Theory of Small Reflections for Conjugately Characteristic-Impedance Transmission Lines

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***Abstract*—In this letter, the standard theory of small reflections (TSR) is generalized for the case of multi-section transformers, composed of conjugately characteristic-impedance transmission lines (CCITLs) connected in cascade, with small discontinuities. Using the assumption of small reflections, the total voltage reflection coefficient looking into the input of the transformer can be approximated, resulting in the TSR for multi-section CCITL transformers with small discontinuities. Numerical results, some physical interpretations and potential applications of results will be discussed in this letter.**

***Index Terms*—Conjugately characteristic-impedance transmission lines (CCITLs), theory of small reflections (TSR).**

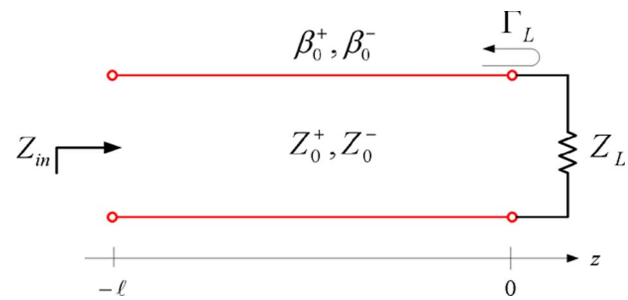

Fig. 1. CCITL terminated in a passive load impedance.

I. Introduction

Impedance matching is often a part of the design process for a microwave system to achieve a maximum power delivered to the load, and to avoid reflections from mismatched loads and junctions that result in echoes and distortion of the information-carrying signal [1]–[5]. Standard simple matching circuits; e.g., a single quarter-wave transformer, and single-stub and double-stub matching networks, normally have rather narrow bandwidth. For applications requiring more bandwidth than the standard simple matching circuits can provide, multi-section matching transformers can be used. These transformers are usually implemented using commensurate cascading sections of reciprocal lossless uniform transmission lines, and are designed based on the theory of small reflections (TSR) [3], [4]. For small reflections, the theory briefly states that the resultant voltage reflection coefficient is obtained by taking only first-order reflections into account.

Conjugately characteristic-impedance transmission lines (CCITLs) are *lossless* and possess different characteristic impedances Z_0^+ and Z_0^-, which are *complex conjugate* of each other, for waves propagating in opposite directions [6]–[9]. Due to potential applications of CCITLs in impedance matching, it is of interest to extend the standard TSR to the case of multi-section CCITL transformers with small discontinuities. Only CCITLs with *passive* characteristic impedances ($\mathrm{Re}\{Z_0^\pm\} \geq 0$) terminated with *passive* loads are considered in this letter.

II. Theory of CCITLs

Fig. 1 illustrates a CCITL terminated in a passive load impedance Z_L possessing the propagation constants, β_0^+ and β_0^-, with corresponding conjugate characteristic impedances, Z_0^+ and Z_0^-, for propagation in the *forward* and *reverse* directions, respectively. The input impedance Z_{in} can be written as [6]

$$Z_{in} = Z_0^+ Z_0^- \frac{1 + \Gamma_L e^{-j2\widetilde{\beta}_0 \ell}}{Z_0^- - Z_0^+ \Gamma_L e^{-j2\widetilde{\beta}_0 \ell}} \tag{1}$$

where Γ_L is the voltage reflection coefficient at the load defined as

$$\Gamma_L = \frac{Z_L Z_0^- - Z_0^+ Z_0^-}{Z_L Z_0^+ + Z_0^+ Z_0^-} \tag{2}$$

and $\widetilde{\beta}_0$ is defined as

$$\widetilde{\beta}_0 = \frac{1}{2}\left(\beta_0^+ + \beta_0^-\right). \tag{3}$$

Note that the characteristic impedances $Z_0^\pm$ of CCITLs must be *complex conjugate* of one another [6], which can be expressed mathematically as

$$Z_0^+ = \left(Z_0^-\right)^* \tag{4}$$

where the superscript "$*$" denotes the complex-conjugate symbol. For convenience, $Z_0^\pm$ are defined in a polar form as

$$Z_0^\pm = |Z_0|\, e^{\mp j\phi_0} \tag{5}$$

where $|Z_0|$ and ϕ_0 are the absolute value and the argument of Z_0^-, respectively. For *passive* characteristic impedances

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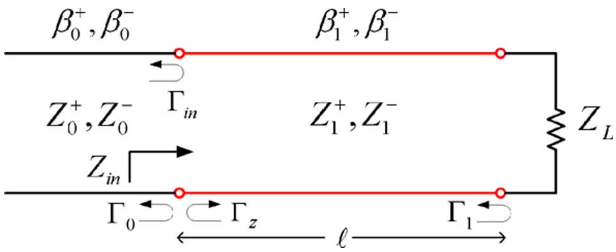

Fig. 2. Single-section CCITL transformer terminated with a load impedance.

