MA 207 - Differential Equations-II

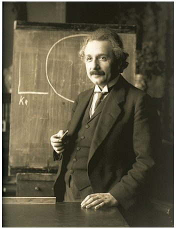
Amiya Kumar Pani

Department of Mathematics,
Indian Institute of Technology Bombay,
Powai, Mumbai 76
akp@math.iitb.ac.in

October 13, 2020

Start with Two Quotations

"Everything should be made as simple as possible, but not simpler."
by Albert Einstein



On Differential Equations:

"Differential Equations are the foundation of natural scientific mathematical view of the world."
by V.I. Arnold

Outline of Lecture 1

- Recall : Methods for solving ODE
- Power series method : Airy's equation
- Power series, some properties
- Real analytic functions
- Outline of power series method- working rule

ODE with constant coefficients

How do we solve second order ODE with constant coefficients ?
For example, consider

$$y'' + ay' + by = 0, \tag{1}$$

where a and b and constant coefficients.
Guess the solution y as

$$y = e^{\lambda x} \tag{2}$$

Substitute (2) in (1) to obtain $\boxed{\lambda^2 + a\lambda + b = 0.}$
This is called the characteristic equation.

If the roots of the characteristic equation are real and distinct, say λ_1 and λ_2, the general solution of (1) is

$$\boxed{y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}, \;\; A \text{ and } B \text{ being arbitrary constants.}}$$

If the roots are complex, then the general solution of (1) is

$$\boxed{y(x) = e^{Re(\lambda)x}\left(A\cos(Im(\lambda)x) + B\sin(Im(\lambda)x)\right).}$$

For real and equal roots, $\lambda = \lambda_1 = \lambda_2$, one solution is $\boxed{y = e^{\lambda x}}$. One can construct a second linearly independent solution using $y_1(x)$ as $\boxed{y_2(x) = xe^{\lambda x}}$.

The general solution in this case is $\boxed{y(x) = (A + Bx)\, e^{\lambda x}.}$

Example 1 : $y'' - y = 0$.
Characteristic equation is $\lambda^2 = 1,\ \lambda = 1, -1$.
General solution is $\boxed{y = Ae^{-x} + Be^{x}}$.

ODE with variable coefficients

How do we solve second order ODE with variable coefficients ?
For example, how do we solve :

$$y'' + p(x)y' + q(x)y = 0? \tag{3}$$

Let $p = \frac{\alpha}{x}$ and $q = \frac{\beta}{x^2}$, where α and β are constants.
We rewrite the given equation as

$$x^2y'' + \alpha xy' + \beta y = 0. \tag{4}$$

This is called the Cauchy-Euler equation.

For solving this equation, we use a change of variable:

$\boxed{x = e^t \quad OR \quad t = \log x}$

to transform (4) as an equation with constant coefficients.

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{x}$,

$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{dy}{dt} \times \frac{1}{x})$

That is, $\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{d^2y}{dt^2}(\frac{1}{x})^2$

Substituting in $x^2y'' + \alpha xy' + \beta y = 0$, we obtain a DE with constant coefficients as :

$$\frac{d^2y}{dt^2} + (\alpha - 1)\frac{dy}{dt} + \beta y = 0. \tag{5}$$

- Corresponding characteristic equation: $\lambda^2 + (\alpha - 1)\lambda + \beta = 0$.
- Find roots λ_1 and λ_2.
- Compute solution y as a function of t and
- Then using $x = e^t$, write the solution y in terms of x.

Note that this is just a particular form of DE with variable coefficients which can be transformed to a DE with constant coefficients.

Consider the *Airy's equation* which is used in physics to model the defraction of light :

$$y'' - xy = 0 \tag{6}$$

Airy's equation cannot be transformed to a DE with constant coefficients.

How do we solve (6)?

English astronomer/mathematician Sir George Biddell Airy (1801-1892) served as astronomer and director of the Greenwich Observatory from 1835 to 1881.

- Step 1 : Assume that the solution of $y'' - xy = 0$ has the form :

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{7}$$

(Is the assumption justified? If yes, for which values of x??)

