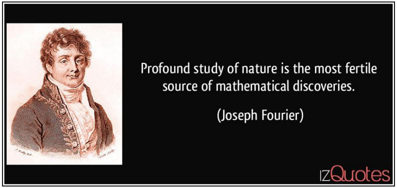
MA 207 - Differential Equations-II

Amiya Kumar Pani

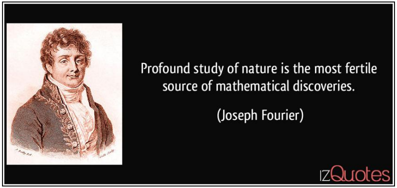
Department of Mathematics,
Indian Institute of Technology Bombay,
Powai, Mumbai 76
akp.ma207.2020@gmail.com

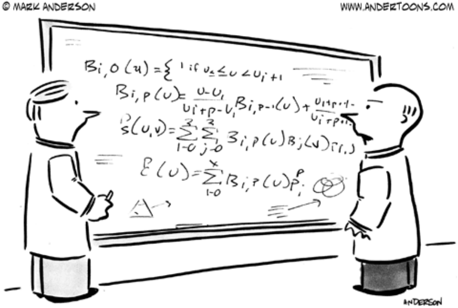
November 20, 2020

Start with Quotation and One Cartoon



Start with Quotation and One Cartoon





"What the hell is *that* supposed to mean?!"

Outline of the lecture

- Conservation Law: Air or Water Pollution Model
- Example
- Exercises

Many PDE models come naturally from:

Conservation Principle

The rate at which a quantity of interest changes in a given domain = the rate at which the quantity flows across the boundary of the domain + the rate at which the quantity is created or destroyed within the domain.

For Air polution model: Consider a tube of cross sectional area A

- $u = u(x, t)$ denotes the concentration of the pollutants. Then, the amount of the concentration in a small section of width dx is given by $u(x, t)Adx$.

- $q = q(x,t)$ denotes the flux

Definition

Flux is defined as the amount of concentration crossing the section x at t and its unit is given by amount per unit area per unit time.

The amount of the concentration that is crossing at x and at t is $Aq(x,t)$. By convention, the flus is positive if the flow is from left to right and negative if it is from right to left.

- $f = f(x,t)$ denotes the rate at the quantity is created or destroyed within the section x and at time t. Then the amount is created or destroyed in a width dx per unit time is $f(x,t)Adx$.

Formulate the conservation law by considering a fixed, but arbitrary section $a \leq x \leq b$ of the tube. By the Conservation principle:

Formulate the conservation law by considering a fixed, but arbitrary section $a \leq x \leq b$ of the tube. By the Conservation principle:

Conservation Principle

$$\frac{d}{dt}\int_a^b u(x,t)A\,dx = Aq(a,t) - Aq(b,t) + \int_a^b f(x,t)A\,dx$$

Conservation Laws——

Formulate the conservation law by considering a fixed, but arbitrary section $a \leq x \leq b$ of the tube. By the Conservation principle:

Conservation Principle

$$\frac{d}{dt}\int_a^b u(x,t)A\,dx = Aq(a,t) - Aq(b,t) + \int_a^b f(x,t)A\,dx$$

Assuming u and q have continuous first derivatives, we rewrite as

$$\int_a^b \Big(u_t(x,t) + q_x(x,t) - f(x,t)\Big)\,dx = 0.$$

Formulate the conservation law by considering a fixed, but arbitrary section $a \leq x \leq b$ of the tube. By the Conservation principle:

Conservation Principle

$$\frac{d}{dt}\int_a^b u(x,t)A\,dx = Aq(a,t) - Aq(b,t) + \int_a^b f(x,t)A\,dx$$

Assuming u and q have continuous first derivatives, we rewrite as

$$\int_a^b \Big(u_t(x,t) + q_x(x,t) - f(x,t)\Big)\,dx = 0.$$

Since the section $[a,b]$ is arbitrary and the integrand is continuous,

Formulate the conservation law by considering a fixed, but arbitrary section $a \leq x \leq b$ of the tube. By the Conservation principle:

Conservation Principle

$$\frac{d}{dt}\int_a^b u(x,t)A\,dx = Aq(a,t) - Aq(b,t) + \int_a^b f(x,t)A\,dx$$

Assuming u and q have continuous first derivatives, we rewrite as

$$\int_a^b \Big(u_t(x,t) + q_x(x,t) - f(x,t)\Big)\,dx = 0.$$

Since the section $[a,b]$ is arbitrary and the integrand is continuous, therefore,

$$u_t(x,t) + q_x(x,t) = f(x,t).$$

Here, there are two unknowns and one equation, therefore, a relation between the flux and the concentration, which should be given by a physical law. In air pollution model $q = \alpha u$.

