Math 257B Week 2 Day 1:

- Gromov compactness + gluing
- $d^2 = 0$, if no sphere + disc bubbling. Disc bubbling as an obstruction..
- invariance; continuation maps:
- Relative gradings.
- Orientations:

Convergence on compact subsets in the $C^{\ell-1}$ topology, ℓ sufficiently ... class

---

Last time: Statement of Gromov compactness (this time, for Riem. surfaces w/ boundary).

Thm: Suppose $\Sigma_n = (\Sigma, j_n)$ has boundary components $\partial_1 \Sigma, \dots, \partial_k \Sigma$, marked points on interior & boundary.

$$u_n : \Sigma \longrightarrow (X, \omega, J_n) \quad J_n\text{-holomorphic} \quad w/ \quad u : \partial_i \Sigma \to L_i \quad \text{Lagrangians}.$$

$$J_n \in \mathcal{J}(X, \omega)$$
↑ compatible.

~~Then~~ w/ energy $E(u_n) = \int_{\Sigma_n} u^* \omega < K$ (indep. of n).

$\Rightarrow \exists$ Σ_∞ nodal surface & subsequence converging to a stable map

$$u_\infty : \Sigma_\infty \longrightarrow X$$

(~~poles~~ w/ marked pts, etc. as before)

(convergence means in the C^∞ or C^ℓ topology; for each smooth component Σ_α of Σ_∞, $\exists$ subregions Σ_n^α & automorphism ~~Möbius transformations~~ reparametrizations ϕ_α^n of Σ_n^α s.t. $\{u_n \circ \phi_\alpha^n\}$ converges on compact ~~sets~~ to Σ_α).

Ex: (images shown in target):

L_0 L_∞

New phenomena: In addition to degenerations of Σ, have (a) degeneration of domain along boundary or interior:

(b) sphere & disc bubbling

(c) arbitrary (but finitely many) trees.

<u>Ideas</u>: Identify <u>bubbling regions</u>: where $\sup |du_n| \longrightarrow \infty$

(away from these pts, standard analytic estimates + "elliptic bootstrapping" $\Rightarrow$ convergence)

- Say have a sequence z_n^0 where $|du_n| \to \infty$ of interior points

  - on these regions, rescale domain:

$$v_n(z) := u_n(z_n^0 + \varepsilon_n z).$$

for $\varepsilon_n \to 0$ suitably chosen.

$\Rightarrow$ a subsequence of v_n converges to a J-hol. map $\mathbb{C} \longrightarrow X$,

which by a <u>removal of singularities thm</u> for J-hol. curves extends to a map $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \longrightarrow X$; the bubble!

- If $|du_n| \to \infty$ at a seq. of boundary points $z_n^0 \in \partial\Sigma_n$; rescaling produces a map $\mathbb{H} \to X$, which similarly extends to a map $D^2 = \mathbb{H} \cup \{\infty\}$: disc <u>bubble</u>. . .

- Intermediate bubbling steps $\Rightarrow$ might need ~~extra~~ various intermediate rescalings, to catch all bubbles. (& moreover need to show these bubbles <u>connect up</u>), (a) energy is preserved under these limits, &

This process is <u>finite</u> because of ~~(b)~~ (b) a priori energy estimate:

<u>Thm</u>: (a priori energy bound). $E(u) = \int (u^* \omega) \geq \hbar > 0$ for all non-constant J-hol. curves possibly w/ boundary on L.

$\hbar$ = min. energy (M, J, ω, L)

(so each non-trivial bubble draws at ~~least a fixed~~ least $\hbar$ energy). (~~…~~)

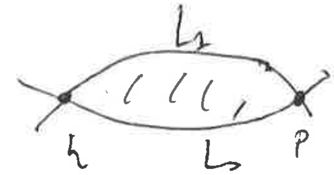
Point: by a mean value inequality satisfied by low energy curves: $\exists\, \delta > 0$ s.t. if $E(u) < \delta$, u constant. . .

<u>Consequence</u>: $\mathcal{M}(\beta)$ space of J-hol. maps in class β has a ~~…~~ <u>stable map</u> <u>compactification</u> $\overline{\mathcal{M}}(\beta)$.. (<u>compact</u> space).

<u>Gromov compactness for Floer ~~trajectories~~.</u>

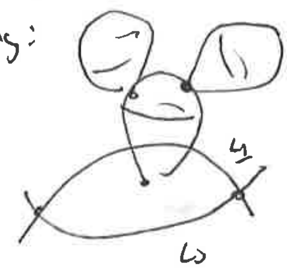
Suppose u_n a seq. of Floer <u>trajectories</u>, w/ finite energy

i.e., J-hol. maps $D^2 \setminus \{\pm 1\} \longrightarrow (X, L_0, L_1)$



3 types of phenomena in energy analysis:

1) sphere bubbling:



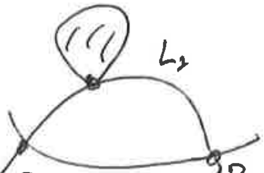
In good cases (if these loci are cut out transversely), this happens in codim ≥ 2 of compactified $\overline{\mathcal{M}}(\ldots)$, or

in ~~the~~ best cases, not at all.

