D Manjunath

Networking Lab
Department of Electrical Engineering
Indian Institute of Technology

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Experiments, Outcomes, and Sample Space

- A random experiment: An experiment with many possible outcomes. The actual outcome is random.
- Sample Space: The set of all outcomes of a random experiment; denoted by Ω.
  - Tossing a coin $\Omega = \{H, T\}$.
  - Throwing a die: Standard is six sided; Pacheesi (the Shakuni version) uses a four-sided version of the die. Astragalus, (the heel bone; favourite of many an ancient civilisations; not unbiased) is four-sided; $\Omega = \{A, B, C, D\}$ or $\Omega = \{1, 2, 3, 4\}$.
  - Selecting numbered (or colored or both) balls from an urn
  - Selecting a set of cards from a deck of cards.
  - Selecting a real number in $[0, 1]$.

Experiments, Outcomes, and Sample Space

- The set denoting the sample space for the random experiment must be
  - Mutually exclusive: only one of the possible outcomes must occur in each experiment.
  - Collectively exhaustive.
  - Omits irrelevant detail in defining the outcome.

Types of Samples Spaces

- Discrete and finite sample spaces
  - Coin toss, tossing two coins simultaneously.
  - Rolling a die, rolling two dice simultaneously (with or w/o order).
- Countable sample spaces
  - Number of coin tosses till the first head.
  - Age in years of the next person you meet
- Uncountable sample spaces
  - Choosing a real number in the interval $(0, 1)$.
  - Choosing a point on the unit square or on a circular target.

Events

- Any susbet E of the sample space Ω, including ϕ the null set and Ω the sample space.
- If the outcome, say $\omega \in \Omega$, is in E, then we say that E has occurred.
  - Even number on a throw of a six-sided die.
  - Odd number on a throw of a six-sided die.
  - Spades from deck of cards
  - A red card from a deck of cards
  - Drawing a black ball from an urn with 10 black balls and 8 brown balls

Events

- The set of all events, denoted by $\mathcal{F}$ is assumed to be a σ-algebra which satisfies the following properties.
  - $\Omega \in \mathcal{F}$.
  - If $A \in \mathcal{F}$ then $\bar{A} \in \mathcal{F}$.
  - If $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$.
  - If $A_1, A_2, \ldots, \in \mathcal{F}$ is a **sequence** of events, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$

Events and Sets

- This leads us to a ‘set theoretic’ view of the outcome of random experiments (and of probability)
- Language of sets can be translated into the language of events.
  - Complement of an event: E did not occur.
    $\bar{E}$ (E^c) is the set of elements of Ω that are not in E.
  - Union of events: Either of E_1 or E_2 or both occurred. $E_1 \cup E_2$ is the set of elements that are either in E_1 or E_2 or both.
  - Intersection of events: E_1 and E_2 both occurred. $E_1 \cap E_2$ is the set of elements that in both E_1 and E_2.

Events and Sets

- Language of sets can be translated into the language of events.
  - Difference between sets: $(E_1 - E_2)$: is the set of elements that are in E_1 but not in E_2.
    Exercise: Express $(E_1 - E_2)$ using complement, union, and intersection operations.
  - Mutually exclusive events: Events E_1 and E_2 are mutually exclusive if the occurrence of one precludes the other; only one of the two can occur. $E_1 \cap E_2 = \phi$.

