Week 4 Day 1 (no day 2 this week!) 4/18/2016

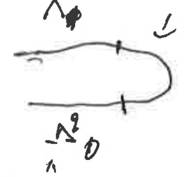
<u>Orientations</u> (a brief redux): how does the (S)Pin str on Lag'ns contribute?

$T_pX \cong \mathbb{C}^n$

$\cup$

T_pL_1

$\shortparallel$



a choice of paths Λ_t^p.

$H = D^2 \setminus \{-1\}$,

$E = \mathbb{C}^n$,

$F \subseteq E|_\partial$

path of Lag'n subspaces.

T_pL_0

$\cap$

T_pX

$\cong$

$\mathbb{C}^n$.

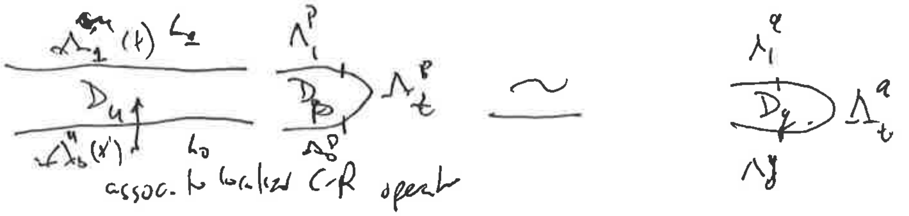
Recall, to $p \in L_0 \cap L_1$

$\rightsquigarrow$ o_p orientation line

$= \det(D_p)$ D_p a local Cauchy-R operator on

(depends on Λ_t, but certainly, ∃ unique choice if L_0, L_1 are <u>graded</u>; L_i come w/ lifts $\tilde{L}_i$) - & then $\deg(p) = \mathrm{ind}(D_p)$.

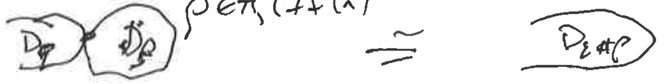
Given p, q, & $u \in \mathcal{M}(p,q)$, wanted to say there was a gluing isomorphism



assoc. to localized C-R operator on u^*TX, u^*TL_i.

~~Choosen~~ Such an isomorphism depends on a homotopy of paths of Lag'ns rel endpoints between Λ_t^q & $\Lambda_1^u(t) \cdot \Lambda_t^p \cdot \Lambda_0^u(t)$.

(note: we're assuming these paths are homotopic rel endpoints, which is only true in general if L_i graded; otherwise, need to factor in local effect of "connect summing in a non-trivial loop")



Problem: $\pi_1(\mathcal{P}_{a,b}\mathcal{L}(n)) = \pi_1(\Omega\mathcal{L}(n)) \neq 0!$

rel fixed endpoints

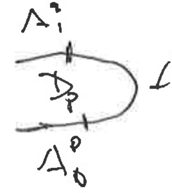
$= \pi_2\mathcal{L}(n)$

$= \mathbb{Z}/2$ stably.

So, there is a non-trivial choice <u>here</u> to make, which ~~often~~ changes the sign of the isomorphism $\det(D_u \# D_p) \cong \det(D_q)$.

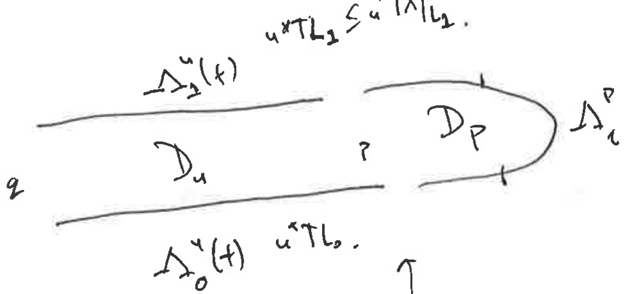
<u>Sol'n</u>: Equip each L_i w/ a Spin (or rel. Pin) structure.

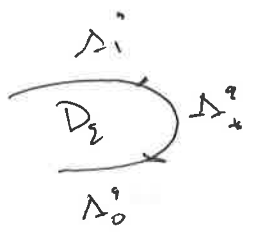
For each $p \in L_0 \cap L_1$, $\rightsquigarrow$ extra choice:



, equip Λ_t^p, think of as $F \subseteq \mathbb{C}^n|_{[0,1]} \times [0,1]$, w/ a Spin structure rel. existing ones at endpoints $\Lambda_1^p \cong T_pL_1$, $\Lambda_0^p \cong T_pL_0$:

