End-sem: CS 754, Advanced Image Processing, 1st May

Instructions: There are 180 minutes for this exam. This exam is worth 10% of your final grade. Attempt all **eight** questions. Write **brief** answers - lengthy answers are not expected. Each question carries 10 points.

1. Define the problem of compressive low rank matrix recovery. Clearly state the meaning of all mathematical terms. Give a mathematical definition of the matrix restricted isometry property. [6 + 4 = 10 points]

2. In blind compressed sensing, recall that we consider N compressive measurements of the form $\boldsymbol{y_i} = \boldsymbol{\Phi_i \Psi \theta_i} + \boldsymbol{\eta_i}, 1 \leq i \leq N$. For each i, $\boldsymbol{y_i}$ is the compressive measurement for the signal $\boldsymbol{x_i} \triangleq \boldsymbol{\Psi \theta_i}$. We want to infer $\boldsymbol{\theta_i}$ as well as $\boldsymbol{\Psi}$ from the compressive measurements. The objective function that is optimized in this application is $J(\boldsymbol{\Psi}, \{\boldsymbol{\theta_i}\}_{i=1}^N) = \sum_{i=1}^N \|\boldsymbol{y_i} - \boldsymbol{\Phi_i} \sum_{k=1}^K \boldsymbol{\Psi_k} \theta_{ik}\|^2$ subject to the constraints $\forall i, \|\boldsymbol{\theta_i}\|_0 \leq T_0; \forall k \boldsymbol{\Psi_k}^t \boldsymbol{\Psi_k} = 1$. Why does the update of the dictionary columns $\{\boldsymbol{\Psi_k}\}_{k=1}^K$ require that the sensing matrices $\{\boldsymbol{\Phi_i}\}_{i=1}^N$ for the different signals $\{\boldsymbol{x_i}\}_{i=1}^N$ be different from each other? You may write an equation to support your answer. [10 points]

3. In parallel bean computed tomography, the projection measurements are represented as a single vector $\boldsymbol{y} \sim \text{Poisson}(I_o \exp(-\boldsymbol{Rf}))$, where $\boldsymbol{y} \in \mathbb{R}^m$ with m = number of projection angles $\times$ number of bins per angle; I_o is the power of the incident X-Ray beam; $\boldsymbol{R}$ represents the Radon operator (effectively a $m \times n$ matrix) that computes the projections at the pre-specified known projection angles; and $\boldsymbol{f}$ represents the unknown signal (actually tissue density values) in $\mathbb{R}^n$. If $m < n$, write down a suitable objective function whose minimum would be a good estimate of $\boldsymbol{f}$ given $\boldsymbol{y}$ and $\boldsymbol{R}$ and which accounts for the Poisson noise in $\boldsymbol{y}$. State the motivation for each term in the objective function. Recall that if $z \sim \text{Poisson}(\lambda)$, then $P(z = k) = \lambda^k e^{-\lambda}/k!$ where k is a non-negative integer. Now suppose that apart from Poisson noise, there was also additive impulse noise (say, salt and pepper noise) in $\boldsymbol{y}$. How would you solve this problem (eg: appropriate preprocessing or suitable change of objective function)? [6+ 4 = 10 points]

4. Explain the relative advantages and disadvantages of overcomplete dictionary representations as compared to orthonormal basis representations. [5 + 5 = 10 points]

5. Consider that you learned a dictionary $\boldsymbol{D}$ to sparsely represent a certain class $\mathcal{S}$ of images - say handwritten alphabet or digit images. How will you convert $\boldsymbol{D}$ to another dictionary which will sparsely represent the following classes of images? Note that you are not allowed to learn the dictionary all over again, as it is time-consuming.
   (a) Class $\mathcal{S}_1$ which consists of images obtained by applying a known derivative filter to the images in $\mathcal{S}$.
   (b) Class $\mathcal{S}_2$ which consists of images obtained by rotating a subset of the images in class $\mathcal{S}$ by a known fixed angle α, and the other subset by another known fixed angle β.
   (c) Class $\mathcal{S}_3$ which consists of images obtained by applying an intensity transformation $I^i_{new}(x, y) = \alpha(I^i_{old}(x, y))^2 + \beta(I^i_{old}(x, y)) + \gamma$ to the images in $\mathcal{S}$, where α, β, γ are known. [3 + 3 + 4 = 10 points]

6. We have studied statistical compressed sensing in class for reconstruction of signal $\boldsymbol{x}$ from compressive measurements of the form $\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{\eta}$. Here every element of $\boldsymbol{\eta}$ is randomly drawn from $\mathcal{N}(0, \sigma^2)$ where σ is known, and $\boldsymbol{y} \in \mathbb{R}^m, \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{\Phi} \in \mathbb{R}^{m \times n}, \boldsymbol{\eta} \in \mathbb{R}^m, m \ll n$. In addition, we assume $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu} \in \mathbb{R}^n$ is the known mean vector, and $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ is the known positive semi-definite covariance matrix. The MAP estimate of $\boldsymbol{x}$ is given as $(\boldsymbol{\Sigma}^{-1} + \boldsymbol{\Phi}^T \boldsymbol{\Phi}/(2\sigma^2))^{-1} \boldsymbol{\Phi}^T \boldsymbol{y}/(2\sigma^2)$. What are the relative advantages and disadvantages of this technique for signal reconstruction compared to regular ℓ_1 norm based approaches for compressed sensing? [10 points]

7. State briefly any one application of compressive RPCA. Write down the main objective function involved in the optimization problem and state the meaning of each term w.r.t. the particular application. [10 points]

8. How will you solve for the minimum of the following objective functions: (1) $J(\boldsymbol{A_r}) = \|\boldsymbol{A} - \boldsymbol{A_r}\|_F^2$, where $\boldsymbol{A}$ is a known $m \times n$ matrix of rank greater than r, and $\boldsymbol{A_r}$ is a rank-r matrix, where $r < m, r < n$. (2) $J(\boldsymbol{R}) = \|\boldsymbol{A} - \boldsymbol{RB}\|_F^2$, where $\boldsymbol{A} \in \mathbb{R}^{n \times m}, \boldsymbol{B} \in \mathbb{R}^{n \times m}, \boldsymbol{R} \in \mathbb{R}^{n \times n}, m > n$ and $\boldsymbol{R}$ is constrained to be orthonormal. Note that $\boldsymbol{A}$ and $\boldsymbol{B}$ are both known.
   In both cases, explain briefly any one situation in image processing where the solution to such an optimization problem is required. [2.5 + 2.5 + 2.5 + 2.5 = 10 points]