Generating Fukaya Categories of LG models

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1   Setup and Geometric constructions (Abouzaid)

1.1   Geometric preliminaries

Let E be a symplectic manifold with convex boundary. This means that we have a primitive λ_E defined in a neighbourhood of ∂E with outwards pointing flow.

We assume that E is modelled after a mapping torus at infinity, i.e. there is a compact symplectic manifold M with contact boundary defined by 1-form λ_M defined near the boundary, and a symplectomorphism $\mu\colon M \to M$ which preserves λ_M so that the complement of a compact subset of E agrees with the mapping torus

$$E_{(M,\mu)} \equiv \frac{M \times [0, 2\pi] \times [1, \infty)}{(x, 0, r) \sim (\mu(x), 2\pi, r)}, \tag{1.1}$$

which is equipped with the obvious symplectic form and primitive λ_M.

Definition 1. *An* admissible *Lagrangian submanifold $L \subset E$ is a properly embedded submanifold such that $\pi(L)$ agrees, outside the unit disc, with a finite collection arcs which are asymptotic to radial rays disjoint from the negative real axis. We write $A(L) \subset (-\pi, \pi)$ for the corresponding set of angular directions.*

Impose whatever condition you need to get gradings, etc...

1.2   The dualising symplectomorphism

Define *a boundary rotation* of angle b to be the identity inside the unit disc, increasing rotation outside the unit disc, which eventually becomes rotation by angle b. We can achieve this as the Hamiltonian flow of the product by
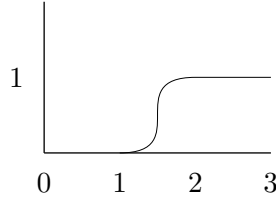
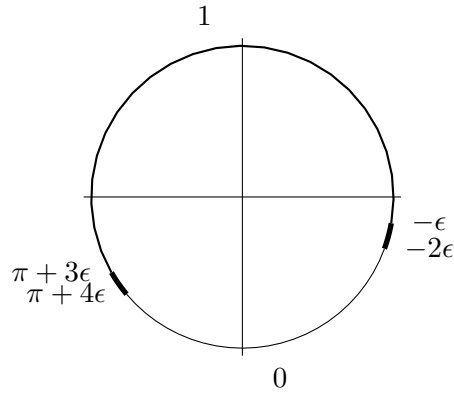Figure 1: Derivative of the hamiltonian for rotation

Figure 2: The function on the circle

b by the function h whose derivative is shown in Figure 1. We write ϕ_b for this.

Consider a function χ on the circle as in Figure 2:

$$\chi(\theta) = \begin{cases} 1 & \theta \in [-\epsilon, \pi + 3\epsilon] \\ 0 & \theta \in [-\pi + 4\epsilon, -2\epsilon] \end{cases} \tag{1.2}$$

The Hamiltonian flow ϕ_σ of $(\pi+2\epsilon)h(\rho_D-1)\cdot\chi(\theta)$ is shown in Figure 3. By construction, ϕ_σ is radially invariant in the sense that the graph $\Delta_\sigma \subset \mathbb{C}^2$ is invariant under dilation outside a compact set. The following is obvious from the picture:

Lemma 2. *The symplectomorphism ϕ_σ is the identity in the sector $\theta \in [-\pi + 4\epsilon, -2\epsilon]$, and agrees with rotation by $\pi + 2\epsilon$ in the sector $\theta \in [-\epsilon, \epsilon]$. No point on a circle of radius greater than 2 is mapped to an antipodal point (on the circle of the same radius).*
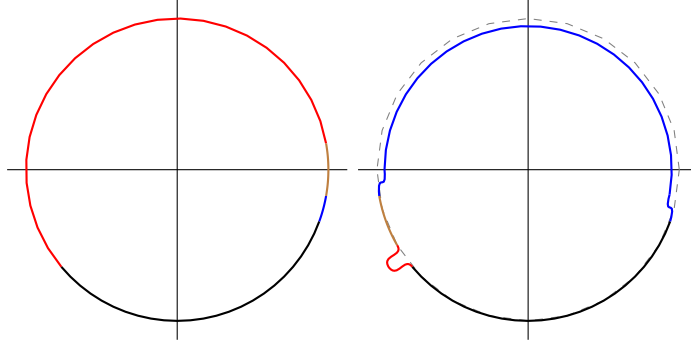
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Figure 3: The Hamiltonian flow on the circle: the left picture is the domain, decomposed into 4 regions which are color coded. The right picture is the image of the circle under the Hamiltonian flow. The large black region is mapped identically to itself. The brown region is rotated by $\pi + 2\epsilon$, and the images of the red and blue regions are as shown.

Let $\phi_\sharp$ denote the composition $\phi_\pi \circ \phi_\sigma$.

Proposition 3. *The symplectomorphism $\phi_\sharp$ agrees with*

1. *ϕ_π in the sector $\theta \in [-\pi + 4\epsilon, -2\epsilon]$*

2. *ϕ_π in the disc of radius 2.*

3. *Rotation by $2\pi + 2\epsilon$ in the sector $\theta \in [-\epsilon, \epsilon]$*

Moreover, $\phi_\sharp$ is radially invariant, and has no fixed points outside a compact set.

Corollary 4. *If L is admissible, and $A_L \cap (-2\epsilon, -\epsilon) = \emptyset$, $\phi_\sharp L$ is admissible.*

1.3   Some Floer cohomology groups

For the generation criterion, we shall also need a Floer cohomology group that plays the rôle of the Hochschild cohomology of the category. This should be the fixed point Floer cohomology group $HF^*(\phi_\sharp)$. To get around technical trouble, we consider the graph $\Delta_\sharp = (x, \phi_\sharp x)$ as a Lagrangian in $E \times E^-$, and define Floer cohomology as:

$$CF^*(\Delta, \Delta_\sharp). \tag{1.3}$$

Note that this is easy to set up because $\Delta_\sharp$ is radially invariant, so the maximum principle for $|z_1|^2 + |z_2|^2$ (as a function on $\mathbb{C}^2$) can be used.

If L and K are admissible, it is easy to define $HF^*(K, L)$ by standard methods (compactly supported Hamiltonian perturbations, and maximum principle for holomorphic curves) if the directions at infinity are disjoint. It is also easy to define this Floer group if $K = L$ (use a proper Morse function). This means that we have Floer cohomology groups $HF^*(\phi_\sharp K, L)$ whenever the angular directions satisfy some inequalities.

