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CORRELATION AND SPECTRAL DENSITY

CORRELATION AND SPECTRAL DENSITY

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- Another approach to signal and system analysis - CORRELATION FUNCTIONS
- Focus on time averages and signal power or energy
- $\mathcal{F}\{\text{Correlation function}\} \rightarrow$ Frequency domain representation in terms of 'Spectral Density Functions'

  ESD - energy signal
  PSD - power signal
- Signals need not be Fourier transformable
- Spectral Density $\longrightarrow$ Broader range of signal models including random signals

- Convolution Theorem:

$$\int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau)\, d\tau \longleftrightarrow G_1(f) G_2(f)$$

Roles of t & τ interchanged →

$$\int_{-\infty}^{\infty} g_1(t) g_2(\tau-t)\, dt \longleftrightarrow G_1(f) G_2(f)$$

$$\Rightarrow \int_{-\infty}^{\infty} g_1(t) g_2\{-(t-\tau)\}\, dt \longleftrightarrow G_1(f) G_2(f)$$

$$\Rightarrow \int_{-\infty}^{\infty} g_1(t) g_2^*(t-\tau)\, dt \longleftrightarrow G_1(f) G_2^*(f)$$

(reflection + conjugation rule

CORRELATION THEOREM

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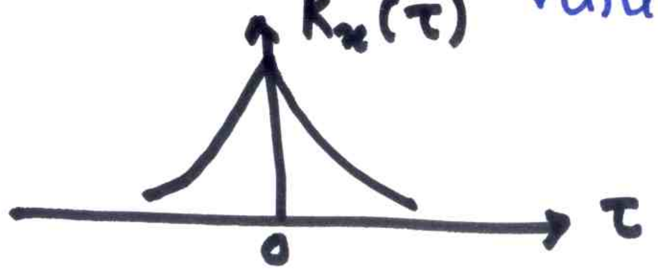
Autocorrelation Function

- Consider an energy signal $x(t)$
  (w.l.o.g. $x(t) \rightarrow$ complex valued)
- $$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$
  ↳ provides a measure of similarity between $x(t)$ and $\underbrace{x(t-\tau)}_{\text{delayed version}}$
- $R_x(\tau) \rightarrow$ complex valued if $x(t)$ complex valued
- $$R_x(0) = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$



CORRELATION THEOREM

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt \longleftrightarrow X_1(f)\, X_2^*(f)$$

$x_1(t) \uparrow x(t)$, $x_2^*(t-\tau) \uparrow x^*(t-\tau)$

$$\Rightarrow R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt \longleftrightarrow X(f)\, X^*(f)$$

$$= |X(f)|^2$$

$$\triangleq \Psi_x(f) \text{ (Energy Spectral Density)}$$

RAYLEIGH'S ENERGY THEOREM

$$\left.\begin{aligned} &\bullet\ \Psi_x(f) = \int_{-\infty}^{\infty} R_x(\tau) \exp(-j2\pi f\tau)\, d\tau \\ &\bullet\ R_x(\tau) = \int_{-\infty}^{\infty} \Psi_x(f) \exp(j2\pi f\tau)\, df \end{aligned}\right\} \text{WIENER-KHITCHINE Relation for energy signal}$$

PROPERTIES :

(1) $$\int_{-\infty}^{\infty} R_x(\tau)\,d\tau = \Psi_x(0)$$

(2) $$\int_{-\infty}^{\infty} \underbrace{\Psi_x(f)}_{\text{ESD}}\,df = \underbrace{R_x(0)}_{\text{(Energy of the signal)}} = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

Ex: $$x(t) = A\,\mathrm{sinc}(2Wt) \longleftrightarrow X(f) = \frac{A}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right)$$

$$R_x(t) = \frac{A^2}{2W}\,\mathrm{sinc}(2Wt) \longleftrightarrow \Psi_x(f) = \left(\frac{A}{2W}\right)^2 \mathrm{rect}\left(\frac{f}{2W}\right)$$

