Some computations of Chern classes and Chern numbers.

One source of complex vector bundles comes from the tangent bundle to a complex manifold, as we'll now explain.

Some (fiber-wise) linear algebra.

V real vec. space of dim. $2n$. A cplex str. on V is $J: V \to V$ w/ $J^2 = -\mathrm{id}$.

Using J, V inherits str. of a $\mathbb{C}$-n-dim'l vec. space via $(a+bi)(v) := (a+bJ)(v)$, call this cplx. vector space (V, J).

Given a real vector bundle $E \to X$ of rank $2n$, a $J \in \Gamma(\mathrm{End}(E))$ (i.e., $J_x : E_x \to E_x$) w/ $J^2 = -\mathrm{id}$ (meaning $J_x^2 = -\mathrm{id}$ for all x) induces a complex vec. bundle str. on E, call it (E, J) and hence Chern classes

Call such a J a (fiberwise) complex structure on E.

Call a pair (X, J) an almost complex manifold & such a J on TX an almost complex structure.

manifold fiberwise complex structure on TX

An almost cplx manifold (X, J) has $c_j(X) := c_j(TX, J)$.

A complex manifold (dim. n) is a space equipped w/ (equiv. class of, or maximal) atlas $\{U_\alpha, \phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \underset{\text{open}}{\subset} \mathbb{C}^n\}_\alpha$, ($U_\alpha \subset X$) whose transition functions

$$\phi_\beta \circ \phi_\alpha^{-1} : \underset{\underset{\mathbb{C}^n}{\cap \text{ open}}}{\phi_\alpha(U_\alpha)} \longrightarrow \underset{\underset{\mathbb{C}^n}{\cap \text{ open}}}{\phi_\beta(U_\beta)} \quad \text{are holomorphic,}$$

meaning that $d(\phi_\beta \circ \phi_\alpha^{-1}) \circ i = i \circ d(\phi_\beta \circ \phi_\alpha^{-1})$.

Lem: Any complex manifold X has a canonical almost complex structure, hence TX is a cplx. vec. bdle (& has Chern classes).

Sketch: At a given $p \in X$, pick a chart U_α around p, giving

$$T_p X \xrightarrow[(d\phi_\alpha)_p]{\cong} T_{\phi_\alpha(p)}(\phi_\alpha(U_\alpha)) \cong \mathbb{C}^n \circlearrowleft i$$

Define J_p to be $(d\alpha)_p^{-1}\, i\, (d\phi_\alpha)_p$; check independent of choice & smoothly varying. (uses holomorphicity of transition functions). ☒.

Ex: $G_k(\mathbb{C}^n)$. We can construct a complex differentiable atlas parallel to the (real) atlas we constructed for $G_k(\mathbb{R}^n)$.

(i.e., around $E_0 \in G_k(\mathbb{C}^n)$, obtain (inverse to) a chart map by using the standard Hermitian inner prod. on $\mathbb{C}^n$.

$$\Psi : \mathrm{Hom}_{\mathbb{C}}(E_0, E_0^{\perp}) \longrightarrow G_k(\mathbb{C}^n)$$

$\cong \mathbb{C}^{k(n-k)}$

$$a \longmapsto \mathrm{graph}(a)(E_0). \quad)$$

$\mathrm{graph}(a) : E_0 \hookrightarrow E_0 \oplus E_0^{\perp} \cong \mathbb{C}^n$
$\quad \| $
(id, a)

(exercise: complex manifold)

The same analysis previously applied to $G_k(\mathbb{R}^n)$ implies that as complex vector bundles

$$T\,Gr_k(\mathbb{C}^n) \cong \underline{\mathrm{Hom}}_{\mathbb{C}}(E, E^{\perp})$$

tautological bundle over $G_k(\mathbb{C}^n)$

inside $\underline{\mathbb{C}}^n$ using $\langle -,- \rangle_{\text{Euclidean}}$ Hermitian metric.

$\underline{k=1}$ $\quad (G_1(\mathbb{C}^n) \cong \mathbb{C}P^{n-1})$

$$\Rightarrow T\mathbb{C}P^{n-1} \oplus \underline{\mathbb{C}} \underset{\text{as before}}{\cong} \underline{\mathrm{Hom}}_{\mathbb{C}}(L_{taut}, L_{taut}^{\perp}) \oplus \mathrm{Hom}(L_{taut}, L_{taut})$$

$\mathrm{Hom}(L_{taut}, L_{taut}) \cong L_{taut}^* \otimes L_{taut} \cong \underline{\mathbb{C}}$.

$$= \underline{\mathrm{Hom}}(L_{taut}, \underline{\mathbb{C}}^n) = \underbrace{L_{taut}^* \oplus \cdots \oplus L_{taut}^*}_{n \text{ times}}$$

Now for any cplx line bundle $L \to B$, $\underline{c_1(L^*) = -c_1(L)}$.

$$(b/c \quad c_1(L \otimes L^*) = c_1(\underline{\mathbb{C}}) = 0).$$
$\quad \| $
$c_1(L) + c_1(L^*)$

h canonical generator in $H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$.

