Symplectic integral transforms from open-closed string maps

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ABSTRACT. In this paper, we construct an A_∞ functor from a wrapped Fukaya associated to the product of Liouville manifolds to *bimodules* over the wrapped Fukaya categories of the factors. We show this functor is always full on product Lagrangians, implying an A_∞ refinement of the Künneth formula. Our core construction, which focuses on the subcategory of product Lagrangians and the diagonal, involves the use of families of *open-closed operations*, and includes the *closed-open string map* from symplectic cohomology to Hochschild cohomology as part of its first order data.

In a sequel paper, we use the functor constructed here to verify a conjecture of Kontsevich in the wrapped setting: under suitable hypotheses, the closed-open string map and another so-called open-closed string map, from Hochschild homology to symplectic cohomology, are both isomorphisms.

1. Introduction

The *wrapped Fukaya category* is an important invariant of Liouville manifolds, first introduced in work of Abouzaid and Seidel [**AS**]. It is the open-string, or Lagrangian counterpart to *symplectic cohomology*, introduced by Viterbo and Floer-Hofer [**V**] [**FH2**], and both theories have already seen many (sometimes combined) applications, e.g, [**M2**] [**M3**] [**AS**] [**A2**].

The constructions of the wrapped Fukaya category and symplectic cohomology, which we collectively term *wrapped Floer theory*, formally resemble Floer theory on a compact manifold, and it is natural to ask what structures these invariants possess in analogy with the compact setting. As one crucial difference, the field theory whose operations control wrapped Floer theory is *non-compact*; it does not admit geometric operations lacking outputs (such as trace maps or Poincaré duality-type pairings). Indeed, unlike their compact analogues, wrapped Floer homologies often have infinite rank.

In this paper, we develop geometric and algebraic structures arising from considerations of *functoriality* between wrapped Fukaya categories. Our main result is

THEOREM 1.1. *Let $M = (M, \theta)$ be a Liouville manifold with wrapped Fukaya category $\mathcal{W} := \mathcal{W}(M)$. Let M^- denote the same manifold with opposite Liouville form $(M, -\theta)$, and denote by*

$$\mathcal{W}^2 \tag{1.1}$$

the wrapped Fukaya category of product Lagrangians and the diagonal in $M^- \times M$, constructed with split Hamiltonians. Then, there is a (geometric) A_∞ functor from $\mathcal{W}^2$ to the category of A_∞ **bimodules** *over $\mathcal{W}$:*

$$\mathbf{M} : \mathcal{W}^2 \to \mathcal{W}\text{–mod–}\mathcal{W}, \tag{1.2}$$

which is cohomologically full and faithful on product Lagrangians.

The main intended application is to verify, in the Liouville setting and under suitable hypotheses, a conjecture of Kontsevich equating symplectic cohomology with the Hochschild invariants of the wrapped Fukaya category. This will be addressed in a sequel companion article [**G**]. Theorem 1.1, along with Corollary 9.1 (relating the terms of this functor to open-closed maps) are central inputs into that paper.

Our construction applies directly with minor notation change to the case of $M^- \times N$, with the caveat that one exclude the diagonal Lagrangian (with minor additional effort, we anticipate it is straightforward to add graphs of exact symplectomorphisms $\phi : M \to N$). A *bimodule between wrapped Fukaya categories*

$$\mathcal{B} \in \text{ob } \mathcal{W}(M)\text{–mod–}\mathcal{W}(N) \tag{1.3}$$

should be thought of as a generalization of the notion of a bimodule between algebras to multiple objects; for every pair of objects $X \in \text{ob } \mathcal{W}(M)$, $Y \in \mathcal{W}(N)$, we are provided a chain complex

$$\mathcal{B}(X, Y) \tag{1.4}$$

along with left and right multiplication chain maps

$$\begin{aligned} \mathcal{B}(X, Y_0) \otimes \hom_{\mathcal{W}(N)}(Y_0, Y) &\to \mathcal{B}(X, Y) \\ \hom_{\mathcal{W}(N)}(X, X_0) \otimes \mathcal{B}(X_0, Y) &\to \mathcal{B}(X, Y) \end{aligned} \tag{1.5}$$

and higher, A_∞-type multiplications. For instance, to any functor $\mathcal{F} : \mathcal{W}(M) \to \mathcal{W}(N)$, one can associate a *graph bimodule*

$$\Gamma_{\mathcal{F}}(X, Y) := \hom_{\mathcal{W}(N)}(\mathcal{F}X, Y). \tag{1.6}$$

Via tensor product, also called *convolution*, any bimodule $\mathcal{B}$ induces a functor

$$\mathcal{W}(M)\text{–mod} \to \mathcal{W}(N)\text{–mod} \tag{1.7}$$

between *categories of modules over wrapped Fukaya categories*, an algebraic enlargement of the relevant Fukaya categories. Hence, $\mathcal{W}(M)$–mod–$\mathcal{W}(N)$ can be thought of as an enlargement of the category of functors.

Our approach would be applicable to the compact setting as well (at least under a suitable technical assumption such as monotonicity), so let us for a moment suppress the distinction, and discuss the general problem and context. It is a basic principle that a symplectomorphism $\phi : M \to N$ should induce a functor

$$\phi_* : \mathcal{F}(M) \to \mathcal{F}(N) \tag{1.8}$$

between Fukaya categories. An idea due to Weinstein [**W**] is that one should enlarge the category of morphisms between symplectic manifolds to include *Lagrangian correspondences*, that is, Lagrangian submanifolds

$$\mathcal{L} \subset M^- \times N \tag{1.9}$$

where if $M = (M^{2n}, \omega)$, then $M^- = (M^{2n}, -\omega)$. The graph Γ_ϕ of a symplectomorphism ϕ gives a Lagrangian correspondence, but there are others, such as products $K \times L$ of Lagrangians in M and N. An initial difficulty in implenting Weinstein's proposal for Fukaya categories relates to *composability*; given a general Lagrangian correspondence $\mathcal{L} \subset M^- \times N$ and a Lagrangian $K \in \text{ob } \mathcal{F}(M)$, the geometric composition $\mathcal{L} \circ K$ may fail to be embedded or even have transverse self-intersections. The category of embedded Lagrangians thus is too small for functoriality from a general Lagrangian correspondence. One solution, proposed and implemented in the important work of Wehrheim-Woodward [**WW1**] at the homology level, is to enlarge the category of objects to *generalized Lagrangians*, those of the form

$$pt \stackrel{\mathcal{L}_0}{\to} M_1 \stackrel{\mathcal{L}_1}{\to} M_2 \cdots M_{k-1} \stackrel{\mathcal{L}_k}{\to} M, \tag{1.10}$$

where $M_i \stackrel{\mathcal{L}_i}{\to} M_{i+1}$ denotes a Lagrangian correspondence from M_i to M_{i+1}. Via the analysis of *quilted Floer homology groups* [**WW2**], pairs of generalized Lagrangians can be shown to have a well-defined Floer theory (under certain monotonicity and compactness assumptions), compatible with geometric composition of *composable* adjacent correspondences, and constitute objects of the *extended Fukaya category*

$$\mathcal{F}^\sharp(M). \tag{1.11}$$

By formally concatenating with a Lagrangian correspondence $\mathcal{L}$, Wehrheim and Woodward associate a homology-level functor $\Phi_{\mathcal{L}}$ from $\mathcal{F}^\sharp(M)$ to $\mathcal{F}^\sharp(N)$. This correspondence is functorial in $\mathcal{L}$, so one obtains again at the homology level a functor

$$H^*(\mathcal{F}(M^- \times N)) \to \text{fun}(H^*(\mathcal{F}^\sharp(M)), H^*(\mathcal{F}^\sharp(N))) \tag{1.12}$$

A chain level implementation has been developed by Ma'u-Wehrheim-Woodward [**MWW**].

In the wrapped setting, there is further a *finiteness issue*; *the category of functors from $\mathcal{F}(M)$ to $\mathcal{F}(N)$ (or even triangulated closures of their generalized Fukaya categories) is not large enough* for a general correspondence. Consider for example a product Lagrangian

$$\mathcal{L} = K \times L \subset M^- \times N.$$

A simple calculation in the compact setting [**AbSm**] shows that up to equivalence the associated functor should act on objects as

$$X \mapsto \hom_{\mathcal{F}}(K, X) \otimes L. \tag{1.13}$$

The above target object does not live in $\mathcal{F}(N)$ but rather within the mild enlargement of *twisted complexes* over $\mathcal{F}(N)$, which contains objects of the form $V \otimes Y$, for V a finite-rank chain complex. In the wrapped setting, one expects the product $K \times L$ to act similarly as $X \mapsto \hom_{\mathcal{W}}(K, X) \otimes L$, but as noted earlier, now the chain complex $\hom_{\mathcal{W}}(K, X)$ could infinite homological rank! The image object would therefore need to live in an infinite enlargement of the category of twisted complexes, a category which may have a completely different behavior (for instance, infinite enlargements are not Morita equivalences).

REMARK 1.1. *If X or K is compact, $CW^*(K, X)$ has finite rank, and hence one gets a well-defined functor as before.*

In order to bypass both composability and finiteness issues, we take a different approach, inspired by work of Ma'u ([**M1**] and unpublished): via new moduli spaces we associate to a Lagrangian in the product $M^- \times N$ a *bimodule* between the wrapped Fukaya category of M and that of N. Under appropriate finiteness conditions, one can recover from convolving with a bimodule a functor between (derived split-closed wrapped) Fukaya categories.

At the expense of a more algebraic framework, our approach is technically simpler; for instance, since every Lagrangian in the product induces a genuine bimodule, one does not need to talk about *generalized objects* and their Floer theory; consequently there is no new gluing or transversality analysis for the moduli spaces considered (in the exact setting, at least). The category of bimodules is also large enough to serve as the correct target for product Lagrangians in the wrapped Fukaya category. Under suitable finiteness conditions on these Lagrangians, homological algebra can promote a bimodule to a functor between (derived split-closed) Fukaya categories. This approach is also backwards compatible for implementation in various compact settings (under the usual technical hypotheses such as monotonicity or M/L not containing spheres/bounding discs, etc.).

On the other hand, some direct control of the geometry is lost. For instance, given an exact symplectomorphism $\phi : M \to N$, a priori the bimodule Γ_{ϕ_*} sends K (rather, a module representing K) to some infinite complex, which is only quasi-equivalent to (a module representing) ϕK. It is not at all obvious more generally without the strip-shrinking analysis of [**WW2**] that the composition of bimodules associated to $\mathcal{L}_0$ and $\mathcal{L}_1$ is equivalent to the bimodule associated to $\mathcal{L}_0 \circ \mathcal{L}_1$ (though this is known for graph bimodules). Finally, our approach is technically easiest for Lagrangians which are products, the diagonal, and graphs of exact symplectomorphisms, and we particularly focus on the former two classes of examples. On the other hand, whenever product Lagrangians split generate the (wrapped) Fukaya category of the product, one can recover the full functoriality, at least at the level of twisted complexes.

REMARK 1.2. *In fact, using even more homological algebra, we can promote our functor defined just for product Lagrangians to a functor from any collection of admissible Lagrangians which, along with product Lagrangians, form an A_∞ category. See Remark 9.1. The resulting functor is technically even cheaper to define, at the expense of less geometric control over functoriality. This seems to be formally related to the approach used in a different non-compact setting in work of Nadler* [**N**].

It is worth saying a few words about the methods of our construction. Let $\mathcal{W}$ denote the wrapped Fukaya category of M and

$$\mathcal{W}^2 \tag{1.14}$$

the wrapped Fukaya category of $M^- \times M$ with objects consisting of product Lagrangians and the diagonal Δ, with operations coming from *split data*, that is split Hamiltonians and almost complex structures (part of our work is constructing this category). It is an well-known observation, dating back to Floer, that the Lagrangian Floer chain complex of the diagonal $\Delta \subset M^- \times M$ with split data is exactly equal to the Hamiltonian Floer complex of M for the same Hamiltonian, via a suitable *unfolding,* or gluing process. We build upon this, showing in Section 6 that one can glue components of discs, realizing A_∞ operations for Δ and product Lagrangians $M^- \times M$ via operations in M associated to certain families of bordered Riemann surfaces with interior and boundary marked points, or *open-closed strings.* As a special case, we construct an A_∞ algebra structure on symplectic cohomology, which may be of independent interest (see Remark 6.2).

