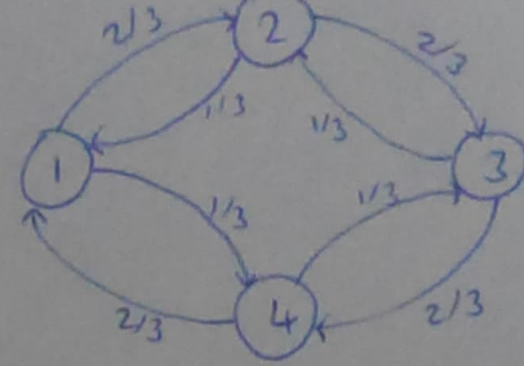
1) (i) $\hat{P} = \begin{bmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix}$

a)



b) $\pi = \pi \hat{P}$

$\therefore \pi_1 = \frac{2}{3}\pi_2 + \frac{\pi_4}{3}$

$\pi_3 = \frac{1}{3}\pi_2 + \frac{2\pi_4}{3}$

$\pi_2 = \frac{1}{3}\pi_1 + \frac{2}{3}\pi_3$

$\pi_4 = \frac{2}{3}\pi_1 + \frac{1}{3}\pi_4$

As is easily visible, $\pi_1 = \pi_2 = \pi_3 = \pi_4$ satisfies the equations

Also $\sum \pi_i = 1$

$\therefore \pi = [\, 1/4 \quad 1/4 \quad 1/4 \quad 1/4 \,]$

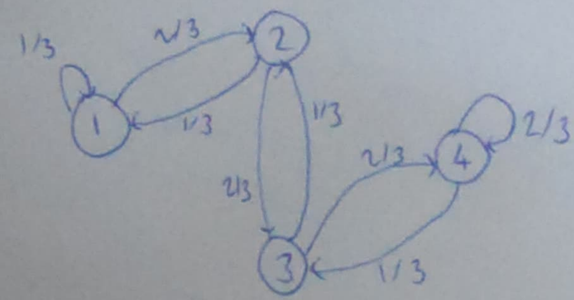
c) $\pi_1 \cdot p_{12} = \frac{1}{4} \cdot \frac{2}{3}$

$\pi_2 \cdot p_{21} = \frac{1}{4} \cdot \frac{1}{3}$

Since these are not equal, this is not a time reversible change.

1) (ii) $\tilde{P} = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix}$

a)



b) $\pi = \pi \tilde{P}$

$\therefore \pi_1 = \frac{\pi_1}{3} + \frac{2\pi_2}{3} \Rightarrow \cancel{\pi_1 = \pi_2}$

$\pi_2 = \frac{2\pi_1}{3} + \frac{\pi_3}{3}$

$\pi_3 = \frac{2\pi_2}{3} + \frac{\pi_4}{3}$

$\pi_4 = \frac{2\pi_3}{3} + \frac{2\pi_4}{3}$

$\therefore \pi_2 = 2\pi_1$

$\therefore \pi_3 = 4\pi_1$

$\therefore 4\pi_1 = \frac{4\pi_1}{3} + \frac{\pi_4}{3}$

$\therefore \pi_4 = 8\pi_1$

$\therefore \pi_1 + 2\pi_1 + 4\pi_1 + 8\pi_1 = 1$

$\therefore \pi_1 = 1/15$

$\therefore \pi = \begin{bmatrix} 1/15 & 2/15 & 4/15 & 8/15 \end{bmatrix}$

c) we only have to check for:

(1,2) (2,3) (3,4)

for each of these $x_i = x_j/2$

$p_{ij} = p_{ji} \times 2$

$\therefore p_{ij} \cdot x_i = p_{ji} \cdot x_j$

$\therefore$ this chain is time reversible.

