EE 621 **Spring 2021**

Problem Set (Week 6)

1 Battery renewals

Avi has a battery-run radio that is always ON. Suppose that each battery has a lifetime (in hours) that is uniformly distributed over the interval $[30, 60]$. Once a battery runs out, Avi goes and buys a new one. The time it takes for Avi to replace the used battery with a new one (in hours) is uniformly distributed over $[0, 2]$. What is the long term rate at which Avi buys batteries?

2 Threshold queue

We define a threshold queue with parameter T via a DTMC over the non-negative integers as follows: The state of the DTMC represents the number of jobs in the queue at any time. When the number of jobs is $< T$, the number of jobs decreases by 1 with probability 0.4 and increases by 1 with probability 0.6 at each time step. However, when the number of jobs $\geq T$, then the reverse is true; the number of jobs decreases by 1 with probability 0.6 and increases by 1 with probability 0.4 at each time step. This is illustrated in Fig. 1, for the case $T = 3$.

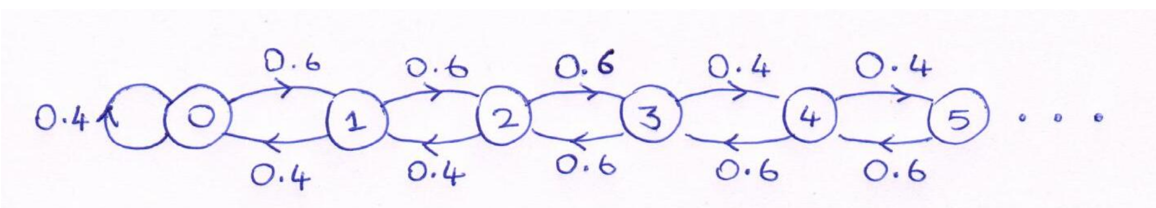


Figure 1: Example of threshold queue with $T = 3$.

Prove that the DTMC corresponding to the threshold queue is positive recurrent.

3 Hitting times

Consider an irreducible DTMC over a countable state space S. Recall the following notation from the class:

f_{ij} is the probability of ever hitting state j starting from state i. When $f_{ij} = 1$, we denote by ν_{ij} the mean hitting time of state j, starting at state i.

Prove or disprove the following statement: If the DTMC is null recurrent, then $\nu_{ij} = \infty$ for all $i, j \in S$.

4 Hitting times reloaded[1]

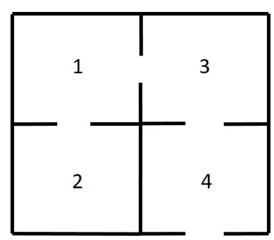
Consider an irreducible positive recurrent DTMC over a countable state space S. With the same notation as in the previous problem, prove or disprove the following conjecture:

$$\nu_{ii} \leq \nu_{ij} + \nu_{ji}.$$

5 Mouse play

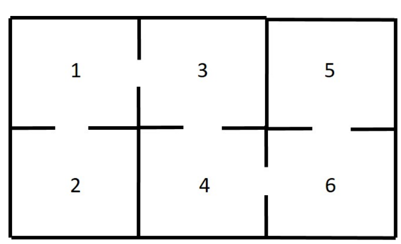
A mouse is placed in Room 1 of the following maze. Note that each room has openings to other rooms, and there is an escape opening to the outside world from Room 4. The mouse suffers from short-term memory loss; i.e., after entering any room, it chooses one of the openings out of that room uniformly at random, independent of its past movements. For example, once in Room 4, it is equally likely to escape or return to Room 3.

1. Is the mouse guaranteed to escape eventually? Justify your answer.
2. Compute the expected number of moves it makes to escape.



6 More mouse play

Our mouse (from the previous problem) is now placed in the following closed maze with six rooms. Recall that the mouse suffers from short-term memory loss: After entering any room, it chooses any of the openings out of the room uniformly at random, independent of its past movements.

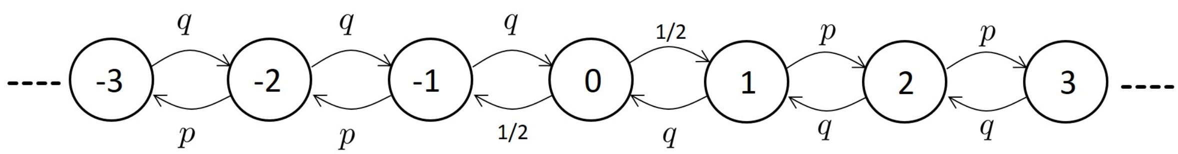


1. Suppose you spot the mouse in Room 1 at some time. From this point on, how many moves will it take for the mouse, on average, to return to Room 1?
2. Suppose you spot the mouse in Room 1 at some time. From this point on, how many moves will it take for the mouse, on average, to visit Room 6 for the first time?

[1]This is Problem 9.5 in Mor Harchol-Balter's book.

7 Stability issues

Consider the following DTMC over the set of integers.



Note that for positive states, the DTMC transitions right with probability p and left with probability q. However, for negative states, the DTMC transitions left with probability p and right with probability q. Of course, $p+q = 1$. Assume that $0 < p < 1/2$.

Prove that this DTMC is positive recurrent.

8 Interval of arrival

For a renewal process with iid inter-renewal times $\{X_i\}_{i\geq 1}$, consider the length of the inter-renewal period into which a random observer arrives. (Going back to our "bus arrivals" example from class, this corresponds to the time between the arrival of the last bus before you and the first bus after you.)

Prove, using renewal reward arguments, that average length of this interval equals $\frac{\mathbb{E}[X_1^2]}{\mathbb{E}[X_1]}$.

Notice that $\frac{\mathbb{E}[X_1^2]}{\mathbb{E}[X_1]} \geq \mathbb{E}[X_1]$, with equality only when X_1 is constant with probability 1.

9 Reinterpreting the stationary equations in DTMCs

Consider an irreducible, positive recurrent DTMC. For $i, j \in S$, prove that the long run rate of $i \to j$ transitions equals $\pi_i p_{ij}$.

Note: The stationary equations for a DTMC can be rewritten as

$$\sum_j \pi_i p_{i,j} = \sum_j \pi_j p_{ji}.$$

With the above result, you can interpret the stationary equations as balancing the rate of outgoing transitions from state i to the rate of incoming transitions into state i. You should also be able to re-interpret the definition of time-reversibility using this result.