MA 207 - Differential Equations-II

Amiya Kumar Pani

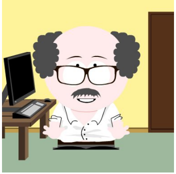
Department of Mathematics,
Indian Institute of Technology Bombay,
Powai, Mumbai 76
akp.ma207.2020@gmail.com

October 22, 2020

Start with Quotation and One Cartoon

"If God has made Nature, Man made Mathematics to study it "

by Anonymous



Outline of the lecture

- Initial value problems
- Boundary value problems
- Sturm Liouville Problems
- Example
- Exercises

RECALL :

$$\mathcal{L}y \equiv p(x)y'' + q(x)y' + r(x)y = f(x) \quad (1)$$
$$y(x_0) = y_0, \; y'(x_0) = y_1 \quad (2)$$

is a linear second order initial value problem with initial conditions given at one point $x = x_0$.

Theorem (Existence and Uniqueness result for IVP)

If $p(x)$, $q(x)$, $r(x)$, and $f(x)$ are continuous in some interval I with $p(x) \neq 0$ in I and $x_0 \in I$, then the IVP (1)-(2) has a unique solution $y(x)$ in the interval I.

If $f(x) \neq 0$, the general solution of (1) has the following form :

$$y(x) = y_h(x) + y_p(x) \tag{3}$$

where

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) \tag{4}$$

is the solution of the associated homogeneous problem ($f = 0$) and $y_p(x)$ is a particular solution of (1).

(2) helps in picking up a unique solution for (1) by giving initial conditions at a particular point called the initial point x_0.

1. Instead of providing two conditions at one point $x = x_0$, CAN we impose two conditions at two different points, usually called boundary points?
2. If so, CAN we state a general result like theorem on IVP for the boundary value problem ?

Definition

Boundary Value Problems (BVP): *Consider a general second order linear differential equation :*

$$p(x)y'' + q(x)y' + r(x)y = f(x) \quad a < x < b \tag{5}$$

$$with \quad boundary \quad conditions$$

$$b_1(y) \equiv k_1 y(a) + k_2 y'(a) = \alpha \tag{6}$$

$$b_2(y) \equiv l_1 y(b) + l_2 y'(b) = \beta \tag{7}$$

where k_1, k_2, l_1, l_2, α, β are real constants with $k_1 \& k_2$ both not equal to zero, $l_1 \& l_2$ both not equal to zero.

Regular BVP

$$
(BVP) \quad \begin{aligned} & p(x)y'' + q(x)y' + r(x)y = f(x) \quad a < x < b \\ & b_1(y) \equiv k_1 y(a) + k_2 y'(a) = \alpha \\ & b_2(y) \equiv l_1 y(b) + l_2 y'(b) = \beta \end{aligned}
$$

where $k_1,\ k_2,\ l_1,\ l_2,\ \alpha,\ \beta$ are real constants with $k_1 \& k_2$ both not equal to zero, $l_1 \& l_2$ both not equal to zero.

Definition

The linear BVP is called HOMOGENEOUS if $f(x) \equiv 0$ and $b_1(y) \equiv \alpha = 0,\ b_2(y) \equiv \beta = 0$.

Otherwise, non-homogeneous BVP.

Definition

The linear BVP is called REGULAR, if

1. *a and b are finite,*
2. *the coefficient $p(x) \neq 0$ for all $x \in [a, b]$.*

SINGULAR BVP

A linear BVP is called SINGULAR, if it is not REGULAR.
(Example : either $a = -\infty$, or $b = \infty$ or both; $p(x) = 0$ for at least one $x \in [a, b]$.)

1. $-xy'' + y' = f, \ x \in (0, 1), \ y(0) = y(1) = 0.$ Here $p(x) = -x$ is zero at $x = 0$.
2. Hermite's equation : $y'' - xy' + \lambda y = 0, \ -\infty < x < \infty$, with $\lim_{x \to \pm\infty} e^{-x^2/2} y(x) = 0$.

