Math 257B — Week 1 Day 3.

Last time: L_0, L_1 Lag'ns in (X,ω) $L_0 \pitchfork L_1$ transverse.

Λ field (most likely $\{\sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}\}$ Novikov field), T

$T \in \Lambda$ elt. maybe $(\mathbb{C}, 1)$, so forget this).

tentatively defined, based on the idea of doing Morse theory for $\mathcal{A} : \tilde{\mathcal{P}}(L_0, L_1) \longrightarrow \mathbb{R}$,

$CF^*(L_0, L_1) := \Lambda^{|L_0 \cap L_1|}$, & picking an almost cplx. structure J,

tentatively defined a "differential"

$$\delta p = \sum_{\substack{u \in \mathcal{M}(p,q,J)/\mathbb{R} \\ \text{"rigid" elements}}} T^{\omega(u)} \operatorname{sign}(u) \cdot q$$

$\operatorname{sign}(u) = \pm 1$ (assuming $\mathcal{M}(p,q)/\mathbb{R}$ oriented! (maybe $\Lambda = \mathbb{Z}/2$).

where $\mathcal{M}(p,q,J) = \coprod_\beta \mathcal{M}(p,q,\beta,J)$, $\beta \in \pi_2(-)$ &

$$= \{u : (\mathbb{R}\times[0,1])_{s,t} \longrightarrow (M, J) \text{ satisfying } (*)\}$$

$$(*)\ \begin{cases} \bar\partial_J u = \tfrac12 (du + J\circ du \circ j) = 0. \\ u(s, i) \in L_i \quad i = 0,1. \\ \lim_{s\to+\infty} u(s,t) = p \\ \lim_{s\to-\infty} u(s,t) = q. \\ E(u) = \int |du|^2 < \infty \\ \quad = \int u^*\omega. \end{cases}$$

It will turn out: ~~will see~~ ~~Theorem~~:

(a) For generic J, each $\mathcal{M}(p,q,\beta,J)$ is a finite-dim'l manifold w/ a fixed ~~expected~~ dimension := expected dim $\mathcal{M}(p,q,\beta,J) =: \text{"ind}(\beta)\text{"}$

w/ free $\mathbb{R}$ action (unless dim $=0$) (so "rigid" means 0 dim'l. components of $\mathcal{M}/\mathbb{R}$).

(b) $\mathcal{M}(p,q)$ is compact(ifiable). (in particular, the set of 0 dim'l components of $\mathcal{M}$ is compact).

(c) $\delta^2 = 0$.

these may fail for related reasons.

(a) Last time: observed we could express $\mathcal{M}(p,q, \underline{\ \ \beta\ \ })$ as $\bar\partial_J^{-1}(0)$,
$$\mathcal{E} \underset{\bar\partial_J}{\overset{\pi}{\rightleftarrows}} \mathcal{B}$$
∞ dim'l vector bundle

$\bar\partial_J$: ~~$C^\infty(\mathbb{R}\times[0,1], M)$~~ $\mathcal{B}$

$\mathcal{B} = C^\infty(\underbrace{\mathbb{R}\times[0,1]}_{S}, M, L_0, L_1, p, q)$, $\mathcal{E}_u = C^\infty(S, \Omega^{0,1}_S \otimes u^*TM)$.

Floer: ~~$\bar\partial_J$ has a Fredholm extension~~, meaning ~~$\mathcal{B}^{1,2} \xrightarrow{\bar\partial_J} \mathcal{E}^{0,2}$ w/ $\ker(\bar\partial_J)$ ...~~

~~(k,p)~~ ~~$(k-1,p)$~~

at $u \in \bar\partial_J^{-1}(0)$,

Linearizing, have $D^u_{\bar\partial_J}$:

Toy model from finite diff'l topology:

$f: M^m \longrightarrow N^n$ is a submersion at $p \in M$ if $Df_p : T_pM \twoheadrightarrow T_{f(p)}N$.

~~Ifso~~ ~~Implicit fcn~~ Implicit fcn. thm $\Rightarrow$ ~~a~~ ~~$f^{-1}(f(p))$~~ $f^{-1}(q)$ near p is ($q = f(p)$) a smooth manifold dim ~~...~~ $m-n$.