Here, there are two unknowns and one equation, therefore, a relation between the flux and the concentration, which should be given by a physical law. In air pollution model $q = \alpha u$.
Hence, we obtain

$$u_t + \alpha u_x = f(x, t).$$

α is the wind velocity.
The situation like spread of pollutants from an industrial chimney mainly due to a strong wind (with negligible diffusion) with velocity say α and no other external input, that is, with $f = 0$ can be described by

Here, there are two unknowns and one equation, therefore, a relation between the flux and the concentration, which should be given by a physical law. In air pollution model $q = \alpha u$.
Hence, we obtain

$$u_t + \alpha u_x = f(x, t).$$

α is the wind velocity.
The situation like spread of pollutants from an industrial chimney mainly due to a strong wind (with negligible diffusion) with velocity say α and no other external input, that is, with $f = 0$ can be described by

$$u_t + \alpha u_x = 0, \quad x \in \mathbb{R}, \; t > 0$$

and initial concentration $u(x, 0) = u_0(x)$, $x \in \mathbb{R}$,

Here, there are two unknowns and one equation, therefore, a relation between the flux and the concentration, which should be given by a physical law. In air pollution model $q = \alpha u$.
Hence, we obtain

$$u_t + \alpha u_x = f(x,t).$$

α is the wind velocity.
The situation like spread of pollutants from an industrial chimney mainly due to a strong wind (with negligible diffusion) with velocity say α and no other external input, that is, with $f = 0$ can be described by

$$u_t + \alpha u_x = 0, \quad x \in \mathbb{R}, \quad t > 0$$

and initial concentration $u(x,0) = u_0(x)$, $x \in \mathbb{R}$,
u : the concentration of the pollutants
the wind velocity $\alpha > 0$ means wind is blowing from left to right.

In water pollution model: α is the velocity of the flow of water

In water pollution model: α is the velocity of the flow of water
This model is also known as *transport model*.

In water pollution model: α is the velocity of the flow of water
This model is also known as *transport model*.
By a solution u of the above first order PDE (as the highest derivative is of first order), we mean that u is continuously differentiable and it satisfies the PDE and the initial condition.

In water pollution model: α is the velocity of the flow of water
This model is also known as *transport model*.
By a solution u of the above first order PDE (as the highest derivative is of first order), we mean that u is continuously differentiable and it satisfies the PDE and the initial condition.
This is the mathematical model of air pollution (in the absence of diffusion). How does one solve it?

In water pollution model: α is the velocity of the flow of water
This model is also known as *transport model*.
By a solution u of the above first order PDE (as the highest derivative is of first order), we mean that u is continuously differentiable and it satisfies the PDE and the initial condition.
This is the mathematical model of air pollution (in the absence of diffusion). How does one solve it?
A use of change of variable which makes it an ODE will help.

Solution Process–

Use a change of variable $\xi = x - \alpha t$ and $\tau = t$.

Solution Process–

Use a change of variable $\xi = x - \alpha t$ and $\tau = t$.In the new variables :

$$U(\xi, \tau) = u(\xi + \alpha\tau, \tau),$$

that is, $u(x, t) = U(x - \alpha t, t)$.

Solution Process–

Use a change of variable $\xi = x - \alpha t$ and $\tau = t$.In the new variables :

$$U(\xi, \tau) = u(\xi + \alpha\tau, \tau),$$

that is, $u(x, t) = U(x - \alpha t, t)$.
Then, using chain rule, we write :

$$u_t = U_\xi \xi_t + U_\tau \tau_t = -\alpha U_\xi + U_\tau,$$

and

$$u_x = U_\xi \xi_x + U_\tau \tau_x = U_\xi.$$

On substituting in the main PDE, we obtain

$$U_\tau = 0.$$

It is an ordinary differential equation in the new transformed dependent variable U. On integration:

Solution Process——

It is an ordinary differential equation in the new transformed dependent variable U. On integration:

$$U(\xi, \tau) = C(\xi),$$

where C is a function of ξ and independent of τ.

It is an ordinary differential equation in the new transformed dependent variable U. On integration:

$$U(\xi, \tau) = C(\xi),$$

where C is a function of ξ and independent of τ.
At $\tau = t = 0$, $U(\xi, 0) = u_0(\xi)$.

