~~Thurs~~ week 8, wednesday:

- (E^{2n+2}, W) symplectic LG model.

$W: E^{2n+2} \longrightarrow \mathbb{C}$ symplectic fibration away from $K_{cpct} \subseteq \mathbb{C}$. ~~satisfying~~ (**) w/ well-defined $\parallel$ transport maps in this region.

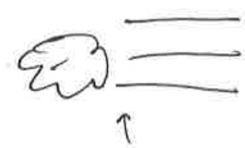
- $M := W^{-1}(p)$ general fiber. (all such fibers are symplectomorphic).

Towards $\mathcal{F}(E, W)$:

objects: properly embedded $L \subseteq E$, s.t. $W(L)$ is contained in

$\nwarrow$ union of radial rays w/ angle $\neq -\pi$.

OR :



$\uparrow$ $\cup$ straight rays

$\rightsquigarrow D_L \subseteq (-\pi, \pi)$ "directions of L"

say $D_K < D_L$ if any $\theta_K < \theta_L$ $\forall$ $\theta_K \in D_K$, $\theta_L \in D_L$.

$\leadsto D_L \subseteq \mathbb{R}$ "height"

$H^0(hom_{\mathcal{F}(E,W)}(K,L))$

$\overset{\cdot\cdot}{=}$ "$Hom(K, L) := HF^*(\phi_\varepsilon K, L)$"

$\uparrow$ time ε "counterclockwise bend"

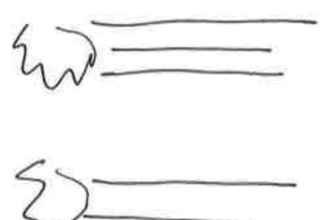
$\varepsilon > 0$ sufficiently large that all ends of $\phi_\varepsilon K$ above the ends of L.

meaning $D_{\phi_\varepsilon K} > D_L$,

~~but $\phi_\varepsilon K$ doesn't~~ & $\phi_t K$ admissible $\forall\, 0 \leq t \leq \varepsilon$ (so doesn't cross $-\pi$.

What is a "time ε counterclockwise bend?"

straight ray setup:



$\swarrow$ consider the vector field on $\mathbb{C}$, for $x >> 0$ equal to ∂_y

(= ham. flow of $h(x,y) = x$)

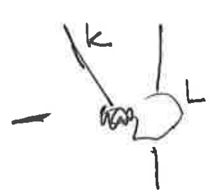
& any ~~Hamiltonian~~ time ε flow of $h \circ W$.

In horizontal setup, Lem: $\exists\, \varepsilon$ s.t. $D_{\phi_\varepsilon K} > D_L$.

- Note that in the angular setup, the rotating ~~cdot~~ flows gen. by $h \circ W$ $\quad h(\theta) = r$.

may not be enough.

Ex:



$\leftarrow$ there's no strictly rotating flow pushing L past K.

~~objects~~

Define an Auxiliary category $\mathcal{O}_W$:

objects $\mathcal{O}$:= admissible Lagrangians branes in $(\bar{F}, W)$

$$\hom(K,L) := \begin{cases} CF^\bullet(K,L) & \text{if } D_K > D_L \quad \text{(or a cptly supported perturbation)} \\ \mathbb{Z}\langle e_L^+\rangle & \text{if } D_K = D_L \text{ and } K = L \\ 0 & \text{otherwise} \end{cases}$$

this is a "directed category;" objects live over some poset

(K compact: can adopt convention that $D_K < D_L \ \forall\ L$,

or add objects (K, I) for all $I \subseteq (-\pi, \pi)$ non-empty)

in other words, make objects pairs (L, I) w/ L admissible & $\dot{D}_L \subseteq I$ a finite subset of $\mathbb{R}$..

Define the A_∞ structure on $\mathcal{O}$ as usual: e_L^+ is a strict unit, &

$$\mu^k : \hom(L_{k-1}, L_k) \otimes \cdots \otimes \hom(L_0, L_1) \to \hom(L_0, L_k)$$

$= 0$ unless $L_0 > L_1 > \cdots > L_k$

& then counts discs as usual. (not problematic, b/c L_i all pairwise disjoint near ∞)

(works b/c inductively for any subset of morphisms, this condition is preserved

& satisfies A_∞ relations.

