1) Say there are two zeros $0_1, 0_2$, $0_1 \neq 0_2$.

$$\therefore \quad 0_1 + 0_2 = 0_2 \qquad (\because \; 0 + x = x)$$
$$0_2 + 0_1 = 0_1$$

But $+$ is commutative.

$\therefore \; 0_1 + 0_2 = 0_2 + 0_1 \Rightarrow 0_2 = 0_1$

$\therefore$ contradiction.

$\therefore$ There can't be two zeros.

Similarly:

$1_1 = 1_1 \cdot 1_2 = 1_2 \cdot 1_1 = 1_2$ gives contradiction for two ones.

2) $0 = a \cdot \bar{a} = a \cdot (0 + \bar{a})$

$= a \cdot 0 + a \cdot \bar{a}$

$= a \cdot 0 + 0$

$= a \cdot 0$

3) $1 = a + \bar{a} = a + (1 \cdot \bar{a})$

$= (a+1) \cdot (a + \bar{a})$

$= (a+1) \cdot 1$

$= a + 1$

$$
\begin{aligned}
4)\quad a\cdot a &= a\cdot a + 0 \\
&= a\cdot a + a\cdot\bar{a} \\
&= a\cdot(a+\bar{a}) \\
&= a\cdot 1 \\
&= a
\end{aligned}
\left.\vphantom{\begin{aligned} a \\ a \end{aligned}}\right\} \text{distributivity}
$$

$$
a + a = (a+a)\cdot 1 = (a+a)\cdot(a+\bar{a}) \overset{\text{distributivity}}{=} a + a\cdot\bar{a} = a + 0 = a
$$

5) To test that $\bar{a}+\bar{b}$ satisfies complement properties of $a \cdot b$

$$a \cdot b \cdot (\bar{a}+\bar{b}) = a \cdot b \cdot \bar{a} + a \cdot b \cdot \bar{b}$$
$$= a \cdot \bar{a} \cdot b + a \cdot b \cdot \bar{b}$$
$$= 0 \cdot b + a \cdot 0$$
$$= 0 + 0$$
$$= 0$$

$$a \cdot b + (\bar{a}+\bar{b}) = a \cdot b + \bar{a} + \bar{b} = a \cdot b + \bar{a}(b+\bar{b}) + \bar{b}(a+\bar{a})$$
$$= a \cdot b + \bar{a} \cdot b + \bar{a} \cdot \bar{b} + \bar{b} \cdot a + \bar{b} \cdot \bar{a}$$
$$= (a+\bar{a}) \cdot b + (\bar{a}+a) \cdot \bar{b} + \bar{b} \cdot \bar{a}$$
$$= b + \bar{b} + \bar{b} \cdot \bar{a}$$
$$= 1 + \bar{b} \cdot \bar{a}$$
$$= 1$$

6) Say $0=1$

~~$a \cdot 1 = a$~~

$\because$ there are atleast 2 elements, we have some $a \neq 0$.

Now $a \cdot 1 = a$ (by axiom)

But $1=0 \Rightarrow a \cdot 1 = a \cdot 0 = 0$ (by Q2)

$\therefore a = a \cdot 1 = a \cdot 0 = 0$

$\therefore a = 0$

$\therefore$ contradiction.

7) all B where $|\Omega| \geq 2$ has atleast one atom:

$\rightarrow$ $\exists$ $b \neq 0$.

Say there is no non-zero element lesser than b.

TPT $\forall c$, $b \cdot c = 0 / b$.

Say $b \cdot c = d \neq 0/b$

$\therefore d \cdot b = b \cdot c \cdot b = b \cdot b \cdot c = b \cdot c = d$.

$\therefore d \cdot b = d \Rightarrow d \leq b$

But $d \neq b \Rightarrow d < b$. and $d \neq 0$.

$\therefore$ contradiction.

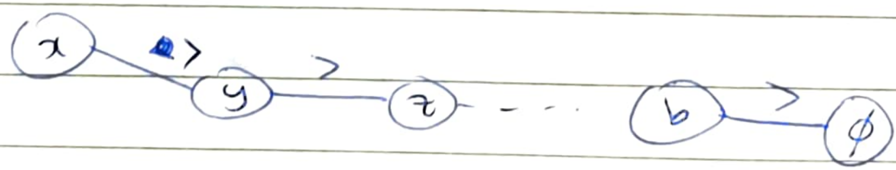
$\therefore$ if $\exists$ b st. there is no non-zero element lesser than it, then b is an atom.

8)

only non-zero elements.

This b can be found by starting with any element x, and choosing an element y from those elements $< x$. If y doesn't exist, then x is an atom. Else, we can repeat for y.

$\because$ Ω is finite, we will eventually get an atom.



9) $a \leq b,\ c \leq b. \Rightarrow a \cdot b = a \ \& \ c \cdot b = c$

$(a+c) \cdot b = a \cdot b + c \cdot b = a + c$

$\therefore (a+c) \leq b$

10) $a \le b \Rightarrow a+b=b$ and $a \cdot b = a$

$\bar{b} \cdot \bar{a} = \overline{(b+a)}$ (demorgan's)

$= \bar{b}$ ($\because a \le b$)

$\therefore \bar{b} \le \bar{a}$

11) $a \leq b \Rightarrow a \cdot b = a$

$a \leq c \Rightarrow a \cdot c = a$

$a \cdot (b \cdot c) = (a \cdot b) \cdot c = a \cdot c = a$

$\therefore a \leq b \cdot c$

12) $\bar{a} \cdot b + a \cdot b$

$= b \cdot \bar{a} + b \cdot a$ commutable

$= b \cdot (\bar{a} + a)$ distributable

$= b \cdot 1$ axiom

$= b$

13) $a \cdot \bar{b} \cdot c + a \cdot \bar{b} \cdot \bar{c}$

$= a \cdot \bar{b} (c + \bar{c}) = a \cdot \bar{b} \cdot 1 = a \cdot \bar{b}$

$\therefore a \cdot \bar{b} \cdot c + a \cdot \bar{b} \cdot \bar{c} + \bar{a} \cdot \bar{b} = a \cdot \bar{b} + \bar{a} \cdot \bar{b} = \bar{b}$

(from q 12)