Sampling and Pulse Modulation

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Introduction

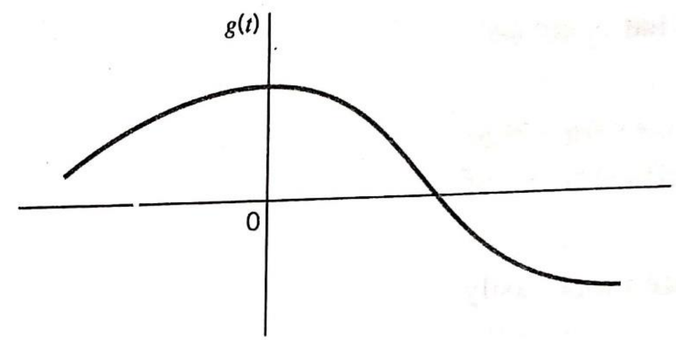
- Recall: digital communication systems have several advantages over analog communication systems
  - ❑ former have replaced or are replacing latter in most contexts, e.g., cellular networks, TV
- "Analog communication" and "digital communication":
  - ❑ in practice, *all* communication is via continuous signals and hence analog in nature
  - ❑ the message signal that is to be transmitted is either analog or digital
  - ❑ E.g., if the source is speech, then:
    - o In analog communication, it is directly used to modulate a high-frequency carrier signal
    - o In digital communication, it is sampled and quantized to obtain a bit stream, which is then used to modulate a high-frequency carrier signal
- First step in digital transmission of analog source (e.g., speech, music) is conversion of source to digital representation
- We now study:
  - ❑ this analog to digital conversion
  - ❑ and representation of the analog information as a sequence of pulses

The Sampling Process

- Sampling is used to convert an analog signal to sequence of samples that are usually spaced uniformly in time
- Sampling rate must be chosen carefully, so that:
  - the sequence of samples uniquely defines the original analog signal
- Sampling theorem tells us how to choose sampling rate
- We now briefly review the sampling process and prove the sampling theorem

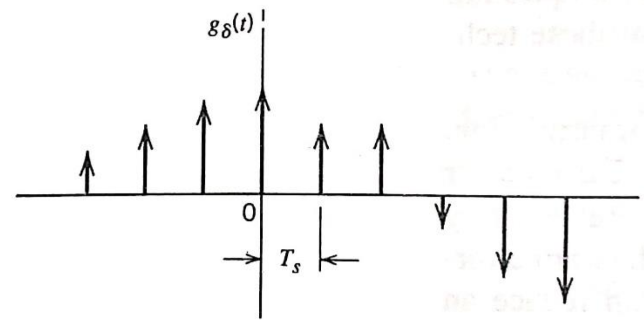
The Sampling Process (contd.)

- Consider an arbitrary signal $g(t)$ of finite energy, which is specified for all time t
- Suppose $g(t)$ sampled at uniform rate:
  - ❑ once every T_s seconds
- Then we obtain an infinite sequence of samples spaced T_s seconds apart:
  - ❑ denoted by $\{g(nT_s)\}$, where n takes on all possible integer values
- We refer to:
  - ❑ T_s as "*sampling period*"
  - ❑ and $f_s = 1/T_s$ as "*sampling rate*"
- Let:
  - ❑ $g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - n\,T_s)$
- $g(t)$ and $g_\delta(t)$ shown in fig.



(a)

- We will show that Fourier transform of sampled signal $g_\delta(t)$ is:
  - *1)* $G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$
  - ❑ where $G(f)$ is Fourier transform of $g(t)$
- 1) shows that *process of uniformly sampling a signal $g(t)$ results in a periodic spectrum with period equal to the sampling rate*



(b)

Ref: "Communication Systems" by Haykin and Moher, 5th ed

Proof of the Claim $G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$

- First, consider a periodic signal $f_{T_0}(t)$ of period T_0
- We can represent it using Fourier series:
  - ❑ $f_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_0 t)$, where
  - ❑ $c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_{T_0}(t)\exp(-j2\pi n f_0 t)dt$ and $f_0 = \frac{1}{T_0}$
- Let $f(t) = \begin{cases} f_{T_0}(t), & -\frac{T_0}{2} \le t \le \frac{T_0}{2}, \\ 0, & \text{else.} \end{cases}$
  - ❑ So $f_{T_0}(t) = \sum_{m=-\infty}^{\infty} f(t - mT_0)$
- Hence, c_n:
  - ❑ $f_0 F(nf_0)$, where
  - ❑ $F(f)$ is the Fourier transform of $f(t)$
- Thus:
  - ❑ $\sum_{m=-\infty}^{\infty} f(t - mT_0) = f_0 \sum_{n=-\infty}^{\infty} F(nf_0)\exp(j2\pi n f_0 t)$