The functor to bimodules is controlled by a family of surfaces with boundary marked points known as (fixed width) *quilted strips*, whose geometry has been studied in [**M1**]. Maps from quilted strips have boundary components on Lagrangians in M and also on Lagrangians in $M^- \times M$ (along any *internal boundaries*, also known as *seams*). Correspondingly, we show that in Section 7 that one can *partially glue* quilted surfaces along seams to obtain an open-closed string. On the level of a single quilted surface, such gluing observations are not new [**AbSm**]; our observation is simply that without developing new analysis for maps from quilted strips, one can instead use these gluings, and operations from open-closed strings to give an a posteriori definition of the functor, which we do in Proposition 7.3. Theorem 1.1 is a combination of this Proposition with the calculations in §8, see particularly Proposition 8.2.

In Sections 2-5, we set up the necessary preliminaries and construct these operations. Specifically, Section 2 reviews various algebraic structures that arise: A_∞ categories, modules and bimodules, functors, and tensor products, and Section 3 reviews the definition of symplectic cohomology and wrapped Floer cohomology. In Section 4, a foundational section used in all later sections, we construct Floer theoretic operations corresponding to (compactified) families of genus 0 open-closed strings. This includes the A_∞ structure maps and the pair of pants product on symplectic cohomology as special cases as well as various *open-closed string maps*, first studied by Seidel [**S1**]. Finally, in Section 5, we construct operations corresponding to families of open-closed strings which arise as *partially glued* pairs of discs, the necessary input to the observations of Section 7.

Lastly, Appendix A discusses the ingredients necessary to orient moduli spaces of maps, in order to obtain operations defined over the integers (or a field of arbitrary characteristic). Our geometric constructions also make use of a new compactness result for open-closed moduli spaces appearing in Appendix B (using methods in the symplectic cohomology case which date back to Cieliebak and Floer-Hofer), which may be of independent interest.

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2. Algebraic preliminaries

We give an overview of the algebraic structures appearing in this paper: A_∞ categories and their modules, bimodules, and Hochschild invariants.

2.1. A_∞ algebras and categories.

Definition 2.1. *An A_∞ **algebra** $\mathcal{A}$ is a graded vector space $\mathcal{A}$ together with maps*

$$\mu^s_{\mathcal{A}} : \mathcal{A}^{\otimes s} \to \mathcal{A}, \;\; s \geq 1 \tag{2.1}$$

of degree 2-s such that the following quadratic relation holds, for each k:

$$\sum_{i,l} (-1)^{\maltese_i} \mu^{k-l+1}_{\mathcal{A}}(x_k, \ldots, x_{i+l+1}, \mu^l_{\mathcal{A}}(x_{i+l}, \ldots, x_{i+1}), x_i, \ldots, x_1) = 0. \tag{2.2}$$

where the sign is determined by

$$\maltese_i := |x_1| + \cdots + |x_i| - i. \tag{2.3}$$

Remark 2.1. *The parity of $\maltese_i$ is the same as the sum of the* **reduced degrees** $\sum_{j=1}^i ||x_j||$. *Here $||x_j|| = |x_j| - 1$ is the degree of x_j thought of as an element of the shifted vector space $\mathcal{A}[1]$. Thus, $\maltese_i$ can be thought of as a Koszul-type sign arising as μ^l acts from the right.*

The first few equations of (2.2) imply that μ^1 is a differential, and (up to a sign change) μ^2 descends to an associative product on $H^*(\mathcal{A}, \mu^1)$. This product is not associative on the chain level; and μ^3 can be thought of as the first in a sequence of higher correcting homotopies.

One can recast the notion of an A_∞ algebra in the following way: Let

$$T\mathcal{A}[1] = \bigoplus_{i>0} \mathcal{A}[1]^{\otimes i} \tag{2.4}$$

be the tensor co-algebra of the shifted $\mathcal{A}[1]$. Given any map $\phi : T\mathcal{A}[1] \to \mathcal{A}[1]$, there is a unique so-called **hat extension**

$$\hat{\phi} : T\mathcal{A}[1] \to T\mathcal{A}[1] \tag{2.5}$$

specified as follows:

$$\hat{\phi}(x_k \otimes \cdots \otimes x_1) := \sum_{i,j} (-1)^{\maltese_i} x_k \otimes \cdots \otimes x_{i+j+1} \otimes \phi^j(x_{i+j}, \ldots, x_{i+1}) \otimes x_i \otimes \cdots \otimes x_1. \tag{2.6}$$

The (shifted) A_∞ operations μ^i fit together to form a map

$$\mu : T\mathcal{A}[1] \to \mathcal{A}[1] \tag{2.7}$$

of total degree 1. Then the A_∞ equations, which can be re-expressed as one equation

$$\mu \circ \hat{\mu} = 0, \tag{2.8}$$

are equivalent to the requirement that $\hat{\mu}$ is a differential on $T\mathcal{A}[1]$

$$\hat{\mu}^2 = 0. \tag{2.9}$$

Remark 2.2. *Actually, the hat extension $\hat{\phi}$ defined above is the unique extension satisfying the graded co-Leibniz rule with respect to the natural co-product $\Delta : T\mathcal{A}[1] \longrightarrow T\mathcal{A}[1] \otimes T\mathcal{A}[1]$, given by*

$$\Delta(x_n \otimes \cdots \otimes x_1) = \sum_i (x_n \otimes \cdots \otimes x_{i+1}) \otimes (x_i \otimes \cdots \otimes x_1). \tag{2.10}$$

In this way, the association of $(T\mathcal{A}, \hat{\mu})$ to the A_∞ algebra $(\mathcal{A}, \mu)$ gives an embedding of A_∞ algebra structures on a vector space to differential-graded co-algebra structures on the tensor algebra over that vector space. The chain complex $(T\mathcal{A}, \hat{\mu})$ is called the **bar complex** *of $\mathcal{A}$.*

The discussion so far generalizes in a straightforward manner to the categorical setting.

Definition 2.2. *An A_∞* **category** *$\mathcal{C}$ consists of the following data:*

- *a collection of objects* ob $\mathcal{C}$
- *for each pair of objects X, X', a graded vector space* $\hom_{\mathcal{C}}(X, X')$
- *for any set of $d+1$ objects $X_0, \ldots, X_d$, higher composition maps*

$$\mu^d : \hom_{\mathcal{C}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{C}}(X_0, X_1) \to \hom_{\mathcal{C}}(X_0, X_d) \tag{2.11}$$

  of degree $2-d$, satisfying the same quadratic relations as equation (2.2).

In this paper, we will work with some A_∞ categories $\mathcal{C}$ with finitely many objects $X_1, \ldots, X_k$. As observed in [**S6**] and [**S3**], any such category $\mathcal{C}$ is equivalent to an A_∞ algebra over the semi-simple ring

$$R = \mathbb{K}e_1 \oplus \cdots \oplus \mathbb{K}e_k,$$

which we also call $\mathcal{C}$. The correspondence is as follows: as a graded vector space this algebra is

$$\mathcal{C} := \bigoplus_{i,j} \hom(X_i, X_j) \tag{2.12}$$

with the idempotents e_i of R acting by

$$e_s \cdot \mathcal{C} \cdot e_t = \hom(X_t, X_s). \tag{2.13}$$

Tensor products are now interpreted as being over R (with respect to composable morphisms), i.e.

$$\mathcal{C}^{\otimes r} := \mathcal{C}^{\otimes_R r} = \bigoplus_{V_0, \ldots, V_r \in \mathrm{ob}\ \mathcal{C}} \hom(V_{r-1}, V_r) \otimes \cdots \otimes \hom(V_0, V_1). \tag{2.14}$$

In this picture, the A_∞ structure on the category $\mathcal{C}$ is equivalent to the data of an A_∞ structure over R on the graded vector space $\mathcal{C}$. Namely, maps

$$\mu^d : \mathcal{C}^{\otimes d} \longrightarrow \mathcal{C} \tag{2.15}$$

are by definition the same data as the higher composition maps (2.11).

Definition 2.3. *Given an A_∞ category $\mathcal{C}$, the* **opposite category**

$$\mathcal{C}^{op} \tag{2.16}$$

is defined as follows:

- *objects of $\mathcal{C}^{op}$ are the same as objects of $\mathcal{C}$,*
- *as graded vector spaces, homs of $\mathcal{C}^{op}$ are reversed homs of $\mathcal{C}$:*

$$\hom_{\mathcal{C}^{op}}(X, Y) = \hom_{\mathcal{C}}(Y, X) \tag{2.17}$$

- *A_∞ operations are, up to a sign, the reversed A_∞ operations of $\mathcal{C}$:*

$$\mu^d_{\mathcal{C}^{op}}(x_1, \ldots, x_d) = (-1)^{\maltese_d} \mu^d_{\mathcal{C}}(x_d, \ldots, x_1) \tag{2.18}$$

where $\maltese_d = \sum_{i=1}^d ||x_i||$ is the usual sign.

The opposite category $\mathcal{C}^{op}$ can be thought of as an algebra over the semi-simple ring R^{op}.

Definition 2.4. *An A_∞ algebra $\mathcal{A}$ is said to be* **homologically unital** *if there is a unit element $[e] \in H^*(\mathcal{A})$ on homology, so $\big(H^*(\mathcal{A}), H^*(\mu^2), [e]\big)$ is an associative unital algebra. Any chain-level representative e is called a* **homology unit** *of $\mathcal{A}$. Similarly, an A_∞ category $\mathcal{C}$ is* **homologically unital** *if, for each object X, there are identity morphisms $[e_X] \in H^* \hom_{\mathcal{C}}(X, X)$ on the homology level.*

The Fukaya category (and its wrapped analogue) is known to be homologically unital. We will assume throughout this paper that the categories we consider are homologically unital.

2.2. Morphisms and functors.

Definition 2.5. *A* **morphism of A_∞ algebras**

$$\mathbf{F} : (\mathcal{A}, \mu^{\cdot}_{\mathcal{A}}) \to (\mathcal{B}, \mu^{\cdot}_{B}) \tag{2.19}$$

is the data of, for each $d \geq 1$, maps of graded vector spaces

$$\mathbf{F}^d : \mathcal{A}^{\otimes d} \to \mathcal{B} \tag{2.20}$$

of degree $1-d$, satisfying the following equation, for each k:

$$\begin{gathered}\sum_{j; i_1 + \cdots + i_j = k} \mu^j_{\mathcal{B}}(\mathbf{F}^{i_j}(x_k, \ldots, x_{k-i_j+1}), \ldots, \mathbf{F}^{i_1}(x_{i_1}, \ldots, x_1)) = \\ \sum_{s \leq k, t} (-1)^{\maltese_t} \mathbf{F}^{k-s+1}(x_k, \ldots, x_{t+s+1}, \mu^s_{\mathcal{A}}(x_{t+s}, \ldots, x_{t+1}), x_t, \ldots, x_1).\end{gathered} \tag{2.21}$$

Here, $\maltese_t$ is the Koszul sign (2.3)*.*

Suppose $\mathcal{A}$ and $\mathcal{B}$ are algebras over semi-simple rings, or equivalently A_∞ categories. Then, unwinding the definition above leads to the following categorical notion of functor:

Definition 2.6. *An A_∞* **functor**

$$\mathbf{F} : \mathcal{C} \longrightarrow \mathcal{C}' \tag{2.22}$$

consists of the following data:

- *For each object X in $\mathcal{C}$, an object $\mathbf{F}(X)$ in $\mathcal{C}'$,*
- *for any set of $d+1$ objects $X_0, \ldots, X_d$, higher maps*

$$\mathbf{F}^d : \hom_{\mathcal{C}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{C}}(X_0, X_1) \longrightarrow \hom_{\mathcal{C}}(\mathbf{F}(X_0), \mathbf{F}(X_d)) \tag{2.23}$$

of degree $1-d$, satisfying the same relations as equation (2.21).

The equations (2.21) imply that the first-order term of any morphism or functor descends to a cohomology level functor $[\mathbf{F}^1]$. We say that a morphism $\mathbf{F}$ is a **quasi-isomorphism** if $[\mathbf{F}^1]$ is an isomorphism. Call a functor $\mathbf{F}$ **quasi-full and faithful** if $[\mathbf{F}^1]$ is an isomorphism of all morphism spaces, inducing an equivalence of the source category with a full subcategory of the image cohomologically. Call $\mathbf{F}$ a **quasi-equivalence** if it is quasi-full and faithful, and also $\mathbf{F}$ is also essentially surjective.

2.3. Categories of modules and bimodules. To an A_∞ algebra or category $\mathcal{C}$ one can associate categories of left A_∞ modules and right A_∞ modules over $\mathcal{C}$. These categories are dg categories, with explicitly describable morphism spaces and differentials. Similarly, to a pair of A_∞ algebras/categories $(\mathcal{C}, \mathcal{D})$, one can associate a dg category of A_∞ $\mathcal{C}-\mathcal{D}$ bimodules. These dg categories can be thought of as 1-morphisms in a 2-category whose objects are A_∞ categories.

