MA 207 - Differential Equations-II

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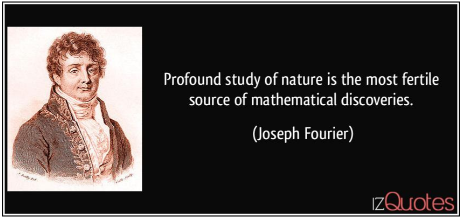
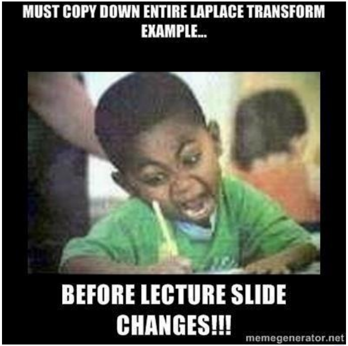
Start with ....



Profound study of nature is the most fertile source of mathematical discoveries.
(Joseph Fourier)
IZQuotes



MUST COPY DOWN ENTIRE LAPLACE TRANSFORM EXAMPLE...
BEFORE LECTURE SLIDE CHANGES!!!
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Outline of the lecture

- Linear PDE
- Classification of II order linear PDE (2 variables - constant/variable coefficients) (RECALL)
- Canonical Form/ Normal Form
- Classification of II order linear PDE (n variables- constant coefficients)

PDE: CAN you guess—What it is?

Definition

A partial differential equation (PDE) is an equation involving an unknown function of two or more variables and its derivative(s).

Examples:

- -
  $$\Delta u \equiv -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = f(x), \;\; u = u(x_1, x_2, \cdots, x_n), \; x \in \Omega \subset \mathbb{R}^n$$
  (Poisson equation)
- $\dfrac{\partial u}{\partial t} - \Delta u = f(x,t) \;\; x \in \Omega \subset \mathbb{R}^n, \; t > 0, \; u = u(x,t)$ is the temperature; (Heat Conduction Equation).
- $\dfrac{\partial^2 u}{\partial t^2} - \Delta u = f(x,t) \;\; x \in \Omega \subset \mathbb{R}^n, \; t > 0$ (Wave equation)

Definition

The **order** *of a PDE is the order of the highest derivative that occurs in the PDE.*

Examples : • $u_{xx} + 2xu_{xy} + 3yu_{yy} = e^y$ (second order)
• $u_{xxy} + u_{xy} + xu_x + yu_y + 3u = 5$ (third order).

Definition

Linear/Non-linear PDE : A PDE is said to be **linear**, *if it is linear in the unknown functions and its derivatives with coefficients depending on the independent variables $x_1, x_2, \cdots, x_n$. An equation that is not linear is called a* **non-linear** *equation.*

That is, in the examples of the PDE, set the left hand side as $L(u)$; and write the equation as $L(u) = f$. L is linear if

$$L(u + \alpha v) = L(u) + \alpha L(v),\ \alpha \in \mathbb{R}.$$

Exercise : Are the examples of PDE stated in the previous slide linear?

- Eikonal Equation in optics: $|\nabla u| := \sqrt{u_x^2 + u_y^2 + u_z^2} = 1.$
- Minimal surface Equation:
  $u_t - \nabla \cdot \left((1 + |\nabla u|^2)^{-1/2} \nabla u \right) = f(x, y, z, t).$
  (Long time behaviour is related to Poincare conjecture which was solved by Gregory Pereleman)
- Navier-Stokes Equation (NSE): Equation of motion for a viscous incompressible fluids is described by the following sets of equations:

$$\vec{u}_t - \nu \Delta \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nabla p = \vec{f}(x, y, z, t), \; t > 0, x \in R^3$$

with incompressible condition : $\nabla \cdot \vec{u} = 0$ and initial condition.

NSE: A Million Dollar Open Problem (by Clay Mathematical Institute)

'Prove or Disprove that the 3D- Navier Stokes Equation has a unique global solution for a given forcing function and initial condition'

For the last 150 or more, it has been a basic model of any viscous incompressible fluid in applications and based on this most of the scientists work for fluid flow problems.

Q. Why one should bother about it?

It is related to self consistency which is like: Even if we conduct numerous experiments, but unless proved by Mathematically, it remains as a thumb rule, that is, in some situations it may or may not work.

That means, if proved it provides confidence to the user's community.