Evaluation easier than using the autocorrelation formula directly

Effect on Filtering on ESD

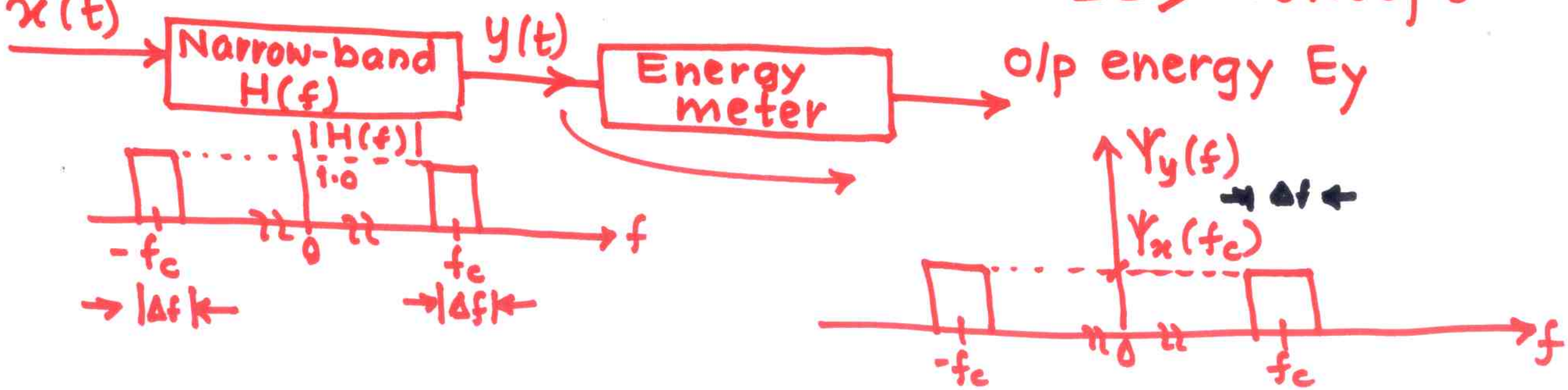
$$x(t) \longleftrightarrow X(f)$$

$$h(t) \longleftrightarrow H(f) \quad (\text{LTI system})$$

o/p: $$y(t) \longleftrightarrow Y(f) = H(f)\,X(f)$$

$$\Psi_y(f) = |Y(f)|^2 = |H(f)|^2\,\Psi_x(f)$$

* provides a basis for the physical interpretation of ESD concept



$$|H(f)| = \begin{cases} 1, & f_c - \frac{\Delta f}{2} \le |f| \le f_c + \frac{\Delta f}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$|Y(f)| = |H(f)||X(f)|$$

$$= \begin{cases} |X(f_c)|, & f_c - \frac{\Delta f}{2} \le |f| \le f_c + \frac{\Delta f}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\Psi_y(f) = \begin{cases} \Psi_x(f_c), & f_c - \frac{\Delta f}{2} \le |f| \le f_c + \frac{\Delta f}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$E_y = \int_{-\infty}^{\infty} \Psi_y(f)\,df = 2\int_{0}^{\infty} \Psi_y(f)\,df = 2\Psi_x(f_c)\,\Delta f$$

$$\Rightarrow \Psi_x(f_c) = \frac{E_y}{2\Delta f}$$

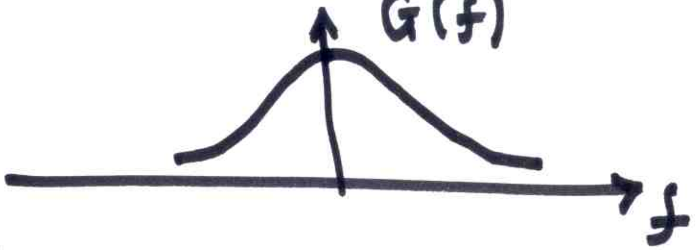
* ESD of the filter i/p at f_c equals the energy of the filter o/p divided by $2\Delta f$, where Δf is the filter BW centered on f_c

$\Rightarrow$ ESD of an energy signal for any frequency 'f' as the energy per unit BW, which is contributed by frequency components of the signal around the frequency 'f'

ENERGY AND POWER SPECTRAL DENSITY

$$g(t) \xleftrightarrow{ESD} |G(f)|^2$$

$$R_g(\tau) \longleftrightarrow |G(f)|^2$$



"Essential Bandwidth"

BW — B Hz

<u>Ex:</u> $g(t) = \text{rect}\left(\frac{t}{T}\right) \longleftrightarrow T\,\text{sinc}(fT)$

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = T \qquad ESD \rightarrow |T|^2 \text{sinc}^2(fT)$$

$$\int_{-B}^{+B} T^2 \operatorname{sinc}^2(fT)\,df = 0.95T$$

$$s(t) = g(t)\cos 2\pi f_c t$$

Energy of modulated signal ?

$$s(t) \longleftrightarrow S(f) = \frac{1}{2}\left[G(f-f_c) + G(f+f_c)\right]$$

$$\Big\updownarrow \text{ESD}$$

$$|S(f)|^2 = \frac{1}{4}\left[G(f-f_c) + G(f+f_c)\right]^2$$

$$= \frac{1}{4}\left[|G(f-f_c)|^2 + |G(f-f_c)|^2\right]$$

$$|S(f)|^2 = \frac{1}{4}\left[E_{ESD_g}(f+f_c) + E_{ESD_g}(f-f_c) \right]$$

$$(\mathcal{E}nergy)_{s(t)} = \frac{1}{2}(\mathcal{E}nergy)_{g(t)}$$

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Power Spectral Density

Average power of a signal $g(t)$ is

$$P \triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2 dt$$

$$P < \infty$$

$$g_T(t) = g(t)\,\mathrm{rect}\left(\frac{t}{T}\right)$$

$$= \begin{cases} g(t), & -T/2 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases}$$