Solution Process——

It is an ordinary differential equation in the new transformed dependent variable U. On integration:

$$U(\xi, \tau) = C(\xi),$$

where C is a function of ξ and independent of τ.
At $\tau = t = 0$, $U(\xi, 0) = u_0(\xi)$. Therefore,

$$U(\xi, \tau) = u_0(\xi)$$

It is an ordinary differential equation in the new transformed dependent variable U. On integration:

$$U(\xi, \tau) = C(\xi),$$

where C is a function of ξ and independent of τ.
At $\tau = t = 0$, $U(\xi, 0) = u_0(\xi)$. Therefore,

$$U(\xi, \tau) = u_0(\xi)$$

and in the original variable

$$u(x, t) = u_0(x - \alpha t).$$

(Check this is a solution!)

Solution Process——

It is an ordinary differential equation in the new transformed dependent variable U. On integration:

$$U(\xi, \tau) = C(\xi),$$

where C is a function of ξ and independent of τ.
At $\tau = t = 0$, $U(\xi, 0) = u_0(\xi)$. Therefore,

$$U(\xi, \tau) = u_0(\xi)$$

and in the original variable

$$u(x, t) = u_0(x - \alpha t).$$

(Check this is a solution!)
If $\alpha > 0$, the solution is a right travelling wave which preserves the initial profile.

It is an ordinary differential equation in the new transformed dependent variable U. On integration:

$$U(\xi, \tau) = C(\xi),$$

where C is a function of ξ and independent of τ.
At $\tau = t = 0$, $U(\xi, 0) = u_0(\xi)$. Therefore,

$$U(\xi, \tau) = u_0(\xi)$$

and in the original variable

$$u(x, t) = u_0(x - \alpha t).$$

(Check this is a solution!)
If $\alpha > 0$, the solution is a right travelling wave which preserves the initial profile.
In the context of air pollution model that too if the wind velocity is strong (with negligible diffusion), one can feel the same effect in near by areas which are in the direction of the wind even after a later time.

This model can be modified to include the effect of diffusion so that a distance far away from the chimney one does not feel the strong impact of the pollutant-a case of Bhopal gas tragedy is an example.

This model can be modified to include the effect of diffusion so that a distance far away from the chimney one does not feel the strong impact of the pollutant-a case of Bhopal gas tragedy is an example.

Question

Why should we bother about such a model?

In case such a model with a simulator is available, then one can take a decision to evacuate people so that the damaged can be minimal and hence, it helps in level of planning.

Exercise. Solve using the transformation technique:

$$u_t + \alpha u_x + au = f(x,t), \quad x \in \mathbb{R}, \quad t > 0,$$

with initial condition : $u(x,0) = u_0(x), \quad x \in \mathbb{R}$, where α and a are constants and f is a given function.

Exercise. Solve using the transformation technique:

$$u_t + \alpha u_x + au = f(x,t), \quad x \in \mathbb{R}, \quad t > 0,$$

with initial condition : $u(x,0) = u_0(x), \quad x \in \mathbb{R}$, where α and a are constants and f is a given function.
Using the transformation technique $\xi = x - \alpha t$ and $\tau = t$, we now arrive at:

$$U_\tau + aU = f(\xi + \alpha\tau, \tau)$$

with $U(\xi) = u_0(\xi)$.

Exercise. Solve using the transformation technique:

$$u_t + \alpha u_x + au = f(x,t), \quad x \in \mathbb{R}, \quad t > 0,$$

with initial condition : $u(x,0) = u_0(x), \quad x \in \mathbb{R}$, where α and a are constants and f is a given function.

Using the transformation technique $\xi = x - \alpha t$ and $\tau = t$, we now arrive at:

$$U_\tau + aU = f(\xi + \alpha\tau, \tau)$$

with $U(\xi) = u_0(\xi)$.

Its solution can be written as:

$$U(\xi,\tau) := u_0(\xi)e^{-a\tau} + \int_0^\tau e^{-a(\tau - s)} f(\xi + \alpha\tau, \tau)\, d\xi.$$

In the original variable, we obtain the solution as

$$u(x,t) := u_0(x - \alpha t)e^{-at} + \int_0^t e^{-a(t-s)} f(x,t)\, dx.$$

One may be curious to know

Question

'How do we choose the transformations?'

One may be curious to know

Question

'How do we choose the transformations?'

Note that the straight lines $\xi = x - \alpha t \equiv$ constant in space-time domain are solutions of

$$\frac{dx}{dt} = \alpha,$$

One may be curious to know

Question

'How do we choose the transformations?'

Note that the straight lines $\xi = x - \alpha t \equiv$ constant in space-time domain are solutions of

$$\frac{dx}{dt} = \alpha,$$

This is called characteristic equation and the straight lines $\xi = C$, where C is an arbitrary constant are called characteristics. Thus it is possible to get the transformation by solving the characteristic equation.