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Summary

1. ω is an outcome of a random experiment
2. Ω is the sample space, the set of all outcomes from a random experiment.
3. E, a subset of Ω, is an event
4. Operations on Events: Complement, Union (OR, also called inclusive OR), Intersection (AND), Difference

Axioms of Probability

- Denote the probability of an event $E \in \mathcal{F}$ by $\mathsf{Pr}(E)$.
- We can assign probabilities to events $E \in \mathcal{F}$ that satisfy the following properties/axioms.
  1. For all events E, $0 \leq \mathsf{Pr}(E) \leq 1$.
  2. $\mathsf{Pr}(\Omega) = 1$; $\mathsf{Pr}(\phi) = 0$.
  3. If E_1 and E_2 are two disjoint events, i.e., if $E_1 \cap E_2 = \phi$, then

$$\mathsf{Pr}(E_1 \cup E_2) = \mathsf{Pr}(E_1) + \mathsf{Pr}(E_2)$$

- Thus probabilities are a function (or a map) $\mathcal{P} : \mathcal{F} \to [0, 1]$
- The triplet $(\Omega, \mathcal{F}, \mathcal{P})$ is called the probability space.

Axioms of Probability (Contd.)

- Consider $\Omega = \{1, 2, \ldots, \}$.
- Let $\mathsf{Pr}(i) = \frac{1}{2^i}$.
- Verify that this satisfies the axioms of probability.
- Now consider the event $A = \{2, 4.6, \ldots\}$. We see that

$$\mathsf{Pr}(A) = \mathsf{Pr}(2) + \mathsf{Pr}(4) + \cdots$$

  i.e., it is a sum of a countably infinite number of events.
- The axioms do not allow us to make that equality;
- But such events are clearly of interest.
- Hence we add the following extension to the third axiom
- Axiom 3a: If $E_1, E_2, \ldots$ is a **sequence** of mutually exclusive events, then

$$\mathsf{Pr}(\cup_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \mathsf{Pr}(E_i)$$

- Note that it applies to a sequence of events.

Probability with Discrete Sample Space

- Recall that we defined probabilities on events and not on outcomes.
- For discrete sample space, we can define probabilities on outcomes.
- Let ω_i be outcome i, $i = 1, 2, 3, \ldots,$
- This will also be called an elementary event.
- The probabilities for the occurrence of ω_i will satisfy the following:
  1. For all $i > 0$, $0 \leq \mathsf{Pr}(\omega_i) \leq 1$.
  2. Let E be an event; then

$$\mathsf{Pr}(E) = \sum_{\omega_i \in E} \mathsf{Pr}(\omega_i)$$

  3. $\sum_i \mathsf{Pr}(\omega_i) = 1$.

Probability with Uncountable Sample Space

- For uncountable sample space, we cannot assign probabilities to probabilities to outcomes that satisfy the axioms.
- Recall that axiom 3a was for a countable sum.
- Here $\mathsf{Pr}(()\,\omega) = 0$.
- Let the set of real numbers, $\Re$, be the sample space.
- Here we can define events of the type $x_1 \leq x \leq x_2$ as events and assign probabilities to such events.

Properties of Probability

- $E_1 \subset E_2 \Rightarrow \mathsf{Pr}(E_2) \geq \mathsf{Pr}(E_1)$

$$\begin{aligned} E_2 &= E_1 \cup (\bar{E}_1 \cap E_2) \\ \mathsf{Pr}(E_2) &= \mathsf{Pr}(E_1) + \mathsf{Pr}(\bar{E}_1 \cap E_2) \\ &\geq \mathsf{Pr}(E_1) \end{aligned}$$

- $\mathsf{Pr}(\bar{E}) = 1 - \mathsf{Pr}(E)$

$$\begin{aligned} \Omega &= E \cup \bar{E} \\ \mathsf{Pr}(\Omega) &= \mathsf{Pr}(E) + \mathsf{Pr}(\bar{E}) \\ \mathsf{Pr}(\bar{E}) &= 1 - \mathsf{Pr}(E) \end{aligned}$$

The second equation follows because E and $\bar{E}$ are mutually exclusive.

Properties of Probability

- **Exercise:** For mutually exclusive events E_1, E_2, and E_3,

$$\mathsf{Pr}(E_1 \cup E_2 \cup E_3) = \mathsf{Pr}(E_1) + \mathsf{Pr}(E_2) + \mathsf{Pr}(E_3)$$

- **Exercise:** Prove that $\mathsf{Pr}(\phi) = 0$.