Ex: if $\pi_2(M) = 0$ or more generally if $\langle \omega, \pi_2(M) \rangle = 0$.

(why? ~~by energy identity~~ any non-trivial J-hol. sphere needs some energy. $\Rightarrow \langle \omega, [S] \rangle > 0$!

2) bubbling of discs:



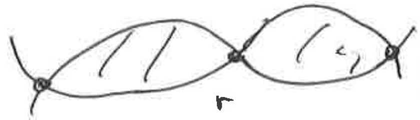
Serious issue: can occur in codimension 1, even when transversally cut out, & contribute to $\partial \overline{\mathcal{M}}$. (& often tough to ~~be~~ cut out transversely).

Again, if $\pi_2(M, L_i) = 0$, or L_i doesn't bound J-hol. discs, this is a priori excluded, $\langle \omega, \pi_2(M, L_i) \rangle = 0$.

3) Breaking of strips: (energy escapes to $s \to \pm\infty$); $\longleftrightarrow$ "bubbling at ± 1 on D^2"

i.e. reparametrizing $u_n(\cdot - \delta_n, \cdot)$ gives non-equal limits

Limiting configurations



---- $L_0, L_1 \subseteq X$, transverse, Λ, $T \in \Lambda$.

$CF^*(L_0, L_1) = \Lambda^{(L_0 \cap L_1)}$

Fix generic J.

shorthand for $\mathcal{M}(X, L_0, L_1, p, q, \beta)$.

$$\partial p = \sum_{\substack{q, \beta \in \pi_2(p,q) \\ \mathrm{ind}(\beta) = 1}} \sum_{u \in (\mathcal{M}(p,q,J))/\mathbb{R}} T^{\omega(u)} \cdot q \ \mathrm{sgn}(u)$$

$$(\text{if } T = 1) \quad = \sum_{\substack{q, \beta \\ \mathrm{ind}(\beta) = 1}} \left(\# \mathcal{M}(p, q, J)^{\beta} / \mathbb{R}\right) T^{\omega(\beta)} \cdot q.$$

this # is finite for each β & well defined

by Gen transversality + Gromov compactness

~~Theory~~ This one is analogous to Morse theory, where have:

Transversality $\Rightarrow$ this configuration, is codimension 1, so doesn't appear if ind $= 1$.

How to prove $\partial^2 = 0$ assuming no disc/sphere bubbling:

consider $\mathcal{M}(p, q, \beta, J)/\mathbb{R}$ where $\beta \in \pi_2$ has ind$(\beta) = 2$.

J-generic.

This should be a 1-manifold, which can be compactified to $\overline{\mathcal{M}}$

by adding in broken trajectories

$$\coprod_{\substack{r \in L_0 \cap L_1 \\ \beta_1 \# \beta_2 = \phi}} \left(\mathcal{M}(p, r, \beta_1, J)/\mathbb{R}\right) \times \left(\mathcal{M}(r, q, \beta_2, J)/\mathbb{R}\right).$$

(no bubbling $\Rightarrow$ no other limiting sequences!)

~~Gluing theorem~~. For example, if $\pi_2(M) = 0$ and $\pi_2(M, L_i) = 0$, cannot have any J-hol. spheres or discs. (why?) b/c energy identity $\Rightarrow$ for a disc or sphere, $\langle [\omega], u_*[\Sigma] \rangle \geq 0$ $\Rightarrow u_*[\Sigma] \neq 0$ in $\pi_2(M, L)$.

Thm [Gluing]: the resulting $\overline{\mathcal{M}(p, q, \beta, J)/\mathbb{R}}$ is a 1-mfld w/ boundary. Ex: all boundary strata ~~have~~ have gluing charts $\mathbb{R}^k \times [0,1) \to \overline{\mathcal{M}}$.

Now, signed # ends of 1-mfld, oriented (or counted mod 2) $= 0$.