- Step 2 : Formally differentiate (7):

$$y' = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} a_n \frac{d}{dx}(x^n) = \sum_{n=1}^{\infty} n a_n x^{n-1} \tag{8}$$

(Justification for this term by term differentiation?)

$$y'' = \frac{d}{dx} y' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \tag{9}$$

(Shifted the index to bring in x^n term in the summation)

$y'' - xy = 0$ ...

Now,

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n \tag{10}$$

Note that here also we have shifted index so that we bring in x^n term in the summation !

- Step 3 : Substitute (9) and (10) in the Airy's equation $y'' - xy = 0$ to obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0. \tag{11}$$

  Common range in the summation is from 1 to ∞, write out the terms which do not belong to the common range to obtain :

$$2 \cdot a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-1}] x^n = 0$$

For the expression to hold true $\forall x$, where the expansion is valid,

$$\begin{aligned} a_2 &= 0 \\ a_{n+2} &= \frac{a_{n-1}}{(n+1)(n+2)} \; n \geq 1, \;\; (\textit{RECURRENCE FORMULA}) \end{aligned}$$

That is,

$$n = 1: \; \textit{yields} \;\; a_3 = \frac{a_0}{2 \cdot 3}; \;\; n = 2: \; \textit{yields} \;\; a_4 = \frac{a_1}{3 \cdot 4};$$

and so on.

Since a_{n+2} is given in terms of a_{n-1}, for $n = 1, 2, \cdots$, we have $a_2 = 0$ yielding $a_5 = a_8 = a_{11} = \cdots = 0$;

$$\boxed{a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}}$$

a_0 determining $a_3 = a_6 = a_9 = \cdots$ as

$$a_3 = \frac{a_0}{2 \cdot 3}, \ a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}; \cdots$$

that is,

$$\boxed{a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-4) \cdot (3n-3) \cdot (3n-1) \cdot (3n)} \ (n = 1, 2, \cdots)}$$

a_1 determining $a_4 = a_7 = a_{10} = \cdots$ as

$$a_4 = \frac{a_1}{3 \cdot 4}, \ a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}; \cdots$$

that is,

$$\boxed{a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n-3) \cdot (3n-2) \cdot (3n) \cdot (3n+1)} \ (n = 1, 2, \cdots)}$$

Here a_0 and a_1 are arbitrary constants.

- Step 4 : The general solution of $y'' - xy = 0$ can be written by substituting the values of coefficients in $y = \sum_{n=0}^{\infty} a_n x^n$ as
  - $y_1(x)$

General Solution

$$y = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdot \dots \cdot (3n-1) \cdot (3n)} \right] +$$

$$a_1 \left[x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdot \dots \cdot (3n) \cdot (3n+1)} \right]$$

That is, $y(x) = a_0 y_1(x) + a_1 y_2(x)$, a_0 and a_1 being arbitrary constants.

- Step 4 : The general solution of $y'' - xy = 0$ can be written by substituting the values of coefficients in $y = \sum_{n=0}^{\infty} a_n x^n$ as

  - $y_1(x)$

General Solution

$$y = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdot \dots \cdot (3n-1) \cdot (3n)} \right] +$$

$$a_1 \left[x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdot \dots \cdot (3n) \cdot (3n+1)} \right]$$

  - $y_2(x)$

That is, $y(x) = a_0 y_1(x) + a_1 y_2(x)$, a_0 and a_1 being arbitrary constants.

Linearly independent solutions of Airy's equation

1. Choose $a_0 = 1,\ a_1 = 0$ to obtain
$$y_1(x) = \left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdot \cdots \cdot (3n-1) \cdot (3n)}\right],$$ is a solution;
similarly, choose $a_0 = 0,\ a_1 = 1$ to obtain
$$y_2(x) = \left[x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdot \cdots \cdot (3n) \cdot (3n+1)}\right]$$ is a solution.
2. $y_1(0) = 1,\ y_2(0) = 0,\ y_1'(0) = 0,\ y_2'(0) = 1.$
The Wronskian
$$W(y_1, y_2)(0) = y_1(0)y_2'(0) - y_2(0)y_1'(0) = 1$$

$\Longrightarrow y_1(x)$ & $y_2(x)$ are linearly independent solutions of Airy's equation.

