Signal Processing - 1 by One

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Outline

- So Far: Impulse, Sampling, Replacement
- Previous Week: Convolution ($\circledast$) and Interpolation
- Previous Class: Drawing with Sinusoids
- Today: Fourier Series

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Function Approximation

Aim: approx. $y(t)$ by $f(t)$ s.t. the error $\|y(t) - af(t)\|_{\ell_2}^2$ is minimum.

Solution:

$$y_{approx}(t) = \frac{\langle y(t), f(t)\rangle}{\|f(t)\|_{\ell_2}^2} f(t),$$

where

$$\langle y(t), f(t)\rangle = \int_{\mathbb{T}} y(t) f^*(t) dt.$$

Joint Approximation by Two Functions

Definition: Two functions $f_1(t)$ and $f_2(t)$ are said to be **orthogonal** over the interval $[0, T_d]$ if

$$\langle f_1(t), f_2(t) \rangle = 0.$$

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Examples:

1. $f_1(t) = \mathbb{I}_{\{0 \le t \le T_d\}}$ and $f_2(t) = \sin(\frac{2\pi}{T_d} t)$.
2. $f_1(t) = \mathbb{I}_{\{0 \le t \le T_d\}}$ and $f_2(t) = \cos(\frac{2\pi}{T_d} t)$.
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Minimizing MSE

$$\min \int_{\mathbb{T}} |y(t) - c_1 f_1(t) - c_2 f_2(t)|^2 dt$$ with $f_1(t)$ and $f_2(t)$ orthogonal in $\mathbb{T}$.

The objective can be written as

$$\|y(t) - c_1 f_1(t)\|_{\ell_2}^2 + |c_2|^2 \|f_2(t)\|_{\ell_2}^2 - 2\,\mathrm{Real}\big(c_2 \langle y(t) - c_1 f_1(t), f_2(t) \rangle\big)$$

$$= \|y(t) - c_1 f_1(t)\|_{\ell_2}^2 + |c_2|^2 \|f_2(t)\|_{\ell_2}^2 - 2\,\mathrm{Real}\big(c_2 \langle y(t), f_2(t) \rangle\big).$$

The partial derivative w.r.t c_1 is zero at the minimum.

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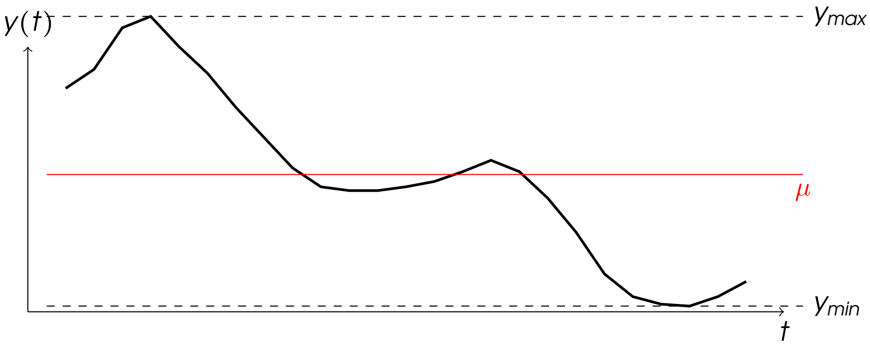
The partial derivative w.r.t c_1 is zero at the minimum.

$$c_1 = \frac{\langle y(t), f_1(t)\rangle}{\|f_1(t)\|_{\ell_2}^2} \text{ and } c_2 = \frac{\langle y(t), f_2(t)\rangle}{\|f_2(t)\|_{\ell_2}^2}$$

Drawing Our Envelope



$\alpha_0 = 1.8125,$

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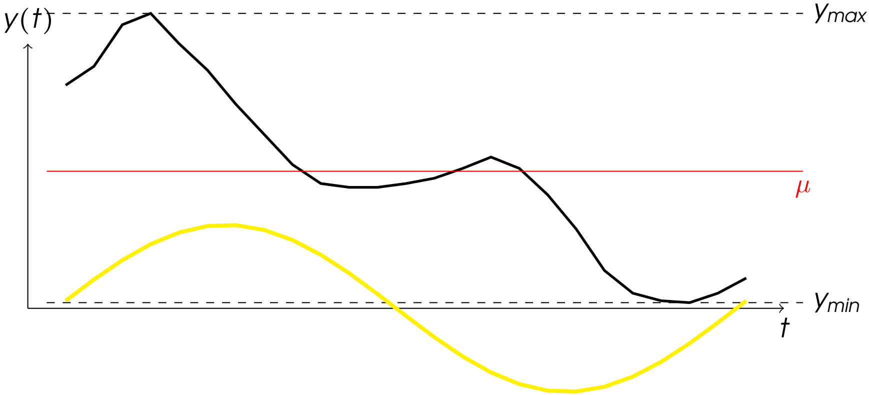