~~But, the ends are counts associated to~~

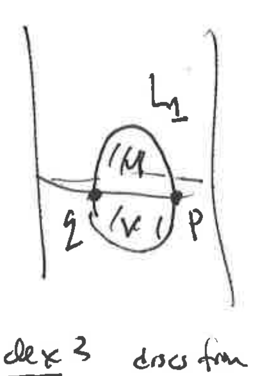
$$\Rightarrow 0 = \sum T^{\omega(\beta) = \omega(\beta_1) + \omega(\beta_2)} \left(\# \mathcal{M}(p, r, \beta, J)/\mathbb{R}\right) \cdot \left(\# \mathcal{M}(r, q, \beta_2, J)/\mathbb{R}\right) q$$

$$= \langle \partial^2(p), q \rangle_{\beta\text{-component}}.$$

Unlike sphere bubbles (technical problem) Bubbling of discs ~~is not~~ actually obstructs the definition of Floer homology.

Example:

$T^* S^1 \supset$



$CF(L_0, L_1) = \Lambda p \oplus \Lambda q$

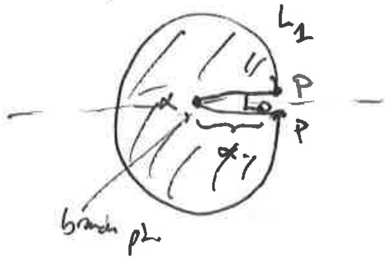
$\partial p = \pm T^{\text{area}(u)} q$

$\partial q = \pm T^{\text{area}(v)} p$.

Look at moduli space of index 2 discs from p to itself.

$\partial^2 p = T^{\text{area}(u) + \text{area}(v)} \cdot p \neq 0$!

It's an interval:

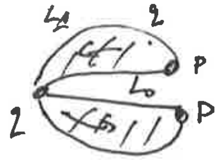


$\alpha \in (-1, 1)$

can find explicit disk for each α

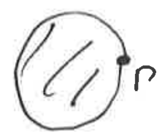
Ex: $\dfrac{z^2+\alpha}{1+\alpha z^2}$

Two end points: $\alpha \to -1$:



$p \to q \to p$

(contributes to $\partial^2 p$)



constant strip at p + disc bubble w/ boundary in L_1.

Disc bubble prevents $\partial^2 = 0$.

Suppose no disc + sphere bubbles, so $\partial^2 = 0$.

⇝ get a group $HF^*(L_0, L_1; J)$

Thms: 1) $HF^*(L_0, L_1; J)$ independent of J, ~~& Hamil~~

$=: HF^*(L_0, L_1)$.

2) $HF^*(L_0, L_1)$ Hamiltonian isotopy invariant,

so $HF^*(L_0, L_1) = HF^*(\phi_{H_0} L_0, \phi_{H_1} L_1)$ provided they are transverse.

In either case, we'll construct continuation maps between Floer chain complexes. ~~map for J_0, J_1 different a.c. structures~~.

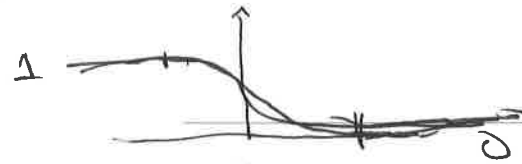
E.g., say $J_0 \neq J_1$. Pick

$L_0, \phi^1_{H_0} L_0$.

Say $H : [0,1) \times M \to \mathbb{R}$ generic

$$\phi_H^t = \text{flow of } X_H \quad (i_{X_H}\omega = dH)$$

Pick a cutoff fcn. $\beta : \mathbb{R} \to [0,1]$



& pick a path J_s of ~~almost complex structures~~ compatible almost complex structures

w/ $J_s = \begin{cases} J_0 & s \gg 0 \\ J_1 & s \ll 0. \end{cases}$

& consider, for each $p, q \in \phi_H(L_0) \pitchfork L_1$, finite energy sol'ns of

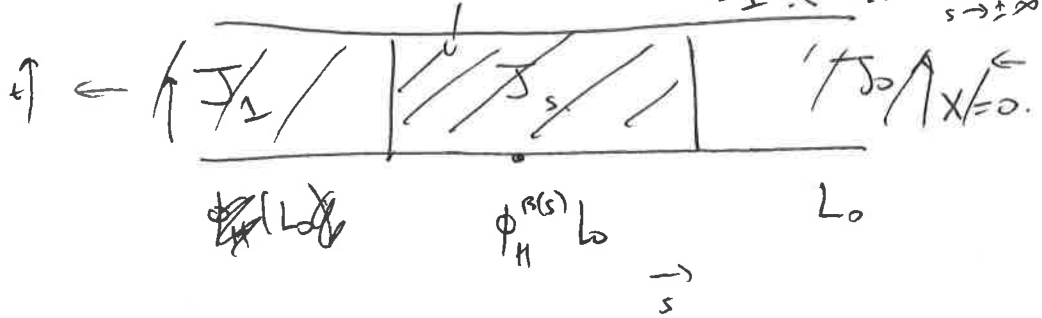
$$(*) \begin{cases} u : \mathbb{R} \times [0,1] \to M \\ \partial u/\partial s + J_s\left(\frac{\partial u}{\partial t} - \beta(s) X_{H_t}\right) = 0 \\ u(s,0) \in \phi_H^{\beta(s)} L_0 \\ u(s,1) \in L_1 \\ \lim_{s \to \pm\infty} u = \begin{cases} p & +\infty \\ q & -\infty \end{cases} \end{cases}$$

"s-dependent J" inhom. C-R eq'n "Floer's eq'n."

