MA 205 Complex Analysis: Power Series

August 24, 2020

Introduction

In the last lecture, we saw holomorphic functions in some detail. If $f = u + \imath v$ is holomorphic in Ω, then (i) both u and v satisfy CR equations, and (ii) $f(x, y) = (u(x, y), v(x, y))$ is real differentiable. We also saw that though neither (i) nor (ii) is sufficient to guarantee holomorphicity, both (i) and (ii) together do guarantee holomorphicity of f. We also studied harmonic functions which are closely related to holomorphic functions. The notion of a harmonic conjugate of a given harmonic function was defined, and it was stated that the existence of a harmonic conjugate for every harmonic function is guaranteed if and only if the domain is "nice". The precise mathematical condition replacing "nice" is **simply connected**. We shall come to this notion soon.

Cauchy (1789-1857) & Riemann (1826-1866); Wiki

"More concepts and theorems have been named for Cauchy than for any other mathematician (in elasticity alone there are sixteen concepts and theorems named for Cauchy). Cauchy was a prolific writer; he wrote approximately eight hundred research articles and five complete textbooks."

Berhard Riemann(1826-1866) is one of the greatest mathematician of all times. After Newton, he is possibly the mathematician who has had the greatest impact on the development of mathematics ever since. In a relatively short mathematical career he wrote under 20 paper, most of them worth their weight in gold. His areas of contribution include - Real Analysis, Complex Analysis, Function Theory, Riemannian Geometry, Number theory - most importantly the distribution of primes, various areas of mathematical physics ... A conjecture he made in 1859, now known as the Riemann hypothesis is considered to be the most important open problem today. Proving it would also fetch one a million dollars from the Clay Mathematics Foundation !

Polynomials

Today, we'll discuss the so called analytic functions. To warm up, let's first look at the simplest of all functions. What's the most trivial function? $f(z) = a_0$, i.e, constant functions. The next easiest class of functions are polynomials:
$f(z) = a_0 + a_1 z + \ldots + a_n z^n$, $a_i \in \mathbb{C}$.

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$$b_0 + b_1(z - z_0) + \ldots + b_n(z - z_0)^n.$$

The complex numbers b_i can be easily calculated. Of course, you could expand powers of $(z - z_0)$ using binomial theorem and compare coefficients to get b_i in terms of a_i.

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The complex numbers b_i can be easily calculated. Of course, you could expand powers of $(z - z_0)$ using binomial theorem and compare coefficients to get b_i in terms of a_i. Perhaps a smarter way would be to notice that $b_i = \frac{f^{(i)}(z_0)}{i!}$, which then can be calculated from $f(z) = \sum_{i=0}^{n} a_i z^i$.

Power Series

A polynomial, by definition, is a "finite" polynomial; i.e., it comes with a finite degree. As the next class of functions, we consider functions defined by their power series. They look like:

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or more generally,

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Of course one has to be careful; there are convergence issues. For example, $f(z) = 1 + z + z^2 + \dots$ makes sense for all z such that $|z| < 1$, but not when $|z| > 1$. (Why?) We say that this power series has radius of convergence $= 1$.

Power Series

It's a beautiful fact that the radius of convergence exists for any power series; i.e., there exists a real number R such that $\sum_{i=0}^{\infty} a_i(z - z_0)^i$ converges when $|z - z_0| < R$, and diverges when $|z - z_0| > R$.

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Before we prove the existence of radius of convergence, let's recall a few definitions and observations.

We write $a = \sum_{n=1}^{\infty} a_i$, $a_i \in \mathbb{C}$, if the sequence of partial sums $\{s_n\}$, where $s_n = a_1 + \ldots + a_n$, converges, and $\lim_{n\to\infty} s_i = a$. The series $\sum_{n=1}^{\infty} a_i$ is said be <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_i|$ is convergent.

<u>Exercise</u>:

1. Absolute convergence $\implies$ convergence.

2. (Comparison Test) If $\sum_{n=1}^{\infty} b_i$ is absolutely convergent, and if $|a_i| \leq |b_i|$ for all large enough i, then $\sum_{n=1}^{\infty} a_i$ is absolutely convergent.

The series $1 - \frac{1}{2} + \frac{1}{3} + \ldots$ converges to $\ln 2$ but does not converge absolutely. After all the harmonic series $\sum \frac{1}{n}$ diverges.

Recall **Supremum**: Let $\{x_n\}$ be a sequence of real numbers. We say that a real number M is the supremum of this sequence if every term of the sequence is less than or equal to M and there exists terms of the sequence which are arbitrarily close to M. Equivalently it is the smallest real number having the property that it is greater than or equal to all the terms of the sequence. The supremum may or may not exist and if it exists it may or may not be equal to any of the terms of the sequence (that is the supremum may not be attained by the sequence). For example, supremum of the sequence 1.4, 1.41, 1.414, 1.4142 ,... is $\sqrt{2}$.

Limit Supremum

Upper Limit/ Limit Supremum: for a sequence of real numbers $x_1, x_2, \ldots$, let y_n be the supremum of the set $\{x_n, x_{n+1}, \ldots\}$. Then the sequence $y_1, y_2, \ldots$ is a monotonically decreasing sequence which diverges to ∞ or has a finite limit. This is called the upper limit (also called limit superior, denoted lim sup) of the sequence $\{x_i\}$. It can be ∞. If limit exists, then the upper limit coincides with the usual limit.

Examples:

1. If $\{x_n\}$ is a convergent sequence converging to l, then l is the lim sup. This follows immediately from the definition.
2. the sequence $1, 2, 3, \ldots$ has lim sup ∞.
3. the sequence $1, -1, 1, -1, ...$ has lim sup 1.
4. the sequence $1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4...$ has upper limit ∞.

