MA 205 Complex Analysis: Introduction

Sudarshan Gurjar
IIT Bombay

August 17, 2020

Introduction

Welcome to the first lecture of MA 205!

Introduction

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

Introduction

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$,

Introduction

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$,

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105.

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$.

Introduction

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions,

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better,

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable,

Introduction

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable, functions.

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable, functions. Later in MA 105, you even studied $f : \mathbb{R}^m \mapsto \mathbb{R}^n$,

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable, functions. Later in MA 105, you even studied $f : \mathbb{R}^m \mapsto \mathbb{R}^n$, their continuity, differentiability

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable, functions. Later in MA 105, you even studied $f : \mathbb{R}^m \mapsto \mathbb{R}^n$, their continuity, differentiability (both partial and total).

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable, functions. Later in MA 105, you even studied $f : \mathbb{R}^m \mapsto \mathbb{R}^n$, their continuity, differentiability (both partial and total). Similarly, we'll be interested in $f : \mathbb{C} \mapsto \mathbb{C}$ which are differentiable in the "complex sense".

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable, functions. Later in MA 105, you even studied $f : \mathbb{R}^m \mapsto \mathbb{R}^n$, their continuity, differentiability (both partial and total). Similarly, we'll be interested in $f : \mathbb{C} \mapsto \mathbb{C}$ which are differentiable in the "complex sense". If f fails to be differentiable at some points,

Introduction

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable, functions. Later in MA 105, you even studied $f : \mathbb{R}^m \mapsto \mathbb{R}^n$, their continuity, differentiability (both partial and total). Similarly, we'll be interested in $f : \mathbb{C} \mapsto \mathbb{C}$ which are differentiable in the "complex sense". If f fails to be differentiable at some points, we'll also investigate such failure.

Introduction

Thus, in a way,

> *in going from MA 105 to MA 205, we're <u>just</u> going from $\mathbb{R}$ to $\mathbb{C}$.*

Thus, in a way,

in going from MA 105 to MA 205, we're <u>just</u> going from $\mathbb{R}$ to $\mathbb{C}$.

But, as you'll see, the tone of trivialization in the above sentence is quite unjust.

Introduction

Thus, in a way,

in going from MA 105 to MA 205, we're just going from $\mathbb{R}$ to $\mathbb{C}$.

But, as you'll see, the tone of trivialization in the above sentence is quite unjust. $\mathbb{C}$ is a thing of beauty,

Thus, in a way,

in going from MA 105 to MA 205, we're <u>just</u> going from $\mathbb{R}$ to $\mathbb{C}$.

But, as you'll see, the tone of trivialization in the above sentence is quite unjust. $\mathbb{C}$ is a thing of beauty, and analysis/calculus over here is immensely beautiful and charming as well as extremely useful,

Thus, in a way,

in going from MA 105 to MA 205, we're just going from $\mathbb{R}$ to $\mathbb{C}$.

But, as you'll see, the tone of trivialization in the above sentence is quite unjust. $\mathbb{C}$ is a thing of beauty, and analysis/calculus over here is immensely beautiful and charming as well as extremely useful, that you do want to go from $\mathbb{R}$ to $\mathbb{C}$,

Introduction

Thus, in a way,

in going from MA 105 to MA 205, we're just going from $\mathbb{R}$ to $\mathbb{C}$.

But, as you'll see, the tone of trivialization in the above sentence is quite unjust. $\mathbb{C}$ is a thing of beauty, and analysis/calculus over here is immensely beautiful and charming as well as extremely useful, that you do want to go from $\mathbb{R}$ to $\mathbb{C}$, and you don't want to go any further!

Thus, in a way,

in going from MA 105 to MA 205, we're <u>just</u> going from $\mathbb{R}$ to $\mathbb{C}$.

But, as you'll see, the tone of trivialization in the above sentence is quite unjust. $\mathbb{C}$ is a thing of beauty, and analysis/calculus over here is immensely beautiful and charming as well as extremely useful, that you do want to go from $\mathbb{R}$ to $\mathbb{C}$, and you don't want to go any further! In other words, there is MA 205 after MA 105,

Thus, in a way,

in going from MA 105 to MA 205, we're just going from $\mathbb{R}$ to $\mathbb{C}$.

But, as you'll see, the tone of trivialization in the above sentence is quite unjust. $\mathbb{C}$ is a thing of beauty, and analysis/calculus over here is immensely beautiful and charming as well as extremely useful, that you do want to go from $\mathbb{R}$ to $\mathbb{C}$, and you don't want to go any further! In other words, there is MA 205 after MA 105, but no MA 305 or 405!

Thus, in a way,

> *in going from MA 105 to MA 205, we're just going from $\mathbb{R}$ to $\mathbb{C}$.*

But, as you'll see, the tone of trivialization in the above sentence is quite unjust. $\mathbb{C}$ is a thing of beauty, and analysis/calculus over here is immensely beautiful and charming as well as extremely useful, that you do want to go from $\mathbb{R}$ to $\mathbb{C}$, and you don't want to go any further! In other words, there is MA 205 after MA 105, but no MA 305 or 405! Furthermore complex analysis introduces techniques which are often useful in answering questions which a priori dont have anything to do with complex numbers. We will see examples of this later..

