Hw-1) Clearly, For $N<k$, Probability is $\boxed{0}$.

For $k \le N$:
Let E_i be the event that i^{th} instructor takes at least one class in first N classes

To find: $\left|\bigcap_{i=1}^{k} E_i\right|$

Using De-morgan's law on principle of inclusion exclusion:

$$\left|\bigcap_{i=1}^{k} E_i\right| = |S| - \sum_{i=1}^{k} |\bar{E_i}| + \sum_{1\le i<j\le k} |\bar{E_i} \cap \bar{E_j}| + \cdots (-1)^k |\bar{E}_1 \cap \cdots \bar{E}_k|$$

where S is union of all possible events $\Rightarrow |S| = 1$.

For some $m \le k$, $i_1, i_2 \ldots i_m$ such that $1 \le i_1 < i_2 \cdots < i_m \le k$

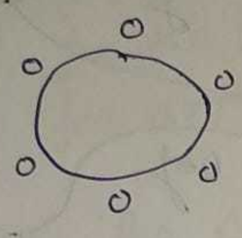
$|\bar{E}_{i_1} \cap \bar{E}_{i_2} \cdots \cap \bar{E}_{i_m}|$ means the probability that none of the $i_1, i_2 \ldots i_m$ instructors took at least one class.
Probality for this = $\left(\frac{k-m}{k}\right)^N$

No. of ways to choose $i_1, i_2 \ldots i_m$ = ${}^kC_m$

$$\therefore \boxed{\left|\bigcap_{i=1}^{k} E_i\right| = 1 - {}^kC_1\left(\frac{k-1}{k}\right)^N + {}^kC_2\left(\frac{k-2}{k}\right)^N + \cdots + (-1)^{k-1} \cdot {}^kC_{k-1}\left(\frac{1}{k}\right)^N + 0}$$

For $k=4$, this is P_N =: $\boxed{1 - 4\cdot\left(\frac{3}{4}\right)^N + 6\cdot\left(\frac{1}{2}\right)^N - 4\cdot\left(\frac{1}{4}\right)^N}$

HW, Q2

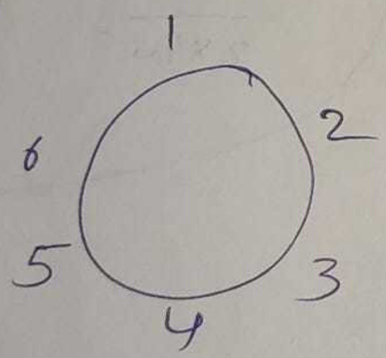


In circular arrangement, total ways of arranging 6 numbers are:

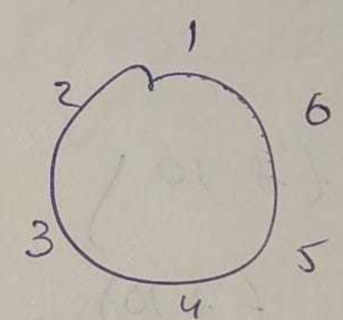
$$|\underline{6-1} = |\underline{5}$$

(As anticlockwise and clockwise are different)

Favourable ways:



and



$= 2$

$$\boxed{P = \frac{2}{|\underline{5}}}$$

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③ For A to win the game, his value should always be greater than or equal to B's.

We find probability of A's winning for each value which B obtains

(I) $B = 1$ $\quad A = 1|2|3|4|5|6$

$P(B) = \frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6}$, $P(A) = 1$

gets one in 1^{st} throw

and again gets one

$\rightarrow P(A \cap B) = \frac{1}{36}$

(II) $B = 2$ $\quad A = 2|3|4|5|6$

$P(B) = \frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6}$, $P(A) = \frac{5}{6}$

$\rightarrow P(A \cap B)$

$\left(\frac{5}{6} \times \frac{7}{36}\right)$

gets in 1^{st} try

gets 1 in 1^{st} try & 2 in 2^{nd} try

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Similiarly doing for all six cases, we get

$$P(E) = \frac{1}{36} + \left(\frac{1}{6} + \frac{1}{36}\right)\left(\frac{5}{6} + \frac{4}{6} + \frac{3}{6} + \frac{2}{6} + \frac{1}{6}\right)$$

$$= \frac{1}{36} \times \frac{21}{6} + \frac{1}{6} \times \frac{15}{6}$$

$$P(E) = \frac{37}{72}$$

Where E is the event of A's winning

H4) R brown balls & B Black balls.

