Module 2-HW

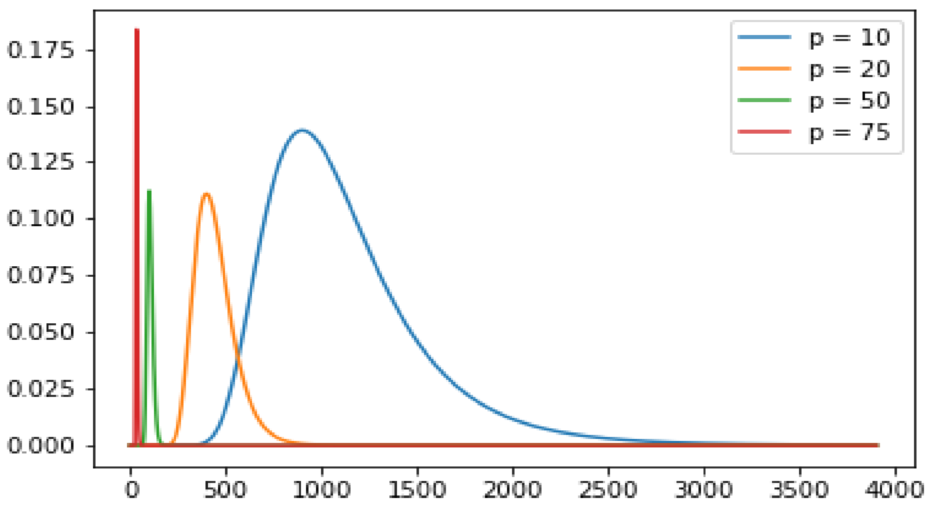
Group 1

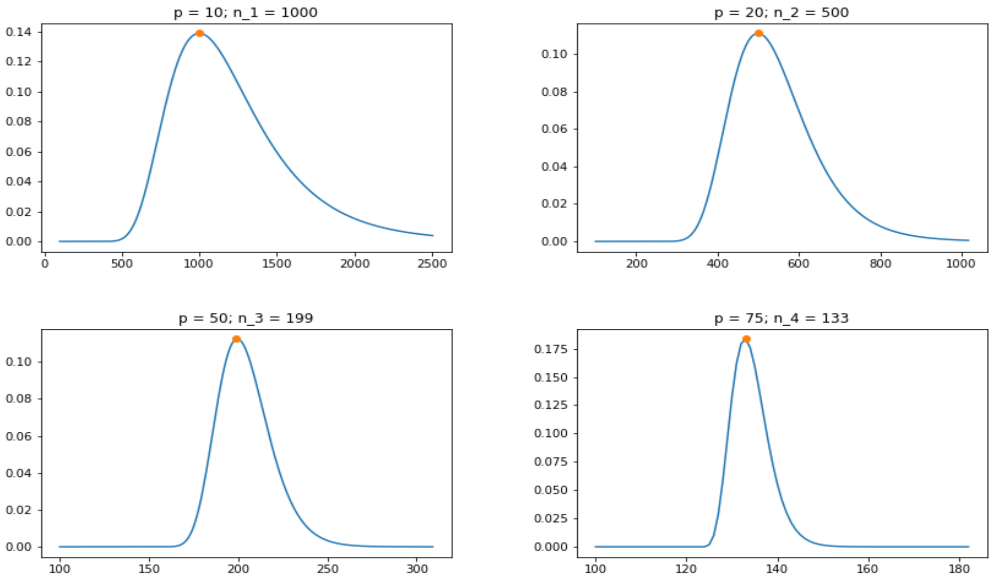
September 20, 2020

3.

4.

5. $P_{m,p}(n)$ is the probability of p marked fish being caught, given m fish out of n were marked in the first place.
Total favourable outcomes are given by choosing p out of m marked fish and $m-p$ out of $n-m$ unmarked fish.
Total possible outcomes are given by choosing m out of n fish.
Hence, $P_{m,p}(n) = \frac{\binom{m}{p}\binom{n-m}{m-p}}{\binom{n}{m}}$
Here are the plots of $P_{m,p}(n)$ plotted as a function of n, for the given values of p :





Hence, the n_i obtained are 1000, 500, 199, 133.

6. After having run the simulation for the n_i calculated above, the following p values were obtained:

   - $p = 9.972$ for $n_i = 1000$
   - $p = 19.706$ for $n_i = 500$
   - $p = 50.386$ for $n_i = 199$
   - $p = 75.332$ for $n_i = 133$

   These values are very close to the values of p that yielded these n_i in the first place.
   The following statement can now be concluded.
   Keeping m fixed, if n_0 total fish result in the expected value of p to be p_0, then the most probable value of n, given p_0 fish out of m were marked in the second catch, is indeed n_0.

7. Probability of getting exactly n 6's in the first $6n$ tosses :

$$\rho_n = \binom{6n}{n} \left(\frac{1}{6}\right)^n \left(\frac{5}{6}\right)^{5n}$$

Proving that ρ_n is monotonically decreasing by showing $\dfrac{\rho_{n+1}}{\rho_n} < 1 \ \forall n \geq 1$:

$$\begin{aligned}
\frac{\rho_{n+1}}{\rho_n} &= \left(\frac{5}{6}\right)^5 \left(\frac{1}{6}\right) \frac{(6n+6)(6n+5)\cdots(6n+1)}{(n+1)(5n+5)\cdots(5n+1)} \\
&= \left(\frac{5}{6}\right)^5 \frac{(6n+5)\cdots(6n+1)}{(5n+5)\cdots(5n+1)} \\
&= \left(\frac{5}{6}\right)^5 \left(\frac{n}{5n+5}+1\right)\left(\frac{n}{5n+4}+1\right)\cdots\left(\frac{n}{5n+1}+1\right)
\end{aligned}$$

$\forall n \geq 1,\ \dfrac{n}{5n+i}$ is strictly decreasing $\forall i \in \{1,2,3,4,5\}$

$$\therefore \frac{n}{5n+i} < \lim_{n\to\infty} \frac{n}{5n+i} = \frac{1}{5} \quad \forall i \in \{1,2,3,4,5\}$$

Hence $\left(\dfrac{n}{5n+5}+1\right)\left(\dfrac{n}{5n+4}+1\right)\cdots\left(\dfrac{n}{5n+1}+1\right)$ is strictly decreasing and its value is bounded above by $\left(\dfrac{6}{5}\right)^5$

Hence $\dfrac{\rho_{n+1}}{\rho_n}$ is strictly decreasing and is bounded above by $\lim_{n\to\infty} \dfrac{\rho_{n+1}}{\rho_n}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{\rho_{n+1}}{\rho_n} &= \left(\frac{5}{6}\right)^5 \left(\frac{6}{5}\right)^5 = 1 \\
\therefore \frac{\rho_{n+1}}{\rho_n} &< 1 \\
\therefore \rho_{n+1} &< \rho_n
\end{aligned}$$

Hence ρ_n is monotonic function.