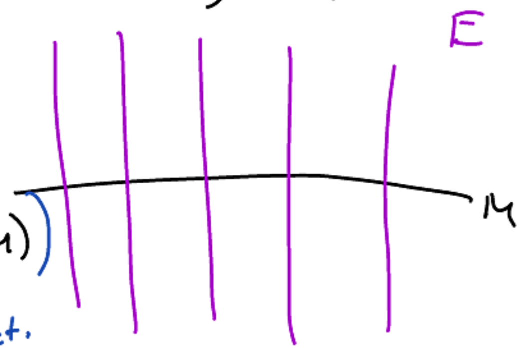
Relation between Thom classes & Poincaré duality

Say M^m oriented cpct (connected for simplicity) manifold, $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ oriented vector bundle (rank k), then E is a non-compact oriented manifold dimension $m+k$, so we have non-compact formulation Poincaré duality for E:



$$H_c^*(E) \xrightarrow[\text{P.D.}]{\cong} H_{k+m-*}(E) \left(\xrightarrow{\cong} H_{k+m-*}(M) \right)$$

relate this to other groups we've studied.

Picking a metric $\langle -,- \rangle$ on E, get an exhaustion of E by cpct. sets (b/c M is cpct.) $D_R(E) = \{(x,v) \mid \|v\| \le R\}$.

$$\Rightarrow H_c^*(E) \cong \varinjlim_R H^*(E, E \setminus D_R(E))$$

note
$$\begin{tikzcd}
(E, E\setminus D_R(E)) \arrow[r, hook, "\sim"', "\text{h.e.}"] \arrow[d, hook] & (E, E^0) \\
(E, E\setminus D_{R'}(E)) \arrow[ur, "\text{h.e.}"'] &
\end{tikzcd}$$
for $R > R'$

$$\cong H^*(E, E^0) \quad \text{(by } \to \text{)}.$$

So therefore the Thom class $u \in H^k(E, E_0)$ corresponds to an element of $H_c^k(E) \cong H_m(E)$ via Poincaré duality for E. what element?

$i_M : M \underset{\sim}{\overset{0}{\hookrightarrow}} E$ homotopy inverse to $\pi: E \to M$

Claim: P.D. for E sends Thom class u to $(i_M)_*[M]$ in $H_m(E)$.

(P.D. for E: determined by orientation on E as a manifold (which in turn is determined by orientation on M and one on E as a bundle). u: determined by orientation on E as a bundle. $[M]$: det. by orientation on M.)

How to use this fact?

If $K^k \subseteq M^m$ any submanifold (assume smooth), K, M oriented, cpct $\Rightarrow$ get a class $i_*[K] \in H_k(M)$. ($[K]$)

Can ask: how to think about $\text{P.D.}([K]) \in H^{m-k}(M)$ explicitly?

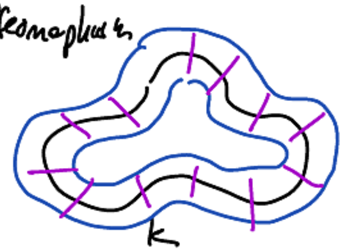
(1) (Tubular neighborhood theorem):

Thm: $\exists$ open nhood $\mathcal{U}$ of K in M and a diffeomorphism

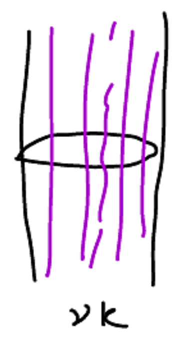
$$\begin{tikzcd}
\nu_K \arrow[rr, "\cong", "G"'] & & \mathcal{U} \\
& K \arrow[ul, "0"] \arrow[ur, "\text{incl.}"'] &
\end{tikzcd}$$

normal bundle of K in M ($TM|_K / TK$)

e.g.,



$\cong$



(2) Given an open inclusion (codim. 0) of m-manifolds $W \overset{i}{\subset} W'$, we prev. asserted one gets

$$\begin{tikzcd}
H^*_c(W) \arrow[r, "i_!"] \arrow[d, "\wr\|"', "\text{P.D.}"] & H^*_c(W') \arrow[d, "\wr\|", "\text{P.D.}"'] \\
H_{m-*}(W) \arrow[r, "i_*"] & H_{m-*}(W')
\end{tikzcd}$$

"extend by zero" (covariant).

