Review of Signals and Systems: Part 1

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Introduction

- Deterministic signal:
  - signal whose waveform is known exactly as a function of time
  - e.g.: $g(t) = \sin(2\pi f_0 t)$, $h(t) = \exp(-at)\, u(t)$
- Random process:
  - waveform not known exactly
  - distribution often known
  - e.g.: AC voltage from wall socket measured starting from a random instant:
    - $X(t) = R\cos(\omega t + \Theta)$, where R, ω and Θ are random variables
- For now, we focus on deterministic signals
- We review "*Fourier transform*"
  - provides a link between the time-domain and frequency-domain description
- We also study the transmission of deterministic signals through linear time-invariant (LTI) and other systems
  - e.g., filters, communication channels

Fourier Transform

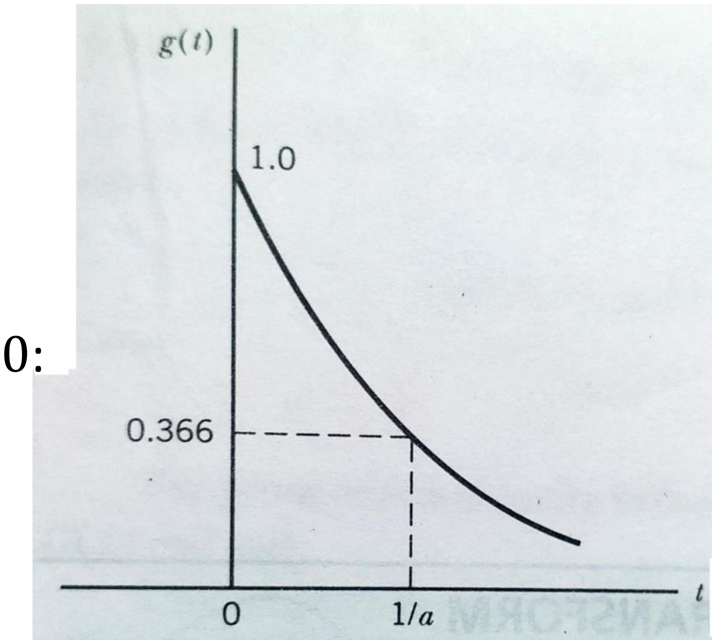
- Let $g(t)$: a deterministic signal
- Fourier transform of $g(t)$:

  1) $G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi f t)\, dt$
- Inverse Fourier transform:

  2) $g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi f t)\, df$
- **Notation**:
  - ❑ In above formulas, frequency f is measured in Hz
  - ❑ f is related to angular frequency ω by $\omega = 2\pi f$
  - ❑ ω is measured in rad/ s
  - ❑ Throughout this course, we will use f instead of ω since the frequency content of message signals (e.g., audio, video) and bandwidth of communication channels are usually expressed in Hz
- **Shorthand**: We will often use the following shorthand:
  - ❑ $G(f) = F[g(t)]$ for 1)
  - ❑ $g(t) = F^{-1}[G(f)]$ for 2) (note: $F[.]$ and $F^{-1}[.]$ are *linear* operators)
  - ❑ $g(t) \rightleftharpoons G(f)$ if $g(t)$ and $G(f)$ form a Fourier transform pair

Amplitude and Phase Spectrum

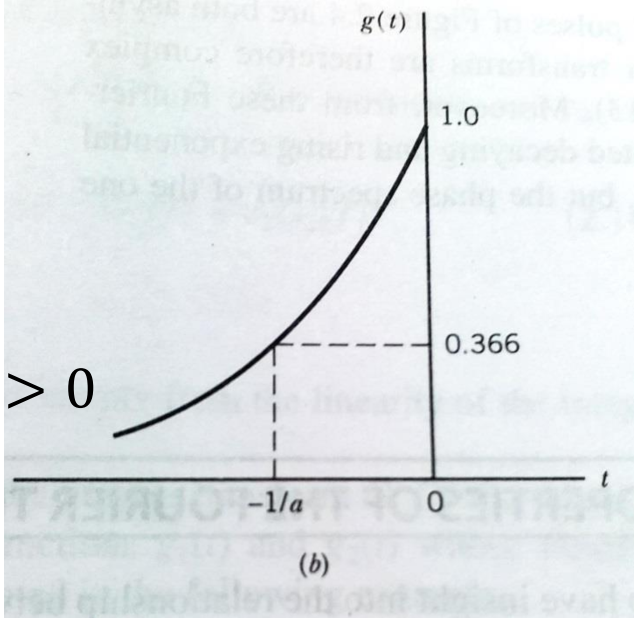
- $G(f)$ is in general a complex function
- So we express it in the form:
  - ❑ $G(f) = |G(f)|e^{j\theta(f)}$, where
  - ❑ $|G(f)|$ is called "*amplitude spectrum*" and
  - ❑ $\theta(f)$ is called "*phase spectrum*"
- E.g.:
- Fourier transform of $g(t) = e^{-at}u(t)$, where $a > 0$:
  - ❑ $G(f) = \frac{1}{a+j2\pi f}$
- Amplitude spectrum:
  - ❑ $|G(f)| = \frac{1}{\sqrt{a^2+4\pi^2 f^2}}$
- Phase spectrum:
  - ❑ $\theta(f) = \tan^{-1}\left(\frac{-2\pi f}{a}\right)$
- Fourier transform of a *real-valued* function $g(t)$ has the property:
  - ❑ $G(-f) = G^*(f)$
- So amplitude and phase spectrum have the properties:
  - ❑ $|G(-f)| = |G(f)|$
  - ❑ $\theta(-f) = -\theta(f)$
- It can be seen that these properties hold for the above example



