1) Say the boolean formula is over n-variables.
Then, we can create the corresponding boolean function, which is just a truth table, by replacing the string 'x_i' by the value of the variable $x_1 \in B_2$, and so on for all x_2 to x_n. Hence, for each possible input to the function, we have an expression that evaluates to 0 or 1.
Hence, we can form a boolean function from a boolean formula.

2) We simply evaluate the formula:

x_1 x_2 x_3	$(x_1 \cdot (x_2 + x_3)) + (\bar{x}_1 \cdot (\bar{x}_2 + \bar{x}_3))$
0 0 0	$0 \cdot (0+0) + 1 \cdot (1+1) = 1$
0 0 1	$0 \cdot (0+1) + 1 \cdot (1+0) = 1$
0 1 0	$0 \cdot (1+0) + 1 \cdot (0+1) = 1$
0 1 1	$0 \cdot (1+1) + 1 \cdot (0+0) = 0$
1 0 0	$1 \cdot (0+0) + 0 \cdot (1+1) = 0$
1 0 1	$1 \cdot (0+1) + 0 \cdot (1+0) = 1$
1 1 0	$1 \cdot (1+0) + 0 \cdot (0+1) = 1$
1 1 1	$1 \cdot (1+1) + 0 \cdot (0+0) = 1$

3) From (1), we know that we can get a boolean function from a boolean formula. This function maps B_2^n (boolean algebra) to B_2. Consider the collection of all elements of B_2^n for which our bolean function evaluates to 1. This collection has a corresponding subset in 2^u, since boolean algebra is equivalent to set algebra.

$\therefore$ formula $\rightarrow$ function $\rightarrow$ boolean algebra set $\rightarrow$ subset

4) We construct boolean function for LHS and RHS and verify that they are equivalent

(a)

x_1	x_2	x_3	$x_1 \cdot (x_2 + x_3)$	$x_1 \cdot x_2 + x_1 \cdot x_3$
0	0	0	$0 \cdot (0+0) = 0$	$0 \cdot 0 + 0 \cdot 0 = 0$
0	0	1	$0 \cdot (0+1) = 0$	$0 \cdot 0 + 0 \cdot 1 = 0$
0	1	0	$0 \cdot (1+0) = 0$	$0 \cdot 1 + 0 \cdot 0 = 0$
0	1	1	$0 \cdot (1+1) = 0$	$0 \cdot 1 + 0 \cdot 1 = 0$
1	0	0	$1 \cdot (0+0) = 0$	$1 \cdot 0 + 1 \cdot 0 = 0$
1	0	1	$1 \cdot (0+1) = 1$	$1 \cdot 0 + 1 \cdot 1 = 1$
1	1	0	$1 \cdot (1+0) = 1$	$1 \cdot 1 + 1 \cdot 0 = 1$
1	1	1	$1 \cdot (1+1) = 1$	$1 \cdot 1 + 1 \cdot 1 = 1$

b)

x_1	x_2	x_3	$x_1 + (x_2 \cdot x_3)$	$(x_1 + x_2) \cdot (x_1 + x_3)$
0	0	0	$0 + (0 \cdot 0) = 0$	$(0+0) \cdot (0+0) = 0$
0	0	1	$0 + (0 \cdot 1) = 0$	$(0+0) \cdot (0+1) = 0$
0	1	0	$0 + (1 \cdot 0) = 0$	$(0+1) \cdot (0+0) = 0$
0	1	1	$0 + (1 \cdot 1) = 1$	$(0+1) \cdot (0+1) = 1$
1	0	0	$1 + (0 \cdot 0) = 1$	$(1+0) \cdot (1+0) = 1$
1	0	1	$1 + (0 \cdot 1) = 1$	$(1+0) \cdot (1+1) = 1$
1	1	0	$1 + (1 \cdot 0) = 1$	$(1+1) \cdot (1+0) = 1$
1	1	1	$1 + (1 \cdot 1) = 1$	$(1+1) \cdot (1+1) = 1$