Remark 2.3. *The fact that module categories over $\mathcal{C}$ are dg categories comes from an interpretation of left/right module categories over $\mathcal{C}$ as categories of (covariant/contravariant) A_∞ functors from $\mathcal{C}$ into chain complexes. The A_∞ structure on module categories is the natural one on A_∞ functor categories, and the fact that is is in fact dg is due to the dg nature of the target category, chain complexes. Similarly, $\mathcal{C}-\mathcal{D}$ bimodules can be thought of as A_∞* **bilinear functors** *from $\mathcal{C} \times \mathcal{D}^{op}$ into chain complexes. We will not pursue this viewpoint further, and instead refer the reader to* [**S5**, §(1j)].

Definition 2.7. *A* **left $\mathcal{C}$-module** $\mathcal{N}$ *consists of the following data:*

- *For $X \in \operatorname{ob} \mathcal{C}$, a graded vector space $\mathcal{N}(X)$.*
- *For $r \geq 0$, and objects $X_0, \ldots, X_r \in \operatorname{ob} \mathcal{C}$, module structure maps*

$$\mu_{\mathcal{N}}^{r|1} : \hom_{\mathcal{C}}(X_{r-1}, X_r) \otimes \cdots \hom_{\mathcal{C}}(X_0, X_1) \otimes \mathcal{N}(X_0) \longrightarrow \mathcal{N}(X_r) \tag{2.24}$$

of degree $1-r$, satisfying the following analogue of the A_∞ equations, for each k:

$$\begin{split}\sum (-1)^{\maltese_0^s} \mu_{\mathcal{N}}^{k-j+1|1}(x_k, \ldots, x_{s+j+1}, \mu_{\mathcal{C}}^j(x_{s+j}, \ldots, x_{s+1}), x_s, \ldots, x_1, \mathbf{n}) \\ + \sum \mu_{\mathcal{N}}^{s|1}(x_k, \ldots, x_{s+1}, \mu_{\mathcal{N}}^{s|1}(x_s, \ldots, x_1, \mathbf{n})) = 0.\end{split} \tag{2.25}$$

Here, the sign

$$\maltese_0^s := |\mathbf{n}| + \sum_{i=1}^{s} ||x_i|| \tag{2.26}$$

is given by the sum of the degree of $\mathbf{n}$ plus the reduced degrees of $x_1, \ldots, x_s$.

The first two equations

$$\begin{gathered}(\mu_{\mathcal{N}}^{0|1})^2 = 0 \\ \mu_{\mathcal{N}}^{1|1}(a, \mu_{\mathcal{N}}^{0|1}(\mathbf{m})) \pm \mu_{\mathcal{N}}^{1|1}(\mu_{\mathcal{C}}^1(a), \mathbf{m}) = \pm \mu_{\mathcal{N}}^{0|1}(\mu_{\mathcal{N}}^{1|1}(a, \mathbf{m}))\end{gathered} \tag{2.27}$$

imply that $\mu_{\mathcal{N}}^{0|1}$ is a differential and that the first module multiplication $\mu_{\mathcal{N}}^{1|1}$ descends to homology. Right modules have an essentially identical definition, up direction reversal and sign change.

Definition 2.8. *A* **right $\mathcal{C}$-module** $\mathcal{M}$ *consists of the following data:*

- *For $X \in \operatorname{ob} \mathcal{C}$, a graded vector space $\mathcal{M}(X)$.*
- *For $r \geq 0$, and objects $X_0, \ldots, X_r \in \operatorname{ob} \mathcal{C}$, module structure maps*

$$\mu_{\mathcal{M}}^{1|r} : \mathcal{M}(X_r) \otimes \hom_{\mathcal{C}}(X_{r-1}, X_r) \otimes \cdots \hom_{\mathcal{C}}(X_0, X_1) \longrightarrow \mathcal{M}(X_0) \tag{2.28}$$

of degree $1-r$, satisfying the following analogue of the A_∞ equations, for each k:

$$\begin{split}\sum (-1)^{\maltese_s} \mu_{\mathcal{M}}^{1|k-j+1}(\mathbf{m}, x_k, \ldots, x_{s+j+1}, \mu_{\mathcal{C}}^j(x_{s+j}, \ldots, x_{s+1}), x_s, \ldots, x_1) \\ + \sum (-1)^{\maltese_{k-s}} \mu_{\mathcal{M}}^{1|s}(\mu_{\mathcal{M}}^{1|k-s}(\mathbf{m}, x_k, \ldots, x_{k-s+1}), x_{k-s}, \ldots, x_1) = 0.\end{split} \tag{2.29}$$

Here, the signs $\maltese_s$ as usual denote the sum of the reduced degrees of elements to the right of the inner operation, as in (2.3).

Again, the first two equations imply that $\mu_{\mathcal{M}}^{1|0}$ is a differential and that the first module multiplication $\mu_{\mathcal{M}}^{1|1}$ descends to homology. Thus, for right or left modules, one can talk about unitality.

Definition 2.9 (Compare [**S5**, §(2f)]). *A left (right) module is* **homologically-unital** *if the underlying cohomology left (right) modules are unital; that is, for any $X \in \mathrm{ob}\ \mathcal{C}$ with homology unit e_X, the cohomology level module multiplication by $[e_X]$ is the identity.*

Now, let $\mathcal{C}$ and $\mathcal{D}$ be A_∞ categories.

Definition 2.10. *An A_∞ $\mathcal{C}-\mathcal{D}$* **bimodule** *$\mathcal{B}$ consists of the following data:*

- *for $V \in \mathrm{ob}\ \mathcal{C}$, $V' \in \mathrm{ob}\ \mathcal{D}$, a graded vector space $\mathcal{B}(V, V')$*
- *for $r, s \geq 0$, and objects $V_0, \ldots, V_r \in \mathrm{ob}\ \mathcal{C}$, $W_0, \ldots, W_s \in \mathrm{ob}\ \mathcal{C}'$, bimodule structure maps*

$$\begin{aligned}\mu_{\mathcal{B}}^{r|1|s} :\ & \hom_{\mathcal{C}}(V_{r-1}, V_r) \otimes \cdots \hom_{\mathcal{C}}(V_0, V_1) \otimes \mathcal{B}(V_0, W_s) \otimes \\ & \otimes \hom_{\mathcal{D}}(W_{s-1}, W_s) \otimes \cdots \otimes \hom_{\mathcal{D}}(W_0, W_1) \longrightarrow \mathcal{B}(V_r, W_0)\end{aligned} \tag{2.30}$$

of degree $1 - r - s$,

such that the following equations are satisfied, for each $r \geq 0$, $s \geq 0$:

$$\begin{aligned}&\sum (-1)^{\maltese^w_{s-j}} \mu_{\mathcal{B}}^{r-i|1|s-j}(v_r, \ldots, v_{i+1}, \mu_{\mathcal{B}}^{i|1|j}(v_i, \ldots, v_1, \mathbf{b}, w_s, \ldots, w_{s-j+1}), w_{s-j} \ldots, w_1) \\ &+ \sum (-1)^{\maltese^w_s + |\mathbf{b}| + \maltese^v_k} \mu_{\mathcal{B}}^{r-i+1|1|s}(v_r, \ldots, v_{k+i+1}, \mu_{\mathcal{C}}^i(v_{k+i}, \ldots, v_{k+1}), v_k, \ldots, v_1, \mathbf{b}, w_s, \ldots, w_1) \\ &+ \sum (-1)^{\maltese^w_l} \mu_{\mathcal{B}}^{r|1|s-j+1}(v_r, \ldots, v_1, \mathbf{b}, w_s, \ldots, w_{l+j+1}, \mu_{\mathcal{D}}^j(w_{l+j}, \ldots, w_{l+1}), w_l \ldots, w_1) \\ &\quad = 0.\end{aligned} \tag{2.31}$$

The signs above are given by the sum of the degrees of elements to the right of the inner operation, with the convention that we use **reduced degree** *for elements of $\mathcal{C}$ or $\mathcal{D}$ and* **full degree** *for elements of $\mathcal{B}$. In particular,*

$$\maltese^w_s := \sum_{i=1}^{s} ||w_i|| \tag{2.32}$$

$$\maltese^v_t := \sum_{i=1}^{t} ||v_i||. \tag{2.33}$$

Once more, the first few equations imply that $\mu^{0|1|0}$ is a differential, and the left and right multiplications $\mu^{1|1|0}$ and $\mu^{0|1|1}$ descend to homology.

Definition 2.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be homologically-unital A_∞ categories, and $\mathcal{B}$ a $\mathcal{C}-\mathcal{D}$ bimodule. $\mathcal{B}$ is* **homologically-unital** *if the homology level multiplications $[\mu^{1|1|0}]$ and $[\mu^{0|1|1}]$ are unital, i.e. homology units in $\mathcal{C}$ and $\mathcal{D}$ act as the identity.*

We will frequently refer to $\mathcal{C}-\mathcal{C}$ bimodules as simply $\mathcal{C}$-bimodules, or bimodules over $\mathcal{C}$.

Now, we define the *dg*-category structure on various categories of modules and bimodules. For the sake of brevity, we assume that A_∞ categories $\mathcal{C}$ and $\mathcal{D}$ have finitely many objects, and can thus be thought of as algebras over semi-simple rings R and R' respectively. In this language, a left (right) $\mathcal{C}$-module $\mathcal{N}$ ($\mathcal{M}$) is the data of an R (R^{op}) vector space $\mathcal{N}$ ($\mathcal{M}$) together with maps

$$\begin{aligned}\mu_{\mathcal{N}}^{r|1} &: \mathcal{C}^{\otimes_R r} \otimes_R \mathcal{N} \longrightarrow \mathcal{N},\ r \geq 0 \\ \mu_{\mathcal{M}}^{1|s} &: \mathcal{M} \otimes_R \mathcal{C}^{\otimes_R s} \longrightarrow \mathcal{M},\ s \geq 0\end{aligned} \tag{2.34}$$

satisfying equations (2.25) and (2.29) respectively. Similarly, a $\mathcal{C}-\mathcal{D}$ bimodule $\mathcal{B}$ is an $R \otimes R'^{op}$ vector space $\mathcal{B}$ together with maps

$$\mu_{\mathcal{B}}^{r|1|s} : \mathcal{C}^{\otimes_R r} \otimes_R \mathcal{B} \otimes_{R'} \mathcal{D}^{\otimes_{R'} s} \longrightarrow \mathcal{B} \tag{2.35}$$

satisfying (2.31). We can combine the structure maps $\mu_{\mathcal{B}}^{r|1|s}$, $\mu_{\mathcal{M}}^{1|s}$, $\mu_{\mathcal{N}}^{r|1}$ for all r, s to form total (bi)-module structure maps

$$\begin{aligned} \mu_{\mathcal{B}} &:= \oplus \mu_{\mathcal{B}}^{r|1|s} : T\mathcal{C} \otimes \mathcal{B} \otimes T\mathcal{D} \longrightarrow \mathcal{B} \\ \mu_{\mathcal{N}} &:= \oplus \mu_{\mathcal{N}}^{r|1} : T\mathcal{C} \otimes \mathcal{N} \longrightarrow \mathcal{N} \\ \mu_{\mathcal{M}} &:= \oplus \mu_{\mathcal{M}}^{1|s} : \mathcal{M} \otimes T\mathcal{C} \longrightarrow \mathcal{M}. \end{aligned} \tag{2.36}$$

