Signal Processing - 1 by One

Sibi Raj B. Pillai
Dept of Electrical Engineering
IIT Bombay

22-10-2020, Class-21

Outline

- So Far: Sampling, Fourier Analysis
- Previous Week: DTFT
- Previous Class: Circular Convolution
- Today: DFT and Properties

Outline

- So Far: Sampling, Fourier Analysis
- Previous Week: DTFT
- Previous Class: Circular Convolution
- Today: DFT and Properties

DFT: Matrix Form

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp(-j\frac{2\pi}{N}kn).$$

$$F = \begin{bmatrix} \alpha_0^0 & \alpha_1^0 & \alpha_2^0 & \cdots & \alpha_M^0 \\ \alpha_0^1 & \alpha_1^1 & \alpha_2^1 & \cdots & \alpha_M^1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_M^2 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \alpha_0^{M'} & \alpha_1^M & \alpha_2^M & \cdots & \alpha_M^M \end{bmatrix},$$

$$\alpha = \exp(-j\frac{2\pi}{N})$$

$$\alpha_i = \alpha^i\,,\ 0 \le i \le N-1$$

$$M = N - 1$$

DFT : $\bar{X} = F\,\bar{x}.$

Inverse DFT

Proposition:

$$F^H F = N \, \mathbb{I}_N .$$

Inverse DFT

Proposition:

$$F^H F = N\,\mathbb{I}_N.$$

Proof:

$$\left(F^H F\right)_{k_1,k_2} = \sum_{m=0}^{N-1} \exp(j\frac{2\pi}{N}k_1 m)\exp(-j\frac{2\pi}{N}mk_2)$$

Inverse DFT

Proposition:

$$F^H F = N\,\mathbb{I}_N.$$

Proof:

$$\begin{aligned}
\left(F^H F\right)_{k_1,k_2} &= \sum_{m=0}^{N-1} \exp(j\frac{2\pi}{N}k_1 m)\exp(-j\frac{2\pi}{N}mk_2) \\
&= \sum_{m=0}^{N-1} \exp(-j\frac{2\pi}{N}(k_2-k_1)m)
\end{aligned}$$

Inverse DFT

Proposition:

$$F^H F = N\,\mathbb{I}_N.$$

Proof:

$$\begin{aligned}
\left(F^H F\right)_{k_1,k_2} &= \sum_{m=0}^{N-1} \exp(j\frac{2\pi}{N}k_1 m)\exp(-j\frac{2\pi}{N}mk_2) \\
&= \sum_{m=0}^{N-1} \exp(-j\frac{2\pi}{N}(k_2-k_1)m) \\
&= \frac{1-\exp(-j2\pi(k_2-k_1))}{1-\exp(-j\frac{2\pi}{N}(k_2-k_1))}
\end{aligned}$$

Inverse DFT

Proposition:

$$F^H F = N\,\mathbb{I}_N.$$

Proof:

$$\begin{aligned}
\left(F^H F\right)_{k_1,k_2} &= \sum_{m=0}^{N-1} \exp(j\frac{2\pi}{N}k_1 m)\exp(-j\frac{2\pi}{N}mk_2) \\
&= \sum_{m=0}^{N-1} \exp(-j\frac{2\pi}{N}(k_2-k_1)m) \\
&= \frac{1-\exp(-j2\pi(k_2-k_1))}{1-\exp(-j\frac{2\pi}{N}(k_2-k_1))} \\
&= N\delta[k_2-k_1].
\end{aligned}$$

Inverse DFT

Proposition:

$$F^H F = N\,\mathbb{I}_N.$$

Proof:

$$\begin{aligned}
\left(F^H F\right)_{k_1,k_2} &= \sum_{m=0}^{N-1} \exp(j\frac{2\pi}{N}k_1 m)\exp(-j\frac{2\pi}{N}mk_2) \\
&= \sum_{m=0}^{N-1} \exp(-j\frac{2\pi}{N}(k_2-k_1)m) \\
&= \frac{1-\exp(-j2\pi(k_2-k_1))}{1-\exp(-j\frac{2\pi}{N}(k_2-k_1))} \\
&= N\delta[k_2-k_1].
\end{aligned}$$

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\exp(j\frac{2\pi}{N}kn).}$$

DFT and Circular Convolution

Proposition

$$x[n] \circledast h[n] \xrightleftharpoons{DFT} X[k]H[k].$$

In other words, $y[n] = x[n] \circledast h[n] \Rightarrow Y[k] = X[k]H[k]$.

