Signal Processing - 1 by One

Sibi Raj B. Pillai
Dept of Electrical Engineering
IIT Bombay

Outline

- So Far: Sampling, Convolution, Interpolation
- Previous Week: Fourier Series
- Previous Class: Fourier Transform
- Today: Series and Transform

Outline

- So Far: Sampling, Convolution, Interpolation
- Previous Week: Fourier Series
- Previous Class: Fourier Transform
- Today: Series and Transform

Convolution-Multiplication Formula

For integrable functions $x(t)$ and $h(t)$, let $y(t) = x(t) \ast h(t)$.

$$Y(f) = \int_{\mathbb{R}} y(t) \exp(-j2\pi f t) dt$$

Convolution-Multiplication Formula

For integrable functions $x(t)$ and $h(t)$, let $y(t) = x(t) \ast h(t)$.

$$Y(f) = \int_{\mathbb{R}} y(t) \exp(-j2\pi f\, t) dt = \int_{\mathbb{R}} \big(x(t) \ast h(t)\big) \exp(-j2\pi f\, t) dt$$

Convolution-Multiplication Formula

For integrable functions $x(t)$ and $h(t)$, let $y(t) = x(t) \ast h(t)$.

$$
\begin{aligned}
Y(f) &= \int_{\mathbb{R}} y(t) \exp(-j2\pi f t) dt = \int_{\mathbb{R}} \big(x(t) \ast h(t)\big) \exp(-j2\pi f t) dt \\
&= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} h(\tau) x(t-\tau) d\tau \Big] \exp(-j2\pi f t)\, dt
\end{aligned}
$$

Convolution-Multiplication Formula

For integrable functions $x(t)$ and $h(t)$, let $y(t) = x(t) \ast h(t)$.

$$
\begin{aligned}
Y(f) &= \int_{\mathbb{R}} y(t) \exp(-j2\pi f t) dt \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} h(\tau) x(t-\tau) d\tau \right] \exp(-j2\pi f t)\, dt
\end{aligned}
$$

Convolution-Multiplication Formula

For integrable functions $x(t)$ and $h(t)$, let $y(t) = x(t) * h(t)$.

$$\begin{aligned}
Y(f) &= \int_{\mathbb{R}} y(t) \exp(-j2\pi f\, t) dt \\
&= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} h(\tau) x(t-\tau) d\tau \Big] \exp(-j2\pi f\, t)\, dt \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(t-\tau) \exp(-j2\pi f\, t) dt\, d\tau \quad \text{(Fubini's Theorem)}
\end{aligned}$$

Convolution-Multiplication Formula

For integrable functions $x(t)$ and $h(t)$, let $y(t) = x(t) \ast h(t)$.

$$
\begin{aligned}
Y(f) &= \int_{\mathbb{R}} y(t) \exp(-j2\pi f\, t) dt \\
&= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} h(\tau) x(t-\tau) d\tau \Big] \exp(-j2\pi f\, t)\, dt \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(t-\tau) \exp(-j2\pi f\, t) dt\, d\tau \quad \text{(Fubini's Theorem)} \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(t-\tau) \exp(-j2\pi f(t-\tau)) dt \exp(-j2\pi f \tau) d\tau
\end{aligned}
$$

Convolution-Multiplication Formula

For integrable functions $x(t)$ and $h(t)$, let $y(t) = x(t) \ast h(t)$.

$$
\begin{aligned}
Y(f) &= \int_{\mathbb{R}} y(t) \exp(-j2\pi f\, t) dt \\
&= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} h(\tau) x(t-\tau) d\tau \Big] \exp(-j2\pi f\, t)\, dt \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(t-\tau) \exp(-j2\pi f\, t) dt\, d\tau \quad \text{(Fubini's Theorem)} \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(t-\tau) \exp(-j2\pi f(t-\tau)) dt \exp(-j2\pi f \tau) d\tau \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(u) \exp(-j2\pi f\, u) du\ \exp(-j2\pi f \tau) d\tau \text{(Variable change)}
\end{aligned}
$$

Convolution-Multiplication Formula

For integrable functions $x(t)$ and $h(t)$, let $y(t) = x(t) \circledast h(t)$.

