# Mid-sem: CS 754, Advanced Image Processing, 24th February

**Instructions:** There are 120 minutes for this exam. This exam is worth 10% of your final grade. Attempt all questions. Write brief answers - lengthy answers are not expected. Each question carries 10 points.

1. State the advantages and disadvantages of mutual coherence over the restricted isometry property (RIP) for a sensing matrix in compressed sensing.

2. For successful reconstruction of a $k$-sparse signal $\boldsymbol{\theta} \in \mathbb{R}^n$ from compressive measurements $\boldsymbol{y} = \boldsymbol{A\theta}, \boldsymbol{A} \in \mathbb{R}^{m \times n}, m \ll n$, what is the smallest number of measurements required in terms of $k$ and/or $n$ using $L_0$ minimization? If you switched over to $L_1$ minimization, what is the smallest number of measurements required in terms of $k$ and/or $n$?

3. We have seen the following theorem (Theorem 3) in class: Consider compressive measurements of the form $\boldsymbol{y} = \boldsymbol{A\theta} + \boldsymbol{\eta}$ where $\boldsymbol{y} \in \mathbb{R}^m, \boldsymbol{\theta} \in \mathbb{R}^n, \boldsymbol{A} \in \mathbb{R}^{m \times n}, m \ll n$. Suppose $\boldsymbol{A}$ obeys the Restricted isometry property with restricted isometry constant $\delta_{2s}$ (of order $2s$) such that $\delta_{2s} < \sqrt{2} - 1 \approx 0.414$. Let $\boldsymbol{\theta}^\star$ be the solution to the following optimization problem (P1): $\min \|\boldsymbol{\theta}\|_1$ such that $\|\boldsymbol{y} - \boldsymbol{A\theta}\|_2 \leq \varepsilon$ where $\|\boldsymbol{\eta}\|_2 \leq \varepsilon$. Then, we have the following error bound: $\|\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|_2 \leq \dfrac{C_1}{\sqrt{s}}\|\boldsymbol{\theta} - \boldsymbol{\theta_s}\|_1 + C_2\varepsilon$ where $C_1, C_2$ are monotonically increasing functions of $\delta_{2s}$ (in the range $[0, 1]$). Also the vector $\boldsymbol{\theta_s}$ is defined such that $\forall i \in \mathcal{S}, \theta_s(i) = \theta_i, \forall i \notin \mathcal{S}, \theta_s(i) = 0$ where the set $\mathcal{S}$ consists of indices of the $s$ largest absolute-value elements of $\boldsymbol{\theta}$.
   Now consider that I gave you another theorem (called Theorem 3A), which is the same as Theorem 3 except that it requires that $\delta_{2s} < 0.6246$. Out of Theorem 3 and Theorem 3A, which is the more powerful theorem? Why?

4. Given $Q$ noiseless tomographic projections of a 2D image $f$, each in a different angle, explain how you can determine the zero-th and first order moments of the image directly without reconstruction. Recall that the moment of image $f$ of order $(a, b)$ is given by $M_{ab} = \int\int f(x, y)x^a y^b dxdy$, and the $r$-order moment of the tomographic projection $R_\theta f(\rho) = \int\int f(x, y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy$ is given by $m_r^{(\theta)} = \int R_\theta f(\rho)\rho^r d\rho = \int\int f(x, y)\delta(x\cos\theta + y\sin\theta - \rho)\rho^r dxdyd\rho$. Here $\delta(.)$ denotes the Dirac delta function.

5. State the Fourier slice theorem for a 3D image $f(x, y, z)$. In the tomography under unknown angles problem, what is the additional source of information available if the underlying image is a 3D image instead of a 2D image?

6. State whether true or false and justify: In orthogonal matching pursuit for estimating sparse $\boldsymbol{\theta} \in \mathbb{R}^n$ from measurements of the form $\boldsymbol{y} = \boldsymbol{A\theta}$ where $\boldsymbol{y} \in \mathbb{R}^m, \boldsymbol{A} \in \mathbb{R}^{m \times n}$, the same column of the matrix $\boldsymbol{A}$ never gets selected in more than one iteration.

7. The Hitomi video compressive camera acquires a coded snapshot image of the form $Y = \sum_{t=1}^T X_t \cdot \Phi_t$ where $Y \in \mathbb{R}^{n_1 \times n_2}$; $\{X_t\}_{t=1}^T$ are the $T$ consecutive frames of the underlying unknown video and $\forall t, X_t \in \mathbb{R}^{n_1 \times n_2}$; and $\{\Phi_t\}_{t=1}^T$ are the corresponding modulation functions and $\forall t, \Phi_t \in \{0, 1\}^{n_1 \times n_2}$. The operation " $\cdot$ " denotes an entry-wise multplication. To recover $\{X_t\}_{t=1}^T$ from $Y$ and $\{\Phi_t\}_{t=1}^T$, we need to solve problem (P1) as defined in Q3 of this paper. Clearly state the relationship between $\boldsymbol{y}, \boldsymbol{A}, \boldsymbol{\theta}$ on one hand as defined in (P1), with the quantities $Y, \{X_t\}_{t=1}^T, \{\Phi_t\}_{t=1}^T$.