On Wronskian: a slight departure

Two functions $y_1(x)$ and $y_2(x)$ defined on the interval I are said to be linearly independent in I, if their linear combinations: $k_1 y_1(x) + k_2 y_2(x) = 0,\ x \in I$, for any real constant k_1 and k_2 imply that $k_1 = 0$ and $k_2 = 0$. (work out some examples).

If two continuously differentiable functions f and g defined on an open interval I and if their Wronskian $W(f,g)(x_0) \neq 0$ for some $x_0 \in I$, then f and g are linearly independent on I, where Wronskian is given by a 2×2-matrix:

$$W(f,g)(x) = \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}$$

Now, two solutions y_1 and y_2 of the general 2nd order ODE: $y'' + p(x)y' + q(x)y = 0,\ x \in I$ are linearly independent in I, if their Wronskian $W(y_1, y_2)(x_0) \neq 0$, for any $x_0 \in I$.

Questions..

1. Under what conditions on the coefficients of the second order linear DE can one write down the solution as $y = \sum_{n=0}^{\infty} a_n x^n$? (known as power series expansion)
2. Does the series $y = \sum_{n=0}^{\infty} a_n x^n$ converge to the solution y of the ODE? If so, for which values of x?
3. How do we justify the term by term differentiation under infinite sum?
4. Is the product of two power series again a power series? If so how do we find its coefficients?
5. If $\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n$ for each x, then is $a_n = b_n$?

We start with the definition of a power series.

Definition

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{12}$$

is called a power series.

The center x_0, coefficients $a_0,\ a_1, \cdots$, the variable x are assumed to be real.

The power series $\sum\limits_{n=0}^{\infty} a_n x^n$ has center at origin.

Recall: A series converges if the sequence of its partial sums converges.

Therefore, the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges to say $s(x)$, if sequence $\{s_m\}$ converges to s, where $s_m := \sum_{n=0}^{m} a_n(x-x_0)^n$ and $s(x)$ is the sum of the series.
In other words,

$$\lim_{m \longrightarrow \infty} s_m(x) = s(x),$$

that is, Given any $\epsilon > 0$, there is a natural number $N_0 > 0$ such that

$$\text{for } m \geq N_0, |s_m(x) - s(x)| \leq \epsilon.$$

Augustin Louis Cauchy: the man who gave the notion of Convergence

(21st August 1789-23rd May, 1857): French Mathematician, Engineer and Physicist wrote close to 800 papers. Gave rigorous proves of theorems in Calculus and Analysis, DEs, almost founded Complex Analysis, etc.

More concepts of theorems have been named after Cauchy than any other Mathematicians.

Ratio Test The series $\displaystyle\sum_{n=0}^{\infty} a_n(x-x_0)^n$

$$\begin{cases} \text{converges (absolutely)} & \text{if } \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}(x-x_0)\right| = L < 1; \\ \text{diverges} & \text{if } \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}(x-x_0)\right| = L > 1; \end{cases}$$

The test fails if $L = 1$.

OR

The power series converges absolutely for all those values of x for which $|x - x_0| < R$,

where $\boxed{R = \dfrac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}}$ is called radius of convergence.

Note that the power series diverges for all x for which $|x - x_0| > R$.

Examples

1. Find the radii of convergence of $(i) \sum_{n=0}^{\infty} n!x^n$ $(ii) \sum_{n=1}^{\infty} \frac{x^n}{n}$.

(i) $\frac{a_{n+1}}{a_n} = (n+1) \to \infty$ as $n \to \infty$. Hence, $R = 0$ and the series diverges for all values of x except $x = 0$.

(ii) $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$ as $n \to \infty$. Now, $R = 1$. Therefore, the series converges (absolutely) for all $|x| < 1$ and diverges for all $|x| > 1$. What happens at $x = \pm 1$?

At $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$ and it diverges.

Now, at $x = -1$, the series becomes $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ and this series converges, but not absolutely.

