_n - a_n - a_{n-1} \Big) x^n - a_0 - (a_1 + a_0)x = 0$$

For the expression to hold true for all x in the interval of convergence,

$$\begin{aligned} a_0 &= 0, \; a_1 = -a_0 = 0, \\ (n^2 - n - 1)a_n &= a_{n-1}, \;\; n = 0, 1, \cdots \end{aligned}$$

Hence, we obtain $a_0 = a_1 = a_2 = a_3 = a_4 = \cdots = 0.$

Hence we obtain only the trivial solution $\boxed{y=0}$ for the DE.

WHAT WENT WRONG?

The normalized form of the DE $x^2y'' - (1+x)y = 0$ is

$\boxed{y'' + p(x)y' + q(x)y = 0}$, where $p(x) = 0$ and $q(x) = -\dfrac{1+x}{x^2}$.

Note that $q(x)$ is not REAL ANALYTIC at $x = 0$ and hence we could'nt apply the existence of power series solution theorem.

Recall,

Definition

A real function $f(x)$ is called real analytic at the point $x = x_0$ if it can be represented by a power series in powers of $(x - x_0)$ with non-zero radius of convergence $R > 0$.

Can we obtain non trivial solutions to DE's which have coefficients which are not real analytic?

Ordinary Point of a DE

Consider the second order homogeneous linear DE:

$$\boxed{A_0(x)y'' + A_1(x)y' + A_2(x)y = 0}$$

The equivalent normalized form is $\boxed{y'' + p(x)y' + q(x)y = 0}$,

where $p(x) = \dfrac{A_1(x)}{A_0(x)}$, $q(x) = \dfrac{A_2(x)}{A_0(x)}$.

The point x_0 is called an ORDINARY POINT of the DE, if both the functions $p(x)$ and $q(x)$ are analytic at x_0.

If either (or both) of these functions are not analytic at x_0, then x_0 is called a SINGULAR POINT of $A_0(x)y'' + A_1(x)y' + A_2(x)y = 0$.

Examples

Consider $(x-1)y'' + xy' + \frac{1}{x}y = 0$.
The normalized form yields

$$\boxed{y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}y = 0}$$

Here $p(x) = \frac{x}{x-1}$, $q(x) = \frac{1}{x(x-1)}$.
$p(x)$ is not analytic at $x = 1$, $q(x)$ is not analytic at $x = 0$ and $x = 1$.
Hence $x = 0$ and $x = 1$ are SINGULAR POINTS of the DE (even if $p(x)$ is analytic at $x = 0$).
All the other points in $\mathbb{R}$ are ordinary points.

For $\boxed{y'' + xy' + x^2 y = 0}$, $p(x) = x,\ q(x) = x^2$.
Being polynomial functions, $p(x)$ and $q(x)$ are analytic everywhere in $\mathbb{R}$.
Hence all the points in $\mathbb{R}$ are ORDINARY POINTS of the DE.

Consider the Cauchy-Euler equation : $\boxed{y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0.}$

$x = 0$ is a singular point of this equation, but we know that
$y_1(x) = x$ and $y_2(x) = \frac{1}{x}$, $x > 0$ are solutions of this equation.
HOPES OF GETTING NON-TRIVIAL SOLUTION TO A DE WITH SINGULAR POINTS !
MOTIVATION FOR CLASSIFYING SINGULAR POINTS..

REGULAR SINGULAR POINT

Definition

A singular point x_0 of the DE $y'' + p(x)y' + q(x)y = 0$ is called a REGULAR SINGULAR POINT if both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are ANALYTIC at $x = x_0$. If either (or both) of the functions $(x - x_0)p(x)$, $(x - x_0)^2 q(x)$ is NOT ANALYTIC at x_0, then x_0 is called an IRREGULAR SINGULAR POINT of the DE.

Examples:

- Consider $\boxed{x^2(x-2)y'' + 2(x-2)y' + (x+1)y = 0.}$

The normalized form is $\boxed{y'' + \dfrac{2}{x^2(x-2)}y' + \dfrac{x+2}{x^2(x-2)}y = 0.}$

The singular points of the DE are $x = 0, \; x = 2.$

$xp(x) = \dfrac{2}{x(x-2)}$ is NOT ANALYTIC at $x = 0$, though

$x^2q(x) = \dfrac{x+1}{(x-2)^2}$ is analytic at $x = 0$.

Hence, $\boxed{x = 0}$ is an IRREGULAR SINGULAR POINT of the given DE.

Also, $(x-2)p(x) = \dfrac{2}{x^2}$ and $(x-2)^2q(x) = \dfrac{x+1}{x^2}$ are ANALYTIC at $x = 2$.

Hence, $\boxed{x = 2}$ is a REGULAR SINGULAR POINT of the given DE.

The point at infinity is an irregular singular point of the Airy's equation :

$\boxed{y'' - xy = 0}$.

