Homework 1 Solutions

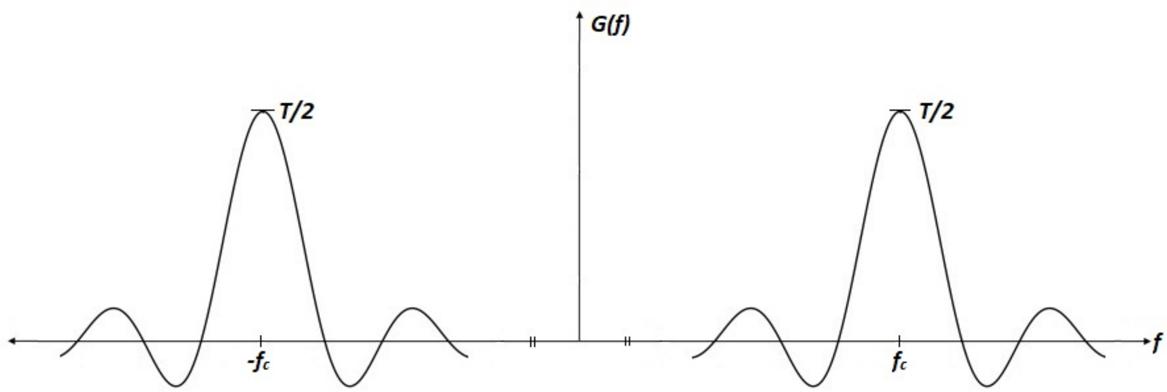
Communication Systems I (EE 341), Autumn 2021

2) $G(f)$ evaluation

a) $g(t) = \cos(2\pi f_c t).rect\left(\frac{t}{T}\right)$

$$g(t) \leftrightarrow G(f)$$

$$G(f) = T\text{sinc}(fT) * \left(\frac{1}{2}\right)[\delta(f-f_c) + \delta(f+f_c)]$$
$$= \left(\frac{T}{2}\right)[\text{sinc}((f-f_c)T) + \text{sinc}((f+f_c)T)]$$



b) $g(t) = \cos(2\pi f_c t).u(t)$

$$g(t) \leftrightarrow G(f)$$

$$G(f) = \left(\frac{1}{2}\right)[\delta(f-f_c) + \delta(f+f_c)] * \left(\frac{\delta(f)}{2} + \frac{1}{j2\pi f}\right)$$
$$= \left(\frac{1}{4}\right)[\delta(f-f_c) + \delta(f+f_c)] + \left(\frac{1}{j4\pi}\right)\left(\frac{1}{(f-f_c)} + \frac{1}{(f+f_c)}\right)$$

c) $g(t) = e^{-\alpha t}.\cos(2\pi f_c t).u(t)$

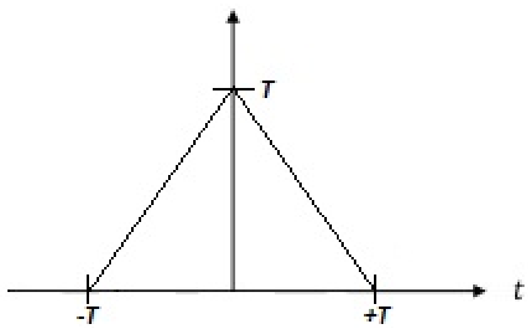
$$g(t) \leftrightarrow G(f)$$

$$G(f) = \mathcal{F}\{e^{-\alpha t}.\text{u}(\text{t})\} * \mathcal{F}\{\cos(2\pi f_c t)\} \; \mathcal{F}\{e^{-\alpha t}.\text{u}(\text{t})\} = \left(\frac{1}{\alpha + j2\pi f}\right)$$
$$\therefore \; G(f) = \left(\frac{1}{\alpha + j2\pi f}\right) * \left(\frac{1}{2}\right)[\delta(f-f_c) + \delta(f+f_c)]$$
$$= \left(\frac{1}{2}\right)\left[\frac{1}{\alpha + j2\pi(f-f_c)} + \frac{1}{\alpha + j2\pi(f+f_c)}\right] = \left[\frac{\alpha + j2\pi f}{(\alpha + j2\pi f)^2 + 4\pi^2 f_c^2}\right]$$

d) $g\left(t\right) = rect\left(\frac{t}{T}\right) \ * \ rect\left(\frac{t}{T}\right) \ = \ T\varDelta\left(\frac{t}{2T}\right)$

$$g\left(t\right) \ \leftrightarrow G\left(f\right)$$

$G\left(f\right) = \ \left[Tsinc(fT)\right]^2 = \ \ T^2 \ sinc^2(fT)$



e) $g\left(t\right) = \ e^{\frac{-t^2}{2\sigma^2}}$

$$e^{-\pi t^2} \ \leftrightarrow \ e^{-\pi f^2}$$

$$f\left(t\right) \ \leftrightarrow F\left(f\right)$$

$$f\left(\alpha t\right) \ \leftrightarrow \frac{1}{|\alpha|}F\left(\frac{f}{\alpha}\right)$$