Another Example.

Solve : $u_t + 2tu_x = 0, \quad x \in \mathbb{R}, \quad t > 0$ with $u(x,0) = F(x)$.

Another Example.

Solve : $u_t + 2tu_x = 0, \quad x \in \mathbb{R}, \quad t > 0$ with $u(x,0) = F(x)$.

As $a = 2t$, the characteristic equation is given by : $\dfrac{dx}{dt} = 2t$.

Another Example.

Solve : $u_t + 2tu_x = 0, \quad x \in \mathbb{R}, \quad t > 0$ with $u(x,0) = F(x)$.

As $a = 2t$, the characteristic equation is given by : $\dfrac{dx}{dt} = 2t$.

On solving $x - t^2 = C$. Thus, set $\xi = x - t^2$ and $\tau = t$.

Using chain rule we obtain

$$u_t = U_\xi(-2t) + U_\tau, \quad u_x = U_\xi$$

Another Example.

Solve : $u_t + 2tu_x = 0, \quad x \in \mathbb{R}, \quad t > 0$ with $u(x,0) = F(x)$.

As $a = 2t$, the characteristic equation is given by : $\dfrac{dx}{dt} = 2t$.

On solving $x - t^2 = C$. Thus, set $\xi = x - t^2$ and $\tau = t$.

Using chain rule we obtain

$$u_t = U_\xi(-2t) + U_\tau, \quad u_x = U_\xi$$

Therefore, we obtain $U_\tau = 0$.

Another Example.

Solve : $u_t + 2tu_x = 0, \quad x \in \mathbb{R}, \quad t > 0$ with $u(x, 0) = F(x)$.

As $a = 2t$, the characteristic equation is given by : $\dfrac{dx}{dt} = 2t$.

On solving $x - t^2 = C$. Thus, set $\xi = x - t^2$ and $\tau = t$.
Using chain rule we obtain

$$u_t = U_\xi(-2t) + U_\tau, \quad u_x = U_\xi$$

Therefore, we obtain $U_\tau = 0$.
Its general solution is now given by $U = F(\xi)$.In the original variable, we obtain : $u(x, t) = F(x - t^2)$.

Another Example.

Solve : $u_t + 2tu_x = 0, \quad x \in \mathbb{R}, \quad t > 0$ with $u(x,0) = F(x)$.

As $a = 2t$, the characteristic equation is given by : $\dfrac{dx}{dt} = 2t$.

On solving $x - t^2 = C$. Thus, set $\xi = x - t^2$ and $\tau = t$.

Using chain rule we obtain

$$u_t = U_\xi(-2t) + U_\tau, \quad u_x = U_\xi$$

Therefore, we obtain $U_\tau = 0$.

Its general solution is now given by $U = F(\xi)$.In the original variable, we obtain : $u(x,t) = F(x - t^2)$.

If $F(x) = e^{x^2}$, then the solution is $u(x,t) = e^{(x-t^2)^2}$.

Another Example.

Solve : $u_t + 2tu_x = 0, \quad x \in \mathbb{R}, \quad t > 0$ with $u(x,0) = F(x)$.

As $a = 2t$, the characteristic equation is given by : $\dfrac{dx}{dt} = 2t$.

On solving $x - t^2 = C$. Thus, set $\xi = x - t^2$ and $\tau = t$.

Using chain rule we obtain

$$u_t = U_\xi(-2t) + U_\tau, \quad u_x = U_\xi$$

Therefore, we obtain $U_\tau = 0$.

Its general solution is now given by $U = F(\xi)$.In the original variable, we obtain : $u(x,t) = F(x - t^2)$.

If $F(x) = e^{x^2}$, then the solution is $u(x,t) = e^{(x-t^2)^2}$.

Exercises: Q.1, Q.2, Q.3, Q.4

Heat Equation in Half Space

Consider

$$u_t - Du_{xx} = 0, 0 < x < \infty, \quad t > 0$$

with initial condition : $u(x, 0) = u_0(x), \quad x > 0$ and boundary condition: $u(0, t) = 0, \ t > 0.$

Heat Equation in Half Space

Consider

$$u_t - Du_{xx} = 0, 0 < x < \infty, \quad t > 0$$

with initial condition : $u(x, 0) = u_0(x), \quad x > 0$ and boundary condition: $u(0, t) = 0, \; t > 0.$
$D > 0$: diffusivity constant.

Heat Equation in Half Space

Consider

$$u_t - Du_{xx} = 0, 0 < x < \infty, \quad t > 0$$

with initial condition : $u(x, 0) = u_0(x), \quad x > 0$ and boundary condition: $u(0, t) = 0, \ t > 0$.

