Solutions To Tutorial Sheet 5, Tutorial Sheet 6

Tut Sheet 5 : Wave equation based - Qns 4, 5, 6, 9(ii), 10(ii)

4 (i) Set $u(x,t) = F(x)G(t) \Longrightarrow FG'' = c^2F''G$

$$F'' + \lambda F = 0, \tag{1}$$
$$G'' + c^2\lambda G = 0 \tag{2}$$

$u_x(0,t) = u_x(l,t) = 0 \Longrightarrow F'(0) = 0 = F'(l) \Longrightarrow \lambda < 0$ doesn't produce a non-trivial solution for 1.

For $\lambda = 0,\ F \equiv 1$ is a solution.

For $\lambda > 0,\ F(x) = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x$

$F'(0) = 0 \Longrightarrow b = 0,\ F'(l) = 0 \Longrightarrow \sqrt{\lambda}l = n\pi \Longrightarrow \lambda = \dfrac{n^2\pi^2}{l^2}$.

Hence, $F_n(x) = \cos(n\pi x/l),\ n \geq 1$.

For $\lambda = 0,\ G(t) = a_0 + b_0t \Longrightarrow u_0(x,t) = a_0 + b_0t$,

$u_t(x,0) = 0 \Longrightarrow b_0 = 0 \Longrightarrow u_0(x,t) = a_0.\ \lambda = n^2\pi^2/l^2$
$\Longrightarrow G_n(t) = a_n\cos(n\pi ct/l) + b_n\sin(n\pi ct/l)$.

Hence, $u_n(x,t) = (a_n\cos(n\pi ct/l) + b_n\sin(n\pi ct/l))\cos(n\pi x/l)$.

$u(x,t) = a_0 + \displaystyle\sum_{n=1}^{\infty}(a_n\cos(n\pi ct/l) + b_n\sin(n\pi ct/l))\cos(n\pi x/l)$.

$u_t(x,t) = \displaystyle\sum_{n=1}^{\infty}(-\frac{n\pi c}{l}a_n\sin(n\pi ct/l) + b_n\frac{n\pi c}{l}\cos(n\pi ct/l))\cos(n\pi x/l)$.

$u_t(x,t) = 0 \Longrightarrow \displaystyle\sum_{n=1}^{\infty} b_n(\frac{n\pi c}{l})\cos(n\pi x/l) = 0 \Longrightarrow b_n = 0$.

$u(x,0) = x^2(x^2 - l^2) \Longrightarrow a_0 + \displaystyle\sum_{n=1}^{\infty} a_n\cos(n\pi x/l) = x^2(x^2 - l^2)$
$\Longrightarrow a_0 - l^4/12,\ a_n = 12l^4n^4\pi^4 + \frac{2l^4}{n^2\pi^2}(1 - \frac{1}{n^2\pi^2})\cos n\pi$.

4 (iii). $u(x,t) = a_0 + \displaystyle\sum_{n=1}^{\infty}(a_n\cos(n\pi ct/l) + b_n\sin(n\pi ct/l))\cos(n\pi x/l)$.
$u(x,0) = 0 \Longrightarrow a_n = 0\ \forall n \geq 0, \quad u_t(x,0) = 1 \Longrightarrow b_n = \frac{2l^2}{\pi^2n^2c}(1 - \cos n\pi),\ n \geq 1$.

9 (ii)
- Find a function $\phi(x,t)$ such that $\phi_x(0,t) = t$ and $\phi_x(l,t) = 0$. Note that $\phi(x,t) = -\frac{(x-l)^2t}{2l}$ satisfies this.

- Now $\phi(x,t)$ satisfies :

$$\phi_{tt} - c^2\phi_{xx} = -c^2 t/l; \phi_x(0,t) = t,\ \phi_x(l,t) = 0$$
$$\phi(x,0) = 0, \phi_t(x,0) = -\frac{(x-l)^2}{2l}$$

- The required solution is $u(x,t) = \phi(x,t) + w(x,t)$, where $w(x,t)$ solves

$$w_{tt} - c^2 w_{xx} = c^2 t/l;\ \ w_x(0,t) = 0,\ w_x(l,t) = 0$$
$$w(x,0) = 0, w_t(x,0) = \frac{(x-l)^2}{2l}$$

- Consider the following eigenvalue problem:

$$y'' + \lambda y = 0,\ \ 0 < x < l,$$
$$y'(0) = y'(l) = 0$$

We know that $\lambda_n = (\frac{n\pi}{l})^2$ are the eigenvalues and $y_n = \cos(\frac{n\pi}{l}x)$ are the corresponding eigenfunctions. Expand $w(x,t) = \sum_{n=0}^{\infty} \alpha_n(t) y_n(x)$.

- $\alpha_0''(t) + \sum_{n=1}^{\infty} \alpha_n''(t) y_n(x) + \sum_{n=1}^{\infty} \frac{n^2\pi^2c^2}{l^2}\alpha_n(t) y_n(x) = c^2 t/l$
- $\alpha_0''(t) = c^2 t/l,\ \alpha_0(0) = 0,\ \alpha_0'(0) = a_0,$
$\alpha_n''(t) + \frac{n^2\pi^2c^2}{l^2}\alpha_n(t) = 0\ \alpha_n(0) = 0,\ \alpha_n'(0) = a_n,$
where $a_0 = \int_0^l (x-l)^2/2l\, dx,\ a_n = 2/l \int_0^l \frac{(x-l)^2}{2l} \cos(n\pi x/l)\, dx.$

Tutorial Sheet 6

1. dropped
2. dropped
3. dropped
4. Lecture class done
5. Lecture class done
6. (i), (ii),(iii),(v), (vi): done in the Lecture class
7. Lecture class done
8. Lecture class done

10. $u_t = -v_y\beta + v_\tau,\ u_x = v_y,\ u_{xx} = v_{yy}$, substituting in the given equation, we obtain the required result. For solution, proceed as in Qn. 7,8.

