EE 621 **Spring 2021**

Homework 1

Assigned: 25/01/21 Due: 11.59 pm, 01/02/21 (on BodhiTree)

1 Practice with stationary equations[1] [20 marks]

Consider the DTMCs defined by the following transition probability matrices.

$$\hat{P} = \begin{bmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

(a) Draw the corresponding transition probability diagrams.

(b) Compute the stationary distribution for each chain.

(c) Which DTMC, if any, is time-reversible?

2 Random walk on an undirected graph [10 marks]

Consider a finite, connected, undirected graph (V, E), where V is the set of nodes, and E is the set of edges. Let d_i denote the degree of node $i \in V$.

Now, consider a particle making a random walk over the nodes of this graph, where at each time step, the particle moves from the previous node to one of its neighbors, chosen uniformly at random. Note that the position of the particle evolves as a DTMC over V.

(a) What is the stationary distribution corresponding to the above DTMC?

(b) Is the DTMC reversible?

3 Phat-phati [10 marks]

You own an old, khatara car. On any day, the car has a major breakdown with probability 1/10, and a minor breakdown with probability 1/5. If the car has a minor breakdown, it will get repaired with probability 1 the following day. If the car has had a major breakdown, then the probability of it getting repaired after each passing day is 1/4.

1. Define a DTMC to describe the car's state, draw the corresponding transition probability diagram.

[1] This is Problem 9.2 in Mor Harchol-Balter's book.

2. Is the DTMC irreducible? Aperiodic?
3. What is the limiting probability that your car is running on any day?

4 Randomised chess[2] [20 marks]

In chess, a rook can either move horizontally within its row (left or right), or vertically within its column (up or down) any number of squares. In the 8×8 chess board, imagine a rook starting at the lower left corner of the board. At each move, a bored child decides to move the rook to a random, uniformly chosen, legal location. Let T denote the time until the rook first lands in the upper right corner of the board. Compute $\mathbb{E}[T]$.

Extra credit: Compute $variance(T)$. (10 marks)

5 Randomised chess revisited[3] [15 marks]

This time, consider a lone king starting on the bottom left corner square of an $8{\times}8$ chess board. Recall that a king can move one square in any direction (vertically, horizontally, or diagonally). On each move, our (still) bored child moves the king to a random, uniformly chosen, legal location.

(a) Is the DTMC describing the king's position aperiodic?

(b) Compute the limiting probability that the king is in either of the four corner squares. (Hint: Problem 2.)

6 Card shuffling [10 marks]

Consider the following approach to shuffling a deck of n cards. Starting with any initial ordering of the cards, we pick a number in $\{1, 2, \cdots, n\}$ uniformly at random. If the number i is selected, we take the card in position i of the deck and put it at the top of the deck, i.e., we put that card in position 1. We then repeatedly perform the same operation. Prove that in the limit (as the number of shuffles goes to infinity), any card is equally likely to be in any of the positions.

7 Umbrella management [20 marks]

Prof. X lives on campus. He walks from his house to his office every morning and back every evening. Being absent-minded, Prof. X does not check the weather forecast to decide if he should carry an umbrella on any day. Instead, he has the following strategy.

Prof. X has a total of m umbrellas, split between his office and his house. At the start of each commute, if it is raining, and if he has an umbrella at the current location, Prof. X uses it. The umbrella then remains at the desination until its next use. Of course, if there isn't an umbrella in the current location, Prof. X gets wet. Assume that at the time of each commute, it is raining independently with probability p.

[2] This is Problem 8.5 in Mor Harchol-Balter's book
[3] This is part of Problem 9.9 in Mor Harchol-Balter's book.

1. Assume $m = 2$. Let $p(n)$ denote that the probability that Prof. X gets wet on the nth commute. Define a DTMC to calculate the limiting value of $p(n)$.

2. Repeat the above calculation for $m = 3$.

Extra credit: (20 marks) Obtain the limiting value of $p(n)$ for general m. Prof. X wants to figure out the least number of umbrellas he needs to keep so that the limiting probability of getting wet ≤ 0.05? Compute this number for $p = 0.9, 0.5$, and 0.1. Do you observe anything interesting?