1) So Fourier transform of $\sum_{m=-\infty}^{\infty} f(t - mT_0)$ is:
   - ❑ $f_0 \sum_{n=-\infty}^{\infty} F(nf_0)\delta(f - nf_0)$

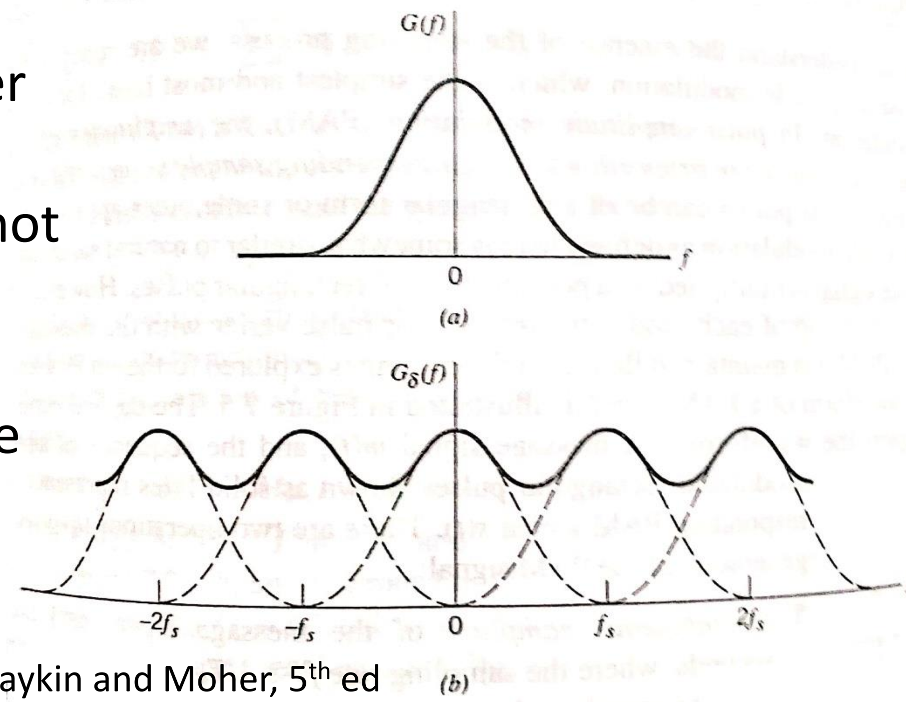
- Now, in the sampling context: $g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - n\,T_s)$
- Fourier transform of $g_\delta(t)$ is $G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$ by:
  - ❑ Duality theorem and the fact that the $\delta(.)$ function is an even function

The Sampling Process (contd.)

- Recall:
  1) $g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$
  2) $G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$
- Taking Fourier transforms on both sides of 1), we get:
  3) $G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)\exp(-j2\pi nfT_s)$
  - This relation is called:
    - *discrete-time Fourier transform*
  - Can be viewed as Fourier series representation of the periodic frequency function $G_\delta(f)$
- Next, suppose the signal $g(t)$ is strictly bandlimited:
  - $G(f) = 0$ for $|f| \geq W$
- Also, suppose we choose the sampling period $T_s = \frac{1}{2W}$
- Then by 3), we get:
  4) $G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(\frac{-j\pi nf}{W}\right)$
- Also, by 2), we get:
  5) $G(f) = \frac{1}{2W} G_\delta(f)$, for $-W < f < W$
- Substituting 4) into 5), we get:
  6) $G(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(\frac{-j\pi nf}{W}\right)$, for $-W < f < W$
- 6) shows that if sample values $g\left(\frac{n}{2W}\right)$ of signal $g(t)$ are specified for all n, then signal $g(t)$ is completely determined for all values of t
- Taking inverse Fourier transform of 6), we get:
  7) $g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\mathrm{sinc}(2Wt - n)$ for $t \in (-\infty, \infty)$
- Equation 7) provides an *interpolation formula* for reconstructing the original signal $g(t)$ from the sequence of sample values $\left\{g\left(\frac{n}{2W}\right)\right\}$
- Thus, we have derived the "*Sampling Theorem*", which states the following:
  - A band-limited signal which only has frequency components in the range $-W < f < W$ is completely described by specifying the values of the signal at instants of time separated by $1/2W$ seconds
  - Such a signal can be completely recovered from a knowledge of its samples taken at the rate of $2W$ samples per second
- Sampling rate of $2W$ samples per second, for a signal bandwidth of W Hz, called *Nyquist rate*; its reciprocal $\frac{1}{2W}$ called *Nyquist interval*

