End-sem: CS 754, Advanced Image Processing

Instructions: There are 180 minutes for this exam. This exam is worth 10% of your final grade. Attempt all **eight** questions. Write **brief** answers - lengthy answers are not expected. Each question carries 10 points.

1. In blind compressed sensing, recall that we consider N compressive measurement vectors of the form $\boldsymbol{y_i} = \boldsymbol{\Phi_i \Psi \theta_i} + \boldsymbol{\eta_i}, 1 \leq i \leq N$. For each i, $\boldsymbol{y_i}$ is the compressive measurement vector for the signal $\boldsymbol{x_i} \triangleq \boldsymbol{\Psi \theta_i}$. We want to infer $\boldsymbol{\theta_i}$ as well as $\boldsymbol{\Psi}$ from $\{\boldsymbol{y_i}, \boldsymbol{\Phi_i}\}_{i=1}^N$. The objective function that is optimized in this application is $J(\boldsymbol{\Psi}, \{\boldsymbol{\theta_i}\}_{i=1}^N) = \sum_{i=1}^N \|\boldsymbol{y_i} - \boldsymbol{\Phi_i} \sum_{k=1}^K \boldsymbol{\Psi_k} \theta_{ik}\|^2$ subject to the constraints $\forall i, \|\boldsymbol{\theta_i}\|_0 \leq T_0; \forall k \boldsymbol{\Psi_k}^t \boldsymbol{\Psi_k} = 1$. Why does the update of the dictionary columns $\{\boldsymbol{\Psi_k}\}_{k=1}^K$ require that the sensing matrices $\{\boldsymbol{\Phi_i}\}_{i=1}^N$ for the different signals $\{\boldsymbol{x_i}\}_{i=1}^N$ be different from each other? You may write an equation to support your answer. [10 points]

2. What is the significance of common lines in cryo-electron microscopy? Explain very briefly. We know that the particle orientations (in 3D) are unknown in cryo-electron microscopy. Why are the particle shifts also unknown? Are these shifts in 2D or 3D? Explain. [5+5 = 10 points]

3. In parallel bean computed tomography, the projection measurements are represented as a single vector $\boldsymbol{y} \sim \text{Poisson}(I_o \exp(-\boldsymbol{Rf}))$, where $\boldsymbol{y} \in \mathbb{R}^m$ with $m =$ number of projection angles $\times$ number of bins per angle; I_o is the power of the incident X-Ray beam; $\boldsymbol{R}$ represents the Radon operator (effectively a $m \times n$ matrix) that computes the projections at the pre-specified known projection angles; and $\boldsymbol{f}$ represents the unknown signal (actually tissue density values) in $\mathbb{R}^n$. If $m < n$, write down a suitable objective function whose minimum would be a good estimate of $\boldsymbol{f}$ given $\boldsymbol{y}$ and $\boldsymbol{R}$ and which accounts for the Poisson noise in $\boldsymbol{y}$. State the motivation for each term in the objective function. Recall that if $z \sim \text{Poisson}(\lambda)$, then $P(z = k) = \lambda^k e^{-\lambda}/k!$ where k is a non-negative integer. Now suppose that apart from Poisson noise, there was also iid additive Gaussian noise with mean 0 and known standard deviation σ, in $\boldsymbol{y}$. How would you solve this problem (eg: appropriate preprocessing or suitable change of objective function)? [6+ 4 = 10 points]

4. In non-negative sparse coding, we seek to minimize the cost function $J(\boldsymbol{W}, \boldsymbol{H}) = \|\boldsymbol{Y} - \boldsymbol{W}\boldsymbol{H}^T\|_F^2 + \lambda \|\boldsymbol{H}\|_1$, where $\boldsymbol{Y} \in \mathbb{R}_{\geq 0}^{d \times N}$ is a known data matrix, $\boldsymbol{W} \in \mathbb{R}_{\geq 0}^{d \times r}$ is a dictionary, $\boldsymbol{H}^T \in \mathbb{R}_{\geq 0}^{r \times N}$ is a matrix of coefficients, and $\lambda > 0$ is a regularization parameter. This cost function has a scaling ambiguity which could cause $\boldsymbol{H}$ to become a zero matrix. Elaborate on this scaling ambiguity, and explain what modifications you would introduce to the cost function to avoid this shrinkage of $\boldsymbol{H}$ to an all-zeros matrix. [5 + 5 = 10 points]