DFT and Circular Convolution

Proposition

$$x[n] \circledast h[n] \xrightleftharpoons{DFT} X[k]H[k].$$

In other words, $y[n] = x[n] \circledast h[n] \Rightarrow Y[k] = X[k]H[k]$.

Proof

$$\sum_{n=0}^{N-1}\Big(\sum_{m=0}^{N-1} h[m]x_c[n-m]\Big)\exp(-j\frac{2\pi}{N}kn)$$

$$= \sum_{m=0}^{N-1} h[m]\sum_{n=0}^{N-1} x_c[n-m]\exp(-j\frac{2\pi}{N}kn)$$

DFT and Circular Convolution

Proposition

$$x[n] \circledast h[n] \overset{DFT}{\rightleftharpoons} X[k]H[k].$$

In other words, $y[n] = x[n] \circledast h[n] \Rightarrow Y[k] = X[k]H[k]$.

Proof

$$\sum_{n=0}^{N-1}\Big(\sum_{m=0}^{N-1} h[m]x_c[n-m]\Big)\exp(-j\frac{2\pi}{N}kn)$$

$$= \sum_{m=0}^{N-1} h[m]\sum_{n=0}^{N-1} x_c[n-m]\exp(-j\frac{2\pi}{N}kn)$$

$$= \sum_{m=0}^{N-1} h[m]\sum_{n=0}^{N-1} x_c[n-m]\exp(-j\frac{2\pi}{N}k(n-m))\exp(-j\frac{2\pi}{N}km)$$

DFT and Circular Convolution

Proposition

$$x[n] \circledast h[n] \overset{DFT}{\rightleftharpoons} X[k]H[k].$$

In other words, $y[n] = x[n] \circledast h[n] \Rightarrow Y[k] = X[k]H[k]$.

Proof

$$\sum_{n=0}^{N-1} \Big(\sum_{m=0}^{N-1} h[m] x_c[n-m] \Big) \exp(-j\frac{2\pi}{N} kn)$$

$$= \sum_{m=0}^{N-1} h[m] \sum_{n=0}^{N-1} x_c[n-m] \exp(-j\frac{2\pi}{N} kn)$$

$$= \sum_{m=0}^{N-1} h[m] \sum_{n=0}^{N-1} x_c[n-m] \exp(-j\frac{2\pi}{N} k(n-m)) \exp(-j\frac{2\pi}{N} km)$$

$$= \sum_{m=0}^{N-1} h[m] \exp(-j\frac{2\pi}{N} km) \sum_{l=-m}^{N-1-m} x_c[l] \exp(-j\frac{2\pi}{N} kl)$$

DFT and Circular Convolution

Proposition

$$x[n] \circledast h[n] \xrightleftharpoons{DFT} X[k]H[k].$$

In other words, $y[n] = x[n] \circledast h[n] \Rightarrow Y[k] = X[k]H[k]$.

Proof

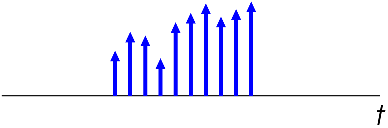
$$\sum_{n=0}^{N-1}\Big(\sum_{m=0}^{N-1} h[m]x_c[n-m]\Big)\exp(-j\frac{2\pi}{N}kn)$$

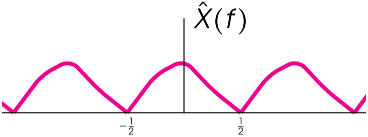
$$= \sum_{m=0}^{N-1} h[m]\sum_{n=0}^{N-1} x_c[n-m]\exp(-j\frac{2\pi}{N}kn)$$

$$= \sum_{m=0}^{N-1} h[m]\sum_{n=0}^{N-1} x_c[n-m]\exp(-j\frac{2\pi}{N}k(n-m))\exp(-j\frac{2\pi}{N}km)$$

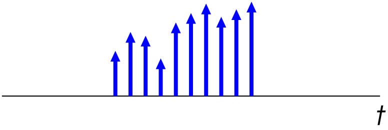
$$= \sum_{m=0}^{N-1} h[m]\exp(-j\frac{2\pi}{N}km)\sum_{l=-m}^{N-1-m} x_c[l]\exp(-j\frac{2\pi}{N}kl) = H[k]X[k]$$