Note this is invariant under cptly supported Ham. isotopies up to q. isomorphism (usual argument)

• but not under admissible flows

Note that $\hom_{\mathcal{O}}(\phi_\varepsilon L, L) := CF^\circ(\phi_\varepsilon L, L)$ but

$\hom_{\mathcal{O}}(L, \phi_\varepsilon L) := 0$. In particular,

in $\mathcal{O}$, L & $\phi_\varepsilon L$ are not isomorphic objects.

[Abouzaid-Seidel]: For every L admissible, every $\varepsilon > 0$, define a quasi-unit

$q \in HF^\circ(\phi_\varepsilon L, L) := H^\circ \hom_{\mathcal{O}}(\phi_\varepsilon L, L)$. by continuation: (a) count $\sum \phi_\varepsilon \bigcap_x L \cdot x$

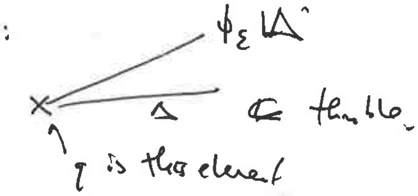
OR (b) if L has one end,

(by the identification $HF^*(\phi_\varepsilon L, L) \cong H^\bullet(L)$ for ε small, which is enough.

get a collection of morphisms

$$Z \subseteq H^0 \mathcal{O}.$$

ex:



<u>Def</u> [Abouzaid-Seidel]: $\mathcal{F} := \mathcal{F}(E, W) := \mathcal{O}[Z^{-1}]$, localized category.

$$:= \text{image}\left(\mathcal{O} \longrightarrow tw^\pi \mathcal{O} \longrightarrow tw^\pi \mathcal{O} / \text{cones}(Z)\right).)$$

By def'n, $\mathcal{F}(E, W)$ comes equipped w/ a functor $\mathcal{O} \xrightarrow{j} \mathcal{F}(E, W)$ which is "initial among functors sending Z to isomorphisms".

In particular, we can think of $\text{ob}\,\mathcal{O} = \text{ob}\,\mathcal{F}$, but note that in $\mathcal{F}$, K & $\phi_\varepsilon K$ become <u>isomorphic</u>.

~~The ... ...~~

The key proposition that makes $\mathcal{O}[Z^{-1}]$ useful is:

<u>Prop</u>: ["correct position lemma"] [Abouzaid-Seidel]: If $D_K > D_L$, then $j^1 : \hom_{\mathcal{O}}(K, L) \longrightarrow \hom_{\mathcal{F}}(K, L)$ <u>is a quasi-isomorphism</u>.

$$\hom_{\mathcal{O}}(K,L) = CF^\bullet(K, L)$$

<u>Conclusion</u>: So, <u>in</u> $H^\bullet(\mathcal{F})$, $\text{Hom}_{\mathcal{F}}(K, L) \cong HF^\bullet(\phi_\varepsilon K, L)$. where $\varepsilon > 0$ large enough so $D_K > D_L$.

$$\text{Hom}_{\mathcal{F}}(K,L) \cong \text{Hom}_{\mathcal{F}}(\phi_\varepsilon K, L) \underset{[\text{Prop.}]}{\cong} \text{Hom}_{\mathcal{O}}(\phi_\varepsilon K, L) = HF^\bullet(\phi_\varepsilon K, L)$$

<u>Prop</u>: ["correct position lemma v2"] (A-S).

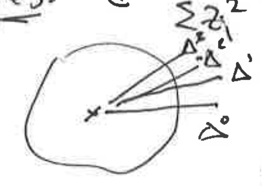
If $D_{L_0} > D_{L_1} > \cdots > D_{L_k}$,

then can set up model of $\mathcal{F}$ so that $\hom_{\mathcal{O}}(L_i, L_j) \to \hom_{\mathcal{F}}(L_i, L_j)$ <u>injective</u> on chain level for $L_i > L_j$.