Well-posed problems

Boundary/initial conditions : In order that a PDE has a unique solution, additional conditions on the solution in the form of initial conditions or boundary conditions or a combination of both needs to be imposed on the solution.

A linear BVP/ IVP will have the equation and the initial/ boundary conditions to be linear.

Well-posed Problem : A problem (PDE + initial/ boundary conditions in this context) is said to be well-posed in the sense of Hadamard provided

1. it has a solution (existence)
2. the solution is unique (uniqueness)
3. the solution depends continuously on the data of the given problem (continuous dependence) [1].

[1] The data may be obtained through experiments or measurements, so a small change/ error in the measurement of data should not cause a drastic change in the solution.

Classification - Constant Coefficients

Consider a general second order algebraic equation in two variables with real coefficients :

$$\underbrace{ax^2 + 2bxy + cy^2}_{PRINCIPAL\ \ PART\ P(x,y)} + dx + ey + f = 0.$$

The *nature of the curves* will be decided by the principal part. Depending on the sign of $b^2 - ac$, we classify the curve as :

$$b^2 - ac \begin{cases} > 0 & \text{hyperbola;} \\ = 0 & \text{parabola;} \\ < 0 & \text{ellipse.} \end{cases}$$

With suitable coordinate transformations, $x \mapsto X$, $y \mapsto Y$, which depends on the roots of the polynomial equation $P(x, y) = 0$, the algebraic equation is transformed to NORMAL FORM (CANONICAL FORM) :

$$\begin{cases} \dfrac{X^2}{A^2} - \dfrac{Y^2}{B^2} = 1 & \text{hyperbola;} \\ Y^2 = 4AX & \text{parabola;} \\ \dfrac{X^2}{A^2} + \dfrac{Y^2}{B^2} = 1 & \text{ellipse.} \end{cases}$$

Classification of II order linear PDE in 2 variables - Constant Coefficients

Consider a general linear second order PDE in two variables with constant real coefficients :

$$\underbrace{au_{xx} + 2bu_{xy} + cu_{yy}}_{PRINCIPAL\ \ PART\ P(\frac{\partial}{\partial x},\frac{\partial}{\partial y})u} \quad + du_x + eu_y + fu + g = 0.$$

The *nature of the PDE* will be decided by the principal part (the part containing the highest derivative terms)

$$P(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \equiv a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\,\partial y} + c\frac{\partial^2}{\partial y^2}.$$

If

$$b^2 - ac \begin{cases} > 0, & \text{then hyperbolic PDE;} \\ = 0 & \text{then parabolic PDE;} \\ < 0 & \text{then elliptic PDE.} \end{cases}$$

Exercise : Tut. Sheet 4 - Qns. 1-2

Normal form - Case 1: Hyperbolic case $b^2 - ac > 0$

To see what is the normal form (canonical form) in this case and the transformation that yields the normal form.

• Form a family of curves

$$\frac{dy}{dx} = -\omega^{\pm}, \qquad (A)$$

where $\omega^{\pm}$ are defined by

$$\boxed{\omega^{\pm} = \frac{-b \pm \sqrt{b^2 - ac}}{a}.}$$

(Obtained from $P(x, y) = 0$).

• Integrating (A), we obtain the family of CHARACTERISTIC CURVES of the PDE,

$\xi(x, y) \equiv y + \omega^{+}x = c_1,$

and $\eta(x, y) \equiv y + \omega^{-}x = c_2,$

c_1 and c_2 being arbitrary constants.

- Change of variables :

$u = u(\xi(x,y), \eta(x,y)),$

$$\xi(x,y) \equiv y + \omega^+ x = c_1, \eta(x,y) \equiv y + \omega^- x = c_2,$$

$$\begin{aligned} u_x &= u_\xi \cdot \xi_x + u_\eta \cdot \eta_x (= u_\xi \omega^+ + u_\eta \omega^-) \\ u_y &= u_\xi \cdot \xi_y + u_\eta \cdot \eta_y (= u_\xi + u_\eta) \end{aligned}$$

$$\begin{aligned} u_{xx} &= (\underbrace{u_\xi \cdot \xi_x + u_\eta \cdot \eta_x}_{u_x})_x = \frac{\partial}{\partial x}[u_\xi \cdot \xi_x + u_\eta \cdot \eta_x] \\ &= \frac{\partial}{\partial \xi}[u_\xi \cdot \xi_x + u_\eta \cdot \eta_x]\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}[u_\xi \cdot \xi_x + u_\eta \cdot \eta_x]\frac{\partial \eta}{\partial x} \end{aligned}$$