So, $c_1(L_{taut}^*) = -c_1(L_{taut}) = -(-h) = h$.

$$\text{so, } c(\mathbb{C}P^{n-1}) := c(T\mathbb{C}P^{n-1}) \underset{\text{whitney sum}}{=} c((L_{taut}^*)^{\oplus n}) \underset{\text{whitney sum}}{=} \prod_{i=1}^{n} \underbrace{c(L_{taut}^*)}_{(1+h)}$$

$$= (1+h)^n \quad \text{in } H^*(\mathbb{C}P^{n-1}; \mathbb{Z}) \cong \mathbb{Z}[h]/h^n.$$

$$1 + nh + \binom{n}{2}h^2 + \cdots + nh^{n-1}.$$

i.e., $c_i(\mathbb{C}P^{n-1}) = \binom{n}{i} h^i \in \underbrace{H^{2i}(\mathbb{C}P^{n-1}; \mathbb{Z})}_{\cong\, \mathbb{Z}\langle h^i \rangle}$.

Above it was convenient to know relationship between c_1's for L, L^*. What about E vs. E^*? (& all c_i's)

<u>Lemma</u>: E rank k complex vector bundle, and let $E^* := \underline{\mathrm{Hom}}_{\mathbb{C}}(E, \mathbb{C})$.
Then for each i, $c_i(E^*) \cong (-1)^i c_i(E)$.

<u>Pf</u>: • true when $\mathrm{rank}(E) = 1$, by above. $\left(c_1(L^*) = -c_1(L), \quad c_i(L^*) = 0 = (-1)^i c_i(L) \text{ for } i > 1, \ \& \ c_0(L^*) = 1 = c_0(L) \right)$.

• true when $E \cong L_1 \oplus \cdots \oplus L_k$. $(\Rightarrow E^* \cong L_1^* \oplus \cdots \oplus L_k^*)$.

$$\Rightarrow \quad c(E^*) = \prod_{i=1}^{k} c(L_i^*) = \prod_{i=1}^{k} (1 - c_1(L_i))$$

$$\text{vs. } c(E) = \prod_{i=1}^{k} (1 + c_1(L_i))$$
; now check in deg $2i$: these differ by $(-1)^i$.

• In general, by splitting principle, $\exists$ $s : Z \to X$ (with $E \downarrow X$) w/ $s^* E \cong L_1 \oplus \cdots \oplus L_k$.

So it follows from previous case that

$$\begin{array}{ccc} (-1)^i c_i(s^* E) & = & c_i((s^* E)^*) \\ \parallel & & \parallel \\ s^*\left((-1)^i c_i(E)\right) & \text{- - - - -} & s^*\left(c_i(E^*)\right) \end{array}$$

s^* is injective so $(-1)^i c_i(E) = c_i(E^*)$ ✓. $\boxed{\text{Z}}$.

V vector space $/ \mathbb{C}$, have $\overline{(-)} : \mathbb{C} \xrightarrow{\cong} \mathbb{C}$ real-linear involution; pulling back the action
$a + ib \mapsto a - ib$
of $\mathbb{C}$ on V by $\overline{(-)}$ gives a new complex vector space $\overline{V}$; as real vector spaces $V_{\mathbb{R}} = (\overline{V})_{\mathbb{R}}$,
but

$$\underset{\substack{\uparrow \\ \text{scalar mult. in } \overline{V}}}{(a+bi) \cdot v} := \underset{\substack{\uparrow \\ \text{scalar mult. in } V}}{(a - ib) \cdot v}$$

Observe that a Hermitian inner product $\langle -, - \rangle$ on V induces an isomorphism $V^* \overset{\mathbb{C}-}{\cong} \overline{V}$.

complex linear in this factor

complex antilinear in this factor, meaning linear when thought of as a map from $\overline{V}$.

Similarly, from a complex vector bundle $E \downarrow X$, can construct $\overline{E} \downarrow X$, & a choice of (fiberwise) Hermitian metric gives an iso. $\overline{E} \cong E^*$

<u>Cor</u>: $c_i(\overline{E}) = (-1)^i c_i(E)$.

We've studied the effect of $\oplus$, $(-)^*$, $\otimes$ on char. classes, but $\otimes$ only for line bundles so far.

What about $\otimes$ for other vector bundles? In general, there's not a clean formula; however can use splitting principle to deduce formulae in each degree in any given example.

<u>Ex:</u> $G := Gr_2(\mathbb{C}^4)$, calculate $c_i(G) := c_i(TG)$ in terms of $c_1(\overset{E}{\overbrace{E_{taut}}})$, $c_2(E_{taut})$

(using $TG \cong \underline{Hom}_{\mathbb{C}}(E_{taut}, E_{taut}^{\perp}) \cong E_{taut}^* \otimes (E_{taut}^{\perp})$.)