$\therefore$

$$Pr(A_K) = \frac{B}{B+R} \times \frac{B-1}{B+R-1} \times \cdots \times \frac{R}{B+R-K+1} = p_K$$

Now, we want to find the limit as $B+R \to \infty$.

Given $\alpha = \dfrac{R}{B+R}$

$$\Rightarrow B+R = R/\alpha \quad \& \quad B = \frac{(1-\alpha) \times R}{\alpha}$$

Let $t_i := \dfrac{B-i}{B+R-i}$; $i \in \{0, 1, 2, \cdots, K-2\}$

$$\& \, T := \frac{R}{B+R-K+1}$$

$$\Rightarrow t_i = \frac{\frac{1-\alpha}{\alpha} R - i}{R/\alpha - i} \quad \& \, T = \frac{1}{(1/\alpha) - \frac{K-1}{R}}$$

$$\therefore \lim_{R \to \infty} t_i = 1-\alpha \quad \& \quad \lim_{R \to \infty} T = \alpha$$

$$\Rightarrow p_K = \alpha (1-\alpha)^{K-1}$$

HW-5) For k : $1 \leq k \leq n$

U_k be the event of picking k^{th} urn out of n urns randomly

$P(U_k) = \frac{1}{n}$. Also : $i \neq j$: $U_i \cap U_j = \phi$

kth urn has : $k-1$ brown ; $n-k$ black, $n-1$ total balls.

E_1 : Event of picking a black ball ~~and~~ twice.

E_2 : Event of picking a brown ball, then a black ball.

Note : $E_1 \cup E_2$ $\Rightarrow$ ~~Picking second ball black~~ second picked ball is black.

Note : $E_1 \cap E_2 = \phi$

$$P(U_k \cap E_1) = \frac{1}{n} \cdot \frac{(n-k)}{(n-1)} \cdot \frac{(n-k-1)}{(n-2)}$$

$$P(U_k \cap E_2) = \frac{1}{n} \cdot \frac{(k-1)}{(n-1)} \cdot \frac{(n-k)}{(n-2)}$$

$$P(U_k \cap (E_1 \cup E_2)) = P(U_k \cap E_1) + P(U_k \cap E_2)$$
$$= \frac{n-k}{n \cdot (n-1)}$$

To find: $P(E_1 \cup E_2) = P((U_1 \cup U_2 \cup \cdots U_n) \cap (E_1 \cup E_2))$

$$= P\left(\bigcup_{i=1}^{n} U_i \cap (E_1 \cup E_2)\right)$$

$$= \sum_{i=1}^{n} P(U_i \cap (E_1 \cup E_2))$$

$$= \sum_{i=1}^{n} \frac{n-i}{n \cdot (n-1)} = \boxed{\frac{1}{2}}$$

Now total probability of first ball being black:

$$P(\text{first black}) = \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{(n-i)}{n-1} = \frac{1}{2}$$

Probability of both balls being black:

$$P(E_1) = \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{(n-i)}{(n-1)} \cdot \frac{(n-i-1)}{(n-2)} = \frac{1}{3}$$

~~P(E₁| first black~~

$$P(E_1 \cup E_2 \mid \text{first black}) = \frac{P(\text{first black} \cap (E_1 \cup E_2))}{P(\text{first black})}$$

$$= \frac{P(E_1)}{P(\text{first black})}$$

$$= \boxed{\frac{2}{3}}$$

Q6) $P(G|T) = \frac{P(G \cap T)}{P(T)} = \frac{P(G \cap T)}{P(G)} \cdot \frac{P(G)}{P(T)}$