Ref: "Communication Systems" by S. Haykin and M. Moher, 5th ed

Examples

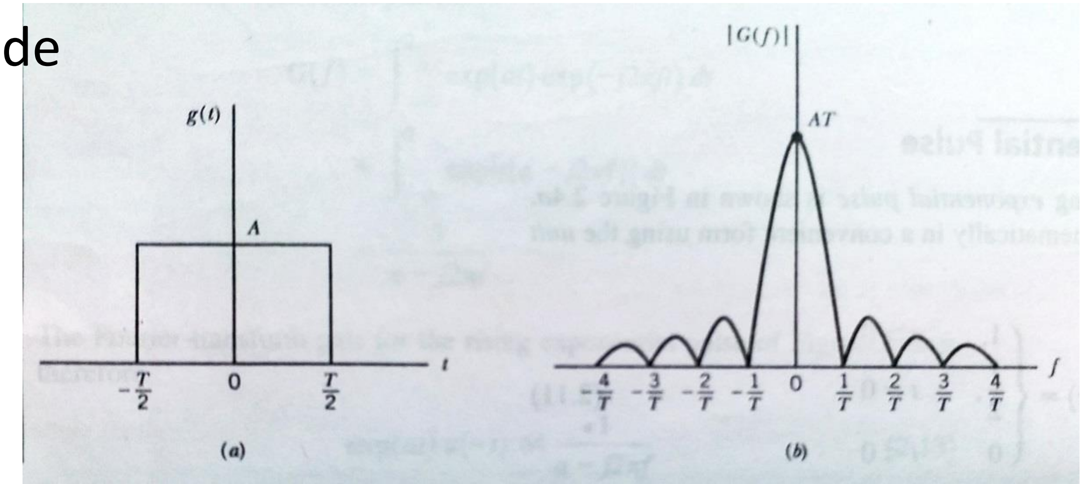
1) Fourier transform of $g(t) = e^{at}u(-t)$, where $a > 0$:
   - ❑ $G(f) = \frac{1}{a - j2\pi f}$



- Let $\mathrm{rect}(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2}, \\ 0, & |t| \geq \frac{1}{2}. \end{cases}$

2) Fourier transform of $g(t) = A\mathrm{rect}\left(\frac{t}{T}\right)$, where $A > 0$ and $T > 0$:
   - ❑ $G(f) = AT\mathrm{sinc}(fT)$,
   - ❑ where $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

- Above three examples show that if a signal is narrow in time, then it has significant content over a wide range of frequencies and vice-versa



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Properties of Fourier Transform

1) **Linearity**: If $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$, then for all constants c_1 and c_2, $F[c_1 g_1(t) + c_2 g_2(t)]$:
   - ❑ $c_1 G_1(f) + c_2 G_2(f)$
2) **Time Scaling**: If $g(t) \rightleftharpoons G(f)$, then $F[g(at)]$:
   - ❑ $\frac{1}{|a|} G\left(\frac{f}{a}\right)$
3) **Duality**: If $g(t) \rightleftharpoons G(f)$, then $F[G(t)]$:
   - ❑ $g(-f)$

- **Exercise**: Find the Fourier transform of $g(t) = A\mathrm{sinc}(2Wt)$

4) **Time Shifting**: If $g(t) \rightleftharpoons G(f)$, then $F[g(t - t_0)]$:
   - ❑ $G(f)e^{-j2\pi f t_0}$

Properties of Fourier Transform (contd.)

5) **Frequency Shifting**: If $g(t) \rightleftharpoons G(f)$, then $F^{-1}[G(f -$

Properties of Fourier Transform (contd.)

10) Multiplication in Time Domain: If $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$, then $F[g_1(t)g_2(t)]$:

- ❑ $G_1(f) * G_2(f)$

11) Convolution in Time Domain: If $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$, then $F[g_1(t) * g_2(t)]$:

- ❑ $G_1(f)G_2(f)$

12) Rayleigh's Energy Theorem (Parseval's Theorem): If $g(t) \rightleftharpoons G(f)$ and $\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$, then:

- ❑ $\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$
- ❑ $|G(f)|^2$ known as "*energy spectral density*" of the signal $g(t)$

- **Exercise**: Find the value of $A^2 \int_{-\infty}^{\infty} \mathrm{sinc}^2(2Wt) dt$