Signal Processing - 1 by One

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Outline

- So Far: Sampling, Convolution, Interpolation
- Previous Week: Fourier Series and Fourier Transform
- Previous Class: Parseval's Relation
- Today: Inverse Fourier Transform

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Inverse Fourier Transform

$$x(t) = \int_{\mathbb{R}} X(f)\exp(+j2\pi f\, t)\,df,$$

1. For $x(t) \in \mathcal{L}_1$, $X(f) \in \mathcal{L}_1$. (Proof will be discussed)
2. For $x(t) \in \mathcal{L}_1$, $X(f) \in \mathcal{L}_2$. Proof known, but not discussed here
3. Symbolization: $x(t) = \delta(t)$.

Inverse Fourier Transform

$$x(t) = \int_{\mathbb{R}} X(f)\exp(+j2\pi f\,t)\,df,$$

$$\text{where } X(f) = \int_{\mathbb{R}} x(t)\exp(-j2\pi f\,t)\,dt.$$

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Strategy: Holds good for GAUSSIANS, hence several others too.

F.T. of Derivative

Theorem

If $x(t) \in \mathcal{L}_1$ *and* $x(t) \overset{F.T.}{\rightleftharpoons} X(f)$*, then*

$$F.T.\left[x'(t)\right] = j2\pi f\,X(f).$$

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Proof:

$$\begin{aligned} F.T.\left[x'(t)\right] &= \int_{\mathbb{R}} x'(t) \exp(-j2\pi f t) dt \\ &= \left[x(t) \exp(-j2\pi f t)\right]_{-\infty}^{\infty} + j2\pi f \int_{\mathbb{R}} x(t) \exp(-j2\pi f t) dt \\ &= j2\pi f X(f). \end{aligned}$$

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Theorem

$$x(\alpha t) \overset{F.T.}{\rightleftharpoons} \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right), \quad \alpha \in \mathbb{R}.$$

Symmetry Relations

If $x(t) \in \mathbb{R} \Rightarrow X(f) = X^*(-f)$ Time Real $\rightleftharpoons$ Freq Symmetry.

If $x(t) = x^*(-t) \Rightarrow X(f) \in \mathbb{R}$ Time Symmetry $\rightleftharpoons$ Freq Real.

If $x(t) = -x^*(-t) \Rightarrow jX(f) \in \mathbb{R}$. Anti-Symmetry $\rightleftharpoons$ Freq Imaginary.

Gaussian Functions

Theorem

$$\exp\left(-\pi t^2\right) \xrightleftharpoons{F.T.} \exp\left(-\pi f^2\right)$$

Proof: For $g(t) = \exp(-\pi t^2)$ and $G(f) = \exp(-\pi f^2)$,

$$F.T.[g'(t)] = \int_{\mathbb{R}} g(t)(-2\pi t)\exp(-j2\pi ft)\,dt = -j\frac{d}{df}\int_{\mathbb{R}} g(t)\exp(-j2\pi ft)\,dt.$$

Inverse Fourier Transform Formula Holds for $g(t)$ and $G(f)$

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Thus

$$j2\pi f G(f) = -jG'(f)$$

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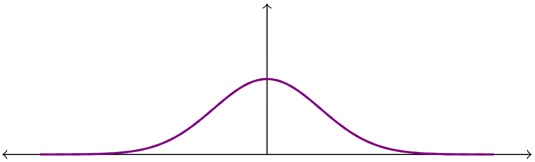
Inverse Fourier Transform Formula Holds for $g(t)$ and $G(f)$

$$g_\delta(t) := \frac{1}{\sqrt{\delta}}\exp(-\pi\frac{t^2}{\delta}) \xrightleftharpoons{F.T.} \exp(-\pi f^2\delta) := G_\delta(f)\,,\ \delta > 0.$$

Gaussian Plots



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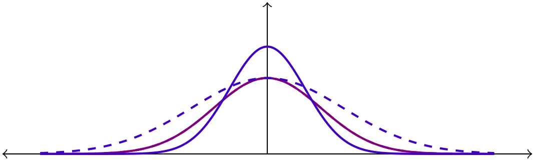

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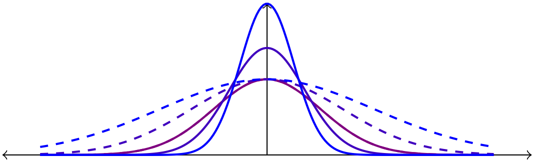

