EE325 Module 5

(Unofficial)

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Topics in Module 5

- Sequences of Events and the Borel-Cantelli Lemma
- Sequences of random variables and convergence concepts.
- Weak law of large numbers
- Strong law of large numbers
- Central limit theorem
- Reiteration: these are just outlines; you are expected to take notes during the lecture and read from the text book. There could also be typographical in these errors notes.

Sequences of Events

- Let E_n be sequence of events.
  - Toss a coin n times. E_n is the indicator variable of there being at least one heads in the n tosses.
  - Toss a coin n times. E_n is the indicator variable that all the tosses yield heads.
  - Keep tossing a coin; E_n is the event that there is at least one heads from the first n tosses.
- Consider the following events

$$
\begin{aligned}
A_n &= \cup_{k\geq n}^{\infty} E_k \\
A = \limsup_{n\to\infty} E_n &:= \cap_{n=1}^{\infty} \cup_{k\geq n}^{\infty} E_k
\end{aligned}
$$

  - A_n occurs if any of $E_n, E_{n+1}, \ldots$ occur.
  - A occurs if for all n, E_k occurs at least once for $k \geq n$.
  - If A occurs, we say that E_n occurs **infinitely often** or **i.o.**

Borel-Cantelli Lemma

Theorem

Let $p_n = \Pr(E_n)$ *for* $n = 1, 2, \ldots$. *Suppose*

$$\sum_{n=1}^{\infty} p_n < \infty.$$

Then with probability 1, A does not occur,

If the probabilities p_n are summable then

- With probability 1, E_n does not occur infinitely often.
- With probability 1, only a finitely many of $E_1, E_2, \ldots$ will occur.

Theorem

Let $p_n = \Pr(E_n)$ *for* $n = 1, 2, \ldots$. *Suppose that the* E_n *are independent and*

$$\sum_{n=1}^{\infty} p_n = \infty.$$

Then with probability 1, A occurs

Continuity of Probability

Theorem

Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$, i.e., E_n is an inreasing sequence of events. Let $A = \bigcup_{k=1}^{\infty} E_k = \lim_{n\to\infty} E_n$.

$$\Pr\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \Pr(E_n) .$$

Proof Let $A_k = E_k\overline{E}_{k-1} = E_k \setminus E_{k-1}$ and $A_0 = \phi$. Clearly $\bigcup_{k=1}^{n} A_k = E_n$

$$\Pr\left(\bigcup_{k=1}^{\infty} E_k\right) = \Pr\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \Pr(A_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \Pr(A_k)$$

$$= \lim_{n\to\infty} \Pr\left(\bigcup_{k=1}^{n} A_k\right) = \lim_{n\to\infty} \Pr(E_k)$$

Theorem

Let $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$, i.e., E_n is an inreasing sequence of events. Let $A = \bigcap_{k=1}^{\infty} E_k = \lim_{n\to\infty} E_n$.

$$\Pr\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \Pr(E_n) .$$

Borel-Cantelli Lemma

- Consider the following events

$$A_n = \bigcup_{k \geq n}^{\infty} E_k$$

$$A = \limsup_{n \to \infty} E_n := \bigcap_{n=1}^{\infty} \bigcup_{k \geq n}^{\infty} E_k$$

  - A_n occurs if any of $E_n, E_{n+1}, \ldots$ occur.
  - A occurs if for all n, E_k occurs at least once for $k \geq n$.
  - If A occurs, we say that E_n occurs **infinitely often** or **i.o.**

Lemma

Let $p_n = \Pr(E_n)$ for $n = 1, 2, \ldots$.

