Orientations [following de Silva + FOOO, Floer-Hofer (high case), Seidel] : Week 3 Day 2

Wants to associate a signed count $\#\left(\mathcal{M}(p,q)/\mathbb{R}\right)$ in the Floer differential for $CF^*(L_0, L_1)$.

$\longleftrightarrow$ an orientation of $\mathcal{M}(p,q)/\mathbb{R}$ $\underset{\text{(invariant) } \mathbb{R}\text{-action}}{\longleftrightarrow}$ an orientation of $\mathcal{M}(p,q)$

~~Meta~~ Meta statement: (not most general) : say L_0, L_1 oriented.

$\mathcal{M}(p,q)$ is canonically oriented relative to orientations of certain 1-D real vector spaces associated to p & q; ~~...~~ after fixing Spin structures on L_0, L_1.

the orientation lines o_p, o_q.

~~... associated ...~~

Notation: V f.d. v.s. $\rightsquigarrow$ $\det(V) \cong \Lambda^{\dim V} V$, this is a $\mathbb{Z}/2$ graded 1-d vector space (grading = dim V)

$(\lambda(V))$

M manifold $\rightsquigarrow$ $\lambda(TM)$ determinant line bdle.

orientations $\longleftrightarrow$ trivializations of $\lambda(TM)$ (mod $\mathbb{R}_+$ rescaling).

$D: X \to Y$ Fredholm operator

$\rightsquigarrow \det(D) = \lambda(\mathrm{coker}(D))^* \otimes \lambda(\ker D)$, given the fibers of

a natural line bundle $\underline{\det} \to \mathrm{Fred}(X,Y)$.

More specific to our setting,

Banach bdle $\begin{matrix}\mathcal{E}\\ \downarrow \pi \uparrow s\\ \mathcal{B}\end{matrix}$ section whose linearization (vert. part) is Fredholm $\rightsquigarrow$ $\begin{matrix}\underline{\det}\\ \downarrow\\ \mathcal{B}\end{matrix}$ and $\begin{matrix}\underline{\det}\\ \downarrow\\ \mathcal{M} = s^{-1}(0)\end{matrix}$

Note: If $\mathcal{M} = s^{-1}(0)$ transversely cut out, then $\underline{\det}_u \cong \lambda(\ker D_u) = \lambda(T_u \mathcal{M})$.

So an orientation/trivialization of $\begin{matrix}\underline{\det}\\ \downarrow\\ \mathcal{M}\end{matrix}$ $\longleftrightarrow$ orientation of $\mathcal{M}$.

Local orientation operators :

Given $p \in L_0 \cap L_1$, (or ~~path~~ two-λ chord of X_H $L_0 \xrightarrow{\gamma} L_1$), after trivializing $TM|_p$ (or $TM|_\gamma$)

p/γ ~~...~~ $\rightsquigarrow$ path ~~of~~ Λ_t between Lagrangian subspaces Λ_0 & Λ_1 of $\mathbb{C}^n$.

(if L_i are graded, this path is uniquely specified up to htpy rel. endpoints)

~~Takes a~~ $\rightsquigarrow$ a bundle pair (E,F) over $(D^2 \setminus \{+1\}, \partial D^2 \setminus \{1\}) \cong H$.

$E \cong \mathbb{C}^n$.

or rather fixed ~~strip-like~~ negative strip-like end ~~...~~ fixed neg. strip-like end near ∞.

$F = \Lambda_1 \quad \Lambda_t, \quad \Lambda_0$ choose a local J, δ ~~...~~ $\bar\partial_{F}$-operator on $\mathbb{C}^n$, asymptote near ∞ to the pull back of $\bar\partial_{J,H} = 0$ ~~...~~ which,

$\rightsquigarrow$ ~~[illegible]~~ acts on sections asymptotic to P.

namely: $\bar\partial_{J,H}: W^{k,p}(\mathbb{H}, \xi^* F, \text{"}\overset{\circ}{p}\text{"}) \to W^{k-1}(\mathbb{H}, \Omega^{0,1}_{\mathbb{H}} \otimes E)$.

Linearize $\rightsquigarrow D_p$.

