MA 207 - Differential Equations-II

Amiya Kumar Pani

Department of Mathematics,
Indian Institute of Technology Bombay,
Powai, Mumbai 76
akp.ma207.2020@gmail.com

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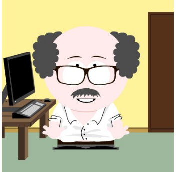
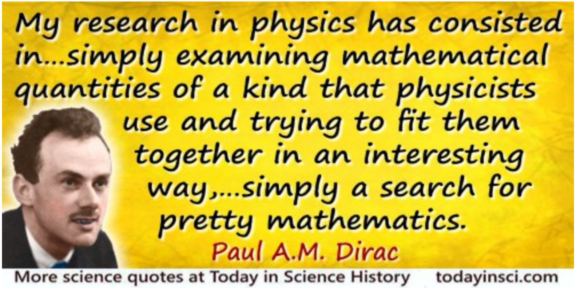
Start with Two Quotations

"Engineers think that equations approximate the real world. Physicists think that the real world approximates equations. Mathematicians are unable to make the connection. "

by Anonymous





Outline of the lecture

- Bessel's equation
- Bessel functions of first kind
- Exercises

Bessel's equation

Friedrich Wilhelm Bessel (22 July 1784 - 17 March 1846) was a German mathematician, astronomer, and systematizer of the Bessel functions. The asteroid 1552 Bessel was named in his honour.

Bessel functions, first defined by Daniel Bernoulli and generalized by Friedrich Bessel, who proposed it while studying planetary perturbation in 1824 , are solutions of Bessel's differential equation: $\boxed{x^2y'' + xy' + (x^2 - p^2)y = 0,}$ where $p \geq 0$ is a real parameter. Bessel functions are also known as cylindrical harmonics, as these are found in the solution to Laplace's equation in cylindrical coordinates.

Bessel's equation of order p

$$\boxed{x^2y'' + xy' + (x^2 - p^2)y = 0,}$$

where $p \geq 0$ is a real parameter.
The normalized form of the equation is

$$\boxed{y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0}$$

$x = 0$ is a REGULAR SINGULAR POINT of the Bessel's equation as $xp(x) = x \times \dfrac{1}{x} = 1$ and $\displaystyle\lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2 \times \frac{x^2 - p^2}{x^2} = -p^2$ are real-analytic.

Frobenius series solution to Bessel's equation

Seek a solution to $\boxed{x^2y'' + xy' + (x^2 - p^2)y = 0}$ in the form :

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n, \text{ valid in } 0 < x < R,\ R > 0,\ a_0 \neq 0.$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1},\ xy' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2},$$

$$x^2y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r}$$

Substituting in the Bessel's equation, we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+2}}$$

$$- \sum p^2 a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \underbrace{\sum_{n=2}^{\infty} a_{n-2}x^{n+r}}$$

$$-\sum_{n=0}^{\infty} p^2 a_n x^{n+r} = 0$$

Grouping the terms appropriately, we obtain

$$\sum_{n=2}^{\infty}\Big((n+r)(n+r-\cancel{1})a_n + \cancel{(n+r)a_n} + a_{n-2} - p^2 a_n\Big)x^{n+r}$$

$$+\Big(\underbrace{r(r-1)a_0}\, x^r + \overbrace{(r+1)ra_1}\, x^{r+1}\Big) + \Big(\underbrace{ra_0}\, x^r + \overbrace{(r+1)a_1}\, x^{r+1}\Big)$$

$$+\Big(\underbrace{-p^2 a_0}\, x^r + \overbrace{-p^2 a_1}\, x^{r+1}\Big) = 0$$

That is,

$$\left(r^2 - p^2\right)a_0x^r + \left((r+1)^2 - p^2\right)a_1x^{r+1} + \sum_{n=2}^{\infty}\left(\left((n+r)^2 - p^2\right)a_n + a_{n-2}\right)x^{n+r} = 0$$

