EE 621 **Spring 2021**

Week 2 problem set: DTMCs

1 DTMC definition

Consider the DTMC $\{X_n\}_{n\geq 0}$ over countable state space S. Prove that for any $i_5, i_3, i_1 \in S$,

$$P\left(X_5 = i_5 \mid X_3 = i_3, X_1 = i_1\right) = P\left(X_5 = i_5 \mid X_3 = i_3\right).$$

Note: From the above proof, you should be able to convince yourself that the following more general result holds. Given $0 \leq n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots < n_{k+m}$, *and* $i_1, i_2, \cdots, i_{k+m} \in S$, *we have*

$$\begin{aligned} &P\left(X_{n_{k+1}} = i_{k+1}, \cdots, X_{n_{k+m}} = i_{k+m} \mid X_{n_1} = i_1, \cdots, X_{n_k} = i_k\right) \\ &= P\left(X_{n_{k+1}} = i_{k+1}, \cdots, X_{n_{k+m}} = i_{k+m} \mid X_{n_k} = i_k\right). \end{aligned}$$

2 Reviewing the basics

Consider DTMCs with the following transition probability matrices. In each case, check if the DTMC is irreducible. If irreducible, also check if the chain is aperiodic.

1. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3 Doubly stochastic transition matrix

A doubly stochastic matrix is one in which each row and column sum equals 1. Consider a finite, irreducible, aperiodic DTMC with a doubly stochastic probability transition matrix. What can you say about the stationary distribution of this DTMC?

4 Markovian rains[1]

Suppose the probability of rain today is 0.6 if it rained yesterday, but only 0.2 if it did not, where these probabilities do not depend on the weather on previous days.

1. Given rain fell today, what is the probability that it will rain day-after tomorrow?
2. What is the average duration (number of days) of a rainy period?
3. What is the limiting probability of a rainy day?

5 Markovian coin-flips

You have two biased coins. Coin 1, when flipped comes up heads with probability 0.6, and Coin 2, when flipped comes up heads with probability 0.5.

A coin is flipped repeatedly until it comes up tails, at which point we set that coin aside, and start flipping the other coin. This process is repeated indefinitely.

Let $p(n)$ denote the probability that the nth flip comes up heads. What is the limiting value of $p(n)$ as $n \to \infty$?

[1] This is Problem 3.13 in "Stochastic Modeling and the Theory of Queues" by R. W. Wolff