Closer: $V \xrightarrow{\pi} M^m$, V = rank k vc. bdle, s a section; want $s^{-1}(0)$ smooth manifold of "right dimension" $(m-k)$ $\Longleftrightarrow$ $s(M) \pitchfork 0$-section in V.

Implicit fcn. thm at $p \in s^{-1}(0)$ applies, as we check

$d^v s : T_pM \longrightarrow T^{vert}_{s(p)} V = \ker d\pi$ is surjective. Moreover, $\boxed{T_p(s^{-1}(0)) = \ker d^v s.}$

Even if $d^v s$ not surjective at p, can determine the "right" ("expected") dimension: its $m-k = \operatorname{rk} \ker d^v s_p - \operatorname{rk} \operatorname{coker} d s_p$.

For us, M & V are infinite dimensional, but in the nicest possible way!

Thm [Floer]: If to A_2 After extending $\bar\partial_J$ to a suitable Sobolev completion, solving $\bar\partial_J u = 0$ is a Fredholm problem

$\bar\partial_J \circlearrowleft \mathcal{E} \downarrow \mathcal{B}^{k,2}$, $\mathcal{E}$ ~~...~~ 2 Banach bdl, meaning $\mathcal{B}^{k,2}$ Banach mfld (or (k,p))

the linearization (vertical part of deriv.) $D^u_{\bar\partial_J} : W^{k,2}(S, u^*TX, u^*TL_i) \longrightarrow L^2(S, u^*TM)$, $S = \mathbb{R}\times[0,1]$

is Fredholm,

meaning (image closed &) $\operatorname{ind}(D^u_{\bar\partial_J}) = \operatorname{rk} \ker D^u_{\bar\partial_J} - \operatorname{rk} \operatorname{coker} D^u_{\bar\partial_J}$.

expected dimension of $\mathcal{M}(p, q, \beta, J)$.

As in the toy case, ~~…~~ what this index is is independent of point u, we're at, & can be computed topologically.. (c.f. Atiyah-Singer index theorem)

Thm 2: [Floer]: ~~...~~ In nice cases (such as when $\pi_2(M,L)$...) $(*)$, there is a set of second Baire category ($\Rightarrow$ dense set) of J such that $D^u_{\bar\partial_J}$ is onto for all u. ($\Longleftrightarrow$ u is regular)

(specifically when u is simple, means somewhere injective).

Idea: Have extended $\bar\partial_J$ operator: $\mathcal{B} \times \mathcal{J} \xrightarrow{\bar\partial^{ex}} \mathcal{E} \times$ ~~...~~ (space of J's)

$(u, J) \longmapsto \bar\partial_J u$.

Floer: proves ~~...~~ this is submersion. ~~then~~ so

IFT $\Rightarrow$ $\mathcal{M}^{ex} = \bar\partial^{ex\,-1}(0) \subseteq \mathcal{B} \times \mathcal{J}$ Banach submfld

Consider
$$\begin{array}{ccc} \mathcal{M}^{ex} & \hookrightarrow & \mathcal{B}\times\mathcal{J} \\ \downarrow \pi' & & \downarrow \pi \\ \mathcal{J} & = & \mathcal{J} \end{array}$$

Sard-Smale $\Rightarrow$ Reg. values of π' dense

But at reg. values, $\pi'^{-1}(\tilde J) := \mathcal{M}_{\tilde J}$ submfld, right dimension.

What's $\mathrm{ind}(\bar\partial_J^u)$? Turns out to depend only on $[u]$;

~~Maslov~~

Maslov index:

~~Let~~ Let $\Lambda(n)$ denote the Lag'n grassmannian $\{L^n \subseteq \mathbb{C}^n$ Lag'n ~~planes through 0~~ linear subspaces$\}$.

Have: $\Lambda(n) \cong U(n)/O(n)$, & hence $H^1(\Lambda(n);\mathbb{Z}) \cong \mathbb{Z}$ generated by Maslov class.

$\pi_1(\Lambda(n)) \cong \mathbb{Z}$; explicitly $U(n)/O(n) \xrightarrow{\det^2} S^1$ is a π_1 isomorphism.