The **hat extensions** of these maps

$$\begin{aligned} \hat{\mu}_{\mathcal{B}} &: T\mathcal{C} \otimes \mathcal{B} \otimes T\mathcal{D} \longrightarrow T\mathcal{C} \otimes \mathcal{B} \otimes T\mathcal{D} \\ \hat{\mu}_{\mathcal{N}} &: T\mathcal{C} \otimes \mathcal{N} \longrightarrow T\mathcal{C} \otimes \mathcal{N} \\ \hat{\mu}_{\mathcal{M}} &: \mathcal{M} \otimes T\mathcal{C} \longrightarrow \mathcal{M} \otimes T\mathcal{C}. \end{aligned} \tag{2.37}$$

sum over all ways to collapse subsequences with either module/bimodule or A_∞ structure maps, as follows:

$$\begin{aligned} &\hat{\mu}_{\mathcal{B}}(c_k, \dots, c_1, \mathbf{b}, d_l, \dots, d_1) := \\ &\qquad \sum (-1)^{\maltese^d_{l-t}} c_k \otimes \cdots \otimes c_{s+1} \otimes \mu_{\mathcal{B}}^{s|1|t}(c_s, \dots, c_1, \mathbf{b}, d_l, \dots, d_{l-t+1}) \otimes d_{l-t} \otimes \cdots \otimes d_1 \\ &\qquad + \sum (-1)^{\maltese^d_l + |b| + \maltese^c_s} c_k \otimes \cdots \otimes c_{s+i+1} \otimes \mu_{\mathcal{C}}^i(c_{s+i}, \dots, c_{s+1}) \otimes c_s \otimes \cdots \otimes c_1 \otimes \\ &\qquad\qquad \mathbf{b} \otimes d_l \otimes \cdots \otimes d_1 \\ &\qquad + \sum (-1)^{\maltese^d_j} c_k \otimes \cdots \otimes c_1 \otimes \mathbf{b} \otimes d_l \otimes \cdots \otimes d_{j+t+1} \otimes \\ &\qquad\qquad \mu_{\mathcal{D}}^t(d_{j+t}, \dots d_{j+1}) \otimes d_j \otimes \cdots \otimes d_1 \end{aligned} \tag{2.38}$$

$$\begin{aligned} &\hat{\mu}_{\mathcal{N}}(c_k, \dots, c_1, \mathbf{n}) := \\ &\qquad \sum c_k \otimes \cdots \otimes c_{s+1} \otimes \mu_{\mathcal{N}}^{s|1}(c_s, \dots, c_1, \mathbf{n}) \\ &\qquad + \sum (-1)^{\maltese^s_0} c_k \otimes \cdots \otimes c_{s+i+1} \otimes \mu_{\mathcal{C}}(c_{s+i}, \dots, c_{s+1}) \otimes c_s \otimes \cdots \otimes c_1 \otimes \mathbf{n} \end{aligned} \tag{2.39}$$

$$\begin{aligned} &\hat{\mu}_{\mathcal{M}}(\mathbf{m}, d_l, \dots, d_1) := \\ &\qquad \sum (-1)^{\maltese_{l-t}} \mu_{\mathcal{B}}^{1|t}(\mathbf{m}, d_l, \dots, d_{l-t+1}) \otimes d_{l-t} \otimes \cdots \otimes d_1 \\ &\qquad + \sum (-1)^{\maltese_t} \mathbf{m} \otimes d_l \otimes \cdots \otimes d_{t+j+1} \otimes \mu_{\mathcal{C}}^j(d_{t+j}, \dots, d_{t+1}) \otimes d_t \otimes \cdots \otimes d_1, \end{aligned} \tag{2.40}$$

with signs as specified in Definitions 2.10, 2.7, 2.8. Then the A_∞ bimodule and module equations, which can be concisely written as

$$\begin{aligned} \mu_{\mathcal{B}} \circ \hat{\mu}_{\mathcal{B}} &= 0, \\ \mu_{\mathcal{N}} \circ \hat{\mu}_{\mathcal{N}} &= 0, \\ \mu_{\mathcal{M}} \circ \hat{\mu}_{\mathcal{M}} &= 0, \end{aligned} \tag{2.41}$$

are equivalent to requiring that the hat extensions (2.37) are differentials.

REMARK 2.4. *The hat extensions of the maps $\mu_{\mathcal{N}}$, $\mu_{\mathcal{M}}$, $\mu_{\mathcal{B}}$ are the unique extensions of those maps which are a bicomodule co-derivation with respect to the structure of $T\mathcal{C} \otimes \mathcal{M} \otimes T\mathcal{D}$ as a bicomodule over differential graded co-algebras $(T\mathcal{C}, \hat{\mu}_{\mathcal{C}})$, $(T\mathcal{D}, \hat{\mu}_{\mathcal{D}})$. A good reference for this perspective, which we will not spell out more, is* [**T**].

DEFINITION 2.12. *A* **pre-morphism** *of left $\mathcal{C}$ modules of degree k*

$$\mathcal{H} : \mathcal{N} \longrightarrow \mathcal{N}' \tag{2.42}$$

is the data of maps

$$\mathcal{H}^{r|1} : \mathcal{C}^{\otimes r} \otimes \mathcal{N} \longrightarrow \mathcal{N}', \ r \geq 0 \tag{2.43}$$

of degree $k - r$. These can be packaged together into a total pre-morphism map

$$\mathcal{H} = \oplus \mathcal{H}^{r|1} : T\mathcal{C} \otimes \mathcal{N} \longrightarrow \mathcal{N}'. \tag{2.44}$$

Definition 2.13. *A* **pre-morphism** *of right* $\mathcal{C}$ *modules of degree* k

$$\mathcal{G} : \mathcal{M} \longrightarrow \mathcal{M}' \tag{2.45}$$

is the data of maps

$$\mathcal{G}^{1|s} : \mathcal{M} \otimes \mathcal{C}^{\otimes s} \longrightarrow \mathcal{M}',\ s \geq 0 \tag{2.46}$$

of degree $k - s$. *These can be packaged together into a total pre-morphism map*

$$\mathcal{G} = \oplus \mathcal{G}^{1|s} : \mathcal{M} \otimes T\mathcal{C} \longrightarrow \mathcal{M}'. \tag{2.47}$$

Definition 2.14. *A* **pre-morphism of** $\mathcal{C}-\mathcal{D}$ **bimodules** *of degree* k

$$\mathcal{F} : \mathcal{B} \longrightarrow \mathcal{B}' \tag{2.48}$$

is the data of maps

$$\mathcal{F}^{r|1|s} : \mathcal{C}^{\otimes r} \otimes \mathcal{B} \otimes \mathcal{D}^{\otimes s} \longrightarrow \mathcal{B}',\ r, s \geq 0. \tag{2.49}$$

of degree $k - r - s$. *These can be packaged together into a total pre-morphism map*

$$\mathcal{F} := \oplus \mathcal{F}^{r|1|s} : T\mathcal{C} \otimes \mathcal{B} \otimes T\mathcal{D} \longrightarrow \mathcal{B}'. \tag{2.50}$$

Remark 2.5. *Such morphisms are said to be degree* k *because the induced map*

$$\mathcal{F} : T\mathcal{C}[1] \otimes \mathcal{B} \otimes T\mathcal{D}[1] \longrightarrow \mathcal{B}' \tag{2.51}$$

has graded degree k.

Now, any collapsing maps of the form

$$\begin{aligned} \phi &: T\mathcal{C} \otimes \mathcal{B} \otimes T\mathcal{D} \longrightarrow \mathcal{B}' \\ \psi &: T\mathcal{C} \otimes \mathcal{N} \longrightarrow \mathcal{N}' \\ \rho &: \mathcal{M} \otimes T\mathcal{C} \longrightarrow \mathcal{M}' \end{aligned} \tag{2.52}$$

admit, in the style of (2.6), **hat extensions**

$$\begin{aligned} \hat{\phi} &: T\mathcal{C} \otimes \mathcal{B} \otimes T\mathcal{D} \longrightarrow T\mathcal{C} \otimes \mathcal{B}' \otimes T\mathcal{D} \\ \hat{\psi} &: T\mathcal{C} \otimes \mathcal{N} \longrightarrow T\mathcal{C} \otimes \mathcal{N}' \\ \hat{\rho} &: \mathcal{M} \otimes T\mathcal{C} \longrightarrow \mathcal{M}' \otimes T\mathcal{C} \end{aligned} \tag{2.53}$$

which sum over all ways (with signs) to collapse a subsequence with ϕ, ψ, and ρ respectively:

$$\begin{aligned} &\hat{\phi}(c_k, \ldots, c_1, \mathbf{b}, d_l, \ldots, d_1) := \\ &\qquad \sum (-1)^{|\phi| \cdot \maltese^d_{l-t}} c_k \otimes \cdots \otimes c_{s+1} \otimes \phi(c_s, \ldots, c_1, \mathbf{b}, d_l, \ldots, d_{l-t+1}) \otimes d_{l-t} \otimes \cdots \otimes d_1. \\ &\hat{\psi}(c_k, \ldots, c_1, \mathbf{n}) := \\ &\qquad \sum c_k \otimes \cdots \otimes c_{s+1} \otimes \psi(c_s, \ldots, c_1, \mathbf{m}). \\ &\hat{\rho}(\mathbf{m}, d_l, \ldots, d_1) := \\ &\qquad \sum (-1)^{|\rho| \cdot \maltese^d_{l-t}} \rho(\mathbf{m}, d_l, \ldots, d_{l-t+1}) \otimes d_{l-t} \otimes \cdots \otimes d_1. \end{aligned} \tag{2.54}$$

Remark 2.6. *Once more, the hat extensions are uniquely specified by the requirements that* $\hat{\psi}$ *and* $\hat{\rho}$ *be (left and right) co-module homomorphisms over the co-algebra* $(T\mathcal{C}, \Delta_{\mathcal{C}})$*, and that* $\hat{\phi}$ *be a bi-co-module homomorphism over the co-algebras* $(T\mathcal{C}, \Delta_{\mathcal{C}})$ *and* $(T\mathcal{D}, \Delta_{\mathcal{D}})$*. In this manner, categories of modules and bimodules over* A_∞ *algebras give categories of dg comodules and dg bicomodules over the associated dg co-algebras. See* [**T**].

It is now easy to define composition of pre-morphisms:

Definition 2.15. *If $\mathcal{F}_1$ is a pre-morphism of A_∞ left modules/right modules/bimodules from $\mathcal{M}_0$ to $\mathcal{M}_1$ and $\mathcal{F}_2$ is a pre-morphism of A_∞ left modules/right modules/bimodules from $\mathcal{M}_1$ to $\mathcal{M}_2$, define the* **composition** $\mathcal{F}_2 “\circ” \mathcal{F}_1$ *as:*

$$\mathcal{F}_2 “\circ” \mathcal{F}_1 := \mathcal{F}_2 \circ \hat{\mathcal{F}}_1. \tag{2.55}$$

Remark 2.7. *Observe the hat extension of the composition agrees with the composition of the hat extensions, e.g. $\widehat{\mathcal{F}_2 “\circ” \mathcal{F}_1} = \hat{\mathcal{F}}_2 \circ \hat{\mathcal{F}}_1$. i.e. this notion agrees with usual composition of homomorphisms of comodules/bi-comodules.*

Similarly, there is a differential on pre-morphisms.

Definition 2.16. *If $\mathcal{F}$ is a pre-morphism of left modules/right modules/bimodules from $\mathcal{M}$ to $\mathcal{N}$ with associated bimodule structure maps $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, define the* **differential** $\delta\mathcal{F}$ *to be:*

$$\delta(\mathcal{F}) := \mu_{\mathcal{N}} \circ \hat{\mathcal{F}} - (-1)^{|\mathcal{F}|} \mathcal{F} \circ \hat{\mu}_{\mathcal{M}}. \tag{2.56}$$

The fact that $\delta^2 = 0$ is a consequence of the A_∞ module or bimodule equations for $\mathcal{M}$ and $\mathcal{N}$. As one consequence of $\delta(\mathcal{F}) = 0$, the first order term $\mathcal{F}^{0|1}$, $\mathcal{F}^{1|0}$ or $\mathcal{F}^{0|1|0}$ descends to a cohomology level module or bimodule morphism. Call any pre-morphism $\mathcal{F}$ of bimodules or modules a **quasi-isomorphism** if $\delta(\mathcal{F}) = 0$, and the resulting cohomology level morphism $[\mathcal{F}]$ is an isomorphism.

Remark 2.8. *We have developed modules and bimodules in parallel, but note now that modules are a special case of bimodules in the following sense: a left A_∞ module (right A_∞ module) over $\mathcal{C}$ is a $\mathcal{C}-\mathbb{K}$ ($\mathbb{K}-\mathcal{C}$) bimodule $\mathcal{M}$ with structure maps $\mu^{r|1|s}$ trivial for $s > 0$ ($r > 0$). Thus we abbreviate $\mu^{r|1|0}$ by $\mu^{r|1}$ (and correspondingly, $\mu^{0|1|s}$ by $\mu^{1|s}$).*

Thus, we have seen that $\mathcal{C}-\mathcal{D}$ bimodules, as well as left and right $\mathcal{C}$ modules form dg categories which will be denoted

$$\begin{gathered} \mathcal{C}\text{–mod–}\mathcal{D} \\ \mathcal{C}\text{–mod} \\ \text{mod–}\mathcal{C} \end{gathered} \tag{2.57}$$

respectively.

2.4. Tensor products. There are several relevant notions of tensor product for modules and bimodules. The first notion, that of tensoring two bimodules over a single common side, can be thought of as composition of 1-morphisms in the 2-category of A_∞ categories.