$\Rightarrow$ in our setting above for $U \overset{i}{\underset{\text{open}}{\subset}} M$, get:

nbhd of K

u Thom class of ν_K

$$\begin{tikzcd}
H^*_c(\nu_K) \arrow[r, phantom, "\cong"] \arrow[d, "\wr\|"', "\text{P.D.}"] & H^*_c(U) \arrow[r, "i_!"] \arrow[d, "\wr\|"', "\text{P.D.}"] & H^*_c(M) \arrow[r, phantom, "\underset{M\text{ cpct}}{=}"] & H^*(M) \arrow[d, "\wr\|", "\text{P.D.}"'] \\
H_{m-*}(\nu_K) \arrow[r, phantom, "\cong"] & H_{m-*}(U) \arrow[rr, "i_*"] & & H_{m-*}(M) \\
H_{m-*}(K) \arrow[u, "\wr\|"] \arrow[ur, "i_*"'] & & &
\end{tikzcd}$$

Q: what α?

$[K] \in H_k(K) = H_{m-(m-k)}(K)$ $\longmapsto$ $i_*[K]$

Cor: $i_*[K]$ (sometimes just called $[K]$) $\in H_k(M)$ is P.D. in M to pushforward of Thom class of the normal bundle ν_K to K in M.

Cobordism rings Using what we've developed so far (plus some further indicated topics), want to compute cobordism rings.

Recall: M, M' m-dim'l cpct manifolds. (smooth)

- say M, M' are cobordant if $\exists$ cpct $(m+1)$-manifold-with-∂ W s.t. $\partial W \overset{\text{fixed diffeo}}{\cong} M \amalg M'$.
- If M, M' oriented, say M, M' oriented cobordant if $\exists$ cpct. oriented —— W s.t.

$$\partial W \cong \overline{M} \amalg M' \quad \text{as oriented manifolds.}$$

$\overline{M}$: M w/ reversed orientation

Lemma: The relation of being cobordant resp. oriented cobordant is an equiv. relation. (exercise)

Define:
$$\Omega_n := \{\text{cpct oriented } n\text{-m'flds}\} / \text{oriented cobordism} \qquad \text{oriented cobordism grp.}$$
$$\mathfrak{N}_n := \{\text{cpct } n\text{-m'flds}\} / \text{cobordism}. \qquad \text{(unoriented) cobordism grp.}$$

Group structure? (Focus on Ω_n, η_n case is same – and simpler).

identity? $0 = [\emptyset]$

addition? $[M]+[M'] := [M \amalg M']$

inverses? note that $M \amalg \overline{M} = \underset{\substack{\uparrow \\ \text{oriented sense}}}{\partial} (M \times [0,1])$ i.e., $[M] + [\overline{M}] = 0$.

$\Rightarrow [\overline{M}]$ additive inverse to $[M]$.

note: if $M \underset{\text{oriented diff}}{\cong} M'$ then $[M] = [M']$ in Ω_m, similarly in η_m if $M \underset{\text{diff.}}{\cong} M'$.

(in η_n, note that $M \amalg M = \partial(M \times [0,1]) \Rightarrow [M]+[M] = 0$. i.e., $2[M]=0$

$\Rightarrow [M]$ is its own additive inverse $\Rightarrow \eta_n$ is 2-torsion)

Product structure:

$$M^m, N^n \longmapsto (M \times N)^{m+n},$$ inherits a canonical orientation if M, N oriented.

induces $\Omega_m \times \Omega_n \longrightarrow \Omega_{m+n}$ $\rightsquigarrow$ $\Omega_* = \bigoplus_{i \geq 0} \Omega_i$ into a graded ring.

(& $\eta_m \times \eta_n \longrightarrow \eta_{m+n}$.

$\rightsquigarrow \eta_* := \bigoplus \eta_i$ similarly.)

(graded commutative? $M^m \times N^n \underset{\text{oriented diff}}{\cong} (-1)^{mn} N^n \times M^m$)

$(-1)^{mn}$ means orientation reverse if mn is odd.

$\Rightarrow [M] \times [N] = (-1)^{mn} [N] \times [M]$.