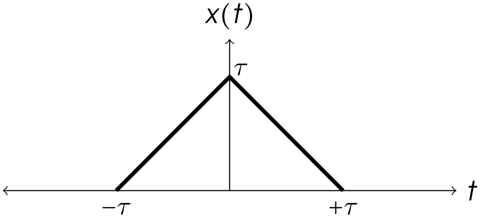
$$
\begin{aligned}
Y(f) &= \int_{\mathbb{R}} y(t) \exp(-j2\pi f\, t) dt \\
&= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} h(\tau) x(t-\tau) d\tau \Big] \exp(-j2\pi f\, t)\, dt \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(t-\tau) \exp(-j2\pi f\, t) dt\, d\tau \ \text{(Fubini's Theorem)} \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(t-\tau) \exp(-j2\pi f(t-\tau)) dt \exp(-j2\pi f\tau) d\tau \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(u) \exp(-j2\pi f\, u) du \ \exp(-j2\pi f\tau) d\tau \text{(Variable change)} \\
&= X(f) \int_{\mathbb{R}} h(\tau) \exp(-j2\pi f\tau) d\tau
\end{aligned}
$$

Convolution-Multiplication Formula

For integrable functions $x(t)$ and $h(t)$, let $y(t) = x(t) \circledast h(t)$.

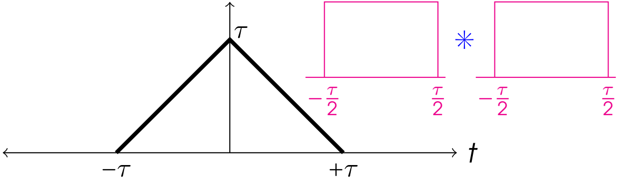
$$
\begin{aligned}
Y(f) &= \int_{\mathbb{R}} y(t) \exp(-j2\pi f\, t) dt \\
&= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} h(\tau) x(t-\tau) d\tau \Big] \exp(-j2\pi f\, t)\, dt \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(t-\tau) \exp(-j2\pi f\, t) dt\, d\tau \ \text{(Fubini's Theorem)} \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(t-\tau) \exp(-j2\pi f(t-\tau)) dt \exp(-j2\pi f\tau) d\tau \\
&= \int_{\mathbb{R}} h(\tau) \int_{\mathbb{R}} x(u) \exp(-j2\pi f\, u) du\ \exp(-j2\pi f\tau) d\tau \text{(Variable change)} \\
&= X(f) \int_{\mathbb{R}} h(\tau) \exp(-j2\pi f\tau) d\tau \\
&= X(f)\, H(f) \ \text{(Product of Fourier Transforms)}.
\end{aligned}
$$

Example-3:Triangles



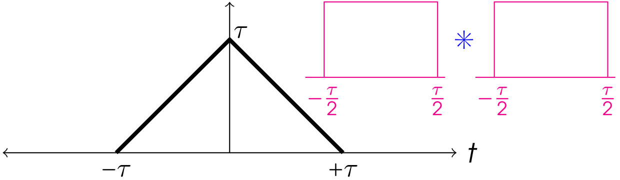
Example-3:Triangles

$$x(t) \;=\; rect_\tau(t) * rect_\tau(t)$$



Example-3:Triangles

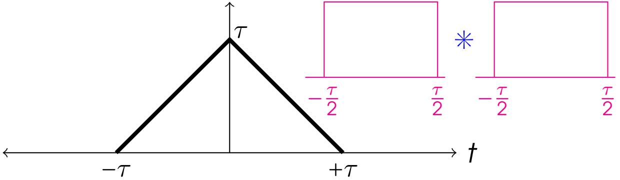
$$x(t) = rect_\tau(t) \ast rect_\tau(t)$$



$X(f) = \tau\,\mathrm{sinc}(f\tau)\ \tau\,\mathrm{sinc}(f\tau).$ "Convolution-Multiplication"

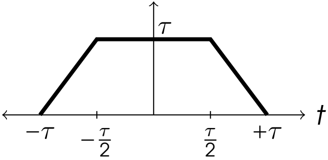
Example-3:Triangles

$$x(t) = rect_\tau(t) \ast rect_\tau(t)$$



$X(f) = \tau\, \mathrm{sinc}(f\tau)\ \tau\, \mathrm{sinc}(f\tau).$ "Convolution-Multiplication"

$u(t)$



$$u(t) = rect_{\frac{\tau}{2}}(t) \ast \left[2\, rect_{1.5\tau}(t)\right]$$

$$U(f) = \frac{\tau}{2} \mathrm{sinc}(f\frac{\tau}{2}) \times 3\tau\, \mathrm{sinc}(f\frac{3\tau}{2}).$$