$D > 0$: diffusivity constant.

Question

How does one solve it?

Heat Equation in Half Space

Consider

$$u_t - Du_{xx} = 0, 0 < x < \infty, \quad t > 0$$

with initial condition : $u(x,0) = u_0(x), \quad x > 0$ and boundary condition: $u(0,t) = 0, \ t > 0.$
$D > 0$: diffusivity constant.

Question

How does one solve it?

Can we extend the boundary condition appropriately so that the problem is defined in whole real line in x direction?

Heat Equation in Half Space

Consider

$$u_t - Du_{xx} = 0, 0 < x < \infty, \quad t > 0$$

with initial condition : $u(x, 0) = u_0(x), \quad x > 0$ and boundary condition: $u(0, t) = 0, \ t > 0.$
$D > 0$: diffusivity constant.

Question

How does one solve it?

Can we extend the boundary condition appropriately so that the problem is defined in whole real line in x direction?

If so, we know how to solve the problem in te whole of the real line.

Heat Equation in Half Space

Consider

$$u_t - Du_{xx} = 0, 0 < x < \infty, \quad t > 0$$

with initial condition : $u(x,0) = u_0(x), \quad x > 0$ and boundary condition: $u(0,t) = 0, \; t > 0$.
$D > 0$: diffusivity constant.

Question

How does one solve it?

Can we extend the boundary condition appropriately so that the problem is defined in whole real line in x direction?

If so, we know how to solve the problem in te whole of the real line.
Extension: Set $v(x,t)$ as

Heat Equation in Half Space

Consider

$$u_t - Du_{xx} = 0, 0 < x < \infty, \quad t > 0$$

with initial condition : $u(x,0) = u_0(x), \quad x > 0$ and boundary condition: $u(0,t) = 0, \ t > 0.$
$D > 0$: diffusivity constant.

Question

How does one solve it?

Can we extend the boundary condition appropriately so that the problem is defined in whole real line in x direction?

If so, we know how to solve the problem in te whole of the real line.
Extension: Set $v(x,t)$ as

$$v(x,t) = \begin{cases} u(x,t), & \text{if } \ x \geq 0, \\ -u(-x,t), & \text{if } \ x < 0, . \end{cases}$$

Heat Equation in Half Space—

Now v satisfies:

Now v satisfies:

$$v_t - Dv_{xx} = 0, \ -\infty < x < \infty, \ t > 0$$

with initial condition

Now v satisfies:

$$v_t - Dv_{xx} = 0, \ -\infty < x < \infty, \ t > 0$$

with initial condition

$$v(x,0) = \begin{cases} u_0(x), & \text{if} \ \ x \geq 0, \\ -u_0(-x), & \text{if} \ \ x < 0, . \end{cases}$$

Now v satisfies:

$$v_t - Dv_{xx} = 0, \; -\infty < x < \infty, \; t > 0$$

with initial condition

$$v(x,0) = \begin{cases} u_0(x), & \text{if } \; x \geq 0, \\ -u_0(-x), & \text{if } \; x < 0, . \end{cases}$$

Using Fourier transform technique, we have solved it earlier and therefore, the solution v becomes:

Now v satisfies:

$$v_t - Dv_{xx} = 0, \ -\infty < x < \infty, \ t > 0$$

with initial condition

$$v(x,0) = \begin{cases} u_0(x), & \text{if } \ x \geq 0, \\ -u_0(-x), & \text{if } \ x < 0, . \end{cases}$$

Using Fourier transform technique, we have solved it earlier and therefore, the solution v becomes:

$$\begin{aligned} v(x,t) &= \frac{1}{2\sqrt{D\pi t}} \int_{-\infty}^{\infty} v(y,0) \exp\Big(- \frac{(x-y)^2}{4Dt}\Big) \, dy \\ &= \frac{1}{2\sqrt{D\pi t}} \Big(\int_{-\infty}^{0} -u_0(-y) \exp\big(- \frac{(x-y)^2}{4Dt}\big) \\ &+ \int_{0}^{\infty} u_0(y) \exp\big(- \frac{(x-y)^2}{4Dt}\big)\Big) \, dy. \end{aligned}$$

For the first integral, use change of variable $s = -y$ that is $-dy = ds$ and obtain

$$\int_{-\infty}^{0} -u_0(-y) \exp\Big(-\frac{(x-y)^2}{4Dt}\Big)\, dy = -\int_{0}^{\infty} u_0(s) \exp\Big(-\frac{(x+s)^2}{4Dt}\Big)\, ds$$