Basic tools for studying Ω_* / η_*: Pontryagin / Stiefel–Whitney numbers:

For any unordered partition $I = \{i_1, \dots, i_r\}$ of k, have Pontryagin numbers

$\{1,2,1\} = \{1,1,2\} \rightsquigarrow p_1 p_1 p_2 = p_1^2 p_2$

$$[M^{4k}] \longmapsto p_I [M^{4k}] := \langle \prod_{j=1}^{r} p_{i_j}(M), \underset{\mathbb{Z}\text{-fund. class}}{[M]} \rangle \in \mathbb{Z}$$

& Stiefel-Whitney numbers

$$[N^k] \longmapsto w_I [N^k] := \langle \prod_{j=1}^{r} w_{i_j}(N), \overset{\mathbb{Z}/2\text{-fund. class}}{[N]} \rangle \in \mathbb{Z}/2$$

what we've shown is that those associations give group homomorphisms:

$$p_I : \Omega_{4k} \longrightarrow \mathbb{Z}, \qquad w_I : \eta_k \longrightarrow \mathbb{Z}/2.$$ respectively.

HW had an application of w_I to bounding below size of η_k, for instance.

Similarly, a computation reveals that:

Thm: (e.g., Milnor–Stasheff): The collection $\{\underbrace{\overbrace{\mathbb{CP}^{2j_1}\times\cdots\times\mathbb{CP}^{2j_r}}^{4k\text{-dim'l real manifolds, oriented.}}}_{\boxed{\mathbb{CP}_J}}\}_{J=(j_1,\dots,j_r)\in p(k)}$ ← (unordered) partitions of k

are all linearly independent and non-zero in Ω_{4k}.

$\Rightarrow \operatorname{rank}(\Omega_{4k}) \geq \# p(k)$.

A couple ideas of proof:

Basic idea is to show that the matrix $\left(p_I(\mathbb{CP}_J)\right)_{I,J\in p(k)}$ is non-singular.

$\Rightarrow \prod_{I\in p(k)} p_I : \Omega_{4k} \longrightarrow \mathbb{Z}^{\#p(k)}$

sends $\{\mathbb{CP}_J\}_{J\in p(k)}$ to a linearly independent collection.

Input: • $p(\mathbb{CP}^{2k}) = (1+h^2)^{2k+1}$ in $\mathbb{Z}[h]/h^{2k+1}$

(so for $i<k$ $p_i(\mathbb{CP}^{2k}) = \binom{2k+1}{i} h^{2i} \Rightarrow p_I[\mathbb{CP}^{2k}] = \binom{2n+1}{i_1}\cdots\binom{2n+1}{i_r}$ ← $I=\{i_1,\dots,i_r\}$

• formula for products. ☐

A few oriented cobordism groups (Milnor–Stasheff p. 203): share —.

The key to computing cobordism ring is the following fundamental theorem of Thom:

Def: $G_k(\mathbb{R}^\infty)$ Grassmannian of k-planes, $\underline{E^k_{taut}} \to G_k(\mathbb{R}^\infty)$.
$\;$
$\overset{\shortparallel}{BO(k)}$

oriented rank k vector bundle.

$\widetilde{G}_k(\mathbb{R}^\infty)$ Grassmannian of oriented k-planes, $\underline{\widetilde{E}^k_{taut}} \to \widetilde{G}_k(\mathbb{R}^\infty)$. (2:1 cover of $BO(k)$ via

$\overset{\shortparallel}{BSO(k)}$ (classifying space for oriented vector bundles) $\;\;\;\;(V,\omega) \longmapsto V.$)
$\;$ $\overset{\cap}{\mathbb{R}^\infty}$

Thm: ([Thom]) Fix n. For any $k > n+1$, there is an isomorphism

$$\pi_{n+k}(T(\widetilde{E}^k_{taut}), t_0) \xrightarrow{\cong} \Omega_n$$

$$\pi_{n+k}(T(E^k_{taut}), t_0) \xrightarrow{\cong} \mathfrak{N}_n$$

($(n+k)^{th}$ homotopy group) — "Thom space" — as above — canonical basepoint in any Thom space.

Goal: explain ingredients of proof, & a way of computing LHS in terms of homology groups.

