Math 641 Homework 3: Characteristic classes

Not due (these are suggested problems only, practice problems for the most recent material)

We will refer to pages/sections from Milnor-Stasheff's *Characteristic classes* by [MilnorStasheff], and sections in Hatcher's *Algebraic Topology* by [HatcherAT].

1. *[MilnorStasheff 4-D]* If M^n can be immersed into $\mathbb{R}^{n+1}$ show that the ith Stiefel-Whitney class $w_i(M)$ is equal to the i-th self-cup product of $w_1(M)$; that is $w_i(M) = w_1(M)^i$. Then show that if $\mathbb{RP}^n$ can be immersed in $\mathbb{R}^{n+1}$ then n must be of the form $2^r - 1$ or $2^r - 2$.

2. *[MilnorStasheff 4-E]* Show that the set Ω consisting of all unoriented cobordism classes of smooth closed n-manifolds can be made into an additive group under disjoint union. This cobordism group is finite by Thom's result (that M is unoriented cobordant to N iff all their Stiefel-Whitney numbers are equal) and is a module over $\mathbb{Z}/2\mathbb{Z}$ (how?). Using the manifolds $\mathbb{RP}^2 \times \mathbb{RP}^2$ and $\mathbb{RP}^4$, show that Ω_4 consists of at least 4 elements.

3. *w_1 and orientability* Define the kth exterior power of a (real) vector bundle $E \to X$ of rank n for each k, and prove that $\wedge^n E$ has a section i.e., is trivial if and only if E is *orientable.* (A vector bundle is orientable if the associated bundle $Frame(E) \times_{GL(n)} (GL(n))/(GL^+(n))$ has a section). Then prove, using e.g., the splitting principle, that if E is rank n then $w_1(\wedge^n E) = w_1(E)$. Using these ingredients, conclude that $w_1(E) = 0$ if and only if E is orientable.

4. *Relationship between c_i and w_i* Let S^{2m+1} be the unit sphere in $\mathbb{C}^{m+1}$, and $S^1 \subset \mathbb{C}^*$ the unit circle of complex numbers, acting on S^{2m+1} by complex multiplication. There is a principal $S^1 = U(1)$-bundle, $S^{2m+1} \to \mathbb{CP}^m$ obtained by quotienting by the S^1 action. Consider now $\mathbb{RP}^{2m} = S^{2m+1}/\{\pm 1\}$; the induced projection map $\mathbb{RP}^{2m} \to \mathbb{CP}^m$ can be thought of as an $S^1/\{\pm 1\} = \mathbb{RP}^1$ bundle. We formally allow $m = \infty$ by taking the uniof such fibrations for increasing m, thinking of $\mathbb{C}^{m+1} \subset \mathbb{C}^{m+2} \subset$
dots
   (a) Prove that the Leray-Hirsch theorem with $\mathbb{Z}/2$-coefficients applies to the fibration $\mathbb{RP}^{2m} \xrightarrow{\pi} \mathbb{CP}^m$ (with fibers $\mathbb{RP}^1$). Conclude that if $\bar{h} \in H^2(\mathbb{CP}^m; \mathbb{Z}_2)$ is the mod-2 reduction of the canonical generator $h \in H^2(\mathbb{CP}^m; \mathbb{Z})$, and if $H^*(\mathbb{RP}^{2m}; \mathbb{Z}_2) \cong \mathbb{Z}_2[w]$ with $|w| = 1$, that $\pi^* \bar{h} = w$.

   (b) Prove that $\mathbb{RP}^{2m} \to \mathbb{CP}^m$ (including the case $m = \infty$) is the real projectivization of the tautological complex line bundle $L_{taut} \to \mathbb{CP}^m$, thought of as a real rank 2 bundle. Conclude from the definition of Stiefel-Whitney classes that $w_1((L_{taut})_{\mathbb{R}}) = 0$ and $w_2((L_{taut})_{\mathbb{R}}) = \bar{h} = c_1(L_{taut})$ (mod 2).

   (c) Using the previous problem, prove more generally that for any complex vector bundle E, the total Stiefel-Whitney class of E thought of as a real bundle is the mod 2 reduction of the total Chern class, so $w(E_{\mathbb{R}}) = c(E)$ (mod 2). In particular, $w_{2k+1}(E_{\mathbb{R}}) = 0$ and $w_{2k}(E_{\mathbb{R}} = c_k(E)$ (mod 2). *Hint: Prove this for all complex line bundles first using* *the previous calculation and universality. Now appeal to the splitting principle.*

5. Show that $\mathbb{CP}^4$ (thought of as an 8-dimensional real manifold) does not embed in $\mathbb{R}^{11}$ (hint: use Pontryagin classes).

6. *Computations of cohomology using Leray-Hirsch, [HatcherAT] §4.D (p. 447) #2.* Consider the action of $\mathbb{Z}_p \subset S^1$ on $S^{2n+1} \subset \mathbb{C}^{2n}$ by multiplying by complex roots of unity; stabilizing (under the inclusions $S^{2n+1} \subset S^{2n+3}$) we obtain an action of $\mathbb{Z}_p$ on S^∞. Denote by $B\mathbb{Z}_p := S^\infty/\mathbb{Z}_p$ (this is the classifying space for principal $G = \mathbb{Z}_p$-bundles, and also the *first Eilenberg-Maclane space* $B\mathbb{Z}_p = K(\mathbb{Z}_p, 1)$; note that $B\mathbb{Z}_2 = \mathbb{RP}^\infty$).

   Quotienting by the residual action of $S^1/\mathbb{Z}_p \overset{z\mapsto z^p}{\cong} S^1$ gives a fibration $S^\infty/\mathbb{Z}_p \to \mathbb{CP}^\infty$ with fiber S^1. Use this fibration and the Leray-Hirsch theorem to compute $H^*(B\mathbb{Z}_p; \mathbb{Z}_p)$ from $H^*(\mathbb{CP}^\infty; \mathbb{Z}_p)$.