Properties of Probability (Contd)

- Let E_1 and E_2 be two arbitrary events (not necessarily mutually exclusive). Then

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2)$$

Proof:

- Let $A = E_1 - E_2$, $B = E_2 - E_2$, and $C = E_1 \cup E_2$.
- Clearly, A, B, and C are mutually exclusive events.
- Also, $E_1 = A \cup C$, $E_2 = B \cup C$, and $E_1 \cup E_2 = A \cup B \cup C$. Therefore,
- $\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C)$
- $\Pr(E_1) = \Pr(A) + \Pr(C)$; $\Pr(E_2) = \Pr(B) + \Pr(C)$
- $\Pr(E_1) + \Pr(E_2) = \Pr(A) + \Pr(B) + 2\Pr(C)$

$$\begin{aligned}
\Pr(E_1) + \Pr(E_2) - \Pr(C) &= \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2) \\
&= \Pr(A) + \Pr(B) + 2\Pr(C) - \Pr(C) \\
&= \Pr(A) + \Pr(B) + \Pr(C) \\
&= \Pr(E_1 \cup E_2)
\end{aligned}$$

Properties of Probability (Contd.)

- Union Bound

$$\mathsf{Pr}(E_1 \cup E_2) \leq \mathsf{Pr}(E_1) + \mathsf{Pr}(E_2)$$

$$\mathsf{Pr}\left(\cup_{i=1}^{N} E_i\right) \leq \sum_{i=1}^{N} \mathsf{Pr}(E_i)$$

- Bonferroni inequality

$$\mathsf{Pr}(E_1 \cap E_2) \geq \mathsf{Pr}(E_1) + \mathsf{Pr}(E_2) - 1$$

$$\mathsf{Pr}\left(\cap_{i=1}^{N} E_i\right) \geq \sum_{i=1}^{N} \mathsf{Pr}(E_i) - (n-1)$$

Interpreting probability

- Intuitively, probability of an event is a numerical measure of the likelihood of the occurrence of the event.
- In one interpretation of probability, called the frequentist view, the probability of an event is the relative frequency of the occurrence of the event when the experiment is repeated a large number of times under identical conditions.
- In a second interpretation, the probability is the strength of the belief that an event will occur.
- Probability theory takes the belief and provides a consistent way to make probabilistic predictions of the phenomena being modeled.
- Needless to say, the predictions are as good as the initial assumptions.

Summary

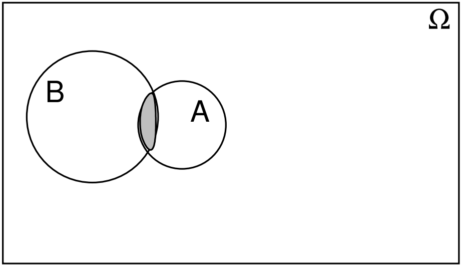
- The three probability axioms
- Added the countable additivity extension to Axiom 3
- Developed some consequences of the axioms
- Finally, gave an interpretation.

Conditional probability: Motivating examples

Probability of an event when given that another event has occurred.

- Throw a six-sided fair die twice. What is the probability that the sum of the two numbers is 7: $1/6$
  - If you know that the first number is 4: 1/6
  - If you know that the first number is k, $(k = 1, \ldots, 6.)$???
- On a throw of a fair six-sided die, what is the probability that a 1 comes on top: 1/6
  - If you know that an odd number has come up: 1/3
  - If you know that an even number has come up: 0
- Toss two fair coins simultaneously. Given that the first coin came up heads, what is the probability that the second coin came up heads.
- Consider families with exactly two children. Assume boys and girls are equally likely. If a family is chosen at random, and it is found that there is one girl in the family, what is the probability that there is another girl in the family?