First note that $TG = \underline{Hom}_{\mathbb{C}}(E, E^{\perp})$ so $TG \oplus Hom(E,E) \cong Hom_{\mathbb{C}}(E, E^{\perp} \oplus E)$

$$\cong Hom_{\mathbb{C}}(E, \mathbb{C}^4)$$

$$\boxed{TG \oplus Hom(E,E) = (E^*)^{\oplus 4}.}$$

so, need to compute c of this bundle.

$c_i(E^*) = (-1)^i c_i(E)$

Let's assume can find L_1, L_2 with $E \cong L_1 \oplus L_2$. (actually we cannot, but in that case, denoting by $\ell_i := c_1(L_i)$, $E^* \cong L_1 \oplus L_2$)

by splitting principle, we can pull back to such a Z where splitting holds, derive identities which remain true on original manifold

$$c(E) = (1+\ell_1)(1+\ell_2) = 1 + \underbrace{(\ell_1+\ell_2)}_{c_1(E)} + \underbrace{\ell_1\ell_2}_{c_2(E)}$$

$$c((E^*)^{\oplus 4}) = (1-\ell_1)^4(1-\ell_2)^4.$$

$$c(E^*) = (1-\ell_1)(1-\ell_2)$$

$$c(hom(E,E)) = c(E^* \otimes E) = c\left(\bigoplus_{i,j=1}^{2} L_i^* \otimes L_j\right)$$

$$c(L_i^* \otimes L_j) = 1 + \ell_j - \ell_i$$

$$= \prod_{\substack{i,j=1 \\ i \neq j}}^{2} (1+\ell_j-\ell_i) = (1+\ell_1-\ell_2)(1+\ell_2-\ell_1)$$

or $1-(\ell_1-\ell_2)^2$

The Whitney sum formula now implies that

$$\underset{\substack{\| \\ (1+c_1(TG)+c_2(TG) \\ +c_3(TG)+c_4(TG))}}{c(TG)} \cup \left(\prod_{i\neq j} 1+\ell_j-\ell_i\right) = (1-\ell_1)^4(1-\ell_2)^4.$$

Using this equation, can solve for $c_i(G)$:

$$c_1(G) = -4(\ell_1+\ell_2) = -4c_1(E)$$

$$c_2(G) = 7c_1(E)^2 = 7(\ell_1+\ell_2)^2$$

$$c_3(G) = -6c_1(E)^3$$

[illegible]

$c_4(G) = 3c_1(E)^4 - 4c_1(E)^2 c_2(E) + 4c_2(E)^2$.

One can use this to calculate Chern #'s of G, at least in terms of integrals of Chern classes of $E_{(taut)}$.

$$\text{e.g.,} \quad c_2^2[G] = \langle 49c_1(E)^4, [G] \rangle.$$

$c_2^2(G) \in H^8(G;\mathbb{Z})$

G is real 8-dim'l / cplx 4 dimensional.

<u>Linear algebra of complexifications</u>

V real vec. space dim n.

$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ complexification.. complex vec. space of dim n.

observe: in contrast to arbitrary complex vec. space, V comes equipped w/ a canonical <u>conjugation action</u>: $\mathbb{C} \xrightarrow{\overline{(-)}} \mathbb{C}$ induces (by $V \otimes_{\mathbb{R}} -$) $V_{\mathbb{C}} \xrightarrow[\cong]{\overline{(-)}} V_{\mathbb{C}}$ complex anti-linear isomorphism, i.e., induces a complex-linear isomorphism $V_{\mathbb{C}} \xrightarrow{\cong} \overline{V_{\mathbb{C}}}$.

Can recover V as $\mathrm{Fix}(V_{\mathbb{C}} \circlearrowleft \overline{(-)})$ (i.e., $+1$-eigenspace: note $\overline{(-)}^2 = id$)

If W is a complex vector space ($\dim_{\mathbb{C}} = n$), denote by $W_{\mathbb{R}}$ the underlying real vector space ($\dim_{\mathbb{R}} 2n$).

Multiplication by i on W $\longleftrightarrow$ $J: W_{\mathbb{R}} \to W_{\mathbb{R}}$, $w \mapsto iw$ (scalar mult. in W) w/ $J^2 = -id$ ("complex structure on $W_{\mathbb{R}}$")

Lem: If W ($\dim_{\mathbb{C}} n$) complex vector space then $\underbrace{(W_{\mathbb{R}})_{\mathbb{C}}}_{\dim_{\mathbb{R}} 2n} := \overbrace{(W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})}^{\dim_{\mathbb{C}} 2n} \cong W \oplus \overline{W}$.

Pf <u>sketch</u>: mult. by i on W induces as above $J: W_{\mathbb{R}} \to W_{\mathbb{R}}$ w/ $J^2 = -id$.