Lemma 5. *Assume that $A_K \subset [-\pi + 4\epsilon, -2\epsilon]$ and $A_L \subset [-\epsilon, \epsilon]$. We have natural isomorphisms:*

$$HF^*(\phi_\sharp K, L) = HF^*(\phi_\pi K, L) = HF^*(K \times L, \Delta_\pi) \cong HF^*(K \times L, \Delta_\sharp) \tag{1.4}$$

If $A_{L'} \subset [-\epsilon, \epsilon]$, we have natural isomorphisms:

$$HF^*(\phi_\sharp L', L) = HF^*(\phi_{2\pi+2\epsilon} L', L) = HF^*(L' \times L, \Delta_{2\pi+2\epsilon}) \cong HF^*(L' \times L, \Delta_\sharp). \tag{1.5}$$

Proof. We need to make sure that the second two groups are well defined. The problem is that $K \times L$, is not radially invariant. However, we can construct a (weakly) psh function ψ, which vanishes in a compact set and agrees with $|z|^2$ outside a compact set. We can then apply the maximum principle for $\psi(z_1) + \psi(z_2)$ when computing the Floer groups of these Lagrangians with $\Delta_\sharp$.

The first equality follows from $\phi_\sharp K = \phi_\pi K$. The second from "doubling," and the third from either the "integrated maximum principle" or from the existence of a Hamiltonian isotopy between Δ_π and $\Delta_\sharp$), among radially invariant Lagrangians, so that no intersection points with $K \times L$ are introduced. $\square$

Remark 1. *It is delicate to directly use the doubling idea for $\phi_\sharp$ because it is not holomorphic outside a compact set.*

1.4 Maps between Floer cohomology groups

To simplify the discussion, we set $L = L'$. There are four basic maps between the Floer groups that we consider:

$$HF^*(L \times L, \Delta_\sharp) \to HF^*(L \times K, \Delta) \otimes HF^*(K \times L, \Delta_\sharp)$$
$$\mathcal{OC}\colon HF^*(L \times L, \Delta_\sharp) \to HF^*(\Delta, \Delta_\sharp)$$
$$\mathcal{CO}\colon HF^*(\Delta, \Delta_\sharp) \to HF^*(K \times K, \Delta_\sharp)$$
$$\mu\colon HF^*(K \times L, \Delta_\sharp) \otimes HF^*(L \times K, \Delta) \to HF^*(K \times K, \Delta_\sharp)$$

(a) Doubly doubled curve
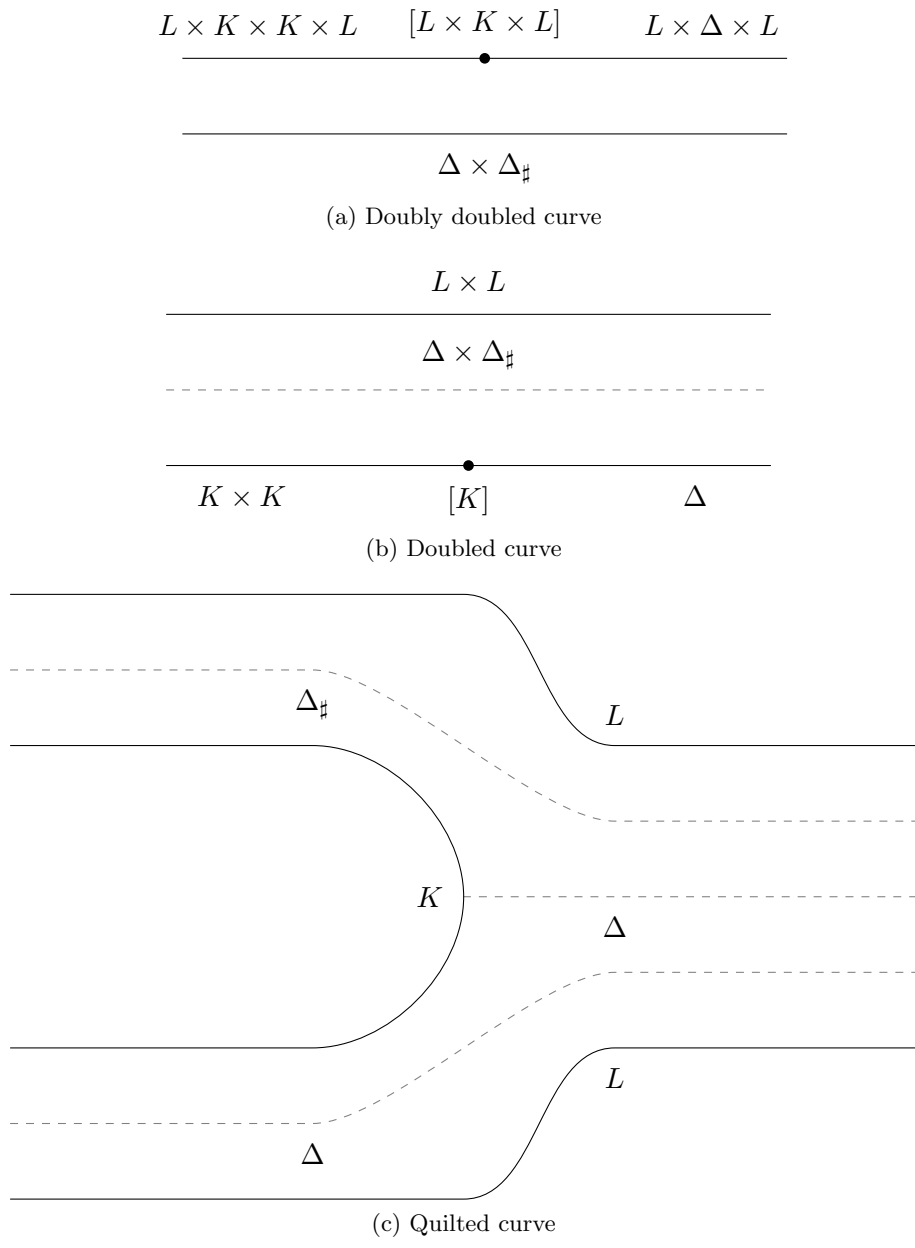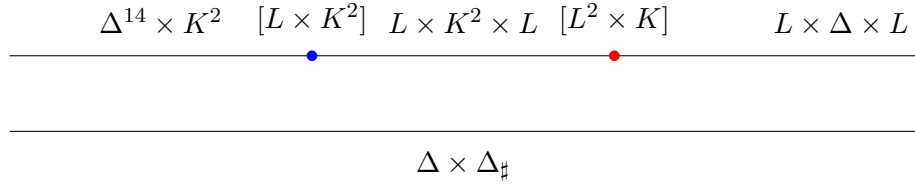