INDICIAL EQUATION : $\boxed{r^2 - p^2 = 0 \ \ (a_0 \neq 0)}$

$\boxed{((r+1)^2 - p^2)a_1 = 0}$

$\boxed{((n+r)^2 - p^2)a_n + a_{n-2} = 0, \quad n \geq 2.}$

The roots of the indicial equation are given by $\boxed{r_1 = p, \ r_2 = -p.}$
Let us assume r_1 to be the larger root. Several cases arise.
Let us first form a solution corresponding to $\boxed{r_1 = p}$.
Substituting $r = r_1 = p$ in the relations,
$((r+1)^2 - p^2)a_1 = 0 \Longrightarrow ((p+1)^2 - p^2)a_1 = 0 \Longrightarrow (2p+1)a_1 = 0 \Longrightarrow a_1 = 0, \text{ since } p \geq 0.$

Using $r = r_1 = p$
in $\left((n+r)^2 - p^2\right) a_n + a_{n-2} = 0, \quad n \geq 2$, we obtain,

$$\left((n+p)^2 - p^2\right) a_n + a_{n-2} = 0 \implies n(n+2p) a_n + a_{n-2} = 0$$

$$\boxed{\text{RECURRENCE RELATION} : a_n = -\frac{a_{n-2}}{n(n+2p)}, \ n \geq 2}$$

Since $a_1 = 0$, all ODD COEFFICIENTS are zero.
That is,

$$\boxed{a_3 = a_5 = \cdots = 0.}$$

Recurrence Relation

$$\boxed{a_n = -\frac{1}{n(n+2p)} a_{n-2}, \quad n \geq 2}$$

$\boxed{n = 2}$

$$a_2 = -\frac{1}{2 \cdot (2+2p)} a_0 = \frac{(-1)^1}{2 \cdot 2 \cdot (p+1)} a_0;$$

$\boxed{n = 4}$

$$\begin{aligned} a_4 = -\frac{1}{4 \cdot (4+2p)} a_2 &= \frac{(-1)^2}{(2 \cdot 4)2^2 \cdot (p+1)(p+2)} a_0 \\ &= \frac{(-1)^2}{2^2(1 \cdot 2)2^2 \cdot (p+1)(p+2)} a_0; \end{aligned}$$

$\vdots$

$\boxed{n = 2k}$

$$a_{2k} = \frac{(-1)^k}{2^{2k} k!(p+1)(p+2)\cdots(p+k)} a_0.$$

Hence the solution of the DE corresponding to the larger root $r_1 = p$ is

$$\boxed{y_1(x) = a_0 x^p + a_0 x^p \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n} n!(p+1)(p+2)\cdots(p+n)} x^{2n}.}$$

Case 1: p is a positive integer

$$\begin{aligned} y_1(x) &= a_0 x^p + a_0 x^p \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n} n!(p+1)(p+2)\cdots(p+n)} x^{2n} \\ &= a_0 x^p + a_0\, 2^p p! \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}. \end{aligned}$$

Case 1: p is a postive integer

Choosing $\boxed{a_0 = \frac{1}{2^p p!}}$,

$$\begin{aligned} J_p(x) &= \frac{x^p}{2^p p!} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p} \end{aligned}$$

Using $\boxed{(p+n)! = \Gamma(p+n+1),}$ we obtain the Bessel's function of FIRST KIND of ORDER p as :

$$\boxed{J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}}.$$

For $z > 0$, $\boxed{\Gamma(z) = \lim_{t\to\infty} \int_0^t e^{-x} x^{z-1}\, dx = \int_0^\infty e^{-x} x^{z-1}\, dx.}$

Using the definition and integration by parts, we obtain

$$\boxed{\Gamma(z+1) = z\,\Gamma(z).} \quad \text{Also, } \Gamma(1) = 1.$$

Hence, if z is a positive integer; say n, we have

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 1.$$

Hence, $\boxed{\Gamma(n+1) = n!}$.

If $\boxed{z \text{ is positive}}$, but not an integer, we can use $\boxed{\Gamma(z+1) = z!}$ to define $z!$.

$\Gamma(z)$ values are available for $1 \le z \le 2$. Using these, we can compute $\Gamma(z)$ for all real numbers, except for $z = 0, -1, -2, \cdots$ using the recursion formula $\boxed{\Gamma(z+1) = z\Gamma(z)}$.