Therefore, the solution becomes

$$\begin{aligned} & u(x,t) = v(x,t) \\ = \quad & \frac{1}{2\sqrt{D\pi t}} \int_0^\infty u_0(y)\Big(\exp\Big(-\frac{(x-y)^2}{4Dt}\Big) - \exp\Big(-\frac{(x+y)^2}{4Dt}\Big)\Big)\, dy. \end{aligned}$$

Find a relation between the solution and the initial data in terms of L^2 (square integrable norm) of

$$u_t + u_{xxxx} = 0, \ -\infty < x < \infty, t > 0$$

with initial data $u(x,0) = u_0(x), \ \ -\infty < x < \infty.$
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega, t) + \hat{u}_{xxxx}(\omega, t) = 0$$

Use of Plancherel's Identity

Find a relation between the solution and the initial data in terms of L^2 (square integrable norm) of

$$u_t + u_{xxxx} = 0, \; -\infty < x < \infty, t > 0$$

with initial data $u(x,0) = u_0(x), \;\; -\infty < x < \infty.$
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega, t) + \hat{u}_{xxxx}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u}_0(\omega)$$

Find a relation between the solution and the initial data in terms of L^2 (square integrable norm) of

$$u_t + u_{xxxx} = 0, \ -\infty < x < \infty, t > 0$$

with initial data $u(x,0) = u_0(x), \ \ -\infty < x < \infty.$
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega, t) + \hat{u}_{xxxx}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xxxx} = (i\omega)^4 \hat{u}(\omega, t)$

Use of Plancherel's Identity

Find a relation between the solution and the initial data in terms of L^2 (square integrable norm) of

$$u_t + u_{xxxx} = 0, \ -\infty < x < \infty, t > 0$$

with initial data $u(x,0) = u_0(x), \ \ -\infty < x < \infty.$
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega, t) + \hat{u}_{xxxx}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xxxx} = (i\omega)^4 \hat{u}(\omega, t)$
On substituting in the equation, we obtain

Use of Plancherel's Identity

Find a relation between the solution and the initial data in terms of L^2 (square integrable norm) of

$$u_t + u_{xxxx} = 0, \; -\infty < x < \infty, t > 0$$

with initial data $u(x,0) = u_0(x), \;\; -\infty < x < \infty.$
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega, t) + \hat{u}_{xxxx}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xxxx} = (i\omega)^4 \hat{u}(\omega, t)$
On substituting in the equation, we obtain

$$\hat{u}_t(\omega, t) + \omega^4 \hat{u}(\omega, t) = 0$$

Use of Plancherel's Identity

Find a relation between the solution and the initial data in terms of L^2 (square integrable norm) of

$$u_t + u_{xxxx} = 0, \; -\infty < x < \infty, t > 0$$

with initial data $u(x,0) = u_0(x), \;\; -\infty < x < \infty.$
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega, t) + \hat{u}_{xxxx}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xxxx} = (i\omega)^4 \hat{u}(\omega, t)$
On substituting in the equation, we obtain

$$\hat{u}_t(\omega, t) + \omega^4 \hat{u}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Use of Plancherel's Identity

Find a relation between the solution and the initial data in terms of L^2 (square integrable norm) of

$$u_t + u_{xxxx} = 0, \; -\infty < x < \infty, t > 0$$

with initial data $u(x,0) = u_0(x), \;\; -\infty < x < \infty.$
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega, t) + \hat{u}_{xxxx}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xxxx} = (i\omega)^4 \hat{u}(\omega, t)$
On substituting in the equation, we obtain

$$\hat{u}_t(\omega, t) + \omega^4 \hat{u}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Solving the above ODE (IVP) in time,

Use of Plancherel's Identity

Find a relation between the solution and the initial data in terms of L^2 (square integrable norm) of

$$u_t + u_{xxxx} = 0, \ -\infty < x < \infty, t > 0$$

with initial data $u(x,0) = u_0(x), \ \ -\infty < x < \infty.$
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega, t) + \hat{u}_{xxxx}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xxxx} = (i\omega)^4 \hat{u}(\omega, t)$
On substituting in the equation, we obtain

$$\hat{u}_t(\omega, t) + \omega^4 \hat{u}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Solving the above ODE (IVP) in time,

$$\boxed{\hat{u}(\omega, t) = \hat{u_0}(\omega) e^{-\omega^4 t}.}$$

Use of Plancherel's Identity

Find a relation between the solution and the initial data in terms of L^2 (square integrable norm) of

$$u_t + u_{xxxx} = 0, \ -\infty < x < \infty, t > 0$$

with initial data $u(x, 0) = u_0(x), \ \ -\infty < x < \infty.$
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega, t) + \hat{u}_{xxxx}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xxxx} = (i\omega)^4 \hat{u}(\omega, t)$
On substituting in the equation, we obtain

$$\hat{u}_t(\omega, t) + \omega^4 \hat{u}(\omega, t) = 0$$
$$\hat{u}(\omega, 0) = \hat{u_0}(\omega)$$