(b) Doubled curve

(c) Quilted curve

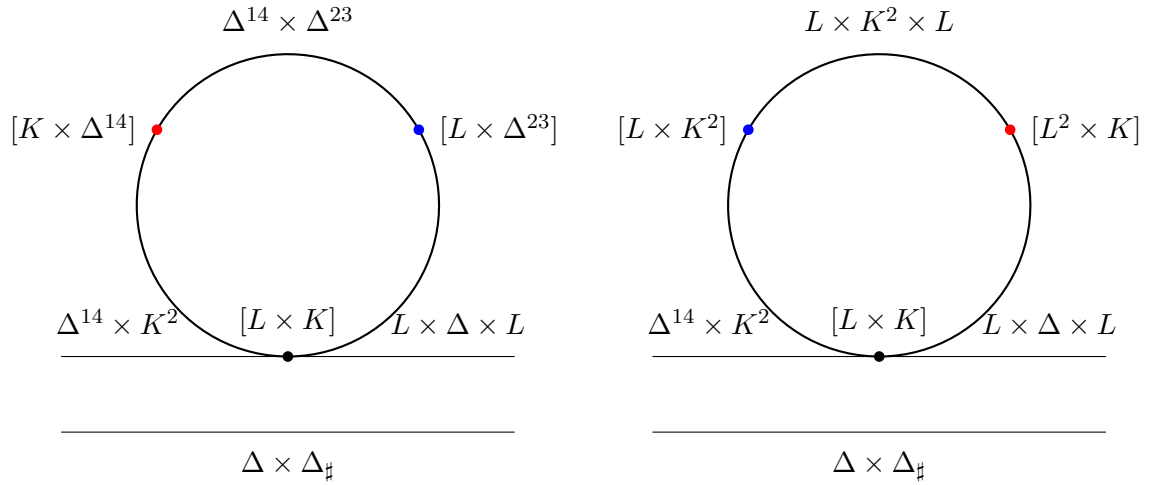Figure 4: Three points of view on the coproduct

Figure 4 shows three points of view on the coproduct map: we can either express such maps using "quilted Riemann surfaces" or using maps to products of E. We use the former point of view for intuition, and the

latter to state the results, although we use the "undoubling" idea to prove that the maximum principle holds at the "corners."
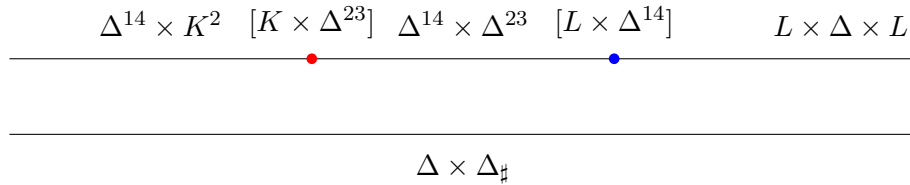
The key relation that is needed is the Cardy relation, which says that the two compositions are homotopic. Figure 5 shows the Cardy relation in the "doubly doubled" picture. When it is not clear from the context, we label a graph Lagrangian Δ_ϕ^{ij} if it is contained in the product of the ith and j^{th} factors of $E \times E^- \times E \times E^-$.

$\Delta^{14} \times K^2 \quad [L \times K^2] \quad L \times K^2 \times L \quad [L^2 \times K] \qquad L \times \Delta \times L$

$\Delta \times \Delta_\sharp$

(a) The composition of product and coproduct

$\Delta^{14} \times \Delta^{23}$

$[K \times \Delta^{14}]$

$[L \times \Delta^{23}]$

$\Delta^{14} \times K^2 \quad [L \times K] \quad L \times \Delta \times L$

$\Delta \times \Delta_\sharp$

$L \times K^2 \times L$

$[L \times K^2]$

$[L^2 \times K]$

$\Delta^{14} \times K^2 \quad [L \times K] \quad L \times \Delta \times L$

$\Delta \times \Delta_\sharp$

(b) The identification of moduli spaces when the nodes meet (the disc bubble is constant)

$\Delta^{14} \times K^2 \quad [K \times \Delta^{23}] \quad \Delta^{14} \times \Delta^{23} \quad [L \times \Delta^{14}] \qquad L \times \Delta \times L$

$\Delta \times \Delta_\sharp$

(c) The composition of $\mathcal{OC}$ and $\mathcal{CO}$

Figure 5: The Cardy relation

2 The pre-localised generation criterion (Break)

2.1 The Fukaya category

In order to define a category, introduce the following partial order: We say that $L_- < L_+$ if $A_{L_-} < A_{L_+}$.

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Definition 6. *The* directed category of π *is the category* $\mathcal{O}$ *with objects admissible Lagrangians and*

$$\mathcal{O}(L_0, L_1) = \begin{cases} CF^*(L_0, L_1) & \text{if } L_1 \leq L_0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

For any sub-interval $I \in (-\pi, \pi)$, we have an inclusion $\mathcal{O}^I \subset \mathcal{O}$. By abuse of notation, we write $\mathcal{O}^b$ for the subcategory of objects corresponding to a fixed (small) neighbourhood of $b \in (-\pi, \pi)$. For any $\beta \in \mathbb{R} \setminus \{2\pi k + \pi - b\}$, we have a functor

$$\phi_b \colon \ \mathcal{O}^b \to \mathcal{O}^{b+\beta}. \tag{2.2}$$

Assuming that $0 \leq \beta$, and $b + \beta < \pi$, we can construct a canonical element $Z_\beta \in H^*\mathcal{O}(\phi_b L, L)$, which we call a *quasi-unit*.

Remark 2. *There is a new "analytic" ingredient in the construction of quasi-units: as currently implemented, the construction uses continuation equations, so we need a maximum principle for them. This can be replaced by a homotopy-method argument, at the cost of also verifying that intersection points which are created at infinity always generate a quotient complex.*

The reason we introduce quasi-units is the following:

Proposition 7 (A-Seidel)**.** *The localisation $\mathcal{F}_\pi$ of $\mathcal{O}$ with respect to all morphisms Z_β satisfies the following properties:*

1. *Hamiltonian isotopies of admissible Lagrangians induce quasi-equivalences.*

2. *If $A_{L_1} < A_{L_0}$, there is a natural quasi-equivalence $CF^*(L_0, L_1) \to \mathcal{F}_\pi(L_0, L_1)$.*

2.2 The Cardy relation

We associate to the symplectomorphism $\phi_\sharp$ the following data:

1. The Hamiltonian Floer complex $CF^*(\Delta, \Delta_\sharp)$

2. A bimodule $\mathcal{O}_\sharp$ over $\mathcal{O}^0$ given by

$$\mathcal{O}_\sharp(L, L') \equiv CF^*(L \times L', \Delta_\sharp) \quad \left(\cong CF^*(\phi_\sharp L', L) \right). \tag{2.3}$$

3. A left module $\mathcal{Y}_K^\sharp$ for $K \in \mathcal{O}^{-\pi/2}$ given by

$$\mathcal{Y}_K^\sharp(L) \equiv CF^*(K \times L, \Delta_\sharp) \quad \left(\cong CF^*(\phi_\sharp K, L) \right). \tag{2.4}$$

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4. A Floer cohomology group $CF^*(K \times K, \Delta_\sharp) \cong CF^*(\phi_\sharp K, K)$.