Conditional probability



- You are interested in the event A. Let $p_A := \mathsf{Pr}(A)$ be the probability of the occurrence of A.
- You are told that the event B has occurred.
- This information will change the probability of occurrence of A—it is not p_A any more.
- Without the additional information, you were measuring the occurrence of A with respect to Ω.
- With the additional information, you were measuring the occurrence of A with respect to B

Conditional probability (Contd)

- $\mathsf{Pr}(A)$ is the **unconditional probability** of event A.
- $\mathsf{Pr}(A \mid B)$ is the **conditional probability** of the event A conditioned on the occurrence of the event B.

Conditional Probability Formula

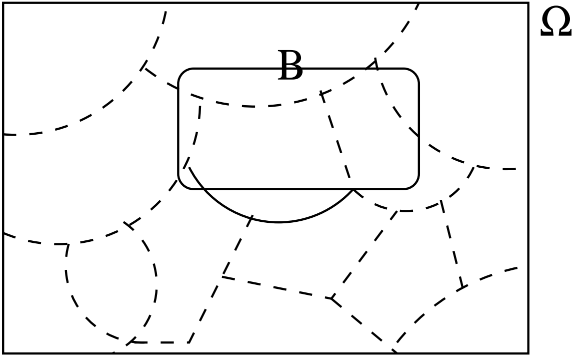
$$\mathsf{Pr}(A \mid B) = \frac{\mathsf{Pr}(A \cap B)}{\mathsf{Pr}(B)}$$

Some immediate consequences

- Intersection from the conditional

$$\begin{aligned} \Pr(A \cap B) &= \Pr(A \mid B) \times \Pr(B) \\ \Pr(A_1 \cap A_2 \cap \cdots A_n) &= \Pr(A_1) \times \\ &\quad \Pr(A_2 \mid A_1) \times \Pr(A_3|A_2, A_1) \times \cdots \end{aligned}$$

- Partition of Ω.



Some immediate consequences (Contd)

- Unconditional from a partition and conditionals
- Let $\Omega = \cup_{n=1}^{N} A_n$.

$$B = \Omega \cap B = \left(\cup_{n=1}^{N} A_n\right) \cap B = \cup_{n=1}^{N} (A_n \cap B)$$

$$\Pr(B) = \sum_{n=1}^{N} \Pr(A_n \cap B) = \sum_{n=1}^{N} \Pr(A_n) \Pr(B \mid A_n)$$

Total Probability Formula

Let $\{A_n\}$ be a partition of Ω. Then

$$\Pr(B) = \sum_{n=1}^{N} \Pr(A_n) \Pr(B \mid A_n) .$$

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Bayes' Theorem

Let $\{A_n\}$ be a partition of Ω.

$$\begin{aligned} \mathsf{Pr}(A_n \cap B) &= \mathsf{Pr}(B \mid A_n)\,\mathsf{Pr}(A_n) \\ \mathsf{Pr}(A_n \mid B) &= \frac{\mathsf{Pr}(A_n \cap B)}{\mathsf{Pr}(B)} \end{aligned}$$

Bayes' Theorem

Let $\{A_n\}$ be a partition of Ω and B be an event. Then

$$\mathsf{Pr}(A_n \mid B) = \frac{\mathsf{Pr}(B \mid A_n)\,\mathsf{Pr}(A_n)}{\sum_{n=1}^{N} \mathsf{Pr}(A_n)\,\mathsf{Pr}(B \mid A_n)}$$

Summary

- $\mathsf{Pr}(A \mid B)$ is the **conditional probability** of the event A conditioned on the occurrence of the event B.

$$\mathsf{Pr}(A \mid B) = \frac{\mathsf{Pr}(A \cap B)}{\mathsf{Pr}(B)}$$

- Total Probability Formula: Unconditional probability of event B from probability of elements of a partition $\{A_n\}$ of Ω and conditional probabilities of B.