$= \frac{P(T \cap G)}{P(G)} \cdot \frac{P(G)}{P(T)} = P(T|G) \cdot \frac{P(G)}{P(T)} \quad \therefore \frac{P(G)}{P(T)} = \frac{P(G|T)}{P(T|G)}$

$\therefore \quad P(G|T) = P(T|G)$

$\iff \frac{P(G|T)}{P(T|G)} = 1 \iff \frac{P(G)}{P(T)} = 1 \iff P(G) = P(T)$

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(7) Let E be the event that atleast one of the vertices of the inscribed cube is white

i.e. $P(E) = P(V_1 \cup V_2 \ldots V_8) \rightarrow (1)$

where $P(V_i)$ denotes the probability of i^{th} white vertex

& $P(V_i) = 0.1 \ \forall i \in [1,8]$ (Given)

We know that (1) satisfies the following inequality

$\rightarrow P(E) \leq P(V_1) + P(V_2) \ldots P(V_8)$

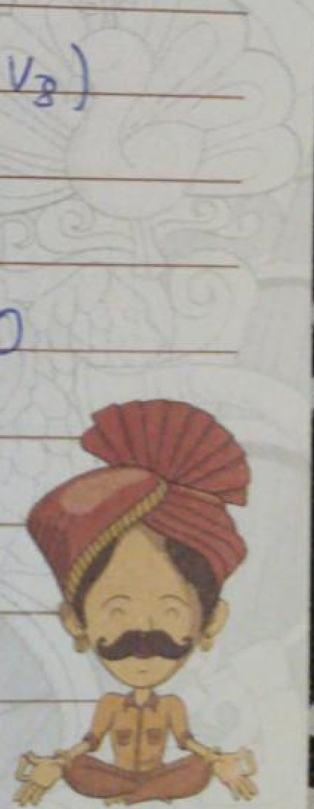
$\leq 0.1 + 0.1 \ldots 0.1$

$\leq 0.8 < 1$

$\therefore P(\bar{E}) = 1 - P(E) \geq 0.2 > 0$

where $\bar{E}$ is the event that all vertices are black

Hence, proved.



Q5) <u>Chapter 2</u>

$\boxed{\text{Note: } P(A \cdot B) = P(A \cap B)}$

To Prove: $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cdot B) - P(B \cdot C) - P(C \cdot A) + P(ABC)$

We know that $P(A \cup B) = P(A) + P(B) - P(A \cdot B)$ — (I)

Using (I), we get:

$$P(A \cup B \cup C) = P(A) + P(B \cup C) - P(A \cdot (B \cup C)) \quad \text{— (II)}$$

$$P(B \cup C) = P(B) + P(C) - P(BC) \quad \text{— (III)}$$

$$P(A \cdot (B \cup C)) = P(AB) + P(AC) - P(A \cdot BC) \quad \text{— (IV)}$$

$$\left[\begin{aligned} \text{As } (A \cap B) \cap (A \cap C) &= A \cap B \cap C \\ (A \cdot B) \cap (A \cdot C) &= A \cdot B \cdot C \end{aligned}\right]$$

Using II, III & IV, we get:

$$\boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(A \cap B \cap C)}$$

Using II, III, IV, we can expand any of A, B or C into a n-term result.

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \dots \cup A_n) = {} & P(A_1) + P(A_2) \dots + P(A_n) \\ & - P(A_1 A_2) - P(A_2 A_3) \cdots \\ & + P(A_1 A_2 A_3) + P(A_2 A_3 A_4) \dots + P(A_{n-2} A_{n-1} A_n) \\ & \vdots \\ & \pm P(A_1 A_2 A_3 \dots A_n) \end{aligned}$$

2-10) We use induction to prove this.

