PROBLEMS

2.1

(a) Find the Fourier transform of the half-cosine pulse shown in Figure P2.1*a*.

(b) Apply the time-shifting property to the result obtained in part (*a*) to evaluate the spectrum of the half-sine pulse shown in Figure P2.1*b*.

(c) What is the spectrum of a half-sine pulse having a duration equal to aT?

(d) What is the spectrum of the negative half-sine pulse shown in Figure P2.1*c*?

(e) Find the spectrum of the single sine pulse shown in Figure P2.1*d*.

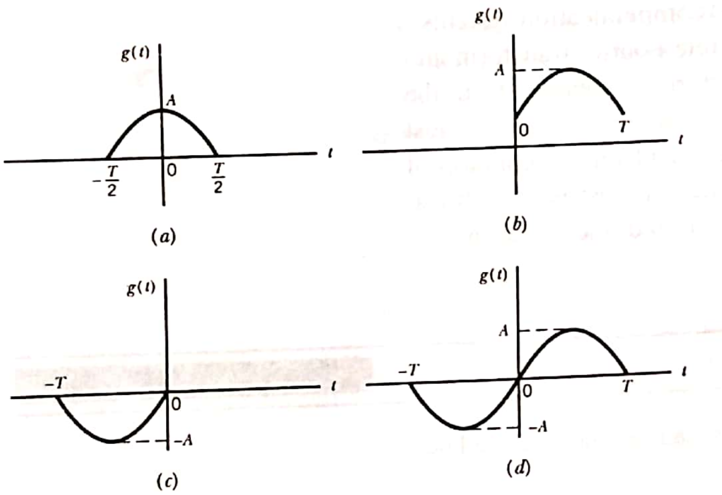

Figure P2.1

2.2 Evaluate the Fourier transform of the damped sinusoidal wave

$$g(t) = \exp(-t)\sin(2\pi f_c t)u(t)$$

where $u(t)$ is the unit step function.

2.3 Any function $g(t)$ can be split unambiguously into an *even part* and an *odd part*, as shown by

$$g(t) = g_e(t) + g_o(t)$$

The even part is defined by

$$g_e(t) = \frac{1}{2}[g(t) + g(-t)]$$

and the odd part is defined by

$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]$$

(a) Evaluate the even and odd parts of a rectangular pulse defined by

$$g(t) = A\,\text{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$$

(b) What are the Fourier transforms of these two parts of the pulse?

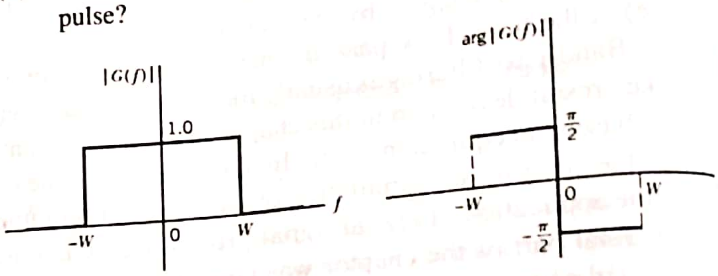

Figure P2.4

2.4 Determine the inverse Fourier transform of the frequency function $G(f)$ defined by the amplitude and phase spectra shown in Figure P2.4.

2.5 The following expression may be viewed as an approximate representation of a pulse with finite rise time:

$$g(t) = \frac{1}{\tau}\int_{t-T}^{t+T} \exp\left(-\frac{\pi u^2}{\tau^2}\right) du$$

where it is assumed that $T >> \tau$. Determine the Fourier transform of $g(t)$. What happens to this transform when we allow τ to become zero? *Hint*: Express $g(t)$ as the superposition of two signals, one corresponding to integration from $t - T$ to 0, and the other from 0 to $t + T$.

2.6 The Fourier transform of a signal $g(t)$ is denoted by $G(f)$. Prove the following properties of the Fourier transform:

(a) If a real signal $g(t)$ is an even function of time t, the Fourier transform $G(f)$ is purely real. If a real signal $g(t)$ is an odd function of time t, the Fourier transform $G(f)$ is purely imaginary.

(b)

$$t^n g(t) \rightleftharpoons \left(\frac{j}{2\pi}\right)^n G^{(n)}(f)$$

where $G^{(n)}(f)$ is the nth derivative of $G(f)$ with respect to

(c) $\int_{-\infty}^{\infty} t^n g(t)\, dt = \left(\frac{j}{2\pi}\right)^n G^{(n)}(0)$

(d) $g_1(t)g_2^*(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda - f)\, d\lambda$

(e) $\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\, dt = \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\, df$

2.7 The Fourier transform $G(f)$ of a signal $g(t)$ is bounded by the following three inequalities:

$$|G(f)| \le \int_{-\infty}^{\infty} |g(t)| dt$$

$$|j2\pi f G(f)| \le \int_{-\infty}^{\infty} \left|\frac{dg(t)}{dt}\right| dt$$

and

$$|(j2\pi f)^2 G(f)| \le \int_{-\infty}^{\infty} \left|\frac{d^2 g(t)}{dt^2}\right| dt$$

where it is assumed that the first and second derivatives of $g(t)$ exist.