In this case, $\alpha = \ \frac{1}{\sqrt{2\pi\sigma^2}}$

$$\therefore g\left(t\right) \ \leftrightarrow G\left(f\right)$$

$G\left(f\right) = \sqrt{2\pi\sigma^2}e^{-\pi 2\pi\sigma^2 f^2} \ = \ \sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2 f^2} \ = \ \sigma\sqrt{2\pi}e^{-2(\pi\sigma f)^2}$

f) $g\left(t\right) = \ rect\left(\frac{t- \ \frac{T}{2}}{T}\right) - \ rect\left(\frac{t+ \ \frac{T}{2}}{T}\right)$

$$g\left(t\right) \ \leftrightarrow G(f)$$

$G\left(f\right) = \ T\text{sinc}\left(fT\right) \ \ [e^{\frac{j2\pi fT}{2}} - e^{\frac{-j2\pi fT}{2}}]$

$= \ T\text{sinc}\left(fT\right) \ \ [2j\text{sin}(\pi fT) \ ]$

$= \left(j2\pi fT\right) T \ \text{sinc}^2\left(fT\right)$

3) Inverse Fourier Transform evaluations:

a) $G(f) = exp(-\frac{f^2}{2\sigma^2})rect(\frac{f}{2B})$

$G(f) \longleftrightarrow g(t) = \mathcal{F}^{-1}\{exp(-\frac{f^2}{2\sigma^2})\} * \mathcal{F}^{-1}\{rect(\frac{f}{2B})\}$

$\mathcal{F}^{-1}\{rect(\frac{f}{2B})\} = 2Bsinc(2Bt)$

$exp(-\pi t^2) \longleftrightarrow exp(-\pi f^2)$

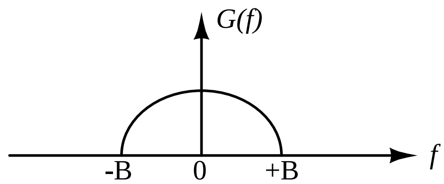
$\Rightarrow exp\{-\pi(\alpha t)^2\} \longleftrightarrow \frac{1}{|\alpha|}exp\{-\pi(\frac{f}{\alpha})^2\}$

$\Rightarrow \sqrt{2\pi\sigma^2}exp(-\pi.2\pi\sigma^2t^2) \longleftrightarrow exp(-\frac{f^2}{2\sigma^2})$

$\Rightarrow \sqrt{2\pi\sigma^2}exp\{-2\pi^2\sigma^2t^2\} \longleftrightarrow exp(-\frac{f^2}{2\sigma^2})$

$\therefore G(f) \longleftrightarrow g(t) = \sqrt{2\pi\sigma^2}exp\{-2\pi^2\sigma^2t^2\} * 2Bsinc(2Bt)$

b) $G(f) = -\cos(\frac{\pi f}{2B})rect(\frac{f}{2B})$



$g(t) = \int_{-\infty}^{\infty} G(f)exp(j2\pi ft)df$

$g(t) = -\int_{-B}^{B} cos(\frac{\pi f}{2B})exp(j2\pi ft)df$

$g(t) = -\frac{1}{2}\{\int_{-B}^{B} e^{\frac{j\pi f}{2B}}e^{j2\pi ft}df + \int_{-B}^{B} e^{-\frac{j\pi f}{2B}}e^{j2\pi ft}df\}$

c) $G(f) = \begin{cases} 1.e^{j2\pi ft_0} & |\ f\ | \leq B \\ 0 & \text{otherwise} \end{cases}$

$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df$

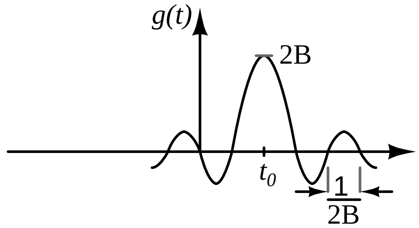
$= \int_{-B}^{+B} e^{-j2\pi ft_0}e^{j2\pi ft}df$