That is, $\boxed{u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2.}$

$$
\begin{aligned}
u_{xy} &= (\underbrace{u_\xi \cdot \xi_x + u_\eta \cdot \eta_x}_{u_x})_y = \frac{\partial}{\partial y}[u_\xi \cdot \xi_x + u_\eta \cdot \eta_x] \\
&= \frac{\partial}{\partial \xi}[u_\xi \cdot \xi_x + u_\eta \cdot \eta_x]\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}[u_\xi \cdot \xi_x + u_\eta \cdot \eta_x]\frac{\partial \eta}{\partial y}
\end{aligned}
$$

That is, $\boxed{u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y.}$

$$
\begin{aligned}
u_{yy} &= (\underbrace{u_\xi \cdot \xi_y + u_\eta \cdot \eta_y}_{u_y})_y = \frac{\partial}{\partial y}[u_\xi \cdot \xi_y + u_\eta \cdot \eta_y] \\
&= \frac{\partial}{\partial \xi}[u_\xi \cdot \xi_y + u_\eta \cdot \eta_y]\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}[u_\xi \cdot \xi_y + u_\eta \cdot \eta_y]\frac{\partial \eta}{\partial y}
\end{aligned}
$$

That is, $\boxed{u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2.}$

Substituting the expressions for u_{xx}, u_{yy}, u_{yy}, u_x, & u_y in the PDE, we obtain

$$
\begin{aligned}
& au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu + g \\
= \quad & a\Big[\underbrace{u_{\xi\xi}\xi_x^2} + 2u_{\xi\eta}\xi_x\eta_x + \overbrace{u_{\eta\eta}\eta_x^2}\Big] \\
& +2b\Big[\underbrace{u_{\xi\xi}\xi_x\xi_y} + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + \overbrace{u_{\eta\eta}\eta_x\eta_y}\Big] \\
& c\Big[\overbrace{u_{\xi\xi}\xi_y^2} + 2u_{\xi\eta}\xi_y\eta_y + \overbrace{u_{\eta\eta}\eta_y^2}\Big] + d\big[u_\xi\xi_x + u_\eta\eta_x\big] \\
& e[u_\xi\xi_y + u_\eta\eta_y]
\end{aligned}
$$

Substituting the expressions for u_{xx}, u_{yy}, u_{yy}, u_x, & u_y in the PDE, we obtain

$$
\begin{aligned}
& au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu + g \\
= \quad & a\Big[\underbrace{u_{\xi\xi}\xi_x^2} + 2u_{\xi\eta}\xi_x\eta_x + \overbrace{u_{\eta\eta}\eta_x^2} \Big] \\
& +2b\Big[\underbrace{u_{\xi\xi}\xi_x\xi_y} + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + \overbrace{u_{\eta\eta}\eta_x\eta_y} \Big] \\
& c\Big[\overbrace{u_{\xi\xi}\xi_y^2} + 2u_{\xi\eta}\xi_y\eta_y + \overbrace{u_{\eta\eta}\eta_y^2} \Big] + d\Big[u_\xi\xi_x + u_\eta\eta_x\Big] \\
& e[u_\xi\xi_y + u_\eta\eta_y] + fu + g \\
= \quad & u_{\xi\xi}\cancel{\Big[a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2\Big]} \\
& +2\, u_{\xi\eta}\Big[a\xi_x\eta_x + b(\xi_x\eta_y + \eta_x\xi_y) + c\xi_y\eta_y\Big] \\
& +u_{\eta\eta}\cancel{\Big[a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2\Big]} + u_\xi\Big[d\xi_x + e\xi_y\Big] \\
& +u_\eta\Big[d\eta_x + e\eta_y\Big] + fu + g
\end{aligned}
$$

Explanation of why $a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$:

$\xi(x,y) = c_1, \;\; \eta(x,y) = c_2.$

That is, $\xi_x + \xi_y \dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{\xi_x}{\xi_y} = -\omega^+.$

Now, $\omega^\pm = \dfrac{-b \pm \sqrt{b^2 - ac}}{a} \Longrightarrow \omega^\pm$ satisfies the quadratic equation $\omega^2 + \dfrac{2b}{a}\omega + \dfrac{c}{a} = 0.$