Since we know,

$P(A_2|A_1) \cdot P(A_1) = P(A_2 A_1)$

So, this is true for $n=2$

Let it be true for any n,

Then, $P(A_{n+1} A_n \ldots A_1)$

$= P(\underbrace{A_{n+1}|A_n \ldots A_1}_{1}) \cdot \underbrace{P(A_n \ldots A_1)}_{2}$

Since (2) $= P(A_n|A_{n-1} \ldots A_1) \ldots P(A_2|A_1)P(A_1)$

which we had assumed,

Therefore, our assumption is correct

$P(A_n \ldots A_1) = P(A_n|A_{n-1} \ldots A_1) \ldots P(A_2|A_1)P(A_1)$

holds true.

Hence, proved.

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2-14) If A and B are mutually exclusive, it follows that

$$P(A \cap B) = \phi \rightarrow 1$$

If A and B are independent

$$\rightarrow \cancel{P(A} \; P(A \cap B) = P(A) \cdot P(B) \rightarrow 2$$

From 1 and 2, it follows that - they can be mutually exclusive if and only if $P(A) \cdot P(B) = 0$

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2-16)

a) For $m<k$: Clearly, probability is 0.

For $m \geq k$:

Let E be the event of 'm' being largest among 'k' drawn numbers.
Exactly 1 drawn ball should be 'm'. Rest should be $\leq m-1$. In other words, rest 'k-1' should be picked from balls numbered '1' to 'm-1'
No. of ways to choose 'k-1' out of 'm-1' balls is ${}^{m-1}C_{k-1}$.

No. of ways to pick k balls = ${}^{n}C_{k} \cdot (k!)$

No. of ways to achive E = ${}^{m-1}C_{k-1} \, (k!)$

(Here, we multiply k! because balls are picked successively)

$$\text{Probability of } E = \frac{{}^{m-1}C_{k-1} \cdot (k!)}{{}^{n}C_{k} \cdot (k!)} = \boxed{\frac{{}^{m-1}C_{k-1}}{{}^{n}C_{k}}}$$

b) For the largest number to be within 'm', all 'k' numbers must be picked from balls numbered '1' to 'm'.
No. of ways to pick 'k' balls from 'm' balls = ${}^{m}C_{k} \cdot (k!)$
Probability of largest no. $\leq k$ is $= \dfrac{{}^{m}C_{k} \cdot (k!)}{{}^{n}C_{k} \cdot (k!)} = \boxed{\dfrac{{}^{m}C_{k}}{{}^{n}C_{k}}}$

17) Let E_1 be the event of largest number being drawn out of 'k' boxes be ~~$\leq$~~ $\leq m$
Let E_2 be the event of largest number being drawn out of 'k' boxes be $\leq m-1$.

No. of ways to pick k balls all numbered $\leq m$:
$= m^k$ $\rightarrow$ corresponds to E_1
No. of ways to pick k balls all numbered $\leq m-1$:
$= (m-1)^k$ $\rightarrow$ corresponds to E_2

$E_1 \cap E_2 \Rightarrow$ All cases where largest number is $\leq m-1$
$\therefore$ $E_1 - E_2 \Rightarrow$ All cases where largest number is m.

As $E_2 \subset E_1$, $E_1 - E_2 = E_1 - (E_1 \cap E_2)$

Total no. of ways to pick k balls $= n^k$

Reqd. probability $= P(E_1 - E_2) = \boxed{\dfrac{m^k - (m-1)^k}{n^k}}$

2-19) m white, n black, k drawn

$$P(\text{all black}) = \frac{n}{m+n} \cdot \frac{n-1}{m+n-1} \cdots \frac{n-k+1}{m+n-k+1}$$

$$= \frac{n!\,/\,(n-k)!}{(m+n)!\,/\,(m+n-k)!} = \frac{{}^{n}C_k}{{}^{m+n}C_k}$$

$$\therefore P(\text{atleast one white}) = 1 - P(\text{all black})$$

$$= 1 - \frac{{}^{n}C_k}{{}^{m+n}C_k}$$

2-24) Box 1: 1000 bulbs, 10% = 100 defective.
Box 2: 2000 bulbs, 5% = 100 defective.