$\Rightarrow$ get $J_{\mathbb{C}} = J \otimes_{\mathbb{R}} id_{\mathbb{C}} : (W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \to (W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$ w/ $(J_{\mathbb{C}})^2 = -id$,

i.e., $J_{\mathbb{C}}$ has $(+i)$ and $(-i)$ eigenspaces, which together give a decomposition $(W_{\mathbb{R}})_{\mathbb{C}}$.

i.e., $W_{\mathbb{R}} \otimes \mathbb{C} \cong W^+ \oplus W^-$ (as $\mathbb{C}$-vec. spaces) $\quad W^{\pm} := \pm i$ eigenspace.

So need to show $W^+ \cong W$ (& $W^- \cong \overline{W}$, & $\overline{(-)}$ on $(W_{\mathbb{R}})_{\mathbb{C}}$ swaps W & $\overline{W}$ factors). — exercise

Define $W \xrightarrow{T} W^+$; on the level of real vector spaces $W_{\mathbb{R}} \xhookrightarrow{v \mapsto v \otimes 1} (W_{\mathbb{R}})_{\mathbb{C}} \xrightarrow{pr_{+i}} W^+$

which all together sends $w \xmapsto{T} \frac{1}{2}(w \otimes 1 - Jw \otimes i)$. $\quad \alpha \longmapsto \frac{1}{2}(\alpha - iJ_{\mathbb{C}}\alpha)$

check: $Jw \xmapsto{T} i(Tw)$; in particular. $\quad \alpha = (\alpha^+, \alpha^-) \in W^+ \oplus W^-$.

T is a complex-linear map $W \to W^+$, isomorphism (check). $iJ\alpha = (-\alpha^+, +\alpha^-)$

$\frac{1}{2}(\alpha - iJ\alpha) = (\alpha^+, 0)$.

Ex.

Pontryagin classes of real vector bundles

$E \to X$ real vec. bundle of rank k.

Form $E \otimes_{\mathbb{R}} \mathbb{C} \to X$ (fiberwise) complexification, complex rank k vec. bundle w/ an iso. $E \otimes_{\mathbb{R}} \mathbb{C} \underset{\text{conjugation}}{\cong} \overline{E \otimes_{\mathbb{R}} \mathbb{C}} \underset{\substack{\text{using} \\ \text{Hermitian metric} \\ \text{as in last time}}}{\cong} (E \otimes_{\mathbb{R}} \mathbb{C})^*$. $(\star)$

Taking Chern classes $c_i(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{2i}(X;\mathbb{Z})$, and $(\star)$ implies:

$$\underline{c_i(E \otimes_{\mathbb{R}} \mathbb{C})} = c_i((E \otimes_{\mathbb{R}} \mathbb{C})^*) \underset{\text{last time}}{=\!=} (-1)^i \underline{c_i(E \otimes_{\mathbb{R}} \mathbb{C})}.$$

If i is odd, this tells us that $2c_i(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$ in $H^{4k+2}(X;\mathbb{Z})$.
$i = 2k+1$

Def: $E \to X$ real vec. bundle of rank k, define its k^{th} Pontryagin class by

$$p_k(E) := (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}(X;\mathbb{Z}).$$

By definition, $p_k(E) = 0$ if $2k > \text{rank}(E)$.

Whitney sum formula E, E' two vector bundles, then

$$p_k(E \oplus E') := (-1)^k c_{2k}((E \otimes_{\mathbb{R}} \mathbb{C}) \oplus (E' \otimes_{\mathbb{R}} \mathbb{C}))$$

(Whitney sum for Chern classes)
$$= (-1)^k \underbrace{\sum_{\substack{i+j=2k \\ i \geq 0 \\ j \geq 0}} c_i(E \otimes_{\mathbb{R}} \mathbb{C}) \smile c_j(E' \otimes \mathbb{C})}$$
(convention $c_0 = 1$)

terms where both i, j even

terms where one of i or j is odd.

$$= \sum_{r+s=k} (-1)^{k=r+s} c_{2r}(E \otimes \mathbb{C}) \smile c_{2s}(E \otimes \mathbb{C}) + (2\text{-torsion terms})$$

$$= \sum_{\substack{r+s=k \\ r\geq 0,\, s\geq 0}} p_r(E) \smile p_s(E') + (\text{2-torsion terms}).$$

$\leftarrow$ convention that $p_0 = 1$.

So denoting $p(E) := \underbrace{1}_{= p_0(E)} + p_1(E) + p_2(E) + \cdots$ total Pontryagin class,

get
$$p(E \oplus E') = p(E)p(E') + \text{2-torsion terms}.$$

Special case:

Say $E = F_{\mathbb{R}}$ rank $2n$ real vec. bundle for $F \to X$ a complex rank n vec. bundle.

