MA 207 - Differential Equations-II

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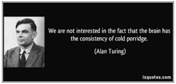
Start with One Quotations and One Cartoon

"Science is a Differential Equation. Religion is a Boundary Condition. "

by Alan M. Turing



Outline of Lecture 2

- Power series method - Recall
- Legendre equation-power series solution
- Legendre polynomials
- Exercises

Outline of the power series method

The power series method of solution of a second order DE with variable coefficients (which are real analytic about $x = x_0$) can be outlined as follows:

1. Assume power series solution is of the form
$\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n}$ about $x = x_0$ convergent in say, $|x - x_0| < R$, $R > 0$.
2. Since the series can be differentiated term by term on this interval, we have
$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \qquad |x - x_0| < R.$$
$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} \qquad |x - x_0| < R$$
Substitute y' and y'' in the DE after expressing the coefficients of the DE as Taylor series about x_0.

3. Simplify the resulting expression combining the like powers of x so that it takes the form $\boxed{\sum_{n=1}^{\infty} K_n(x-x_0)^n = 0}$, where $K_n's$ are functions of certain coefficients $a_n's$.
4. In order that the above expression holds true, we have $K_n = 0, \; n = 0, 1, 2, \cdots$, which leads to a set of conditions satisfied by $a_n's$.
5. Choose $a_n's$ appropriately so that the conditions are satisfied.
6. Verify whether there are two linearly independent power series solutions.
7. Compute the radius of convergence.

Some remarks

- It may be difficult to determine the general coefficient a_n in terms of a_0 and a_1; but we may determine as many coefficients as we want.
- The partial sums provide local approximations to the solution in the neighbourhood of the center $x = x_0$.
- The quality of approximation improves as the number of terms increases.
- The task of calculating several coefficients can be tedious and can be done using a symbolic manipulation package.

The Legendre equation

Portrait debacle : For two centuries, until the recent discovery of the error in 2009, books, paintings and articles have incorrectly shown a side-view portrait of the obscure French politician Louis Legendre (1752-1797) as that of the mathematician Legendre.

Adrien Marie Legendre (1752-1833) was a French mathematician. He made important contributions to statistics, number theory, abstract algebra and mathematical analysis. On the Eiffel Tower, seventy-two names of French scientists, engineers and some other notable people are engraved in recognition of their contributions. His name is one of the 72 names inscribed on the Eiffel Tower.

Legendre and Fourier (caricatures)

$$\boxed{(1-x^2)y'' - 2xy' + p(p+1)y = 0}$$

the parameter p being a given real number.

An application of the method of **separation of variables** to the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in spherical coordinates yields the **Legendre's equation.**

Dividing the equation by $(1-x^2)$ with $x \neq 1$, we obtain

$$\boxed{y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0 \text{ (normalized DE)}}$$

The coefficients of the normalized DE $p(x) = \dfrac{2x}{1-x^2}$ and $q(x) = \dfrac{p(p+1)}{1-x^2}$ are analytic at $x = 0$.

Since the coefficients are analytic at $x = 0$, we seek a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

in $|x| < R, \ R > 0$ for the Legendre equation.

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ -2xy' &= -2\sum_{n=1}^{\infty} n a_n x^n \end{aligned} \tag{2}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \quad (3)$$

(Shifted the index to bring in x^n term in the summation)

Now,

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^n \quad (4)$$

Note that here also we have shifted index so that we bring in x^n term in the summation !

Substitute (3) and (4) in the Legendre equation
$y'' - x^2 y'' - 2xy' + p(p+1)y = 0$, to obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$
$$-\sum_{n=1}^{\infty} 2na_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0. \tag{5}$$

Since the second and third terms of the above expression are 0 for $n = 0,\ 1$ and $n = 0$ respectively, the above expression can be rewritten as

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \underbrace{\sum_{n=0}^{\infty} n(n-1)a_n x^n}_{}$$
$$-\underbrace{\sum_{n=0}^{\infty} 2na_n x^n}_{} + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0. \tag{6}$$

Grouping the terms, we obtain

$$\sum_{n=0}^{\infty}\Big((n+2)(n+1)a_{n+2} - \{n(n-1)+2n-p(p+1)\}a_n\Big)x^n = 0$$

$$\sum_{n=0}^{\infty}\Big((n+2)(n+1)a_{n+2} - \{n(n+1)-p(p+1)\}a_n\Big)x^n = 0$$

Since the relation holds true for all x in $|x| < R$,

$$\begin{aligned} a_{n+2} &= \frac{n(n+1)-p(p+1)}{(n+2)(n+1)}a_n \\ &= \frac{n^2-p^2+n-p}{(n+2)(n+1)}a_n \end{aligned}$$

$$\boxed{a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)}a_n \qquad \text{RECURRENCE RELATION}}$$

$$\boxed{a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)} a_n}$$

$\boxed{n = 0}$

$$a_2 = -\frac{p(p+1)}{1 \cdot 2} a_0 = -\frac{p(p+1)}{2!} a_0;$$

$\boxed{n = 2}$

$$a_4 = -\frac{(p-2)(p+3)}{3 \cdot 4} a_2 = (-1)^2 \frac{p(p-2)(p+1)(p+3)}{4!} a_0;$$

$\vdots$

$\boxed{n = 2k}$

$$a_{2k} = (-1)^k \frac{p(p-2)\cdots(p-2k+2)(p+1)(p+3)\cdots(p+2k-1)}{(2k)!} a_0;$$

$$\boxed{a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)} a_n}$$

$\boxed{n=1}$

$$a_3 = -\frac{(p-1)(p+2)}{2 \cdot 3} a_1 = -\frac{(p-1)(p+2)}{3!} a_1;$$

$\boxed{n=3}$

$$a_5 = -\frac{(p-3)(p+4)}{4 \cdot 5} a_3 = (-1)^2 \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1;$$

