Lagrangian Floer Homology _ I:

Recall: (M^{2n}, ω). a.c.s.

Def: An almost cplx. structure J is an endomorphism $J \in \mathrm{End}(TM)$ w/ $J^2 = -\mathbb{1}$.

J is compatible w/ ω if $\omega(-, J-)$ is a metric. $\Longleftrightarrow$ $\omega(Jv, Jw) = \omega(w, J(Jv)) = \omega(v, w)$.

Recall: Ex: X^n complex manifold; X^{2n} underlying real manifold, w/ $J = i \cdot$ $+ \times S$, integrable almost cplx structure: $\Leftrightarrow$ vanishing of $N(X,Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$ $\forall X, Y$

- Any metric g determines uniquely compatible a.c.s.

Prop: Any (M, ω) has a compatible a.c.s. & space of them is connected/contractible.

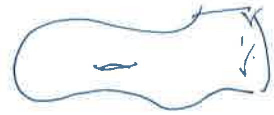
Rmk: could work w/ larger class of tame a.c. structures: J s.t. $\omega(v, Jv) > 0$ $\forall v \neq 0$.

In this case, $\frac{1}{2}[\omega(x, Jy) + \omega(Jx, y)]$ is a metric]

- often preferred b/c tameness is open in space of almost cplx structures; may more to choose from

J-holomorphic curves:

(S, j) Riemann surface: (maybe bordered).



(M, J) m'fold w/ A.C.S. J.

Def: $f: S \to M$ is J-holomorphic if $df \circ j = J \circ df$

e.g., df is $\mathbb{C}$-linear w.r.t. j on source, J on target.

Define: $\bar{\partial}_J f = (df)^{0,1} := \frac{1}{2}(df + J \circ df \circ j) = 0 \in \Gamma(S, \Omega^{0,1}_{\Sigma} \otimes f^* TM)$

f J-holomorphic $\Leftrightarrow$ f satisfies the non-linear PDE $\bar{\partial}_J f = 0$. In local coords $s + it$ on Σ, $\boxed{\begin{array}{l}\partial_t u = J \partial_s u \\ -\partial_s u = J \partial_t u\end{array}}$

Given a metric g on Σ, M, have an induced one on the bundle, $\mathrm{Maps}(T\Sigma, u^* TM)$, & hence can talk about

Energy of u: $E(u) := \int_{\Sigma} |du|^2$.

So far, nothing used ω, but the crucial point will be:

Thm: [Energy identity]: If ω, J, g compatible triple, u $\bar{\partial}_J$-hol. then

$$E_{(g)}(u) = \int u^* \omega.$$

Pf: With respect to local hol. unit-length coordinates $(s + it)$ on Σ, near p $|du|^2_p = \frac{1}{2}(g(\partial_s u, \partial_s u) + g(\partial_t u, \partial_t u)) \, ds \wedge dt$.

$= \omega(\partial_s u, J \partial_s u) = \omega(\partial_s u, \partial_t u) \, ds \wedge dt = u^* \omega$. ☐

$L_0, L_1 \subseteq (M, \omega)$ compact Lag'n submanifolds.

Floer homology: Formally, the Morse theory for an "action functional" on the path space

$\mathcal{P}(L_0, L_1) = \{ \gamma: [0,1] \to M, \gamma(0) \in L_0, \gamma(1) \in L_1 \}$.

"$A: \mathcal{P}(L_0, L_1) \to \mathbb{R}$"

• crit ~~points are~~ / constant paths

• gradient ~~flowlines~~: J-holomorphic strips. (with respect to some auxiliary J, which induces a metric on $\mathcal{P}(L_0, L_1)$ by $\langle \xi_0, \xi_1 \rangle_{L^2} = \int_\gamma \omega(\xi_0, J\xi_1)$.)

Actually, only dA is well-defined in general; (good enough to get "∇A flow").

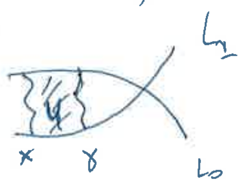
Define $T_\gamma \mathcal{P}(L_0, L_1) =$ – –

Note: $T_\gamma \mathcal{P}(L_0, L_1) = \{$ vector fields v on $\gamma^* TM$ w/ $v(0) \in T_{\gamma(0)} L_0$, $v(1) \in T_{\gamma(1)} L_1 \}$ "smooth tangent space".

More precisely, $A : \widetilde{\mathcal{P}}(L_0, L_1) \to \mathbb{R}$,

univ. cover,

$(\gamma, [u]) \longmapsto \int u^* \omega$.