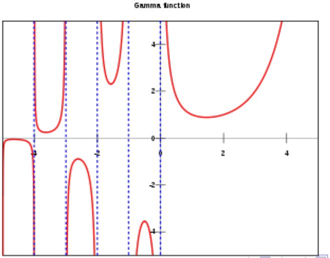
$$\Gamma(2\frac{1}{2}) = \Gamma(5/2) = (3/2)\,\Gamma(3/2) = (3/2)(1/2)\Gamma(1/2) = (3/4)\sqrt{\pi}.$$

For negative real numbers, $(z \neq -1, -2, \cdots)$, we use

$$\boxed{\Gamma(z) = \frac{\Gamma(z+1)}{z}}.$$

$$\begin{aligned}
\Gamma(-5/2) &= \frac{\Gamma(-3/2)}{-5/2} = \frac{\Gamma(-1/2)}{(-5/2)(-3/2)} \\
&= \frac{\Gamma(1/2)}{(-5/2)(-3/2)(-1/2)} = -\frac{8}{15}\sqrt{\pi}.
\end{aligned}$$

The Gamma function. Note that it blows up at 0 and negative integers.



Case 2 : p is NOT a positive integer.

$$\boxed{y_1(x) = a_0 x^p + a_0 x^p \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n} n! \underbrace{(p+1)(p+2)\cdots(p+n)}} x^{2n}.}$$

Using $\boxed{\Gamma(z+1) = z\Gamma(z),}$

$$\begin{aligned} \Gamma(n+p+1) &= (n+p)\Gamma(n+p) = (n+p)(n+p-1)\Gamma(n+p-1) \\ &= (n+p)(n+p-1)\cdots(p+1)\Gamma(p+1) \end{aligned}$$

Hence, $\boxed{\dfrac{\Gamma(n+p+1)}{\Gamma(p+1)} = (n+p)(n+p-1)\cdots(p+1).}$

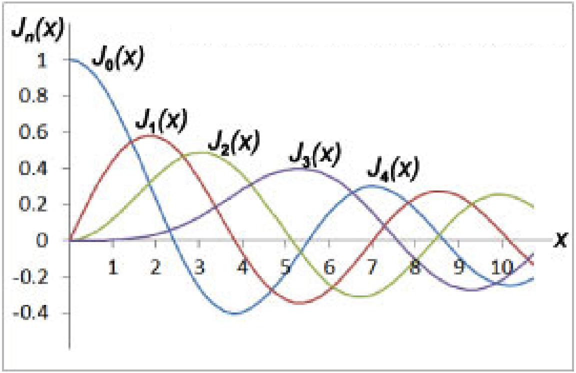
Using the gamma notation and $\boxed{a_0 = \dfrac{1}{2^p\,\Gamma(p+1)}}$, we obtain

$$\begin{aligned} J_p(x) &= \frac{1}{2^p\Gamma(p+1)} x^p \\ &\quad + \frac{1}{2^p\Gamma(p+1)} 2^p\Gamma(p+1) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p} \end{aligned}$$

For $p \geq 0$, Bessel function of FIRST KIND of ORDER p is defined as

$$\boxed{J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p} = \sum_{n=0}^{\infty} A_n \left(\frac{x}{2}\right)^{2n+p}}$$

Using ratio test, the radius of convergence $R = \infty$.



Cases that arise - Second solution

In Frobenius method, the following cases were discussed:

$$\text{Roots are} \begin{cases} \text{distinct \& not differing by an integer}, r_1 - r_2 = 2p \neq integer; \\ \text{equal}, \quad p = 0, \\ \text{distinct \& differing by an integer}, \quad r_1 - r_2 = 2p = integer, \end{cases}$$

When $\boxed{r_1 - r_2 = 2p \neq integer}$, the second linearly independent solution can be obtained from Frobenius theory as

$$\boxed{J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(-p+n+1)} \left(\frac{x}{2}\right)^{2n-p}} \quad \text{(EXERCISE)},$$

just by replacing p by $-p$ in the definition of $J_p(x)$. Moreover, as long as p is NOT AN INTEGER, even if $r_1 - r_2 = 2p$ is AN INTEGER, the functions $J_p(x)$ and $J_{-p}(x)$ are LINEARLY INDEPENDENT.

- Hence, the general solution for the Bessel's equation when $\boxed{p \geq 0 \text{ IS NOT AN INTEGER}}$ is
  $$\boxed{y(x) = C_1 J_p(x) + C_2 J_{-p}(x)}$$
- In this case, $J_p(x)$ and $J_{-p}(x)$ are linearly independent.