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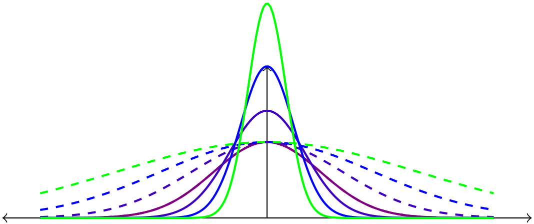

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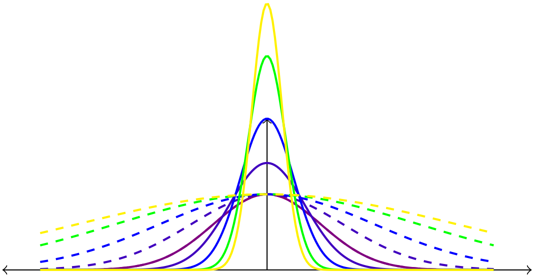

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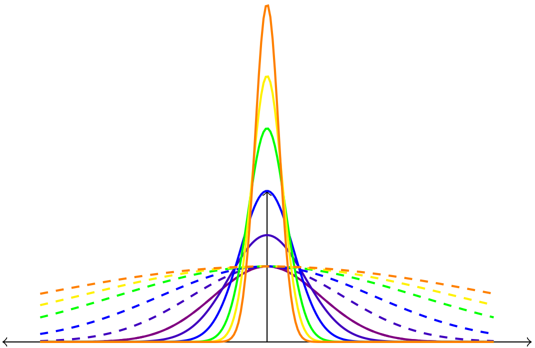

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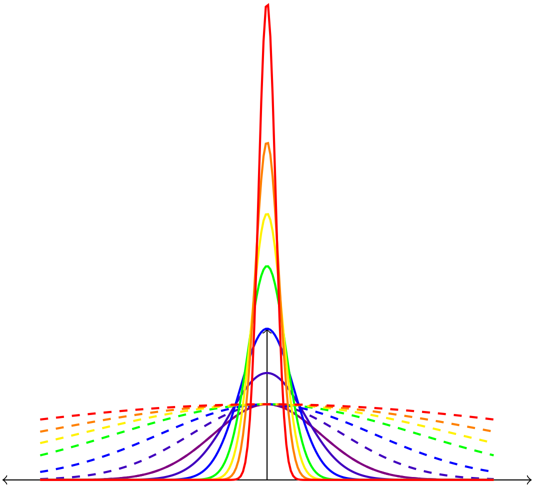
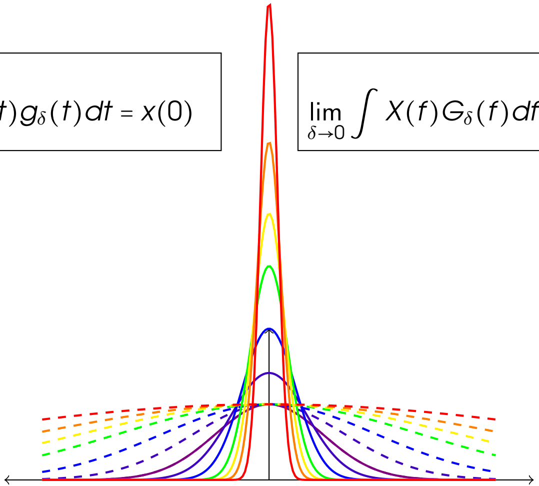


Gaussian Plots

$$\lim_{\delta \to 0} \int x(t) g_\delta(t) dt = x(0)$$

$$\lim_{\delta \to 0} \int X(f) G_\delta(f) df = \int X(f) df$$



Fourier Inverse

$$\int_{\mathbb{R}} x(t) g_\delta(t) dt = \int_{\mathbb{R}} x(t) \int_{\mathbb{R}} G_\delta(f) \exp(j2\pi f t) df dt$$

(A)

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When the Fourier Transform $X(f)$ of $x(t)$ is integrable

$$\int_{\mathbb{R}} X(f)df = x(0).$$

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Proof: Take $\delta \to 0$ in Equation (A).

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Proof: Take $\delta \to 0$ in Equation (A).

If $x(t+\tau) = y(t)$, then $\int Y(f) df = \int X(f) \exp(j2\pi f\tau) df = y(0) = x(\tau)$.