The boundary conditions given as

$$\begin{aligned} b_1(y) &\equiv k_1 y(a) + k_2 y'(a) = \alpha \\ b_2(y) &\equiv l_1 y(b) + l_2 y'(b) = \beta \end{aligned}$$

are of general type. Depending on k_1, k_2, l_1, l_2, we now classify the boundary conditions (BC) as

- $\boxed{\text{DIRICHLET BC's}}$: $y(a) = \alpha, \; y(b) = \beta$.
  (Value of the unknown function is prescribed at the end points of the interval).
- $\boxed{\text{NEUMANN BC's}}$: $y'(a) = \alpha, \; y'(b) = \beta$.
  (Value of the derivatives of the unknown function is prescribed at the end points of the interval).

- MIXED BC's :
  $y(a) = \alpha,\ y'(b) = \beta$ OR $y'(a) = \alpha,\ y(b) = \beta$.
  (Value of the unknown function is prescribed at one end point and derivative at the other end point of the interval).
- ROBIN BC's :

$$y(a) + ky'(a) = \alpha, \quad (k \neq 0)$$
$$y(b) + ly'(b) = \beta, \quad (l \neq 0)$$

  (Combination of value of the unknown function and derivatives at the end points).
- PERIODIC BC's : $y(a) = y(b),\ y'(a) = y'(b)$.

Does a unique solution exist for a BVP?

Consider the example :

$$y'' + y = 0, \;\; 0 < x < \pi$$
$$y(0) = 0, \; y(\pi) = \beta \neq 0.$$

The general solution is $\boxed{y(x) = C_1 \cos x + C_2 \sin x.}$
$y(0) = 0 \implies C_1 = 0.$
$y(\pi) = \beta \implies 0 \neq \beta = C_2 \times 0 = 0.$
Unless $\beta = 0$, the BVP doesn't have a solution.
Hence, even if the coefficients in the DE are continuous with $p(x) \neq 0$ in the interval, the BVP need not have a solution.

Homogeneous BVP

A homogeneous linear BVP defined by

$$\mathcal{L}(y) \equiv p(x)y'' + q(x)y' + r(x)y = 0 \quad a < x < b$$
$$b_1(y) \equiv k_1 y(a) + k_2 y'(a) = 0$$
$$b_2(y) \equiv l_1 y(b) + l_2 y'(b) = 0$$

has a TRIVIAL SOLUTION $y = 0$ but there can be non-trivial solutions which are of interest in applications.
We now state a result which tells us the conditions for a **homogeneous BVP** to have only **a unique solution**, that is the trivial solution.

Result

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of $p(x)y'' + q(x)y' + r(x)y = 0$.
Then, the homogeneous BVP has only one trivial solution iff

$$W = \begin{vmatrix} b_1(y_1) & b_1(y_2) \\ b_2(y_1) & b_2(y_2) \end{vmatrix} \neq 0.$$

Proof:

$y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $p(x)y'' + q(x)y' + r(x)y = 0$.
The general solution of the DE can be written as

$\boxed{y(x) = c_1 y_1(x) + c_2 y_2(x).}$

For $y(x)$ to be a solution of the homogeneous BVP, we should be able to determine c_1 and c_2 uniquely, such that $b_1(y) = 0$ and $b_2(y) = 0$.
That is,

$$\begin{aligned} b_1(y) &\equiv b_1(c_1 y_1 + c_2 y_2) = c_1 b_1(y_1) + c_2 b_1(y_2) = 0 \\ b_2(y) &\equiv b_2(c_1 y_1 + c_2 y_2) = c_1 b_2(y_1) + c_2 b_2(y_2) = 0 \end{aligned}$$

$$\Longleftrightarrow \begin{bmatrix} b_1(y_1) & b_1(y_2) \\ b_2(y_1) & b_2(y_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longleftrightarrow W = \begin{vmatrix} b_1(y_1) & b_1(y_2) \\ b_2(y_1) & b_2(y_2) \end{vmatrix} \neq 0.$$

The homogeneous BVP has infinitely many non-trivial solutions if and only if $W = 0$.