Then, a fiberwise version of the lemma at the start of lecture implies:
$$(F_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \cong F \oplus \overline{F} \underset{\substack{\text{Hermitian} \\ \text{metric on } T}}{\cong} F \oplus F^*.$$

So, $\boxed{p_k(F_{\mathbb{R}})} = (-1)^k c_{2k}(F_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^k c_{2k}(F \oplus F^*)$

$$\underset{\substack{\text{Whitney sum} \\ \text{for Chern classes}}}{=} (-1)^k \sum_{\substack{i+j=2k \\ i\geq 0\; j\geq 0}} c_i(F) \smile c_j(F^*) = \boxed{(-1)^k \sum_{\substack{i+j=2k \\ i\geq 0\; j\geq 0}} (-1)^j c_i(F) \smile c_j(F)}$$

As usual if Q a (real) smooth manifold denote $p_k(Q) := p_k(TQ)$.

Example: Compute $p_k(\mathbb{CP}^n)$.

$L :=$ taut. line bundle

We previously computed as complex vector bundles, $T\mathbb{CP}^n \oplus \underline{\mathbb{C}} \cong \underbrace{L^* \oplus \cdots \oplus L^*}_{n+1 \text{ copies}}$ (✱)

$$\Rightarrow c(T\mathbb{CP}^n) = (\underbrace{1+h}_{c(L^*)})^{n+1} \quad \text{in } H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}.$$

complex conjugating (✱), we get:
$$\overline{T\mathbb{CP}^n \oplus \underline{\mathbb{C}}} \cong \underbrace{\overline{L^* \oplus \cdots \oplus L^*}}_{n+1} \cong \underbrace{L \oplus \cdots \oplus L}_{n+1}$$

$$\overline{T\mathbb{CP}^n} \oplus \underline{\mathbb{C}} \quad (\overline{\mathbb{C}} \cong \mathbb{C}).$$

$$\Rightarrow c(\overline{T\mathbb{CP}^n}) = c(L)^{n+1} = (1-h)^{n+1}, \text{ in same ring}$$

So, $p_k(\mathbb{CP}^n) = p_k(T\mathbb{CP}^n) = (-1)^k c_{2k}(T\mathbb{CP}^n \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^k c_{2k}(T\mathbb{CP}^n \oplus \overline{T\mathbb{CP}^n})$.

$= (-1)^k \cdot (\deg 2k \text{ part of } (1+h)^{n+1}(1-h)^{n+1})$.

So, $p(\mathbb{CP}^n) = \sum_{k \geq 0} (-1)^k \left((1+h)^{n+1}(1-h)^{n+1} \right)_{\deg 2k \text{ part}}$

$= \sum_{k \geq 0} (-1)^k \left(\underbrace{(1-h^2)^{n+1}} \right)_{\deg 2k \text{ part}}$

deg $2k$ part is $(-1)^k \cdot$ deg $2k$ part of $(1+h^2)^{n+1}$, & no odd degree parts of this expression, hence.

$\boxed{= (1+h^2)^{n+1}}$ (again in $H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}$).

Special case: $n = 2m$ is even. We get $p(\mathbb{CP}^{2m}) = (1+h^2)^{2m+1}$.

In particular $p_m(\mathbb{CP}^{2m}) := p_m(T\mathbb{CP}^{2m}) = \binom{2m+1}{m} h^{2m} \in H^{4m}(\mathbb{CP}^{2m};\mathbb{Z}) \cong \mathbb{Z}\langle h^{2m} \rangle$

↑ top degree cohomology ($\dim_{\mathbb{R}} \mathbb{CP}^{2m} = 4m$).

Pairing w/ the fundamental class $[\mathbb{CP}^{2m}]$ (using complex orientation) sends $h^{2m} \longmapsto +1$, hence we get

$$\underbrace{\langle p_m(\mathbb{CP}^{2m}), [\mathbb{CP}^{2m}] \rangle}_{\text{the Pontryagin number } p_m[\mathbb{CP}^{2m}]} = \binom{2m+1}{m}$$

More generally, if X compact oriented manifold, for any collection $\{n_i \geq 0\}$ with $\sum 4i n_i = \dim X$, can define

$$\prod p_i^{n_i}[X] := \langle \underbrace{\prod p_i(TX)^{n_i}}_{\substack{\in \\ H^{\dim X}(X;\mathbb{Z}) \\ \text{by hypothesis}}}, [X] \rangle \in \mathbb{Z}.$$

Pontryagin numbers.

Observe: If $X \xrightarrow[f]{\cong} Y$ oriented diffeo. (so $f_*[X] = [Y]$) then naturality $\Rightarrow$

$\prod p_i^{n_i}[X] = \prod p_i^{n_i}[Y]$.

On the other hand, $\prod p_i^{n_i}[\overline{X}] = -\prod p_i^{n_i}[X]$.

means X w/ opposite orientation, $-[X]$.

Cor: If a single Pontryagin # is non-zero, then $X \underset{\text{oriented diff.}}{\not\cong} \overline{X}$.

Cor: $\mathbb{CP}^{2m} \not\cong \overline{\mathbb{CP}^{2m}}$.