$$\mathsf{Pr}(B) \; = \; \sum_{n=1}^{N} \mathsf{Pr}(A_n) \, \mathsf{Pr}(B \mid A_n) \, .$$

- Bayes' Theorem: Let $\{A_n\}$ be a partition of Ω and B be an event. Then

$$\mathsf{Pr}(A_n \mid B) \quad = \quad \frac{\mathsf{Pr}(B \mid A_n) \, \mathsf{Pr}(A_n)}{\sum_{n=1}^{N} \mathsf{Pr}(A_n) \, \mathsf{Pr}(B \mid A_n)}$$

Independent Events

- Two events are independent if the the occurrence of one does not influence the other.
- Intuitively, we want $\Pr(B|A) = \Pr(B)$
- This implies $\Pr(A \cap B) = \Pr(A)\Pr(B)$. And that is the formal deifinition:
- Two events A and B are independent if

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

- If A and B are independent, then what is $\Pr(A \mid B)$?

Independent Events (Contd.)

- If A and B are independent events then A and $\bar{B}$ are also independent. Write $A = AB + A\bar{B}$ Clearly, (AB) and $(A\bar{B})$ are mutually exclusive.

$$\begin{aligned}
\Pr(A) &= \Pr(AB) + \Pr(A\bar{B}) \\
&= \Pr(A)\Pr(B) + \Pr(A\bar{B}) \\
\Pr(A \cap \bar{B}) &= \Pr(A) - \Pr(A)\Pr(B) \\
&= \Pr(A)\,(1 - \Pr(B)) \\
&= \Pr(A) \times \Pr(\bar{B})
\end{aligned}$$

- Events A, B, and C are independent if A, B, and C are mutually (or pairwise) independent, i.e.,

$$\begin{aligned}
\Pr(AB) &= \Pr(A)\Pr(B) \\
\Pr(BC) &= \Pr(B)\Pr(C) \\
\Pr(AC) &= \Pr(C)\Pr(A)
\end{aligned}$$

and

$$\Pr(ABC) = \Pr(A)\Pr(B)\Pr(C)$$

Independent Events (Contd.)

- Pairwise independence does not imply independence.

Example

- An urn contains b black balls and r brown balls. A ball is drawn from the urn and discarded without observing the color of the ball. A second ball is now drawn from the urn. What is the probability that the second ball is black?
- **Solution**: B_1 and B_2 denote the event that the first and second balls were black respectively. Want $\Pr(B_2)$.

$$
\begin{aligned}
\Pr(B_1) &= \frac{b}{b+r} \\
\Pr(B_2 \mid B_1) &= \frac{b-1}{b+r-1} \\
\Pr\left(B_2 \mid \overline{B_1}\right) &= \frac{b}{b+r-1} \\
\Pr(B_2) &= \Pr(B_2 \mid B_1) \Pr(B_1) + \Pr\left(B_2 \mid \overline{B_1}\right) \Pr\left(\overline{B_1}\right) \\
&= \left(\frac{b-1}{b+r-1} \frac{b}{b+r}\right) + \left(\frac{b}{b+r-1} \frac{r}{b+r}\right) \\
&= \frac{b}{b+r}
\end{aligned}
$$

Example

Revisiting an old problem.

- A test is designed to identify 'Brilliant,' (designated as B) students. At the end of the test the student is pronounced to be B or not-B.
- The test can give a false positive: Declare a not-B student to be B (say, 5% chance)
- The test can give a false negative: Declare a B student to be not-B (say 2% chance)
- Question: The test says a student, nicknamed *2B-or-not-2B,* is B. What are the chances that he is indeed B.
- Need additional data: Assume that the 3% of the test takers are indeed brilliant.