Lemma 8 (A-Ganatra). *There is commutative diagram*

$$
\begin{array}{ccc}
\mathrm{CC}_*(\mathcal{O}^0, \mathcal{O}_\sharp) & \longrightarrow & \mathcal{Y}^r_K \otimes_{\mathcal{O}^0} \mathcal{Y}^\sharp_K \\
\downarrow & & \downarrow \\
CF^*(\Delta, \Delta_\sharp) & \longrightarrow & CF^*(K \times K, \Delta_\sharp)
\end{array}
\tag{2.5}
$$

In order for this to be useful, we need to identify the module $\mathcal{Y}^\sharp_K$ and some of the maps:

Lemma 9. *There is a natural quasi-isomorphism of bimodules:*

$$
\mathcal{Y}^\sharp_K \cong \mathcal{Y}^l_{\phi_\pi K}.
\tag{2.6}
$$

Moreover, we have a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{Y}^r_{\phi_\pi K} \otimes_{\mathcal{O}^0} \mathcal{Y}^l_K & \xrightarrow{\ \mu\ } & \mathcal{O}(\phi_\pi K, K) & \xleftarrow{\ Z_\pi\ } & \mathbb{Z} \\
\downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow \\
\mathcal{Y}^r_K \otimes_{\mathcal{O}^0} \mathcal{Y}^\sharp_K & \longrightarrow & CF^*(K \times K, \Delta_\sharp) & \longleftarrow & CF^*(\Delta, \Delta_\sharp)
\end{array}
\tag{2.7}
$$

3 The generation criterion (Ganatra)

Denote by $\mathcal{F}_{big}$ the localization of $\mathcal{O}$ with respect to all quasi-units Z_β constructed above. For any subcategory $\mathcal{O}^I, \mathcal{O}^b \subset \mathcal{O}$, we can localize by all quasi-units which remain within I to obtain a category $\mathcal{F}^I$, $\mathcal{F}^b$ (where b once more by abuse of notation refers to a small neighborhood around b). The inclusion $i_0 : \mathcal{O}^0 \subset \mathcal{O}$ send quasi-units to quasi-units, and in particular induces a functor $\bar{i}_0 : \mathcal{F}^0 \to \mathcal{F}_{big}$ fitting into a commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}^0 & \xrightarrow{\ i_0\ } & \mathcal{O} \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_{big}} \\
\mathcal{F} & \xrightarrow{\ \bar{i}_0\ } & \mathcal{F}_{big}
\end{array}
\tag{3.1}
$$

Lemma 10. $\bar{i}_0$ *is a quasi-equivalence.*

Define the *Fukaya category of the fibration* π to be

$$\mathcal{F} := \mathcal{F}^0. \tag{3.2}$$

(it is quasi-equivalent to any other $\mathcal{F}^I$). We will discuss the homological algebra of localization more shortly, but essentially $\mathcal{F}$ (and $\mathcal{F}_{big}$) is the universal category, with ob $\mathcal{F} = $ ob $\mathcal{O}^0$ (ob $\mathcal{F}_{big} = $ ob $\mathcal{O}$), satisfying the following properties:

- There is a functor $j : \mathcal{O}^0 \to \mathcal{F}$ (resp. $j_{big} : \mathcal{O} \to \mathcal{F}_{big}$).

- Quasi-units are sent via $j_{(big)}$ to isomorphisms.

We will argue, using the properties of localization and geometric position arguments, that the Cardy relation implies:

Theorem 11 (Abouzaid-G.)**.** *Let $\tilde{K} \in $ ob $\mathcal{O}^0 = $ ob $\mathcal{F}$ be a test object. Then:*

1. *There exists a chain map $\mathcal{CO}_{\mathcal{F}} : CF^*(\pi) \to \hom_{\mathcal{F}}(\tilde{K}, \tilde{K})$, sending 1 to the homology unit.*

2. *There exists a $\mathcal{F}$-bimodule, denoted $\mathcal{F}_{rot}$, satisfying $\mathcal{F}_{rot}(A, B) = HF^*(\phi_{2\pi+2\epsilon} A, B)$, for $A, B \in $ ob $\mathcal{F} = $ ob $\mathcal{O}^0$.*

3. *There is a chain map $\mathcal{OC}_{\mathcal{F}} : CC_*(\mathcal{F}, \mathcal{F}_{rot}) \to CF^*(\pi)$.*

4. *There is a morphism of bimodules $\Delta_{\mathcal{F}} : \mathcal{F}_{rot} \to \mathcal{Y}_{\tilde{K}}^l \otimes_{\mathbb{C}} \mathcal{Y}_{\tilde{K}}^r$ fitting into a commutative diagram*

$$
\begin{array}{ccc}
CC_*(\mathcal{F}, \mathcal{F}_{rot}) & \xrightarrow{(\Delta_{\mathcal{F}})_*} & \mathcal{Y}_{\tilde{K}}^r \otimes_{\mathcal{F}} \mathcal{Y}_{\tilde{K}}^l \\
\downarrow{\scriptstyle \mathcal{OC}_{\mathcal{F}}} & & \downarrow{\scriptstyle \mu_{\mathcal{F}}} \\
CF^*(\pi) & \xrightarrow{\mathcal{CO}_{\mathcal{F}}} & \hom_{\mathcal{F}}(\tilde{K}, \tilde{K})
\end{array}
, \tag{3.3}
$$

Corollary 12. *The generation criterion in this setting. If $\mathcal{OC}_{\mathcal{F}}$ restricted to any subcategory $\mathcal{B}$ of $\mathcal{F}$ hits 1, then $\mathcal{B}$ split-generates.*

Let's put the diagram from geometry side-by-side: for K an object in sector $-\pi/2$:

$$
\begin{array}{ccc}
CC_*(\mathcal{O}^0, \mathcal{O}_{rot}) & \longrightarrow & \mathcal{Y}_K^r \otimes_{\mathcal{O}^0} \mathcal{Y}_{\phi_\pi K}^l \\
\downarrow & & \downarrow{\scriptstyle \mu_{\mathcal{O}}} \\
CF^*(\pi) & \longrightarrow & CF^*(\phi_\pi K, K)
\end{array}
\tag{3.4}
$$

Here, we are composing the geometric commutative diagram with a few helpful identifications: we have that $\mathcal{O}_\sharp \simeq \mathcal{O}_{2\pi+2\epsilon}$, where $\mathcal{O}_{2\pi+2\epsilon}$ is an $\mathcal{O}^0$ bimodule satisfying $\mathcal{O}_{2\pi+2\epsilon}(K,L) = \hom_{\mathcal{O}}(\phi_{2\pi+2\epsilon}K, L)$ (the correspondence uses a souped up version of the identification (1.5)). For consistencey let's call this bimodule

$$\mathcal{O}_{rot} := \mathcal{O}_{2\pi+2\epsilon}. \tag{3.5}$$

We have implicitly also applied the commutative diagram (2.7) to characterize the right vertical arrow.