Definition 2.17. *Given a $\mathcal{C}-\mathcal{D}$ bimodule $\mathcal{M}$ and an $\mathcal{D}-\mathcal{E}$ bimodule $\mathcal{N}$, the* **(convolution) tensor product over** $\mathcal{D}$

$$\mathcal{M} \otimes_{\mathcal{D}} \mathcal{N} \tag{2.58}$$

is the $\mathcal{C}-\mathcal{E}$ bimodule given by

- *underlying graded vector space*

$$\mathcal{M} \otimes T\mathcal{D} \otimes \mathcal{N}; \tag{2.59}$$

- *differential*

$$\mu^{0|1|0}_{\mathcal{M} \otimes_{\mathcal{D}} \mathcal{N}} : \mathcal{M} \otimes T\mathcal{D} \otimes \mathcal{N} \to \mathcal{M} \otimes T\mathcal{D} \otimes \mathcal{N} \tag{2.60}$$

  given by

$$\begin{aligned} &\mu^{0|1|0}(\mathbf{m}, d_k, \dots, d_1, \mathbf{n}) = \\ &\qquad \sum (-1)^{\maltese^d_{k-t} + |\mathbf{n}|} \mu^{0|1|t}_{\mathcal{M}}(\mathbf{m}, d_k, \dots, d_{k-t+1}) \otimes d_{k-t} \otimes \cdots \otimes d_1 \otimes \mathbf{n} \\ &\qquad + \sum \mathbf{m} \otimes d_k \otimes \cdots\cdots \otimes d_{s+1} \otimes \mu^{s|1|0}_{\mathcal{N}}(d_s, \dots, d_1, \mathbf{n}) \\ &\qquad + \sum (-1)^{\maltese^d_j + |\mathbf{n}|} \mathbf{m} \otimes d_k \otimes \cdots \otimes d_{j+i+1} \otimes \mu^i_{\mathcal{A}}(d_{j+i}, \dots, d_{j+1}) \otimes \\ &\qquad\qquad d_j \otimes \cdots \otimes d_1 \otimes \mathbf{n}. \end{aligned} \tag{2.61}$$

- *for r or $s > 0$, higher bimodule maps*

$$\mu^{r|1|s}_{\mathcal{M}\otimes_{\mathcal{D}}\mathcal{N}} : \mathcal{C}^{\otimes r} \otimes \mathcal{M} \otimes T\mathcal{D} \otimes \mathcal{N} \otimes \mathcal{E}^{\otimes s} \longrightarrow \mathcal{M} \otimes T\mathcal{D} \otimes \mathcal{N} \tag{2.62}$$

given by:

$$\begin{aligned} &\mu^{r|1|0}(c_r, \ldots, c_1, \mathbf{m}, d_k, \ldots, d_1, \mathbf{n}) = \\ &\qquad \sum_t (-1)^{|\mathbf{n}| + \maltese^d_{k-t}} \mu^{r|1|t}_{\mathcal{M}}(c_r, \ldots, c_1, \mathbf{m}, d_k, \ldots, d_{k-t+1}) \otimes d_{k-t} \otimes \cdots \otimes d_1 \otimes \mathbf{n} \end{aligned} \tag{2.63}$$

$$\begin{aligned} &\mu^{0|1|s}(\mathbf{m}, d_k, \ldots, d_1, \mathbf{n}, e_s, \ldots, e_1) = \\ &\qquad \sum_j \mathbf{m} \otimes d_k \otimes \cdots \otimes d_{j+1} \otimes \mu^{j|1|s}_{\mathcal{N}}(d_j, \ldots, d_1, \mathbf{n}, e_s, \ldots, e_1) \end{aligned} \tag{2.64}$$

and

$$\mu^{r|1|s} = 0 \text{ if } r > 0 \text{ and } s > 0. \tag{2.65}$$

In all equations above, the sign is the sum of degrees of all elements to the right of the applied A_∞ operation, using reduced degree for elements of $\mathcal{A}$ and full degree for elements of $\mathcal{N}$; in particular once more $\maltese^d_t := \sum_{i=1}^t ||d_i||$.

One can check that these maps indeed give $\mathcal{M} \otimes_{\mathcal{D}} \mathcal{N}$ the structure of an $\mathcal{C}-\mathcal{E}$ bimodule. As one would expect from a two-categorical perspective, convolution with $\mathcal{N}$ gives a dg functor

$$\cdot \otimes_{\mathcal{D}} \mathcal{N} : \mathcal{C}\text{–mod–}\mathcal{D} \longrightarrow \mathcal{C}\text{–mod–}\mathcal{E}. \tag{2.66}$$

Namely, there is an induced map on morphisms, which we will omit for the time being.

As a special case, suppose $\mathcal{M}^r$ is a right $\mathcal{A}$ module and $\mathcal{N}^l$ is a left $\mathcal{A}$ module. Then, thinking of $\mathcal{M}$ and $\mathcal{N}$ as $\mathbb{K}-\mathcal{A}$ and $\mathcal{A}-\mathbb{K}$ modules respectively, there two possible one-sided tensor products. The tensor product over $\mathcal{A}$

$$\mathcal{M}^r \otimes_{\mathcal{A}} \mathcal{N}^l \tag{2.67}$$

is by definition the graded vector space

$$\mathcal{M}^r \otimes T\mathcal{A} \otimes \mathcal{N}^l \tag{2.68}$$

with differential

$$d_{\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}} : \mathcal{M}^r \otimes T\mathcal{A} \otimes \mathcal{N}^l \to \mathcal{M}^r \otimes T\mathcal{A} \otimes \mathcal{N}^l \tag{2.69}$$

given by

$$\begin{aligned} &d(\mathbf{m}, a_k, \ldots, a_1, \mathbf{n}) = \\ &\quad \sum (-1)^{|\mathbf{n}| + \maltese^a_{k-r}} \mu^{1|r}_{\mathcal{M}}(\mathbf{m}, a_k, \ldots, a_{k-r+1}) \otimes a_{k-r} \otimes \cdots \otimes a_1 \otimes \mathbf{n} \\ &\quad + \sum \mathbf{m} \otimes a_k \otimes \cdots\cdots a_{s+1} \otimes \mu^{s|1}_{\mathcal{N}}(a_s, \ldots, a_1, \mathbf{n}) \\ &\quad + \sum (-1)^{|\mathbf{n}| + \maltese^a_s} \mathbf{m} \otimes a_k \otimes \cdots \otimes a_{s+j+1} \otimes \mu^j_{\mathcal{A}}(a_{s+j}, \ldots, a_{s+1}) \otimes a_s \otimes \cdots \otimes a_1 \otimes \mathbf{n}. \end{aligned} \tag{2.70}$$

In the opposite direction, tensoring over $\mathbb{K}$, we obtain the *product $\mathcal{A}-\mathcal{B}$ bimodule*

$$\mathcal{N}^l \otimes_{\mathbb{K}} \mathcal{M}^r, \tag{2.71}$$

which equals $\mathcal{N}^l \otimes_{\mathbb{K}} \mathcal{M}^r$ on the level of graded vector spaces and has

$$\begin{aligned} &\mu^{r|1|s}_{\mathcal{N}\otimes_{\mathbb{K}}\mathcal{M}}(a_1, \ldots, a_r, \mathbf{n} \otimes \mathbf{m}, b_1, \ldots, b_s) := \\ &\qquad \begin{cases} (-1)^{|\mathbf{m}|} \mu^{r|1}_{\mathcal{N}}(a_1, \ldots, a_r, \mathbf{n}) \otimes \mathbf{m} & s = 0, r > 0 \\ \mathbf{n} \otimes \mu^{1|s}_{\mathcal{M}}(\mathbf{m}, b_1, \ldots, b_s) & r = 0, s > 0 \\ (-1)^{|\mathbf{m}|} \mu^{1|0}_{\mathcal{N}}(\mathbf{n}) \otimes \mathbf{m} + \mathbf{n} \otimes \mu^{0|1}_{\mathcal{M}}(\mathbf{m}) & r = s = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2.72}$$

The A_∞ bimodule equations follow from the A_∞ module equations for $\mathcal{M}$ and $\mathcal{N}$.

Finally, given an $\mathcal{A}-\mathcal{B}$ bimodule $\mathcal{M}$ and a $\mathcal{B}-\mathcal{A}$ bimodule $\mathcal{N}$, we can simultaneously tensor over the $\mathcal{A}$ and $\mathcal{B}$ module structures to obtain a chain complex.

Definition 2.18. *The* **bimodule tensor product** *of* $\mathcal{M}$ *and* $\mathcal{N}$ *as above, denoted*

$$\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N} \tag{2.73}$$

is a chain complex defined as follows: As a vector space,

$$\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N} := (\mathcal{M} \otimes T\mathcal{B} \otimes \mathcal{N} \otimes T\mathcal{A})^{diag}, \tag{2.74}$$

where the diag superscript means to restrict to cyclically composable elements. The differential on $\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}$ *is*

$$\begin{aligned}
d_{\mathcal{M}\otimes_{\mathcal{A}-\mathcal{B}}\mathcal{N}} : \mathbf{m} \otimes b_k \otimes \cdots \otimes b_1 \otimes \mathbf{n} \otimes a_l \otimes \cdots \otimes a_1 \longmapsto &\\
\sum_{r,s} (-1)^{\#_{r,s}} \mu_{\mathcal{M}}^{r|1|s}(a_r, \ldots, a_1, \mathbf{m}, b_k, \ldots, b_{k-s+1}) \otimes b_{k-s} \otimes \cdots \otimes b_1 \otimes &\\
\mathbf{n} \otimes a_l \otimes \cdots \otimes a_{r+1} &\\
+ \sum_{i,r} (-1)^{\maltese_l^a + |\mathbf{n}| + \maltese_i^b} \mathbf{m} \otimes b_k \otimes \cdots b_{i+r+1} \otimes \mu_{\mathcal{B}}^r(b_{i+r}, \ldots, b_{i+1}) \otimes b_i \otimes \cdots \otimes b_1 \otimes &\\
\mathbf{n} \otimes a_l \otimes \cdots \otimes a_1 &\\
+ \sum_{j,s} (-1)^{\maltese_j^a} \mathbf{m} \otimes b_k \otimes \cdots b_1 \otimes \mathbf{n} \otimes a_l \otimes \cdots \otimes a_{j+s+1} \otimes &\\
\mu_{\mathcal{A}}^s(a_{j+s}, \ldots, a_{j+1}) \otimes a_j \otimes \cdots \otimes a_1 &\\
+ \sum_{r,s} (-1)^{\maltese_{l-s}^a} \mathbf{m} \otimes b_k \otimes \cdots \otimes b_{r+1} \otimes \mu_{\mathcal{N}}^{r|1|s}(b_r, \ldots, b_1, \mathbf{n}, a_l, \ldots, a_{l-s+1}) \otimes &\\
a_{l-s} \otimes \cdots \otimes a_1 &
\end{aligned} \tag{2.75}$$

with signs given by:

$$\maltese_i^a := \sum_{n=1}^{i} ||a_n|| \tag{2.76}$$

$$\maltese_j^b := \sum_{n=1}^{j} ||b_n|| \tag{2.77}$$

$$\#_{r,s} := \left(\maltese_r^a\right) \cdot \left(|\mathbf{m}| + \maltese_k^b + |\mathbf{n}| + \sum_{n=r+1}^{l} ||a_n||\right) + \maltese_{k-s}^b + |\mathbf{n}| + \sum_{n=r+1}^{l} ||a_n||. \tag{2.78}$$

The sign (2.78) is the sum of the Koszul sign coming from reordering $a_r, \ldots, a_1$ *to the left of the other elements with the usual Koszul sign coming from applying* $\mu_{\mathcal{M}}$ *(thought of as acting from the right).*

The bimodule tensor product is functorial in the following sense. If

$$\mathcal{F} : \mathcal{N} \longrightarrow \mathcal{N}' \tag{2.79}$$

is a morphism of $\mathcal{B}-\mathcal{A}$ bimodules, then there is an induced morphism

$$\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N} \xrightarrow{\mathcal{F}_\#} \mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}' \tag{2.80}$$

given by summing with signs over all ways to collapse some of the terms around the element of $\mathcal{N}$ by the various $\mathcal{F}^{r|1|s}$, which can be concisely written as

$$\mathcal{F}_\#(\mathbf{m} \otimes b_k \otimes \cdots \otimes b_1 \otimes \mathbf{n} \otimes a_l \otimes \cdots \otimes a_1) := \mathbf{m} \otimes \hat{\mathcal{F}}(b_k, \ldots, b_1, \mathbf{n}, a_l, \ldots, a_1). \tag{2.81}$$

One can then see that

Proposition 2.1. *Via (2.81), quasi-isomorphisms of bimodules induce quasi-isomorphisms of complexes.*

REMARK 2.9. *There are identically induced morphisms* $\mathcal{G}_\# : \mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N} \to \mathcal{M}' \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}$ *from morphisms* $\mathcal{G} : \mathcal{M} \to \mathcal{M}'$. *One simply needs to add additional Koszul signs coming from moving elements of* $\mathcal{B}$ *to the beginning in order to apply* $\mathcal{G}$.