  Consider $c_1 J_p(x) + c_2 J_{-p}(x) = 0$, let $x \to 0^+$ and hence, it must happened that $c_2 = 0$. But if $c_2 = 0$, c_1 must be identically 0 as $J_p(0) \neq 0$.
- When $\boxed{p \geq 0 \text{ IS AN INTEGER}}$, $\boxed{J_{-p}(x) = (-1)^p J_p(x)}$
  Try to prove it. if not able to do it, then, see, Theorem 2 of pages 222-223 (Kreyszig) for a proof.
  Therefore, the two solutions $J_p(x)$ and $J_{-p}(x)$ are LINEARLY DEPENDENT.
  In this case, the second solution needs to be constructed using method of reduction of order or any other suitable method.

Example - Frobenius method ($r_1 - r_2 \notin \mathbb{N}$) - HW

Use the method of Frobenius to solve $2x^2y'' + 3xy' - (x^2+1)y = 0$ around the regular singular point $x = 0$.

Hints:

- $y = x^r \sum_{n=0}^{\infty} a_n x^n$
- Indicial equation : $2r^2 + r - 1 = 0$, $\boxed{r_1 = 1/2,\ r_2 = -1}$.
- $(2r(r+1) + 3(r+1) - 1)a_1 = 0 \Longrightarrow a_1 = 0$
- $2(n+r)(n+r-1)a_n + 3(n+r)a_n - a_{n-2} - a_n = 0,\ n \geq 2.$
- $$\boxed{a_n = \frac{a_{n-2}}{2(n+r)^2 + (n+r) - 1}, \quad n \geq 2}$$

  Hence, $a_1 = a_3 = a_5 = \cdots = 0$.
- $r = 1/2$; $\boxed{a_n = \frac{a_{n-2}}{2n^2 + 3n},\ n \geq 2.}$

  $r = -1$; $\boxed{a_n^* = \frac{a_{n-2}^*}{2n^2 - 3n},\ n \geq 2.}$

- $$y_1(x) = x^{1/2}\left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \cdots\right),$$
  $$y_2(x) = x^{-1}\left(1 + \frac{x^2}{2} + \frac{x^4}{40} + \cdots\right)$$ (CHECK THIS!)
  are linearly independent solutions.

Example - Frobenius method ($r_1 - r_2 \in \mathbb{N}$), 2 l.i. solutions without $\ln x$ term

Use the method of Frobenius to solve $x^2y'' + (6x + x^2)y' + xy = 0$ around the regular singular point $x = 0$.

- $y = x^r \sum_{n=0}^{\infty} a_n x^n$
- Substituting in the DE,

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 6(n+r)a_n x^{n+r}$$

$$+ \sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}x^{n+r} = 0$$

$\boxed{\text{INDICIAL EQUATION:}}$

$(r(r-1)+6r)a_0 = 0 \implies r(r+5) = 0 \implies \boxed{r = 0,\ r = -5}$.

$$\sum_{n=0}^{\infty}(n+r)(n+r-\cancel{1})a_nx^{n+r}+\sum_{n=0}^{\infty}\cancel{6(n+r)}\;5(n+r)\,a_nx^{n+r}$$

$$+\sum_{n=1}^{\infty}(n+r-\cancel{1})a_{n-1}x^{n+r}+\sum_{n=1}^{\infty}\cancel{a_{n-1}}x^{n+r}=0$$

- $(n+r)(n+r+5)a_n+(n+r)a_{n-1}=0,\ n\geq 1.$

  If we begin with the smaller root $\boxed{r=-5}$ we get :

  $n(n-5)\,a_n+(n-5)\,a_{n-1}=0 \Longrightarrow a_n=-\dfrac{a_{n-1}}{n},\ n\geq 1,\ n\neq 5.$

  $a_1=-\dfrac{a_0}{1},\ a_2=-\dfrac{a_1}{2}=\dfrac{a_0}{2},\ a_3=-\dfrac{a_2}{3}=-\dfrac{a_0}{6},$

  $a_4=-\dfrac{a_3}{4}=\dfrac{a_0}{24}.$

  $\boxed{\text{For } n=5, \text{ any choice of } a_5 \text{ works}}$ $\Longrightarrow$ a_5 can be chosen arbitrarily.