Motivation

God made integers, all else is the work of man - Leopold Kronecker

Motivation

God made integers, all else is the work of man - Leopold Kronecker

So let's ask ourselves:

Motivation

God made integers, all else is the work of man - Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105?

Motivation

God made integers, all else is the work of man - Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}$,

Motivation

God made integers, all else is the work of man - Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}, \mathbb{Z}$, or

Motivation

God made integers, all else is the work of man - Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$?

Motivation

God made integers, all else is the work of man - Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$? $\mathbb{N}$ and $\mathbb{Z}$ are ruled out at the very start; they fail the basic "algebra test", namely division.

Motivation

God made integers, all else is the work of man - Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$? $\mathbb{N}$ and $\mathbb{Z}$ are ruled out at the very start; they fail the basic "algebra test", namely division. $\mathbb{Q}$ passes the "algebra test" alright, after all it was constructed only to pass this test,

Motivation

God made integers, all else is the work of man - Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$? $\mathbb{N}$ and $\mathbb{Z}$ are ruled out at the very start; they fail the basic "algebra test", namely division. $\mathbb{Q}$ passes the "algebra test" alright, after all it was constructed only to pass this test, but it fails the "analysis test"! , namely Cauchy sequences of rational numbers need not converge to a rational number.

Motivation

God made integers, all else is the work of man - Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$? $\mathbb{N}$ and $\mathbb{Z}$ are ruled out at the very start; they fail the basic "algebra test", namely division. $\mathbb{Q}$ passes the "algebra test" alright, after all it was constructed only to pass this test, but it fails the "analysis test"! , namely Cauchy sequences of rational numbers need not converge to a rational number.

$\mathbb{R}$ passes both these tests and hence analysis over $\mathbb{R}$ is rich and exciting.

Motivation

God made integers, all else is the work of man - Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$? $\mathbb{N}$ and $\mathbb{Z}$ are ruled out at the very start; they fail the basic "algebra test", namely division. $\mathbb{Q}$ passes the "algebra test" alright, after all it was constructed only to pass this test, but it fails the "analysis test"! , namely Cauchy sequences of rational numbers need not converge to a rational number.

$\mathbb{R}$ passes both these tests and hence analysis over $\mathbb{R}$ is rich and exciting. But it fails another "algebra test" namely obtaining roots of polynomials.

Fundamental theorem of Algebra

Theorem

Every non-constant polynomial with complex coefficients has a complex root.

This theorem fails over all the other "number systems" we know, namely $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Carl Friedrich Gauss (1777-1855) gave some of the earliest proofs of this theorem. Today this theorem has more than a hundred proofs, many of them using complex analysis. We will see at least one proof of this in this course.

Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is

Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1 = 0$.

Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1 = 0$. Now, suppose it has a root somewhere, Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1 = 0$. Now, suppose it has a root somewhere, and suppose we denote it by $\imath$, Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1 = 0$. Now, suppose it has a root somewhere, and suppose we denote it by $\imath$, then of course $-\imath$ is also a root.

Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1 = 0$. Now, suppose it has a root somewhere, and suppose we denote it by $\imath$, then of course $-\imath$ is also a root.

Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1 = 0$. Now, suppose it has a root somewhere, and suppose we denote it by $\imath$, then of course $-\imath$ is also a root. In other words, you are imagining an $\imath$, Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1 = 0$. Now, suppose it has a root somewhere, and suppose we denote it by $\imath$, then of course $-\imath$ is also a root. In other words, you are imagining an $\imath$, which has this property that $\imath^2 = -1$.

Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1 = 0$. Now, suppose it has a root somewhere, and suppose we denote it by $\imath$, then of course $-\imath$ is also a root. In other words, you are imagining an $\imath$, which has this property that $\imath^2 = -1$. And then we can write $x^2 + 1 = (x - \imath)(x + \imath)$.

Numbers

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

Numbers

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$,

Numbers

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part.

Numbers

We imagine $\mathbb{C}$ as
$$\mathbb{R} \oplus \imath\mathbb{R}.$$
i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

Numbers

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

Numbers

We imagine $\mathbb{C}$ as
$$\mathbb{R} \oplus \imath\mathbb{R}.$$
i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:
$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$
If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$,

Numbers

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition.

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

$(a+\imath b)(c+\imath d) =$

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

$(a+\imath b)(c+\imath d) = ac$

Numbers

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

$(a+\imath b)(c+\imath d) = ac+a\imath d$

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

$$(a+\imath b)(c+\imath d) = ac+a\imath d+\imath bc$$

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

$$(a+\imath b)(c+\imath d) = ac+a\imath d+\imath bc+(\imath b)\cdot(\imath d)$$

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

$$(a+\imath b)(c+\imath d) = ac+a\imath d+\imath bc+(\imath b)\cdot(\imath d) = (ac-bd)+\imath(ad+bc),$$

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

$$(a+\imath b)(c+\imath d) = ac+a\imath d+\imath bc+(\imath b)\cdot(\imath d) = (ac-bd)+\imath(ad+bc),$$

which is another complex number.