Arnold: Geom. interpretation of Maslov class:

$\Lambda_\beta \subseteq \Lambda = \{$ Lag'n planes that are not transverse to $\mathbb{R}^n\}$ "Maslov cycle"

$\mathbb{R}^n \subseteq \mathbb{C}^n$.

[Arnold]: μ (a loop of $\{L_t\}$)

$= \text{winding } \#(\det^2 \circ \{L_t\})$

$= \{L_t\} \cdot \Lambda_0$

signed intersection #.

The relevance to index theory is the following toy case of what we want to do:

Note that given a trivial $\mathbb{C}^n$ bundle $E \to D^2$, equipped w/ a Lag'n sub-bundle $F \subseteq E|_{S^1} \to S^1$,

$\rightsquigarrow$ loop ρ in $\Lambda(n)$. Maslov index $= \mu(E,F) = \mu(\rho)$:= obstruction to trivializing $F \subseteq E$.

Think of this as a real version of the "chern class" (Can relate chern class); Exercise: $E \cong \mathbb{C}^n \to S^2$, $F \subseteq E|_{S^1}$ Lag'n sub-bundle, then $\Sigma\mu(E_i,F) = 2c_1(E)$.

Thm: (Riemann-Roch w/ boundary)

Σ, j Riem. surface w/ boundary, $u: \Sigma \to M$ map to J-hol. w/ bdl. sending $\partial\Sigma \mapsto$ Lag'ns.

Then, the index of the linearized operator $D_u\bar\partial_J$ is

$$\mathrm{ind}_{\mathbb{R}}(D_u\bar\partial_J) = (\dim_{\mathbb{C}} M)(\chi(S)) + \sum \mu(B_i)$$

Maslov class of B_i w.r.t. trivialization of u^*TM.

a trivialization of u^*TM. (

Two steps: $L_0, L_1(t)$ Lagr. subspaces

Differential topology

Toy models:

$f: M^m \to \mathbb{R}^n$ 'submersive' at p if $Df: T_pM \to T_{f(p)}\mathbb{R}^n$ surj.

If so, implicit fn. theorem $\Rightarrow$ $\exists$ a nbhd U of $f^{-1}(0)$ w/ a smooth mfld structure of dim. $m-n$.

Somewhat closer: V rank k vector bundle, s, π, M^m. want $s^{-1}(0)$ smooth manifold of the "right dimension" $(m-k)$, i.e. equiv. $s(M) \pitchfork 0(M)$ in V.

Implicit fn. thm applies, once we check that

$ds: TM \to T^{vert}V$ is surjective. (vertical part; $T^{vert}V = \ker d\pi$)

Even if ds not surjective at p, there's a way of knowing what the "right dimension" should be:

$\mathrm{ind}(df) = \mathrm{rk} \ker df_p - \mathrm{rk}\,\mathrm{coker}\, df_p = n-m$, indep. of p.

$(m - \mathrm{rk}\,\mathrm{im})$

If $V \to W \cong Q \oplus K$, surjective, Q

To

Today: Maslov index, statement of transversality, $\mathbb{C}^n$. $e^{\pi i \theta}$

Maslov index: Let $L_0, L_1(t)$ Lagr. subspaces of $\mathbb{R}^{2n}$ s.t. $t \in [0,1]$

$L_1(0), L_1(1) \pitchfork L_0$.

The Maslov index of the path $L_1(t)$:= # times that $L_1(t)$ fails to be transverse to L_0 (counted w/ signs & multiplicities):

Ex: $L_0 = \mathbb{R}^n \subseteq \mathbb{C}^n$, L_1 = path

$$(e^{\pi i \theta_1 t}\mathbb{R}) \times \dots \times e^{i\pi\theta_n t}\mathbb{R}.$$

If all $\theta_i \neq 0, 1$ this is transverse at 0, 1 & if θ_i distinct, $\neq 0$ $\to \mu(L_0, L_1(t)) = n$

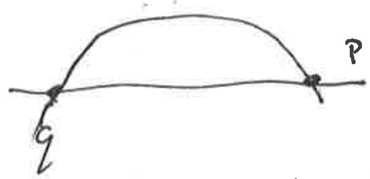
~~The relation to permutation~~

~~Given~~ Now: Given a strip $u: \mathbb{R} \times [0,1] \to X$, L_0, L_1 transverse $u^*TM \cong \Sigma \times \mathbb{C}^n$. Get u^*TL_0, u^*TL_1 paths of Lagns.

