Signal Processing - 1 by One

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Outline

- So Far: Sampling and Convolution
- Previous Week: Fourier Series and Fourier Transform
- Previous Class:Shannon Sampling Theorem
- Today: Frequency Concepts, Laplace Transform

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Sampling Revisited

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$$\cos(2\pi f t + \theta) = \cos(2\pi f t)\cos(\theta) - \sin(2\pi f t)\sin(\theta).$$

Notice that

$$\int_{k\frac{T}{2}}^{(k+1)\frac{T}{2}} \cos(2\pi f t)\sin(2\pi f t)dt = 0 \text{ for } T = \frac{1}{f}, k \in \mathbb{Z}.$$

Furthermore

$$\lim_{T_s\to\infty}\int_{-T_s}^{T_s} \cos(2\pi f t)\sin(2\pi f t)dt = 0 \text{ (In a generalized sense)}$$

$$\lim_{T_s\to\infty}\int_{-T_s}^{T_s} \cos(2\pi f_1 t)\sin(2\pi f_2 t)dt = 0 \text{ (as a generalized integral)} .$$

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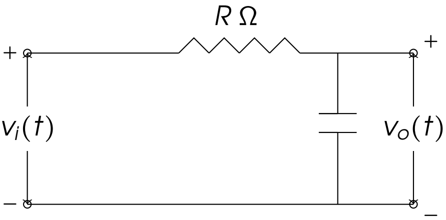
For $f > 0$, if the component $\exp(-j2\pi ft)$ corresponds to positive frequency, then $\exp(j2\pi ft)$ corresponds to negative frequency

Notice that $\exp(-j2\pi f\, t)$ and $\exp(j2\pi f\, t)$ are *orthogonal* for $f > 0$.

Complex Circuits

Complex numbers in electrical circuits suggest the presence of both $\cos(2\pi ft)$ and $\sin(2\pi ft)$ inside, even when $\cos(2\pi ft)$ is input.



Generalizing the Fourier Transform: For $s = \sigma + j2\pi f$,

$$X(s) = \int_{\mathbb{R}} x(t)\exp(-st)dt$$ (Two-sided Laplace Transform).

Region of Convergence (ROC) : $\{Real(\sigma)\}$ s.t. Integral exists.

$$\lim_{\sigma \to 0} X(s) = X(f).$$

Kirchoff's Voltage Law:

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Taking Laplace Transform (Two-sided)

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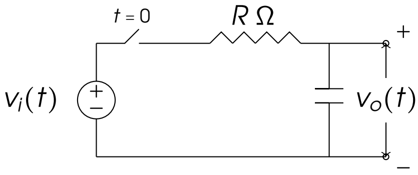
At $s = j2\pi f$,

$$V_o(f) = H(f)V_i(f) \text{ where } H(f) = \frac{1}{1+j2\pi fRC}$$

$$v_o(t) = h(t) \ast v_i(t) \text{ where } h(t) = \frac{1}{RC}\exp(-\frac{t}{RC}), t \geq 0.$$

Laplace with Initial Conditions



$$X(s) = \int_0^{\infty} x(t)\exp(-jst)\,dt$$

$$v_i(t) = i(t)R + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau.$$

$$v_i'(t) = i'(t)R + \frac{1}{C}i(t)$$

$$sV_i(s) - v_i(0) = sRI(s) - i(0) + \frac{1}{C}I(s)$$