We imagine $\mathbb{C}$ as

$$\mathbb{R} \oplus \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. a is called the real part and b is called the imaginary part. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

$$(a+\imath b)(c+\imath d) = ac+a\imath d+\imath bc+(\imath b)\cdot(\imath d) = (ac-bd)+\imath(ad+bc),$$

which is another complex number. So coming back to the fundamental theorem of algebra, it is interesting that just adding one root of one real polynomial, namely $X^2 + 1$ gives you all the roots of all the complex polynomials !

Numbers

Assume FTA for now.

Assume FTA for now.

Exercise: A complex polynomial of degree n has exactly n roots.

Assume FTA for now.

Exercise: A complex polynomial of degree n has exactly n roots.

Exercise: Show that a real polynomial that is irreducible has degree at most two. i.e., if

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n, \; a_i \in \mathbb{R},$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$.

Some basic notions of topology

Let $\Omega \subseteq \mathbb{C}$ be a subset. We say that Ω is an open subset of $\mathbb{C}$ is given any point $z_0 \in \Omega$, there exists $\delta > 0$ such that the set $\{z \in \mathbb{C} \text{ such that } |z - z_0| < \delta\} \subset \Omega$. Here $|z - z_0|$ is the distance between z and z_0;

Some basic notions of topology

Let $\Omega \subseteq \mathbb{C}$ be a subset. We say that Ω is an open subset of $\mathbb{C}$ is given any point $z_0 \in \Omega$, there exists $\delta > 0$ such that the set $\{z \in \mathbb{C} \text{ such that } |z - z_0| < \delta\} \subset \Omega$. Here $|z - z_0|$ is the distance between z and z_0; thus, if $z = x + \imath y$ and $z_0 = a + \imath b$, then $|z - z_0| = \sqrt{(x-a)^2 + (y-b)^2}$.

Some basic notions of topology

Let $\Omega \subseteq \mathbb{C}$ be a subset. We say that Ω is an open subset of $\mathbb{C}$ is given any point $z_0 \in \Omega$, there exists $\delta > 0$ such that the set $\{z \in \mathbb{C} \text{ such that } |z - z_0| < \delta\} \subset \Omega$. Here $|z - z_0|$ is the distance between z and z_0; thus, if $z = x + \imath y$ and $z_0 = a + \imath b$, then $|z - z_0| = \sqrt{(x-a)^2 + (y-b)^2}$.Thus, z is in the δ-neighbourhood of z_0 means that z is inside the circle with center z_0 and radius δ.

Some basic notions of topology

Let $\Omega \subseteq \mathbb{C}$ be a subset. We say that Ω is an open subset of $\mathbb{C}$ is given any point $z_0 \in \Omega$, there exists $\delta > 0$ such that the set $\{z \in \mathbb{C} \text{ such that } |z - z_0| < \delta\} \subset \Omega$. Here $|z - z_0|$ is the distance between z and z_0; thus, if $z = x + \imath y$ and $z_0 = a + \imath b$, then $|z - z_0| = \sqrt{(x-a)^2 + (y-b)^2}$.Thus, z is in the δ-neighbourhood of z_0 means that z is inside the circle with center z_0 and radius δ.

By a δ-neighbourhood of a point $z_0 \in \mathbb{C}$, denoted $B_\delta(z_0)$, we mean the set of points $\{z \in \mathbb{C} \text{ such that } |z - z_0| < \delta\}$.

Let $\Omega \subseteq \mathbb{C}$ be a subset. We say that Ω is an open subset of $\mathbb{C}$ is given any point $z_0 \in \Omega$, there exists $\delta > 0$ such that the set $\{z \in \mathbb{C}$ such that $|z - z_0| < \delta\} \subset \Omega$. Here $|z - z_0|$ is the distance between z and z_0; thus, if $z = x + \imath y$ and $z_0 = a + \imath b$, then $|z - z_0| = \sqrt{(x-a)^2 + (y-b)^2}$.Thus, z is in the δ-neighbourhood of z_0 means that z is inside the circle with center z_0 and radius δ.

By a δ-neighbourhood of a point $z_0 \in \mathbb{C}$, denoted $B_\delta(z_0)$, we mean the set of points $\{z \in \mathbb{C}$ such that $|z - z_0| < \delta\}$.
A subset $Z \subseteq \mathbb{C}$ is said to be **closed** if its complement is open. It is a basic fact that there exists no subset of $\mathbb{C}$ that is both open and closed other than $\emptyset$ and $\mathbb{C}$.

Some basic notions of topology

Let $\Omega \subseteq \mathbb{C}$ be a subset. We say that Ω is an open subset of $\mathbb{C}$ is given any point $z_0 \in \Omega$, there exists $\delta > 0$ such that the set $\{z \in \mathbb{C} \text{ such that } |z - z_0| < \delta\} \subset \Omega$. Here $|z - z_0|$ is the distance between z and z_0; thus, if $z = x + \imath y$ and $z_0 = a + \imath b$, then $|z - z_0| = \sqrt{(x-a)^2 + (y-b)^2}$.Thus, z is in the δ-neighbourhood of z_0 means that z is inside the circle with center z_0 and radius δ.