2.5. The diagonal bimodule. For any A_∞ category $\mathcal{A}$, there is a natural $\mathcal{A}-\mathcal{A}$ bimodule quasi-representing the identity convolution endofunctor.

DEFINITION 2.19. *The* **diagonal bimodule** $\mathcal{A}_\Delta$ *is specified by the following data:*

$$\mathcal{A}_\Delta(X,Y) := \hom_{\mathcal{A}}(Y,X) \tag{2.82}$$

$$\mu^{r|1|s}_{\mathcal{A}_\Delta}(c_r,\dots,c_1,\mathbf{c},c'_s,\dots,c'_1) := (-1)^{\maltese'_s+1}\mu^{r+1+s}_{\mathcal{A}}(c_r,\dots,c_1,\mathbf{c},c'_s,\dots,c'_1). \tag{2.83}$$

with sign

$$\maltese'_s := \sum_{i=1}^{s} ||c'_i||. \tag{2.84}$$

One of the standard complications in theory of bimodules is that tensor product with the diagonal is only quasi-isomorphic to the identity. However, these quasi-isomorphisms are explicit, at least in one direction.

PROPOSITION 2.2. *Let* $\mathcal{M}$ *be a homologically unital right* A_∞ *module* $\mathcal{M}$ *over* $\mathcal{A}$. *Then, there is a quasi-isomorphism of modules*

$$\mathcal{F}_{\Delta,right} : \mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_\Delta \longrightarrow \mathcal{M} \tag{2.85}$$

given by the following data:

$$\begin{aligned}
&\mathcal{F}^{1|l}_{\Delta,right} : \mathcal{M} \otimes T\mathcal{A} \otimes \mathcal{A}_\Delta \otimes \mathcal{A}^{\otimes l} \longrightarrow \mathcal{M}\\
&\quad(\mathbf{m},a_k,\dots,a_1,\mathbf{a},a^1_l,\dots,a^1_1) \longmapsto (-1)^{\circ^k_{-l}}\mu^{1|k+l+1}_{\mathcal{M}}(\mathbf{m},a_k,\dots,a_1,\mathbf{a},a^1_l,\dots a^1_1),
\end{aligned} \tag{2.86}$$

where the sign is

$$\circ^k_{-l} = \sum_{n=1}^{l} ||a^1_n|| + |\mathbf{a}| - 1 + \sum_{m=1}^{k} ||a_m||. \tag{2.87}$$

There are similar quasi-isomorphisms of homologically unital left-modules

$$\begin{aligned}
&\mathcal{F}_{\Delta,left} : \mathcal{A}_\Delta \otimes_{\mathcal{A}} \mathcal{N} \longrightarrow \mathcal{N}\\
&\mathcal{F}^{l|1} : (a^1_l,\dots,a^1_1,\mathbf{a},a_k,\dots a_1,\mathbf{n}) \longmapsto (-1)^{\bullet^k_0}\mu^{k+l+1|1}(a^1_l,\dots,a^1_1,\mathbf{a},a_k,\dots,a_1,\mathbf{n}).
\end{aligned} \tag{2.88}$$

and quasi-isomorphisms of homologically unital bimodules

$$\begin{aligned}
&\mathcal{F}_{\Delta,right} : \mathcal{B}\otimes_{\mathcal{A}}\mathcal{A}_\Delta \longrightarrow \mathcal{B}\\
&\mathcal{F}^{r|1|s}_{\Delta,right} : \mathcal{A}^{\otimes r} \otimes \mathcal{B} \otimes T\mathcal{A}\otimes\mathcal{A}_\Delta \otimes \mathcal{A}^{\otimes s} \longrightarrow \mathcal{B}\\
&(a_r,\dots,a_1,\mathbf{b},a^1_l,\dots,a^1_1,\mathbf{a},a^2_s,\dots,a^2_1) \longmapsto\\
&\qquad\qquad (-1)^{\circ^l_{-s}}\mu^{r|1|l+s+1}_{\mathcal{B}}(a_r,\dots,a_1,\mathbf{b},a^1_l,\dots,a^l_1,\mathbf{a},a^2_s,\dots,a^2_1)
\end{aligned} \tag{2.89}$$

$$\begin{aligned}
&\mathcal{F}_{\Delta,left} : \mathcal{A}_\Delta\otimes_{\mathcal{A}}\mathcal{B} \longrightarrow \mathcal{B}\\
&\mathcal{F}^{r|1|s}_{\Delta,left} : \mathcal{A}^{\otimes r} \otimes \mathcal{A}_\Delta \otimes T\mathcal{A}\otimes\mathcal{B} \otimes \mathcal{A}^{\otimes s} \longrightarrow \mathcal{B}\\
&(a_r,\dots,a_1,\mathbf{a},a^1_l,\dots,a^1_1,\mathbf{b},a^2_s,\dots,a^2_1) \longmapsto\\
&\qquad\qquad (-1)^{\bigstar^l_{-s}}\mu^{r+l+1|1|s}_{\mathcal{B}}(a_r,\dots,a_1,\mathbf{a},a^1_l,\dots,a^1_1,\mathbf{b},a^2_s,\dots,a^2_1)
\end{aligned} \tag{2.90}$$

with signs

$$\bullet_0^k := |\mathbf{n}| - 1 + \sum_{i=1}^{k} ||a_i|| \tag{2.91}$$

$$\circ_{-s}^l := \sum_{n=1}^{s} ||a_n^2|| + |\mathbf{a}| - 1 + \sum_{m=1}^{l} ||a_m^1|| \tag{2.92}$$

$$\bigstar_{-s}^l := \sum_{n=1}^{s} ||a_n^2|| + |\mathbf{b}| - 1 + \sum_{m=1}^{l} ||a_m^1||. \tag{2.93}$$

Proof. This will just be a proof sketch; a detailed argument for Proposition 2.2 in which $\mathcal{A}$ is strictly unital also appears in [**S3**, §2]. It suffices establish that (2.89) is a quasi-isomorphism; the other cases are analogous or special cases.

The cone of $\mathcal{F}^{0|1|0}$ is the complex

$$(\mathcal{B} \otimes T\mathcal{A} \otimes \mathcal{A}_\Delta) \oplus \mathcal{B}[1] \tag{2.94}$$

with differential

$$\begin{pmatrix} d_{\mathcal{B}\otimes_{\mathcal{A}}\mathcal{A}_\Delta} & 0 \\ \mathcal{F}^{0|1|0} & \mu_{\mathcal{B}}^{0|1|0}, \end{pmatrix}. \tag{2.95}$$

Here $d_{\mathcal{B}\otimes_{\mathcal{A}}\mathcal{A}_\Delta}$ is the internal bimodule differential. This differential respects the *length filtration* of the complex, and thus we can look at the associated spectral sequence. The only terms that preserve length involve the differentials $\mu_{\mathcal{B}}^{0|1|0}$ and $\mu_{\mathcal{A}}^1$ and thus the first page of the spectral sequence is the complex

$$(H^*(\mathcal{B}) \otimes T(H^*(\mathcal{A})) \otimes H^*(\mathcal{A}_\Delta)) \oplus H^*(\mathcal{B}), \tag{2.96}$$

with first page differential given by all of the homology-level terms involving $\mu_{\mathcal{A}}^2$ and $\mu_{\mathcal{B}}^{1|1}$.

This cone complex is visibly identical to the classical right-sided *bar complex* for the bimodule $H^*(\mathcal{B})$ over the associative algebra $H^*(\mathcal{A})$, which is acyclic for $H^*(\mathcal{A})$ and $H^(\mathcal{B})$ unital (with explicit contracting homotopy given by inserting $[e]$ to the right). □

2.6. The Yoneda embedding.

Objects in $\mathcal{C}$ provide a natural source for left and right $\mathcal{C}$-modules.

Definition 2.20. *Given an object $X \in \text{ob}\ \mathcal{C}$, the* **left Yoneda-module** $\mathcal{Y}_X^l$ *over $\mathcal{C}$ is defined by the following data:*

$$\mathcal{Y}_X^l(Y) := \hom_{\mathcal{C}}(X, Y) \text{ for any } Y \in \text{ob}\ \mathcal{C} \tag{2.97}$$

$$\begin{aligned} \mu^{r|1} : \hom_{\mathcal{C}}(Y_{r-1}, Y_r) \otimes \hom_{\mathcal{C}}(Y_{r-2}, Y_{r-1}) \otimes \cdots \otimes \hom_{\mathcal{C}}(Y_0, Y_1) \otimes \mathcal{Y}_X^l(Y_0) &\longrightarrow \mathcal{Y}_X^l(Y_r) \\ (y_r, \ldots, y_1, \mathbf{x}) &\longmapsto (-1)^{\maltese_0^r} \mu^{r+1}(y_r, \ldots, y_1, \mathbf{x}), \end{aligned} \tag{2.98}$$

with sign

$$\maltese_0^r = ||\mathbf{x}|| + \sum_{i=1}^{r} ||y_i||. \tag{2.99}$$

Similarly, the **right Yoneda-module** $\mathcal{Y}_X^r$ *over $\mathcal{C}$ is defined by the following data:*

$$\mathcal{Y}_X^r(Y) := \hom_{\mathcal{C}}(Y, X) \text{ for any } Y \in \text{ob}\ \mathcal{C} \tag{2.100}$$

$$\begin{aligned} \mu^{1|s} : \mathcal{Y}_X^r(Y_s) \otimes \hom_{\mathcal{C}}(Y_{s-1}, Y_s) \otimes \hom_{\mathcal{C}}(Y_{s-2}, Y_{s-1}) \otimes \cdots \otimes \hom_{\mathcal{C}}(Y_0, Y_1) &\to \mathcal{Y}_X^r(Y_0) \\ (\mathbf{x}, y_s, \ldots, y_1) &\mapsto \mu^{s+1}(\mathbf{x}, y_s, \ldots, y_1). \end{aligned} \tag{2.101}$$

These modules are associated respectively to the **left** and **right Yoneda embeddings**, A_∞ functors which we will now describe.

DEFINITION 2.21. *The* **left Yoneda embedding** *is a contravariant A_∞ functor*

$$\mathbf{Y}_L : \mathcal{C}^{op} \longrightarrow \mathcal{C}\text{–mod} \tag{2.102}$$

defined as follows: On objects,

$$\mathbf{Y}_L(X) := \mathcal{Y}^l_X. \tag{2.103}$$

On morphisms

$$\begin{aligned} \mathbf{Y}^d_L : \hom(X_{d-1}, X_d) \otimes \cdots \otimes \hom(X_0, X_1) &\longrightarrow \hom_{\mathcal{C}\text{–mod}}(\mathcal{Y}^l_{X_d}, \mathcal{Y}^l_{X_0}) \\ (x_d, \ldots, x_1) &\longmapsto \phi_{(x_1,\ldots,x_d)} \end{aligned} \tag{2.104}$$

where $\phi_{\vec{x}} := \phi_{x_1,\ldots,x_d}$ is the morphism given by

$$\begin{aligned} \hom(Y_{f-1}, Y_f) \otimes \cdots \otimes \hom(Y_0, Y_1) \otimes \mathcal{Y}^l_{X_d}(Y_0) &\longrightarrow \mathcal{Y}^l_{X_0}(Y_f) \\ (y_f, \ldots, y_1, \mathbf{m}) &\longmapsto (-1)^{\maltese^f_{-d}} \mu^{f+d+1}_{\mathcal{C}}(y_f, \ldots, y_1, \mathbf{m}, x_d, \ldots, x_1) \end{aligned} \tag{2.105}$$

with sign

$$\maltese^f_{-d} = \sum_{i=1}^d ||x_i|| + ||\mathbf{m}|| + \sum_{j=1}^f ||y_j||. \tag{2.106}$$

DEFINITION 2.22. *The* **right Yoneda embedding** *is a (covariant) A_∞ functor*

$$\mathbf{Y}_R : \mathcal{C} \longrightarrow \text{mod–}\mathcal{C} \tag{2.107}$$

defined as follows: On objects,

$$\mathbf{Y}_R(X) := \mathcal{Y}^r_X. \tag{2.108}$$

On morphisms

$$\begin{aligned} \mathbf{Y}^d_R : \hom(X_{d-1}, X_d) \otimes \cdots \otimes \hom(X_0, X_1) &\longrightarrow \hom_{\text{mod–}\mathcal{C}}(\mathcal{Y}^r_{X_0}, \mathcal{Y}^r_{X_d}) \\ (x_d, \ldots, x_1) &\mapsto \psi_{(x_1,\ldots,x_d)} \end{aligned} \tag{2.109}$$

where $\psi_{\vec{x}} := \psi_{x_1,\ldots,x_d}$ is the morphism given by

$$\begin{aligned} \mathcal{Y}^r_{X_0}(Y_f) \otimes \hom(Y_{f-1}, Y_f) \otimes \cdots \otimes \hom(Y_0, Y_1) &\longrightarrow \mathcal{Y}^r_{X_d}(Y_0) \\ (\mathbf{m}, y_f, \ldots, y_1) &\longmapsto \mu^{f+d+1}_{\mathcal{C}}(x_d, \ldots, x_1, \mathbf{m}, y_f, \ldots, y_1). \end{aligned} \tag{2.110}$$

An important feature of these modules, justifying the use of module categories, is that the A_∞ Yoneda embedding is full. In fact, a slightly stronger result is true, which we will need.

