MA 205 Complex Analysis: Laurent Series and Examples

September 14, 2020

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Isolated singularities are of 3 types: Removable singularity, Pole and Essential Singularity.

A singularity at z_0 is **removable** if $\lim_{z \to z_0} f(z)$ exists. In particular $f(z)$ is bounded in a neighborhood of z_0. Also, if $f(z)$ is bounded in a neighborhood of z_0, then f has a removable singularity at z_0.
A singularity at z_0 is a **pole** if $f(z) \to \infty$ as $z \to z_0$. In particular the function takes unbounded values in any punctured neighborhood of z_0.
A singularity at z_0 is an **essential singularity** if it is neither a removable singularity nor a pole.

We also discussed the Casorati Weierstrass theorem which said that restricted to any punctured neighborhood of an essential singularity, the image of $f(z)$ is dense, i.e, comes arbitrarily close to every complex number.

$e^{1/z}$ has an essential singularity at 0. (Why ?)

The Casorati-Weierstrass theorem has a deep generalization, namely the Big Picard Theorem.

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The Big Picard theorem implies another striking and beautiful theorem, namely the **Little Picard Theorem** which states that the image of a non-constant entire function can atmost miss one point. Of course I sketched a proof of the fact that if it does miss one pont then it is of the form $e^{g(z)}$ for some entire function $g(z)$.

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I will make some more comments about these theorem at the end of the course, although a complete proof is beyond the scope of this course.

Laurent Series

Recall how we derived the power series representation of a holomorphic function on a disc centered around z_0. We used

$$f(z) = \frac{1}{2\pi \imath} \int_{|w-z_0|=r} \frac{f(w)}{w-z} dw,$$

and manipulated $\frac{1}{w-z}$ as

$$\frac{1}{w-z_0} \cdot \frac{1}{1 - \frac{z-z_0}{w-z_0}}.$$

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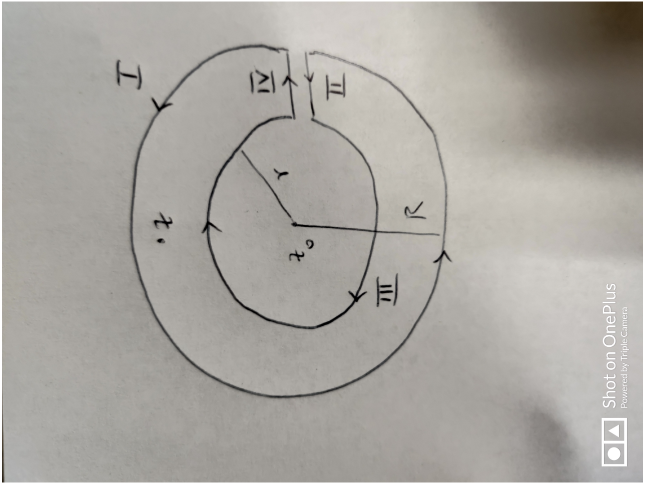

The first integral gives rise to $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ with

$$a_n = \frac{1}{2\pi\imath}\int_{|z-z_0|=R}\frac{f(z)}{(z-z_0)^{n+1}}dz,$$

exactly as before.
In the second integral, write

$$\frac{-1}{w-z} = \frac{1}{z-z_0}\cdot\frac{1}{1-\frac{w-z_0}{z-z_0}},$$

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Note that $\left|\ \frac{w-z_0}{z-z_0}\ \right| < 1$ for all w with $|w-z_0| = r$.

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Residue

We write both together as $\displaystyle\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. This is the Laurent series around the isolated singularity z_0. The negative part is called the principal part of the Laurent series. If z_0 is an isolated singularity of f, then f is holomorphic in an annulus $0 < |z-z_0| < R$ for some R. The corresponding Laurent expansion is called the Laurent expansion around z_0.

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$$a_{-1} = \frac{1}{2\pi\imath}\int_\gamma f(z)dz.$$

If you integrate a Laurent series, only a_{-1} remains; other terms vanish. What remains is usually called a residue.

$$a_{-1} = \mathrm{Res}(f; z_0).$$

Often a_{-1} is easy to compute from $f(z)$ and if that's the case integration has become easy.