$\left(\mathrm{Re}\left\{Z_0^{\pm}\right\} \geq 0\right)$, (5) simply implies that the argument ϕ_0 must lie in the following range:

$$-90^\circ \leq \phi_0 \leq 90^\circ. \tag{6}$$

III. Theory of Small Reflections for CCITLs

Consider a single-section CCITL transformer of length ℓ, connected between a CCITL feedline and a passive load impedance Z_L, as shown in Fig. 2. Note that the feedline possesses the characteristic impedances $Z_0^{\pm} = |Z_0|\, e^{\mp j\phi_0}$ and the propagation constants $\beta_0^{\pm}$, and the characteristic impedances and the propagation constants of the matching transformer are equal to $Z_1^{\pm} = |Z_1|\, e^{\mp j\phi_1}$ and $\beta_1^{\pm}$, respectively.

In Fig. 2, Γ_{in} is the *total* input voltage reflection coefficient, and Z_{in} is the input impedance seen at the input of the single-section CCITL transformer. Note that Γ_{in} can be derived by using the impedance method as shown below. Using (1), Z_{in} can be written in terms of Γ_1 as

$$Z_{in} = Z_1^+ Z_1^- \frac{\left[1 + \Gamma_1 e^{-j2\widetilde{\theta}}\right]}{\left[Z_1^- - Z_1^+ \Gamma_1 e^{-j2\widetilde{\theta}}\right]} \tag{7}$$

where

$$\Gamma_1 = \frac{Z_L Z_1^- - Z_1^+ Z_1^-}{Z_L Z_1^+ + Z_1^+ Z_1^-} \tag{8}$$

$$\widetilde{\beta}_1 = \frac{1}{2}\left(\beta_1^+ + \beta_1^-\right) \tag{9}$$

$$\widetilde{\theta} = \widetilde{\beta}_1 \ell. \tag{10}$$

Using (2), Γ_{in} can be expressed in terms of Z_{in} *exactly* as

$$\Gamma_{in} = \frac{Z_{in} Z_0^- - Z_0^+ Z_0^-}{Z_{in} Z_0^+ + Z_0^+ Z_0^-}. \tag{11}$$

Define the *partial* voltage reflection coefficients Γ_0 and Γ_z as

$$\Gamma_0 = \frac{Z_1^+ Z_0^- - Z_0^+ Z_0^-}{Z_1^+ Z_0^+ + Z_0^+ Z_0^-} \tag{12}$$

$$\Gamma_z = \frac{Z_0^- Z_1^+ - Z_1^- Z_1^+}{Z_0^- Z_1^- + Z_1^- Z_1^+}. \tag{13}$$

Substituting (7) into (11) and using (12) and (13), Γ_{in} can be expressed compactly as

$$\Gamma_{in} = \Gamma_0 \frac{1 + \Gamma_x^{-1} \Gamma_1 e^{-j2\widetilde{\theta}}}{1 - \Gamma_z \Gamma_1 e^{-j2\widetilde{\theta}}} \tag{14}$$

where

$$\Gamma_x = \frac{Z_1^+ Z_1^- - Z_0^+ Z_1^-}{Z_1^+ Z_1^- + Z_0^+ Z_1^+}. \tag{15}$$

From (8) and (13), it is observed that $|\Gamma_1|$ is small when $Z_1^+ \approx Z_L$, and $|\Gamma_z|$ is small when $Z_1^- \approx Z_0^-$. If the discontinuities between the impedances Z_1^+, Z_L and Z_0^-, Z_1^- are small; i.e., $|\Gamma_z \Gamma_1| \ll 1$, (14) can be approximated as

$$\Gamma_{in} \approx \Gamma_0 + \left(\frac{\Gamma_0}{\Gamma_x}\right) \Gamma_1 e^{-j2\widetilde{\theta}} \tag{16}$$

which is the TSR for the single-section CCITL transformer.

