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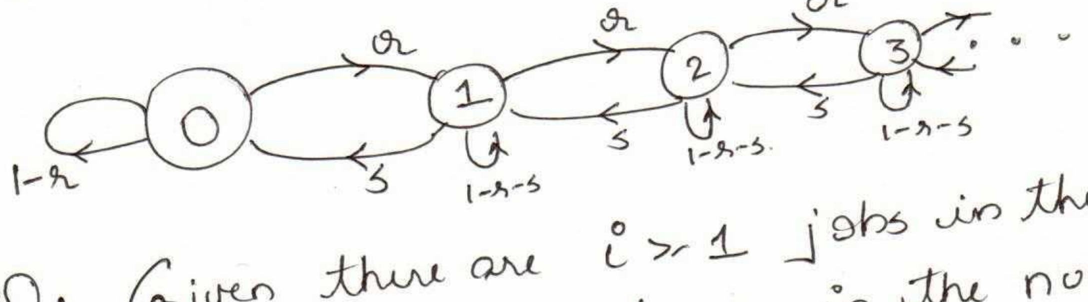
# Lyapunov stability analysis

For an irreducible DTMC, we know that if we can find a stationary distribution, then the chain is positive recurrent. However, often, we come across chains for which the stationary equations are intractable.

In such cases, one can still establish recurrence via a drift analysis. This involves guessing a Lyapunov function such that the DTMC has a negative drift on the function.

To motivate this, let us return to our discrete-time M/M/1 queue.




Q: Given there are $i \geq 1$ jobs in the system, what is the average change in the number of jobs over 1 time step?

A: $E[X_{k+1} - X_k \mid X_k = i] = r - s$.

$\Rightarrow$ When $r < s$, the DTMC tends to reduce the # in system, on average.

Turns out this implies positive recurrence of the chain.

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<u>Theorem:</u> Consider an irreducible DTMC over countable state space $S$. Suppose $V : S \to \mathbb{R}_+$ and $C$ is a finite subset of $S$. If there exists $\varepsilon > 0$, $b < \infty$ s.t.

i) $E\left[V(X_{k+1}) - V(X_k) \mid X_k = i\right] \le -\varepsilon \quad \forall\, i \in S - C,$

ii) $E\left[V(X_{k+1}) - V(X_k) \mid X_k = i\right] \le b \quad \forall\, i \in C,$

then the DTMC is positive recurrent.

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↳ attributed to Foster; also called Foster's / ~~Fostest~~ Foster-Lyapunov theorem.

Proof & generalizations: Hajek notes.

<u>Note:</u> In our M/M/1 example, set $V(i) = i$.

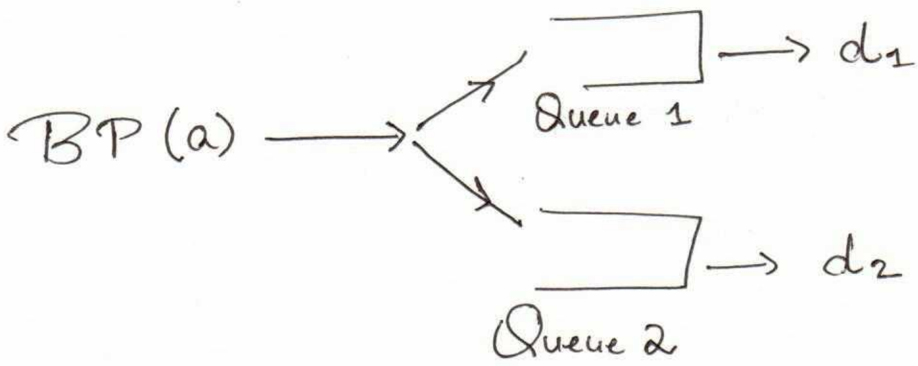
$$\Rightarrow E\left[V(X_{k+1}) - V(X_k) \mid X_k = i\right] = E\left[X_{k+1} - X_k \mid X_k = i\right] = \begin{cases} -(s - r) & \text{if } i \ge 1 \\ r & \text{if } i = 0 \end{cases}$$

~~Our V satisfies~~ For $r < s$ ($\Leftrightarrow p < q$), our choice of $V$ satisfies the criterion of Foster's Theorem, with $C = \{0\}$, $\varepsilon = (s - r)/2$, $b = 1$.

③

$\Rightarrow$ The DTMC is positive recurrent.

<u>Example 2</u>: Routing to multiple queues.




Two queues are fed by a single arrival stream BP(a) of jobs/packets/files. When a job arrives, we have to route it immediately to one of the queues.

Each queue $i$ when non-empty, completes a job with probability $d_i$. This might model:

- Jobs/files with service time Geometric($d_i$)
- Wireless setup where each queue ~~sees~~ sees a good channel in a slot with some probability.

We allow an arriving packet/job to depart in the same slot.

Suppose $a < d_1 + d_2$. Intuitively, this is a necessary condition for stability.

(4)

Q: How can you route so that the system is stable?

A: Find $p \in (0,1)$ s.t. $a p < d_1$,
$a(1-p) < d_2$.