$\vdots$

$\boxed{n=2k+1}$

$$a_{2k+1} = (-1)^k \frac{(p-1)(p-3)\cdots(p-2k+1)(p+2)(p+4)\cdots(p+2k)}{(2k+1)!} a_1;$$

- $y_1(x)$

$y =$

$a_0 \; (1 - \frac{p(p+1)}{2!}x^2 + .. + (-1)^n \frac{p \cdots (p-2n+2)(p+1) \cdots (p+2n}{(2n)!}$

$a_1 \; (x - \frac{(p-1)(p+2)}{3!}x^3 + \cdots + (-1)^n \frac{(p-1) \cdots (p-2n+1)(p+}{(2n+1)!}$

That is, $y(x) = a_0 y_1(x) + a_1 y_2(x)$, a_0 and a_1 being arbitrary constants.

- $y_1(x)$

$y =$

$$a_0\ (1 - \frac{p(p+1)}{2!}x^2 + .. + (-1)^n \frac{p\cdots(p-2n+2)(p+1)\cdots(p+2n}{(2n)!}$$

$$a_1\ (x - \frac{(p-1)(p+2)}{3!}x^3 + \cdots + (-1)^n \frac{(p-1)\cdots(p-2n+1)(p+}{(2n+1)!}$$

- $y_2(x)$

That is, $y(x) = a_0 y_1(x) + a_1 y_2(x)$, a_0 and a_1 being arbitrary constants.

Use ratio test show that

$$\boxed{y_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{p \cdots (p-2n+2)(p+1)\cdots(p+2n-1)}{(2n)!} x^{2n}}$$

and

$$\boxed{y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{(p-1)\cdots(p-2n+1)(p+2)\cdots(p+2n)}{(2n+1)!} x^{2n+1}}$$

are **convergent** in $|x| < 1$.

(If p is such that $y_1(x)$ (resp. $y_2(x)$) terminates and becomes a polynomial, then series converges for $x = \pm 1$).

Linear independence of solutions

Note that

$y_1(0) = 1,\ y_1'(0) = 0,\ y_2(0) = 0,\ y_2'(0) = 1.$

$$\boxed{W(y_1, y_2)(0) = y_1(0)y_2'(0) - y_2(0)y_1'(0) = 1 \neq 0}$$

$\implies y_1(x)$ and $y_2(x)$ are linearly independent solutions of the Legendre equation.

Legendre Polynomials

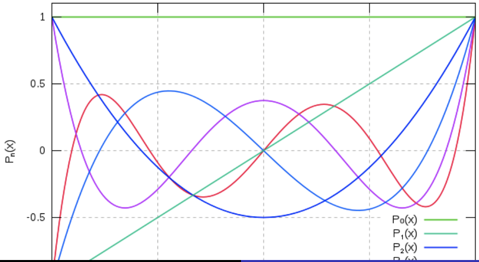
Consider the Legendre DE: $\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0,}$
where n is a non-negative integer.
Recall that

$$
\begin{aligned}
y_1(x) &= 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \cdots \\
&\textit{and} \\
y_2(x) &= x - \frac{(n-1)(n+2)}{3!}x^2 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \cdots
\end{aligned}
$$

are linearly independent solutions of the Legendre DE.
If n is a non-negative even integer (resp. an odd integer), the solution $y_1(x)$ (resp. $y_2(x)$) terminates.
That is, we obtain polynomial solutions to Legendre equation.
The polynomial solutions $P_n(x)$ which satisfy $P_n(1) = 1$, for non-negative integers n are called $\boxed{\text{LEGENDRE POLYNOMIALS}}$.

Examples

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$\frac{1}{8}(63x^5 - 70x^3 + 15x)$



Exercise

Assuming that $a_n = 1$ for the case $n = 0$ and

$$a_n = \frac{(2n)!}{2^n (n!)^2}, \tag{7}$$

prove that the polynomial solutions

$$P_n(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots \tag{8}$$

(the sum terminating with a_0 if n is even and $a_1 x$ if n is odd) of the Legendre equation $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ can be expressed as

$$\boxed{P_n(x) = \sum_{k=0}^{M} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}},$$ where $M = n/2$ or $(n-1)/2$, whichever is an integer.

Proceed to derive the Rodrigue's formula (Q9, Tut. Sheet 1) :

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \ .$$

Hint : Express $a_{n-2}, a_{n-4}, \cdots$ in terms of a_n using $a_{k+2} = \dfrac{(k-n)(k+n+1)}{(k+2)(k+1)} a_k$, use the expression for a_n and substitute in (8). (The choice of a_n as in (7) ensures $P_n(1) = 1$).

Tutorial Sheet Exercises

Tutorial Sheet 1.

- Qn. 6 (ii)-(iv) - Power series solution.
- Qn. 8 - Will be done in class.
- Qn. 9 - in class.
- Qn. 10 - Tutorial class : generating function, Legendre polynomials are coefficients of a Taylor series expansion.
- Qn. 11 - Try using Rolle's theorem.
- Qn. 12 - Try by differntiating the generating function in (Q.10 w.r.t h, x; equate coefficients of h^n etc.)
- Qn. 13 - Work out, (Hint: Rodrigues's formula)
- Qn. 14, 15 - Orthogonality - Tutorial class.