2. Find the values of x for which the power series solutions of Airy's equation converges.

$\boxed{\text{Root Test}}$ The series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$

$$\begin{cases} \text{converges (absolutely)} & \text{if } \lim_{n\to\infty} (|a_n(x-x_0)^n|)^{1/n} = L < 1; \\ \text{diverges} & \text{if } \lim_{n\to\infty} (|a_n(x-x_0)^n|)^{1/n} = L > 1; \end{cases}$$

The test fails if $L = 1$.

OR

The power series converges absolutely for all those values of x for which $|x - x_0| < R$,

where $\boxed{R = \dfrac{1}{\lim_{n\to\infty} |a_n|^{1/n}}}$ is called radius of convergence.

Note that the power series diverges for all x for which $|x - x_0| > R$.

Show that if $\sum_{1}^{\infty} a_n x^n$ has radius of convergence R, then $\sum_{1}^{\infty} a_n x^{2n}$ has radius of convergence $\sqrt{R}$ and $\sum_{1}^{\infty} a_n^2 x^n$ has the radius of convergence R^2.

Solution: Set $x^2 = u$.

Then $\sum_{n=0}^{\infty} a_n u^n$ has radius of convergence R (given),

that is, the series $\sum_{n=0}^{\infty} a_n u^n$ converges if $|u| < R$.

Now $|u| = |x^2| = |x|^2$.

That is, the series $\sum_{n=0}^{\infty} a_n x^{2n}$ converges if

$|u| = |x^2| = |x|^2 < R$ or $\boxed{|x| < \sqrt{R}}$.

Now we show that $\sum_{1}^{\infty} a_n^2 x^n$ has the radius of convergence R^2.

Note that

$$\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}.$$

$$\implies \lim_{n\to\infty} |a_n^2|^{1/n} = \lim_{n\to\infty} (|a_n|^{1/n})^2 = \frac{1}{R^2}.$$

Thus the radius of convergence for $\sum_{n=0}^{\infty} a_n^2 x^n$ is R^2.

Q.1 Under what conditions on the coefficients of the second order linear DE can one write down the solution as $y = \sum_{n=0}^{\infty} a_n x^n$?

Theorem

If $\tilde{h}(x)$, $\tilde{p}(x)$, $\tilde{q}(x)$, $\tilde{r}(x)$, in $\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = \tilde{r}(x)$ are analytic at $x = x_0$, then every solution of the DE is analytic at $x = x_0$ (and thus can be represented by a power series in powers of $(x - x_0)$ with radius of convergence $\tilde{R} > 0$, provided $\tilde{h}(x_0) \neq 0$).

- Does the series $y = \sum_{n=0}^{\infty} a_n x^n$ converge to the solution y of the ODE? If so, for which values of x?
  Yes, in the interval of convergence.

Q.3 How do we justify the term by term differentiation under infinite sum?

If a power series $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges for $|x - x_0| < R$, where $R > 0$, then the series obtained by differentiating term by term also converges for $|x - x_0| < R$ and represents the derivative $y'(x)$ of $y(x)$ for those x.

- Can the series be added or subtracted termwise?

  If $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x - x_0)^n$ are two power series with positive radii of convergence R_1 and R_2 respectively, then the power series

- $\sum_{n=0}^{\infty}(a_n + b_n)(x - x_0)^n$ represents the sum with $\boxed{\text{radius of convergence } \min(R_1, R_2)}$.

Q.4 Also, $\sum_{n=0}^{\infty} c_n(x-x_0)^n$, where $c_n = \sum_{m=0}^{n} a_m b_{n-m}$ represents the product with radius of convergence as $\min(R_1, R_2)$.

Q.5 If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$ for each x, then is $a_n = b_n$?

Yes. (Hint: Let $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$, then $a_n = b_n = \dfrac{f^n(x_0)}{n!}$).

Work out the following problems from **Tutorial Sheet 1**.

- Qn. 1 (i)-(ix) - Radius of convergence, use ratio test.
- Qn. 2 - Radius of convergence, use ratio test, root test.
- Qn. 4 (i)- (iv) - Solve DE using power series method, compute radius of convergence.
- Qn. 5 - power series solution about $x = 1$ (start with
$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n)$$
- Qn. 7 Initial conditions are $y(0) = 0, \ y'(0) = 0$.