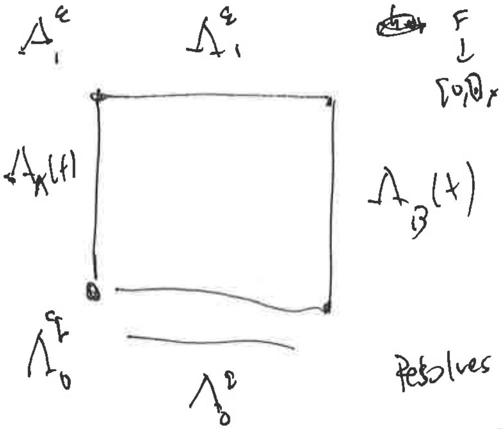
Then, in this situation:





Both paths $\Lambda_A(t)$ & $\Lambda_B(t)$ are equipped w/ Spin structures. (rel. Pin)

Now, there is a unique choice of homotopy of paths rel. endpoints $\Lambda_A \sim \Lambda_B$ which itself possesses a Spin str. extending the choices at the boundary rel endpts



$$\begin{tikzcd} F \arrow[d] \\ {[0,1]\times[0,1]} \end{tikzcd}$$

uniquely determined, by

Then note: the choices of Spin structures made at p & q have minimal effect; at most a global sign flip for trajectories going from/going to p; (get isomorphic chain cplx).

Resolves ambiguity in isomorphism.

Product structures: Say we have defined $\underset{\underset{CF^*(L_0,L_1)}{\|}}{CF^*(L_0,L_1;H_{01},J_{01})}$, $\overset{\overset{CF^*(L_1,L_2)}{\|}}{CF^*(L_1,L_2;H_{12},J_{12})}$, $\underset{\underset{CF^*(L_0,L_2)}{\|}}{CF^*(L_0,L_2,H_{02},J_{02})}$.

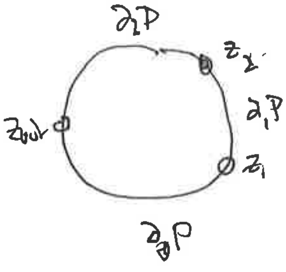
Let's recall the construction of a product map

$$\mu^2: CF^*(L_1,L_2)\otimes CF^*(L_0,L_1)\to CF^*(L_0,L_2)$$

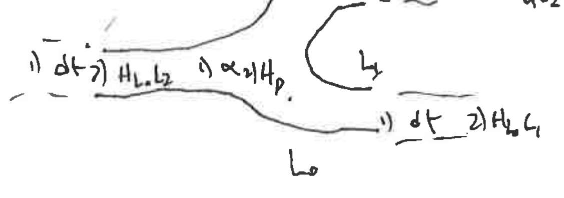
Consider $\overset{P=}{D^2}\setminus 3$ pts., equipped w/ three "strip-like ends," 2 pos., 1 neg.

$\epsilon_i^{\pm}:[0,\infty)\times[0,1]\to P$.

Equip P w/
1) one-form α
2) Ham. perturbation term (could be zero sometimes)
3) a.c. structure J (P dependent)
4) "Lagrangian labels"



$\longleftrightarrow$



Rmk: In nice cases, may accept • $H_p = 0$ and/or • J P independent (and/or integrable)
in order to actually count solutions to those equations!)

Now, given $z_2 \in$ ~~$L_1 \cap L_2$~~, $z_1 \in L_0 \cap L_1$ & $\cdot z_{out} \in \chi^{H_{L_0,L_2}}_{L_0,L_2}$,
$L_1 \cap L_2$ $\qquad \chi^{H_{L_0,L_1}}_{L_0,L_1}$

(or rather $\chi^{H_{L_1 L_2}}_{L_1, L_2}$)

↑ set of traj chords $L_1 \to L_2$.

consider $\mathcal{M}(z_{out}; z_1, z_2) =$

$$\left\{ \begin{array}{l} u : P \longrightarrow X \\ (*)\left(du - X_{H_S} \otimes \alpha \right)^{0,1} = 0 \\ \cdot\ u|_{\partial_i P} \subseteq L_i \\ \bullet\ \lim_{s \to +\infty} (\epsilon_i^{\pm})^* u(s,t) = z_i \qquad i = 1, 2. \\ \cdot\ \lim_{s \to -\infty} ((\epsilon_i^{-})^* u)(s,t) = z_{out} \end{array} \right.$$

↑ w.r.t. J_S : $(K)^{0,1} = K + \frac{1}{2}(J_S \circ K \circ j)$

$\mathcal{M}_{\cancel{z_{out}}}(z_{out}; z_1, z_2)$ decomposes as

$$\coprod_{\beta \in \pi_2(-)} \mathcal{M}(z_{out}, z_1, z_2).$$

Rmks: strip-like ends help (a) analytically in ~~taking~~ setting up a space of maps

$W^{k,p}(P, X, L_i)$ (need to specify "exponential convergence near non-compact ends" in order to get Fredholm problem)

(b) establishing in compactness analysis that when a strip breaks off, it corresponds exactly to a solution to Floer's eqn for H_{L_i, L_j}, J_{L_i, L_j}.

consistency condition.