$\alpha_0 = 1.8125$, $b_1 = 1.1$,

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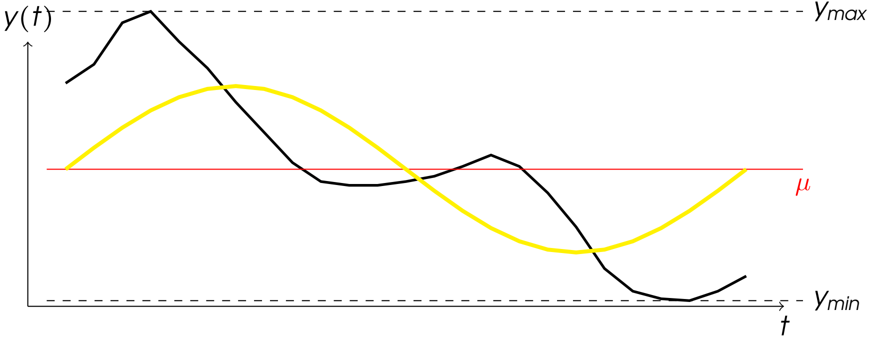

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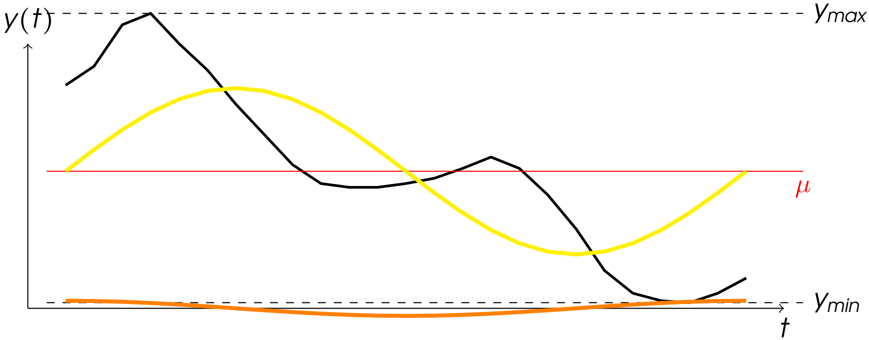

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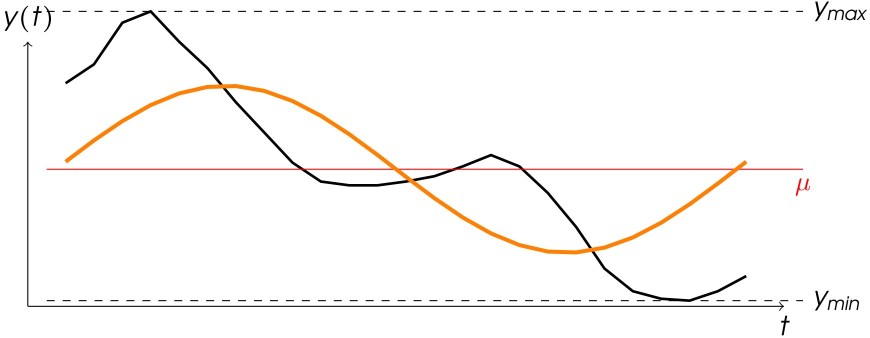

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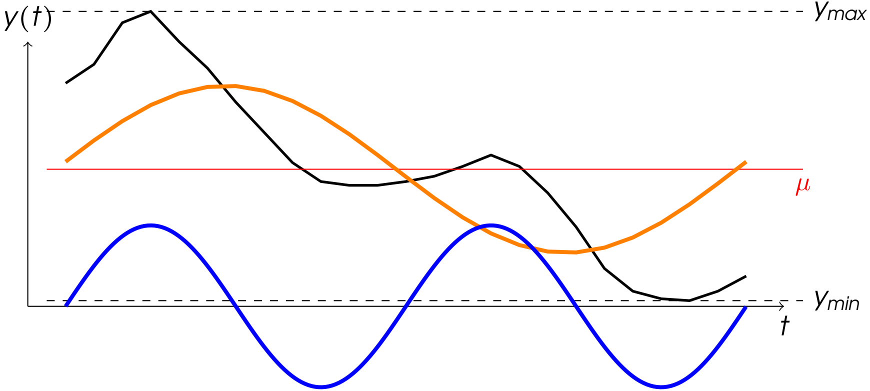