Construct these three bounds for the triangular pulse shown in Figure P2.7 and compare your results with the actual amplitude spectrum of the pulse.

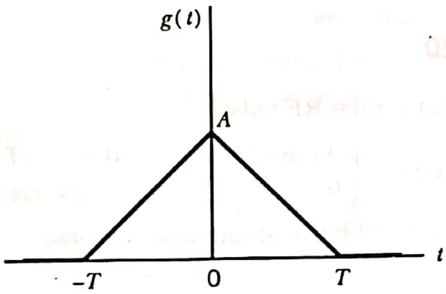

Figure P2.7

2.8 Prove the following properties of the convolution process:

(a) The commutative property:

$$g_1(t) \star g_2(t) = g_2(t) \star g_1(t)$$

(b) The associative property:

$$g_1(t) \star [g_2(t) \star g_3(t)] = [g_1(t) \star g_2(t)] \star g_3(t)$$

(c) The distributive property:

$$g_1(t) \star [g_2(t) + g_3(t)] = g_1(t) \star g_2(t) + g_1(t) \star g_3(t)$$

2.9 Consider the convolution of two signals $g_1(t)$ and $g_2(t)$. Show that

(a) $\frac{d}{dt}[g_1(t) \star g_2(t)] = \left[\frac{d}{dt} g_1(t)\right] \star g_2(t)$

(b) $\int_{-\infty}^{t} [g_1(\tau) \star g_2(\tau)] d\tau = \left[\int_{-\infty}^{t} g_1(\tau) d\tau\right] \star g_2(t)$

2.10 A signal $x(t)$ of finite energy is applied to a square-law device whose output $y(t)$ is defined by

$$y(t) = x^2(t)$$

The spectrum of $x(t)$ is limited to the frequency interval $-W \le f \le W$. Hence, show that the spectrum of $y(t)$ is limited to $-2W \le f \le 2W$. *Hint*: Express $y(t)$ as $x(t)$ multiplied by itself.

2.11 Evaluate the Fourier transform of the delta function by considering it as the limiting form of (1) a rectangular pulse of unit area, and (2) a sinc pulse of unit area.

2.12 The Fourier transform $G(f)$ of a signal $g(t)$ is defined by

$$G(f) = \begin{cases} 1, & f > 0 \\ \frac{1}{2}, & f = 0 \\ 0, & f < 0 \end{cases}$$

Determine the signal $g(t)$.

2.13 Show that the two different pulses defined in parts (*a*) and (*b*) of Figure P2.1 have the same energy spectral density:

$$\mathcal{E}_g(f) = \frac{4A^2T^2\cos^2(\pi T f)}{\pi^2(4T^2f^2 - 1)^2}$$

2.14

(a) The *root mean-square (rms) bandwidth* of a low-pass signal $g(t)$ of finite energy is defined by

$$W_{\text{rms}} = \left[\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2\, df}{\int_{-\infty}^{\infty} |G(f)|^2\, df}\right]^{1/2}$$

where $|G(f)|^2$ is the energy spectral density of the signal. Correspondingly, the *root mean-square (rms) duration* of the signal is defined by

$$T_{\text{rms}} = \left[\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2\, dt}{\int_{-\infty}^{\infty} |g(t)|^2\, dt}\right]^{1/2}$$

Using these definitions, show that

$$T_{\text{rms}} W_{\text{rms}} \ge \frac{1}{4\pi}$$

Assume that $|g(t)| \to 0$ faster than $1/\sqrt{|t|}$ as $|t| \to \infty$.

(b) Consider a Gaussian pulse defined by

$$g(t) = \exp(-\pi t^2)$$

Show that, for this signal, the equality

$$T_{\text{rms}} W_{\text{rms}} = \frac{1}{4\pi}$$

can be reached.

2.15 Let $x(t)$ and $y(t)$ be the input and output signals of a linear time-invariant filter. Using Rayleigh's energy theorem, show that if the filter is stable and the input signal $x(t)$ has finite energy, then the output signal $y(t)$ also has finite energy. That is, given that

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$$

then show that

$$\int_{-\infty}^{\infty} |y(t)|^2 \, dt < \infty$$