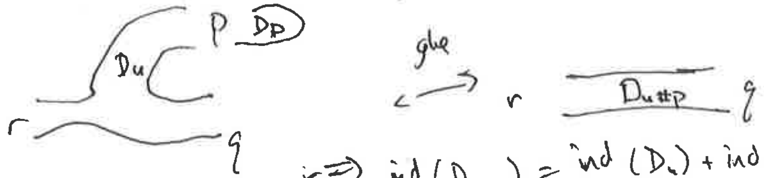
In nice cases analogous to those already described:

(1) $\mathcal{M}$ is a manifold of dimension $\deg(z^{out}) - \deg(z_1) - \deg(z_2)$

↑ deg defined using fixed choices of grading.

(~~Pf~~ transversality argument + index calculation: can calculate the index of a homotopy class via a local gluing argument:

Ex:



$\underset{\text{(gluing)}}{\Longrightarrow}$ $\mathrm{ind}(D_{u\#p}) = \mathrm{ind}(D_u) + \mathrm{ind}(D_p)$

$\deg(r) - \deg(q) = \mathrm{ind}(D_u) + \deg(p)$.).

~~edge~~ **Def:** ~~A Lag'n Fix~~ Say $2c_1(X)=0$, & fix a choice of ~~[illegible]~~ fibrewise ~~[illegible]~~ over (univ. or $\mathbb{Z}/N$).

$$\begin{tikzcd} \tilde{\mathcal{L}} \arrow[d] \\ \mathcal{L} \arrow[d, dashed] \\ X \end{tikzcd}$$

$\mathcal{L}$ Lag'n Grass'man

A Lag'n <u>brane</u> is a triple $L^{\#} = (L, \tilde{\alpha}_L, P)$.

$\underset{\text{it}}{}$ $\uparrow$ grading structure ($\mathbb{Z}$ or $\mathbb{Z}/N$) $\nwarrow$ Spin structure (if L oriented) or rel. Pin str.

$$\text{ob}\,DFuk(X) = \begin{cases} \text{Lag'n branes ~~fix a choice~~ } L^{\#}. \\ Hom(L_0, L_1) = HF^{0}(L_0, L_1) = H^0(~~hfm~~ CF^*(L_0, L_1; H_{L_0,L_1}; J_{L_0,L_1})) \\ \text{Composition } Hom(L_1, L_2) \otimes Hom(L_0, L_1) \to Hom(L_0, L_2) \end{cases}$$

pick some generic choice.

Given by $[\mu^2](-,-)$. (we've not verified $[\mu^2]$ indep. of choice, but follows from a "homotopy of homotopies" argument)

Rmk: Note that ~~[illegible]~~ even though $L \pitchfork\!\!\!\!/\ L$, w/ a Ham. term, can define $HF^*(L,L) = H^0(~~CF~~ CF^*(L, L; H_{L,L}; J_{L,L}))$.

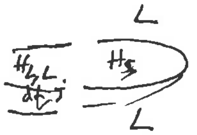
<u>Need</u>: (a) identity morphisms?

$[e_L] \in Hom(L,L)$ for each L.
$\quad\quad\quad\quad \overset{\shortparallel}{HF^*(L,L)}$

Define
$$e_L = \bigoplus \sum_{\substack{z_{out};\\ \beta;\, ind(\beta)=0}} \#\mathcal{M}(z_{out}, \beta)\, T^{\omega(\beta)} \cdot z_{out}$$

where

$$\mathcal{M}(z_{out}) = \begin{cases} u : \text{[domain]} \longrightarrow X \\ (du - X \otimes \alpha)^{0,1} = 0 \\ u|_{\partial S} \in L. \end{cases}$$



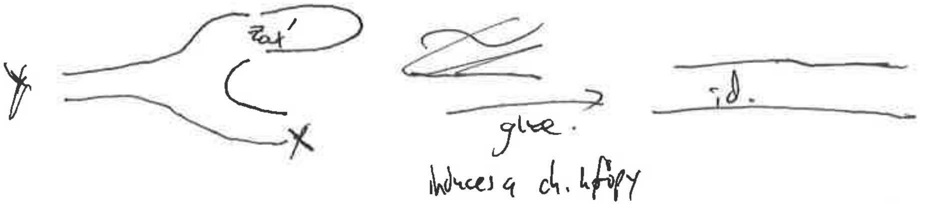
note: $ind(u) =$
~~[illegible]~~ z_{out} graded, ~~ind(u)~~ for any $u \in \mathcal{M}$,
$ind(u) = ind(D_\beta)$
$= deg(\beta)$.

why is there a unit?