Time Shifts

$$x(t) \xrightleftharpoons{F.T.} X(f)$$

$$x(t-\tau) \xrightleftharpoons{F.T.} X(f)\exp(-j2\pi f\tau)$$

$$x(t)+x(t-\tau) \xrightleftharpoons{F.T.} X(f)\big[1+\exp(-j2\pi f\tau)\big]$$

$$\sum_{n\in\mathbb{I}} x(t-nT) \xrightleftharpoons{F.T.} X(f)\Big[\sum_{n\in\mathbb{I}} \exp(-j2\pi fnT)\Big].$$

$$\sum_{n\in\mathbb{Z}} x(t-nT) \xrightleftharpoons{F.T.?} X(f)\Big[\sum_{n\in\mathbb{Z}} \exp(-j2\pi f nT)\Big].$$

$$x(t)*\sum_{n\in\mathbb{Z}} \delta(t-nT) \xrightleftharpoons{F.T.?} X(f)\,\left[\frac{1}{T}\Delta_{\frac{1}{T}}(f)\right] \text{ (Intuitively!)}$$

Time Shifts

$$x(t) \xrightleftharpoons{F.T.} X(f)$$

$$x(t-\tau) \xrightleftharpoons{F.T.} X(f)\exp(-j2\pi f\tau)$$

$$x(t) + x(t-\tau) \xrightleftharpoons{F.T.} X(f)\left[1+\exp(-j2\pi f\tau)\right]$$

$$\sum_{n\in\mathbb{I}} x(t-nT) \xrightleftharpoons{F.T.} X(f)\left[\sum_{n\in\mathbb{I}} \exp(-j2\pi fnT)\right].$$

$$\sum_{n\in\mathbb{Z}} x(t-nT) \xrightleftharpoons{F.T.?} X(f)\left[\sum_{n\in\mathbb{Z}} \exp(-j2\pi f\,nT)\right].$$

$$x(t) \ast \sum_{n\in\mathbb{Z}} \delta(t-nT) \xrightleftharpoons{F.T.?} X(f)\ \left[\frac{1}{T}\Delta_{\frac{1}{T}}(f)\right] \text{ (Intuitively!)}$$

Time Shifts

$$x(t) \overset{F.T.}{\rightleftharpoons} X(f)$$

$$x(t-\tau) \overset{F.T.}{\rightleftharpoons} X(f)\exp(-j2\pi f\tau)$$

$$x(t) + x(t-\tau) \overset{F.T.}{\rightleftharpoons} X(f)\big[1 + \exp(-j2\pi f\tau)\big]$$

$$\sum_{n\in\mathbb{I}} x(t-nT) \overset{F.T.}{\rightleftharpoons} X(f)\Big[\sum_{n\in\mathbb{I}} \exp(-j2\pi fnT)\Big].$$

$$\sum_{n\in\mathbb{Z}} x(t-nT) \overset{F.T.?}{\rightleftharpoons} X(f)\Big[\sum_{n\in\mathbb{Z}} \exp(-j2\pi f nT)\Big].$$

$$x(t) * \sum_{n\in\mathbb{Z}} \delta(t-nT) \overset{F.T.?}{\rightleftharpoons} X(f)\ \left[\frac{1}{T}\Delta_{\frac{1}{T}}(f)\right] \text{ (Intuitively!)}$$

Time Shifts

$$x(t) \xrightleftharpoons{F.T.} X(f)$$

$$x(t-\tau) \xrightleftharpoons{F.T.} X(f)\exp(-j2\pi f\tau)$$

$$x(t)+x(t-\tau) \xrightleftharpoons{F.T.} X(f)\big[1+\exp(-j2\pi f\tau)\big]$$

$$\sum_{n\in\mathbb{I}} x(t-nT) \xrightleftharpoons{F.T.} X(f)\Big[\sum_{n\in\mathbb{I}}\exp(-j2\pi fnT)\Big].$$

$$\sum_{n\in\mathbb{Z}} x(t-nT) \xrightleftharpoons{F.T.?} X(f)\Big[\sum_{n\in\mathbb{Z}}\exp(-j2\pi f\,nT)\Big].$$

$$x(t)\ast\sum_{n\in\mathbb{Z}}\delta(t-nT) \xrightleftharpoons{F.T.?} X(f)\ \Big[\frac{1}{T}\Delta_{\frac{1}{T}}(f)\Big]\ \text{(Intuitively!)}$$