$= \frac{e^{j2\pi f(t-t_0)}}{j2\pi(t-t_0)}\Bigg|_{-B}^{+B}$

$= \frac{e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)}}{j2\pi(t-t_0)}$

$= \frac{2B\sin(2\pi B(t-t_0))}{2B\pi(t-t_0)}$

$= 2Bsinc\{2B(t-t_0)\}$



d) $G(f) = \frac{1}{(2\pi f)^2}(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1)$

$j2\pi f G(f) = \frac{1}{-j2\pi f}(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1)$

$= e^{j2\pi f} + \frac{1-e^{j2\pi f}}{j2\pi f}$

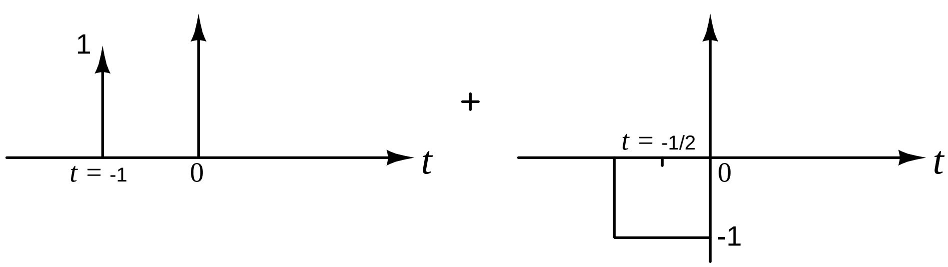
$= e^{j2\pi f} - \frac{(e^{j\pi f}-e^{-j\pi f})e^{j\pi f}}{j2\pi f}$

$= e^{j2\pi f} - sinc(f)e^{j2\pi f \times \frac{1}{2}}$

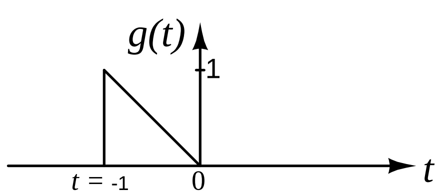
This means that the derivative of $g(t)$ has an impulse at $t = -1$ and also the function $\{-rect(\frac{t-1/2}{1})\}$.

Let us denote the derivative of $g(t)$ as $g^{'}(t)$

$\therefore g^{'}(t) \Rightarrow$



$\therefore g(t) \Rightarrow$



$g_1(t) = \frac{dg(t)}{dt} = \delta(t+1) - rect(\frac{t-1/2}{1})$

$j2\pi f G(f) = G_1(f) = e^{j2\pi f} - sinc(f)e^{j2\pi f \times \frac{1}{2}}$

$= e^{j2\pi f} - (\frac{\sin \pi f}{\pi f})e^{j\pi f}$

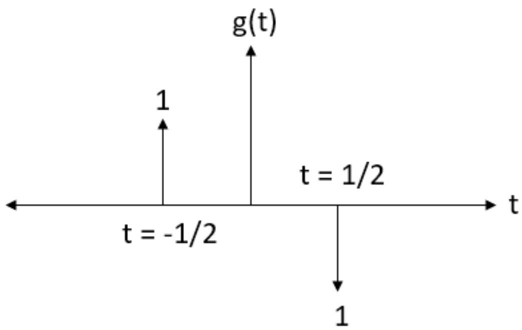
$= e^{j2\pi f} - \frac{(e^{j\pi f}-e^{-j\pi f})}{j2\pi f}e^{j\pi f}$

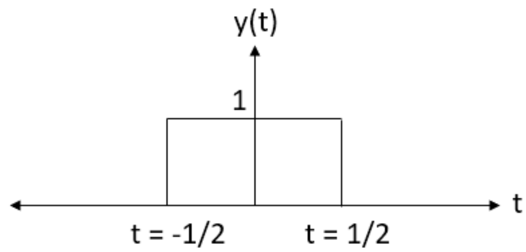
$= \frac{j2\pi f e^{j2\pi f}+1-e^{j2\pi f}}{j2\pi f}$

$\therefore G(f) = \frac{j2\pi f e^{j2\pi f}-e^{j2\pi f}+1}{-4\pi^2 f^2}$

$G(f) = \frac{1}{(2\pi f)^2}(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1)$