Solution

- Let B denote the event that a student is brilliant and P denote the event that a test-taker passes.
- Given data

$$\begin{aligned} \Pr(B) &= 0.03 \\ \Pr(\overline{P} \mid B) &= 0.02 \\ \Pr(P \mid \overline{B}) &= 0.05 \end{aligned}$$

- Can infer

$$\begin{aligned} \Pr(P \mid B) &= 0.98 \\ \Pr(\overline{P} \mid \overline{B}) &= 0.95 \end{aligned}$$

- Want $\Pr(B \mid P)$.

$$\begin{aligned} \Pr(B \mid P) &= \frac{\Pr(PB)}{\Pr(P)} \\ &= \frac{\Pr(P|B) \times \Pr(B)}{\Pr(P|B) \times \Pr(B) + \Pr(P|\overline{B}) \times \Pr(\overline{B})} \end{aligned}$$

Solution (Contd)

- Continuing,

$$\begin{aligned} \mathsf{Pr}(B \mid P) &= \frac{0.98 \times 0.03}{0.98 \times 0.03 + 0.05 \times 0.97} \\ &= 0.377! \end{aligned}$$

- The example works with medical tests too! Next time you get a test done and the doctor says the "test is positive and you have the ailment" you should know how to interpret that statement.
- Question: How would you analyse the effect of improving the accuracy of the test?

Experiments wth Equally Likely Outcomes

- In many experiments, especially those with discrete sample space, a convenient model, usually reasonable, is to assume that all outcomes are equally likely.
- This means that if there are N items in the sample space, denoted by $|\Omega| = N$, then for $i = 1, 2, \ldots, N$,

$$\mathsf{Pr}(\omega_i) = \frac{1}{N}$$

- In such experiments, calculating the probability of an event is essentially a counting problem—if we want to know the probability of an event A, we just count the number of elements in A and divide by N. This gives us

$$\mathsf{Pr}(A) = \frac{\text{Number of Elements in } A}{N}$$

Counting Techniques

- Familiar territory: Permutations and combinations.
- Basic principle of counting: If R experiments are to be performed, and experiment r has n_r possible outcomes, then the total number of possible outcomes is $n_1 \times n_2 \times \cdots n_R$.
- Use this to construct sampling models as follows: An urn (matka) contains m distinguishable balls marked 1 to m. n balls are sampled from the urn. We can define the following types of sampling.
  - Sampling with replacement and with ordering
  - Sampling without replacement and with ordering
  - Sampling without replacement and without ordering
  - Sampling with replacement and without ordering
- An ordered subset is called a *permutation* and an unordered subset is called a *combination.*
- A large class of interesting (and useful) problems can be cast as a counting problem.

Counting Techniques (Contd)

- Sampling with replacemnet and with ordering: m^n
- Sampling without replacement and with ordering: $m(m-1)\cdots(m-n+1)$ denoted by $(m)_n$. Special case of $(m)_m = m!$.
- Sampling without replacement and without ordering: $\binom{m}{n}$
- **Proof:** Let the sample size be x. Number of ordered samples is $(m)_n$; A sample of size n can be ordered in $n!$ ways. Therefore

$$
\begin{aligned}
n!x &= (m)_n \\
x &= \frac{(m)_n}{n!} \\
&= \frac{(m)_n\,(m-n)!}{n!\,(m-n)!} \\
&= \frac{m!}{n!(m-n)!}
\end{aligned}
$$

Immediate Extensions

- The number of permutations when m balls are arranged in a circle. This is also called circular permutation. Two permutations are not different if corresponding objects are not followed and preceded by different objects.
  $(m-1)!$.
- Number of permutations of m balls when there are m_i balls of colour i for $i = 1, \ldots, K$; Of course $\sum_{i=1}^{K} m_i = m$.

$$\frac{m!}{m_1!\, m_2! \ldots m_K!}$$

Counting Techniques (Contd)

- Sampling with replacement and without ordering. This is essentially the number of times each element is picked.

$$\binom{m+n-1}{n} = \binom{m+n-1}{m-1}$$

Example

There are 100 questions in a question bank 10 of which are on probability. What is the probability that the 5th question is a probability question? What is the probability that at least one of the first five questions is a probability question?