2) (i) Using finiteness and theorem 1.4, we have that the stationary distribution is unique

Claim: $\pi = \left[\frac{d_i}{\sum d_i}\right]$ is a stationary matrix

$\rightarrow$ Say i is connected to $a_1, a_2 \cdots a_{d_i}$ only

$\therefore$ The only connections to i are from $a_1, a_2 \cdots a_{d_i}$

$\therefore$ i^{th} column of P looks like $\begin{bmatrix} 0 \\ 1/d_{a_1} \\ \vdots \\ 0 \\ 1/d_{a_{d_i}} \end{bmatrix}$

$$\therefore [\pi P]_i = \sum_j \pi_j p_{ji}$$

$$= \sum_{j=1}^{d_i} \left(\frac{1}{d_{a_j}} \cdot \frac{d_{a_j}}{\sum d_i}\right)$$

$$= \sum_1^{d_i} \frac{1}{\sum d_i} = \frac{d_i}{\sum d_i} = \pi_i$$

Hence proved

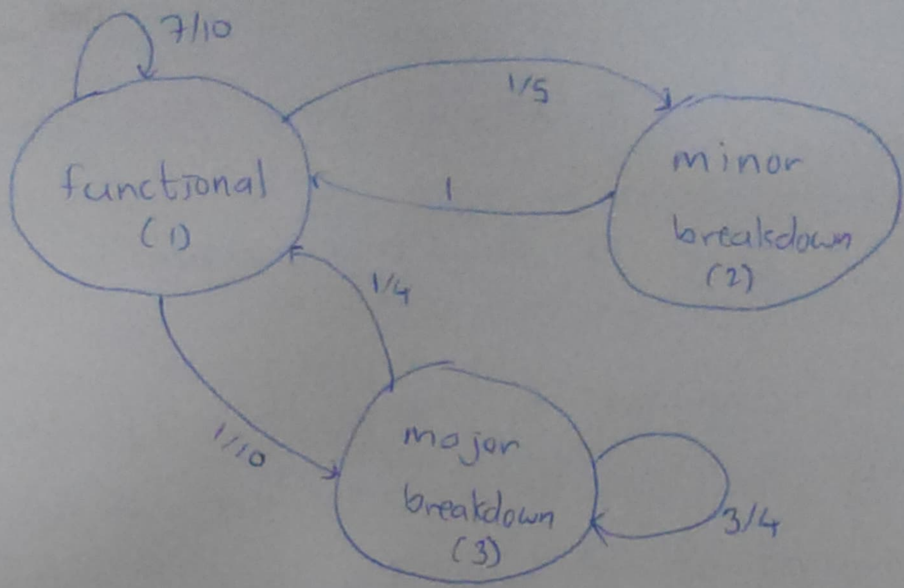
(ii) $\pi_i p_{ij} = \pi_j \cdot p_{ji}$ is true if $p_{ij} = 0 \Rightarrow p_{ji} = 0$

else: $\text{lhs} = \frac{d_i}{\sum d_i} \cdot \frac{1}{d_i} = \frac{1}{\sum d} = \frac{d_j}{\sum d_j} \cdot \frac{1}{d_j}$

$$= \pi_j \cdot p_{ji} = \text{rhs}$$

Hence the dtmc is reversible.

3) (i)



each step is one day

(ii) The dtmc is irreducible, since each state is reachable ~~to~~ from every other state.
Being irreducible and having a self-loop ⇒ it is aperiodic.

(iii) From thrm-4, ℓ exists.
$\therefore$ ℓ is a unique stationary distribution.