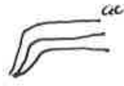
and $\mu^k_{\mathcal{F}}$ (for this order) can be computed in $\mathcal{O}$..."

$\Rightarrow$ [μ^2] product agrees with the one defined above.

<u>Examples</u>: $\mathbb{C}^n \xrightarrow{\sum z_i^2} \mathbb{C}$ model Lefschetz fibration,



Let Δ be a Lefschetz thimble, actually, we can admissible L Δ^i for each i.



In $\mathcal{F}(E,W)$ these objects all become isomorphic:

up to isomorphism one object $\underline{\Delta}$

Ex $H^\bullet \hom_{\mathcal{F}(E,W)}(\underline{\Delta}, \underline{\Delta}) := HF^\bullet(\phi_\epsilon \Delta, \Delta) = \mathbb{K}$, ^ degree zero, ~~gr~~ key

The A_∞ algebra structure on $A := \hom_{\mathcal{F}}(\underline{\Delta}, \Delta)$ is q-equivalent to one on its minimal model, $H^\bullet A \cong \mathbb{K}$. But in $\mathbb{K}$, there is no room (for degree reasons) for nontrivial μ^k, $k \geq 3$ (μ^k has deg $2-k$)

product: ~~cross from~~ the ~~object~~ generator of $\mathbb{K}$ is a unit.

$\Rightarrow \mathrm{End}_{\mathcal{F}}(\underline{\Delta}, \underline{\Delta}) \simeq (\mathbb{K})$

^ usual algebra

Thm: [Seidel, Abouzaid-G.]. $\underline{\Delta}$ split-generates $\mathcal{F}(\mathbb{C}^n, \sum z_i^2)$.

Hence, $\mathrm{perf}(\mathcal{F}(E,W)) \cong \mathrm{perf}(\mathbb{K}) \simeq Ch^{fin}(\mathbb{K})$

^ the algebra; ^ ch. complexes w/ finite rank cohomology.

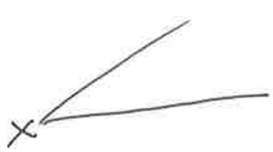
More generally consider $M \times \mathbb{C}^n$, $W = \sum_{i=1}^n z_i^2$, M monotone or exact

& $L \subseteq M$ a Lagrangian.

⇝ generalized thimble $\underline{\Delta}^L := \underline{\Delta} \times L$ in E, W.

$H^\bullet \hom(\underline{\Delta}^L, \underline{\Delta}^K) := HF^\bullet$

compute:



Thm: [Abouzaid-Auroux-Katzarkov, Abouzaid-G.], $\underline{\Delta}$ extends to an A_∞ functor

$\underline{\Delta}^{(\cdot)}$:

Given X ~~a compact~~ a sympl. manifold, there is an Abouzaid–Seidel construction of its Fukaya category: ~~Fix finite collection $(L_1, \ldots, L_m)$ of Lagns.~~ Say S finite collection of Lag'n branes

For each ~~object~~ brane $L \in S$, choose $L^{(i)}$ Hamiltonian perturbations for each $i \in \mathbb{Z}$ pairwise transverse, w/

- each $L^{(i)}$ Ham. isotopic to L
- any finite subset $(L_0^{(k_0)}, \ldots, L_m^{(k_m)})$ w/ $k_0, \ldots, k_m$ pairwise distinct, is in general position.

Then, define a category $\mathcal{O}_X$

- objects = $(L^{(k)}, L \in S)$ or rather pairs (L, k).

$$\hom_{\mathcal{O}_X}(K^{(i)}, L^{(j)}) := \begin{cases} CF^\bullet(K, L) & i > j \\ \mathbb{K}\cdot\langle e_L^+ \rangle & i = j \quad K = L \\ 0 & \text{else.} \end{cases}$$

directed.

$\rightsquigarrow$ strictly unital category.

