Math 520 Take-home Final

Due Wednesday, May 13, 2020 at 5 pm (by e-mail to Instructor and TA)

There are 5 problems, worth 45 points total. (point values are indicated below) Please remember to write down your name on your take-home final. You are welcome to use any results we have used/stated/proved in class or the homework. You may consult the textbook, course notes, and previous homework assignments. (You can consult other references for background, but you must solve these problems completely on your own) *You may not discuss your solutions with other people except for the course instructor and TA.*

1. *Rigidity of holomorphic functions* (10 points, 5 points per part):
   (a) Let $f : \Omega \to \Omega$ be holomorphic and satisfy $f \circ f = f$. Show that either f is constant or $f = id_\Omega$. (In contrast, there are non-constant and non-identity real differentiable functions on domains with $g \circ g = g$, for instance $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $g(x, y) = (x, 0)$).

   (b) Weierstrass's theorem for a real interval states that a continuous real-valued function on any interval $[a, b]$ can be uniformly approximated by real polynomials. Prove that the analogous theorem in the complex category is *false*. That is, show that there is a continuous function on e.g., the closed unit disk which *cannot* be uniformly approximated by complex polynomials.

2. *Properties of some specific complex analytic functions* (10 points, 5 points per part):
   (a) Give the Taylor series expansion about $z = 0$, and its radius of convergence, for the function $\frac{z^5}{(z^2+1)(z-1)}$.

   (b) Let f be a function which is complex differentiable in $B_r(a)\backslash\{a\} = \{0 < |z - a| < r\}$, except at a sequence $a_n \in B_r(a)\backslash\{a\}$ tending to a (so $\lim_n a_n = a$) along each of which f has poles. Show that for any $y \in \mathbb{C}$, there exists a sequence of points in the domain of f, $z_n \in B_r(a)\backslash\{a\}$ with $\lim_{n\to\infty} f(z_n) = w$.

   Bonus (optional, worth 1 point): Write down an example of such a function, for some r and a.

3. *On biholomorphisms* (10 points, 5 points per part):
   (a) Prove that the punctured disc $A = B_1(0)\backslash\{0\}$ and the annulus $B = \{1 < |z| < 2\}$ are not biholomorphic. In particular, (seeing as $H_1(A) = H_1(B)$) this shows that the analogue of the Riemann mapping theorem for non-simply connected domains with the same homology is false.

   (b) Map the region $\{|z - \frac{1}{2}| > \frac{1}{2}, \mathrm{Re}(z) > 0\}$ biholomorphically onto the unit disk. You can express your answer as a composition of elementary biholomorphisms.

4. *Counting zeroes* (5 points). How many roots does the polynomial $p(z) = z^5 + 12z^3 + 3z^2 + 20z + 3$ have in the annulus $\{1 < |z| < 2\}$?

5. *Definite integrals* (10 points, 5 points per part).

(a) Evaluate the definite integral $\int_{-\infty}^{\infty} \frac{\sin(\frac{\pi}{4}x)dx}{x(x^2+1)}$.

(b) Let a be a real number with $0 < a < 3$. Evaluate the definite integral $\int_0^{\infty} \frac{x^{a-1}dx}{1+x^3}$.