11. Let them work out. (dropped if they like)

12. Proceed as in Qn. 8, with the given value of $u_0(x)$.

13. (i) Take Fourier transforms to obtain

$$\begin{aligned} \hat{u}_t + (i\omega + b)\hat{u} = 0,\ \hat{u}(\omega, 0) = \hat{f}(\omega) \\ \Longrightarrow \hat{u}(\omega, t) = \hat{f}(\omega)e^{-(i\omega+b)t}. \end{aligned}$$

Planceherel's identity implies

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x,t)|^2\, dx = \int_{-\infty}^{\infty} |\hat{u}(\omega,t)|^2\, d\omega = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 e^{-2bt}\, d\omega \\ = e^{-2bt}\int_{-\infty}^{\infty} |f(x)|^2\, dx \qquad \text{(Plancherel's identity)} \end{aligned}$$

(ii) Take Fourier transforms to obtain

$$\begin{aligned} \hat{u}_t - i\omega^3\hat{u} = 0,\ \hat{u}(\omega, 0) = \hat{u}_0 \\ \Longrightarrow \hat{u}(\omega, t) = \hat{u_0}(\omega)e^{(i\omega^3)t}. \end{aligned}$$

Planceherel's identity implies

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x,t)|^2\, dx = \int_{-\infty}^{\infty} |\hat{u}(\omega,t)|^2\, d\omega = \int_{-\infty}^{\infty} |\hat{u}_0(\omega)|^2 |e^{i\omega^3 t}|\, d\omega \\ = \int_{-\infty}^{\infty} |u_0|^2\, dx \qquad \text{(Plancherel's identity)} \end{aligned}$$

(iv) (Done in the Lecture class) Take Fourier transforms to obtain

$$\begin{aligned} \hat{u}_{tt} + \omega^2\hat{u} = 0,\ \hat{u}(\omega,0) = \hat{u}_0,\ \hat{u}_t(\omega,0) = \hat{u}_1 \\ \Longrightarrow \hat{u}(\omega,t) = Ae^{i\omega t} + Be^{-i\omega t} \\ \hat{u}(\omega,0) = \hat{u}_0 \Longrightarrow A + B = \hat{u}_0 \\ \hat{u}_t(\omega,0) = \hat{u}_1 \Longrightarrow i\omega(Ae^{i\omega t} - Be^{-i\omega t})\Big|_{t=0} = \hat{u}_1 \Longrightarrow A - B = -\frac{i}{\omega}\hat{u}_1 \\ \hat{u}(\omega,t) = \frac{1}{2}(\hat{u}_0 - \frac{i}{\omega}\hat{u}_1)e^{i\omega t} + \frac{1}{2}(\hat{u}_0 + \frac{i}{\omega}\hat{u}_1)e^{-i\omega t} \end{aligned}$$

When $u_1 = 0$, then $\hat{u}_1 = 0$. Hence, $\hat{u}(\omega,t) = \hat{u}_0\frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) = \hat{u}_0 \cos\omega t$.

Thus

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x,t)|^2\, dx = \int_{-\infty}^{\infty} |\hat{u}(\omega,t)|^2\, d\omega = \int_{-\infty}^{\infty} |\hat{u}_0(\omega)|^2 \frac{1}{2}(|e^{i\omega t}| + |e^{i\omega t}|)^2\, d\omega \\ \leq \int_{-\infty}^{\infty} |u_0|^2\, dx \qquad \text{(Plancherel's identity)} \end{aligned}$$

14. We know $\hat{u}_x = (i\omega)\hat{u}$.

$$\int_{-\infty}^{\infty} |u_x|^2\, dx = \int_{-\infty}^{\infty} |\hat{u}_x|^2\, d\omega \qquad (\text{Plancherel's identity})$$
$$= \int_{-\infty}^{\infty} |(i\omega)^2||\hat{u}|^2\, dx \leq C\int_{-\infty}^{\infty} \frac{\omega^2}{(1+\omega^4)^2}\, d\omega < \infty.$$

$\hat{u}_{xx} = (i\omega)^2\hat{u}$.

$$\int_{-\infty}^{\infty} |u_{xx}|^2\, dx = \int_{-\infty}^{\infty} |\hat{u}_{xx}|^2\, d\omega \qquad (\text{Plancherel's identity})$$
$$= \int_{-\infty}^{\infty} |(i\omega)^4||\hat{u}|^2\, dx \leq C\int_{-\infty}^{\infty} \frac{\omega^4}{(1+\omega^4)^2}\, d\omega \leq C\int_{-\infty}^{\infty} \frac{1}{(1+\omega^4)}\, d\omega < \infty.$$

15.
$$\hat{u}(\omega, t) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} u(x,t)e^{-i\omega x}\, dx.$$

Taking FT, $\hat{u}_t(\omega,t) + ia\omega\hat{u}(\omega,t) = 0,\ \omega \in \mathbb{R},\ \hat{u}(\omega,0) = \hat{u}_0(\omega)$.
On solving, $\hat{u}(\omega,t) = \hat{u}_0 e^{-ia\omega} = \mathfrak{F}(u_0(x-at))$, since $\mathfrak{F}(f(x-a)) = e^{-i\omega a}\mathfrak{F}(f)$.

Taking inverse FT, $u(x,t) = u_0(x-at)$.