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Gradings

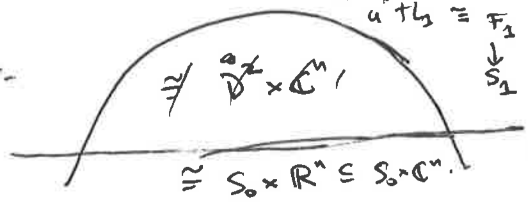
Recall, for $[u] \in \pi_2(X, L_0, L_1, p, q)$, we defined its index $\mathrm{ind}([u])$ as follows:

for a representative $u: \overbrace{\mathbb{R}\times[0,1]}^{S} \longrightarrow (X, L_0, L_1, p, q)$,

Simultaneously trivialize u^*TX, $u^*TL_0 \subseteq u^*TX|_{\underbrace{\mathbb{R}\times\{0\}}_{S_0}}$:

$u^*TL_1 \cong F_1 \subseteq \mathbb{C}^n$ Lag'n.
$\downarrow \quad \downarrow$
$S_1 \quad S_1$

$u^*TX =$



$\cong S_0 \times \mathbb{R}^n \subseteq S_0 \times \mathbb{C}^n$.
$u^*TL_0 \cong$

Gives map $S_1 \longrightarrow \Lambda(n)$ Lag'n Grassmannian; we looked at its intersection # w/

$$\Lambda_1(n) = \{ L \mid L \cap \mathbb{R}^n \text{ has rank} \geq 1 \}.$$

<u>Want</u>: to be able to assign ($\mathbb{Z}$-) gradings to p, q, call them $\widetilde{\mathrm{ind}}(p)$, $\widetilde{\mathrm{ind}}(q)$, so

$\mathrm{ind}([u]) = \widetilde{\mathrm{ind}}(p) - \widetilde{\mathrm{ind}}(q)$.. Or at least a relative grading $\widetilde{\mathrm{ind}}(p, q)$.

Problem: ind really depends on $\pi_2(-, p, q)$, & there could exist $\beta_0, \beta_1 \in \pi_2(p,q)$ w/ different indices!

Sources of
<u>Ambiguity</u>:

Given a $[u]$, & a nontrivial htpy class

$$v: (D^2, S^1) \longrightarrow (M, L_i)$$

or $\quad w: S^2 \longrightarrow M$.

can connect sum w/ u:

gives a new htpy class, & a gluing formula

$\Rightarrow$ <u>Disk</u>: $\mu(p, q, u\#v) = \mu(p, q, u) + \mu(v)$ Maslov index of disk. (defined prev.)

sphere: $\mu(p, q, u\#w) = \text{———} + 2\langle c_1(TM), w_*[S^2]\rangle$

In general, we can only get relative gradings in $\mathbb{Z}/N\mathbb{Z}$, $N\mathbb{Z}$ = subgp gen. by ambiguity terms.

If L_0, L_1 are oriented, then $\mu(v) \in 2\mathbb{Z}$, so (exercise) $2 \mid N$ & $CF^*(L_0, L_1)$ relative $\mathbb{Z}/2$ grades. Ex:

Absolute gradings: [Kontsevich, Seidel] ~~In some cases~~ Fixing some extra data of L_i, M can associate in some cases <u>absolute</u> gradings to $p \in L_0 \cap L_1$.

(Idea: in earlier picture, there would be a unique path F_1 of Lag'n subspaces in $\Lambda(n)$ between $u^* T_p L_0$ & $u^* T_p L_1$ if $T\mathbb{C}^n$ ~~case~~ case w/ preferred lifts in $\tilde{\Lambda}(n)$ univ. covers.

<u>Recall</u>: $\Lambda(n) :=$ Lag'n Grassmannian, $H^1(\Lambda(n);\mathbb{Z}) = \mathbb{Z}\langle \mu \rangle$
$\cong U(n)/O(n)$ & $\pi_1(\Lambda(n)) \cong \mathbb{Z}$ ↑ Maslov class.

w/ $\det^2 : \Lambda(n) \to S^1$ the π_1 isomorphism which "classifies μ"

$\tilde{\Lambda}(n) \to \Lambda(n)$ universal cover.

<u>Global case</u>: X sympl mfld. $\mathcal{L} \to X$ bundle of Lagr. subspaces of $(T_x X, \omega)$

<u>Prop</u>: TFAE:

(i) $\exists$ global $\mu_{glob} \in H^1(\mathcal{L};\mathbb{Z})$ w/ restricting to the generator μ on each fiber $\mathcal{L}_x \cong \Lambda(n)$. (why?)

"Global Maslov class"

(ii) $\exists$ a universal cover over $\tilde{\mathcal{L}} \to \mathcal{L}$ whose restriction to each $\mathcal{L}_x$ is ~~the~~ a universal cover

(iii) $\exists$ a trivialization of the square of the determinant bundle $K_X^{\otimes -2}$ ~~is trivial~~.
$K_X \cong \Lambda^n_{\mathbb{C}} TX$ ↑ uses some J

($\Rightarrow 2c_1(X) = 0$).

(In fact, the choices of (i), (ii), (iii) are canonically identified)

Choices of such trivializations, for instance, are an affine space over $H^1(X;\mathbb{Z}) \cong [X, S^1]$. Fix <u>such a choice</u>.