$$g_T(t) \longleftrightarrow G_T(f)$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |g_T(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |G_T(f)|^2 df \longrightarrow (\mathrm{I})$$

$$P = \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{1}{T} \left|G_T(f)\right|^2 \right) df$$

$$S_g(f) \triangleq \lim_{T\to\infty} \frac{1}{T} \left|G_T(f)\right|^2 \longleftarrow \text{Power Spectral Density.}$$

$$P = \int_{-\infty}^{\infty} S_g(f)\, df$$

(Periodogram)

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Energy Signal

- $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$
- $R_g(\tau) = \int_{-\infty}^{\infty} g(t) g(t-\tau) dt$
- $R_g(\tau) \longleftrightarrow |G(f)|^2$ ESD
- $E_g = \int_{-\infty}^{\infty} |G(f)|^2 df$

Power Signal

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{E_{g_T}}{T}$$

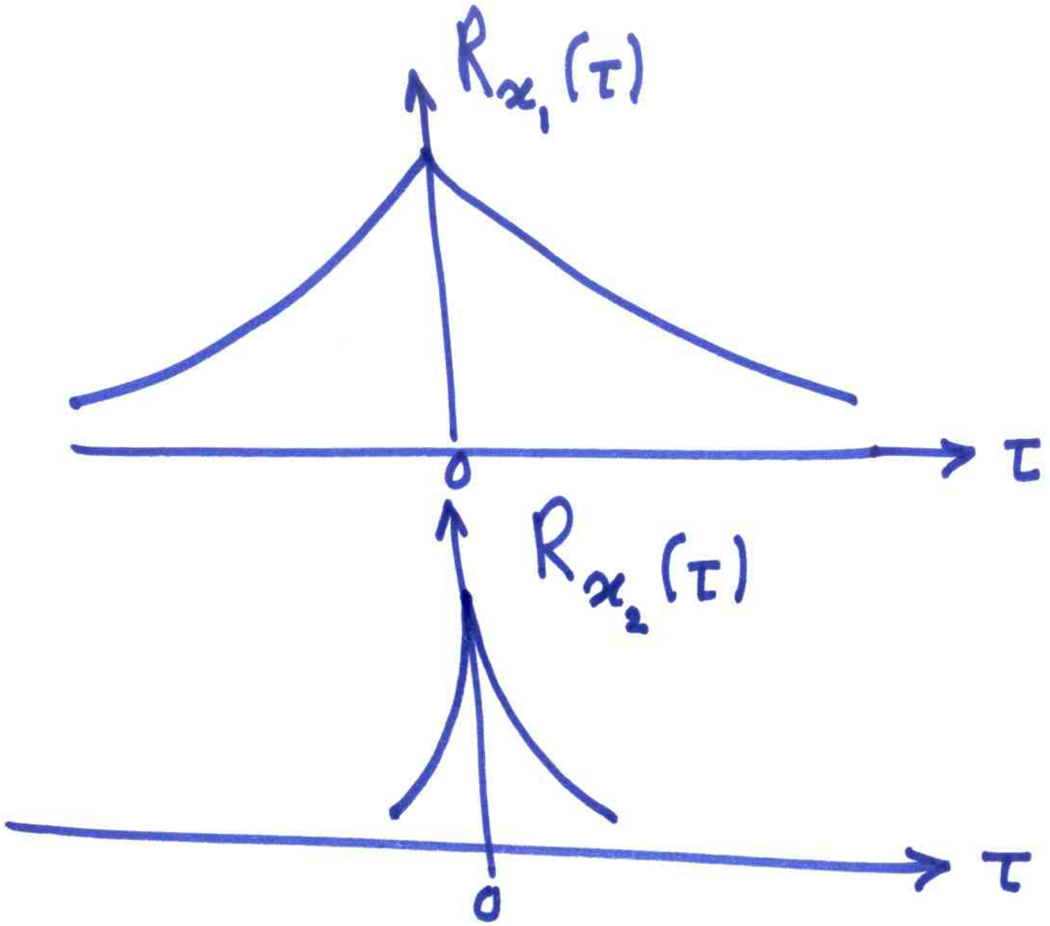
$$R_g(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t) g(t-\tau) dt$$

$$= \lim_{T\to\infty} \frac{R_{g_T}(\tau)}{T}$$

$$R_g(\tau) \longleftrightarrow \lim_{T\to\infty} \frac{|G_T(f)|^2}{T} = S_g(f)$$

(PSD)

$$P_g = \int_{-\infty}^{\infty} S_g(f) df$$


$R_{x_1}(\tau)$
τ
0
$R_{x_2}(\tau)$
τ
0