Aliasing

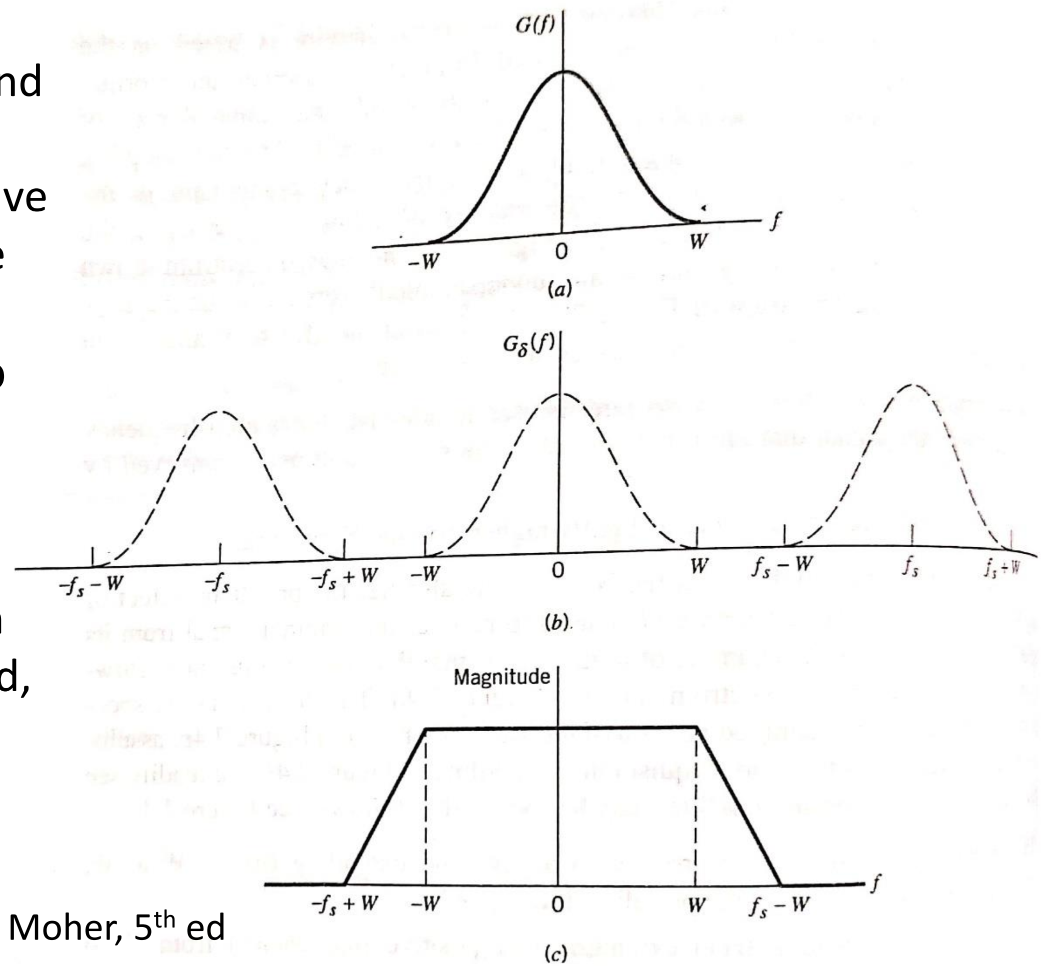
- In above derivation of sampling theorem, we assumed that signal $g(t)$ is strictly band-limited
- However, in practice, an information-bearing signal is *not* strictly band-limited
  - so some *undersampling* occurs
- So sampling process produces some "*aliasing*" as shown in fig
- To combat the effects of aliasing in practice:
  - Prior to sampling, a low-pass filter used to attenuate those high-frequency components that are not essential to information being conveyed by signal
  - Filtered signal is sampled at a rate slightly higher than Nyquist rate



Ref: "Communication Systems" by Haykin and Moher, 5th ed

Aliasing (contd.)