PROPOSITION 2.3 (Seidel [**S5**, Lem. 2.12]). *Let $\mathcal{C}$ be a homologically unital, and let $\mathcal{M}$ and $\mathcal{N}$ be homologically unital left and right $\mathcal{C}$ modules respectively. Then, for any object X of $\mathcal{C}$ there are quasi-isomorphisms of chain complexes*

$$\lambda_{\mathcal{M},X} : \mathcal{M}(X) \xrightarrow{\sim} \hom_{\mathcal{C}\text{–mod}}(\mathcal{Y}^l_X, \mathcal{M}) \tag{2.111}$$

$$\lambda_{\mathcal{N},X} : \mathcal{N}(X) \xrightarrow{\sim} \hom_{\text{mod–}\mathcal{C}}(\mathcal{Y}^r_X, \mathcal{N}). \tag{2.112}$$

When $\mathcal{M} = \mathcal{Y}^l_Z$ or $\mathcal{N} = \mathcal{Y}^r_Z$, the quasi-isomorphisms defined above

$$\lambda_Z : \hom_{\mathcal{C}}(X, Z) \xrightarrow{\sim} \hom_{\mathcal{C}\text{–mod}}(\mathcal{Y}^l_Z, \mathcal{Y}^l_X) \tag{2.113}$$

$$\lambda_X : \hom_{\mathcal{C}}(X, Z) \xrightarrow{\sim} \hom_{\mathcal{C}\text{–mod}}(\mathcal{Y}^r_X, \mathcal{Y}^r_Z) \tag{2.114}$$

are exactly the first order terms of the Yoneda embeddings $\mathbf{Y}^l_L$ and $\mathbf{Y}^1_R$, implying that

COROLLARY 2.1 ([**S5**, Cor. 2.13]). *The Yoneda embeddings $\mathbf{Y}_L$ and $\mathbf{Y}_R$ are full.*

We will need a bimodule extension of Proposition 2.3. In what follows, for X, Y a pair of objects, denote by

$$\mathcal{Y}_{X,Y} := \mathcal{Y}^l_X \otimes_{\mathbb{K}} \mathcal{Y}^r_Y := \hom_{\mathcal{C}}(X, -) \otimes \hom_{\mathcal{C}}(-, Y) \tag{2.115}$$

the split bimodule obtained by tensoring the left module over X with the right module over Y.

Proposition 2.4. *Let $\mathcal{B}$ be an A_∞ bimodule over a category $\mathcal{C}$. Then, there is a natural quasi-isomorphism of chain complexes*

$$\lambda = \lambda_{\mathcal{B},X,Y} : \mathcal{B}(X,Y) \xrightarrow{\sim} \hom_{\mathcal{C}\text{–}\mathcal{C}}(\mathcal{Y}_{XY}, \mathcal{B}) \tag{2.116}$$

Proof. Given an element $\mathbf{b} \in \mathcal{B}(X,Y)$, the associated bimodule pre-morphism

$$\lambda(\mathbf{b}) = \oplus_{k,l} \lambda(\mathbf{b})^{k|1|l} : T\mathcal{C} \otimes \mathcal{Y}_{X,Y} \otimes T\mathcal{C} \to \mathcal{B}, \tag{2.117}$$

given an input

$$\begin{aligned} \vec{v} &:= (x_k, \ldots, x_1, \mathbf{x} \otimes \mathbf{y}, y_l, \ldots, y_1) \in \\ &\qquad \hom(X_{k-1}, X_k) \otimes \cdots \otimes \hom(X_0, X_1) \otimes \mathcal{Y}_{XY}(X_0, Y_l) \otimes \hom(Y_{l-1}, Y_l) \otimes \cdots \otimes \hom(Y_0, Y_1) \end{aligned} \tag{2.118}$$

returns

$$\lambda(\mathbf{b})(\vec{v}) = \sum_s (-1)^{\maltese^{y}_{l-s}} \mu_{\mathcal{B}}(x_k, \ldots, x_1, \mathbf{x}, \mu_{\mathcal{B}}(\mathbf{b}, \mathbf{y}, y_l, \ldots, y_{l-s+1}), y_{l-s}, \ldots, y_1). \tag{2.119}$$

It is an exercise to verify this pre-morphism is closed; we will sketch a proof that it is in fact a quasi-isomorphism, or rather equivalently that the cone of λ is acyclic.

First, we consider the case that $C := \mathcal{C}$ has no higher products or differential, and $B := \mathcal{B}$ is an ordinary bimodule over C (so $\mu^{1|1|0}$ and $\mu^{0|1|1}$ are the only non-zero bimodule multiplications). Via the correspondence (2.12), we can view C as an associative algebra over a semi-simple ring $R = \oplus_{X \in \mathrm{ob}\, C} \mathbb{K} e_X$, B as an ordinary bimodule, and $\mathcal{Y}_{X,Y}$ as the free simple bimodule $Ce_X \otimes e_Y C$. Then, as $Ce_X \otimes e_Y C$ is projective, we have that

$$B(X,Y) = e_X B e_Y \xrightarrow{\sim} \hom_{C\text{–}C}(Ce_X \otimes e_Y C, B) \xrightarrow{\sim} \mathrm{Ext}_{C\text{–}C}(Ce_X \otimes e_Y C, B), \tag{2.120}$$

with explicit maps given by sending an element b to the bimodule homomorphism $\phi(a,c) := a \cdot (b \cdot c)$, first thought of as ordinary bimodule homomorphism, and then as living in the relevant Ext group (this agrees with (2.119)). Using the bar resolution of $Ce_X \otimes e_Y C$ to compute Ext, it is possible to give an explicit contracting homotopy for the cone of this map by viewing it as a composition of first evaluating on the right

$$B(\cdot, Y) = Be_Y \xrightarrow{\sim} \mathrm{Ext}_C(e_Y C, B) \tag{2.121}$$

and then evaluating on the left; for the left module $M = \hom_C(e_Y, C, B)$,

$$B(X,Y) \simeq M(X) \xrightarrow{\sim} \mathrm{Ext}_C(Ce_X, M). \tag{2.122}$$

The general case, as in [**S5**, Lemma 2.12] follows from a length-filtration spectral sequence argument. Namely, we filter the cone of λ

$$\mathcal{B}(X,Y) \oplus \hom_{\mathcal{C}\text{–mod–}\mathcal{C}}(\mathcal{Y}_{X,Y}, \mathcal{B})[-1], d = \begin{pmatrix} \mu^1_{\mathcal{B}} & 0 \\ \lambda & -\mu^1_{\mathcal{C}\text{–mod–}\mathcal{C}} \end{pmatrix} \tag{2.123}$$

first by the subcomplex $\hom_{\mathcal{C}\text{–mod–}\mathcal{C}}(\mathcal{Y}_{X,Y}, \mathcal{B})[-1]$, followed by the usual length filtration on this subcomplex (where a pure pre-morphism $\phi : \mathcal{C}^{\otimes k} \otimes \mathcal{Y}_{X,Y} \otimes \mathcal{C}^{\otimes l} \to \mathcal{B}$, has length $k + l$).

The first page of the associated spectral sequence involves taking the homology of the differential which acts by preserving the length, e.g. simply by terms involving $\mu^1_{\mathcal{C}}$, $\mu^1_{\mathcal{B}}$, and $\mu^1_{\mathcal{Y}_{X,Y}}$, thereby passing to the associated homology-level complex.

Writing $B = H^*(\mathcal{B})$, $N_{XY} = H^*(\mathcal{Y}_{X,Y})$ and $C = H^*(\mathcal{C})$ for the homology level bimodules and categories (so $N_{X,Y}(A,B) = \hom_C(X,A) \otimes \hom_C(B,Y)$), the first page can be written as

$$\begin{aligned} B(X,Y) \oplus \prod \hom_{\mathbb{K}} \Big(&\hom_C(X_{k-1}, X_k) \otimes \cdots \otimes \hom_C(X_0, X_1) \otimes \\ &N_{XY}(X_0, Y_l) \otimes \hom_C(Y_{l-1}, Y_l) \otimes \cdots \otimes \hom_C(Y_0, Y_1), B(X_k, Y_0) \Big), \end{aligned} \tag{2.124}$$

with differential acting on an element in the first summand $b \in B(X,Y)$ by sending it to the pre-morphism

$$[\lambda](b)(\mathbf{a}_1 \otimes \mathbf{a}_2) := (\mathbf{a_1} \cdot (b \cdot \mathbf{a}_2)), \tag{2.125}$$

denoting all homology level algebra and bimodule multiplications by $\cdot$. On in the second summand the differential acts by a slight variant on the bar-differential from [**S5**, Lemma 2.12] where one does not collapse the two central input morphisms, thought of as an element of $N_{XY}(X_0, Y_l)$:

(2.126)

$$\begin{aligned} d\phi(x_{k+1}, \dots, x_1, \mathbf{x}_0 \otimes \mathbf{y}_{l+2}, y_{l+1}, \dots, y_l) = \\ \sum_i \phi(x_{k+1}, \dots, x_{i+1} \cdot x_i, \dots, x_1, \mathbf{x}_0 \otimes \mathbf{y}_{l+2}, y_{l+1}, \dots, y_1) + \phi(x_{k+1}, \dots, x_1 \cdot \mathbf{x}_0 \otimes \mathbf{y}_{l+2}, y_{l+1}, \dots, y_1) \\ + \phi(x_{k+1}, \dots, x_1, \mathbf{x}_0 \otimes \mathbf{y}_{l+2} \cdot y_{l+1}, \dots, y_1) + \sum_j \phi(x_{k+1}, \dots, x_1, \mathbf{x}_0 \otimes \mathbf{y}_{l+2}, y_{l+1}, \dots, y_{j+1} \cdot y_j, \dots, y_1) \\ + x_{k+1} \cdot \phi(x_k, \dots, x_1, \mathbf{x}_0 \otimes \mathbf{y}_{l+2}, y_{l+1}, \dots, y_1) + \cdot\phi(x_{k+1}, \dots, x_1, \mathbf{x}_0 \otimes \mathbf{y}_{l+2}, y_{l+1}, \dots, y_2) \cdot y_1. \end{aligned}$$

As exhibited, this page is the cone of the isomorphism $[\lambda] : B(X,Y) \to \mathrm{Ext}_{C-C}(Ce_X \otimes e_Y C, B)$, and therefore is acyclic. $\square$

See e.g., [**S3**, Lemma 5.1] for an appearance of this Lemma for A_∞ algebras and their bimodules.

Remark 2.10. *Thinking of the map (2.119) as multiplying on the right followed by collapsing, there is a homotopic map given by multiplying on the left and then collapsing.*

An easy corollary of the above Proposition is the following Künneth-type result for bimodules:

Corollary 2.2 (Künneth decomposition of A_∞ bimodules). *Let K_0, L_0, K_1, and L_1 be objects of $\mathcal{C}$. Then the inclusion of hom complexes*

$$i : \hom_{\mathrm{mod}-\mathcal{C}}(\mathcal{Y}^l_{K_0}, \mathcal{Y}^l_{K_1}) \otimes \hom_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}^r_{L_0}, \mathcal{Y}^r_{L_1}) \hookrightarrow \hom_{\mathcal{C}-\mathrm{mod}-\mathcal{C}}(\mathcal{Y}_{K_0,L_0}, \mathcal{Y}_{K_1,L_1}) \tag{2.127}$$

is a quasi-isomorphism.