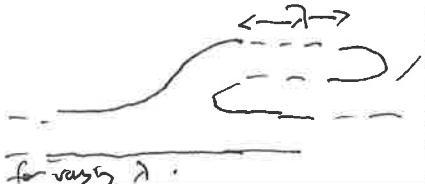
Need e.g. $[\mu^2](X, e_L) = X$

this composition counts

(in fact, $(-1)^{(deg(x))}[\mu^2(x, e_L)] = \mu^2(e_L, ~~x~~)$
$= x$
for any $x \in Hom(k,L)$ or $Hom(L,k)$ resp.)



induces a ch. htpy
if one carefully steps through ~~the~~ counting sol'ns



$\Rightarrow$ can define

$$\mu^2(x,y) = \sum_{z,\beta} \#(\mathcal{M}_2(z; x,y))\, T^{\omega(\beta)} \cdot z$$

$\uparrow$ in the sense before (we'll be impressionistic about signs),

<u>Claim:</u> in fact,

$$\mu^2(y,x) = \sum_{z,\beta} (-1)^{|y|} \#(\underline{\qquad\qquad}) \, z.$$

<u>Prop:</u>

$$\mu^2(y,\mu^1(x)) + (-1)^{|x|-1} \mu^2(\mu^1(y), x) + \mu^1\mu^2(y,x) = 0.$$

The count of sol'ns to K as λ varies gives a ch. htpy between $\mu^2(x, e_L)$ and ~~the~~ ~~$I(x)$~~ $I(x)$

where $I(x)$ counts index 0

(b) Associativity:

given ~~$L_0 \xrightarrow{x_{01}}$~~ $L_0 \xrightarrow{X_{01}} L_1 \xrightarrow{X_{12}} L_2 \xrightarrow{X_{23}} L_3$

need

$$[\mu^2]\big(X_{23}, [\mu^2](X_{12}, X_{01})\big)$$

But there are only const. sol'ns in index 0 ⇒ identity map.!

$$= [\mu^2]\big([\mu^2](X_{23}, X_{12}), X_{01}\big).$$

as maps

$$HF^*(L_2,L_3) \otimes HF^*(L_1,L_2) \otimes HF^*(L_0,L_1) \to HF^*(L_0,L_3)$$

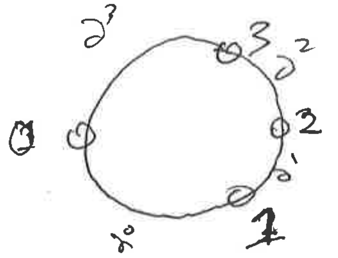
On chain level: we'll show the difference is null htpic.

Namely, we'll construct $P: CF^*(L_2,L_3) \otimes CF^*(L_1,L_2) \otimes CF^*(L_0,L_1) \to CF^*(L_0,L_3)$

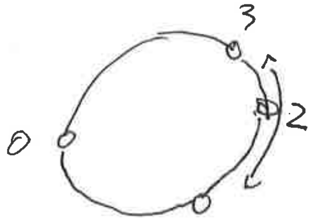
w/ $$P(\mu^1 \otimes id \otimes id) \pm P(id \otimes \mu^1 \otimes id) \pm P(id \otimes id \otimes \mu^1)$$
$$\pm \mu^1 P = \mu^2(\mu^2(-,-) \otimes id) - \mu^2(id \otimes \mu^2(-,-))$$

We'll call it μ^3. ($3 \longleftrightarrow$ "3 inputs")

Let $\mathcal{R}^3$ denote the space of discs with ~~three~~ four boundary points removed, and a biholomorphism up to bihol., the position of 0, 1, 3 can be fixed at ~~0 1~~ $-e^{2\pi i k/3}$

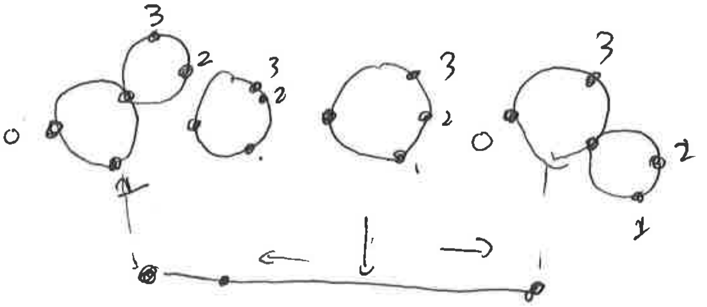


⇒ one real modulus.