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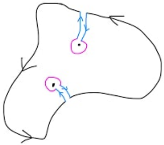
Theorem (Cauchy Residue Theorem)

$$\int_\gamma f(z)dz = 2\pi\imath \cdot \sum_{i=1}^{n} \mathrm{Res}(f, z_i).$$

Proof : We have already seen the proof in the previous lectures. The following figure should remind you of the proof. (Here the case of 2 singularities is considered; similarly one can handle more singular points)



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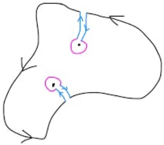

Joke

Why did the mathematician name his dog Cauchy?

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B'coz it left a residue at every pole!

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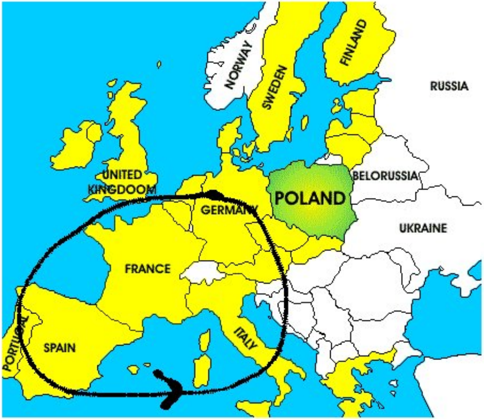

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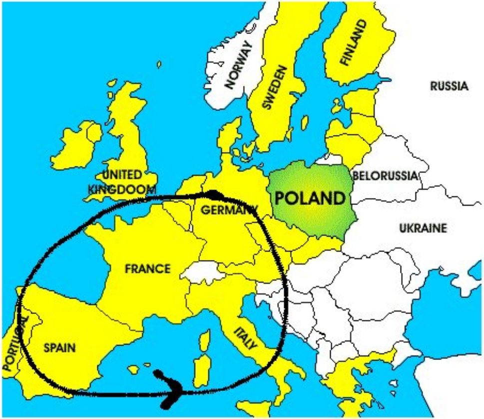

Zero. All the poles are in Eastern Europe.

Modification: Actually there are poles in Western Europe but they are all removable !!

Principal Part of the Laurent Series

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Note that:

- removable iff principal part is zero.
- pole iff principal part is finite.
- essential iff principal part is infinite.

Proof (Easy exercise using previous slides).

Residue at a Pole

If the isolated singularity is removable, then the residue is trivial. If the isolated singularity is a pole, then the residue is trivial to compute. If z_0 is a pole, can write

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \ldots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots.$$

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Thus,

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Thus, g is holomorphic and

$$a_{-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

Example

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$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$.

For the first term, $\frac{1}{z-2} = -\frac{1}{2}(\frac{1}{1-\frac{z}{2}}) = -\frac{1}{2}\sum_{0}^{\infty}(\frac{z}{2})^n$.

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Putting the two together we get the desired Laurent Series Expansion.

Example

Determine the Laurent series of $e^{1/z}$ around the point 0.

$$e^{1/z} = \sum_{0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$$

$$= \sum_{-\infty}^{0} \frac{z^n}{(-n)!}$$

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$= \frac{1}{z} + \frac{1}{z^2} + \frac{5}{6}\frac{1}{z^3} + \frac{1}{2}\frac{1}{z^4} + \cdots$

Example

Determine the Laurent series of $f(z) = \frac{1}{z+2}$ around $z = 1$. The given function has a pole at -2. So we will break up the Laurent series computation for the regions $|z-1| < 3$ and the region $|z-1| > 3$. In the region $|z-1| < 3$, the Laurent series will coincide with the Taylor series since the function is holomorphic there.

For the region $|z-1| < 3$, write $f(z) = \frac{1}{3+(z-1)} = \frac{1}{3}\frac{1}{1+\frac{z-1}{3}}$.

Since $|z-1| < 3$, we can expand this into a geometric series and get $f(z) = \sum_0^\infty \frac{(-1)^n(z-1)^n}{3^{n+1}}$ in this region. For the region

$|z-1| > 3$, write $f(z) = \frac{1}{3+(z-1)} = \frac{1}{z-1}\frac{1}{1+\frac{3}{z-1}} = \frac{1}{z-1}\sum_0^\infty \frac{(-1)^n 3^n}{(z-1)^n}$
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