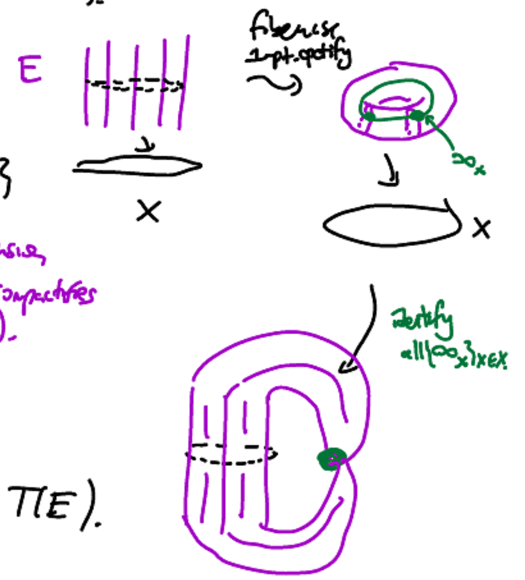
Thom spaces of vector bundles

$E \to X$ real rank k vector bundle. Fixing metric, get
$D(E)$ unit disc bundle, $S(E)$ unit sphere bundle, $S(E) \subseteq D(E)$.

Define the Thom space of E to be $T(E) := D(E)/S(E) \cong E/\{(x,v) \mid \|v\| \geq 1\}$.

There's a preferred basepoint $t_0 \in T(E)$ given by $[S(E)] =: t_0$ in $T(E)$.

Can think of forming $T(E)$ in two steps:



- in fiber E_x (rank k), take quotient $D(E_x)/S(E_x) \cong E_x \cup \{\infty_x\}$
  - $D(E_x)$: k-disk; $S(E_x)$: $k-1$ sphere
  - sphere of dimension k (one point compactifies E_x).
- identify all $\{\infty_x\}_{x \in X}$ to a single point.

We'd like to develop some tools for studying homology (+homotopy groups!) of $T(E)$.

Lemma: E oriented (rank k), then $\exists$ canonical iso

$$H_{k+i}(T(E), t_0) \cong H_i(B)$$

(homology variant of Thom isomorphism, for Thom spaces).

($\exists$ such an iso. w/ $\mathbb{Z}/2$ coeffs w/o assuming E oriented).

(note as sets $T(E) \cong E \cup t_0$ 'one point compactification of E' (provided X is compact))

Pf sketch:

$$H_{k+i}(T(E), t_0) \cong H_{k+i}(T(E), \overbrace{T(E) - 0_X}^{T(E)^\circ})$$

(b/c $T(E)^\circ = \amalg E_x^\circ \cup t_0$ is contractible, htpy equiv. to t_0)

$$\underset{\substack{\text{excision} \\ \text{(exercise too)}}}{\cong} H_{k+i}(E, E^\circ) \underset{\substack{\text{Thom iso.} \\ \text{(homology version)}}}{\cong} H_i(X).$$

given Thom class $u \in H^k(E, E^\circ)$, $- \cap u : H_{k+i}(E, E^\circ) \to H_i(E) \cong H_i(X)$.

Homotopy groups:

$k > 0$, by def'n $\pi_k(X, x_0) := [(S^k, s_0), (X, x_0)]$. in analogy w/ π_1. ('based space')

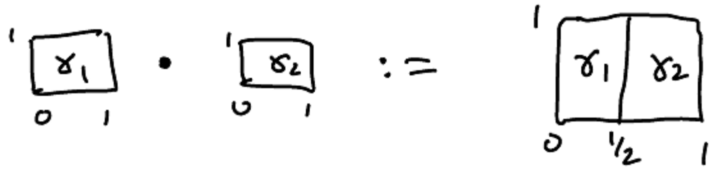
(write $\pi_k(X)$ if x_0 'implicit')

have $(S^k, s_0) \simeq (I^k, \partial I^k)$, so can think of:

$I = [0,1]^k$

$$\pi_k(X, x_0) := [(I^k, \partial I^k), (X, x_0)] \quad \& \quad \pi_k(X, A) := [(I^k, \partial I^k), (X, A)].$$

There's a group structure on each π_k, most natural to see via 'concatenate in first coordinate'



π_k is abelian for $k \geq 2$, via following homotopy:

— 4/14/2021

To<u>day</u>: More about Thom's cobordism theorem:

- homotopy groups + methods of computing (to compute LHS of theorem in some cases) — sketch.
- proof of Thom's thm (sketch)
- applications (sketch)

Homotopy groups are much more difficult to compute (failure of excision). However, in suitable ranges, they can be (partly) computed in terms of homology via (versions of the) <u>Hurewicz theorem</u>.