Example

In a true/false question, you randomly choose true or false. There are five questions. What is the probability that you will get exactly one right? What is the probability that you will get at least one right?

Solution:

- Number of ways in which answers can be chosen:
  - Sampling with replacement from an urn with two balls labelled *correct* and *wrong*.
  - Five picks of the balls: 2^5 ways to pick the balls
- Number of ways to get exactly k right
  - You are now told that exactly k answers are correct. Any answers.
  - Put five balls in an urn and choose k samples without replacement, without order. $\binom{5}{k}$ ways of doing it.

$$\mathsf{Pr}(\text{exactly } k \text{ right}) = \frac{\binom{5}{k}}{2^5}$$

For $k = 1$

$$\mathsf{Pr}(\text{exactly 1 right}) = \frac{5}{32}$$

Example (Contd)

A related problem:: Good students get 80% of the answers correct. You are given 10 questions and you get 7 right.

- Can it be claimed that you are a good student.
  Need to know: If you are a good student, the probability that the above outcome would be observed
- Alternatively, can it be claimed that you are not a good student?
  Need to know: If you are **not** a good student, the probability that the above outcome would be observed.

Above probabilities will help provide a quantitative basis to the 'binary' answer that one expects.

Example

There are 500 components in a box. You know 15% are defective But you want to try your luck in palming off the box, whereas the inspector is looking for 8% defects. A QC inspector inspects the box by randomly choosing 25 components. Find the probability that exactly k samples are defective. If more than m are found defective, the box is rejected. Find the probability that the box will be rejected.

Solution

- There are 75 defective components in the box.
- Number of ways to pick 25 samples from 500:
  - Sampling without replacement, and without ordering. $\binom{500}{25}$.
- Number of ways to choose k defective samples:
  - Choose k defective samples out of 75 defective ones in the box
  - Choose $(25-k)$ good samples from the 425 good ones in the box.
  - Both are sampling without replacement and without ordering.
  - The counts are $\binom{75}{k}$ and $\binom{425}{25-k}$.
  - Total number of ways is $\binom{75}{k} \times \binom{425}{25-k}$.

Example (Contd)

Define

$$p_k = \frac{\binom{75}{k} \times \binom{425}{25-k}}{\binom{500}{25}}$$

The probability that the box is rejected is

$$\sum_{k=m+1}^{25} p_k$$

- *The Design Question:* What should m be?
- Need to qualify the above statement better
- In fact, m depends on the 'presumption':
- For example, what should be m for me to be 'reasonably' sure that the box has less than 8% defective components.
- **Exercise:** How would your solution change if you knew that every component that you manufacture is defective with probability 0.15 but do not the number of defective components in the box.

Example

You want to estimate the fish population in Powai lake. You do the following: Catch 100 fish, mark them and release them back into the lake. Allow the fish to mix well and then you catch 100 fish again. Of these 10 are those that were marked before. Assume that the fish population is n and has not changed between the catches. Find the probability of the outcome of your experiment.

- Using the same procedure as in the previous example, we can show that the probability of the event is

$$\frac{\binom{n-100}{90} \times \binom{100}{10}}{\binom{n}{100}}$$

- Possible questions that lead to the above experiment
  - BMC claims that there are more than 1500 fish in the lake.
  - Environmentalists claim it is much less than a 1000.

Summary: Counting Formulae

- An urn with m distinguishable balls; sample n times.
- Interest is in counting the number of different samples under different assumptions
- Sampling with replacement and with ordering: m^n
- Sampling without replacement and with ordering: $mP_n := m \times (m-1) \times (m-2) \times \cdots \times (m-n+1)$.
- Sampling without replacement and without ordering: ${}^mC_n = mC_n = mCn = \binom{m}{n}$
- Sampling with replacement and without ordering: $\binom{m+n-1}{n}$.