While discussing the existence of solution of BVP is difficult, it is possible to prove the uniqueness of solution to

$$\mathcal{L}(y) \equiv p(x)y'' + q(x)y' + r(x)y = f(x) \quad a < x < b,$$
$$b_1(y) \equiv k_1 y(a) + k_2 y'(a) = \alpha,$$
$$b_2(y) \equiv l_1 y(b) + l_2 y'(b) = \beta.$$

Result

The non-homogeneous linear BVP has a unique solution if and only the corresponding homogeneous linear BVP has only the trivial solution.

Proof : If possible, let the BVP have two solutions, say y_1 and y_2.

That is, for $i = 1,\ 2$

$$\mathcal{L}(y_i) = f(x) \;\; a < x < b,$$
$$b_1(y_i) = \alpha \text{ at } x = a$$
$$b_2(y_i) = \beta \text{ at } x = b.$$

Set $\boxed{u = y_1 - y_2}$.
Then u satisfies

$$\mathcal{L}(u) = 0 \;\; a < x < b,$$
$$b_1(u) = 0 \text{ at } x = a$$
$$b_2(u) = 0 \text{ at } x = b.$$

Since the corresponding homogeneous problem has only unique solution, $u \equiv 0 \iff y_1 = y_2$.

Sturm-Liouville Problems

- Represents a class of BVP's.
- Applications in Physics and Enginnering.
- Generate a set of orthogonal functions.
- Used in obtaining solutions of BVP's involving PDEs.
- They are eigenvalue problems involving a parameter λ (may be related to frequencies, energies or other physical quantities.)
- Non-trivial solutions to these problems exhibit the orthogonality property leading to eigen function expansions such as those involving cosine, sine series (Fourier series), Legendre polynomials, Bessel functions etc.

Consider the second order linear DE of the form:

$$(r(x)y')' + (q(x) + \lambda p(x))y = 0 \quad a < x < b, \tag{8}$$
$$k_1 y(a) + k_2 y'(a) = 0 \tag{9}$$
$$l_1 y(b) + l_2 y'(b) = 0 \tag{10}$$

Here k_1, k_2 not both are zero, l_1, l_2 not both are zeros, λ is a parameter; k_1, k_2, l_1, l_2 are given constants. (11)-(13) is referred to as STURM-LIOUVILLE BVP.

Often (11) is represented as

$$\boxed{\mathcal{L}(y) + \lambda p(x)y = 0, \ \mathcal{L}(y) = (r(x)y')' + q(x)y.}$$

$y \equiv 0$ is always a solution of (11)-(13).

We are interested to determine the values of the parameter λ for which the BVP has non-trivial solutions.

On Strum-Liouvelle Theory

Jacques Charles-François Sturm (born 29th September,1803-died 15th December,1855),French Mathematician was a professor of Mechanics.

Jacques Charles-François Sturm and Joseph Liouville in a series of papers during 1936-37 developed eigen-value problem, which later called 'Strum-Liouville Problem'. It had tremendous impact on DE and Mathematical Physics.

Joseph Liouville, (born March 24, 1809, Saint-Omer, France-died September 8, 1882, Paris), French mathematician known for his work in analysis, mathematical physics, differential geometry, and number theory

Definition

The values of λ for which (11)-(13) has non-trivial solutions are called as eigenvalues and the corresponding non-trivial solutions are called as eigenfunctions.

Example (Tut. Sheet 3, 2 (ii)) : Consider

$$y'' + \lambda y = 0,\ 0 < x < l$$
$$y(0) = 0,\ y(l) = 0$$

$\lambda \in \mathbb{R}$ denotes the frequency of the vibrating string.
Case 1: $\boxed{\lambda = 0}$.
General solution of the DE is $\boxed{y(x) = Ax + B}$.
$y(0) = 0 \implies B = 0.$
$y(l) = 0 \implies A = 0.$
Hence $\lambda = 0$ yields only the trivial solution $y \equiv 0$ (discard).