Here will be our approach, with maps we have yet to define (or from sources/targets not yet constructed) parenthesized and in bold:

$$\tag{3.6}$$

If this diagram commutes, then $\mathcal{OC}_{\mathcal{F}}$ is defined (at least homologically) as $\mathcal{OC}$ composed with the inverse of $\mathbf{j}_{1*}$ (which works because $\mathbf{j}_{1*}$ is a quasi-isomorphism), and so on. $(\Delta_{\mathcal{F}})_*$ could be defined via inverting $(\mathbf{j}_1)_*$ (the approach we take in this note, for simplicity), but there is also an intrinsic characterization mentioned in Remark 6.

3.1   General nonsense about localization

There are two types of localization:

- Localization with respect to a class of morphisms $[Z] \subset \mathcal{C}$ (our situation): this produces a category $\mathcal{Q}$ and a functor $L : \mathcal{C} \to \mathcal{Q}$ sending morphisms $[Z]$ to quasi-isomorphisms (L is the (quasi-)universal such).

- Localization with respect to a full subcategory $\mathcal{A}$, often simply called a *quotient*. This produces a new category $\mathcal{Q}$ and a functor $L : \mathcal{C} \to \mathcal{Q}$ in which objects of the subcategory $\mathcal{A}$ are sent to trivial objects (L is again the (quasi-)universal such functor).

The following construction embeds the first setting (ours) into the second: given $[Z]$, let $\mathcal{A} \subset Tw\mathcal{C}$ denote the sub-category consisting of all cones of morphisms in $[Z]$; localizing in the second sense produces a category $\mathcal{Q}'$ with a functor $Tw\mathcal{C} \to \mathcal{Q}'$, and we define $\mathcal{Q}$ to be the image of $\mathcal{C} \hookrightarrow Tw\mathcal{C} \to \mathcal{Q}'$.

In this second setting, there are explicit chain-level formulae for morphisms and compositions in $\mathcal{Q}$ (due to Drinfeld, and in the A_∞ setting, Lyubashenko-Manzyuk); for instance,

$$\hom_{\mathcal{Q}}(K, L) := \mathrm{Cone}(\mathcal{Y}_L^r \otimes_{\mathcal{A}} \mathcal{Y}_K^l \xrightarrow{\mu} \hom_{\mathcal{C}}(K, L)) \qquad (3.7)$$

Such formulae are useful for general functorial properties of localization, A_∞ structures, etc., but inconvenient for actually computing morphism spaces, Hochschild invariants, etc. To start, we seek simpler formulae to compute Homs in $\mathcal{F}$ terms of Homs in $\mathcal{O}$ and our geometry.

The first observation, due to [**Abouzaid-Seidel**], is that, if one is taking the hom of a pair (L_0, L_1) of objects in the correct order $(L_0 > L_1)$, then the morphism space $\hom_{\mathcal{O}}(L_0, L_1)$ upstairs computes the right hom downstairs:

Lemma 13 ("Correct position lemma"). *Say $L_0, L_1 \in$ ob $\mathcal{O}^{(I)}$. If $L_0 > L_1$, then*

$$j_I \text{ induces a quasi-isomorphism}$$
$$(j_I)_* : CF^*(L_0, L_1) = \hom_{\mathcal{O}_I}(L_0, L_1) \xrightarrow{\sim} \hom_{\mathcal{F}_I}(L_0, L_1). \qquad (3.8)$$

Proof. Quasi-units act on the morphism space upstairs by isomorphisms. . .
$\square$

If condition (3.8) is satisfied, we say L_0 is *left local* for L_1 (or L_1 is right local. . .); this lemma gives a criterion. Of course it is not possible to find an object $L_0 \in$ ob $\mathcal{O}$ that is left local for every other object $L_1 \in$ ob $\mathcal{O}$.

Remark 3. *Given an object L_0, one can always find an infinite twisted complex $L_0 \xrightarrow{q} L_0' \xrightarrow{q} L_0'' \xrightarrow{q} \ldots$ (all arrows are quasi-units) that are eventually left local for every object L_1, leading to a direct limit formulations of morphism spaces/products in $\mathcal{F}$. See* **Abouzaid-Seidel***.*

We will take a simpler approach, and instead observe the following:

$$\text{If } K_+ \text{ is an object in the sector } \pi/2, \text{ then it is left local}$$
$$\text{for every object in the category } \mathcal{O}^0 \text{ (sector 0)}. \qquad (3.9)$$

Similarly, K_- is *right local* for every object in $\mathcal{O}^0$. But of course $\mathcal{O}^0$ contains essentially all of the information in $\mathcal{F}_{big}$, so this gives us some handle on most morphisms to or from a specific object.

3.2 The dualising bimodule

Let $\phi_{rot} := \phi_{2\pi+2\epsilon}$ be the boundary rotation by $2\pi+2\epsilon$ defined earlier, which we can view as an A_∞ functor from $\mathcal{O}^0 \to \mathcal{O}$. Because of the following fact

Lemma 14. ϕ_{rot} *sends quasi-units to quasi-units.*

ϕ_{rot} descends (quasi-)uniquely to a functor $\bar{\phi}_{rot} : \mathcal{F} \to \mathcal{F}_{big}$ fitting into a commutative diagram

$$\begin{array}{ccc} \mathcal{O}^0 & \xrightarrow{\phi_{rot}} & \mathcal{O} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_{big}} \\ \mathcal{F} & \xrightarrow{\bar{\phi}_{rot}} & \mathcal{F}_{big} \end{array} \qquad (3.10)$$

We can take the *graph bimodules* of ϕ_{rot} and $\bar{\phi}_{rot}$; these produce $\mathcal{O}^0-\mathcal{O}$ and $\mathcal{F}-\mathcal{F}_{big}$ bimodules respectively, characterized as

$$\Gamma_{rot}(X,Y) := \hom_{\mathcal{O}}(\phi_{rot}X,Y) \qquad (3.11)$$
$$\bar{\Gamma}_{rot}(X,Y) := \hom_{\mathcal{F}_{big}}(\bar{\phi}_{rot}X,Y) \qquad (3.12)$$