Theorem (Cauchy's Root Test)

For a series $\sum_{n=1}^{\infty} a_i$ *of complex numbers, let* $C = \limsup_{i\to\infty} \sqrt[i]{|a_i|}$. *Then the series converges absolutely if* $C < 1$ *and it diverges if* $C > 1$.

The test is indecisive for $C = 1$

Theorem (Ratio Test)

For a series $\sum_{n=1}^{\infty} a_i$, *let* $L = \limsup_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$. *Then, if* $L < 1$, *the series converges absolutely. The series diverges if there exists* N *such that* $\left|\frac{a_{i+1}}{a_i}\right| > 1$ *for* $i \geq N$.

Remark $L > 1$ in the above test doesn't imply that the series diverges. (Exercise !)

Proof: Let $L < 1$. Let r be such that $L < r < 1$. Then after a stage, say for $i \geq N$, $|a_{i+1}| < r|a_i|$. So $|a_{i+k}| < r^k|a_i|$.

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$$\sum_{n=1}^{\infty} |a_i| = \sum_{0}^{N} |a_i| + \sum_{N+1}^{\infty} |a_i| = \sum_{0}^{N} |a_i| + \sum_{1}^{\infty} |a_{N+i}|$$

$$< \sum_{0}^{N} |a_i| + |a_N| \sum_{1}^{\infty} r^i = \sum_{0}^{N} |a_i| + |a_N| \frac{r}{1-r} < \infty.$$

Theorem (Existence of Radius of Convergence)

For the power series $\sum_{n=1}^{\infty} a_i(z-z_0)^i$, *let* $R = \dfrac{1}{\limsup_{i\to\infty} \sqrt[i]{|a_i|}}$. *Then the power series converges absolutely if* $|z-z_0| < R$ *and diverges if* $|z-z_0| > R$.

Proof: Apply root test. The series absolutely at a complex number z if $\limsup |a_i|^{\frac{1}{i}} |z-z_0| < 1$. That is $|z-z_0| < \dfrac{1}{\limsup \sqrt[i]{|a_i|}}$.

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If $\lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$ exists, then by applying the ratio test instead of the root test, it follows that $R = \lim_{i\to\infty} \left|\frac{a_i}{a_{i+1}}\right|$.

Remark: If a series converges by the ratio test, then it converges by the root test as well. But not conversely. Thus the root test is better than the ratio test. But the ratio test is often easier to use whenever it succeeds.

In fact:

$$\limsup_{i\to\infty} \sqrt[i]{|a_i|} \leq \limsup_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right|$$

Examples:

1. $\sum_{n=1}^{\infty} \frac{z^i}{i}$. Apply ratio test. $\lim_{i\to\infty} \left|\frac{a_i}{a_{i+1}}\right| = \lim_{i\to\infty} \frac{i+1}{i} = 1$. Hence radius of convergence is 1.

2. $\sum_{n=1}^{\infty} \frac{z^i}{i!}$. Apply ratio test. $\lim_{i\to\infty} \left|\frac{a_i}{a_{i+1}}\right| = \lim_{i\to\infty} i = \infty$; i.e., the series converges everywhere.

3. $z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots$ Radius of convergence is 1. Both the tests apply here. Let us apply the root test.
Then $R = \frac{1}{\limsup_{i\to\infty} \sqrt[i]{|a_i|}} = \frac{1}{\limsup_{i\to\infty} \sqrt[i]{\frac{1}{2i+1}}} = 1$ since $\sqrt[i]{2i+1}$ tends to 1 as i tends to ∞.

4. $\frac{1}{2} + \frac{1}{3}z + \left(\frac{1}{2}\right)^2 z^2 + \left(\frac{1}{3}\right)^2 z^3 + \ldots$. Check that the ratio test fails. Lets apply root test to show that the radius of convergence is $\frac{1}{\sqrt{2}}$.

First observe that if $\{x_n\}$ is a sequence of real numbers, then $\limsup\{x_n\} = max\{\limsup x_{2n}, \limsup x_{2n+1}\}$.

Applying this to the above series, we get

$\limsup \sqrt[i]{|a_i|} = max\{\limsup \sqrt[i]{\left(\frac{1}{2}\right)^{\frac{i+1}{2}}}, \limsup \sqrt[i]{\left(\frac{1}{3}\right)^{\frac{i}{2}}}\}$

$= \max \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\}$

$= \frac{1}{\sqrt{2}}$

Power series can be added, subtracted, and multiplied in the obvious way. It can also be differentiated and integrated term by term, in its domain of convergence. Indeed, $f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$, then,

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \sum_n a_n \left(\lim_{h \to 0} \frac{[(z - z_0 + h)^n - (z - z_0)^n]}{h} \right)$$

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Apply root test to check that a given power series, the differentiated series and the integrated series, all have the same radius of convergence.

Analytic Functions

A function $f : \Omega \to \mathbb{C}$ is said to be **analytic** if it is locally given by a convergent power series; i.e., every $z_0 \in \Omega$ has a neighbourhood contained in Ω such that there exists a power series centered at z_0 which converges to $f(z)$ for all z in that neighbourhood.

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holomorphic $\implies$ *analytic.*

This would prove our statement from Lecture 1 that once differentiable is always differentiable!

Analytic Functions

Just as in the complex case, power series and analytic functions can be defined in the real case too. But unlike in the complex case, differentiable does not mean real analytic. In fact, even infinitely differentiable does not mean real analytic. For example, $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

is infinitely differentiable but not real analytic. In this example, $f^{(i)}(0) = 0$ for all i, and thus the Taylor series of f is the zero function.

<u>**Remark:**</u> In fact such functions are in "abundance". (This statement can be made precise). Their existence is of great importance in analysis.