There's a map

$$\pi_r(X) \xrightarrow{h} H_r(X;\mathbb{Z}) \quad \text{for any (based) } X := (X, x_0)$$

$$[\alpha: S^n \to X] \longmapsto \alpha_*[S^n]$$

can check: h is a group homomorphism.

Can't always be an isomorphism, & for $r=1$ h has to factor through abelianization $\pi_1/[\pi_1,\pi_1]$ b/c H_1 abelian.

In Math 540 one proves <u>Thm</u> ($r=1$ Hurewicz thm): h induces an iso. $\pi_1(X)/[\pi_1(X),\pi_1(X)] \xrightarrow{\cong} H_1(X;\mathbb{Z})$.

Now, π_k abelian for $k \geq 2$, & thm states: means $\pi_i(X)=0$ for $i<n$.

<u>Thm</u>: (Hurewicz) If X (based) is $(n-1)$-connected, $n \geq 2$. Then,

$$\tilde{H}_i(X) = 0 \text{ for } i<n \quad \text{and} \quad h: \pi_n(X) \xrightarrow{\cong} H_n(X).$$

reduced homology ($=H_i$ $i>0$, 0 $i=0$)

Hurewicz homomorphism.

(similarly have Hurewicz for pairs: if (X,A) $n-1$ connected, & A is simply connected non-empty, then ...)

$\boxed{\text{Rmk}}$: There are various ways to show that if X CW complex w/ no i-cells (aside from one 0-cell) for $i<k$, then X is $(k-1)$ connected, i.e., $\pi_i(X)=0$ for $i<k$. (consequence of "cellular approximation" of maps,

c.f., Hatcher) !

$\Rightarrow \pi_i(S^k) = 0$ for $i<k$.

(moment you know a given $f: S^i \to S^k$ up to homotopy misses a point, it factors through $\mathbb{R}^k$, which is contractible)

know: • $\pi_1(S^1) = \mathbb{Z}$, know $\pi_k(S^1) = 0$ for $k>1$ (why? any $S^k \to S^1$, $k>1$, b/c S^k simply connected, hence is nullhomotopic). $\mathbb{R}$

The above then tells us: S^k, $k \geq 2$, then

$$\pi_i(S^k) = 0 \quad i<k, \quad \text{and} \quad \pi_k(S^k) \cong H_k(S^k) = \mathbb{Z}$$

(in genl $\pi_j(S^k)$ $j>k$ are quite interesting, unlike $H_j(S^k) = 0$)

e.g., using cup product on $\mathbb{CP}^2$ we argued that [attaching map in $\mathbb{CP}^2$: $\partial e^4 \to S^2 = \mathbb{CP}^1$] $\neq 0$ in $\pi_3(S^2)$ (in fact, $\pi_3(S^2) \cong \mathbb{Z}$). S^3

However, range of Hurewicz can be extended, provided one forgets some information, [Serre, "Serre classes"].

One instance: let $\mathcal{C}$ denote the class of all finite abelian groups (e.g., $\mathbb{Z}/2$, $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$, NOT $\mathbb{Z}$).

$\mathcal{C}$ is an instance of a Serre class (subclass of all groups closed under subgroups, quotients, extensions)

↳ other examples include {fin. gen. abelian groups}, {p-groups},

A homomorphism $h: A \to B$ is said to be a $\mathcal{C}$-iso. (or iso. mod $\mathcal{C}$), $\mathcal{C}$ any Serre class, if ker A and coker A $\in \mathcal{C}$.

(e.g., for $\mathcal{C}$ = {fin. ab. groups}, $\mathbb{Z}/7 \oplus \mathbb{Z}^5 \xrightarrow{h} \mathbb{Z}^5 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$ is a $\mathcal{C}$ iso.) (the $\mathbb{Z}/7$ summand goes to 0; on $\mathbb{Z}^5$, h is id)

Similarly, $A \simeq 0$ mod $\mathcal{C}$ if $A \in \mathcal{C}$.

Thm: (Mod $\mathcal{C}$ Hurewicz) If X (based) is $(n-1)$-connected mod $\mathcal{C}$ (means $\pi_i(X) = 0$ mod $\mathcal{C}$ $\forall\, i \leq n-1$.), $n \geq 2$. Then,

$\tilde{H}_i(X) = 0$ mod $\mathcal{C}$ for $i<n$ and $h: \pi_n(X) \xrightarrow{\cong} H_n(X)$ mod $\mathcal{C}$.

reduced homology ($= H_i$ $i>0$, 0 $i=0$)

Hurewicz homomorphism.