By virtue of (3.10), the localization morphism j_{big} induces a natural morphism of bimodules

$$\tilde{j} : \Gamma_{rot} \to (j, j_{big})^*\bar{\Gamma}_{rot} \qquad (3.13)$$

defined, for a pair of objects as

$$\Gamma_{rot}(X,Y) = \hom_{\mathcal{O}}(\phi_{rot}X,Y) \xrightarrow{j} \hom_{\mathcal{O}}(j_{big}\phi_{rot}X, j_{big}Y) \\ = \hom_{\mathcal{O}}(\bar{\phi}_{rot}jX, j_{big}Y) = (j,j_{big})^*\bar{\Gamma}_{rot}(X,Y). \qquad (3.14)$$

Remark 4. *Such a morphism is unsurprising; recall that any functor* $f :$ $\mathcal{C} \to \mathcal{Q}$ *also induces natural morphisms of Yoneda modules over* $\mathcal{C}$

$$\mathcal{Y}_K^r \to f^*\mathcal{Y}_{fK}^r \qquad (3.15)$$

and morphisms between diagonal bimodules

$$\mathcal{C}_\Delta \to (f,f)^*\mathcal{Q}_\Delta. \qquad (3.16)$$

These are nearly the bimodules $\mathcal{O}_{rot}$ and $\mathcal{F}_{rot}$: we now simply restrict the right inputs to live in $\mathcal{O}^0$ and $\mathcal{F}$ respectively, via the morphisms $i_0 : \mathcal{O}^0 \hookrightarrow \mathcal{O}$ and $\bar{i}_0 : \mathcal{F} \to \mathcal{F}_{big}$:

$$\mathcal{O}_{rot}(X,Y) = (1,i_0)^*\Gamma_{rot}(X,Y) := \hom_{\mathcal{O}}(\phi_{rot}X, i_0Y) \qquad (3.17)$$
$$\mathcal{F}_{rot}(X,Y) = (1,\bar{i}_0)^*\bar{\Gamma}_{rot}(X,Y) = \hom_{\mathcal{F}_{big}}(\bar{\phi}_{rot}X, \bar{i}_0Y) \qquad (3.18)$$

The pullback of (3.13) by i_0 on the right gives a natural morphism

$$\tilde{j} : \mathcal{O}_{rot} \to j^* \mathcal{F}_{rot} = (j, j)^* \mathcal{F}_{rot}. \tag{3.19}$$

(here we are using the fact that $j_{big} \circ i_0 = \bar{i}_0 \circ j$, so $(1, i_0)^*(j, j_{big})^* \bar{\Gamma}_{rot} = (j, j)^*(1, \bar{i}_0)^* \bar{\Gamma}_{rot}$).

We have the following important fact about (3.19):

Proposition 15 (*Locality* for $\mathcal{O}_{rot}$). $\tilde{j}$ *induces a quasi-isomorphism:*

$$\mathcal{O}_{rot} \xrightarrow{\sim} j^* \mathcal{F}_{rot}.$$

Proof. Note that for any $K, L \in \mathrm{ob}\ \mathcal{O}^0$, $\phi_{rot} K > L$ (as we are bending by $2\pi + 2\epsilon$), so by the correct position lemma,

$$
\begin{aligned}
\tilde{j} : \mathcal{O}_{rot}(K, L) &= \hom_{\mathcal{O}}(\phi_{rot} K, i_0 L) \overset{j_{big}}{\underset{\sim}{\to}} \hom_{\mathcal{F}_{big}}(j_{big} \phi_{rot} K, j_{big} i_0 L) \\
&= \hom_{\mathcal{F}_{big}}(\bar{\phi}_{rot} j K, \bar{i}_0 j L) = j^* \mathcal{F}_{rot}(K, L).
\end{aligned}
\tag{3.20}
$$

$\square$

3.3 Hochschild invariants and functoriality

We come to the map $\mathbf{j}_{1*}$; why does it exist (and why is it an isomorphism)? We recall in what sense Hochschild homology with coefficients is functorial:

- If $\mathcal{B}_0$ and $\mathcal{B}_1$ are two $\mathcal{C}$ bimodules and $m : \mathcal{B}_0 \to \mathcal{B}_1$ a closed bimodule morphism, then there is a chain map

$$m_* : \mathrm{CC}_*(\mathcal{C}, \mathcal{B}_0) \to \mathrm{CC}_*(\mathcal{C}, \mathcal{B}_1). \tag{3.21}$$

- If $f : \mathcal{C} \to \mathcal{Q}$ is a functor, and $\mathcal{P}$ a bimodule over $\mathcal{Q}$, then f induces a chain map

$$f_* : \mathrm{CC}_*(\mathcal{C}, f^* \mathcal{P}) \to \mathrm{CC}_*(\mathcal{Q}, \mathcal{P}), \tag{3.22}$$

  where $f^* \mathcal{B} := (f, f)^* \mathcal{B}$ is the *pulled-back bimodule* over $\mathcal{C}$ defined by $f^* \mathcal{B}(X, Y) = \mathcal{B}(fX, fY)$ (multiplication by morphisms in $\mathcal{C}$ is given by first mapping morphisms to $\mathcal{Q}$ and then multiplying).

Remark 5. *Readers wishing to reconcile with the fact that Hochschild homology with diagonal coefficients is genuinely functorial should note the following: any functor $f : \mathcal{C} \to \mathcal{P}$ induces a canonical morphism of $\mathcal{C}$ bimodules, $\tilde{f} : \mathcal{C}_\Delta \to f^* \mathcal{P}_\Delta$; the usual pushforward on the Hochschild complex factors as the composition $f_* \circ \tilde{f}_*$.*

Thus, a map $\mathbf{j}_{1*}$ could be produced as a composition of

1. the morphism $\tilde{j}_* : \mathrm{CC}_*(\mathcal{O}^0, \mathcal{O}_{rot}) \to \mathrm{CC}_*(\mathcal{O}^0, j^*\mathcal{F}_{rot})$ coming from the morphism of bimodules $\tilde{j} : \mathcal{O}_{rot} \to j^*\mathcal{F}_{rot}$ constructed above.

2. the morphism $j_* : \mathrm{CC}_*(\mathcal{O}^0, j^*\mathcal{F}_{rot}) \to \mathrm{CC}_*(\mathcal{F}, \mathcal{F}_{rot})$ induced by the functor j.

Regarding 1, we have already shown using geometry that $\mathcal{O}_{rot}$ is *local*; meaning that the morphism $\tilde{j} : \mathcal{O}_{rot} \to j^*\mathcal{F}_{rot}$ is a quasi-isomorphism; $\tilde{j}_*$ is too. For 2, we apply the following lemma, which we explain in the following section:

Lemma 16 (Localization Lemma/"Computing upstairs"). *If $f : \mathcal{C} \to \mathcal{Q}$ is a localization, and $\mathcal{P}$ is any bimodule over $\mathcal{Q}$, then*

$$f_* : \mathrm{CC}_*(\mathcal{C}, f^*\mathcal{P}) \to \mathrm{CC}_*(\mathcal{Q}, \mathcal{P}) \tag{3.23}$$

is always a quasi-isomorphism.