By a δ-neighbourhood of a point $z_0 \in \mathbb{C}$, denoted $B_\delta(z_0)$, we mean the set of points $\{z \in \mathbb{C} \text{ such that } |z - z_0| < \delta\}$.
A subset $Z \subseteq \mathbb{C}$ is said to be **closed** if its complement is open. It is a basic fact that there exists no subset of $\mathbb{C}$ that is both open and closed other than $\emptyset$ and $\mathbb{C}$. One checks that the following properties hold:
1. $\emptyset$ and $\mathbb{C}$ are both open and closed.
2. Arbitrary unions and finite intersections of open subsets is open.
3. Arbitrary intersections and finite unions of closed sets is closed.

Examples

1. $\mathbb{C}$ is open. Similarly the disc $\{z \in \mathbb{C} \mid |z| < r\}$ is open for any $r > 0$.
2. $\mathbb{C}$ minus the non-positive reals is open.
3. The set $\{1/n; n \in \mathbb{N}\} \cup \{0\}$ is closed.
4. The set $\{1/n; n \in \mathbb{N}\}$ is neither open nor closed.
5. The set $S = \{(x, y) \in \mathbb{R}^2 | y^2 = x\}$ is closed.

We define the **closure** of a subset $S \subseteq \mathbb{C}$ to be the smallest closed set containing S. It is denoted $\overline{S}$. Equivalently, the closure of S is the union of S together with its limit points.
(In other words a point belongs to the closure of S if it is arbitrarily close to points in S).

Connectedness

A subset $S \subseteq \mathbb{C}$ is said to be **path-connected** if given any 2 points $z_0, z_1 \in S$, there exists a continuous path joining them. i.e, a continuous function $f : [0, 1] \to S$ such that $f(0) = z_0$ and $f(1) = z_1$.

Connectedness

A subset $S \subseteq \mathbb{C}$ is said to be **path-connected** if given any 2 points $z_0, z_1 \in S$, there exists a continuous path joining them. i.e, a continuous function $f : [0, 1] \to S$ such that $f(0) = z_0$ and $f(1) = z_1$. An open subset of $\mathbb{C}$ which is path-connected is called a domain.

A subset $S \subseteq \mathbb{C}$ is said to be **path-connected** if given any 2 points $z_0, z_1 \in S$, there exists a continuous path joining them. i.e, a continuous function $f : [0,1] \to S$ such that $f(0) = z_0$ and $f(1) = z_1$. An open subset of $\mathbb{C}$ which is path-connected is called a domain. An arbitrary open set in $\mathbb{C}$ is a disjoint union of domains. In this course we will be mostly interested in complex-valued functions defined over domains.

A subset $S \subseteq \mathbb{C}$ is said to be **path-connected** if given any 2 points $z_0, z_1 \in S$, there exists a continuous path joining them. i.e, a continuous function $f : [0,1] \to S$ such that $f(0) = z_0$ and $f(1) = z_1$. An open subset of $\mathbb{C}$ which is path-connected is called a domain. An arbitrary open set in $\mathbb{C}$ is a disjoint union of domains. In this course we will be mostly interested in complex-valued functions defined over domains.

Examples:

1. $\mathbb{C}$ and the open and closed disc are path-connected. They remain so even after removing finitely many points (in fact even after removing countable many points).
2. The hyperbola $XY = 1 \subset \mathbb{R}^2$ is disconnected.
3. $\mathbb{C}$ minus the non-zero reals is path-connected

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$\lim_{x \to a} f(x) = L \quad \iff$

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$\lim_{x \to a} f(x) = L \iff$ values of $f(x)$ can be made as close to L as we like,

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\lim_{x \to a} f(x) = L \quad \Longleftrightarrow \quad \text{values of } f(x) \text{ can be made as close to } L$$
$$\text{as we like, by taking } x \text{ close enough to } a,$$

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\lim_{x \to a} f(x) = L \iff \text{values of } f(x) \text{ can be made as close to } L$$
$$\text{as we like, by taking } x \text{ close enough to } a,$$
$$\text{on either side of } a,$$

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\lim_{x \to a} f(x) = L \quad \iff \quad \text{values of } f(x) \text{ can be made as close to } L$$

as we like, by taking x close enough to a, on either side of a, but not equal to a.

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$
\begin{aligned}
\lim_{x \to a} f(x) = L \quad &\iff \quad \text{values of } f(x) \text{ can be made as close to } L \\
&\qquad\;\; \text{as we like, by taking } x \text{ close enough to } a, \\
&\qquad\;\; \text{on either side of } a, \text{ but not equal to } a. \\
&\iff \quad |f(x) - L| \text{ can be made as small as we like}
\end{aligned}
$$

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\lim_{x \to a} f(x) = L \iff \text{values of } f(x) \text{ can be made as close to } L$$

as we like, by taking x close enough to a, on either side of a, but not equal to a.