The transformation $x = \dfrac{1}{t}$ yields :

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -t^2 \frac{dy}{dt}, \quad (\text{since } \frac{dt}{dx} = -1/x^2 = -t^2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\Big(-t^2\frac{dy}{dt}\Big) = \frac{d}{dt}\Big(-t^2\frac{dy}{dt}\Big)(-t^2) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}$$

Hence, $t^5 \dfrac{d^2y}{dt^2} + 2t^4 \dfrac{dy}{dt} - y = 0.$

The normalized form of this equation is $\dfrac{d^2y}{dt^2} + \dfrac{2}{t}\dfrac{dy}{dt} - \dfrac{1}{t^5}y = 0$

$t = 0$ is an IRREGULAR SINGULAR POINT of this DE as $tp(t) = t \times \dfrac{2}{t}$ and $t^2q(t) = t^2 \times -\dfrac{1}{t^5}$, $t^2q(t)$ is NOT ANALYTIC at $t = 0$.

Hence, the point at infinity is an IRREGULAR SINGULAR POINT of the Airy's equation.

Let $x = x_0$ be a Regular Singular Point of the DE:
$\boxed{A_0(x)y'' + A_1(x)y' + A_2(x)y = 0}$.
Then, the DE has at least one Non-Trivial Solution of the form

$$\boxed{|x - x_0|^r \sum_{n=0}^{\infty} a_n (x - x_0)^n, \ a_0 \neq 0,}$$

where r is a constant which may be determined and this solution is valid in $0 < |x - x_0| < R$, $(R > 0)$.

Example : For the DE $y'' + \dfrac{b(x)}{x} y' + \dfrac{c(x)}{x^2} y = 0$, where $x = 0$ is a regular singular point, that is, $xp(x) = b(x)$ and $x^2 q(x) = c(x)$ are real analytic at $x = 0$, there exists at least one solution that can be represented by $y(x) = |x|^r \displaystyle\sum_{n=0}^{\infty} a_n x^n, \ a_0 \neq 0$, where r may be determined and this solution is valid in $0 < |x| < R, \ R > 0$.

Frobenius

Ferdinand Georg Frobenius (1849-1917) was a German mathematician, best known for his contributions to the theory of differential equations and to group theory. He also gave the first full proof for the Cayley–Hamilton theorem.

Let us outline the Frobenius method for

$$\boxed{x^2y'' + xb(x)y' + c(x)y = 0,}\ x > 0 \tag{5}$$

where $b(x)$ and $c(x)$ are real-analytic at $x = 0$; that is,

$$b(x) = \sum_{n=0}^{\infty} b_n x^n,\ c(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Note that the solution $y(x)$ of (5) can be written as :

$$\boxed{y(x) = x^r \sum_{n=0}^{\infty} a_n x^n},\ (r \text{ is to be determined})$$

$$\boxed{y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}},$$

$$\boxed{y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}},$$

On substituting in (5), we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}$$
$$+(b_0+b_1x+\cdots+b_nx^n+\cdots)\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}$$
$$+(c_0+c_1x+\cdots+c_nx^n+\cdots)\sum_{n=0}^{\infty}a_nx^{n+r}=0 \qquad (6)$$

For the above expression to hold true for all x in the interval of convergence, we equate the coefficients of x^r, x^{r+1}, $x^{r+2}, \cdots$ to ZERO.

Equating the coefficient of x^r to 0, we obtain a **quadratic equation** in r, called the INDICIAL EQUATION of the DE.

For (6), we obtain the coefficient of x^r as
$(r(r-1)+b_0 r+c_0)a_0=0$.
Since $a_0 \neq 0$, we obtain the $\boxed{\text{INDICIAL EQUATION}}$ as

$$r(r-1)+b_0 r+c_0=0. \tag{7}$$

This is a quadratic equation in r.
Depending on the roots, three cases arise:

$$\text{Roots are} \begin{cases} \text{CASE 1 : distinct, but not differing by an integer;} \\ \text{CASE 2: equal,} \\ \text{CASE 3: distinct differing by an integer,} \end{cases}$$

CASE 1: Roots of the indicial equation are distinct, but do not differ by an integer.

Let the distinct roots be r_1 and r_2.
Two linearly independent solutions $y_1(x)$ and $y_2(x)$ are given by

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \ a_0 \neq 0$$

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} a_n^* x^n, \ a_0^* \neq 0$$

We obtain the coefficients a_n (resp. a_n^*) using recurrence relations.
The solutions $y_1(x)$ and $y_2(x)$ are linearly independent solutions to the DE.

CASE 2: Roots of the indicial equation equal, that is, they do differ by the integer '0'.

The roots are say, $\boxed{r = r_1 = r_2 = \dfrac{1 - b_0}{2}}$.

Now $y_1(x)$ is given by $y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n,\ a_0 \neq 0$

The second linearly independent solution will have the form

$$y_2(x) = x^{r+1} \sum_{n=0}^{\infty} a_n^* x^n + y_1(x) \ln|x|,\ a_0^* \neq 0.$$

(Use method of reduction of order to obtain the second solution,

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\, dx \ )$$

The solutions $y_1(x)$ and $y_2(x)$ are linearly independent solutions to the DE.

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \; a_0 \neq 0$$

The second linearly independent solution can be obtained by say, the method of reduction of order, as

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} a_n^* x^n + C y_1(x) \ln|x|, \; a_0^* \neq 0,$$

C could be zero or non-zero .

Self Study. Example 3 of Section 4.4 (pages:215-216) of the text book : E. Kreyszig.

Exercises

Tutorial Sheet 2:

- Qn. 1 (locate singular points, classify them)
- Qn.2
- Qn.3
- Qn. 4 - Airy's done discussed in class, work out for hypergeometric equation.