Now, each $L \hookrightarrow X$ Lag'n comes equipped w/ canonical $s_L : L \to \mathcal{L}|_L$
$x \longmapsto T_x L \subseteq T_x X$.

Given this map $\begin{array}{ccc} & & \mathcal{L} \\ & \overset{s}{\nearrow} & \downarrow \\ L & \xrightarrow{i} & X \end{array}$, the Maslov class of L is

$\mu_L := s^* \mu_{glob} \in H^1(L;\mathbb{Z})$.

Def: A $\mathbb{Z}$-grading of L is a lift

$$\begin{tikzcd} & \widetilde{\mathcal{L}} \arrow[d] \\ L \arrow[ur, dashed, "\tilde{s}"] \arrow[r, "s"'] & \mathcal{L}|_L \end{tikzcd}$$

Obstruction: $\mu_L \in H^1(L;\mathbb{Z})$.

Hence, gradings exist if L is Maslov zero, i.e. if $\mu_L = 0$.

(Rmk: this is automatic if L is a sphere S^n, or any other space w/ $H^1(L;\mathbb{Z}) = 0$.)

Rmk: There's also a notion of a $\mathbb{Z}/N\mathbb{Z}$ grading structure: replace $\widetilde{\mathcal{L}}$ w/ an N-fold cover $\mathcal{L}^N$, look at global classes $\tilde{\mu} \in H^1(\mathcal{L};\mathbb{Z}/N)$, instead of trivializing $K_X^{-\otimes 2}$, look at N^{th} roots, etc. ($\Rightarrow 2c_1 \bmod N = 0$)

($N=2 \Rightarrow$ any (M,ω) admits a 2-fold cover $\underset{\text{oriented Lag's}}{\mathcal{L}^{or}} \to \mathcal{L}$; L is $\mathbb{Z}/2$-gradable if L orientable)

Given a $\mathbb{Z}$-grading str. of L_i, how define $\deg(p)$, $p \in L_0 \cap L_1$? ($|p|$ = sign of intersection)

Have: $(\widetilde{T_pL_0}, \widetilde{T_pL_1}) \in \widetilde{\Lambda} \cong \widetilde{\mathcal{L}}_p$. Def: $i(\widetilde{\Lambda}_0, \widetilde{\Lambda}_1) = n/2 - \mu(\lambda_0, \lambda_1)$

$\widetilde{T_pL_0} = \tilde{\lambda}_0(t)$, $\widetilde{T_pL_1} = \tilde{\lambda}_1(t)$ w/ $\tilde{\lambda}_0(0) = \tilde{\lambda}_1(0) =$ base point. Note, projects to $\lambda_0(t)$, $\lambda_1(t)$ Maslov paths

(μ: # times λ_0, λ_1 fail to be transverse)

An easier to compute w/ point of view:

Suppose $c_1(X) = 0$, ~~say X trivialize~~.

Take a complex vol. form Ω_X (nowhere vanishing $(n,0)$ form; $\Leftrightarrow$ section of $\Lambda^n_{\mathbb{C}} T^*X$).

$\rightsquigarrow$ classical phase function

$$\alpha_X : \mathcal{L}_X \longrightarrow S^1.$$

($\mathcal{L}_X$: Lag Grassmannian bdle)

given $x \in X$, $\Lambda \in \mathcal{L}_{X_x}$, take any basis $e_1, \dots, e_n \in \Lambda$,

& $\alpha_X(\Lambda) = \dfrac{\Omega(e_1 \wedge \dots \wedge e_n)^2}{|\Omega(e_1 \wedge \dots \wedge e_n)|^2}$. ~~classical~~ phase

Note: α_X^* (gen. of $H^1(S^1)$) is the Global Maslov class

A grading of L is a lift:

$$\begin{tikzcd} & & \mathbb{R} \arrow[d, "\exp(2\pi i\, -)"] \\ L \arrow[r, "s_L"] \arrow[urr, dashed, "\tilde{\alpha}_L"] & \mathcal{L}_X|_L \arrow[r, "\alpha_X"] & S^1 \end{tikzcd}$$

Can call $(L, \tilde{\alpha}_L)$ a graded Lag'n.

- Can shift and $\tilde{L}$ by ~~...~~ $(L, \tilde{\alpha}_L)[1] = (L, \tilde{\alpha}_L - 1)$

- given $p \in \tilde{L}_0 \cap \tilde{L}_1$,

($\alpha_\phi : \mathcal{L}_X \to S^1$
$\alpha_X(D\phi(\Lambda))/\alpha_X(\Lambda)$)

- choose path Λ_t from $T_p\tilde{L}_0$ to $T_p L_1$ which is crossingless, means $\Lambda_t \cap \Lambda_0 = \Lambda_1 \cap \Lambda_0$,
& negative definite at Λ_0. "canonical short path"

$\Lambda_0/\Lambda_0\cap\Lambda_1$
crossing form for $d\Lambda_t/dt|_{t=0}$

Lift $\alpha_X(\Lambda_t)$ to $\tilde{a}_t \in \mathbb{R}$,

& $\deg(p) := (\tilde{\alpha}_{L_1}(p) - \tilde{a}_1) - (\tilde{\alpha}_{L_0}(p) - \tilde{a}_0)$.