$$\iff |f(x) - L| \text{ can be made as small as we like}$$

by taking $|x - a|$ sufficiently small,

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$
\begin{aligned}
\lim_{x\to a} f(x) = L \quad &\Longleftrightarrow \quad \text{values of } f(x) \text{ can be made as close to } L \\
&\qquad\ \ \text{as we like, by taking } x \text{ close enough to } a, \\
&\qquad\ \ \text{on either side of } a, \text{ but not equal to } a. \\
&\Longleftrightarrow \quad |f(x) - L| \text{ can be made as small as we like} \\
&\qquad\ \ \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a.
\end{aligned}
$$

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$
\begin{aligned}
\lim_{x \to a} f(x) = L \quad &\iff \quad \text{values of } f(x) \text{ can be made as close to } L \\
&\qquad\;\; \text{as we like, by taking } x \text{ close enough to } a, \\
&\qquad\;\; \text{on either side of } a, \text{ but not equal to } a. \\
&\iff \quad |f(x) - L| \text{ can be made as small as we like} \\
&\qquad\;\; \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a. \\
&\iff \quad \text{for every } \epsilon > 0,
\end{aligned}
$$

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\begin{aligned}
\lim_{x \to a} f(x) = L \quad &\iff \quad \text{values of } f(x) \text{ can be made as close to } L \\
&\qquad \text{as we like, by taking } x \text{ close enough to } a, \\
&\qquad \text{on either side of } a, \text{ but not equal to } a. \\
&\iff \quad |f(x) - L| \text{ can be made as small as we like} \\
&\qquad \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a. \\
&\iff \quad \text{for every } \epsilon > 0, \text{ there is a number } \delta > 0
\end{aligned}$$

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\lim_{x \to a} f(x) = L \iff \text{values of } f(x) \text{ can be made as close to } L$$

as we like, by taking x close enough to a, on either side of a, but not equal to a.

$\iff$ $|f(x) - L|$ can be made as small as we like by taking $|x - a|$ sufficiently small, for $x \neq a$.

$\iff$ for every $\epsilon > 0$, there is a number $\delta > 0$ such that

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\begin{aligned}
\lim_{x\to a} f(x) = L \quad &\iff \quad \text{values of } f(x) \text{ can be made as close to } L \\
&\qquad\ \ \text{as we like, by taking } x \text{ close enough to } a, \\
&\qquad\ \ \text{on either side of } a, \text{ but not equal to } a. \\
&\iff \quad |f(x) - L| \text{ can be made as small as we like} \\
&\qquad\ \ \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a. \\
&\iff \quad \text{for every } \epsilon > 0, \text{ there is a number } \delta > 0 \text{ such} \\
&\qquad\ \ \text{that } |f(x) - L| < \epsilon
\end{aligned}$$

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\begin{aligned} \lim_{x\to a} f(x) = L \quad &\iff \quad \text{values of } f(x) \text{ can be made as close to } L \\ &\qquad \text{as we like, by taking } x \text{ close enough to } a, \\ &\qquad \text{on either side of } a, \text{ but not equal to } a. \\ &\iff \quad |f(x) - L| \text{ can be made as small as we like} \\ &\qquad \text{by taking } |x-a| \text{ sufficiently small, for } x \neq a. \\ &\iff \quad \text{for every } \epsilon > 0, \text{ there is a number } \delta > 0 \text{ such} \\ &\qquad \text{that } |f(x) - L| < \epsilon \text{ whenever } 0 < |x-a| < \delta. \end{aligned}$$

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$
\begin{aligned}
\lim_{x \to a} f(x) = L \quad &\iff \quad \text{values of } f(x) \text{ can be made as close to } L \\
& \quad \text{as we like, by taking } x \text{ close enough to } a, \\
& \quad \text{on either side of } a, \text{ but not equal to } a. \\
&\iff \quad |f(x) - L| \text{ can be made as small as we like} \\
& \quad \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a. \\
&\iff \quad \text{for every } \epsilon > 0, \text{ there is a number } \delta > 0 \text{ such} \\
& \quad \text{that } |f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.
\end{aligned}
$$

Thus,

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$
\begin{aligned}
\lim_{x\to a} f(x) = L \quad &\iff \quad \text{values of } f(x) \text{ can be made as close to } L \\
&\qquad\ \text{as we like, by taking } x \text{ close enough to } a, \\
&\qquad\ \text{on either side of } a, \text{ but not equal to } a. \\
&\iff \quad |f(x) - L| \text{ can be made as small as we like} \\
&\qquad\ \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a. \\
&\iff \quad \text{for every } \epsilon > 0, \text{ there is a number } \delta > 0 \text{ such} \\
&\qquad\ \text{that } |f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.
\end{aligned}
$$

Thus, for every $\epsilon > 0$,

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$
\begin{aligned}
\lim_{x \to a} f(x) = L \quad &\iff \quad \text{values of } f(x) \text{ can be made as close to } L \\
&\qquad \text{as we like, by taking } x \text{ close enough to } a, \\
&\qquad \text{on either side of } a, \text{ but not equal to } a. \\
&\iff \quad |f(x) - L| \text{ can be made as small as we like} \\
&\qquad \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a. \\
&\iff \quad \text{for every } \epsilon > 0, \text{ there is a number } \delta > 0 \text{ such} \\
&\qquad \text{that } |f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.
\end{aligned}
$$