Interestingly enough, $\mathbb{CP}^{2m+1} \underset{\text{oriented}}{\cong} \overline{\mathbb{CP}^{2m+1}}$. e.g., $\mathbb{CP}^1 = S^2 \xrightarrow{\text{reflection}} S^2 = \overline{\mathbb{CP}^1}$.

Oriented cobordism:

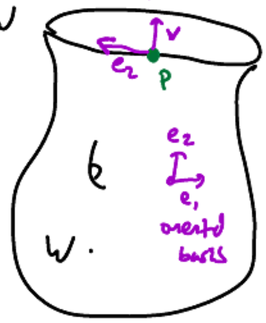
Now we'll consider W^{n+1} := compact smooth $(n+1)$-dim'l mfold w/ boundary, equipped w/ an orientation

$\rightsquigarrow$ get $[W] \in H_{n+1}(W, \partial W; \mathbb{Z})$.

If such a W is orientable (which we're assuming), then ∂W is too, & an orientation on W determines one on ∂W: the convention we'll use is "outward normal first":

(v, e_2) oriented basis in T_pW, so e_2 is an oriented basis for $T_p\partial W$.

∂W



If $p \in \partial W$, $v \in T_pW$ any 'outward' pointing tangent vector, then we declare $(e_2, \ldots, e_n) \in T_p\partial W$ to be positively oriented iff $(v, e_2, \ldots, e_n)$ is positively oriented basis in T_pW.

$\rightsquigarrow$ get using this convention, a class $[\partial W] \in H_n(\partial W; \mathbb{Z})$.

This convention is compatible w/ connecting homomorphism:

Lem: The map $\partial_* : H_{n+1}(W, \partial W; \mathbb{Z}) \longrightarrow H_n(\partial W; \mathbb{Z})$ sends $[W] \longmapsto [\partial W]$.

(omitted).

We'll sometimes denote an oriented manifold by $X := (X, \omega)$ & opposite orientation by $\overline{X} = (X, -\omega)$

choice of orientation

(section of $\widetilde{X} \to X$ if exists or $\Leftrightarrow$ a section of $\text{Frame}(TX) \times_{GL(n,\mathbb{R})} \mathbb{Z}/2$) where $\mathbb{Z}/2 := GL(n)/GL(n)^+$ inherits an action of $GL(n)$ by "sign of det").

oriented cpct m'fld-with-∂.

Say W is an <u>oriented cobordism</u> from $X_0 = (X_0, \omega_0)$ to $X_1 = (X_1, \omega_1)$ is $\partial W \underset{\text{oriented diff}}{=} \overline{X}_0 \sqcup X_1$.

oriented manifold

Example: $W := X \times [0,1]$ is an oriented cobordism from X to X.

- Any W w/ $\partial W = X$ as oriented manifolds can be thought of as an oriented cob. from $\emptyset$ to X.

Given such a W, if $i := \overline{X}_0 \sqcup X_1 = \partial W \hookrightarrow W$, LES of $(W, \partial W)$

$\Rightarrow$ since $\partial_*[W] = [\partial W]$, then $i_*[\partial W] = 0$ in $H_n(W)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\quad i_*([X_1]) - i_*[X_0]$.

$\Rightarrow i_*[X_0] = i_*[X_1]$ in $H_n(W;\mathbb{Z})$.

Using this, as before (for Stiefel-Whitney #'s) we get:

Thm: Pontryagin #'s are invariant under oriented cobordism.

$\Longleftrightarrow$ If $X = \partial W$ as oriented m'folds, then all Pontryagin #'s of X are 0.

Cor: $\underbrace{\mathbb{CP}^{2m}}_{\text{real } \dim_{\mathbb{R}} = 4m}$ is not the oriented boundary of any cpct oriented $(4m+1)$-dim'l m'fold.

(note in contrast that $\mathbb{CP}^1 = S^2 = \partial B^3$).

Also similar cor for $\coprod_r \mathbb{CP}^{2m}$, w/ same orientation for each copy.

(of course $\mathbb{CP}^{2m} \amalg \overline{\mathbb{CP}^{2m}}$ is $\partial(\mathbb{CP}^{2m} \times [0,1])$).

Today: want to compute the cohomology of $BU(k)$ resp. $BO(k)$. (why? any char. class of cplx. resp. real vector bundles of rank k is pulled back from a coh. class in $BU(k)$ resp. $BO(k)$ via classifying map, hence the computation would tell us what all possible such char. classes could be).

($BU(k) = G_k(\mathbb{C}^\infty)$, $BO(k) = G_k(\mathbb{R}^\infty)$)

We'll focus on $BU(k) \simeq G_k(\mathbb{C}^\infty)$ ($BO(k)$ case, as usual is parallel provided we work w/ $\mathbb{Z}_2$ instead of $\mathbb{Z}$-coeffs.)