Have a relative version for (X,A), $A \subset X$, & more genlly a version for maps $A \to X$:

Thm: (generalized Whitehead mod $\mathcal{C}$): $f: A \to X$ map, A, X both simply connected.

(a) $\pi_i(A) \xrightarrow{f_*} \pi_i(X)$ is an iso. mod $\mathcal{C}$ for $i \leq n$ and surjective mod $\mathcal{C}$ for $i = n+1$,

$\Updownarrow$

(b) $H_i(A) \xrightarrow{f_*} H_i(X)$ " " " " " ".

Using this, one can show the following claim:

Cor: (Milnor–Stasheff): Say X $(k-1)$-connected, $k \geq 2$. finite CW complex (in each dimension) $\mathcal{C} = \{\text{finite abelian groups}\}$

Then $\pi_r(X) \xrightarrow{h} H_r(X;\mathbb{Z})$ is a $\mathcal{C}$-iso. for all $r < 2k-1$

(ordinary iso. for $r \leq k$ by usual Hurewicz)

Pf sketch:

(i) True for spheres S^k. (uses the fact that $\pi_i(S^k) = \begin{cases} \mathbb{Z} & i = k \\ \text{finite} & i < 2k-1,\ i \neq k \end{cases}$) (note $\pi_3(S^2) = \mathbb{Z}$; $3 = 2(2)-1$.)

(ii) If true for X, Y, then true for $X \vee Y$

$\Rightarrow$ true for any $S^{i_1} \vee \cdots \vee S^{i_k}$ $(i_1, \ldots, i_k > 1)$

(iii) General X, (fact: $\pi_i(X)$ are finitely generated (omitted)):

Pick generators for all free parts of $\pi_i(X)$, $i < 2k-1$, $\{f_k : S^{n_k} \to X\}$, wedge together:

$S^{n_1} \vee \cdots \vee S^{n_j} \xrightarrow{\vee f_k} X$ induces an iso. $\pi_i(S^{n_1} \vee \cdots \vee S^{n_k}) \xrightarrow{\cong} \pi_i(X)$ mod $\mathcal{C}$ for $i < 2k-1$.

By generalized Whitehead mod $\mathcal{C}$, there's also a homology iso. mod $\mathcal{C}$

$H_i(S^{n_1} \vee \cdots \vee S^{n_k}) \xrightarrow{\cong} H_i(X)$,

reducing us to case (ii). ☒.

— $\mathcal{C} = \{\text{finite abelian groups}\}$ for below:

Cor: (computation of homotopy groups of $T(E \to X)$) (rank k oriented)

If $E \to X$ rank k oriented bundle,

$\pi_{n+k}(T(E), t_0) \xrightarrow{\cong} H_n(X;\mathbb{Z})$ for all $n < k-1$, $\boxed{\text{mod } \mathcal{C}.}$

Pf: $\pi_{n+k}(T(E), t_0) \xrightarrow[\text{(mod } \mathcal{C})]{\cong} H_{n+k}(T(E);\mathbb{Z}) \xrightarrow[\text{Thom iso (last class)}]{\cong} H_n(X;\mathbb{Z})$.

check: $(k-1)$-connected space.

Milnor–Stasheff cor. $(n < k-1)$ so $k+n < 2k-1$.

(bijection i cells of $X \longleftrightarrow i+k$ cells of $T(E)$) for $i > 0$; cell analogue of Thom iso.

Cor: Mod $\mathcal{C}$, $\pi_{n+k}(\widetilde{E}^k_{taut}) \xrightarrow{\cong} H_n(\widetilde{G}_k(\mathbb{R}^\infty);\mathbb{Z})$.

we've computed invariants in class, e.g., we computed $H_*(G_k(\mathbb{R}^\infty), \mathbb{Z}/2)$, $H_*(G_k(\mathbb{C}^\infty), \mathbb{Z})$.

Returning to:

Thm: ([Thom]) Fix n. For any $k > n+1$, there is an isomorphism

$$\pi_{n+k}(T(\widetilde{E}^k_{taut}), t_0) \xrightarrow{\cong} \Omega_n$$

$$\pi_{n+k}(T(E^k_{taut}), t_0) \xrightarrow{\cong} \mathcal{N}_n$$

We'd like to explain some details of the proof. The first is, how to construct a map?