4) a) Integration w.r.t. time yields $y(t) = \int_{-\infty}^{t} g(\tau)d\tau$





b) $y(t) = \int_{-\infty}^{t} g(\tau)d\tau$

$$\begin{aligned}
\therefore y(t) \longleftrightarrow Y(f) &= \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f) \\
&= \frac{1}{j2\pi f}\left[e^{-j2\pi f\left(\frac{-1}{2}\right)} - e^{-j2\pi f\left(\frac{1}{2}\right)}\right] \qquad \because (G(0) = 0) \\
&= \frac{1}{j2\pi f}\left[e^{j\pi f} - e^{-j\pi f}\right] \\
&= \frac{sin(\pi f)}{\pi f} = sinc(f)
\end{aligned}$$

5)

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} G_n e^{jn2\pi f_o t} \qquad f_o = \frac{1}{T_o}$$

$$\begin{aligned}
\therefore g_{T_0}(t) \longleftrightarrow &\sum_{n=-\infty}^{\infty} \mathcal{F}\left\{G_n e^{jn2\pi f_o t}\right\} \\
&= \sum_{n=-\infty}^{\infty} G_n \delta(f - nf_o)
\end{aligned}$$

Now,

$$G_n = \frac{1}{T_o}\int_{T_o} g_{T_0}(t)e^{-jn2\pi f_o t}dt$$
$$= f_o\int_{-\infty}^{+\infty} g(t)e^{-jnf_o 2\pi t}dt$$
$$= f_o G(nf_o)$$

where, $g(t) \longleftrightarrow G(f)$

$$\therefore g_{T_0}(t) = f_o\sum_{n=-\infty}^{\infty} G(nf_o)\delta(f - nf_o)$$

6) a) single-unit RC circuit:

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

$$|H(f)| = \frac{1}{[1 + (2\pi f\tau_0)^2]^{1/2}} \qquad where\ \ \tau_0 = RC$$

Cascade of N-sections

$$|H(f)| = [1 + (2\pi f\tau_0)^2]^{-N/2}$$

b) $\tau_0^2 = T^2/4\pi^2 N$

$$|H(f)|_{N-section} = [1 + \frac{1}{N}(fT)^2]^{-N/2}$$

$$|H(f)|_{N\to\infty} = \lim_{N\to\infty}[1 + \frac{1}{N}(fT)^2]^{-N/2} = exp(-\frac{N}{2}\frac{1}{N}(fT)^2) = exp(-\frac{f^2T^2}{2})$$

7) $y(t) = \int\limits_{t-T}^{t} g(\tau)d\tau$

Let $g(t) \leftrightarrow G(f)$, i.e., $g(t) = \int\limits_{-\infty}^{\infty} G(f)exp(j2\pi ft)df)$

$$y(t) = \int\limits_{t-T}^{t}[\int\limits_{-\infty}^{\infty} G(f)exp(j2\pi ft)df]d\tau$$

Interchanging the order of integration

$$y(t) = \int\limits_{-\infty}^{\infty} G(f)[\int\limits_{t-T}^{t} exp(j2\pi ft)d\tau]df$$

$$y(t) = \int Tsinc(fT)exp(-j2\pi fT)G(f)e^{+j2\pi fT}df$$

$$\Rightarrow H(f) = Tsinc(fT)exp(-j\pi fT)$$

8) a)

$$g(t) = Arect\left(\frac{t-t_0}{T}\right)$$

$$g(t) \longleftrightarrow G(f) = AT\,sinc(Tf)e^{-j2\pi ft_0}$$

$$\therefore E_g(f) = |G(f)|^2 = (AT)^2 sinc^2(fT)$$

$$\underset{(ESD)}{E_g(f)} \longleftrightarrow R_g(\tau) = (AT)^2\left[\frac{1}{T}rect\left(\frac{t}{T}\right) * \frac{1}{T}rect\left(\frac{t}{T}\right)\right] = A^2T\Delta\left(\frac{t}{2T}\right)$$

$$Energy = R_g(0) = A^2T$$

b)

$$g(t) = Ae^{-\alpha t}u(t)$$

$$g(t) \longleftrightarrow G(f) = \frac{A}{\alpha + j2\pi f}$$

$$\underset{(ESD)}{E_g(f)} = |G(f)|^2 = \frac{A^2}{\alpha^2 + (2\pi f)^2} \longleftrightarrow R_g(\tau) = \frac{A^2}{2\alpha}e^{-\alpha|\tau|}$$

$$\therefore Energy = R_g(0) = \frac{A^2}{2\alpha}$$

c)