Time Shifts

$$x(t) \xrightleftharpoons{F.T.} X(f)$$

$$x(t-\tau) \xrightleftharpoons{F.T.} X(f)\exp(-j2\pi f\tau)$$

$$x(t) + x(t-\tau) \xrightleftharpoons{F.T.} X(f)\big[1 + \exp(-j2\pi f\tau)\big]$$

$$\sum_{n\in\mathbb{I}} x(t-nT) \xrightleftharpoons{F.T.} X(f)\big[\sum_{n\in\mathbb{I}} \exp(-j2\pi fnT)\big].$$

$$\sum_{n\in\mathbb{Z}} x(t-nT) \xrightleftharpoons{F.T.?} X(f)\big[\sum_{n\in\mathbb{Z}} \exp(-j2\pi f\,nT)\big].$$

$$x(t) * \sum_{n\in\mathbb{Z}} \delta(t-nT) \xrightleftharpoons{F.T.?} X(f)\left[\frac{1}{T}\Delta_{\frac{1}{T}}(f)\right] \text{ (Intuitively!)}$$

Time Shifts

Fourier Series Sum for a signal of period $\frac{1}{T}$, and all FS coefficients 1

$$x(t) \xrightleftharpoons{F.T.} X(f)$$

$$x(t-\tau) \xrightleftharpoons{F.T.} X(f)\exp(-j2\pi f\tau)$$

$$x(t) + x(t-\tau) \xrightleftharpoons{F.T.} X(f)\left[1 + \exp(-j2\pi f\tau)\right]$$

$$\sum_{n\in\mathbb{I}} x(t-nT) \xrightleftharpoons{F.T.} X(f)\left[\sum_{n\in\mathbb{I}} \exp(-j2\pi fnT)\right].$$

$$\sum_{n\in\mathbb{Z}} x(t-nT) \xrightleftharpoons{F.T.?} X(f)\left[\sum_{n\in\mathbb{Z}} \exp(-j2\pi f\,nT)\right].$$

$$x(t) * \sum_{n\in\mathbb{Z}} \delta(t-nT) \xrightleftharpoons{F.T.?} X(f)\left[\frac{1}{T}\Delta_{\frac{1}{T}}(f)\right] \text{ (Intuitively!)}$$

Time Shifts

Fourier Series Sum for a signal of period $\frac{1}{T}$, and all FS coefficients 1

$$x(t) \xrightleftharpoons{F.T.} X(f)$$

$$x(t-\tau) \xrightleftharpoons{F.T.} X(f)\exp(-j2\pi f\tau)$$

$$x(t) + x(t-\tau) \xrightleftharpoons{F.T.} X(f)\left[1 + \exp(-j2\pi f\tau)\right]$$

$$\sum_{n\in\mathbb{I}} x(t-nT) \xrightleftharpoons{F.T.} X(f)\Big[\sum_{n\in\mathbb{I}} \exp(-j2\pi fnT)\Big].$$

$$\sum_{n\in\mathbb{Z}} x(t-nT) \xrightleftharpoons{F.T.?} X(f)\Big[\sum_{n\in\mathbb{Z}} \exp(-j2\pi f\,nT)\Big].$$

$$x(t) * \sum_{n\in\mathbb{Z}} \delta(t-nT) \xrightleftharpoons{F.T.?} X(f)\ \left[\frac{1}{T}\Delta_{\frac{1}{T}}(f)\right]\ \text{(Intuitively!)}$$

Added Slide

This slide was added after the class to explain the scaling by $\frac{1}{T}$.

Recall that for a $T-$ periodic signal $x(t)$:

$$x(u) = \sum_{m \in \mathbb{Z}} \alpha_m \exp(-j\frac{2\pi}{T}mu) \tag{10a}$$

For a T'-periodic impulse train $\Delta_{T'}(u) = \sum_n \delta(u - nT')$, the Fourier Series coefficients are

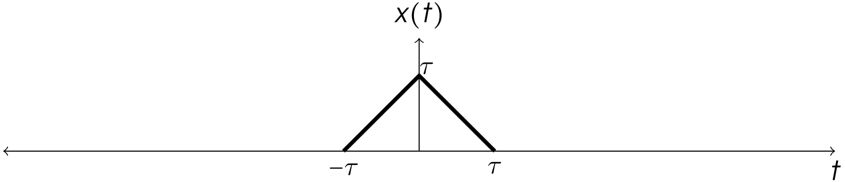
$$\alpha_m = \frac{1}{T'} \int_{-\frac{T'}{2}}^{\frac{T'}{2}} \delta(u) \exp(-j\frac{2\pi}{T'}mu)\, du = \frac{1}{T'}.$$