Thus, for every $\epsilon > 0$, however small it is,

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\begin{aligned}
\lim_{x \to a} f(x) = L \quad &\iff \quad \text{values of } f(x) \text{ can be made as close to } L \\
&\qquad \text{as we like, by taking } x \text{ close enough to } a, \\
&\qquad \text{on either side of } a, \text{ but not equal to } a. \\
&\iff \quad |f(x) - L| \text{ can be made as small as we like} \\
&\qquad \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a. \\
&\iff \quad \text{for every } \epsilon > 0, \text{ there is a number } \delta > 0 \text{ such} \\
&\qquad \text{that } |f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.
\end{aligned}$$

Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$,

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\begin{aligned}
\lim_{x\to a} f(x) = L \quad &\Longleftrightarrow \quad \text{values of } f(x) \text{ can be made as close to } L \\
& \quad \text{as we like, by taking } x \text{ close enough to } a, \\
& \quad \text{on either side of } a, \text{ but not equal to } a. \\
&\Longleftrightarrow \quad |f(x) - L| \text{ can be made as small as we like} \\
& \quad \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a. \\
&\Longleftrightarrow \quad \text{for every } \epsilon > 0, \text{ there is a number } \delta > 0 \text{ such} \\
& \quad \text{that } |f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.
\end{aligned}$$

Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$, such that,

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\begin{aligned}
\lim_{x\to a} f(x) = L \quad &\Longleftrightarrow \quad \text{values of } f(x) \text{ can be made as close to } L \\
&\qquad \text{as we like, by taking } x \text{ close enough to } a, \\
&\qquad \text{on either side of } a, \text{ but not equal to } a. \\
&\Longleftrightarrow \quad |f(x) - L| \text{ can be made as small as we like} \\
&\qquad \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a. \\
&\Longleftrightarrow \quad \text{for every } \epsilon > 0, \text{ there is a number } \delta > 0 \text{ such} \\
&\qquad \text{that } |f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.
\end{aligned}$$

Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$, such that, if $x \in (a - \delta, a + \delta)$,

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$\lim_{x \to a} f(x) = L \iff$ values of $f(x)$ can be made as close to L as we like, by taking x close enough to a, on either side of a, but not equal to a.

$\iff$ $|f(x) - L|$ can be made as small as we like by taking $|x - a|$ sufficiently small, for $x \neq a$.

$\iff$ for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$, such that, if $x \in (a - \delta, a + \delta)$, then, $f(x) \in (L - \epsilon, L + \epsilon)$.

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$\lim_{x \to a} f(x) = L \iff$ values of $f(x)$ can be made as close to L as we like, by taking x close enough to a, on either side of a, but not equal to a.

$\iff$ $|f(x) - L|$ can be made as small as we like by taking $|x - a|$ sufficiently small, for $x \neq a$.

$\iff$ for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$, such that, if $x \in (a - \delta, a + \delta)$, then, $f(x) \in (L - \epsilon, L + \epsilon)$. If $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$,

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$\lim_{x \to a} f(x) = L \iff$ values of $f(x)$ can be made as close to L as we like, by taking x close enough to a, on either side of a, but not equal to a.

$\iff$ $|f(x) - L|$ can be made as small as we like by taking $|x - a|$ sufficiently small, for $x \neq a$.

$\iff$ for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$, such that, if $x \in (a - \delta, a + \delta)$, then, $f(x) \in (L - \epsilon, L + \epsilon)$. If $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$, then $L \in \mathbb{C}$ is the limit of f

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$\lim_{x\to a} f(x) = L \iff$ values of $f(x)$ can be made as close to L as we like, by taking x close enough to a, on either side of a, but not equal to a.

$\iff$ $|f(x) - L|$ can be made as small as we like by taking $|x - a|$ sufficiently small, for $x \neq a$.

$\iff$ for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$, such that, if $x \in (a - \delta, a + \delta)$, then, $f(x) \in (L - \epsilon, L + \epsilon)$. If $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$, then $L \in \mathbb{C}$ is the limit of f as $z \mapsto z_0$, $z_0 \in \mathbb{C}$,

Limit

Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$\lim_{x \to a} f(x) = L \iff$ values of $f(x)$ can be made as close to L as we like, by taking x close enough to a, on either side of a, but not equal to a.

$\iff$ $|f(x) - L|$ can be made as small as we like by taking $|x - a|$ sufficiently small, for $x \neq a$.

$\iff$ for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$, such that, if $x \in (a - \delta, a + \delta)$, then, $f(x) \in (L - \epsilon, L + \epsilon)$. If $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$, then $L \in \mathbb{C}$ is the limit of f as $z \mapsto z_0$, $z_0 \in \mathbb{C}$, if for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Limit

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases,

Limit

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference.

Limit

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this!