$$\mathcal{R}^3 \cong (0,1)$$

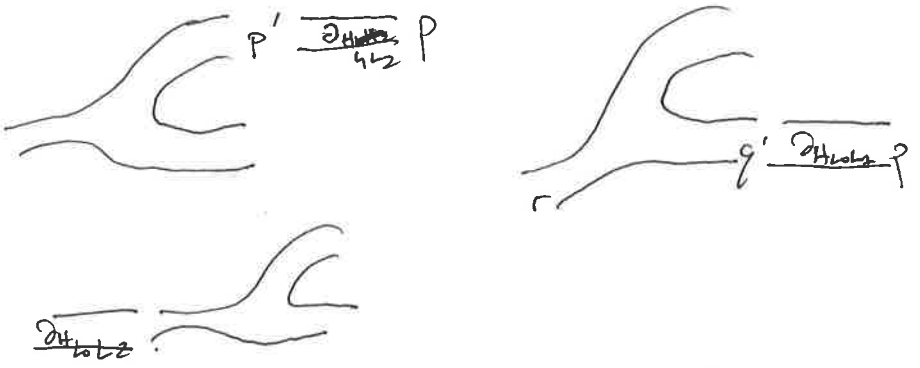
There is a natural compactification of the space of domains to $[0,1]$: $\overline{\mathcal{R}}^3$.



There is a universal family of domains

$$\begin{tikzcd} \overline{\mathcal{S}}^3 \arrow[d] \\ \overline{\mathcal{R}}^3 \end{tikzcd}$$

(2) Gromov compactness & gluing: in absence of sphere / disc bubbling ⇒ $\mathcal{M}$ is compact (if dim 1),
w/ three types of codim-1 boundary: "energy accumulates near punctures":



(*)

(3) $\mathcal{M}$ is orientable (w.r.t. an a priori fixed Spin structure on L_i), with an isomorphism

$$\lambda(T\mathcal{M}(z^{out}; z_1, z_2)) \cong O_{z_{out}} \otimes O_{z_1}^{\vee} \otimes O_{z_2}^{\vee}.$$ (*)

⇒ can define:

$$\mu^2(z_2, z_1) = \sum_{\substack{z,\ \beta \in \pi_2(-) \\ ind(\beta)=0}} \#(\mathcal{M}(z_{out}; z_1, z_2, \beta))\, T^{\omega(\beta)} \cdot z_{out}$$

where # is taken in the signed sense of bebe

$deg(z_{out}) - deg(z_1) - deg(z_2)$

Because the signed count of the elements in the boundary of a dim 1 ~~1-mfld~~ is ~~zero~~ 0, (using K-naturalities of orientation lines & signed maps induced by (*))

~~Prop:~~ components of $\overline{\mathcal{M}}(z_{out}; z_1, z_2)$; (actually $\mu^2(z_2, z_1) = \sum (-1)^{deg(z_1)}$).

we have:

Prop: $\mu^2(z_2, \mu^1(z_1)) + (-1)^{deg(z_1)-1} \mu^2(\mu^1(z_2), z_1) + \mu^1\mu^2(z_2, z_1) = 0.$

$\mu^1 = \delta : CF^*(L_0, L_1)\circlearrowleft$, $\delta : CF^*(L_1, L_2)\circlearrowleft$, $\delta : CF^*(L_0, L_2)\circlearrowleft$

~~i.e.~~ Namely, μ^2 is a chain map, so descends to

$$[\mu^2] : HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \to HF^*(L_0, L_2).$$

Thm: [Donaldson]: ~~$[\mu^2]$ form the morphisms of a category~~

$[\mu^2]$ gives the composition in a category the Donaldson-Fukaya category $DFuk(X) = H^0Fuk(X)$.

Defined: ~~for $CF^*(L_1, L_2)$~~ ~~$\{L_i\} \subset X$~~, $L_i \subseteq X$ Lagn branes (w/ gradings, Spin str., etc.)

$$\mu^2 : \underset{\substack{\uparrow \\ \text{dep. on extra choice} \\ H_{L_1,L_2}, \\ J_{L_1,L_2}}}{CF^*(L_1,L_2)} \otimes CF^*(L_0,L_1) \longrightarrow CF^*(L_0,L_2)$$ which in nice cases was a chain map.