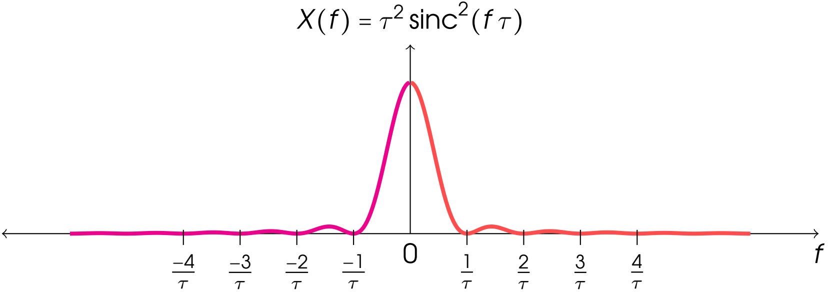
Intuitively, by (10a) and our bravery, makes sense to take

$$\Delta_{T'}(u) = \sum_{n \in \mathbb{Z}} \frac{1}{T'} \exp(-j\frac{2\pi}{T'}nu).$$

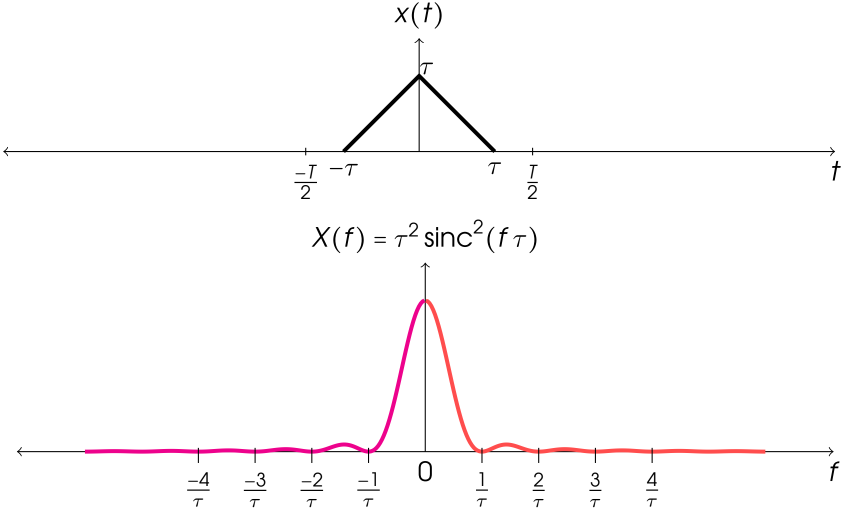
Taking $u = f$ and $T' = \frac{1}{T}$ will justify the idea used in the last slide.