5. Consider that you learned a dictionary $\boldsymbol{D}$ to sparsely represent a certain class $\mathcal{S}$ of images - say handwritten alphabet or digit images. How will you convert $\boldsymbol{D}$ to another dictionary which will sparsely represent the following classes of images? Note that you are not allowed to learn the dictionary all over again, as it is time-consuming.

    (a) Class $\mathcal{S}_1$ which consists of 1D signals obtained by applying a Radon transform in a known angle θ to the images in $\mathcal{S}$.

    (b) Class $\mathcal{S}_2$ which consists of images obtained by translating a subset of the images in class $\mathcal{S}$ by a known fixed offset (x_1, y_1), and the other subset by another known fixed offset (x_2, y_2). Assume appropriate zero-padding and increase in the size of the image canvas owing to the translation.

    (c) Class $\mathcal{S}_3$ which consists of images obtained by applying an intensity transformation $I^i_{new}(x, y) = \alpha(I^i_{old}(x, y))^2 + \beta(I^i_{old}(x, y)) + \gamma$ to the images in $\mathcal{S}$, where α, β, γ are real-valued but unknown. [3 + 3 + 4 = 10 points]

6. How will you solve for the minimum of the following objective functions:

    (a) $J(\boldsymbol{a_r}) = \|\boldsymbol{a} - \boldsymbol{a_r}\|_1$, where $\boldsymbol{a}$ is a known $m \times 1$ vector with r or more than r non-zero-elements, and $\boldsymbol{a_r}$ is a vector with *at the most* r non-zero elements, where $r < m, r < n$.

    (b) $J(\boldsymbol{R}) = \|\boldsymbol{A} - \boldsymbol{RB} - \boldsymbol{C}\|_F^2$, where $\boldsymbol{A} \in \mathbb{R}^{n \times m}, \boldsymbol{B} \in \mathbb{R}^{n \times m}, \boldsymbol{C} \in \mathbb{R}^{n \times m}, \boldsymbol{R} \in \mathbb{R}^{n \times n}, m > n$. Note that $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are all known.

    (c) The same objective function as in the previous part, with the additional constraint that $\boldsymbol{R}$ is orthonormal. Note that $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are all known.

In all three cases, explain briefly any one situation in image/signal processing where the solution to such an optimization problem is required. [(2+2+3) + (1+1+1) = 10 points]

7. State the Fourier slice theorem in 2D, and separately in 3D. Carefully define the meaning of all symbols used. You may use a diagram. [5+5=10 points]

8. Consider compressive measurements of the form $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}$ for sensing matrix $\boldsymbol{A}$, signal vector $\boldsymbol{x}$, noise vector $\boldsymbol{v}$ and measurement vector $\boldsymbol{y}$. Consider the problem P1 done in class: Minimize $\|\boldsymbol{x}\|_1$ w.r.t. $\boldsymbol{x}$ such that $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2 \leq e$. Also consider the problem Q1: Minimize $\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2$ w.r.t. $\boldsymbol{x}$ subject to the constraint $\|\boldsymbol{x}\|_1 \leq t$. Prove that if $\boldsymbol{x}$ is a unique minimizer of P1 for some value $e \geq 0$, then there exists some value $t \geq 0$ for which $\boldsymbol{x}$ is also a unique minimizer of Q1. Note that $\|\boldsymbol{x}\|_1$ and $\|\boldsymbol{x}\|_2$ stand for the L1 and L2 norms of the vector $\boldsymbol{x}$ respectively.
Likewise, consider the problem R1: Minimize $\lambda\|\boldsymbol{x}\|_1 + \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2$ w.r.t. $\boldsymbol{x}$. If $\boldsymbol{x}$ is the unique minimizer of R1 for some $\lambda > 0$, then prove that it is also the unique minimizer of P1 for some $e > 0$. [5+5=10 points]