- What is the benefit of using a sampling rate that is slightly higher than (not equal to) Nyquist rate?
  - ❑ Eases the design of the reconstruction filter used to recover original signal from its sampled version
- E.g., suppose a message signal with bandwidth W is sampled at rate $f_s > 2W$
- Then reconstruction filter:
  - ❑ can be low-pass filter with a passband extending from $-W$ to W and
  - ❑ transition band extending (for positive frequencies) from W to $f_s - W$ (see fig)
- Thus, reconstruction filter allowed to have transition band of width $f_s - 2W > 0$
  - ❑ In contrast, if $f_s = 2W$, then ideal reconstruction filter with zero width of transition band would be required, which is not practically realizable



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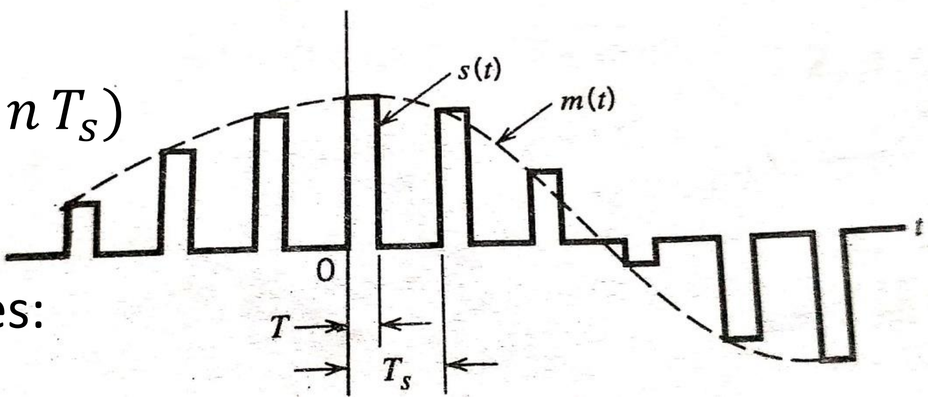
Practical Sampling

- So far, we have considered ideal sampling using an impulse pulse train
- But this sampling process is physically unrealizable
- So next, we consider a practical implementation of sampling
- Called "*Pulse Amplitude Modulation*"

Pulse Amplitude Modulation (PAM)