To analyze space, start w/

$$\begin{array}{c} E_{taut} \\ \downarrow \\ G_k(\mathbb{C}^\infty) \end{array}.$$

The idea will be to use some form of splitting principle (a particular splitting map $s: Z \to BU(k)$) to embed $H^\bullet(G_k(\mathbb{C}^\infty))$ into $H^\bullet$(simpler space which can be computed)

The usual proof of the splitting principle produces a space $Z = \mathbb{F}(E_{taut})$ [fiberwise flags in E_{taut}] $= \mathbb{F}_k(\mathbb{C}^\infty)$. One option would be to use this space & compute $H^\bullet(\mathbb{F}(E_{taut}))$ explicitly by making use of Leray-Hirsch applied to various fibrations

e.g., $\mathbb{F}_k(\mathbb{C}^\infty) \longrightarrow \mathbb{CP}^\infty$ w/ fiber $\mathbb{F}_{k-1}$ --- see Hatcher's Alg. Topology book §4.
$\quad (L_1, \dots, L_k) \longmapsto L_1$

We'll take a shortcut by appealing to a different (somewhat simpler) splitting map (c.f. [Husemoller, Fibre Bundles]).

Consider: $X = \underbrace{\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty}_{k\text{-times}}$. On X we have the rank k vector bundle $E = L_{taut} \times \cdots \times L_{taut}$.

Equivalently, $E := \bigoplus_{i=1}^{k} \pi_i^* L_{taut}$, $\pi_i : X \to \mathbb{CP}^\infty$ proj. to i^{th} factor.

Since $BU(k)$ classifies rank k vector bundles, $\exists!$ (up to homotopy)

$$f_k : X \longrightarrow BU(k) \quad \text{with} \quad f_k^* E_{taut} = E := \bigoplus_{i=1}^{k} \pi_i^* L_{taut}.$$

Prop: f_k is a splitting map for E_{taut}, i.e., ($f_k^* E_{taut}$ splits into line bundles) and f_k^* is injective.

Pf: Let $s : Z \to BU(k)$ be any splitting map for E_{taut} ($\exists$ by splitting principle), i.e.,

$s^* E_{taut} = L_1 \oplus \cdots \oplus L_k$ for $L_i \to Z$ and s^* is injective.

Since each L_i is a complex line bundle, it is classified by a map $g_i : Z \to \mathbb{CP}^\infty$ (so $g_i^* L_{taut} = L_i$). Now consider $g = (g_1, \ldots, g_k) : Z \longrightarrow (\mathbb{CP}^\infty)^k$, and let's observe that $g^*(E = \bigoplus_{i=1}^{k} \pi_i^* L_{taut}) = \bigoplus_{i=1}^{k} g^* \pi_i^* L_{taut} = \bigoplus_{i=1}^{k} g_i^* L_{taut} = \bigoplus_{i=1}^{k} L_i = s^* E_{taut}$.

In particular, $f_k \circ g : Z \to (\mathbb{CP}^\infty)^k \to BU(k)$ classifies $s^*(E_{taut})$, because

$$(f_k \circ g)^*(E_{taut}) = g^* f_k^* E_{taut} = g^* E = s^* E_{taut}.$$

But $s : Z \to BU(k)$ classifies $s^* E_{taut}$ by definition. Since classifying maps are unique up to homotopy,

$\Rightarrow f_k \circ g \simeq s$.

$\Rightarrow s^* = g^* f_k^*$. But s^* is injective. $\Rightarrow f_k^*$ is injective as desired. $\blacksquare$

Using this, we have

Thm: Let $c_i := c_i(E_{taut}) \in H^{2i}(BU(k); \mathbb{Z})$ (rank k). Then, the classes c_i are algebraically independent for $i = 1, \ldots, k$, & moreover $H^\bullet(BU(k); \mathbb{Z}) \underset{\text{as rings}}{\cong} \mathbb{Z}[c_1, \ldots, c_k]$ $(|c_j| = 2j)$.

Pf: Consider the map $f_k : (\mathbb{CP}^\infty)^k \to BU(k)$ which classifies $E = \bigoplus_{i=1}^{k} \pi_i^* L_{taut}$. By prev. Prop, $f_k^* : H^\bullet(BU(k); \mathbb{Z}) \to H^\bullet((\mathbb{CP}^\infty)^k; \mathbb{Z}) \underset{\text{Kunneth}}{\cong} \mathbb{Z}[h_1, \ldots, h_k]$ is injective, $(|h_i| = 2$ for each $i.)$

so need to calculate $\operatorname{im}(f_k^*)$. Now consider the action of the symmetric group Σ_k on $(\mathbb{CP}^\infty)^k$ by permuting factors. The induced action on $H^\bullet((\mathbb{CP}^\infty)^k)$ permutes $(h_1, \ldots, h_n)$. Observe that E is invariant under such an action, that is,

$\sigma^* E = E$ for any $\sigma \in \Sigma_k$. In particular, $f_k \circ \sigma$ still classifies E, so (by uniqueness of classifying maps up to homotopy) $f_k \circ \sigma \simeq f_k$ i.e., $\sigma^* f_k^* = f_k^*$. Hence the image of f_k^* lands in symmetric polynomials in $h_1, \ldots, h_k$.

permutes h_i's.