$\therefore \ \ell = \ell P$

$\therefore \ \ell_1 = \frac{7\ell_1}{10} + \ell_2 + \frac{\ell_3}{4}$

$\ell_2 = \frac{\ell_1}{5}$

$\ell_3 = \frac{3\ell_3}{4} + \frac{\ell_1}{10}$

$\therefore \ \ell_2 = \frac{\ell_1}{5}, \quad \ell_3 = \frac{2\ell_1}{5}$

$\therefore \ \ell_1 \left(1 + \frac{1}{5} + \frac{2}{5}\right) = 1$

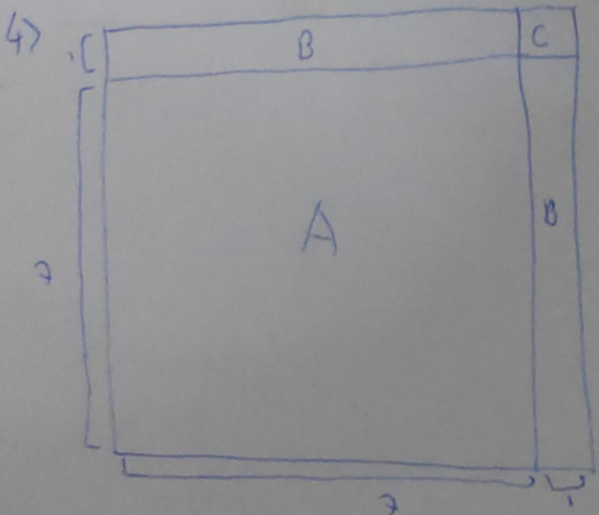
$\therefore \ \ell_1 = 5/8$

$\therefore \ \ell_2 = 1/8$

$\therefore \ \ell_3 = 2/8$

$\therefore \ \ell = [\ 5/8 \quad 1/8 \quad 2/8\ ]$

4)

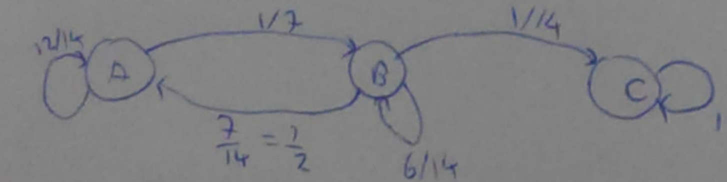


probability of moving from any box in A to any box in B is the same.

Similarly from B to C.

We want E/Var of first arrival in C, so setting $p_{cc} = 1$ won't affect our result.

$\therefore$



~~$P(S_n = c) = P(S_{n-1} = b) \cdot p_{cb}$~~

E_a := expected number of moves from a to c

E_b := " " " " " b " c

$$\therefore \quad E_a = \underbrace{\frac{1}{7}(E_b + 1)}_{\text{[going from } a \to b]} + \underbrace{\frac{6}{7}(E_a + 1)}_{\text{(staying at } a)}$$

$$E_b = \frac{1}{14} \cdot 1 + \frac{3}{7} \cdot (E_b + 1) + \frac{1}{2}(E_a + 1)$$

Solving for E_a : $E_a = 70$.

$\therefore$ ans $= 70$.

5) • The king can do the following routes:

$\therefore$ $\gcd(2,3)=1$

$\therefore$ aperiodic.

• From q2 :

lim. probability $= \dfrac{d_i}{\sum d_i}$

for a corner : $\dfrac{2}{2\times4+3\times24+4\times36}$

$\therefore$ ans $= \dfrac{2\times4}{8+72+144} = \dfrac{1}{1+9+18} = \dfrac{1}{28}$

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5) The following two routes are possible:

, $\gcd(3,4)=1$. $\therefore$ the dtmc is aperiodic

From q2 : limiting probability $= \dfrac{d_i}{\sum d_i}$

$\therefore$ ans $= 4 \times \dfrac{3}{3\times4+5\times24+8\times36} = \dfrac{1}{1+10+24} = \dfrac{1}{35}$

$\therefore$ ans $= 1/35$

6) Since the DTMC is irreducible and aperiodic and finite, from theorem-4, $\exists$ a limiting distribution.

- Here each state of the DTMC is a permutation. (i.e. $n!$ states)

Now from theorem-1, this limiting distribution is a unique stationary distribution.

Hence, if we prove that the uniform distribution is stationary, then we are done.