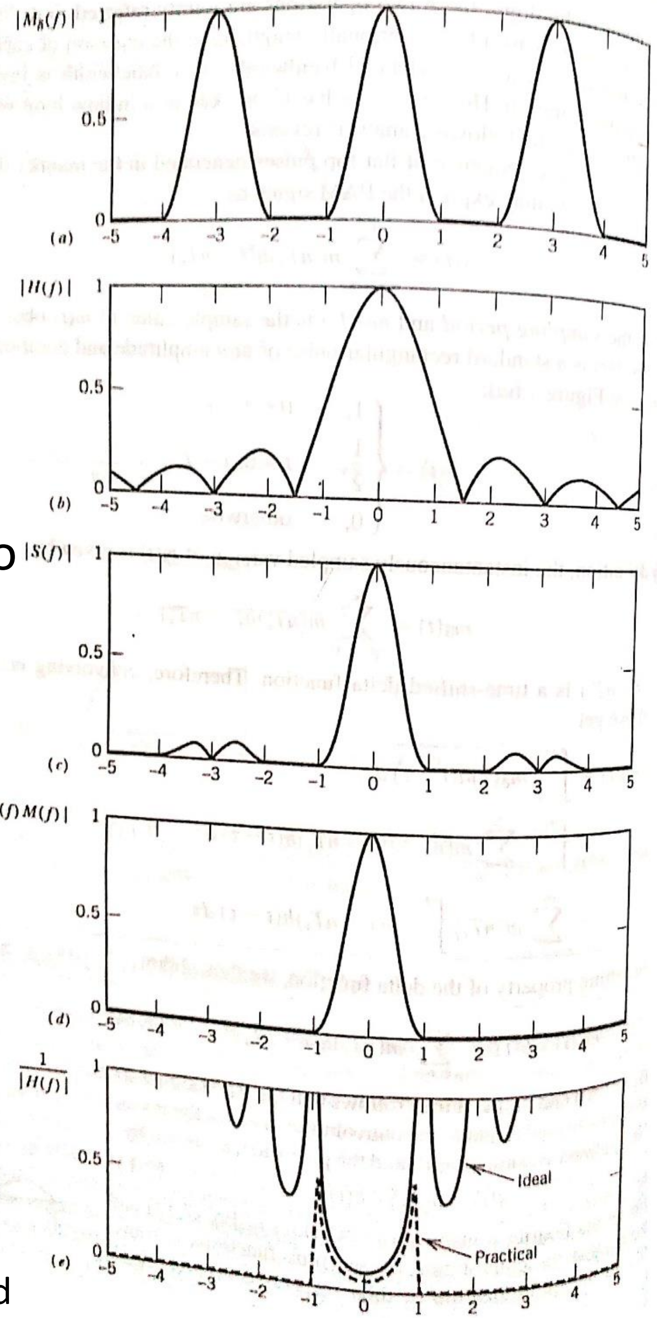
- In PAM, amplitudes of regularly spaced pulses varied in proportion to corresponding sample values of a continuous message signal $m(t)$ as shown in fig.
  - ❑ $s(t)$ is PAM signal obtained from $m(t)$
- PAM signal $s(t)$ can be generated by following operations:
  1) Instantaneous sampling of message signal $m(t)$ every T_s seconds, where sampling rate $f_s = 1/T_s$ chosen in accordance with sampling theorem
  2) Lengthening duration of each sample to some constant value T
- Above two operations jointly referred to as "*sample and hold*"
- Reason for lengthening duration of each sample (step 2):
  - ❑ To avoid use of excessive channel bandwidth
- PAM signal $s(t)$ can be expressed as:
  - ❑ $s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - n\,T_s)$,
  - ❑ where $h(t) = \begin{cases} 1, & 0 \le t \le T, \\ 0, & \text{else.} \end{cases}$
- Recall: $m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - n\,T_s)$
- $s(t)$ in terms of $m_\delta(t)$ and $h(t)$:
  - ❑ $m_\delta(t) * h(t)$
- Taking Fourier transforms on both sides:
  - $S(f) = M_\delta(f)H(f)$



Ref: "Communication Systems" by Haykin and Moher, 5th ed

Pulse Amplitude Modulation (PAM) (contd.)