~~$\mathcal{L}$~~ $K, L \in \text{ob } H^0\mathcal{F}uk(X)$. $\rightsquigarrow$ $CF^*(K,L) \circlearrowleft \mu^1$ ch. cplx.
depending on $H_{K,L}$, $J_{K,L}$

We argued $[\mu^2]$ gave composition in a category:

- identity morphisms: $\exists\, e_K \in HF^0(K,K)$ w/ $[\mu^2](e_K, \eta) = \eta$, for $\eta \in HF(K,L)$. η " $[\mu^2](\eta, e_L)$
- associativity?

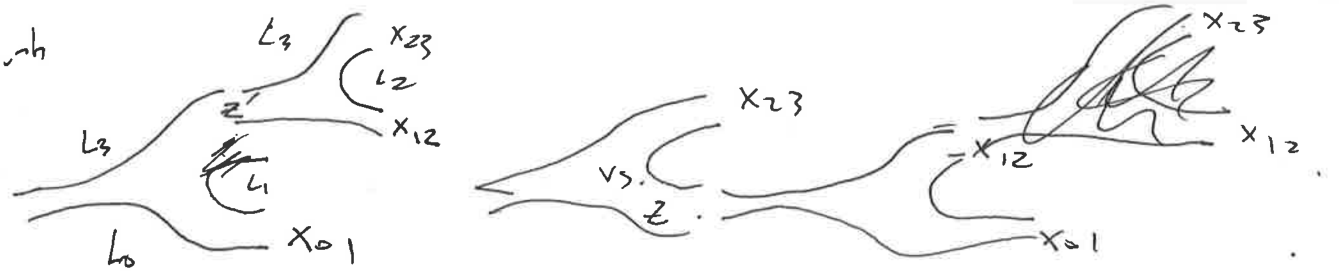
Given $L_0 \xrightarrow{x_{01}} L_1 \xrightarrow{x_{12}} L_2 \xrightarrow{x_{23}} L_3$, need $[\mu^2]([x_{23}], [\mu^2]([x_{12}], [x_{01}])) = [\mu^2]([\mu^2]([x_{23}], [x_{12}]), [x_{01}])$

~~$[\mu^2](x_{23}, [\mu^2]^2$~~

$$\overset{(A)}{[\mu^2(x_{23}, \mu^2(x_{12}, x_{01}))]} = \overset{(B)}{[\mu^2(\cancel{x_{23}}\ \mu^2(x_{23}, x_{12}), x_{01})]}$$ as maps $HF^*(L_2,L_3) \otimes HF^*(L_1,L_2) \otimes HF^*(L_0,L_1) \to HF^*(L_0,L_3)$

Rmk: there is no reason for these to agree on the chain level! Note

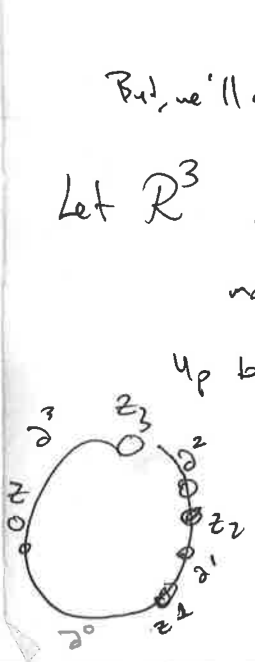
(A) counts



But, we'll show the difference is null htopic via a natural geometric choice of chain homotopy. Moreover, ~~the~~ the same construction produces higher homotopies, $\rightsquigarrow A_\infty$ structure.

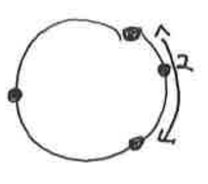
Let $\mathcal{R}^3$ denote the space of discs with four boundary points removed, one ~~output~~ ~~[illegible]~~ output, & 3 inputs, mod biholomorphism.

Up to bihol., the position of z_0, z_1, z_3 can be fixed at $-e^{2\pi i k/3}$



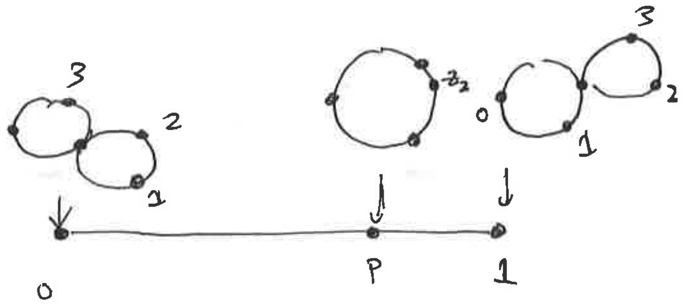
$\Rightarrow$ one real modulus

$\longleftrightarrow$



so $\mathcal{R}^3 \cong (0,1)$.