Roughly the idea is to find in $[f : S^n \to T(\widetilde{E}^k_{taut})]$ a "smooth" rep, & try to take $f^{-1}(\underline{0}\text{ section})$.

First, $\pi_{n+k}(T(\widetilde{E}^k_{taut}), t_0) \cong \pi_{n+k}(T(\widetilde{E}^{k,p}_{taut}), t_0)$,

$$\begin{array}{c} \widetilde{E}^{k,p}_{taut} \\ \downarrow \\ \widetilde{G}_k(\mathbb{R}^{k+p}) \end{array}$$

for $p >> 0$. (by cellular approx)

smooth vector bundle; smooth manifold

so total space of E is a smooth manifold.

Basic useful def'ns/ facts from smooth manifold theory:

M^m, N^n smooth, $f : M \to N$ smooth, recall $y \in N$ <u>regular value of f</u> (or <u>f is transverse to y</u>) if at every $x \in f^{-1}(y)$, $df_x : T_x M \twoheadrightarrow T_y N$. (if $m < n$, this can only happen when $f^{-1}(y) = \emptyset$)

IFT $\Rightarrow$ at a regular value $f^{-1}(y) \subseteq M$ submanifold dimension $m-n$.

More generally, if $Y \subseteq N^n$ submanifold of <u>codimension</u> k (meaning $\dim(Y) = n-k$). f <u>is transverse to Y</u> if at every $x \in f^{-1}(Y)$,

$$df_x(T_x M) + T_{f(x)} Y = T_{f(x)} N, \quad \text{or equivalently}$$

$$\begin{tikzcd} T_x M \arrow[r, "df_x"] \arrow[rr, bend right] & T_{f(x)} N \arrow[r, "pr"] & \nu_{f(x)} Y = T_{f(x)} N / T_{f(x)} Y. \end{tikzcd}$$

<u>is surjective</u>:

(Special case: $M \overset{i}{\subset} N \supset Y$, then i is transverse to $Y \Leftrightarrow M \pitchfork Y$).

<u>IFT</u> $\Rightarrow$ If f transverse to Y, then $f^{-1}(Y) \subset M$ submanifold of dimension $m-k$. (codimension k in M)

(if $f = i$ inclusion, $f^{-1}(Y) = M \cap Y$.)

$(T_x f^{-1}(Y) = \ker(T_x M \to \nu_{f(x)} Y)$, so $0 \to T_x f^{-1}(Y) \to T_x M \to \nu_{f(x)} Y \to 0$,

$\Rightarrow$ if M oriented, νY oriented, then $f^{-1}(Y)$ is.

& <u>Lem</u>: orientation of 2/3 of $0 \to A \to B \to C \to 0 \Rightarrow$ orientation of 3rd canonically.)

Techniques from smooth topology show that transversality is a "generic condition": by either wiggling f or Y a little, can ensure $\tilde{f}$ is transverse to Y or f is transverse to $\tilde{Y}$.

smoothly homotopic to f, and equals f outside a region.

e.g., [Sard's theorem] $\Rightarrow$ Regular values of $f: M \to N$ are open dense

$\Rightarrow$ any $y \in N$, $\exists$ $\tilde{y}$ arbitrarily nearby regular value.

Now, using these techniques, plus 'smooth approximation' (any continuous $f: Q^q \longrightarrow \mathbb{R}^n$ can be approximated up to homotopy by a smooth map (in that region), unchanged outside a nhood of that region).

smooth in some region

All of these techniques imply:

Thm: (Milnor-Stasheff Thm 18.6): $E \to B$ smooth vec. bdle over a smooth manifold, and let $f: S^m \longrightarrow T(E)$ continuous, sending $s_0 \longmapsto t_0$. Then, f is homotopic to a map $g: S^m \to T(E)$

(recall $E \underset{\text{open}}{\subset} T(E)$ w/ $T(E) - E = t_0$.)

smooth manifold

satisfying:

- g is smooth over $g^{-1}(E) = g^{-1}(T(E) - t_0)$.

('smooth approx')

open subset of S^m

- g transverse to $B \subseteq E$, submanifold of codimension k.

('transversality theory')

(zero section inclusion)

(note $\nu B = E$)

$\Rightarrow$ $g^{-1}(B)$ is a submanifold of S^m of dimension $m-k$.