Def: $o_p := \det(D_p)$.

(Claim: this is independent of choices made, and an elaboration of the index calculation which shows $\operatorname{ind}(D_p) = \#$ times Λ_t fails to be transverse to Λ_1, w/ sign.,

also establishes $\det(D_p) \cong \bigotimes_{\substack{t_i \text{ times} \\ \Lambda_t \not\pitchfork \Lambda_1}} V_{t_i}$ where $V_{t_i} = (\Lambda_t \cap \Lambda_1)^{\otimes \text{sign (orientation)}}$

1-D generally; consider generic case.

~~[Floer, FOOO, de Silva, Seidel]~~ .

Thm: L_0, L_1 oriented. Suppose we have fixed a Spin structures on L_0, L_1.

Then, there is a canonical isomorphism

$$\lambda(T\mathcal{M}(p,q)) \cong o_q \otimes o_p^\vee.$$

~~Floer theory w/ signs:~~ How does this work? Consequence of gluing orientation lines:

Given $u \in \mathcal{M}(p,q)$, ~~can glue & homotope ... C-R operators~~ trivialize u^*TM & obtain local C-R operator D_u

Can glue $\underset{q}{D_u} \underset{p}{\#} D_p \xrightarrow[\text{homotopy}]{\sim} D_q$

Make such a choice for each. Gluing theorem: induces isomorphism $\det(D_p) \otimes \det(D_u) \xrightarrow{\sim} \det(D_p)$.

Need to check: indep. of choice of homotopy (uses Spin structures).

(a) ... (b) choice of trivialization of $u^* TM$ ~~...~~

Point: "Spin structure on L determines a ~~unique~~ distinguished (suitable) htpy class of trivializations on of TL on 1-skeleton of L which extends to the two skeleton."

Rmk: not most general; could take rel-Pin

Compare: Thm [de Silva, FOOO]: L orientable. Then orientations of $\mathcal{M}(x,y,J)$ induced by spin str. on L.

Floer theory w/ signs: Two options! (a) coherent orientations! choose isomorphisms $o_x \cong \mathbb{R}\ \forall x$.

(b) canonical orientations.

$$CF^*(L_0, L_1; H, J) := \bigoplus_{x \in L_0 \cap L_1} |o_x|_\Lambda \quad \text{ground field.}$$

Given V real v.s. $|V|_{\mathbb{K}}$ is the "~~[illegible]~~ space of $\mathbb{K}$ orientations"

$$\Rightarrow \bigoplus_{\substack{v \text{ an orientation} \\ \text{of } V}} \mathbb{K}\langle v \rangle \Big/ \mathbb{K}\langle \textstyle\sum_v v \rangle$$

free $\mathbb{K}$-module gen. by orientations of o_x, mod rel'n that sum vanishes.

Orientations [~~Seidel, adapting original approach by Floer–Hofer~~]
de Silva + FOOO Lag'n case, Seidel].

Notation: V fin. dim'l vec. space $\rightsquigarrow \det(V) \cong \Lambda^{\dim} V$ $\big((\dim V) \bmod 2\big)$ - graded vector space.
$\lambda(V)$

M m'fld $\rightsquigarrow \lambda(TM)$ determinant line bundle. orientations $\longleftrightarrow$ trivializations of $\lambda(TM)$ (mod $\mathbb{R}_+$).

$D: X \to Y$ Fredholm operator $\rightsquigarrow \det(D) = \lambda(\operatorname{coker}(D))^* \otimes \lambda(\ker D)$. $(*)$.

fiber of a natural line bdle $\det \longrightarrow \mathrm{Fred}(X, Y)$.

More specific to our setting,

Banach bdle. $\mathcal{E}$, $\pi \downarrow \uparrow s$, $\mathcal{B}$; s section whose ~~vertical~~ linearization (vert. part) is Fredholm.

$\rightsquigarrow$ $\underline{\det} \downarrow \mathcal{B}$ and $\underline{\det} \downarrow \mathcal{M} = s^{-1}(0)$.