3.4 Computing Hochschild invariants upstairs

Recall that any functor, $f : \mathcal{C} \to \mathcal{Q}$, localization or not, induces a $\mathcal{Q} - \mathcal{C}$ bimodule Γ_f, the *graph of f*, and a $\mathcal{C} - \mathcal{Q}$ bimodule, the *transposed graph of f* Γ_f^T. Formally, these are defined as one-sided pullbacks of the diagonal bimodule on $\mathcal{Q}$:

$$\Gamma_f := (f, id)^*\mathcal{Q}_\Delta \ (\text{so } \Gamma_f(X, Y) := \hom_\mathcal{Q}(fX, Y)) \tag{3.24}$$
$$\Gamma_f^T := (id, f)^*\mathcal{Q}_\Delta \ (\text{so } \Gamma_f^T(K, L) := \hom_\mathcal{Q}(K, fL)), \tag{3.25}$$

which, in turn induce (via convolution/one-sided tensor products), functors between module categories

$$f_* := \cdot \otimes_\mathcal{C} \Gamma_f : mod(\mathcal{C}) \to mod(\mathcal{Q}) \tag{3.26}$$
$$f^* := \cdot \otimes_\mathcal{Q} \Gamma_f^T : mod(\mathcal{Q}) \to mod(\mathcal{C}) \tag{3.27}$$

and a functor between bimodules categories

$$f_* = (f, f)_* : \Gamma_f^T \otimes_\mathcal{C} \otimes \cdot \otimes_\mathcal{C} \Gamma_f : \mathcal{C}\text{--mod--}\mathcal{C} \to \mathcal{Q}\text{--mod--}\mathcal{Q} \tag{3.28}$$
$$f^* = (f, f)^* : \Gamma_f \otimes_\mathcal{Q} \otimes \cdot \otimes_\mathcal{Q} \Gamma_f^T : \mathcal{Q}\text{--mod--}\mathcal{Q} \to \mathcal{C}\text{--mod--}\mathcal{C} \tag{3.29}$$

Remark 6. *The morphism $\Delta_{\mathcal{F}} : \mathcal{F}_{rot} \to \mathcal{Y}^l_{\tilde{K}} \otimes \mathcal{Y}^r_{\tilde{K}}$ could be constructed by applying j_* to the morphism Δ. There is something to check though: that $j_*\mathcal{O}_{rot} \simeq \mathcal{F}_{rot}$ and $j_*(\mathcal{Y}^l_K \otimes \mathcal{Y}^r_{\phi_\pi K}) \simeq \mathcal{Y}^l_{\tilde{K}} \otimes \mathcal{Y}^r_{\tilde{K}}$ for suitable $\tilde{K}$. We basically already checked the first, and omit the second (which is straightforward) here.*

Note that there is a simpler definition of pull-back of a module or bimodule, also denoted f^*; this is because these two versions are (quasi-)equivalent via

$$f^*\mathcal{M} \simeq f^*(\mathcal{M} \otimes_{\mathcal{Q}} \mathcal{Q}_\Delta) = \mathcal{M} \otimes_{\mathcal{Q}} \Gamma_f^T.$$

Note that f_* is a *proper functor* (meaning it preserves perfectness), as $f_*\mathcal{Y}^r_K \simeq \mathcal{Y}^r_{fK}$, but f^* may not be.

Lemma 17 (Technical Localization Lemma). *If f is a localization, then $\Gamma_f \otimes_{\mathcal{C}} \Gamma_f^T \simeq \mathcal{Q}_\Delta$. Namely, the pullback f^* is a fully faithful embedding, with left quasi-inverse given by f_*.*

Proof. Using explicit chain-level models, there is a canonical morphism $\Gamma_f \otimes_{\mathcal{C}} \Gamma_f^T \to \mathcal{Q}_\Delta$. Take the cone, length filter, and apply an explicit contracting homotopy on page 1... $\qquad\square$

Remark 7. *A prototypical example of categorical localization arises in algebraic geometry: If X is a smooth projective variety, $j : U \hookrightarrow X$ is an open sub-variety,*

$$Coh(X) \xrightarrow{j^*} Coh(U)$$

presents $Coh(U)$ as the localization of $Coh(X)$ by coherent sheaves $Coh_D(X)$ in X supported along the closed complement $D = X \backslash U$.

The (right) adjoint morphism

$$j_* : Coh(U) \to QCoh(X) = Mod(Coh(X))$$

in general does not preserve coherence, but is a fully faithful embedding, meaning that we can still compute homology/Exts in U after pushing forward to X.

This gives an easy proof of the localization lemma:

Proof of Lemma 16. Let's show that the relevant chain-complexes are quasi-isomorphic via a zig-zag, without worrying about the particular chain map f_*. First, there is a quasi-isomorphism

$$CC_*(\mathcal{C}, f^*\mathcal{B}) \xleftarrow{\sim} CC_*(\mathcal{C}, f^*(\mathcal{Q}_\Delta \otimes_{\mathcal{Q}} \mathcal{B} \otimes_{\mathcal{Q}} \mathcal{Q}_\Delta)) = CC_*(\mathcal{C}, \Gamma_f^T \otimes_{\mathcal{Q}} \mathcal{B} \otimes_{\mathcal{Q}} \Gamma_f) \tag{3.30}$$

coming from an underlying quasi-isomorphism of bimodules. Next, rewrite the latter complex as a two-sided tensor product

$$\mathrm{CC}_*(\mathcal{C}, \Gamma_f^T \otimes_{\mathcal{Q}} \mathcal{B} \otimes_{\mathcal{Q}} \Gamma_f) = (\Gamma_f \otimes_{\mathcal{C}} \Gamma_j^T) \otimes_{\mathcal{Q}-\mathcal{Q}} \mathcal{B}. \tag{3.31}$$

Lemma 17 gives a quasi-isomorphism

$$(\Gamma_f \otimes_{\mathcal{C}} \Gamma_j^T) \otimes_{\mathcal{Q}-\mathcal{Q}} \mathcal{B} \xrightarrow{\sim} \mathcal{Q}_\Delta \otimes_{\mathcal{Q}-\mathcal{Q}} \mathcal{B} \tag{3.32}$$

But the right complex is quasi-isomorphic to $\mathrm{CC}_*(\mathcal{Q}, \mathcal{B})$ via explicit maps (for instance, thinking of it as the the Hochschild complex of $\mathcal{Q}_\Delta \otimes_{\mathcal{Q}} \mathcal{B}$). $\square$