Let's calculate $f_k^*(\underbrace{c(E_{taut})}_{1+c_1+c_2+\cdots+c_k}) = c(f_k^*(E_{taut})) = c\left(\bigoplus_{i=1}^k \pi_i^* L_{taut}\right)$

$$\underset{\text{whitney sum}}{=} \prod_{i=1}^k c(\pi_i^* L_{taut}) = \prod_{i=1}^k \pi_i^*(c(L_{taut})) = \prod_{i=1}^k \pi_i^*(1+h)$$

$$\overset{h_i := \pi_i^* h}{=} \underbrace{(1+h_1)\cdots\cdots(1+h_k)}.$$

Hence $f_k^* c_i = $ deg $2i$ part of $\uparrow$ $= \left(\sum_{\substack{J \subseteq \{1,\ldots,k\} \\ |J|=i}} \prod_{j \in J} h_j\right) = \sigma_i$ ith elementary symmetric polynomial in $h_1, \ldots, h_k$.

Fact: There are no alg. relations between any elementary symmetric polynomials, and any symmetric polynomial in $h_1, \ldots, h_k$ can be uniquely written as a polynomial in $\sigma_1, \ldots, \sigma_k$

Using this, we learn that $\mathrm{im}(f_k^*) = \{$subring of $\mathbb{Z}[h_1,\ldots,h_k]$ gen. by $\sigma_1, \ldots, \sigma_k\} \cong$ all symmetric polynomials

$\cong \mathbb{Z}[\sigma_1, \ldots, \sigma_k]$. deg $\sigma_i = 2i$.

with $c_i \overset{f_k^*}{\longmapsto} \sigma_i$.

Hence $H^\bullet(BU(k);\mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_k]$. $\boxtimes$.

Cor: Each char. class $\phi: \mathrm{Vect}_{\mathbb{C}}^k(-) \to H^\bullet(-;\mathbb{Z})$ [of complex rank k bundles] must have the form $E \longmapsto q(c_1(E), \ldots, c_k(E))$ where q is a polynomial uniquely determined by the class. (q is the element of $\mathbb{Z}[c_1,\ldots,c_k] \cong H^\bullet(BU(k);\mathbb{Z})$ given by taking $\phi(E_{taut})$).

Betti #'s : rank $H^i = b_i$.

Cor: $b_{2k+1}(BU(n)) = 0$, & $b_{2k}(BU(n)) = \mathrm{rk}\, H^{2k}(BU(2k))$

$$= \dim\left(\text{deg. } 2k \text{ part of } \mathbb{Z}[c_1, \ldots, c_n]\right) \quad |c_i| = 2i.$$

$$= \# \text{ of monomials } c_1^{r_1} \cdots c_n^{r_n} \text{ of degree} \qquad r_i \geq 0$$

$$2k = 2(r_1 + 2r_2 + 3r_3 + \cdots + nr_n)$$

$$= \sum z^{(r_1 + 2r_2 + 3r_3 + \cdots + n r_n)}$$

$= \#$ of n-tuples $(r_1, \ldots, r_n)$ w/ $k = \underline{r_1 + 2r_2 + \cdots + n r_n}$.

$= \#$ of unordered partitions of k into at most n integers $\{k_1, \ldots, k_n\}$

$(k_1 \le k_2 \le \cdots \le k_n$ & $\sum k_i = k)$

via $(r_1, \ldots, r_n) \longleftrightarrow \underbrace{r_n}_{k_1} \le \underbrace{r_n + r_{n-1}}_{k_2} \le \underbrace{r_n + r_{n-1} + r_{n-2}}_{k_3} \le \cdots \le \underbrace{r_n + \cdots + r_1}_{k_n}$.

The same arguments apply to compute $H^*(BO(k); \mathbb{Z}/2)$ (using $\mathbb{RP}^\infty$ instead of $\mathbb{CP}^\infty$ etc. as usual)

$\Rightarrow$ Thm: $H^*(BO(k); \mathbb{Z}/2) \cong \mathbb{Z}[w_1, \ldots, w_k]$ where $w_i := w_i(E_{taut})$, $|w_i| = i$.

(in particular w_i are all alg. independent).

$\Rightarrow$ all char. classes of real vect. bundles of rank k taking values in $H^*(-; \mathbb{Z}/2)$ are polynomials in the Stiefel-Whitney classes.

We won't spell out the details, but a more involved computation shows that, modulo certain 2-torsion elements

$H^*(BO(k); \mathbb{Z}) \cong \mathbb{Z}[p_1, \ldots, p_{\lfloor \frac{k}{2} \rfloor}]$ (mod 2-torsion)

certain polynomials in Stiefel-Whitney classes

Pontryagin classes of E_{taut}.

(beginning of another possible paper topic!)