Limit

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this! Exactly as in the real case:

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this! Exactly as in the real case:

Theorem (Limit Laws)

Suppose $c \in \mathbb{C}$ *and* $\lim_{z \to z_0} f(z)$ *and* $\lim_{z \to z_0} g(z)$ *exist.*

Limit

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this! Exactly as in the real case:

Theorem (Limit Laws)

Suppose $c \in \mathbb{C}$ *and* $\lim_{z \to z_0} f(z)$ *and* $\lim_{z \to z_0} g(z)$ *exist. Then,*

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this! Exactly as in the real case:

Theorem (Limit Laws)

Suppose $c \in \mathbb{C}$ *and* $\lim_{z \to z_0} f(z)$ *and* $\lim_{z \to z_0} g(z)$ *exist. Then,*

1. $\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$

Limit

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this! Exactly as in the real case:

Theorem (Limit Laws)

Suppose $c \in \mathbb{C}$ *and* $\lim_{z \to z_0} f(z)$ *and* $\lim_{z \to z_0} g(z)$ *exist. Then,*

1. $\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$
2. $\lim_{z \to z_0} [cf(z)] = c \lim_{z \to z_0} f(z)$

Limit

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this! Exactly as in the real case:

Theorem (Limit Laws)

Suppose $c \in \mathbb{C}$ *and* $\lim_{z \to z_0} f(z)$ *and* $\lim_{z \to z_0} g(z)$ *exist. Then,*

1. $\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$
2. $\lim_{z \to z_0} [cf(z)] = c \lim_{z \to z_0} f(z)$
3. $\lim_{z \to z_0} [f(z)g(z)] = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z)$

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this! Exactly as in the real case:

Theorem (Limit Laws)

Suppose $c \in \mathbb{C}$ *and* $\lim\limits_{z\to z_0} f(z)$ *and* $\lim\limits_{z\to z_0} g(z)$ *exist. Then,*

1. $\lim\limits_{z\to z_0} [f(z)+g(z)] = \lim\limits_{z\to z_0} f(z) + \lim\limits_{z\to z_0} g(z)$
2. $\lim\limits_{z\to z_0} [cf(z)] = c \lim\limits_{z\to z_0} f(z)$
3. $\lim\limits_{z\to z_0} [f(z)g(z)] = \lim\limits_{z\to z_0} f(z) \cdot \lim\limits_{z\to z_0} g(z)$
4. $\lim\limits_{z\to z_0} \dfrac{1}{f(z)} = \dfrac{1}{\lim\limits_{z\to z_0} f(z)}$,

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this! Exactly as in the real case:

Theorem (Limit Laws)

Suppose $c \in \mathbb{C}$ *and* $\lim_{z\to z_0} f(z)$ *and* $\lim_{z\to z_0} g(z)$ *exist. Then,*

1. $\lim_{z\to z_0}[f(z)+g(z)] = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z)$
2. $\lim_{z\to z_0}[cf(z)] = c\lim_{z\to z_0} f(z)$
3. $\lim_{z\to z_0}[f(z)g(z)] = \lim_{z\to z_0} f(z) \cdot \lim_{z\to z_0} g(z)$
4. $\lim_{z\to z_0}\frac{1}{f(z)} = \frac{1}{\lim_{z\to z_0} f(z)}$, *if* $\lim_{z\to z_0} f(z) \neq 0.$

Continuity

In the real case:

Continuity

In the real case:

A function f is continuous at a if

In the real case:

A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

Continuity

In the real case:

A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

The function f is continuous on an interval

In the real case:

A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

The function f is continuous on an interval if it is continuous at every point in the interval.

In the real case:

A function f is continuous at a if

$$\lim_{x\to a} f(x) = f(a).$$

The function f is continuous on an interval if it is continuous at every point in the interval.

Similarly,

A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ is continuous at $z_0 \in \Omega$ if

In the real case:

A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

The function f is continuous on an interval if it is continuous at every point in the interval.

Similarly,

A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ *is continuous at* $z_0 \in \Omega$ *if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Continuity

In the real case:

A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

The function f is continuous on an interval if it is continuous at every point in the interval.

Similarly,

A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ *is continuous at* $z_0 \in \Omega$ *if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The function f is continuous on a domain

Continuity

In the real case:

A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

The function f is continuous on an interval if it is continuous at every point in the interval.

Similarly,

A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ *is continuous at* $z_0 \in \Omega$ *if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The function f is continuous on a domain if it is continuous at every point in the domain.

A subset $S \subseteq \mathbb{C}$ is said to be **compact** if it is closed and bounded.

Theorem

Any continuous complex valued function on a compact subset $S \subseteq \mathbb{C}$ is bounded. i.e,
$\exists M \in \mathbb{R}$ such that $|f(z)| < M$ for all $z \in S$.

Even the converse is true; if a subset $S \subseteq \mathbb{C}$ has the property that every continuous function on it is bounded, then S is compact. (Exercise !)

Differentiation

In the real case:

In the real case: The derivative of a function f at a point a,

Differentiation

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$,

Differentiation

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Differentiation

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Differentiation

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. Equivalently,

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. Equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Differentiation

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. Equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Ditto in the complex case.

Differentiation

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. Equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Ditto in the complex case. Let $\Omega \subset \mathbb{C}$ be open. A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ is said to be **differentiable**, (sometimes called complex-differentiable) at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Notice here that z can approach z_0 along any direction (along any path, that is).