  So we get, $a_5=a_5,\ a_6=-\dfrac{a_5}{6}\ ,\ a_7=-\dfrac{a_6}{7}=\dfrac{(-1)^2}{6\cdot 7}a_5,\cdots.$

That is,

$$y(x) = a_0 x^{-5}(1 - x + x^2/2 - x^3/6 + x^4/24)$$
$$+a_5(1 - \frac{x}{6} + \frac{x^2}{6 \cdot 7} + \cdots + (-1)^n \frac{x^n}{6 \cdot 7 \cdots (n+5)} + \cdots)$$

$\boxed{y(x) = a_0 y_1(x) + a_5 y_2(x)}$, where

$\boxed{y_1(x) = x^{-5}(1 - x + x^2/2 - x^3/6 + x^4/24)}$ and

$$\boxed{y_2(x) = 1 - \frac{x}{6} + \frac{x^2}{6 \cdot 7} + \cdots + (-1)^n \frac{x^n}{6 \cdot 7 \cdots (n+5)} + \cdots}.$$

$r = 0$ yields $n(n+5)a_n + n a_{n-1} = 0$, $a_n = -\dfrac{a_{n-1}}{n+5}$,

Verify that we get $y_2(x)$.

NOTE THAT THE SMALLER ROOT HAS YIELDED TWO LINEARLY INDEPENDENT SOLUTIONS.

Example - Frobenius method ($r_1 - r_2 \in \mathbb{N}$), 2 l.i. solutions with $\ln x$ term (one root doesn't yield a solution!)

$$x^2y'' - xy' + (x^2 - 8)y = 0.$$

- $y = x^r \sum_{n=0}^{\infty} a_n x^n$
- Substituting in the DE,

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$
$$+\sum_{n=2}^{\infty} a_{n-2}x^{n+r} - \sum_{n=0}^{\infty} 8a_n x^{n+r} = 0$$

INDICIAL EQUATION:

$$(r(r-1)-r-8)a_0 = 0 \implies r^2-2r-8 = 0 \implies \boxed{r = 4,\ r = -2.}$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$
$$+\sum_{n=2}^{\infty} a_{n-2}x^{n+r} - \sum_{n=0}^{\infty} 8a_n x^{n+r} = 0$$

- $((r+1)r-(r+1)-8)a_1 = 0 \implies (r^2-9)a_1 = 0 \implies a_1 = 0$ for $r = 4, \ -2$.
- $((n+r)^2 - 2(n+r) - 8)a_n = -a_{n-2}$
  $\implies a_n = -\dfrac{a_{n-2}}{(n+r)^2 - 2(n+r) - 8}, \ n \geq 2.$
  If we begin with the smaller root $\boxed{r=-2}$ we get :
  $a_n = -\dfrac{a_{n-2}}{n(n-6)}, \ n \geq 2, \ n \neq 6.$
  $a_1 = a_3 = a_5 = \cdots = 0,$
  $a_2 = \dfrac{a_0}{8}, \ a_4 = \dfrac{a_0}{64}$

For $r = -2$, $n(n-6)a_n + a_{n-2} = 0$.

$\boxed{\text{For } n = 6,}$ we have $0 \cdot a_6 + a_4 = 0$ from the recurrence relation.
So we get, $a_4 = 0$, and hence we are in trouble if $a_0 \neq 0$!
So, we do not get two Frobenius solutions.

- We only get one, for $\boxed{r = 4}$, which is

  $$y(x) = x^4\left(1 + 6\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(n+3)!}\right) \text{ (CHECK THIS !).}$$

  The second solution is found by the method of reduction of order or any other suitable method.

Second solution for Bessel's equation of order 0 - Reduction of order (see [K] also)