Case 2: $\lambda < 0$

Auxiliary equation is $m^2 + \lambda = 0$. Since $\lambda < 0$, the two roots are real and unequal.

Denoting $\sqrt{-\lambda}$ by β, we have the general solution as

$\boxed{y(x) = Ae^{\beta x} + Be^{-\beta x}.}$

Substituting the boundary conditions,

$y(0) = 0 \implies A + B = 0$

$y(l) = 0 \implies Ae^{\beta l} + Be^{-\beta l} = 0$

$$\iff \begin{bmatrix} 1 & 1 \\ e^{\beta l} & e^{-\beta l} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since

$$\iff \begin{vmatrix} 1 & 1 \\ e^{\beta l} & e^{-\beta l} \end{vmatrix} \neq 0,$$

$A = B = 0$ is the only solution $\implies$ NO NON-TRIVIAL SOLUTIONS when $\lambda < 0$.

Case 2: $\lambda > 0$

The roots of the auxiliary equation are $\boxed{m = \pm\sqrt{\lambda}i}$.

Hence $\boxed{y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x}$.

$y(0) = 0 \implies A = 0$

$y(l) = 0 \implies B\sin\sqrt{\lambda}l = 0 \implies \sin\sqrt{\lambda}l = 0$

(since we are interested in non-trivial solutions).

$\implies \sqrt{\lambda}l = n\pi, \ n = 1, \ 2, \ 3, \cdots$

Hence, $\boxed{\lambda_n = (\frac{n\pi}{l})^2, \ n = 1, \ 2, \ 3, \cdots}$

and the non-trivial solutions are given by

$$\boxed{y_n(x) = B_n \sin\frac{n\pi x}{l}, \ B_n \neq 0, \ n = 1, \ 2, \ 3, \cdots.}$$

Sturm-Liouville Boundary Value Problems

Consider the second order linear DE of the form:

$$(r(x)y')' + (q(x) + \lambda p(x))y = 0 \quad a < x < b, \tag{11}$$
$$k_1 y(a) + k_2 y'(a) = 0 \tag{12}$$
$$l_1 y(b) + l_2 y'(b) = 0 \tag{13}$$

Here $k_1, \ k_2$ not both are zero, $l_1, \ l_2$ not both are zeros, λ is a parameter; $k_1, \ k_2, \ l_1, \ l_2$ are given constants. (11)-(13) is referred to as STURM-LIOUVILLE BVP.

Often (11) is represented as

$$\boxed{\mathcal{L}(y) + \lambda p(x)y = 0, \ \mathcal{L}(y) = (r(x)y')' + q(x)y.}$$

$y \equiv 0$ is always a solution of (11)-(13).

We are interested to determine the values of the parameter λ for which the BVP has non-trivial solutions.

Definition

Let $V = \mathbb{R}^n$ (resp. $\mathbb{C}^n$) and A be an $n \times n$ matrix with real (resp. complex) entries. A is symmetric (resp. Hermitian), if $A = A^T$ (resp. $A = \overline{A^T}$).

Consider $\boxed{AX = \lambda X}$, where A is an $n \times n$ symmetric or Hermitian matrix, λ is a scalar parameter.

Properties :

- All eigenvalues are real.
- Eigenvectors corresponding to distinct eigenvalues are orthogonal.
- If $\lambda_1, \lambda_2, \cdots \lambda_n$ are eigenvalues and $v_1, v_2, \cdots, v_n$ are the corresponding eigenvectors, then any vector z can be written as $z = \sum_{i=1}^{n} \alpha_i v_i$, where α_i is related to z and v_i.

Inner product - Matrices

The usual inner product for real(column) vectors u and v in $\mathbb{R}^n$ is just the dot product, written as $\boxed{< u, v >= u \cdot v = v^T u}$.
For any real square matrix A and any inner product, the adjoint matrix A^* is defined as the matrix which satisfies :

$$\boxed{< Au, v >= v^T(Au) = (A^T v)^T u =< u, A^T v >=< u, A^* v >,}$$

Note that $\boxed{A^* = A^T}$.