That is, $(\dfrac{\xi_x}{\xi_y})$ satisfies :

$$(\frac{\xi_x}{\xi_y})^2 + (\frac{2b}{a})\frac{\xi_x}{\xi_y} + \frac{c}{a} = 0$$
$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0.$$

Similarly, $a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0.$

Hence,

$$\begin{aligned} & au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0 \\ \implies \quad & 2\, u_{\xi\eta}\left[a\xi_x\eta_x + b(\xi_x\eta_y + \eta_x\xi_y) + c\xi_y\eta_y\right] \\ & + u_\xi\left[d\xi_x + e\xi_y\right] + u_\eta\left[d\eta_x + e\eta_y\right] + fu + g = 0 \end{aligned}$$

Since $\xi(x, y) \equiv y + \omega^+ x = c_1$,
and $\eta(x, y) \equiv y + \omega^- x = c_2$,

$$\boxed{\xi_x = \omega^+,\ \xi_y = 1,\ \ \eta_x = \omega^-,\ \eta_y = 1.}$$

Hence,

$$2u_{\xi\eta}\left[a\omega^+\omega^- + 2b(\omega^+ + \omega^-) + c\right] + u_\xi\left[d\omega^+ + e\right] + u_\eta\left[d\omega^- + e\right] + fu + g = 0$$

That is, the PDE can be written in the form

$$\boxed{u_{\xi\eta} + (\bar{a}u_\xi + \bar{b}u_\eta + \bar{c}u) = \bar{d} \text{ (CANONICAL FORM)}}.$$

Alternate form

With $\xi = \alpha + \beta$, $\eta = \alpha - \beta$, where
$\boxed{\alpha = y - \frac{b}{a}x,\ \beta = \frac{\sqrt{b^2 - ac}}{a}x}$, we obtain the canonical form as
$u_{\alpha\alpha} - u_{\beta\beta} + (\bar{\alpha}u_\alpha + \bar{\beta}u_\beta + \bar{c}u) = \bar{d}$. (CHECK !)
Examples :
1. $u_{tt} - u_{xx} + au = f$ is hyperbolic and is in normal form.
$(b^2 - ac = 1 > 0)$
2. The equation $u_{xx} + 4u_{xy} + 3u_{yy} + 3u_x - 3u_y + 2u = 0$ is hyperbolic with canonical form $2u_{\xi\eta} + 2u_\xi + 5u_\eta - 2u = 0$
(EXERCISE!)

In this case, $\boxed{\omega^+ = \omega^- = \omega, \quad \frac{dy}{dx} = -\omega = \frac{b}{a}}$.

Solving the ODE, we obtain one characteristic curve as

$\boxed{\xi(x,y) \equiv y + \omega x = c_1.}$

Choose the characteristic curve $\eta(x,y) = \omega y + \hat{c}x, \;\; \hat{c} \neq \omega^2, \; \hat{c}$ being arbitrary, so that, $\eta(x,y)$ and $\xi(x,y)$ are linearly independent.

$(\hat{c} = \omega^2 \Longrightarrow \eta = \omega y + \omega^2 x = \omega(y + \omega x) = \omega\xi(x,y))$

Exercise : Transform the PDE using the new coordinates of ξ and η as

$$u_{\eta\eta} + (\bar{a}u_\xi + \bar{b}u_\eta + \bar{c}u) = \bar{d} \;\; (\mathit{CANONICAL\ FORM})$$

Example : $\;\; u_t - u_{xx} + u_x = f$ is parabolic.

Normal form - Case 3: Elliptic case $b^2 - ac < 0$

ω^+, ω^- are complex; the characteristic curves are

$$\xi = \underbrace{y - \frac{b}{a}x}_{\alpha} + i \underbrace{\left(\frac{\sqrt{ac - b^2}}{a}\right)x}_{\beta} = \alpha + i\,\beta$$

$$\eta = y - \frac{b}{a}x - i\left(\frac{\sqrt{ac - b^2}}{a}\right)x = \alpha - i\,\beta$$

Using chain rule, the CANONICAL FORM of the PDE is obtained as

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} + (\bar{a}u_\alpha + \bar{b}u_\beta + \bar{c}u) = \bar{d}} \text{ (DERIVE!)}$$

Example : $u_{xx} + u_{yy} = 0$ is elliptic and is in canonical form.