$$g(t) = A_0 + A_1 sin(2\pi f_0 t + \phi) = A_0 + \frac{A_1}{2}e^{j\phi}e^{2\pi f_0 t} + \frac{A_1}{2}e^{-j\phi}e^{-2\pi f_0 t}$$

$$g(t) \Longleftrightarrow G(f) = A_0\delta(f) + \frac{A_1}{2}e^{j\phi}\delta(f - f_0) + \frac{A_1}{2}e^{-j\phi}\delta(f + f_0)$$

$$PSD \equiv S_g(f) = A_0^2\delta(f) + \frac{A_1^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$$

$$R_g(\tau) = A_0^2 + \frac{A_1^2}{2}cos(2\pi f_0\tau),\; P_g \equiv Power = R_g(0) = A_0^2 + \frac{A_1^2}{2}$$

9) The autocorrelation of g(t) = A u(t) = $R_g(t)$

$$R_g(t) = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 u(t)u(t-\tau)dt$$

$$where\ u(t)u(t-\tau) = \begin{cases} 0 & t < \tau \\ 1 & t > \tau \end{cases}$$

Consider $0 \le \tau \le \frac{T}{2}$ so

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)u(t-\tau)dt = \int_{\tau}^{\frac{T}{2}} dt = (\frac{T}{2} - \tau)$$

$$\therefore R_g(\tau) = \lim_{T\to\infty} \frac{A^2}{T}(\frac{T}{2} - \tau) = \frac{A^2}{2} \; for \; all \; \tau$$

The power of the signal and the PSD are given by- $P_g = R_g(0) = \frac{A^2}{2} \longleftrightarrow S_g(f) = \frac{A^2}{2}\delta(f)$

10)

$$g(t) = e^{-\alpha t}u(t)$$

$$g(t) \longleftrightarrow G(f) = \frac{1}{\alpha + j2\pi f} \implies \underset{(ESD)}{E_g(f)} = |G(f)|^2 = \frac{1}{\alpha^2 + (j2\pi f)^2}$$

$$\therefore Energy = \frac{1}{2\alpha}$$

$$\int_{-B}^{B} \frac{df}{\alpha^2 + (j2\pi f)^2} = \frac{0.95}{2\alpha}$$

$$\therefore \frac{0.95}{2\alpha} = \frac{1}{2\pi\alpha}\tan^{-1}\frac{2\pi f}{\alpha}\Big|_{-B}^{B} = \frac{1}{\pi\alpha}\tan^{-1}\frac{2\pi B}{\alpha}$$

$$\implies 2\pi B = 12.7\alpha \; Hz$$

$$\implies B = 2.02\alpha \; Hz$$

11) (a) The absolute bandwidth is b.

(b) The 3-dB bandwidth, say f_{3dB}, is the solution of:

$$\frac{b - f_{3dB}}{b - a} = \frac{1}{2}.$$

So $f_{3dB} = \frac{a+b}{2}$.

(c) Let f_{eq} be the equivalent noise bandwidth. Then:

$$f_{eq}(1) = \int_0^{\infty} H(f)df = \frac{a+b}{2}.$$

So $f_{eq} = \frac{a+b}{2}$.

(d) Let f_{RMS} be the RMS bandwidth. Then:

$$f_{RMS} = \sqrt{\frac{\int_0^{\infty} f^2 H(f)df}{\int_0^{\infty} H(f)df}}.$$

So:

$$f_{RMS} = \sqrt{\frac{\frac{a^3}{3} + \frac{b(a^2+ab+b^2)}{3} - \frac{(a+b)(a^2+b^2)}{4}}{\frac{a+b}{2}}}.$$

(e) The null-to-null bandwidth is b.

(f) Let f_b be the 50 dB bounded spectrum bandwidth. Then:

$$10\log_{10}\left(\frac{1}{H(f_b)}\right) = 50.$$

So $H(f_b) = 10^{-5} = \frac{b-f_b}{b-a}$. Hence:

$$f_b = b - 10^{-5}(b-a).$$

(g) The power of the signal in $[0,\infty)$ is $\int_0^\infty H(f)df = \frac{a+b}{2}$. Let the power bandwidth be f_p. Then:

$$a + \int_a^{f_p} \frac{b-f}{b-a} df = 0.99\left(\frac{a+b}{2}\right).$$

Solving the above, we get:

$$f_p = b - \sqrt{b^2 - [2ab - a^2 + (b-a)(0.99b - 1.01a)]}.$$