Example 4: Periodic Triangle



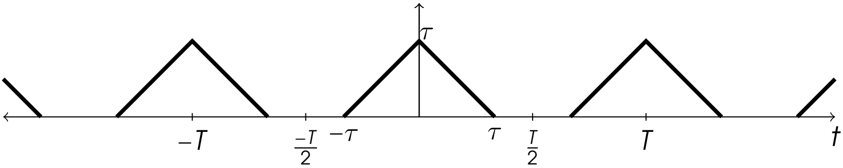


Example 4: Periodic Triangle

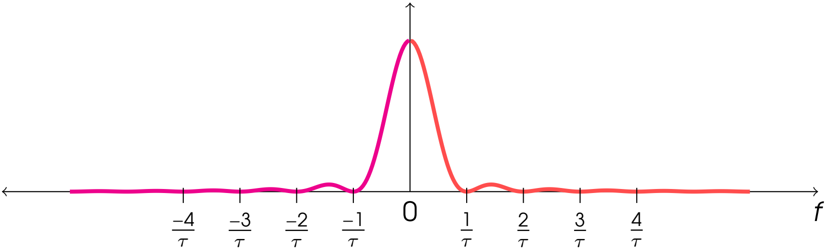


Example 4: Periodic Triangle

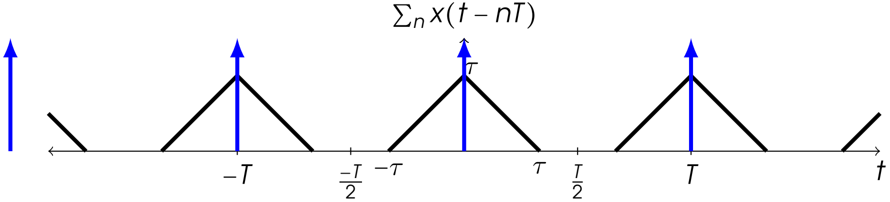
$$\sum_n x(t-nT)$$

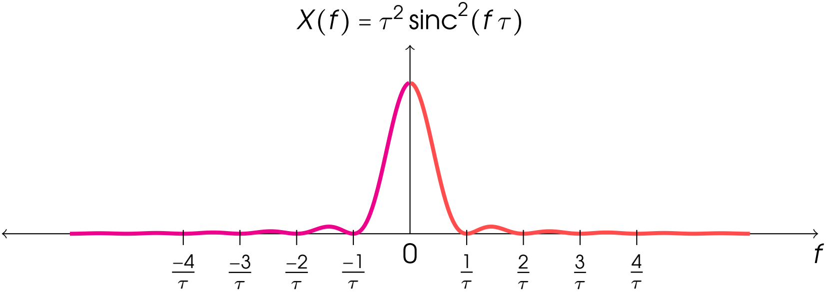


$$X(f) = \tau^2\,\mathrm{sinc}^2(f\tau)$$

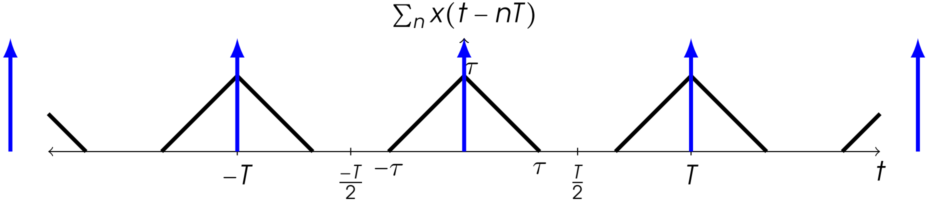


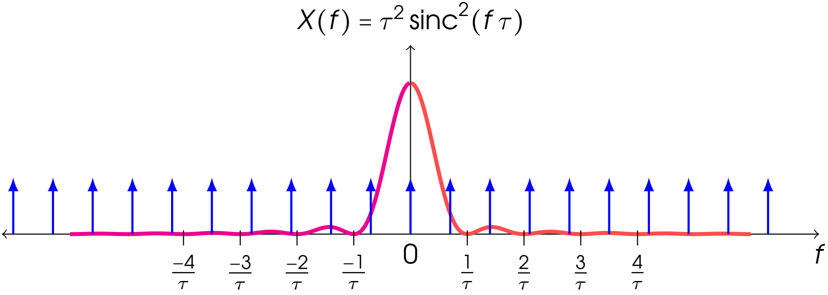
Example 4: Periodic Triangle



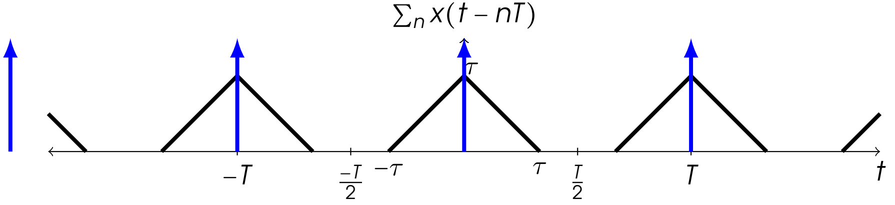


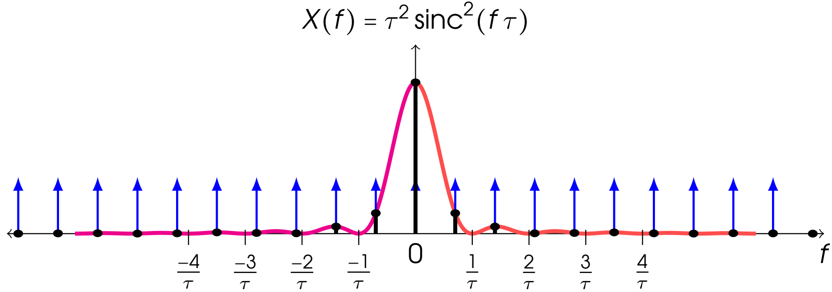
Example 4: Periodic Triangle



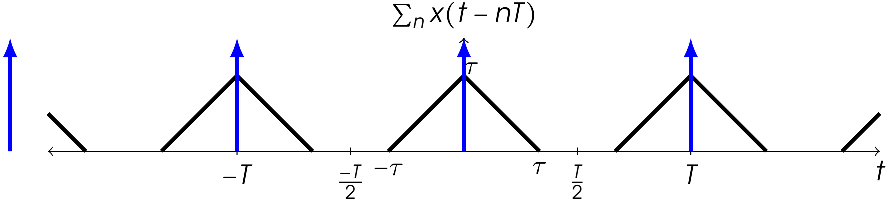


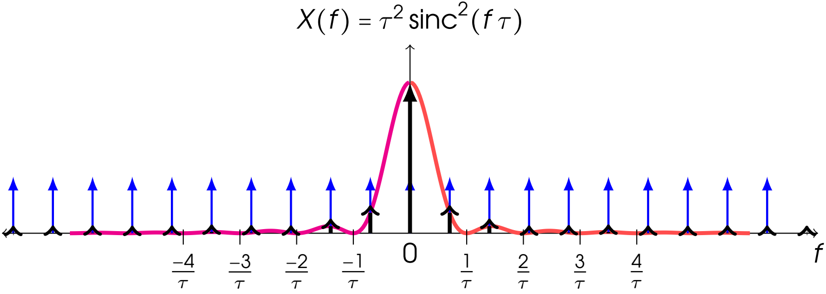
Example 4: Periodic Triangle



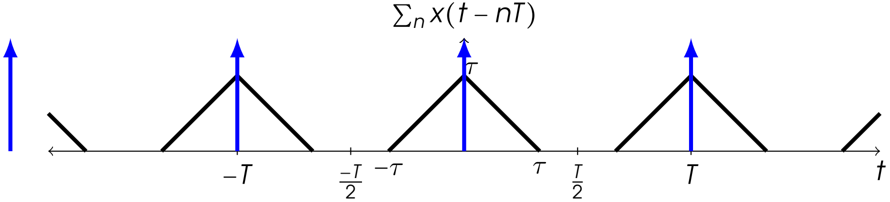


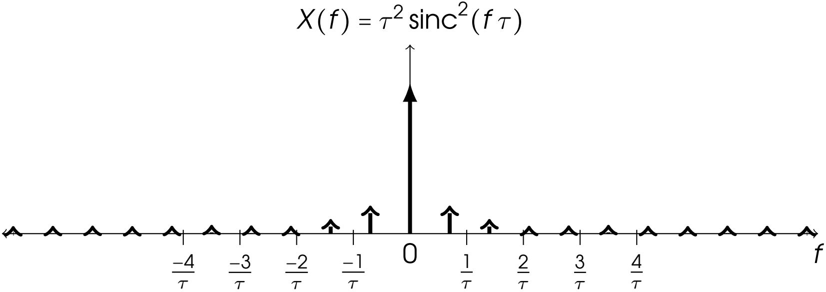
Example 4: Periodic Triangle





Example 4: Periodic Triangle





Poisson Summation Formula

Theorem For an integrable $x(t)$ with Fourier Transform $X(f)$:

$$\sum_{n\in\mathbb{Z}} x(t-nT) = \sum_{m\in\mathbb{Z}} \alpha_m \exp\left(\frac{2\pi}{T}mt\right),$$

at points of continuity of the LHS , where

$$\alpha_m = \frac{1}{T}X(\frac{m}{T}), \ \forall m \in \mathbb{Z}.$$

Proof *(using Uniqueness of FS expansion)*

Poisson Summation Formula

Theorem For an integrable $x(t)$ with Fourier Transform $X(f)$:

$$\sum_{n \in \mathbb{Z}} x(t - nT) = \sum_{m \in \mathbb{Z}} \alpha_m \exp\left(\frac{2\pi}{T} mt\right),$$

at points of continuity of the LHS , where

$$\alpha_m = \frac{1}{T} X(\frac{m}{T}), \ \forall m \in \mathbb{Z}.$$

Proof *(using Uniqueness of FS expansion)*

Poisson Summation Formula

Theorem For an integrable $x(t)$ with Fourier Transform $X(f)$:

$$\sum_{n \in \mathbb{Z}} x(t - nT) = \sum_{m \in \mathbb{Z}} \alpha_m \exp\left(\frac{2\pi}{T} mt\right),$$

at points of continuity of the LHS , where

$$\alpha_m = \frac{1}{T} X\left(\frac{m}{T}\right), \ \forall m \in \mathbb{Z}.$$