$u : [0,1]^2 \to M$

a path in $\mathcal{P}(L_0, L_1)$ from $*$ to γ (assuming we're in the component of $*$). ($*$: ref. path)

Given $(\gamma, [u])$, note that (~~in the~~ using variational calculus)

$$dA(\gamma) \cdot v = \int_{[0,1]} \omega(\dot\gamma, v)\, dt \overset{\text{fixing some } J}{=\!=} \int_{[0,1]} g(J\dot\gamma, v)\, dt = \langle J\dot\gamma, v \rangle_{L^2}.$$

v: vec. field along γ, i.e. in $\gamma^* TM$, w/ $v(0) \in T_{\gamma(0)} L_0$, $v(1) \in T_{\gamma(1)} L_1$.

$\Rightarrow$ gradient $\nabla A_\gamma = -J\dot\gamma$.

Hence, • critical points = constant paths $\dot\gamma \equiv 0$.

• gradient trajectories := J-hol. maps $\frac{\partial \gamma}{\partial s} = -J\dot\gamma$,

It's ~~...~~ rather difficult to define infinite dim'l Morse theory directly using such variational methods. (for instance

• ~~Index~~ For a non-degenerate ($\Leftrightarrow$ transverse intersection) critical point of A, the index (in sense of Morse theory) is infinite). Hess A_γ has ∞'ly many eigenvalues

• We want to study gradient flow in $\Omega(L_0, L_1)$. But it may not be well-defined in general. Note: ∇A_γ is not even tangent to L_0! i.e., $J\dot\gamma(0)$ need not lie in L_0.

Floer: can still make sense of

• gradient flowlines, as sol'n to PDE.

• relative index (in the sense of Morse theory), depending on path between the crit points.

"# eigenvalues which cross from + to −." ~~along it~~

linked to index theory of PDE.

~~Actual setup~~:

Rmk: Say (M,ω) exact if $\omega = d\lambda$ ($\Rightarrow$ M not closed, like $\mathbb{C}^n$),

Given fixed λ, say $L \subseteq M$ exact if $\lambda|_L = df$, some $f : L \to \mathbb{R}$. (often, we choose a fixed f & call (L,f) exact for λ.)

Exercise: If (L_0,f_0), (L_1,f_1) exact, observe that can define

$$\mathcal{A} : \mathcal{P}(L_0,L_1) \longrightarrow \mathbb{R}$$

no cover!

by $\gamma \longmapsto \int \gamma^*\lambda + f_{L_1}(\gamma(1)) - f_{L_0}(\gamma(0))$. (at ~~least assuming $\mathcal{P}(L_0,L_1)$ connected~~).

Actual Setup: Say $L_0 \pitchfork L_1$:

Define Λ field, set of coefficients, $T \in \Lambda$ element

Ex: sometimes $\mathbb{C}$, sometimes $\left\{ \sum a_i T^{\lambda_i} \mid \lambda_i \to +\infty \right\}$.

Floer complex $CF^*(L_0,L_1) = \Lambda^{|L_0 \cap L_1|}$ free Λ module gen. by $L_0 \cap L_1$.

Goal: Define differential: ∂ by counting J-hol. discs:

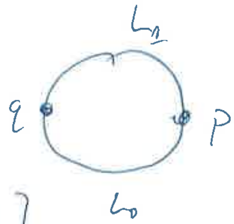
Look at: $u : \mathbb{R}\times[0,1] \longrightarrow M$ equipped w/ J ω compat. a.c.s.

s.t.

$$(*)\begin{cases} \bar\partial_J u = 0, \text{ e.g., } \dfrac{\partial u}{\partial s} + J\dfrac{\partial u}{\partial t} = 0. \\ u(s,0) \in L_0,\quad u(s,1) \in L_1 \\ \bullet \lim_{s\to+\infty} u(s,t) = p \in L_0\cap L_1 \\ \bullet \lim_{s\to-\infty} u(s,t) = q \in L_0 \cap L_1 \\ \bullet \text{The energy } E(u) = \int u^*\omega = \iint \left|\dfrac{\partial u}{\partial s}\right|^2 < \infty \end{cases}$$

Prop: This implies at $\pm\infty$, u exp. converges to some p, q.

Notes $\mathbb{R}\times[0,1] \underset{\text{bihol.}}{\simeq} D^2 \setminus \{\pm 1\}$, so can think of as maps



$\mathcal{M}(p,q,[u],J) = \{ u \text{ sol'ns of } (*) \text{ in homotopy class } [u] \}$, $[u] \in \pi_2(M; L_0, L_1)$.

$\mathcal{M}(p,q,J) = \coprod_\beta \mathcal{M}(p,q,\beta,J)$.

We want to "count rigid sol'ns" ~~of~~ elements of $\mathcal{M}$, but not quite.

Note that given any u satisfying $(*)$, non-constant, ~~there is a free $\mathbb{R}$ action~~ any $\lambda \in \mathbb{R}$, $\tilde{u}(s,t) := u(s+\lambda, t)$ solves $(*)$ too.

induces a free $\mathbb{R}$ action on $\mathcal{M}_{\text{non-constant}}$.