4c)

x_1 x_2	$\overline{x_1 + x_2}$	$\overline{x_1} \cdot \overline{x_2}$
0 0	$\overline{0+0} = 1$	$\bar{0} \cdot \bar{0} = 1$
0 1	$\overline{0+1} = 0$	$\bar{0} \cdot \bar{1} = 0$
1 0	$\overline{1+0} = 0$	$\bar{1} \cdot \bar{0} = 0$
1 1	$\overline{1+1} = 0$	$\bar{1} \cdot \bar{1} = 0$

d)

x_1 x_2	$x_1 \cdot x_2 + x_1 \cdot \overline{x_2}$	x_1
0 0	$0 \cdot 0 + 0 \cdot \bar{0} = 0$	0
0 1	$0 \cdot 1 + 0 \cdot \bar{1} = 0$	0
1 0	$1 \cdot 0 + 1 \cdot \bar{0} = 1$	1
1 1	$1 \cdot 1 + 1 \cdot \bar{1} = 1$	1

5) Each element ~~x~~ in the subset represents the vector $(x_1, x_2, x_3, x_4) \in B_2^4$, and each $x_i \in B_2$

(a) x_1 : $\{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$

(b) $x_1 \cdot x_2$: $\{1100, 1101, 1110, 1111\}$

(c) $x_1 \cdot x_2 \cdot x_3$: $\{1110, 1111\}$

(d) $x_1 \cdot x_2 \cdot x_3 \cdot x_4$: $\{1111\}$

(e) $x_1 + x_2$: $\{0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011,$
$1100, 1101, 1110, 1111\}$

(f) $x_1 + x_2 + x_3 + x_4$:
$\{0001, 0010, 0011, 0100, 0101, 0110, 0111,$
$1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$

6)

$f_{x_1} = f(1, x_2, x_3, x_4)$

$f_{\bar{x}_1} = f(0, x_2, x_3, x_4)$

(a) $f_{x_1} = 1$, $f_{\bar{x}_1} = 0$

$\therefore f = x_1 \cdot 1 + \bar{x}_1 \cdot 0$

(c) $f_{x_1} = x_2 \cdot x_3$, $f_{\bar{x}_1} = 0$

$\therefore f = x_1 \cdot x_2 \cdot x_3 + \bar{x}_1 \cdot 0$

(b) $f_{x_1} = x_2$, $f_{\bar{x}_1} = 0$

$\therefore f = x_1 \cdot x_2 + \bar{x}_1 \cdot 0$

(d) $f_{x_1} = x_2 \cdot x_3 \cdot x_4$, $f_{\bar{x}_1} = 0$

$\therefore f = x_1 \cdot x_2 + x_3 \cdot x_4 + \bar{x}_1 \cdot 0$

(e) $f_{x_1} = 1 + x_2 = 1$

$f_{\bar{x}_1} = 0 + x_2 = x_2$

$\therefore f = x_1 \cdot 1 + \bar{x}_1 \cdot x_2$

(f) $f_{x_1} = 1 + x_2 + x_3 + x_4 = 1$

$f_{\bar{x}_1} = 0 + x_2 + x_3 + x_4 = x_2 + x_3 + x_4$

$\therefore f = x_1 \cdot 1 + \bar{x}_1 \cdot (x_2 + x_3 + x_4)$

7) $\exists x_1(\bar{f}) = \bar{f}_{x_1} + \bar{f}_{\bar{x}_1}$

$= \bar{f}(1, \dots) + \bar{f}(0, \dots)$

Since $\bar{f}_{x_1}$ and $\bar{f}_{\bar{x}_1}$ can me mapped to subsets, we can apply demorgan's ~~law~~ theorem as:

$$\overline{\exists x_1(\bar{f})} = \overline{\bar{f}_{x_1} + \bar{f}_{\bar{x}_1}} = \overline{\bar{f}_{x_1}} \cdot \overline{\bar{f}_{\bar{x}_1}}$$

$$= f_{x_1} \cdot f_{\bar{x}_1} = \forall x_1(f)$$

($\because \bar{\bar{x}} = x$ over boolean algebra

$\Rightarrow \bar{\bar{x}} = x$ over boolean formulae

$\Rightarrow \bar{\bar{x}} = x$ over boolean functions)

Similarly:

$$\overline{\forall x_1(\bar{f})} = \overline{\bar{f}_{x_1} \cdot \bar{f}_{\bar{x}_1}} = \overline{\bar{f}_{x_1}} + \overline{\bar{f}_{\bar{x}_1}}$$

$$= f_{x_1} + f_{\bar{x}_1}$$

$$= \exists x_1(f)$$

Hence proved.