MA 207 - Differential Equations-II

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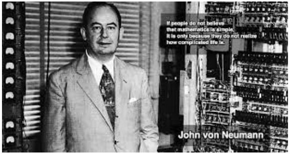
November 7, 2020

Start with ....

" The sciences do not try to explain, they hardly even try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work—that is, correctly to describe phenomena from a reasonably wide area. Furthermore, it must satisfy certain esthetic criteria—that is, in relation to how much it describes, it must be rather simple."

by John von Neumann



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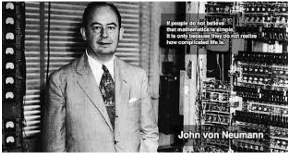

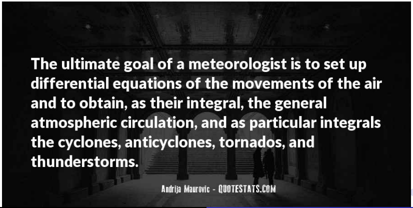

Outline of the lecture

- Elliptic PDE: Laplace Equation, Poisson Equation
- Uniqueness of Solution
- Heat Conduction Equation - Separation of variables
- Non-homogeneous boundary data
- Non-homogeneous equation

- Let Ω be an open, bounded subset of $\mathbb{R}^n$ with boundary $\partial\Omega$.

Poisson equation in a bounded domain

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- Consider

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$$u = g_1 \text{ on } \partial\Omega_D, \; \frac{\partial u}{\partial \nu} = g_2 \text{ on } \partial\Omega_N.$$

Pierre-Simon, marquis de Laplace (23 March 1749 – 5 March 1827)

“ French truely polymath who made important to contributions to engineering, mathematics, statistics, physics, astronomy, and philosophy.
Called French Newton
His five-volumes on Celestial Mechanics (1799–1825) were classic .
Formulated Laplace’s equation, and pioneered the Laplace transform
The Laplacian differential operator, widely used in mathematics, is also named after him.
He restated and developed the nebular hypothesis of the origin of the Solar System and
He was one of the first scientists to postulate the existence of black holes and the notion of gravitational collapse.

Baron Simeon Denis Poisson (21 June 1781 – 25 April 1840)

“ French mathematician, engineer, and physicist who made many scientific advances
Known for his work on definite integrals, electromagnetic theory, and probability.
In probability: Poisson Distribution is named after him.
Worked on Celestial Mechanics (extended works of Laplace and Lagrange)
300 papers published

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$$\underbrace{\int_{\Omega} \nabla \cdot \bar{F}\, dx}_{\sum_{i=1}^{n} \int_{\Omega} \frac{\partial F_i}{\partial x_i}\, dx} = \underbrace{\int_{\partial\Omega} \bar{F} \cdot \nu\, ds}_{\sum_{i=1}^{n} \int_{\partial\Omega} F_i \nu_i\, ds} .$$

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Exercise : 1. Verify for other BC's.
2. Tut. Sheet 4 - Qns. 6, 7(i), 8

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Integrate from 0 to t to obtain

$$\int_\Omega v(x,t)^2 \, dx + 2\int_0^t \int_\Omega |\nabla v(x,t)|^2 \, dx = 0$$

Multiply the PDE by v and integrate over Ω to obtain

$$\begin{aligned} 0 &= \int_\Omega v_t v \, dx - \int_\Omega (\Delta v) v \, dx \\ &= \frac{1}{2}\frac{d}{dt}\left(\int_\Omega v(x,t)^2 \, dx \right) - \cancel{\int_{\partial\Omega} v \frac{\partial v}{\partial \nu} \, ds} + \int_\Omega |\nabla v(x,t)|^2 \, dx \end{aligned}$$

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$\Longrightarrow$ $u_1 = u_2$; hence the solution is unique.

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Exercise : Tut. Sheet 4- Qns. 7 (ii), (iii), (iv), 10

Exercise : Show that the problem

$$
\begin{aligned}
& u_{tt} - \Delta u = f \text{ in } \Omega,\ t \in (0, T] \\
BC: \quad & \frac{\partial u}{\partial \nu} = g \ \text{ on } \partial\Omega,\ t \in [0, T] \\
IC: \quad & u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \ x \in \Omega
\end{aligned}
$$

has a unique solution.

Exercise : Tut. Sheet 4 - Qn. 7(iv) , 9

Heat Conduction Equation (One space variable)

- Consider a homogeneous rod of length l.

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$\lambda > 0$

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

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Let $u_n(x,t) = X_n(x)T_n(t)$. Now, $\sum_{n=1}^{N} u_n(x,t)$ satisfies the heat equation.
Under suitable assumptions on the initial data (which will be specified later), $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ also satisfies the heat equation.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \qquad \text{ASSUMPTION (A1)}$$

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \qquad \textit{ASSUMPTION (A1)} \\
&= \sum_{n=1}^{\infty} X_n(x)\,T_n(t)
\end{aligned}
$$

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \qquad \color{red}{ASSUMPTION\ (A1)} \\
&= \sum_{n=1}^{\infty} X_n(x)T_n(t) \\
&= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{-k\left(\frac{n\pi}{l}\right)^2 t} \qquad (A_n = A_n' B_n)
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$u(x,0) = \phi(x) \Longrightarrow \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}) = \phi(x)$ ASSUMPTION (A2) - Fourier sine series of $\phi(x)$ converges to $\phi(x)$.

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Hence, $\boxed{A_n = \frac{2}{l}\int_0^l \phi(x)\, \sin(\frac{n\pi x}{l})\, dx.}$

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Hence, $\boxed{A_n = \frac{2}{l}\int_0^l \phi(x)\, \sin(\frac{n\pi x}{l})\, dx.}$ The formal solution of the heat equation with Dirichlet BC & IC is given by

$$
\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{l}\int_0^l \phi(x)\, \sin(\frac{n\pi x}{l})\, dx \right) \sin(\frac{n\pi x}{l}) e^{-k(\frac{n\pi}{l})^2 t}.}
$$

If the function $\phi(x)$ satifies the properties:

Justification of (A1) & (A2)

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Exercise :

1. Is the solution unique?

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$$
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Heat Equation - Non-homogeneous Boundary Data

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Let $v(x, t) = u(x, t) - U(x, t)$.

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Let $v(x,t) = u(x,t) - U(x,t)$.

Therefore,

$$
v_t - kv_{xx} \;\; = \;\; (u_t - ku_{xx}) - (U_t - kU_{xx})
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Therefore,

$$
\begin{aligned}
v_t - k v_{xx} &= (u_t - k u_{xx}) - (U_t - k U_{xx}) \\
&= 0 - \left[\frac{1}{l}(l-x)g'(t) + xh'(t)\right] \qquad (U_{xx} = 0)
\end{aligned}
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$$\frac{d}{dt}\alpha_n(t) + k\lambda_n\alpha_n(t) = \gamma_n(t), \quad n = 1,\ 2,\ 3, \cdots,$$

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Exercise : Tut. Sheet 5 - Qns. 7, 8, 9 (i)