Note: if $\mathcal{M} = s^{-1}(0)$ is transversely cut out, then
$$\underline{\det}_u \cong \lambda(\ker D_u) = \lambda(T_u \mathcal{M}).$$

So an orientation of $\mathcal{M}$ $\longleftrightarrow$ orientation of $\underline{\det} \downarrow \mathcal{M}$

Ex (moduli spaces of discs w/ Lag'n boundary):

$L \subseteq X$ Lag'n. J a.c.s.

$\rightsquigarrow \mathcal{M}(X, L; J, \beta)$ [$\beta \in \pi_2(X, L)$] := space of J-hol discs $u: (D^2, S^1) \longrightarrow (X, L)$ in class β.

Thm [de Silva, FOOO]. Suppose L is oriented. Then, orientations of $\mathcal{M}(X, L; J, \beta)$ are induced by Spin structures on L.

Rmk: Not the most general; can e.g. have L ~~relatively~~ "relatively Pin" to some background class $b \in H^2(X; \mathbb{Z}_2)$ (this is the case L orientable, $b = 0$).

Idea: At a given u, the ~~linearized problem involves the ∂̄ operator~~ Cauchy–Riemann operator is of the form:

"bundle pair" $E_{\mathbb{R}}$ u^*TX, F u^*TL $\quad \bar{\partial}^u_J : W^{1,2}(D^2, S^1, u^*TX, u^*L) \longrightarrow L^2(D^2, u^*TX \otimes \Omega^{0,1}_X)$

$\downarrow$ (D^2, S^1); study Linearized operator: D_u: $(E, F) \downarrow$

Note: u^*TL is trivial, but $\downarrow S^1$ not necessarily as a sub-space of u^*TX.

since L oriented,

Can degenerate D_u : $\underline{\mathbb{C}}^n_{\mathbb{R}^n} \subseteq \underline{\mathbb{C}}^n|_{S^2}$ $\quad B \downarrow_E S$ some

There are gluing (homotopy) isomorphisms for local Cauchy Riemann operators.

$\Rightarrow \ker D_u \cong \ker D_{u_1} \#_R \ker D_{u_2} \underset{R \text{ large}}{\cong} \operatorname{ind}(D_{u_1}) \oplus \operatorname{ind}(D_{u_2})$
and coker ... and ... and ...

$\Rightarrow \det(D_u) \cong \det(D_{u_1}) \otimes \det(D_{u_2})$

but now, $\ker D_{u_1}$ is complex oriented

& by explicit calculation,
$$\ker D_{u_2} \cong \mathbb{R}^n, = \text{fiber at any given point } u(\cdot) \in L.$$

so can choose the orientation induced by one on L..

Problem: need to show independence of choices.

Given ~~both~~ two different paths $u \overset{\gamma_1}{\underset{\gamma_2}{\rightrightarrows}} u_1 \# 92$ of operators, show resulting orientation is the same

($\Leftrightarrow$) given a loop, show resulting family of ~~[illegible]~~ det. lines is the trivial family.

Can do this, provided that, given a loop of Cauchy Riemann operators,

$(t \in S^1,\ \gamma_t : (D^2, S^1) \to (X, L))$, $\gamma^* TL \downarrow S^1 \times S^1$ is trivial.

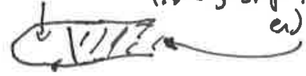
obstruction theory:

Orientation: extend trivialization from 0-skel. $\to$ 1-skel.

Spin str.: extend a trivialization from 1-skel to 2-skel. $\square$

Lagn Floer ~~Cohomology~~ Homology: Let now $p \in L_0, L_1$ intersection point (or time-1 chord of a Hamiltonian X_H) Q: what are asymptotics?

associate a local Cauchy Riemann operator (maybe w/ X_H term?) $D^2 \setminus pt. \cong H$, ~~$F \cong \mathbb{C}^n$~~ fix any strip-like end



F

$\mathbb{C}^n$ / F Lagn sub-bundle.

$\downarrow$ $\downarrow$ $\longleftrightarrow$ "path of Lagrangians in $\Lambda(n)$"

H ∂H

corresponding to $T_pL_0 \to T_pL_1$. (easy to do if L_0, L_1 both have grading structures)

Denote by D_p its linearization, and define the orientation line of p to be

$$O_p := \det(D_p) \ .\ .$$ (a priori, depends on choice of path)

Remark Essentially by Floer's work, there's another way to think about O_p.