Using (12) and (15), the factor $\Gamma_0 \Gamma_x^{-1}$ can be expressed as

$$\frac{\Gamma_0}{\Gamma_x} = e^{j2(\phi_0 - \phi_1)} \left(\frac{Z_1^- + Z_0^+}{Z_1^+ + Z_0^-}\right). \tag{17}$$

If $|\Gamma_0|$ is small, one obtains $Z_1^+ \approx Z_0^+$; i.e., $\phi_1 \approx \phi_0$ and $Z_1^- \approx Z_0^-$. Using this observation in (17) yields $\Gamma_0 \approx \Gamma_x$ when $|\Gamma_0|$ is small. Thus, if $|\Gamma_0| \ll 1$ and $|\Gamma_z \Gamma_1| \ll 1$ can be obtained, Γ_{in} in (16) can be further approximated as

$$\Gamma_{in} \approx \Gamma_0 + \Gamma_1 e^{-j2\widetilde{\theta}} \tag{18}$$

called the *approximate* theory of small reflections (ATSR) for the single-section CCITL transformer. Equation (18) states the intuitive idea that, if the discontinuities between the impedances Z_0^+, Z_1^+ and Z_1^+, Z_L are sufficiently small, the total reflection is dominated by the reflection from the initial discontinuity (Γ_0), and the first reflection from the discontinuity at the load ($\Gamma_1 e^{-j2\widetilde{\theta}}$). The phase term $e^{-j2\widetilde{\theta}}$ accounts for the phase delay when the incident wave travels up and down the CCITL transformer. It is noted that (16) should be more accurate than (18), but (18) is more convenient to work with, especially when dealing with multi-section CCITL transformers. Note that the *approximate* theory in (18) takes only first-order reflections into account and neglects effects of multiple reflections.

One can generalize the result in (18) to multi-section CCITL transformers consisting of N equal-electrical-length ($\widetilde{\theta}$) sections of CCITLs with characteristic impedances $Z_i^{\pm}$, where $i = 1, 2, \ldots, N$, as shown in Fig. 3. Let define *partial* voltage reflection coefficients Γ_n at each junction as follows:

$$\Gamma_n = \frac{Z_{n+1}^+ Z_n^- - Z_n^+ Z_n^-}{Z_{n+1}^+ Z_n^+ + Z_n^+ Z_n^-} \tag{19}$$

where $n = 0, 1, 2, \ldots, N-1$ in (19), and the voltage reflection coefficient Γ_N at the load Z_L is defined as

$$\Gamma_N = \frac{Z_L Z_N^- - Z_N^+ Z_N^-}{Z_L Z_N^+ + Z_N^+ Z_N^-}. \tag{20}$$

If $|\Gamma_n|$ ($n = 0, 1, 2, \ldots, N$) are small, the *total* input voltage reflection coefficient $\Gamma_{in}\left(\widetilde{\theta}\right)$ can be approximated as

$$\Gamma_{in}\left(\widetilde{\theta}\right) \approx \Gamma_0 + \left(\frac{\Gamma_0}{\widetilde{\Gamma}_0}\right) \Gamma_1 e^{-j2\widetilde{\theta}} + \left(\frac{\Gamma_1}{\widetilde{\Gamma}_1}\right) \Gamma_2 e^{-j4\widetilde{\theta}} + \cdots + \left(\frac{\Gamma_{N-1}}{\widetilde{\Gamma}_{N-1}}\right) \Gamma_N e^{-j2N\widetilde{\theta}} \tag{21}$$

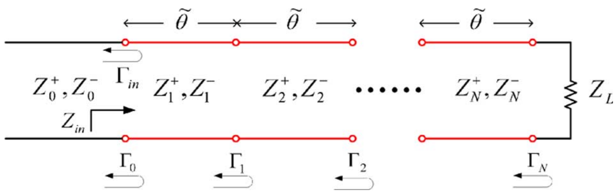

Fig. 3. Multi-section CCITL transformer terminated with a load impedance.