For u and v in $\mathbb{C}^n$ we have $\boxed{< u, v >= \sum_{i=1}^{n} u_i \bar{v}_i = \overline{v^T} u}$.

The adjoint matrix A^* is defined by :

$$\boxed{< Au, v >= \overline{v^T}(Au) =< u, A^* v >,}$$

Note that $\boxed{A^* = \overline{A^T}}$.
Thus once we define an inner product on V, we define the adjoint operator A^* for $A : V \to V$.

A is self-adjoint (symmetric if $V = \mathbb{R}^n$ and Hermitian if $V = \mathbb{C}^n$) if $\boxed{< Au, v >=< u, Av > \quad \forall u, v \in V, \text{ i.e. } A = A^*.}$

CAN WE GENERALIZE THIS CONCEPT FOR LINEAR OPERATOR L?

Set $V = C^2[a, b] =$ space of twice continuously differentiable functions on $[a, b]$.

On V, define an inner product $\boxed{(f, g) = \int_a^b f(x)g(x)\,dx}$.

(Verify that $(\cdot, \cdot)$ defines an inner product on V).

Definition

A linear operator L on V is said to be self-adjoint on V (with respect to the inner product $(\cdot, \cdot)$ on V) if :

$\boxed{(Lu, v) = (u, Lv).}$

Lagrange's Identity

Let V be the space of twice continuously differentiable functions. For $u, v \in V$,

$$
\begin{aligned}
(\mathcal{L}u, v) &= \int_a^b (\mathcal{L}u)v \, dx \qquad (\mathcal{L}(y) = (r(x)y')' + q(x)y) \\
&= \int_a^b \big((r(x)u')' + q(x)u\big) v \, dx \\
&= \int_a^b (r(x)u')'v \, dx + \int_a^b q(x)uv \, dx \\
&= r(x)u'v\Big|_{x=a}^{x=b} - \underbrace{\int_a^b r(x)u'v' \, dx} + \int_a^b q(x)uv \, dx \\
&= r(x)u'v\Big|_{x=a}^{x=b} - \underbrace{r(x)uv'\Big|_{x=a}^{x=b}} \\
&\quad + \int_a^b (r(x)v')'u \, dx + \int_a^b q(x)uv \, dx
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{L}u, v) &= r(x)u'v\Big|_{x=a}^{x=b} - \underbrace{r(x)uv'\Big|_{x=a}^{x=b}} \\
&\quad + \underbrace{\int_a^b (r(x)v')'u\,dx} + \int_a^b q(x)uv\,dx \\
&= r(x)(u'v - uv')\Big|_{x=a}^{x=b} + \int_a^b \Big((r(x)v')'u + q(x)uv \Big)\,dx \\
&= r(x)(u'v - uv')\Big|_{x=a}^{x=b} + \int_a^b u \Big(\underbrace{(r(x)v')' + q(x)v} \Big)\,dx \\
&= r(x)(u'v - uv')\Big|_{x=a}^{x=b} + (u, \mathcal{L}v)
\end{aligned}
$$

Hence, we obtain the LAGRANGE'S IDENTITY :

$$
\boxed{(\mathcal{L}u, v) - (u, \mathcal{L}v) = r(x)(u'v - uv')\Big|_{x=a}^{x=b}.}
$$

When is $\mathcal{L}$ self-adjoint?

Definition

$\mathcal{L}$ is called self-adjoint, if $\boxed{(\mathcal{L}u, v) = (u, \mathcal{L}v)}$.