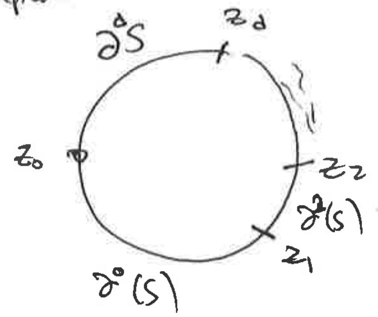
There is a natural "Delgne-Mumford" compactification of the space of such domains to [0,1]; $\overline{\mathcal{R}}^3$.



More generally, let $\mathcal{R}^d$ $(d \geq 2)$ be the space of discs w/ $\underline{d+1}$ boundary marked points, removed, (one marked as an output) / modulo automorphism, ~~Biholomorphism~~ up to biholo. can fix the position of any 3 points

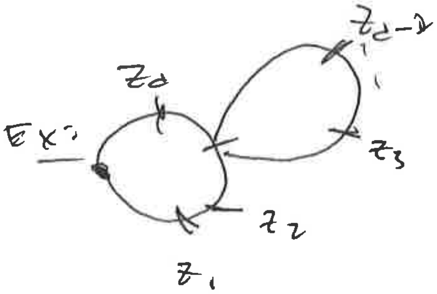
$\Rightarrow$ $\dim \mathcal{R}^d = d-2$. (explicitly, $\mathcal{R}^d = \mathrm{Conf}_{d+1}(\partial D)/\mathrm{Aut}(D)$).

A representative:



There is a natural (Deligne-Mumford) compactification to a $(d-2)$-dim'l polytope, $\overline{\mathcal{R}}^d$, the Stasheff associahedra; top-dimensional boundary faces correspond to a nodal degeneration of D to a pair of discs

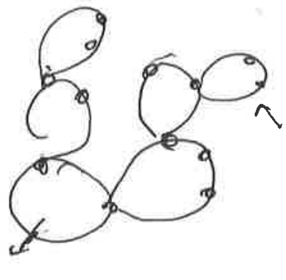
$D_1 \cup D_2$; each component carries at least two of the marked points.



~~Eq~~ There is a corresponding universal family $\overline{\mathcal{S}}^d$ of domains,

$$\overline{\mathcal{S}}^d \downarrow \overline{\mathcal{R}}^d$$

Call the fiber over $r \in \overline{\mathcal{R}}^d$ S_r: A potentially nodal Riemann surface of the form



^ each component stable (# marked pts. including nodes ≥ 2); modulo automorphism of each factor ~~preserving position of points.~~

$Z_+ := [0,1] \times [0,\infty)$

& $Z_- := [0,1] \times (-\infty, 0]$.

Def: A choice of pos./neg. strip-like ends around a given marked pt. is a smoothly varying map $\overline{\mathcal{R}}^d \times Z_\pm \to \mathcal{S}_z$, varying smoothly in $z \in \overline{\mathcal{R}}^d$ and each z_i.

A universal & consistent choice of Floer data is a ~~choice~~ smoothly varying choice, inductively for each d & each d-tuple $(L_0, \ldots, L_d)$, & each $S \in \mathcal{S}^d$

(a) Ham. term: $H : S \longrightarrow C^\infty(M; \mathbb{R})$

w/ $(\varepsilon_i^{\pm})^* H = H_{L_i, L_{i+1}}$, $(\varepsilon_{0-})^* H = H_{L_0, L_d}$.

(b) almost cplx. structure $J : S \longrightarrow \mathcal{J}(X)$

$(\varepsilon_i^{\pm})^* J = J_{L_i, L_j}$, $(\varepsilon_0^-)^* J = J_{L_0, L_d}$.

(c) one-form α on S.

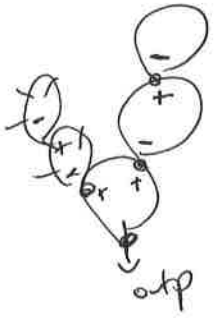
w/ $(\varepsilon_i^{\pm})^* \alpha = dt$.

smoothly varying in S,

Consistency means: the restriction of this data to a boundary ~~stratum~~ component S of w/ induced suitable labeling $(L_{i_0}, \ldots, L_{i_s})$ agrees with choices ~~made on~~ already made on S, & agrees to infinite order w/ the ~~data~~ Floer data induced by gluing maps.