M a module is local if Z acts by q-iso

$$\text{Mod}(\mathcal{O}) \underset{j^*}{\overset{j_*}{\rightleftarrows}} \text{Mod}(\mathcal{P})$$

$$y_k^r \longrightarrow j^* j_* y_k^r.$$

check $y_k^r \to j^* y_{jk}^r$.

Solution: we need to allow flows ~~of the form~~ ~~which~~ of the form pulled back from $\mathbb{C}$ by w

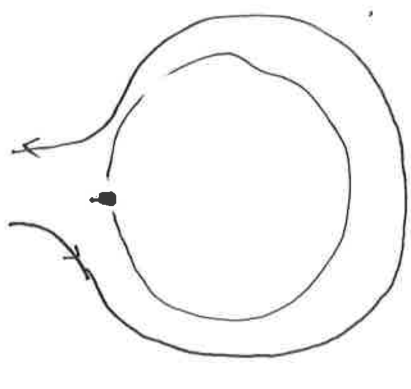
$$h_{r,\theta} \qquad \underset{\substack{\uparrow \\ \text{cutoff fxn, equal to zero} \sout{\text{near}} \text{ at } -\pi}}{\chi(\theta)}\ \underset{\substack{\uparrow \\ r}}{h(r)} \quad \text{near } \infty \text{ – "partially wrapped flow"}$$

"admissible flow"

Lem: Flowing by $h_{r,\theta}$ for any time preserves admissibility.. $\emptyset$.

(check that $h_{r,\theta}$ ~~preser~~ is ~~fractial~~ radial; near ∞ it sends rays to rays.

Ex trajectory:



Cohomological products: Given L_0, L_1, L_2,

define $\mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_2)$

choose $\varepsilon_1, \varepsilon_2$ so that $D\phi_{\varepsilon_2\varepsilon_1} L_0 > D_{\phi_{\varepsilon_1}} L_1 > D_\phi$

(mean $D_{\phi_{\varepsilon_2}} L_0 > D_{\phi_{\varepsilon_1}} L_1$).

& count triangles

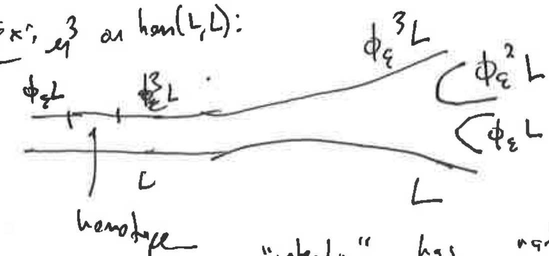
$$HF^\bullet(\phi_{\varepsilon_1} L_1, L_2) \otimes HF^\bullet(\phi_{\varepsilon_2\varepsilon_1} L_0, \phi_{\varepsilon_1} L_1)$$
$$\text{\rotatebox{90}{$\cong$}} \; HF^\bullet(\phi_{\varepsilon_2} L_0, L_1)$$
$$\downarrow$$
$$HF^\bullet(\phi_{\varepsilon_2\varepsilon_1} L_0, L_2) \cong \mathrm{Hom}(L_0, L_2)$$

Desired qualitative behaviour of $\mathcal{F}(E, W)$: cohomologically ~~then~~ $H^\bullet\mathcal{F}(E,W)$ $\otimes$ defined.

- invariance under cptly supported perturbations
- invariance under admissible flows, meaning L_0 & $\phi_t L_0$ should be quasi-isomorphic objects.. admissible flow $\forall t$.

Somewhat difficult to construct A_∞ structure directly: (for each L_i, L_j choose ε_{ij} w/ $D\phi_{\varepsilon_{ij}} L_i > D L_j$ etc — — but at some point, μ^k will have to ~~[illegible]~~ ~~"unbend"~~ make a homotopy which "unbends"

Ex: μ^3 on $\hom(L, L)$:



[Abouzaid–Seidel]: "unbending" has rather bad "maximum principle" properties.

(no u cannot apply Floer eq'n w/ a "neg. radial" ~~[illegible]~~ flow fxn is superharmonic rather than subharmonic).