- Irreducability : say we want to go from $\{a_1, a_2 \cdots a_n\}$ to $\{b_1 \cdots b_n\}$ to do so, we simply have to ~~to~~ pick b_n, then b_{n-1} and so on till b_1

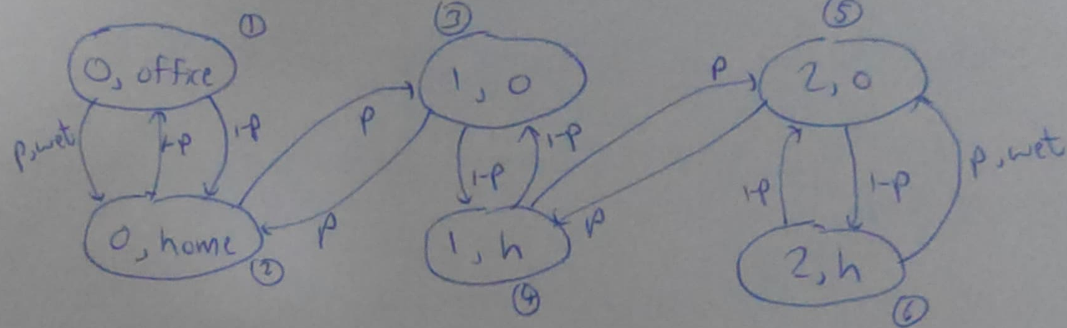
- Aperiodicity:

$$\begin{tikzcd}
a_1, a_2, a_3, \cdots a_n \arrow[r, "(\text{pick } a_2)"] \arrow[d, "(a_3)"] & a_2, a_1, a_3, \cdots a_n \arrow[dd, "(\text{pick } a_1)"] \\
a_3, a_2, a_1, \cdots a_n \arrow[d, "(a_2)"] & \\
a_2, a_3, a_1, \cdots a_n \arrow[r, "(a_1)"] & a_1, a_2, a_3 \cdots a_n
\end{tikzcd}$$

Hence a full loop (for any state) can be done in 2/3 steps. $\Rightarrow$ gcd $= 1$ $\Rightarrow$ aperiodicity.

- To reach to state $\{a_1, a_2, \cdots a_n\}$, a_1 must be picked in the end. Hence, a_1 can perevously be in any 1 of n positions. Hence, the previous state can only be one of these n. Further, the transition probability of every transition like this is $1/n$.
$\therefore$ P is doubly stochastic.
$\therefore$ uniform dist. is a valid stationary dist.

7) 1) the state is the number of umbrellas at the office, and the current location



$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 & 0 \\ 0 & p & 0 & 1-p & 0 & 0 \\ 0 & 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore\ \pi_1 = (1-p)\pi_2$

$\pi_2 = \pi_1 + p\pi_3 \Rightarrow \pi_2 = (1-p)\pi_2 + p\pi_3 \Rightarrow \pi_2 = \pi_3$

$\pi_3 = p\pi_2 + (1-p)\pi_4 \Rightarrow \pi_3 = \pi_4$

$\pi_4 = (1-p)\pi_3 + p\pi_5 \Rightarrow \pi_4 = \pi_5$

$\pi_5 = p\pi_4 + \pi_6 \rightarrow \pi_6 = (1-p)\pi_5$

$\therefore\ \pi = [\pi_1\ \ \pi_2\ \ \pi_2\ \ \pi_2\ \ \pi_2\ \ \pi_1]$

$= [(1-p)\pi_2\ \ \pi_2\ \cdots\ \pi_2\ \ (1-p)\pi_2]$

$\therefore\ \pi_2(4 + 1-p + 1-p) = 1$

$\therefore\ \pi_2 = \dfrac{1}{6-2p}$ $\qquad \therefore\ P(wet) = (\pi_1 + \pi_6)p = \left(\dfrac{2-2p}{6-2p}\right)p$