Note that $\operatorname{ind}(D_p) = \#$ times path Λ_t fails to be transverse to $\Lambda_1(n)$ Maslov cycle "w/ sign."

at each transverse crossing, $\Lambda_{t_0} \pitchfork \Lambda_1$, can associate a real vector space V_{t_0} (normal bundle to Λ_1 at point?).

& $\det(D_p)$ "$\cong$" $V_{t_0} \otimes V_{t_1} \otimes \cdots \otimes V_{t_k}$".

Thm: ~~[illegible]~~ [~~[illegible]~~, F?00, Seidel]: ~~Fa~~ $L_0, L_1 \subseteq X$ graded (?). e.g. de Silva, F?00

A Spin str. on L_0 & L_1 induces ~~an~~ isomorphisms

~~then~~ -- $\lambda(T\mathcal{M}(p,q)) \cong O_q \otimes O_p^\vee$. $\mathbb{R}$?

when $L_0 \& L_1$)

Rmk: have to treat $\mathbb{R}$ action $\mathcal{M}$. (e.g. decide how to orient $\mathbb{R}$) "the moduli space is canonically oriented relative!"

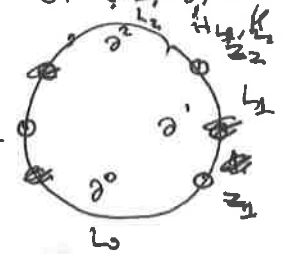
Floer theory w/ signs:

$CF^*(L_0, L_1; \cancel{K}, J) := \bigoplus_{x \in L_0 \cap L_1} |o_x|_\Delta$.

one dimensional ~~k~~ vector space over Δ, generated by choices of orientations of o_x / rel'n that sum of choices vanishes

Now, given rigid $u \in \mathcal{M}(p,q)/\mathbb{R}$,
prev. thm shows that get ~~[illegible]~~ isomorphism

Back to <u>product structures</u>: Wanted to define a product $CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2; H_{L_0,L_2})$, with H_{L_1,L_2}, H_{L_0,L_1}.

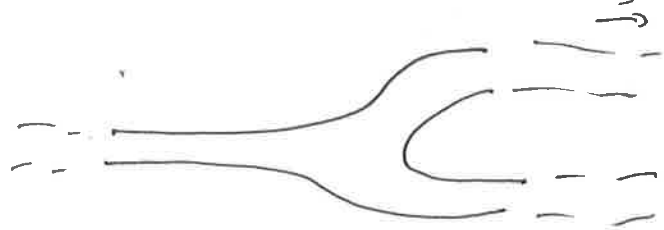


Let $P := D^2 \setminus \{ e^{2\pi i k/3} \}$

Pick ~~a set of~~ pair of positive strip-like ends $\epsilon_+^i : [0,\infty) \times [0,1] \to P$ around z_i, properly embeddings, mapping $\mathbb{R}^+ \times \{0,1\}$ to ∂P.

& $\epsilon_-^{out} : (-\infty, 0]_s \times [0,1] \to P$ around z_{out}

$\Rightarrow$ $P \cong$



(Rmk: as opposed to last time, we'll keep pos. strip-like ends on the right, in same direction as travel as $s \to \infty$).

Given $u \in \mathcal{M}(p,q)/\mathbb{R}$ <u>rigid</u>, get an isomorphism (after translating $\mathbb{R}$ action)

$$\lambda\left(T\frac{\mathcal{M}(p,q)}{\mathbb{R}}\right) \cong \mathbb{R} \qquad \lambda(T\mathcal{M}(p,q)) \cong o_q \otimes o_p^\vee .$$
$$\lambda(T\mathcal{M}(p,q)) \cong \mathbb{R}$$

$\to$ get isomorphism $\mu_u : o_p \xrightarrow{\cong} o_q$.

$\to$ $|\mu_u|_\Lambda : |o_p|_\Lambda \xrightarrow{\cong} |o_q|_\Lambda$.