Proof *(using Uniqueness of FS expansion)*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_n x(t - nT) \exp\left(-j\frac{2\pi}{T} mt\right) dt$$

Poisson Summation Formula

Theorem For an integrable $x(t)$ with Fourier Transform $X(f)$:

$$\sum_{n\in\mathbb{Z}} x(t-nT) = \sum_{m\in\mathbb{Z}} \alpha_m \exp\left(\frac{2\pi}{T}mt\right),$$

at points of continuity of the LHS , where

$$\alpha_m = \frac{1}{T}X(\frac{m}{T}),\ \forall m\in\mathbb{Z}.$$

Proof *(using Uniqueness of FS expansion)*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_n x(t-nT)\exp(-j\frac{2\pi}{T}mt)dt$$

$$= \sum_n \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t-nT)\exp(-j\frac{2\pi}{T}mt)dt \qquad \text{(integrability of function)}$$

Poisson Summation Formula

Theorem For an integrable $x(t)$ with Fourier Transform $X(f)$:

$$\sum_{n\in\mathbb{Z}} x(t-nT) = \sum_{m\in\mathbb{Z}} \alpha_m \exp\left(\frac{2\pi}{T}mt\right),$$

at points of continuity of the LHS , where

$$\alpha_m = \frac{1}{T}X(\frac{m}{T}),\ \forall m \in \mathbb{Z}.$$

Proof *(using Uniqueness of FS expansion)*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_n x(t-nT)\exp(-j\frac{2\pi}{T}mt)dt$$

$$= \sum_n \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t-nT)\exp(-j\frac{2\pi}{T}mt)dt \qquad \text{(integrability of function)}$$

$$= \sum_n \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t-nT)\exp(-j\frac{2\pi}{T}m(t-nT))dt \exp(-j\frac{2\pi}{T}mnT)$$

Poisson Summation Formula

Theorem For an integrable $x(t)$ with Fourier Transform $X(f)$:

$$\sum_{n\in\mathbb{Z}} x(t-nT) = \sum_{m\in\mathbb{Z}} \alpha_m \exp\left(\frac{2\pi}{T}mt\right),$$

at points of continuity of the LHS , where

$$\alpha_m = \frac{1}{T}X(\frac{m}{T}), \ \forall m \in \mathbb{Z}.$$

Proof *(using Uniqueness of FS expansion)*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_n x(t-nT)\exp(-j\frac{2\pi}{T}mt)dt$$

$$= \sum_n \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t-nT)\exp(-j\frac{2\pi}{T}mt)dt \qquad \text{(integrability of function)}$$

$$= \sum_n \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t-nT)\exp(-j\frac{2\pi}{T}m(t-nT))dt\exp(-j\frac{2\pi}{T}mnT)$$

$$= \int_{\mathbb{R}} x(u)\exp(-j2\pi\frac{m}{T}u)du$$

Poisson Summation Formula

Theorem For an integrable $x(t)$ with Fourier Transform $X(f)$:

$$\sum_{n\in\mathbb{Z}} x(t-nT) = \sum_{m\in\mathbb{Z}} \alpha_m \exp\left(\frac{2\pi}{T}mt\right),$$

at points of continuity of the LHS , where

$$\alpha_m = \frac{1}{T}X(\frac{m}{T}), \ \forall m \in \mathbb{Z}.$$

Proof *(using Uniqueness of FS expansion)*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_n x(t-nT)\exp(-j\frac{2\pi}{T}mt)\,dt$$

$$= \sum_n \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t-nT)\exp(-j\frac{2\pi}{T}mt)\,dt \qquad \text{(integrability of function)}$$

$$= \sum_n \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t-nT)\exp(-j\frac{2\pi}{T}m(t-nT))\,dt\,\exp(-j\frac{2\pi}{T}mnT)$$

$$= \int_{\mathbb{R}} x(u)\exp(-j2\pi\frac{m}{T}u)\,du = X(\frac{m}{T}).$$