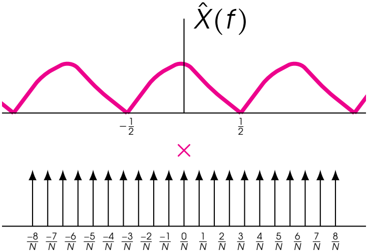
DFT Vs DTFT: Recap



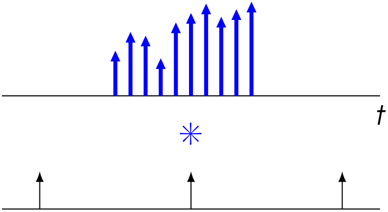


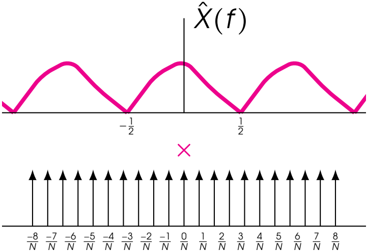
DFT Vs DTFT: Recap



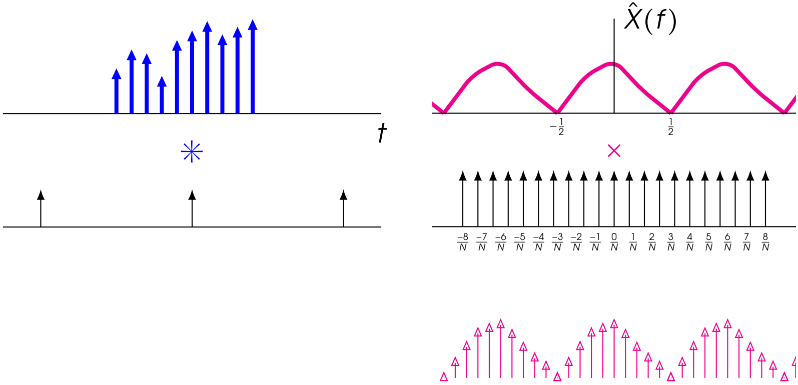


DFT Vs DTFT: Recap

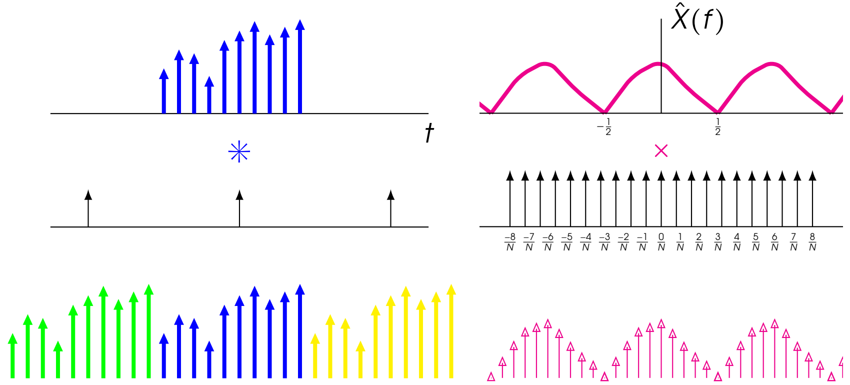




DFT Vs DTFT: Recap



DFT Vs DTFT: Recap

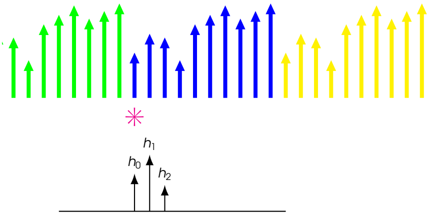


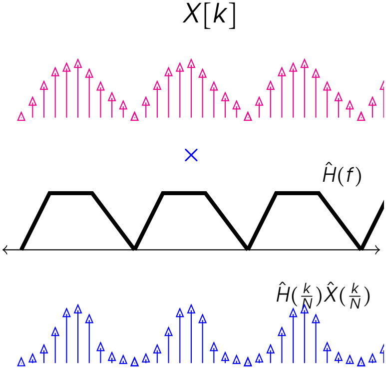
DFT

$$X[k] = \hat{X}\left(\frac{k}{N}\right) = \sum_{n=0}^{N-1} x[n] \exp\left(-j2\pi \frac{k}{N} n\right).$$

DFT-DTFT Product

$$x_c[n] = \sum_{l \in \mathbb{Z}} x[n + lN]$$





$$y[n] = x[n] \circledast h[n]$$

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k] \exp(-j\frac{2\pi}{N}kn)$$