a) P(defective, defective | Box 1) $= \frac{100}{1000} \cdot \frac{99}{999}$

P(defective, defective | Box 2) $= \frac{100}{2000} \cdot \frac{99}{1999}$

P(defective, defective) = P(Box 1) · P(d,d | Box 1) + P(Box 2) · P(d,d | Box 2)

$$= \frac{1}{2} \cdot \left(\frac{99}{9990}\right) + \frac{1}{2}\left(\frac{99}{20 \times 1999}\right)$$

$$= 4.955 \times 10^{-3} + 1.238 \times 10^{-3}$$

∴ P(defective, defective) = $\boxed{6.193 \times 10^{-3}}$

b) $P(\text{Box 1} \mid d,d) = \frac{P(d,d \mid \text{Box 1}) \cdot P(\text{Box 1})}{P(d,d)}$

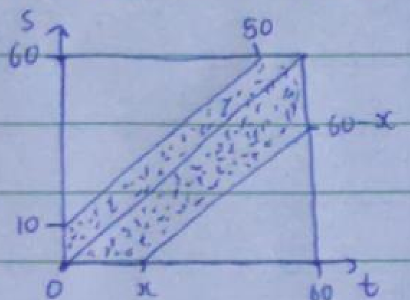
$$= \frac{\frac{1}{2} \times {}^{99}/_{9990}}{6.193 \times 10^{-3}} = \frac{4.955}{6.193} = \boxed{0.8}$$

2-25) Train stops for 10 minutes, bus for x minutes.
Let train arrive at t minutes after 9 am.
Let bus arrive at s minutes after 9 am.

Lets look at limiting cases:

$t = 0. \quad \therefore \quad s \in [0, 10]$ for meeting.
$s = 0. \quad \therefore \quad t \in [0, x]$ " "
$t = 60. \quad \therefore \quad s \in [60-x, 60]$ " "
$s = 60. \quad \therefore \quad t \in [50, 60]$ " ".

We plot these on the t vs s graph. The shaded region indicates that the vehicles meet.



$\therefore$ area of shaded region

$$= \left(\frac{60 \cdot 60}{2} - \frac{(60-10)(50)}{2}\right) + \left(\frac{60 \cdot 60}{2} - \frac{(60-x)(60-x)}{2}\right)$$

$$\therefore \frac{60 \cdot 60 - \frac{(50)^2 + (60-x)^2}{2}}{60 \cdot 60} = \frac{1}{2}$$

$\therefore 60 \cdot 60 = 50^2 + (60-x)^2$

$\therefore (60-x)^2 = 1100$

$\therefore 60 - x = 10\sqrt{11}$

$\therefore \boxed{x = 60 - 10\sqrt{11}}$

2-27) $P(\text{2 heads} \mid \text{coin is fair}) = 1/4$

$P(\text{2 heads} \mid \text{coin is unfair}) = 1.$

$\therefore P(\text{2 heads}) = P(\text{fair}) \cdot P(\text{2 heads} \mid \text{fair}) + P(\text{unfair}) \cdot P(\text{2 heads} \mid \text{unfair})$

$$= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 1$$

$$= 1/8 + 1/2$$

$$= 5/8.$$

$$P(\text{fair} \mid \text{2 heads}) = \frac{P(\text{2 heads} \mid \text{fair}) \cdot P(\text{fair})}{P(\text{2 heads})}$$

$$= \frac{1/4 \cdot 1/2}{5/8} = \frac{1/8}{5/8}$$

$$= \frac{1}{5}$$

$$\therefore \boxed{\text{Ans} = \frac{1}{5} = 0.2}$$