Solving the above ODE (IVP) in time,

$$\boxed{\hat{u}(\omega, t) = \hat{u_0}(\omega) e^{-\omega^4 t}.}$$

Use of Plancherel's Identity–

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega \text{ (Plancherel's identity)}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega \text{ (Plancherel's identity)}$$

Now,

$$\int_{-\infty}^{\infty} |u(x,t)|^2\, dx = \int_{-\infty}^{\infty} |\hat{u}_0(\omega)|^2 e^{-2\omega^4 t}\, d\omega$$

Use of Plancherel's Identity–

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega \text{ (Plancherel's identity)}$$

Now,

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx &= \int_{-\infty}^{\infty} |\hat{u}_0(\omega)|^2 e^{-2\omega^4 t} \, d\omega \\ &\leq \int_{-\infty}^{\infty} |\hat{u}_0(\omega)|^2 d\omega \end{aligned}$$

Use of Plancherel's Identity–

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega \text{ (Plancherel's identity)}$$

Now,

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx &= \int_{-\infty}^{\infty} |\hat{u_0}(\omega)|^2 e^{-2\omega^4 t} \, d\omega \\
&\leq \int_{-\infty}^{\infty} |\hat{u_0}(\omega)|^2 d\omega \\
&= \int_{-\infty}^{\infty} |u_0(x)|^2 \, dx \qquad \text{(Plancherel's identity)}
\end{aligned}$$

$$\boxed{\Longrightarrow \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx \leq \int_{-\infty}^{\infty} |u_0(x)|^2 \, dx.}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega \text{ (Plancherel's identity)}$$

Now,

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx &= \int_{-\infty}^{\infty} |\hat{u_0}(\omega)|^2 e^{-2\omega^4 t} \, d\omega \\ &\leq \int_{-\infty}^{\infty} |\hat{u_0}(\omega)|^2 d\omega \\ &= \int_{-\infty}^{\infty} |u_0(x)|^2 \, dx \qquad \text{(Plancherel's identity)}\end{aligned}$$

$$\boxed{\Longrightarrow \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx \leq \int_{-\infty}^{\infty} |u_0(x)|^2 \, dx.}$$

Stability property

Use of Plancherel's Identity—

Consider Schrodinger wave equation:
$u_t(x,t) + iu_{xx}(x,t) = 0, \;\; -\infty < x < \infty, t > 0$ with initial condition: $u(x,0) = u_0(x,t)$.

Use of Plancherel's Identity—

Consider Schrodinger wave equation:
$u_t(x,t) + iu_{xx}(x,t) = 0, \;\; -\infty < x < \infty, t > 0$ with initial condition: $u(x,0) = u_0(x,t)$.
Here, u with domain $R \times R_+$ and range in complex plane.

Consider Schrodinger wave equation:
$u_t(x,t) + iu_{xx}(x,t) = 0, \quad -\infty < x < \infty, t > 0$ with initial condition: $u(x,0) = u_0(x,t)$.
Here, u with domain $R \times R_+$ and range in complex plane.
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega,t) + i\,\hat{u}_{xx}(\omega,t) = 0$$

Consider Schrodinger wave equation:
$u_t(x,t) + iu_{xx}(x,t) = 0, \quad -\infty < x < \infty, t > 0$ with initial condition: $u(x,0) = u_0(x,t)$.
Here, u with domain $R \times R_+$ and range in complex plane.
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega,t) + i\,\hat{u}_{xx}(\omega,t) = 0$$
$$\hat{u}(\omega,0) = \hat{u_0}(\omega)$$

Use of Plancherel's Identity—

Consider Schrodinger wave equation:
$u_t(x,t) + iu_{xx}(x,t) = 0, \;\; -\infty < x < \infty, t > 0$ with initial condition: $u(x,0) = u_0(x,t)$.
Here, u with domain $R \times R_+$ and range in complex plane.
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega,t) + i\,\hat{u}_{xx}(\omega,t) = 0$$
$$\hat{u}(\omega,0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xx} = (i\omega)^2\hat{u}(\omega,t)$

Consider Schrodinger wave equation:
$u_t(x,t) + iu_{xx}(x,t) = 0, \;\; -\infty < x < \infty, t > 0$ with initial condition: $u(x,0) = u_0(x,t)$.
Here, u with domain $R \times R_+$ and range in complex plane.
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega,t) + i\,\hat{u}_{xx}(\omega,t) = 0$$
$$\hat{u}(\omega,0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xx} = (i\omega)^2 \hat{u}(\omega,t)$
On substituting in the equation, we obtain