Classification of II order linear PDE in 2 variables - Variable Coefficients

If the coefficients of the II order linear PDE depend on (x, y), then

$$b^2 - ac \begin{cases} > 0; \text{(hyperbolic)} \\ = 0; \text{(parabolic)} \\ < 0; \text{(elliptic)} \end{cases}$$

should be satisfied at each point (x, y) in the region of interest where we want to describe its nature.

The characteristic curves $\boxed{\xi(x, y) = c_1, \ \eta(x, y) = c_2}$ in this case are the solutions of $\boxed{\dfrac{dy}{dx} = -\omega^{\pm}(x, y)}$.

Solutions needn't be $\xi = y + \omega^{+}x = c_1, \ \eta = y + \omega^{-}x = c_2$!

Example : Equation of mixed type - Tricomi problem

Classify $u_{xx} + xu_{yy} = 0$ and derive the canonical/ normal form.
$a = 1,\ b = 0,\ c = x;\ b^2 - ac = -x.$

$$\begin{cases} x < 0 & \text{hyperbola;} \\ x = 0 & \text{parabola;} \\ x > 0 & \text{ellipse.} \end{cases}$$

When $x = 0$, we have $u_{xx} = 0$, which is already in canonical form (parabolic case).
Exercise : For $x < 0$, derive $\xi(x, y) = y - 2/3(-x)^{3/2}$ and $\eta(x, y) = y + 2/3(-x)^{3/2}$ and the canonical form as $u_{\xi\eta} = 0$.
Also, derive the canonical form for the elliptic case.

Classification of II order linear PDE with MORE THAN 2 variables- Constant Coefficients

Consider the general second order linear PDE in n variables :

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_1 \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0,$$

where $a_{ij},\ b_i,\ c,\ d$ are real constants and are functions of $(x_1, x_2, \cdots, x_n),\ u = u(x_1, x_2, \cdots, x_n)$.
The principal part is

$$P(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}) := \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

Classification of II order PDE in 2 variables depends on the sign of $b^2 - ac$. The sign of the discriminant idea cannot be extended to n- dimensional case.

For two variable case, the principal part, that is

$\boxed{a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\partial y} + c\frac{\partial^2}{\partial y^2}}$ can be expressed as $X^T AX$, where

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \ X = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}.$$

Since A is symmetric, it is diagonalizable and its eigenvalues λ_1 and λ_2 are real.

$\lambda_1\lambda_2 = |A| = ac - b^2$.

$b^2 - ac > 0 \Longrightarrow \lambda_1\lambda_2 < 0 \Longrightarrow \lambda_1 \& \lambda_2$ are of different signs.

That is, the PDE is hyperbolic provided eigenvalues are of different signs.

If $b^2 - ac = 0$, then one of the eigenvalues is zero and the PDE is parabolic.

If $b^2 - ac < 0$, the eigenvalues are of the same sign and the PDE is elliptic.

This interpretation of the classification of II order equations in 2 variables can be generalized to the situation of more than 2 variables.

$A = [a_{ij}]$ can be assumed to be symmetric (if not, set $\bar{a}_{ij} = \dfrac{a_{ij} + a_{ji}}{2}$ and rewrite

$$P(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}) := \sum_{i,j=1}^{n} \bar{a}_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Attach a quadratic form P,

$$P(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

Since A is symmetric, it is diagonalizable with real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (counted with multiplicity).

There exists, a corresponding orthonormal set of eigenvectors $r_1,\ r_2, \cdots, r_n$ with $R = (r_1, r_2, \cdots, r_n)$ as columns and

$$R^T A R = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix} = D.$$

The PDE is classified based on the signs of eigenvalues of A :
$\lambda_i > 0\ \ \forall i$ or $\lambda_i < 0\ \ \forall i \Longrightarrow$ $\boxed{\text{ELLIPTIC}}$
Anyone of the eigenvalues is negative & remaining are positive OR
Anyone of $\lambda_i > 0$, remaining $\lambda_i < 0 \Longrightarrow$ $\boxed{\text{HYPERBOLIC}}$.
One or more $\lambda_i = 0 \Longrightarrow$ $\boxed{\text{PARABOLIC}}$.
For the case $n \geq 4$, if two or more eigenvalues have the same sign, others have opposite sign $\Longrightarrow$ $\boxed{\text{ULTRA-HYPERBOLIC}}$.

Exercises :

Tutorial Sheet 4, Qns. 1-5.