- *If $\sum_{n=1}^{\infty} p_n < \infty$, then E_n occurs infinitely often with probability 0.*
- *If E_n are independent and $\sum_{n=1}^{\infty} p_n = \infty$, then E_n occurs infinitely often with probability 1.*

Proof of First Borel-Cantelli Lemma

$$
\begin{aligned}
\Pr\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) &= \Pr\left(\bigcap_{n=1}^{\infty} A_n\right) \\
&= \lim_{n\to\infty} \Pr(A_n) \\
&= \lim_{n\to\infty} \Pr\left(\bigcup_{k=n}^{\infty} E_k\right) \\
&\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} p_k \\
&= 0
\end{aligned}
$$

Proof of Second Borel-Cantelli Lemma

$$
\begin{aligned}
\Pr\left(\bigcap_{k=n}^{\infty} \overline{E}_k\right) &= \prod_{k=n}^{\infty}(1-p_k) \\
\log\left(\prod_{k=n}^{\infty}(1-p_k)\right) &= \sum_{k=n}^{\infty}\log(1-p_k) \le -\sum_{k=n}^{\infty} p_k = -\infty \\
\Pr\left(\bigcap_{k=n}^{\infty} \overline{E}_k\right) &\le e^{-\sum_{k=n}^{\infty} p_k}
\end{aligned}
$$

$$
\begin{aligned}
\Pr\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} \overline{E}_k\right) &\le \sum_{n=1}^{\infty}\Pr\left(\bigcap_{k=n}^{\infty} E_k\right) \\
&\le \sum_{n=1}^{\infty} e^{-\sum_{k=n}^{\infty} p_k} \le \infty
\end{aligned}
$$

$\Longrightarrow$ $\overline{E}_n$ does not occur infinitely often. Hence E_n occurs infinitely often.

Sequences of Random Variables and Convergence

- Seqnece of random variables X_n is a sequence of maps from the sample space Ω with elements ω to the reals.
- For every $\omega \in \Omega$ there is a sequence $\mathsf{X}_n(\omega)$.
- Let X be another random variable, i.e., we also have $\mathsf{X}(\omega)$.
- We want to define $\mathsf{X}_n \to \mathsf{X}$ for problems like
  - What happens to the sample mean and sample variance with increasing number of samples.
  - When different sources of random disturbances get added to a signal, is there a way to characterise the net disturbance.
- Recall sequences and convergence: The sequence $x_n \to x$ if for every $\epsilon > 0$, there exists an $N(\epsilon)$ such that for all $n > N(\epsilon)$, $|x_n - x| < \epsilon$. , informally, x_n gets arbitrarily close to x and stays close.

Convergence concepts

- **Convergence everywhere** or **pointwise convergence:** For every $\omega \in \Omega$

$$\mathsf{X}_n(\omega) \to \mathsf{X}(\omega)$$

- **Almost sure convergence** or **convergence almost everywhere.** The following event has probability 1

$$\{\omega : \mathsf{X}_n(\omega) \to \mathsf{X}(\omega)\}$$

- **Mean square convergence**

$$\mathsf{E}\left((\mathsf{X}_n - \mathsf{X})^2\right) \to 0.$$

- **Convergence in probability:** For $\epsilon > 0$,

$$\mathsf{Pr}(\{\omega : |\mathsf{X}_n(\omega) - \mathsf{X}(\omega)| > \epsilon\}) \to 0$$

- **Convergence in distribution**

$$F_{\mathsf{X}_n}(x) \to F_{\mathsf{X}}(x)$$

Convergence of Random Sequences: Implications

Theorem

- $X_n \xrightarrow{\text{a.s.}} X \implies X_n \xrightarrow{\text{p}} X$
- $X_n \xrightarrow{\text{m.s.}} X \implies X_n \xrightarrow{\text{p}} X$
- *If* $X_n \xrightarrow{\text{d}} X$ *and* X *is a constant* c, *then* $X_n \xrightarrow{\text{p}} c$

Limit Theorems

- X_n is a sequence of i.i.d. random variables with mean μ and variance σ^2.
- S_n is the sample sum, i.e.,

$$S_n = \sum_{i=1}^{n} X_n$$

  and S_n/n is the sample mean.
- Interested in the behaviour of S_n/n.

Theorem

- *Weak Law of Large Numbers:* $\frac{S_n}{n} \xrightarrow{p} \mu$
- *Strong Law of Large Numbers:* $\frac{S_n}{n} \xrightarrow{a.s.} \mu$

Limit Theorems

- S_n has mean $n\mu$ and variance $n\sigma^2$.
- $$Z_n := \frac{S_n - n\mu}{\sqrt{n}\sigma}$$

  has zero mean and unit variance.
- Central Limit Theorem