which is the TSR for multi-section CCITL transformers. In (21), $\widetilde{\Gamma}_n$ is mathematically defined as

$$\widetilde{\Gamma}_n = -\left[\frac{Z_n^+ Z_{n+1}^- - Z_{n+1}^+ Z_{n+1}^-}{Z_n^+ Z_{n+1}^+ + Z_{n+1}^+ Z_{n+1}^-}\right] \quad (22)$$

where $n = 0, 1, 2, \ldots, N-1$. Note that (21) can be simplified further if $|\Gamma_n|$ ($n = 0, 1, 2, \ldots, N$) are *sufficiently small* ($|\Gamma_n| \ll 1$); i.e.

$$\Gamma_{in}\left(\widetilde{\theta}\right) \approx \Gamma_0 + \Gamma_1 e^{-j2\widetilde{\theta}} + \Gamma_2 e^{-j4\widetilde{\theta}} + \cdots + \Gamma_N e^{-j2N\widetilde{\theta}} \quad (23)$$

which is the ATSR for multi-section CCITL transformers. Note that (23) is in the same form as the TSR equation for reciprocal transmission lines, and it can be employed to design multi-section CCITL transformers for passband responses when $|\Gamma_n| \ll 1$; e.g., binomial and Chebyshev responses [4].

IV. Numerical Results

Consider a three-section CCITL transformer ($N = 3$). For convenience in interpretation, let the magnitudes of reflection coefficients ($|\Gamma_n|$), where $n = 0, 1, 2, 3$, are the same. The characteristic impedances and the load impedance are given as follows: $Z_0^{\pm} = 43.30 \mp j25.00\ \Omega$, $Z_1^{\pm} = 47.16 \mp j27.36\ \Omega$, $Z_2^{\pm} = 46.90 \mp j32.07\ \Omega$, $Z_3^{\pm} = 51.77 \mp j32.85\ \Omega$, and $Z_L = 48.74 - j36.87\ \Omega$. These impedances result in $|\Gamma_n| = 0.05$, where $n = 0, 1, 2, 3$. Fig. 4 illustrates the plot of the magnitude of the total input voltage reflection coefficient Γ_{in} versus the electrical length $\widetilde{\theta}$, computed by using the exact method (see (11)), the TSR method (see (21)), and the ATSR method (see (23)). From the plot, it is found that the TSR and ATSR methods yield reasonably accurate results compared to the exact solution. It is observed that the TSR method yields more accurate results than the ATSR method's except for some small regions.

V. Conclusion

The theory of small reflections for multi-section CCITL transformers with small discontinuities is derived in this letter, where characteristic impedances of CCITLs are *passive*. It is found that the *approximate* theory of small reflections with *sufficiently small* discontinuities is similar to the standard TSR for the multi-section reciprocal matching transformer; i.e., it still takes only first-order reflections into account. Numerical results show that both TSR and ATSR methods yield reasonably accurate results when the magnitudes of the voltage reflection coefficients are *sufficiently small* ($|\Gamma_n| \leq 0.05$, where $n = 0, 1, 2, \ldots, N$), and the TSR method generally yields more accurate results than the ATSR method's except for some electrical lengths $\widetilde{\theta}$.

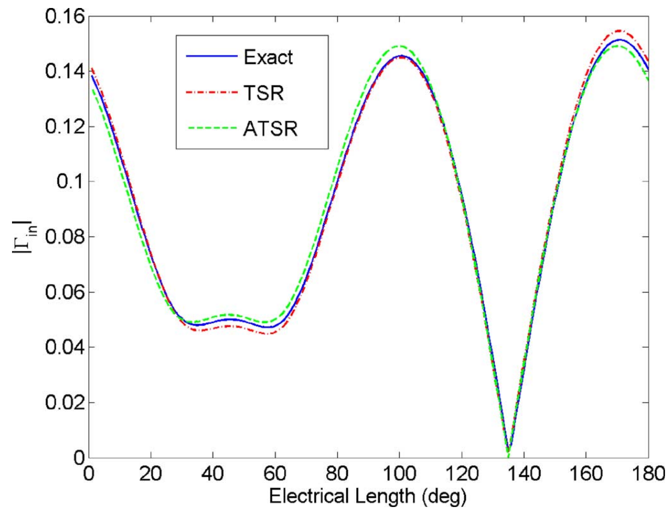

Fig. 4. Plot of the magnitude of the total input voltage reflection coefficient $|\Gamma_{in}|$ versus the electrical length $\widetilde{\theta}$ for three different approaches.

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