• Want to define: for $p \in CF^*(L_0, L_1)$ generator corresp. $p \in L_0 \cap L_1$,

$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ u \in \mathcal{M}(p,q,J) \\ \text{"rigid}/\mathbb{R}\text{"}}} \left(\#\mathcal{M}(p,q,J)/\mathbb{R}\right) T^{u^*\omega} \cdot q$$

translation, weight by area

$$= \sum_{q,\ \beta \in \pi_2} \left(\#\mathcal{M}(p,q,J)/\mathbb{R}\right) T^{u^*\beta} q.$$

Issues: • What does it mean to count # elts. of sol'ns to a PDE on an ∞'dim'l vec. space.

~~and~~ We will actually see: (1) For generic J, smooth finite-dim'l mfld of index ($\text{ind}(\phi)$).

(2) compact (w/ fixable) manifold ~~later compact~~. So can look at 0 dim'l component

• can be oriented ~~(so signed count)~~ so count is signed.

Analytic theory: "$\mathcal{M}$ solves a Fredholm problem" • $\partial^2 = 0$?

Can set up

$$\mathcal{M}(p,q) = \bar{\partial}_J^{-1}(0), \text{ where } \bar{\partial}_J : \mathcal{B} \to \mathcal{E} \text{ section of an } \infty \text{ dim'l vec. bdle}$$

$$\mathcal{B} = C^\infty(\mathbb{R}\times[0,1], M; L_0, L_1, p, q)$$

($t=0$, $t=1$, $-\infty$, $+\infty$)

$\mathcal{E} \to \mathcal{B}$ bundle w/ fiber $\mathcal{E}_u := C^\infty(\mathbb{R}\times[0,1], \Omega^{0,1}_S \otimes u^*TM)$

It turns out $\bar{\partial}_J$ is an elliptic operator ~~tangent~~, which for our purposes means that on suitable ~~introduce~~ Sobolev completions of $\mathcal{B}, \mathcal{E}$:

$$\mathcal{B}_{k,p} := W^{k+1,p}(S, X)$$

$$\mathcal{E}^{k,p}_u := W^{k,p}(S, \Omega^{0,1}_S \otimes u^* TX),$$

"Banach bdle over Banach mfld," have

$$\bar\partial_J : \mathcal{B}_{k+1,p} \longrightarrow \mathcal{E}^{k,p}, \text{ \& its linearization :}$$

$$D^u_{\bar\partial_J} : W^{k,p}(S, u^* TX) \longrightarrow W^{k,p}(S, \Omega^{0,1} \otimes u^* TX)$$

is Fredholm; meaning it has finite dimensional kernel & cokernel.

$$\operatorname{ind}(D^u_{\bar\partial_J}) := \dim(\ker D) - \dim(\operatorname{coker} D) \text{ is stable under spt perturbation}$$

Def: $u \in \mathcal{M}$ is regular if $D_{\bar\partial_J u}$ is onto at u (so ~~coker D~~ coker empty).

~~So the index~~ In this case, $\bar\partial_{J u}^{-1}(0)$ is locally a smooth mfld of dimension $\operatorname{ind}(D_{\bar\partial_J u})$.
(by ∞'dim'l Sard-Smale)

How to compute $\operatorname{ind}(D_{\bar\partial_J u})$ in our case?

Maslov index:
let $L_0, L_1(t)$ $t \in [0,1]$ Lag'n subspaces of $\mathbb{C}^n$ w/ $L_1(0)$ & $L_1(1) \pitchfork L_0$.

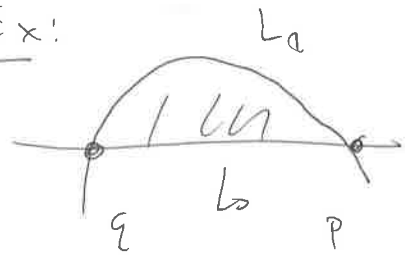
Maslov index of $L_1(t) :=$ # times $L_1(t)$ fails to be transverse to L_0
(counted w/ sign & mult).

Ex: $(e^{i\theta_1} \mathbb{R}) \times \cdots \times (e^{i\theta_n} \mathbb{R})$ if $\theta_i \nearrow$ through 0, then
$\mu(L_0, L_1(t)) = n$.

Given u strip, trivialize $u^* TM$, $\longrightarrow u^* TL_0, TL_1$ paths of Lag'ns.
Can trivialize so TL_0 remains constant

Then, $\operatorname{ind}(u) :=$ Maslov index of path TL_1 relative TL_0 as one goes from p to q.

Ex:



in $\mathbb{R}^2$ has $\operatorname{ind}([u]) = \underline{1}$.

Depends on $[u]$.