Use of Plancherel's Identity—

Consider Schrodinger wave equation:
$u_t(x,t) + iu_{xx}(x,t) = 0, \;\; -\infty < x < \infty, t > 0$ with initial condition: $u(x,0) = u_0(x,t)$.
Here, u with domain $R \times R_+$ and range in complex plane.
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega,t) + i\,\hat{u}_{xx}(\omega,t) = 0$$
$$\hat{u}(\omega,0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xx} = (i\omega)^2\hat{u}(\omega,t)$
On substituting in the equation, we obtain

$$\hat{u}_t(\omega,t) - i\omega^2\hat{u}(\omega,t) = 0$$

Consider Schrodinger wave equation:
$u_t(x,t) + iu_{xx}(x,t) = 0, \quad -\infty < x < \infty, t > 0$ with initial condition: $u(x,0) = u_0(x,t)$.
Here, u with domain $R \times R_+$ and range in complex plane.
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega,t) + i\,\hat{u}_{xx}(\omega,t) = 0$$
$$\hat{u}(\omega,0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xx} = (i\omega)^2\hat{u}(\omega,t)$
On substituting in the equation, we obtain

$$\hat{u}_t(\omega,t) - i\omega^2\hat{u}(\omega,t) = 0$$
$$\hat{u}(\omega,0) = \hat{u_0}(\omega)$$

Use of Plancherel's Identity—

Consider Schrodinger wave equation:
$u_t(x,t) + iu_{xx}(x,t) = 0, \;\; -\infty < x < \infty, t > 0$ with initial condition: $u(x,0) = u_0(x,t)$.
Here, u with domain $R \times R_+$ and range in complex plane.
A use of Fourier transform leads to an ODE in the Fourier or frequency variable w as:

$$\hat{u}_t(\omega,t) + i\,\hat{u}_{xx}(\omega,t) = 0$$
$$\hat{u}(\omega,0) = \hat{u_0}(\omega)$$

Note that $\hat{u}_{xx} = (i\omega)^2\hat{u}(\omega,t)$
On substituting in the equation, we obtain

$$\hat{u}_t(\omega,t) - i\omega^2\hat{u}(\omega,t) = 0$$
$$\hat{u}(\omega,0) = \hat{u_0}(\omega)$$

Its solution : $\hat{u}(\omega,t) = \hat{u_0}(\omega)\; e^{i\omega^2 t}$.

Use of Plancherel's Identity–

Use of $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega$ (Plancherel's identity)

Use of Plancherel's Identity–

Use of $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega$ (Plancherel's identity)

Now,

$$\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx \quad = \quad \int_{-\infty}^{\infty} |\hat{u_0}(\omega)|^2 |e^{i\omega^2 t}|^2 \, d\omega$$

Use of $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega$ (Plancherel's identity)

Now,

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx &= \int_{-\infty}^{\infty} |\hat{u_0}(\omega)|^2 |e^{i\omega^2 t}|^2 \, d\omega \\ &= \int_{-\infty}^{\infty} |\hat{u_0}(\omega)|^2 d\omega \end{aligned}$$

Use of $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega$ (Plancherel's identity)

Now,

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx &= \int_{-\infty}^{\infty} |\hat{u_0}(\omega)|^2 |e^{i\omega^2 t}|^2 \, d\omega \\
&= \int_{-\infty}^{\infty} |\hat{u_0}(\omega)|^2 d\omega \\
&= \int_{-\infty}^{\infty} |u_0(x)|^2 \, dx \qquad \text{(Plancherel's identity)}
\end{aligned}$$

$$\boxed{\Longrightarrow \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx = \int_{-\infty}^{\infty} |u_0(x)|^2 \, dx.}$$

Use of Plancherel's Identity–

Use of $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega$ (Plancherel's identity)

Now,

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx &= \int_{-\infty}^{\infty} |\hat{u}_0(\omega)|^2 |e^{i\omega^2 t}|^2 \, d\omega \\
&= \int_{-\infty}^{\infty} |\hat{u}_0(\omega)|^2 d\omega \\
&= \int_{-\infty}^{\infty} |u_0(x)|^2 \, dx \qquad \text{(Plancherel's identity)}
\end{aligned}$$

$$\boxed{\Longrightarrow \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx = \int_{-\infty}^{\infty} |u_0(x)|^2 \, dx.}$$

Stability property

Wish You Best of Luck!