Signal Processing - 1 by One

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Outline

- So Far: Sampling, Fourier Analysis
- Previous Week: DTFT, DFT and Circular Convolution
- Previous Class: Discrete Fourier Transform
- Today: Fast Fourier Transform (FFT)

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Matrix Multiplication

$$\underset{N\times 1}{y} = \underset{N\times N}{A}\ \underset{N\times 1}{x}.$$

- N^2 mulitplications
- $N(N-1)$ additions.

One addition+multiplication is termed a **flop** or computation.

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Computing DFT as such requires $O(N^2)$ computations.

$$\underset{N\times 1}{\bar{X}} = \underset{N\times N}{F} \underset{N\times 1}{\bar{x}} \text{ with } F_{mn} = \exp(-j\frac{2\pi}{M}mn).$$

Example $N = 4$

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 \\ 1 & \alpha^3 & \alpha^6 & \alpha^9 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

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With $N = 4$, we get $\alpha = \exp(-j\frac{2\pi}{N}) = -j$.

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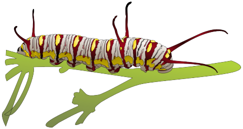
Cater Pillar



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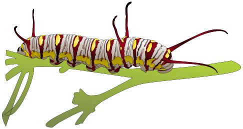

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Pupa Stage

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_2 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -j & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & j \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad \Leftrightarrow$$

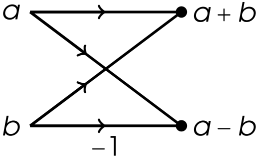

Figure: Butterfly Structure of FFT

Butterflies

$$\begin{bmatrix} X_0 \\ X_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

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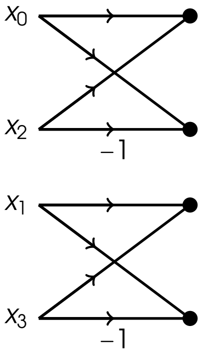

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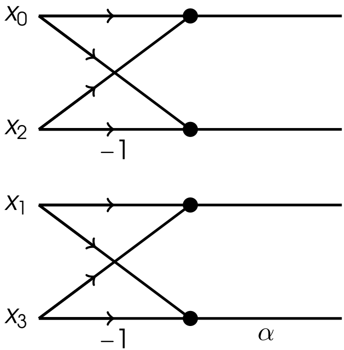

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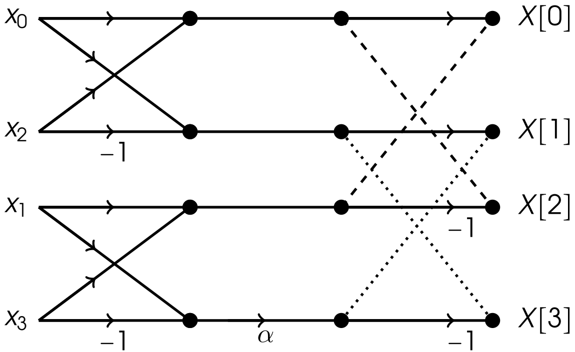

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