Recall

Last time we saw some examples of computing improper integrals using residue calculus. We ended the lecture with the proof of the maximum modulus theorem which stated that a non-constant holomorphic function on a domain never attains its maximum modulus at any point in the domain. I commented that this fails for real analytic functions. The maximum modulus theorem has a nice consequence, namely Schartz lemma. This theorem once again emphasizes the rigid the nature of holomorphic functions.

Schwarz Lemma : Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk and let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map such that $f(0) = 0$ and $|f(z)| \leq 1$ on $\mathbb{D}$.
Then, $|f(z)| \leq |z| \; \forall z \in \mathbb{D}$ and $|f'(0)| \leq 1$.
Moreover, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

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Then $g(z)$ is holomorphic on the whole of $\mathbb{D}$. Now if $D_r = \{z : |z| \leq r\}$ denotes the closed disk of radius r centered at the origin, then the maximum modulus principle implies that, for $r < 1$, given any z in D_r, there exists z_r on the boundary of D_r such that

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$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}.$$

As $r \to 1$ we get $|g(z)| \leq 1$.

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As $r \to 1$ we get $|g(z)| \leq 1$.

Moreover, suppose $|f(z)| = |z|$ for some non-zero z in $\mathbb{D}$, or $|f'(0)| = 1$.Then, $|g(z)| = 1$ at some point of $\mathbb{D}$. Hence by Maximum Modulus Principle, $g(z)$ is a constant , say a with $|a| = 1$. Therefore, $f(z) = az$, as desired.

Open Mapping Theorem

The maximum modulus theorem is a special case of a even more powerful theorem called the Open Mapping Theorem.

Theorem: Any non-constant holomorphic function defined on an open set $\Omega \subseteq \mathbb{C}$ is open; i.e, maps open subsets of $\mathbb{C}$ contained in Ω to open subsets of $\mathbb{C}$.

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The theorem has an interesting proof which unfortunately we will skip due to lack of time. One of the striking applications of this theorem is another proof of the fundamental theorem of algebra. I will list the key steps here without going into details :
1. Let $f(z)$ be a non-constant polynomial. Then $f(z)$ is a proper map. Now its a standard fact in topology (not hard to prove) that any proper map is closed. That is, maps closed sets to closed sets. Hence image of $\mathbb{C}$ under f is closed.
2. By open mapping theorem, it is also an open map.

3. Thus the image of $\mathbb{C}$ under $f(z)$ is both open and closed. Now recall from lecture 1 that the only subsets of $\mathbb{C}$ that are both open and closed are $\emptyset$ and $\mathbb{C}$. This is a consequence of the path-connectedness of $\mathbb{C}$. Hence image of $f(z)$ is all of $\mathbb{C}$ thereby proving the FTA.

Here is an interesting fact whose content suggests that complex analytic functions are not THAT rigid ! First recall that a meromorphic function on an open subset Ω of the complex plane is a function that is holomorphic on all of Ω except for a discrete set of points, which are **poles** of the function.
Mittag-Leffler's Theorem : Given any discrete sequence of points going to infinity, there exists a meromorphic functions with poles exactly along this sequence and having prescribed principal parts at those poles.

Examples

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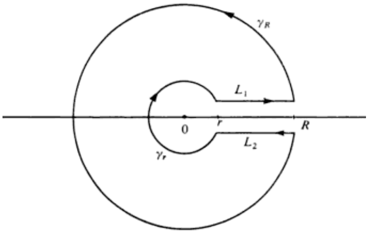
where z^{-c} is the branch corresponding to branch cut being the positive real axis, and integrate along the contour: Note that any open subset of $\mathbb{C}$ on which $log(z)$ has a branch, can be used to define z^{-c} as a holomorphic function. For, $z^{-c} = e^{-clogz}$ and composite of holomorphic functions is holomorphic. Let δ be the vertical distance between the X-axis and each of the horizontal lines L_1 and L_2.

Branch



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$$\int_r^R \frac{t^{-c}}{1+t}dt = \lim_{\delta\to 0}\int_{L_1}\frac{z^{-c}}{1+z}dz.$$

Similarly,

$$\lim_{\delta\to 0}\int_{L_2}\frac{z^{-c}}{1+z}dz = -e^{-2\pi\imath c}\int_r^R \frac{t^{-c}}{1+t}dt.$$

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$$2\pi\imath e^{-\imath\pi c} = (1 - e^{-2\imath\pi c}) \int_0^\infty \frac{t^{-c}}{1+t} dt.$$

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Thus,

$$\int_0^\infty \frac{t^{-c}}{1+t} dt = \frac{2\pi\imath e^{-\imath\pi c}}{1 - e^{-2\imath\pi c}} = \frac{\pi}{\sin \pi c}.$$

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As before, we consider the function $f(z) = \frac{e^{\frac{z}{2}}}{coshz}$. By residue theorem, $\int_\gamma \frac{e^{z/2}dx}{coshz} = 2\pi i Res(f, i\frac{\pi}{2}) = 2\pi e^{i\frac{\pi}{4}}$.

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Now $|cosh(L + iy)| = |e^{L+iy} + e^{-L-iy}|/2 \geq \frac{1}{2}(|e^{L+iy}| - |e^{-L-iy}|) = (e^L - e^{-L})/2 \geq e^L/4$

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When Re $z = L$, $|f(z)| \leq \frac{e^{L/2}}{e^L/4} = \frac{4}{e^{L/2}}$. It thus follows from the ML-inequality that as L tends to ∞, the integral along the right vertical side tends to zero. Similarly one checks that the integral along the left vertical side also tend to zero.
Now since $cosh(x + i\pi) = -coshx$, the integrals along the horizontal sides are related by

$$\int_L^{-L} \frac{e^{(x+i\pi)/2}dx}{cosh(x + i\pi)} = e^{i\pi/2} \int_{-L}^{L} \frac{e^{x^2}dx}{coshx}$$

where the L.H.S. is the integral along the upper horizontal path and the integral in the R.H.S is the path integral along the lower horizontal path.

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$I = \frac{2\pi e^{i\pi/4}}{(1+e^{i\pi/2})} = \frac{\pi}{cos(\pi/4)} = \pi\sqrt{2}$.