Can further trivialize so that u^*TL_0 remains constant. (but not simultaneously u^*TL_1!)
$(\cong (\mathbb{R}\times\{0\})\times\mathbb{R}^n)$

Then, $\mathrm{ind}(u) ::=$ Maslov index of path TL_1 relative TL_0 as one goes from p to q

Ex:



in $\mathbb{R}^2$ has $\mathrm{ind}(L_u)) = 1$.

"(*) is a Fredholm problem, exp. dimension $\mathcal{M} = \mathrm{ind}(\bar\partial_J)$
$\mathrm{ind}(\bar\partial_J) =$ Maslov index;
comes from $\pi_1(\Lambda Gr) = \mathbb{Z}$.

Thm: (Riemann-Roch): Let (S,j) be a cpct. Riemann surface $u: S\to M$ a map to the a.c. mfld (M,J). Then, the index of the linearized operator $D_u\bar\partial_J$ is

$$\mathrm{ind}_{\mathbb{R}}(D_u\bar\partial_J) = (\dim_{\mathbb{C}} M)(\chi(S) + 2\langle c_1(TM), u_*[S]\rangle.$$

(if $\mathbb{C}$-linear: $\mathrm{ind}_{\mathbb{C}} = (\dim_{\mathbb{C}} M)(1-g) + \deg(u^*TM)$.)

Next: Say Σ cpct w/ boundary

$$\partial\Sigma = \coprod_i B_i. \quad \text{w/} \quad u: \Sigma\to M$$
each $B_i \to$ Lag'n L_i.

The linearized $\bar\partial_J$-operator acts on sections of u^*TM w/ the restr. that along B_i, section must lie in u^*TL_i.

cplx. vector bdle u^*TX w/ sub bundles u^*TL_i along B_i.

S not closed $\Rightarrow$ u^*TM can be trivialized over S.

But: subbundles u^*TL_i cannot be simultaneously trivialized.

Non-triviality of u^*TL_i measured by Maslov index
$$\pi_1(U_n/O_n) \xrightarrow{\sim} \mathbb{Z}.$$

Thm: (R.R. w/ boundary):

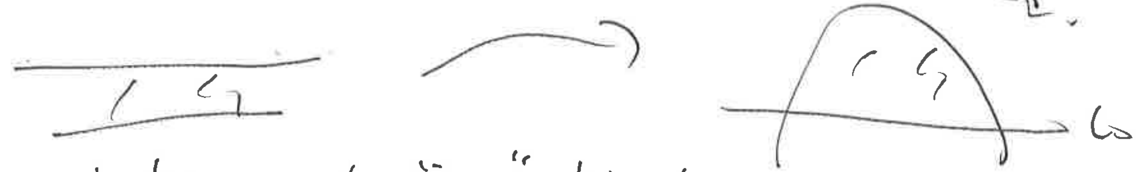
In the set up just described,

$$\mathrm{ind}_{\mathbb{R}}(D_u \bar{\partial}_J) = (\dim_{\mathbb{C}} M)\chi(S) + \sum \mu(B_i)$$

$\mu(u \# v) = \mu(u) + \mu(v)$

$\mu(x, y, u \# w) = \mu(x, y, u) + 2\langle c_1(TM), w[S^2]\rangle$

Consider strips:



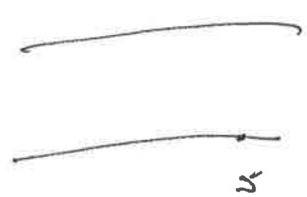
Boundary not a loop, image has "corners" at $L_0 \cap L_1$.

Trivialize & get sub-bundles u^*TL_0 over $\mathbb{R} \times \{0\}$ & u^*TL_1 over $\mathbb{R} \times \{1\}$.