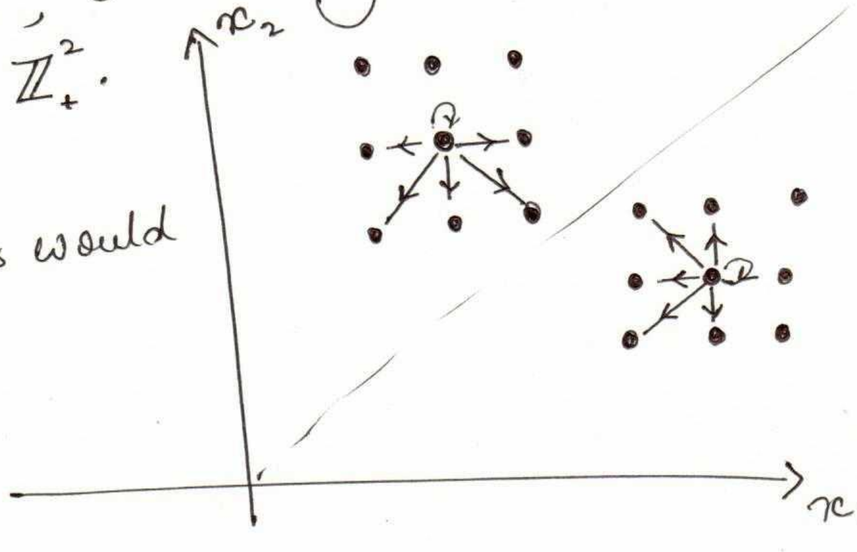
Route each arrival to Queue 1 w.p. $p$, and to Queue 2 w.p. $(1-p)$.

Q: But this assumes you know $a, d_1, d_2$. What if you don't?

A: As we will show, routing to the shortest queue will stabilize the system.

<u>Policy</u>: Let $(x_1, x_2)$ be the vector of queue lengths. We route to Queue 1 if $x_1 \le x_2$, and to Queue 2 if $x_2 < x_1$.

<u>Given the policy</u>, evolution of $(x_1, x_2)$ is a DTMC over $\mathbb{Z}_+^2$.




Stationary equations would be a mess!

(5)

Let $V(x) = \dfrac{x_1^2 + x_2^2}{2}$.

Note:

$$x_i(t+1) = x_i(t) + A_i(t) - D_i(t) + L_i(t)$$

where: $A_1(t) = \overset{\to BP(a)}{A(t)} \cdot \mathbb{1}\{x_1(t) \le x_2(t)\}$

$A_2(t) = A(t)\,\mathbb{1}\{x_1(t) > x_2(t)\}$

$D_i(t) \sim BP(d_i)$

$L_i(t) = \mathbb{1}\{x_i(t) + A_i(t) - D_i(t) = -1\}$

Keeps the queue length non-negative when empty queue dues no arrival & ~~$D_i(t)=1$~~ $D_i(t) = 1$.

Let $\Delta := E\left[V(x(t+1)) - V(x(t)) \,\middle|\, x(t) = \underset{(x_1, x_2)}{x}\right]$

$$= E\left[\frac{x_1^2(t+1) + x_2^2(t+1)}{2} - \frac{x_1^2(t) + x_2^2(t)}{2} \,\middle|\, x(t) = x\right]$$

$$= \frac{1}{2}\sum_{i=1}^{2} E\left[x_i^2(t+1) - x_i^2(t) \,\middle|\, x(t) = x\right]$$

$$= \frac{1}{2}\sum_{i} E\left[(x_i(t) + A_i(t) - D_i(t) + L_i(t))^2 - x_i^2(t) \,\middle|\, x(t) = x\right]$$

⑥ $$\Delta \le \frac{1}{2} \sum_i E\left[\left(x_i(t) + A_i(t) + D_i(t)\right)^2 - x_i^2(t) \,\middle|\, x_i(t) = x\right]$$

$$\le \sum_i E\left[x_i(t)\left(A_i(t) - D_i(t)\right) \,\middle|\, x_i(t) = x\right]$$

$$+ \frac{1}{2} \sum_i E\left[\left(A_i(t) - D_i(t)\right)^2 \,\middle|\, x_i(t) = x\right]$$

$$\le x_1\left(a \mathbb{1}_{\{x_1 \le x_2\}} - d_1\right) + x_2\left(a \mathbb{1}_{\{x_1 > x_2\}} - d_2\right) + 1$$

$$= a\Big(\underbrace{x_1 \mathbb{1}_{\{x_1 \le x_2\}} + x_2 \mathbb{1}_{\{x_1 > x_2\}}}_{\min\{x_1, x_2\}}\Big) - x_1 d_1 - x_2 d_2 + 1$$

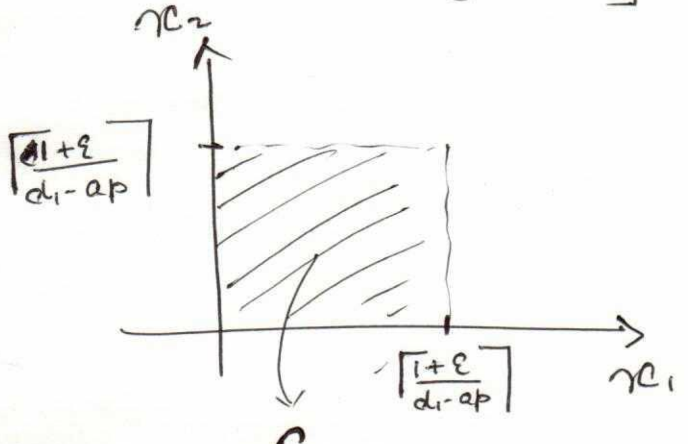
__Note:__ $\exists q, p \in (0,1)$, $\boxed{p + q = 1}$ s.t. $d_1 - ap = d_2 - aq > 0$

$$\therefore \quad \Delta \le a\left(p x_1 + q x_2\right) - x_1 d_1 - x_2 d_2 + 1$$

$$= -x_1(d_1 - ap) - x_2(d_2 - aq) + 1$$

$$\le -\varepsilon$$

↳ so long as __either__ $x_1$ or $x_2 \ge \dfrac{1+\varepsilon}{d_1 - ap}$




$\Rightarrow$ DTMC is positive recurrent, queues are stable.

__Note:__ Our routing policy keeps system stable so long as this is possible, with no knowledge of system parameters!