Remark 8 (General nonsense about locality)**.** *If* $f : \mathcal{C} \to \mathcal{Q}$ *is a localization, and* $\mathcal{B}$ *is any module/bimodule over* $\mathcal{C}$*, we say that* $\mathcal{B}$ *is* **local** *if* $\mathcal{B} \simeq j^* \tilde{B}$*, for some* $\mathcal{Q}$*-bimodule* $\tilde{B}$*. It is a general fact that there is always a morphism* $\mathcal{B} \to j^* j_* \mathcal{B}$*, which need not be a quasi-isomorphism, but is the universal map to* j^* *of a bimodule. In particular, if* $\mathcal{B} \simeq j^* \tilde{B}$ *for some bimodule* $\tilde{B}$*, then* $\tilde{B} \simeq j_* \mathcal{B}$*. So we call* $j_* \mathcal{B}$ *the* **localization** *of* $\mathcal{B}$*.*

There is a straightforward criterion for when a module/bimodule $\mathcal{B}$ *is local, meaning that it is quasi-isomorphic to* $j^* j_* \mathcal{B}$*: if multiplication by quasi-units act (on the left and/or right) by quasi-isomorphisms.*

3.5   Test objects and sectors

There is an obvious (and perhaps less interesting) issue in the right portion of the desired commutative diagram:

$$\begin{array}{ccc} \mathcal{Y}_{\tilde{K}}^r \otimes_{\mathcal{F}} \mathcal{Y}_{\tilde{K}}^l & \longleftarrow & \mathcal{Y}_K^r \otimes_{\mathcal{O}^0} \mathcal{Y}_{\phi_\pi K}^l \\ \downarrow{\scriptstyle \mu_{\mathcal{F}}} & & \downarrow{\scriptstyle \mu_{\mathcal{O}}} \\ \hom_{\mathcal{F}}(\tilde{K}, \tilde{K}) & \longleftarrow & \hom_{\mathcal{O}}(\phi_\pi K, K) \end{array} \tag{3.33}$$

namely, $\tilde{K}$ is an object in the sector defining $\mathcal{O}^0$, whereas K is in the sector defining $\mathcal{O}^{-\pi/2}$ ($\phi_\pi K$ is in the sector defining $\mathcal{O}^{\pi/2}$. The remedy: given a choice of $\tilde{K}$, we pick K differing from $\tilde{K}$ by a time $-\pi/2$ bend, so there are quasi-units in $\mathcal{O}$:

$$K \to \tilde{K} \to \phi_\pi K. \tag{3.34}$$

So in $\mathcal{F}_{big}$, $\tilde{K}$, K, and $\phi_\pi K$ are isomorphic objects. Since $\phi_\pi K > K$, the "correct position lemma" implies that

$$\hom_{\mathcal{O}}(\phi_\pi K, K) \simeq \hom_{\mathcal{F}_{big}}(\phi_\pi K, K) \simeq \hom_{\mathcal{F}_{big}}(\tilde{K}, \tilde{K}) \simeq \hom_{\mathcal{F}}(\tilde{K}, \tilde{K}) \tag{3.35}$$

in a manner sending the quasi-unit to the (homology) unit. This is morally the reason that (3.33) can be made to work. In case details are desired:

Now j_{big} (like any functor), induces a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{Y}^r_K \otimes_{\mathcal{F}_{big}} \mathcal{Y}^l_{\phi_\pi K} & \xleftarrow{(j_{big})_*} & \mathcal{Y}^r_K \otimes_{\mathcal{O}} \mathcal{Y}^l_{\phi_\pi K} & \longleftarrow & \mathcal{Y}^r_K \otimes_{\mathcal{O}^0} \mathcal{Y}^l_{\phi_\pi K} \\
\downarrow{\mu_{\mathcal{F}_{big}}} & & \downarrow{\mu_{\mathcal{O}}} & \swarrow{\mu_{\mathcal{O}}} & \\
\hom_{\mathcal{F}_{big}}(\phi_\pi K, K) & \xleftarrow[j_{big}]{\sim} & \hom_{\mathcal{O}}(\phi_\pi K, K) & &
\end{array}
\qquad (3.36)
$$

In fact, the induced composition $\mathcal{Y}^r_K \otimes_{\mathcal{O}^0} \mathcal{Y}^l_{\phi_\pi K} \to \mathcal{Y}^r_K \otimes_{\mathcal{F}_{big}} \mathcal{Y}^l_{\phi_\pi K}$ is always a quasi-isomorphism: once more we can think of this map as the composition

$$
\mathrm{CC}_*(\mathcal{O}^0, \mathcal{Y}^l_{\phi_\pi K} \otimes \mathcal{Y}^r_K) \to \mathrm{CC}_*(\mathcal{F}, \mathcal{Y}^l_{\phi_\pi K} \otimes \mathcal{Y}^r_K) \to \mathrm{CC}_*(\mathcal{F}_{big}, \mathcal{Y}^l_{\phi_\pi K} \otimes \mathcal{Y}^r_K).
$$

Then, *locality* of the $\mathcal{O}^0$ bimodule $\mathcal{Y}^l_{\phi_\pi K} \otimes \mathcal{Y}^r_K$ (which follows from $\phi_\pi K > L > K$ for any object $L \in \mathcal{O}^0$) along with Lemma 16 will imply that the first map is an isomorphism. Further details are omitted.

In turn, because of the isomorphisms (3.34) in $\mathcal{F}_{big}$, followed by the quasi-equivalence between $\mathcal{F}_{big}$ and $\mathcal{F}$ which sends $\tilde{K}$ to $\tilde{K}$, the left most vertical arrow fits into the homotopy commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{Y}^r_K \otimes_{\mathcal{F}_{big}} \mathcal{Y}^l_{\phi_\pi K} & \xleftarrow{\sim} & \mathcal{Y}^r_{\tilde{K}} \otimes_{\mathcal{F}_{big}} \mathcal{Y}^l_{\tilde{K}} & \xleftarrow{\sim} & \mathcal{Y}^r_{\tilde{K}} \otimes_{\mathcal{F}} \mathcal{Y}^l_{\tilde{K}} \\
\downarrow{\mu_{\mathcal{F}_{big}}} & & \downarrow{\mu_{\mathcal{F}_{big}}} & & \downarrow{\mu_{\mathcal{F}}} \\
\hom_{\mathcal{F}_{big}}(\phi_\pi K, K) & \xleftarrow{\sim} & \hom_{\mathcal{F}_{big}}(\tilde{K}, \tilde{K}) & \xleftarrow{\sim} & \hom_{\mathcal{F}}(\tilde{K}, \tilde{K})
\end{array}
\qquad (3.37)
$$

which all together explain (3.33).