$\alpha_0 = 1.8125$, $b_1 = 1.1$, $a_1 = 0.1$, $b_2 = 1.09$,

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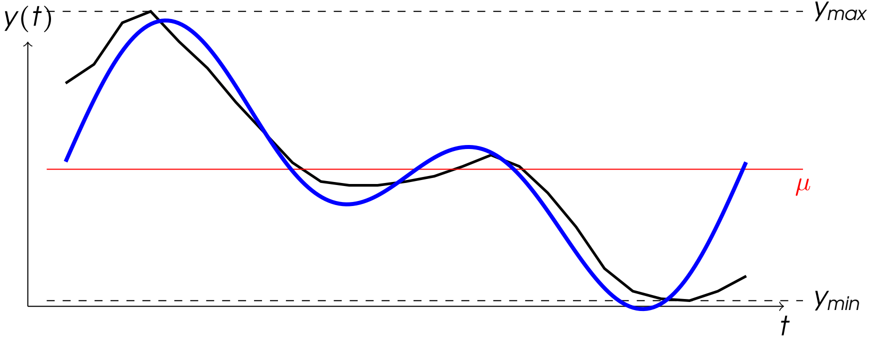

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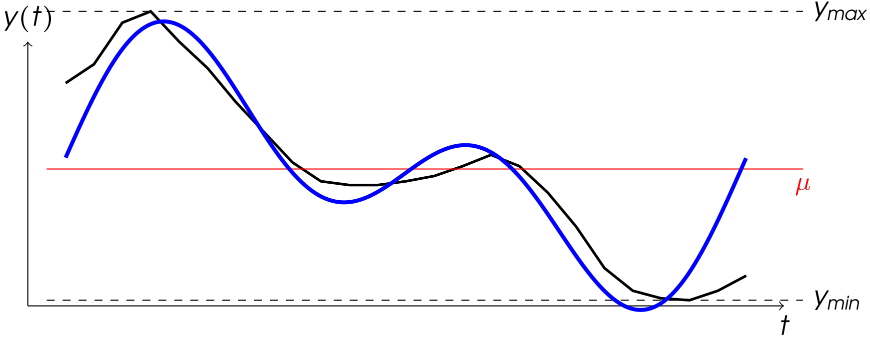

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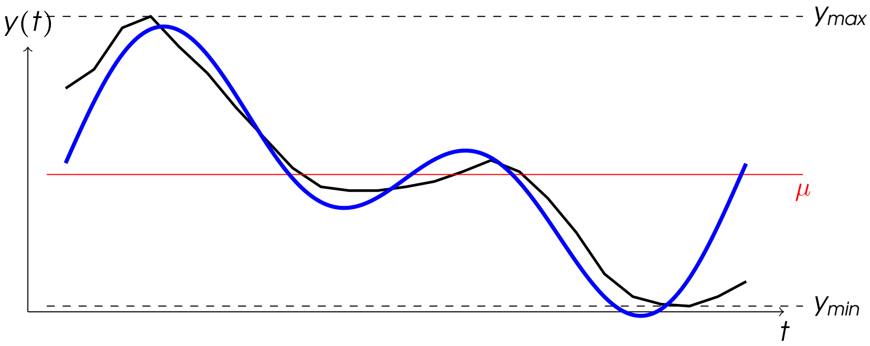

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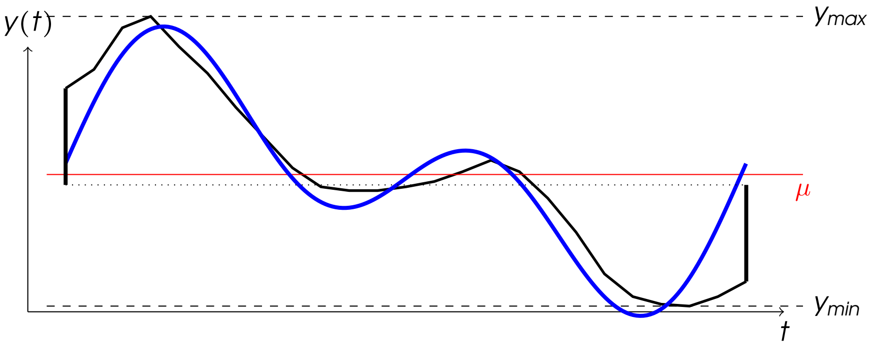

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Fourier Theory

200 years since Joseph Fourier showed (1808) that *many* signals have a '*concise*' representation in frequency domain.

Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi f\, t)\,dt.$$