Moreover, $g^{-1}(B)$ inherits an orientation from an orientation of E. ($\Leftrightarrow$ orientation of νB). (S^m is oriented).

- any homotopic $\tilde{g}$ as above induces a cobordant (oriented cobordant if E oriented) manifold:

$$[\tilde{g}^{-1}(B)] = [g^{-1}(B)] \in \Omega_{m-k} \quad (\text{or } \mathfrak{N}_{m-k}).$$

(why? if $g \simeq \tilde{g}$, smoothly approximate the homotopy to get $H: S^m \times [0,1] \longrightarrow T(E)$ which is smooth over $H^{-1}(E)$, & further perturb H to ensure $H \pitchfork B$.

$\Rightarrow$ $H^{-1}(B)$ m'fold w/ boundary, oriented if E is, w/

$\downarrow$

$[0,1]$

$\partial H^{-1}(B) = \overline{g^{-1}(B)} \amalg \tilde{g}^{-1}(B)$).

over 0 over 1.

smooth rank k bdle over smooth B

gives a map $\pi_m(T(E), t_0) \longrightarrow \Omega_{m-k}$ (or $\mathfrak{N}_{m-k}$ if E unoriented).

This is Thom's map:

$$\pi_{n+k}(\tilde{E}^{k,p}_{taut}) \longrightarrow \Omega_n$$

&

$$\pi_{n+k}(E^{k,p}_{taut}) \longrightarrow \mathfrak{N}_n.$$

Why is the map an iso? <u>Let's just show it's surjective</u>: Focus on oriented case.

Start with any M^n cpct oriented manifold, $[M^n] \in \Omega_n$.

(1) [Whitney embedding]

$\exists$ smooth embedding $M \hookrightarrow \mathbb{R}^{n+k}$, $k >> 0$. ν_M oriented b/c M, $\mathbb{R}^{n+k}$ are.

(2) [Tubular nbhd theorem]: $\exists$ nbhd U of M in $\mathbb{R}^{n+k}$ & diffeo.

$$\begin{tikzcd} U \arrow[rr, "\cong"] & & \nu_M \\ & M \arrow[ul, "i"] \arrow[ur, "\underline{0}"'] & \end{tikzcd}$$

(3) [Classification of ν_M]: ν_M, $\cong (TM)^\perp$ inside $T\mathbb{R}^{n+k}|_M \cong \mathbb{R}^{n+k}$. a rank k oriented bundle over M,

so classified by

$M \xrightarrow{f} \tilde{G}_k(\mathbb{R}^{n+p})$ any $p \geq k$. , and have natural maps.

$x \longmapsto \{\nu_x M = T_x M^\perp \text{ in } \mathbb{R}^{n+k} \subset \mathbb{R}^{n+p}\}$

$$\begin{tikzcd} \nu_M \arrow[r, "\tilde{f}"] \arrow[d] & \tilde{E}^{k,p}_{taut} \arrow[d] \\ M \arrow[r, "f"] & \tilde{G}_k(\mathbb{R}^{n+p}) \end{tikzcd}$$

These maps are smooth, and $\tilde{f} : \nu_M \to \tilde{E}^{k,p}_{taut}$ is transverse to $B = \underline{0} = \tilde{G}_k(\mathbb{R}^{n+p})$.

(check: this follows from the fact that $\tilde{f}$ induces an iso.

$\nu_M \xrightarrow{\cong} f^* \tilde{E}^{k,p}_{taut}$

$T^{vert}_{(x,0)}\nu_M \xrightarrow{\cong} \nu_{(f(x),0)} B$)

(4) The tubular nbhd $U \cong \nu M$ induces a map

$$\begin{tikzcd} S^{n+k} = (\mathbb{R}^{n+k} \cup \{\infty\}) \arrow[r] \arrow[dr, "F"'] & T\nu M \arrow[d, "T\tilde{f}"] \\ & T(\tilde{E}^k_{taut}) \end{tikzcd}$$

(crushing everything outside U to a point t_0)

can check: F is smooth outside $F^{-1}(t_0)$ (i.e., on U) and $F^{-1}(B = \underline{0}) = M$.

so $[F] \longmapsto [M]$ under Thom's map.

(1)

$$\pi_{n+k}\left(T\left(\widetilde{E}^{k,p}_{tact}\right), t_0\right)$$