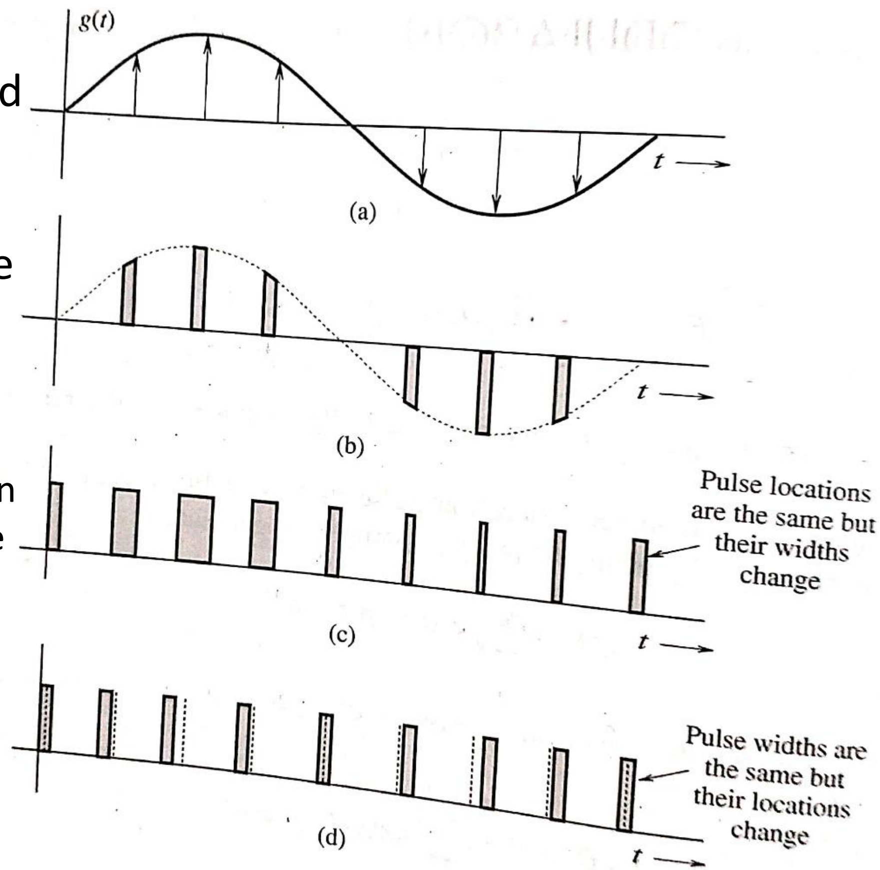
- Recall:
  - ❑ $s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - n\,T_s) = m_\delta(t) * h(t)$
  - ❑ $S(f) = M_\delta(f)H(f)$
  - ❑ $M_\delta(f) = f_s \sum_{m=-\infty}^{\infty} M(f - mf_s)$
- So $S(f)$:
  - ❑ $f_s \sum_{m=-\infty}^{\infty} M(f - mf_s)\,H(f)$
- Given a PAM signal $s(t)$, how can we recover message signal $m(t)$?
- Assuming that sampling rate exceeds Nyquist rate, i.e., $f_s > 2W$, we pass $s(t)$ through low-pass filter to get signal with Fourier transform $M(f)H(f)$
- Recall: $h(t) = \begin{cases} 1, & 0 \le t \le T, \\ 0, & \text{else.} \end{cases}$
- So $H(f)$:
  - ❑ $T\text{sinc}(fT)e^{-j\pi fT}$
- We can recover $m(t)$ by:
  - ❑ passing the above signal with Fourier transform $M(f)H(f)$ through filter with amplitude response $\frac{1}{|H(f)|} = \frac{1}{|T\text{sinc}(fT)|}$
- Fig. shows relevant amplitude spectra

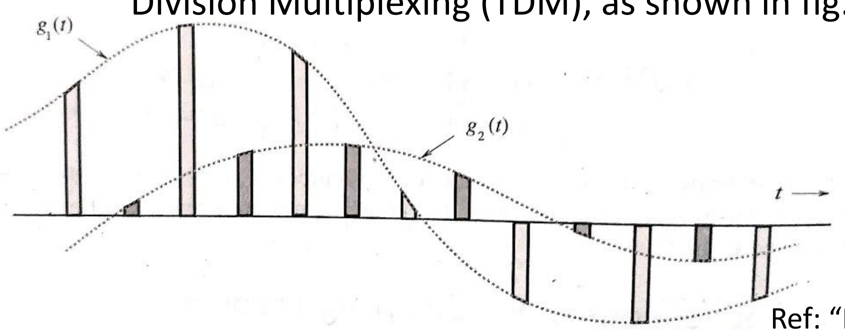


Ref: “Communication Systems” by Haykin and Moher, 5th ed

Communication Using Pulse Modulation

- Suppose a continuous-time message signal $g(t)$ needs to be sent over a baseband channel
- In "*pulse modulation*":
  - ❑ $g(t)$ is sampled
  - ❑ sample values are used to modify certain parameters of a periodic pulse train
- Fig. shows:
  - ❑ PAM signal, in which pulse amplitudes varied
  - ❑ "*Pulse Width Modulation (PWM)*", in which pulse widths varied
  - ❑ "*Pulse Position Modulation (PPM)*", in which pulse positions varied
- In all the above cases, instead of sending $g(t)$, we transmit the corresponding pulse modulated signal over channel
- Recall: previous slide shows that bandwidth of PAM signal is larger than bandwidth of message signal
- Advantage of pulse modulation over sending message signal $g(t)$ itself:
  - ❑ Pulse modulation allows simultaneous transmission of several signals on a time-sharing basis, i.e., Time Division Multiplexing (TDM), as shown in fig.





Ref: "Modern Digital and Analog Comm. Systems" by B.P. Lathi and Z. Ding, 4th ed