~~So~~ choosing a family of strip-like ends, around each boundary marked point (w/ the sign indicated,

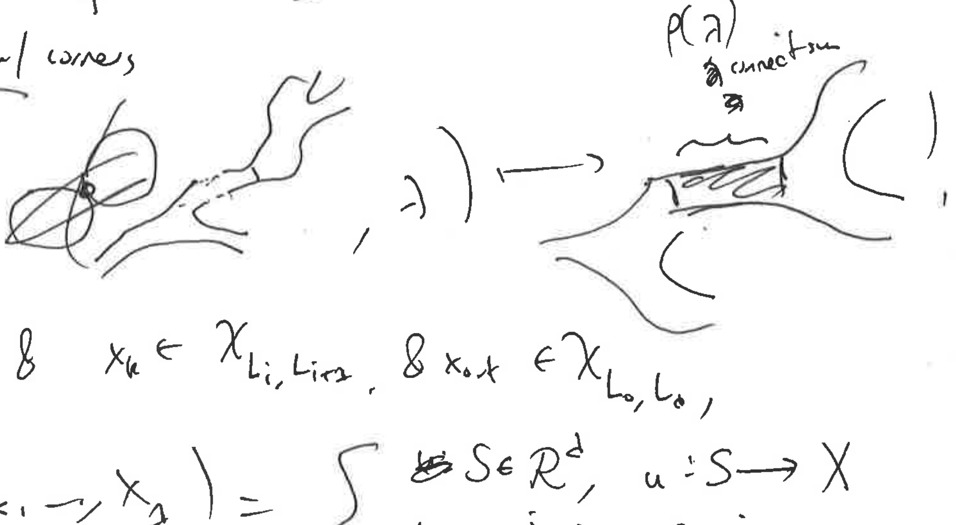
- each ~~component~~ nodal marked point (w/ sign induced by tree structure.



$\Rightarrow$ gluing charts from given strata

$$\partial_T \mathcal{R}^d \times [0,1)^{\#\text{nodes in } T} \longrightarrow \overline{\mathcal{R}}^d,$$

$\Rightarrow \overline{\mathcal{R}}^d$ is a manifold w/ corners



Prop: universal & consistent choices exist.

Given a tuple $(L_0, \ldots, L_d)$, & $x_k \in \mathcal{X}_{L_i, L_{i+1}}$ & $x_{out} \in \mathcal{X}_{L_0, L_d}$,

get
$$\mathcal{R}^d(x_0; x_k, \ldots, x_1) = \left\{ \begin{array}{l} S \in \mathcal{R}^d,\ u : S \longrightarrow X \\ \text{w/ } \bullet\ u(\partial^i S) \subseteq L^i. \\ \bullet\ \lim\limits_{s \to +\infty} (\varepsilon_i^{*}) u(s,t) = x_i \\ (du - X \otimes \alpha)^{0,1} = 0. \end{array} \right.$$

x_0

Remarks: (a) (transversality):

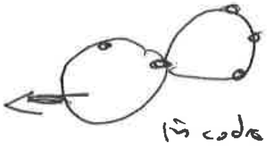
$$\dim = k - 2 + \deg(x_{out}) - \sum \deg(x_i)$$

(b) compactness & gluing: limit configurations allowed by Gromov compactness

= ~~• bubbling of spheres / disks~~ a priori excluded for now

• bubbling of strips at nodal points

• degeneration of domain to $\partial\overline{\mathcal{R}}^d$]



is codim 1

(c) orientability: $\mathcal{R}^d$ (oriented rel. "ends" as below, choice of orientation of $\mathcal{R}^d$)

$$\lambda(T\mathcal{R}^d(\text{—}-)) \cong \lambda(T\mathcal{R}^d) \otimes o_{x_0} \otimes o^\vee_{x_1} \otimes \cdots \otimes o^\vee_{x_d}.$$

get relations when consider ∂ of 1-dim'l families of disks

Define
$$\mu^k(x_d, \text{—}, x_1) = \sum_{\substack{x_0,\ \beta\in\pi_2 \\ |x_0| = \sum|x_i| + 2 - k}} (-1)^{*} \# \mathcal{R}^d(x \text{—}-, \beta)\, T^{E(\beta)} \cdot x_{out}.$$

to get signs to work exactly as desired, require sign correction $\left(\sum_{k=1}^{n} k\,|x_k|\right)$.

For each d,

Prop: $$\sum_{\substack{~~i=1~~ \\ i,k}} (-1)^{\maltese_i} \mu^{d-k+1}(x_d, \text{—}, x_{i+k+1}, \mu^k(x_{i+k}, \text{—}, x_{i+1}), x_i, \text{—}, x_1)$$

where $\maltese_i =$

Def: An A_∞ functor is a map:

$/ \quad / \quad /$