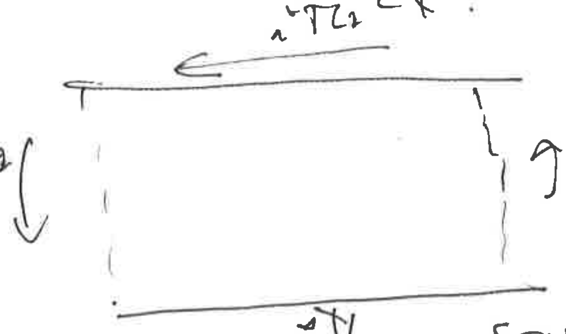
Assume
(i) L_0, L_1 ⋔ at p, q (non-degenerate)
(ii) J on X is such that

$$J \cdot T_p L_0 = T_p L_1 \quad \text{in } T_p X$$
$$J T_q L_0 = T_q L_1 \quad \text{in } T_q X$$

(iii) For $|s| > R$, $u^* TL_0$, $u^* TL_1$ are constant w.r.t. trivialization.



Now, define a loop in $\mathrm{Lag}(\mathbb{C}^n)$.



$u^* TL_1 \subset \mathbb{R}^n$.

$e^{-\pi t/2 J} T_x L_1$

go from $T_y L_0$ to $T_y L_1$ by $e^{\frac{\pi t}{2}} J T_y L_0$

$s = -R$ $\quad u^* TL_0 \quad$ $s = R$

Thm [Floer]: In above situation, $D_u \bar{\partial}_J$ has a Fredholm extension to $L^2_1(S, u^*TM, u^*TL_0, u^*TL_1)$ & index equal to μ

(b) We'll also need a compactness result; otherwise, the numbers we count might be infinite (not finite).

This is one of the key places we use (a).

Statement first for J-hol curves $u: (\Sigma, j) \to X, J$. (not strips / Floer trajectories yet).

Thm: (Gromov compactness) (assume for the moment that Σ has no boundary).

$u_n : \Sigma_n \to X$ sequence of J-holom. curves, $J \in \mathcal{J}(X,\omega)$, say w/ "marked points"

w/ energy $E(u_n) = \int_{\Sigma_n} u^*\omega = \langle [\omega], u_{n*}[\Sigma_n] \rangle$

$\Rightarrow \exists$ subsequence converging to a stable map $u_\infty : \Sigma_\infty \to X$

e.g.) Σ_∞ $\cup$ nodal Riemann surface, w/

all marked pts. & nodes distinct in domain

(if they come together, create a constant bubble to keep them separate).

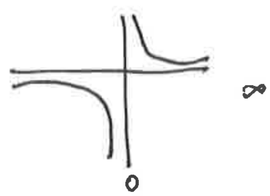
Phenomenon: Besides possible degeneration of domain (Σ_n, j_n) to a nodal curve, main phenomenon here is bubbling of spheres:

Ex: $u_n : S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$

$(x_0 : x_1) \longmapsto (x_0 : x_1), (n x_1 : x_0)$.

(in affine chart $x := x_1/x_0$: $x \mapsto (x, \frac{1}{nx})$, extend to $0, \infty$)

Away from 0, unif. convergence to $x \mapsto (x, 0)$.



so limit seems to be just 1st coord. axis, missing pt.

Reparametrize: $\tilde{x} = nx$, then get $\tilde{x} \mapsto (\frac{\tilde{x}}{n}, \frac{1}{\tilde{x}})$.

uniform convergence away from ∞ to $\tilde{x} \mapsto (0, \frac{1}{\tilde{x}})$ $\Rightarrow$ 2nd coord. axis.

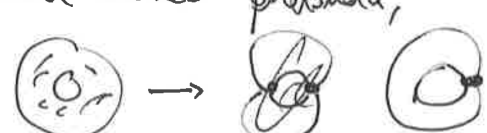
Closer to our application

Suppose Σ has boundary components $\partial_1\Sigma, \dots, \partial_k\Sigma$, $u: \Sigma \to X$ J-hol.

w/ $u: \partial_i \Sigma \to L_i$.

Then, Gromov's compactness thm still holds, but there are new phenomena,

(degeneration of domains along boundary: Ex:



);

& most importantly bubbling of disks