In order to make $\mathcal{L}$ self-adjoint, we could define V as the space of twice continuously differentiable functions satisfying the boundary conditions. That is,

$u,\ v \in V$ satisfies

$k_1 u(a) + k_2 u'(a) = 0,\ l_1 u(b) + l_2 u'(b) = 0,$
$k_1 v(a) + k_2 v'(a) = 0,\ l_1 v(b) + l_2 v'(b) = 0$

$$r(x)(u'v - uv')\Big|_{x=a}^{x=b} = r(b)\big(u'(b)v(b) - v'(b)u(b)\big) - r(a)\big(u'(a)v(a) - v'(a)u(a)\big)$$

$$= r(b)\big(\cancel{-\frac{l_1}{l_2}u(b)v(b)} + \cancel{\frac{l_1}{l_2}v(b)u(b)}\big) - r(a)\big(\cancel{-\frac{k_1}{k_2}u(a)v(a)} + \cancel{\frac{k_1}{k_2}v(a)u(a)}\big) = 0$$

Hence, $\boxed{(\mathcal{L}u, v) = (u, \mathcal{L}v).}$

How to write a DE in self-adjoint form?

QUESTION : Given a second linear DE,

$$a(x)y'' + b(x)y' + c(x)y = 0, \tag{14}$$

can we write it in the self- adjoint form?
$\underbrace{a(x)y'' + b(x)y'}_{(r(x)y')'} + c(x)y = 0.$

That is, we seek an integrating factor $\boxed{\mu(x)}$ such that (14) can be represented as :

$$(\mu(x)a(x)y')' + \mu(x)c(x)y = 0. \tag{15}$$

From (14) (multiplied by $\mu(x)$) and (15), equate the coefficients of y' to obtain

$$\begin{aligned} & \mu(x)a'(x) + \mu'(x)a(x) = \mu(x)b(x) \\ \implies \ & \mu'(x)a(x) = (b(x) - a'(x))\mu(x) \end{aligned}$$

$$\mu'(x) = \frac{(b(x) - a'(x))}{a(x)}\mu(x) \quad \text{(assuming } a(x) \neq 0)$$

On solving for $\mu(x)$, we obtain

$$\boxed{\mu(x) = \frac{1}{a(x)} e^{\int \frac{b(x)}{a(x)}\, dx}} \quad (a(x) \neq 0).$$

Example 1 : Legendre equation

$(1 - x^2)y'' - 2xy' + p(p+1)y = 0, \;\; x \in (-1, 1)$ can be put in the self-adjoint form as

$$\boxed{((1 - x^2)y')' + p(p+1)y = 0}$$

Remark : Self-adjoint as

$$(\mathcal{L}u, v) - (u, \mathcal{L}v) = (1 - x^2)(u'v - uv')\Big|_{x=-1}^{x=1} = 0$$

$(r(x) = 1 - x^2)$.

Example 2: Chebyshev's equation

$(1-x^2)y'' - xy' + \alpha^2 y = 0, \quad x \in (-1,1)$ can be put in the self-adjoint form

$$(\mu(x)a(x)y')' + \mu(x)c(x)y = 0, \tag{16}$$

by choosing $\mu(x) = \dfrac{1}{a(x)} e^{\int \frac{b(x)}{a(x)}\,dx}$.

That is,

$$\begin{aligned}
\mu(x) &= \frac{1}{1-x^2} e^{-\int \frac{x}{1-x^2}\,dx} \quad ({\color{blue}put\ 1-x^2 = t, \quad -2x\,dx = dt}) \\
&= \frac{1}{1-x^2} \times \sqrt{(1-x^2)} = \frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$

Hence, the self-adjoint form for Chebyshev's equation is

$$\boxed{(\sqrt{1-x^2}\,y')' + \frac{\alpha^2}{\sqrt{(1-x^2)}}\,y = 0}.$$

(Check: $\mathcal{L}$ is self-adjoint?).

Tutorial Sheet No. 3.

- Qn. 1
- Qn. 2
- Qn. 3
- Qn. 4
- Qn. 7
- EXERCISE : Express the Laguerre equation $xy'' + (1 - x)y' + \lambda y = 0,\ 0 < x < \infty$ in the self-adjoint form.