1) We have $P(x > t+s \mid x > s) = P(x > t) \quad \forall\, s,t \geq 0$

$\therefore\ \bar{F}_x(t+s) = \bar{F}_x(t) \cdot \bar{F}_x(s) \quad \forall\, s,t \geq 0$

$\therefore\ \bar{F}_x(t+s) - \bar{F}_x(t) = \bar{F}_x(t) \cdot (\bar{F}_x(s) - 1)$

$$\therefore\ \frac{\bar{F}_x(t+s) - \bar{F}_x(t)}{s} = \bar{F}_x(t) \cdot \frac{\bar{F}_x(s) - \bar{F}_x(0)}{s}$$

[$\because \bar{F}_x(0) = 1$ as $X \in R_+$]

Now let $s \to 0$.

$\therefore\ \bar{F}_x'(t) = -\lambda \cdot \bar{F}_x(t)$, with $-\lambda = \bar{F}_x'(0)$. ($\lambda \geq 0$)

$$\therefore\ \int_0^x \frac{\bar{F}_x'(t)}{\bar{F}_x(t)}\, dt = -\int_0^x \lambda \cancel{t}\, dt$$

$\therefore\ \ln(\bar{F}_x(x)) - \ln(\bar{F}_x(0)) = -\lambda x - 0$

But $\bar{F}_x(0) = 1$.

$\therefore\ \bar{F}_x(x) = e^{-\lambda x}$

Now since the CCDF uniquely identifies a distribution of X, we have $X \sim \exp(\lambda)$.

2) Since X is memoryless, the residual life $(X-t)$ is independent of age (t).

$\therefore$ E(residual life) $= 1/\lambda$.

$$\therefore\ E(X \mid X > t) = E(X - t \mid X > t) + E(t \mid X > t) = \underline{(1/\lambda) + t}$$

$$E(X^2 \mid X > t) = E((X-t)^2 \mid X > t) - E(t^2 \mid X > t) + 2E(Xt \mid X > t)$$

$$= 2/\lambda^2 - t^2 + 2t(1/\lambda + t) = 2/\lambda^2 + t^2 + 2t/\lambda$$

3) $D \sim \exp(\lambda)$
$X = \max(D-d, 0)$.

$\therefore P(X \geq x) = P(D \geq d+x) \quad \forall x \geq 0$
$= e^{-\lambda(d+x)}$
$= e^{-\lambda d} \cdot e^{-\lambda x}$
$= c \cdot e^{-\lambda x} \quad ; \quad c := e^{-\lambda d}$.

$\therefore E(X) = c \cdot \frac{1}{\lambda} = \frac{e^{-\lambda d}}{\lambda}$

$$\left[\because E(X) = \int_0^\infty \bar{F}_X(x)dx \right]$$

4) We know that ~~min (X, Y)~~ $X \sim \exp(\lambda_1)$, $Y \sim \exp(\lambda_2)$

$\therefore \min(X, Y) \sim \exp(\lambda_1 + \lambda_2)$ (property seen in class)

$\therefore$ expected waiting time $= \frac{1}{\lambda_1 + \lambda_2}$

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expected serving time $= P(X \text{ finished first}) \cdot E(X)$
$+ P(y \text{ finished first}) \cdot E(Y)$
$= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{1}{\lambda_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \frac{1}{\lambda_2}$
(from property seen in class)

$\therefore E(T) = \frac{3}{\lambda_1 + \lambda_2}$

5) Let there be n people that get on the bus.
$\therefore$ their arrivals are iid uniform $(0, t)$
$\therefore$ wait times $\sim t - U(0,t) \sim U(0,t)$ [iid]

Let T_i be ith persons wait time.
$\therefore \{T_i\} \sim U(0,t)$ and iid.

Now $X = \sum_{i=1}^{n} T_i$, $n \sim \text{poisson}(\lambda t)$.

We have seen that n and $\{T_i\}$ are independent.
$\therefore$ From wald's theorem:

$$E(X) = E(n) \cdot E(T_i) = \lambda t \cdot \frac{t}{2} = \frac{\lambda t^2}{2}$$

Similarly, $X^2 = \left(\sum_i T_i\right)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} T_i \cdot T_j$

~~Consider $E(T_i \sum_{j=1} T_j) = T_i \cdot E(\sum_{j=1}^{n} T_j) = T_i E(n) \cdot E(T_j)$~~

$$\begin{aligned}
\therefore E(X^2) &= E\left(\sum\sum T_i \cdot T_j\right) = \sum E\left(\sum_{1}^{n} T_i T_j\right) \\
&= \sum \left(E(n) \cdot E(T_i T_j)\right) \\
&= E(n)^2 \cdot E(T_i \cdot T_j) \qquad \Big\} \text{wald's} \\
&= \lambda^2 t^2 \cdot \frac{t^2}{3} = \frac{\lambda^2 t^4}{3}
\end{aligned}$$

$$\begin{aligned}
\therefore \operatorname{Var}(X) &= E(X^2) - E(X)^2 \\
&= \frac{\lambda^2 t^4}{12}
\end{aligned}$$

6) $P(A_x = n) = \int_{x=0}^{\infty} P(n \text{ arrivals in } x \text{ interval}) \cdot P(X=x)\,dx$

$$= \int_0^{\infty} \frac{e^{-\lambda x} \cdot (\lambda x)^n}{n!} \cdot f_X(x)\,dx$$

And $E(A_x) = \sum_{n=1}^{\infty} n \cdot P(A_x = n)$

$$E(A_x^2) = \sum_{n=1}^{\infty} n^2 P(A_x = n)$$

$$Var(A_x) = E(A_x^2) - E(A_x)^2$$

Now, from law of total expectation:

$$E(A_x) = E(E(A_x \mid X = x))$$
$$= E(\lambda X)$$
$$= \lambda \cdot E(X)$$

Similarly,

$$Var(A_x) = E(Var(A_x \mid X)) + Var(E(A_x \mid X))$$
$$= E(\lambda X) + Var(\lambda X)$$
$$= \cancel{\lambda(E(X) + Var(X))}$$
$$= \lambda E(X) + \lambda^2 Var(X)$$