For $[x] \in |o_p|_\Lambda$,

define $$\delta([x]) = \sum_q \sum_{\substack{\beta \\ \mathrm{ind}(\beta) = 1}} \sum_{\substack{u \in \mathcal{M}(p,q,\beta)/\mathbb{R} \\ \text{rigid}}} T^{\omega(u)} \mu_u([x]),$$

(strip-like ends are helpful b/c they are helpful analytic tools ~~indeed~~ a ~~...~~ for talking about

- maps $W^{k,p}(P, X, L_i)$ ~~which "exponentially converge" to intersections~~ with asymptotic properties near non-compact ends,
- describing $\bar\partial_J$ a C-R equation over P ~~that~~ whose limits at various ends agree w/ prev. C-R equations.)

Fix: • Hamiltonian term
$$H : P \to C^\infty([0,1]\times X, \mathbb{R})$$
such that $\varepsilon_1^* H = H_{L_0,L_1}$, $\varepsilon_2^* H = H_{L_1 L_2}$, etc.

• almost cplx. structure
$$J : P \to \mathcal{J}(X)$$
w/ $\varepsilon_1^* J_s = J_{L_0,L_1}$, etc.

• one-form α on P w/
$$\varepsilon_1^* \alpha = dt \,..$$

& consider maps $u : P \to X$ w/
given

$$\begin{cases} u(\partial^i P) \subseteq L_i \\ \lim_{s\to+\infty} (\varepsilon_+^i)^* u(s,t) = p^i \quad i=1,2 \\ \lim_{s\to-\infty} (\varepsilon_-^{out})^* u(s,t) = p^{out} \\ \left(du - X_{H_s}\otimes\alpha\right)^{0,1} = 0 \quad \text{w.r.t. } J_s \end{cases}$$

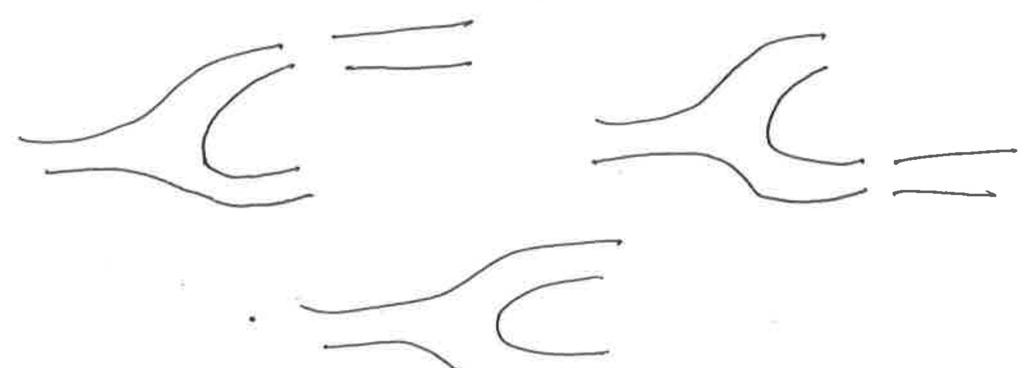
$\rightsquigarrow$ moduli space
$$\mathcal{M}(p^{out}, p_1, p_2) = \coprod_{\beta\in\pi_2(-)} \mathcal{M}(p^{out}; p_1, p_2).$$

~~Fredholm theory~~

In nice cases analogous to those already described:

(a) $\mathcal{M}$ is a ~~smo~~ manifold of dimension $\deg(p^{out}) - \deg(p_1) - \deg(p_2)$.
(when L_i are absolutely graded).

(b) Gromov compactness + gluing: (in absence of sphere/disc bubbles), $\mathcal{M}$ is compactifiable, w/ three types of codim 1 boundary. ∴ "energy accumulates near punctures".



(c) $\mathcal{M}$ is orientable (w.r.t. a priori fixed Spin structures on L_i), w/ an isomorphism
$$\lambda(T\mathcal{M}(p^{out}; p_1, p_2)) \cong \cancel{o_{p_1}\otimes o_{p_2}\otimes} \; o_{p^{out}} \otimes o_{p_1}^\vee \otimes o_{p_2}^\vee$$