Math 171 Midterm Examination

October 18, 2007

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Total	

Name____

Signature______________________________________

Directions:

1. This is closed book/notes exam (only your single 8 1/2 by 11 inch information sheet allowed).
2. Your signature above indicates that you accept the University Honor Code.
3. Write your solutions on the exam sheet; you may use the back side of a page if you run out of space. Throughout the exam you should give complete and clear proofs of your statements, justifying your steps. If you are using a particular theorem, be sure to state clearly what you are using. If you have a question about what you may assume without proof, please be sure to ask.
4. This test is 2 hours and has 5 problems worth a total of 50 pts.
5. Good luck!

Problem 1. Let $\{x_n\}$ be a bounded sequence of real numbers.

(a) (5 pts) Define $\limsup x_n$ and $\liminf x_n$ and show that $\liminf x_n \leq \limsup x_n$.

(b) (5 pts) Show from the definitions that $\liminf x_n = \limsup x_n$ if and only if $\lim x_n$ exists.

Problem 2. Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, we let B denote its closed unit ball; that is, $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

(a) (4 pts) Determine B explicitly for the l^1 and l^∞ norms on $\mathbb{R}^2$. Recall that $\|(x, y)\|_1 = |x| + |y|$ and $\|(x, y)\|_\infty = \max\{|x|, |y|\}$.

(b) (6 pts) Show that for any norm on $\mathbb{R}^n$, B is a convex set which contains a neighborhood of the origin. (Recall that a set is convex if it contains the line segment between any two of its points.)

Problem 3. (a) (3 pts) Give an example to show that the union of a collection of closed subsets of $\mathbb{R}$ need not be closed.

(b) (7 pts) A collection of subsets of a metric space M is said to be *locally finite* if each point of M has a neighborhood which intersects only finitely many sets from the collection. Show that the union of a locally finite collection of closed subsets of a metric space is closed.

Problem 4. Let K be a compact subset of a metric space and O an open subset with $K \subseteq O$.

(a) (5 pts) Prove that there is a number $\epsilon > 0$ so that for all $x \in K$ we have $D(x, \epsilon) \subseteq O$.

(b) (5 pts) Construct an open set U with $K \subseteq U$ and $cl(U) \subseteq O$.

Problem 5. Given a metric space M, we define a relation $\sim$ on M by defining $x \sim y$ if there is a continuous path in M from x to y.

(a) (5 pts) Prove that $\sim$ is an equivalence relation.

(b) (5 pts) Show that each equivalence class is a path connected subset of M. Show that any path connected subset of M is contained in an equivalence class.