Differentiation

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. Equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Ditto in the complex case. Let $\Omega \subset \mathbb{C}$ be open. A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ is said to be **differentiable**, (sometimes called complex-differentiable) at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Notice here that z can approach z_0 along any direction (along any path, that is). You can of course prove product rule, chain rule, etc similar to the real case.

We say that f is holomorphic on Ω if f is differentiable at each point of Ω.

We say that f is holomorphic on Ω if f is differentiable at each point of Ω. f is holomorphic (also called complex analytic) at z_0 if it is holomorphic in some neighbourhood of z_0.

We say that f is holomorphic on Ω if f is differentiable at each point of Ω. f is holomorphic (also called complex analytic) at z_0 if it is holomorphic in some neighbourhood of z_0.
Remark: If $f : \Omega \to \mathbb{C}$ is a function, then f can be thought of as a function from $\mathbb{R}^2 \to \mathbb{R}^2$, i.e two functions of two real variables $f(x, y) = (u(x, y), v(x, y))$, where u and v are the real and imaginary parts of f. It can be shown that if f is a holomorphic function, then thought of as a real function from $\mathbb{R}^2$ to itself in the above sense, it is infinitely differentiable; equivalently, both u and v are infinitely differentiable functions of x and y (all partial derivatives exist up all orders). We will soon see that the converse is not true; most infinitely differentiable functions from the real plane $\mathbb{R}^2$ to itself are **NOT** holomorphic.

Thus the property of holomorphicity is much stronger than infinite differentiability of the associated real functions (in this case denoted u and v).

Thus the property of holomorphicity is much stronger than infinite differentiability of the associated real functions (in this case denoted u and v).

Now, the second surprise.

Thus the property of holomorphicity is much stronger than infinite differentiability of the associated real functions (in this case denoted u and v).

Now, the second surprise. If f is holomorphic in a domain, then f' is also holomorphic there.

Thus the property of holomorphicity is much stronger than infinite differentiability of the associated real functions (in this case denoted u and v).

Now, the second surprise. If f is holomorphic in a domain, then f' is also holomorphic there. Thus, in a domain,

Once differentiable implies infinitely differentiable. In fact something much stronger is true, namely the function is defined by Taylor series.

Thus the property of holomorphicity is much stronger than infinite differentiability of the associated real functions (in this case denoted u and v).

Now, the second surprise. If f is holomorphic in a domain, then f' is also holomorphic there. Thus, in a domain,

Once differentiable implies infinitely differentiable. In fact something much stronger is true, namely the function is defined by Taylor series.

We'll prove this too, but later.

Thus the property of holomorphicity is much stronger than infinite differentiability of the associated real functions (in this case denoted u and v).

Now, the second surprise. If f is holomorphic in a domain, then f' is also holomorphic there. Thus, in a domain,

Once differentiable implies infinitely differentiable. In fact something much stronger is true, namely the function is defined by Taylor series.

We'll prove this too, but later. As you know from MA 105, this is far from true in the real variable case.

Thus the property of holomorphicity is much stronger than infinite differentiability of the associated real functions (in this case denoted u and v).

Now, the second surprise. If f is holomorphic in a domain, then f' is also holomorphic there. Thus, in a domain,

Once differentiable implies infinitely differentiable. In fact something much stronger is true, namely the function is defined by Taylor series.

We'll prove this too, but later. As you know from MA 105, this is far from true in the real variable case. f' needn't even be continuous.

Thus the property of holomorphicity is much stronger than infinite differentiability of the associated real functions (in this case denoted u and v).

Now, the second surprise. If f is holomorphic in a domain, then f' is also holomorphic there. Thus, in a domain,

Once differentiable implies infinitely differentiable. In fact something much stronger is true, namely the function is defined by Taylor series.

We'll prove this too, but later. As you know from MA 105, this is far from true in the real variable case. f' needn't even be continuous. Examples?

Differentiation

Check for differentiability and holomorphicity:

1. $f(z) = c$
2. $f(z) = z$
3. $f(z) = z^n, n \in \mathbb{Z}$
4. $f(z) = \mathrm{Re}(z)$
5. $f(z) = |z|$
6. $f(z) = |z|^2$
7. $f(z) = \bar{z}$
8. $f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

Syllabus

Definition and properties of analytic functions.
Cauchy-Riemann equations, harmonic functions.
Power series and their properties.
Elementary functions.
Cauchys theorem and its applications.
Taylor series and Laurent expansions.
Residues and the Cauchy residue formula.
Evaluation of improper integrals.
Conformal mappings.

1. R. V. Churchill and J. W. Brown, Complex variables and applications (7th Edition), McGraw-Hill (2003).
2. E. Kreyszig, Advanced engineering mathematics (8th Edition), John Wiley (1999).
3. J. M. Howie, Complex analysis, Springer-Verlag (2004).
4. M. J. Ablowitz and A. S. Fokas, Complex Variables-Introduction and Applications, Cambridge University Press, 1998 (Indian Edition).

More advanced references :

1. Lars Ahlfors - Complex Analysis
2. John Conway - Functions of a Complex Variable
3. Serge Lang - Complex Analysis