Bessel's equation of order 0 is $\boxed{y'' + (1/x)y' + y = 0}$

$$y_2(x) = J_0(x)\int \frac{e^{-\int (1/x)\,dx}}{(J_0(x))^2}\,dx = J_0(x)\int \frac{dx}{x(J_0(x))^2}$$

$$\begin{aligned}
\text{Now,}\quad J_0(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n} \\
&= 1 - x^2/4 + x^4/64 - x^6/2304 + \cdots \\
&= 1 - x^2/4(1 - x^2/16 + x^4/576 - \cdots) \\
(J_0(x))^2 &= (1 - x^2/4(1 - x^2/16 + x^4/576 - \cdots))^2 \\
&= 1 + x^4/16(1 - x^2/16 + x^4/576 - \cdots)^2 \\
&\quad - x^2/2(1 - x^2/16 + x^4/576 - \cdots) \\
&= 1 - x^2/2 + 3x^4/32 - 5x^6/576 + \cdots \\
&= 1 - x^2/2(1 - 3x^2/16 + 5x^4/288 - \cdots)
\end{aligned}$$

$$\boxed{y_2(x) = J_0(x)\int \frac{e^{-\int (1/x)\,dx}}{(J_0(x))^2}\,dx = J_0(x)\int \frac{dx}{x(J_0(x))^2}}$$

$$\begin{aligned}
\frac{1}{(J_0(x))^2} &= \frac{1}{1 - x^2/2(1 - 3x^2/16 + 5x^4/288 - \cdots)} \\
&= 1 + x^2/2(1 - 3x^2/16 + 5x^4/288 - \cdots) \\
&\quad + x^4/4(1 - 3x^2/16 + 5x^4/288 - \cdots)^2 + \cdots \\
&= 1 + x^2/2 + 5x^4/32 + 23x^6/576 + \cdots \\
y_2(x) &= J_0(x)\int (\frac{1}{x} + \frac{x}{2} + \frac{5x^3}{32} + \frac{23x^5}{576} + \cdots)\,dx \\
&= J_0(x)\left(\ln x + \frac{x^2}{4} + \frac{5x^4}{128} + \frac{23x^6}{3456} + \cdots\right) \\
&= J_0(x)\ln x + \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right) \times \\
&\qquad \left(\frac{x^2}{4} + \frac{5x^4}{128} + \frac{23x^6}{3456} + \cdots\right)
\end{aligned}$$

General solution

$$\boxed{y_2(x) = J_0(x)\ln x + (\frac{x^2}{4} - \frac{3x^4}{128} + \frac{11x^6}{13824} + \cdots).}$$

A careful observation after computing a few more terms yields

$$\boxed{y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{2^{2n}(n!)^2}(1 + \frac{1}{2} + \cdots \frac{1}{n}).}$$

A special linear combination of $J_0(x)$ and $y_2(x)$ is called as Bessel's equation of SECOND KIND of order 0 is defined by :

$$\boxed{Y_0(x) = \frac{2}{\pi}(y_2(x) + (\gamma - \ln 2)J_0(x)),\ \gamma = \text{Euler's constant} \approx 0.5772.}$$

The general solution for Bessel's equation of ORDER 0 can be written as

$\boxed{y(x) = C_1 J_0(x) + C_2 Y_0(x),}$ in $0 < x < R$, C_1 and C_2 being arbitrary constants.

$J_0(x)$ and $Y_0(x)$

- As $x \to 0$, $J_0(x) \to 0$, while $Y_0(x)$ has a singularity.
- If we desire a solution that is bounded at origin, then choose $C_2 = 0$ (discard $Y_0(x)$).
- Bessel's equation can be written as $y'' + (1/x)y' + (1 - p^2/x^2)y = 0$. For large x, the equation can be approximated by $y'' + y = 0$, whose solutions are $\sin x$ and $\cos x$. Hence, $J_0(x)$ and $Y_0(x)$ appear to be similar to $\sin x$ and $\cos x$ for large values of x.

Exercises: Must try: Tut Sheet 2

1. Qn.1, Qn. 2 (done in the class)
2. Qn. 3 (i), (iii), (iv); Qn. 5 (i), (ii), (iii); Qn. 6, Qn. 7, Qn.8, Qn.9; Qn. 10 (Tut. Class); drop Qn.11.
3. Qn.12, Qn. 13; Qn 14 (done in the class)
4. Prove that, when $\boxed{r_1 - r_2 = 2p \neq integer}$, the second linearly independent solution can be obtained from Frobenius theory as

$$\boxed{J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(-p+n+1)} \left(\frac{x}{2}\right)^{2n-p}},$$

just by replacing p by $-p$ in the definition of $J_p(x)$.

5. When $p \geq 0$ IS AN INTEGER, prove that $\boxed{J_p(x) = (-1)^p J_{-p}(x)}$