"moving Lag'n bdry. condition."

~~with~~

changing region.



q, or rather, the time 1-chords $\{\phi_{H_t}^{1-t} q\}_{t=0}^{1}$ of X_{H_t} from L_0 to L_1:

$\gamma(0) \in L_0$, $\gamma(1) \in L_1$,

$\gamma(1) = q \in \phi_H^1(L_0) \cap L_1$

~~If one worries about the analytics of moving Lag'n boundary conditions:~~

Remark: By a gauge transformation

$$\tilde{u}(s,t) := \phi^{+t}_{\beta(s)H_t} u(s,t) = \phi_{H_t}^{+\beta(s)t} u(s,t).$$

we see that solutions of (*) are in bijection near ∞, with solutions $\tilde{u}$ to:

$$\begin{cases} \tilde{u} : \mathbb{R} \times [0,1] \to M \\ \tilde{u}(s,0) \in L_0 \\ \tilde{u}(s,1) \in L_1 \\ \lim_{s \to -\infty} \tilde{u} = q \\ \bar{\partial}_{J} u = 0 \\ \vdots \end{cases}$$

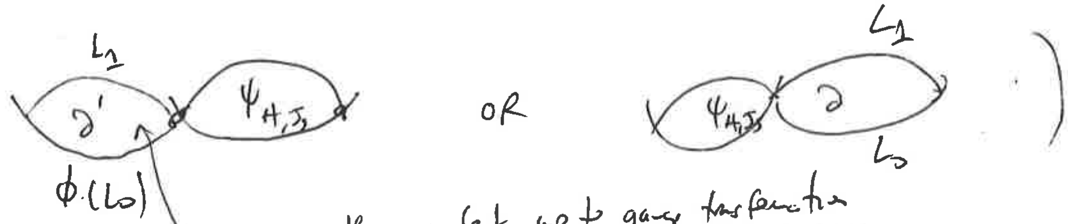
Counting index 0 solutions (weighted by $T^{\text{energy}(u)}$) gives $\Psi_H : CF(L_0, L_1; J_0) \to CF(\phi^1_{H_t}(L_0), L_1; J_1)$.

these are isolated; no $\mathbb{R}$-translation invariance anymore.

Prop: In the absence of disc bubbles (& for generic J_s, H_s) can show this is a chain map: $\Psi_{H,J_s} \circ \partial = \partial' \circ \Psi_{H,J_s}$

Idea: Look at ends of index 1 moduli spaces:

If no disc bubbling, the only ends must be broken trajectories:



or rather something equivalent up to gauge transformation.

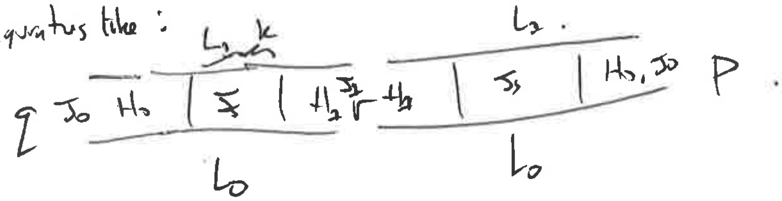
Prop: Under same hypotheses, Ψ_{H,J_s} induces a homology isomorphism.

idea: ~~Look~~ ~~chain~~ build $\Psi_{-H, J_{-s}}$ chain map other way & show

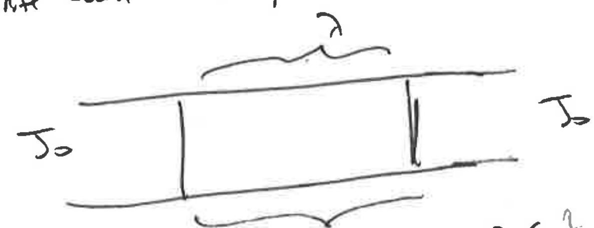
$\Psi_{H,J_s} \circ \Psi_{-H,J_{-s}}$ is chain htpic to id (& vice versa), meaning

$$\Psi_H \Psi_{-H} - \mathrm{id} = \partial' K + K \partial .$$

Point: $\Psi_{H,J_s} \circ \Psi_{-H,J_{-s}}$ counts configurations like:



Write down a family of eqns on the strip



J_0, H_0 if $\lambda \le 1$.

If $\lambda \geq 1$ this regime looks like