Proof. On the level of chain complexes, i sends a pair of pre-module morphisms $\{\mathcal{F}^{s|1}\}$, $\{\mathcal{F}^{1|t}\}$ to $\{\mathcal{F}^{s|1|t} := \mathcal{F}^{s|1} \otimes \mathcal{F}^{1|t}\}$. It is direct to verify that we have a commutative diagram of complexes

$$\begin{tikzcd} \hom(K_1, K_0) \otimes \hom(L_0, L_1) \arrow[r, "\lambda_{K_0} \otimes \lambda_{L_0}"] \arrow[dr, "\lambda_{K_0,L_0}"] & \hom_{\mathrm{mod}-\mathcal{C}}(\mathcal{Y}^l_{K_0}, \mathcal{Y}^l_{K_1}) \otimes \hom_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}^r_{L_0}, \mathcal{Y}^r_{L_1}) \arrow[d, "i"] \\ & \hom_{\mathcal{C}-\mathrm{mod}-\mathcal{C}}(\mathcal{Y}_{K_0,L_0}, \mathcal{Y}_{K_1,L_1}) \end{tikzcd} \tag{2.128}$$

with $\lambda_{K_0} \otimes \lambda_{L_0}$ and λ_{K_0,L_0} quasi-isomorphisms, by Propositions 2.3 and 2.4. It follows that i is as well. $\square$

The above two results can be stated for $\mathcal{C}-\mathcal{D}$ bimodules, with an identical proof.

Remark 2.11. *We note that there is some non-triviality in the content of this corollary, as the complexes* $\hom_{\mathrm{mod}-\mathcal{C}}(\mathcal{Y}^l_{K_0}, \mathcal{Y}^l_{K_1}) \otimes \hom_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}^r_{L_0}, \mathcal{Y}^r_{L_1})$ *and* $\hom_{\mathcal{C}-\mathrm{mod}-\mathcal{C}}(\mathcal{Y}_{K_0,L_0}, \mathcal{Y}_{K_1,L_1})$ *are not identical; it is not necessary for a given premorphism $\{\mathcal{G}^{s|1|t}\}$ of the latter to split. Homologically, this difference turns out to be inessential.*

2.7. Hochschild invariants and bimodules. Hochschild invariants arise as a special case of the hom and tensor constructions for bimodules. This subsection is not strictly necessary for the main thread of the paper, but will only arise in the examples of open-closed operations discussed in Section 4.

In what follows, let $\mathcal{A}$ be an A_∞ algebra or category, and $\mathcal{B}$ an $\mathcal{A}-\mathcal{A}$ bimodule, frequently referred to as simply an $\mathcal{A}$-bimodule. To such a pair ($\mathcal{A}$,$\mathcal{B}$), one can similarly associate the invariants **Hochschild cohomology** and **Hochschild homology**. Instead of taking the more more conceptual route of defining these as bimodule Ext or Tor groups, we give explicit co-chain level models, using the A_∞ bar complex.

Definition 2.23. *The* **(ordinary) Hochschild co-chain complex** *of $\mathcal{A}$ with coefficients in $\mathcal{B}$ is*

$$\mathrm{CC}^*(\mathcal{A}, \mathcal{B}) := \oplus_r \mathrm{CC}^r(\mathcal{A}, \mathcal{B}) \tag{2.129}$$

with graded parts

$$\mathrm{CC}^r(\mathcal{A}, \mathcal{B}) := \hom_{grVect}(\oplus \mathcal{A}^{\otimes j}, \mathcal{B}[r+j]). \tag{2.130}$$

where $\hom_{grVect}$ *denotes graded degree 0 homomorphisms of graded R-modules. Categorically, this can be rewritten as*

$$\mathrm{CC}^r(\mathcal{A},\mathcal{B}) := \prod_{k, X_{i_0},\dots,X_{i_k}} \hom_{grVect}(\hom_{\mathcal{A}}(X_{i_{k-1}}, X_{i_k}) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_{i_0}, X_{i_1}), \mathcal{B}(X_{i_0}, X_{i_k})[r+j]). \tag{2.131}$$

Note that the total complex $\mathrm{CC}^*(\mathcal{A},\mathcal{B})$ *is not quite equal to* $\hom_{Vect}(T\mathcal{A},\mathcal{B})$*, as we require a given Hochschild co-chain to be supported in finitely many degrees. Given a Hochschild co-chain* $\phi \in \mathrm{CC}^l(\mathcal{A},\mathcal{B})$*, one can consider the extension*

$$\hat{\phi} : T\mathcal{A} \longrightarrow T\mathcal{A} \otimes \mathcal{B} \otimes T\mathcal{A}. \tag{2.132}$$

given by

$$\hat{\phi}(x_k, \dots, x_1) := \sum_{i,j} (-1)^{l \cdot \maltese_1^i} x_k \otimes \cdots \otimes x_{i+j+1} \otimes \phi(x_{i+j}, \dots, x_{i+1}) \otimes x_i \otimes \cdots \otimes x_1. \tag{2.133}$$

with sign given by the degree l *of* ϕ *times*

$$\maltese_1^i := \sum_{s=1}^{i} ||x_s||. \tag{2.134}$$

Then the **differential** *is given by:*

$$d(\phi) := \mu_{\mathcal{B}} \circ \hat{\phi} - \phi \circ \hat{\mu}_{\mathcal{A}}. \tag{2.135}$$

With respect to the grading, the differential has degree +1. In an analogous fashion, there is an explicit chain-level model for the Hochschild homology complex $(\mathrm{CC}_*(\mathcal{A},\mathcal{B}), d_{\mathrm{CC}_*})$.

Definition 2.24. *Let* $\mathcal{A}$ *be an* A_∞ *algebra and* $\mathcal{B}$ *an* A_∞ *bimodule. The* **(ordinary) Hochschild homology chain complex** $\mathrm{CC}_*(\mathcal{A},\mathcal{B})$ *is defined to be*

$$\mathrm{CC}_*(\mathcal{A},\mathcal{B}) := (\mathcal{B} \otimes_R T\mathcal{A})^{\mathrm{diag}},$$

where the diag *superscript means we restrict to cyclically composable elements of* $(\mathcal{B} \otimes_R T\mathcal{A})$*. Explicitly this complex is the direct sum of, for any* k *and any* $k+1$*-tuple of objects* $X_0, \dots, X_k \in \mathrm{ob}\ \mathcal{A}$*, the vector spaces*

$$\mathcal{B}(X_k, X_0) \otimes \hom_{\mathcal{A}}(X_{k-1}, X_k) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1). \tag{2.136}$$

The differential d_{CC_*} acts on Hochschild chains as follows:

$$\begin{aligned} &d_{\mathrm{CC}_*}(\mathbf{b} \otimes x_k \otimes \cdots \otimes x_1) = \\ &\qquad \sum (-1)^{\#_j^i} \mu_{\mathcal{B}}^{j|1|k-i}(x_j, \dots, x_1, \mathbf{b}, x_k, \dots, x_{i+1}) \otimes x_i \otimes \cdots \otimes x_{j+1} \\ &\qquad + \sum (-1)^{\maltese_1^s} \mathbf{b} \otimes x_k \otimes \cdots \otimes x_{s+j+1} \otimes \mu_{\mathcal{A}}^j(x_{s+j} \otimes \cdots \otimes x_{s+1}) \otimes x_s \otimes \cdots \otimes x_1 \end{aligned} \tag{2.137}$$

with signs

$$\maltese_s^t := \sum_{i=s}^{t} ||x_i|| \tag{2.138}$$

$$\#_j^i := \left(\sum_{s=1}^{j} ||x_s||\right) \cdot \left(|\mathbf{b}| + \sum_{t=j+1}^{k} ||x_t||\right) + \maltese_{j+1}^i. \tag{2.139}$$

In this complex, Hochschild chains are graded as follows:

$$\deg(\mathbf{b} \otimes x_k \otimes \cdots \otimes x_1) := \deg(\mathbf{b}) + \sum_i \deg(x_i) - k + 1. \tag{2.140}$$

Example 2.1. Let $\mathcal{M}$ be a right $\mathcal{C}$ module and $\mathcal{N}$ a left $\mathcal{C}$ module, and form the product bimodule $\mathcal{N} \otimes_{\mathbb{K}} \mathcal{M}$. Then the Hochschild chain complex

$$\mathrm{CC}_*(\mathcal{C}, \mathcal{N} \otimes_{\mathbb{K}} \mathcal{M}) \tag{2.141}$$

is exactly the bar complex

$$\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N} \tag{2.142}$$

defined in (2.67), up to reordering a term (along with the accompanying Koszul sign change).

There are alternate chain level descriptions of Hochschild invariants that align more closely with our viewpoint of using A_∞ bimodules.

Definition 2.25. *The* **two-pointed complex for Hochschild homology**

$$ {}_2\mathrm{CC}_*(\mathcal{A}, \mathcal{B}) \tag{2.143} $$

is the chain complex computing the bimodule tensor product with the diagonal bimodule:

$$ {}_2\mathrm{CC}_*(\mathcal{A}, \mathcal{B}) := \mathcal{A}_\Delta \otimes_{\mathcal{A}-\mathcal{A}} \mathcal{B}. \tag{2.144} $$

Observe that the complex ${}_2\mathrm{CC}_*(\mathcal{A}, \mathcal{B})$ can be alternatively described as the ordinary Hochschild complex

$$ CC_*(\mathcal{A}, \mathcal{A}_\Delta \otimes_{\mathcal{A}} \mathcal{B}). \tag{2.145} $$

Thus, the quasi-isomorphism of bimodules

$$ \mathcal{A}_\Delta \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\sim} \mathcal{B} \tag{2.146} $$

functorially induces a quasi-isomorphism of complexes

$$ \Phi : \ {}_2\mathrm{CC}_*(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{CC}_*(\mathcal{A}, \mathcal{B}). \tag{2.147} $$

explicitly given by

$$ \begin{aligned} &\Phi(\mathbf{a}, a_k, \ldots, a_1, \mathbf{b}, \bar{a}_l, \ldots, \bar{a}_1) := \\ &\qquad \sum (-1)^{\bowtie} \mu_{\mathcal{B}}^{i+k+1|1|j}(\bar{a}_i, \ldots, \bar{a}_1, \mathbf{a}, a_k, \ldots, a_1, \mathbf{b}, \bar{a}_l, \ldots, \bar{a}_{l-j+1}) \otimes \bar{a}_{l-j} \otimes \cdots \otimes \bar{a}_{i+1}. \end{aligned} \tag{2.148} $$

with sign given by the sum of the sign occuring in the quasi-isomorphism of bimodules (2.93), the accompanying Koszul signs for re-ordering $\bar{a}_1, \ldots, \bar{a}_i$ past other elements, and the Koszul signs appearing from the convention of applying μ on the right:

$$ \bowtie := \left(\sum_{n=l-j+1}^{l} ||\bar{a}_n|| + |\mathbf{b}| - 1 + \sum_{n=1}^{k} ||a_n|| \right) + \left(\sum_{s=1}^{i} ||\bar{a}_s|| \right) \cdot \left(|\mathbf{a}| + |\mathbf{b}| + \sum_{t=1}^{k} ||a_t|| + \sum_{s=i+1}^{l} ||\bar{a}_s|| \right) + \sum_{m=i+1}^{l-j} ||\bar{a}_m||. \tag{2.149} $$

This a special case of the general fact that if $\mathcal{F} : \mathcal{B}_0 \to \mathcal{B}_1$ is a closed morphism of bimodules, there is an induced morphism of Hochschild complexes $\mathrm{CC}(\mathcal{A}, \mathcal{B}_0) \xrightarrow{\mathcal{F}_*} \mathrm{CC}(\mathcal{A}, \mathcal{B}_1)$.

Definition 2.26. *The* **two-pointed complex for Hochschild cohomology**

$$ {}_2\mathrm{CC}^*(\mathcal{A}, \mathcal{B}) \tag{2.150} $$

is the chain complex computing the bimodule hom:

$$ {}_2\mathrm{CC}^*(\mathcal{A}, \mathcal{B}) := \hom_{\mathcal{A}-\mathcal{A}}(\mathcal{A}_\Delta, \mathcal{B}). \tag{2.151} $$

Similarly, as one natural interpretation of Hochschild cohomology is as endomorphisms of the identity functor or the (derived) self-ext of the diagonal bimodule, one expects a quasi-isomorphism of complexes

$$ \Psi : \mathrm{CC}^*(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \hom_{\mathcal{A}-\mathcal{A}}(\mathcal{A}_\Delta, \mathcal{B}) \tag{2.152} $$

Explicitly, if $\phi \in \mathrm{CC}^*(\mathcal{A}, \mathcal{B})$ is a Hochschild co-chain then one such map is given by:

$$ \begin{aligned} &\Psi(\phi)(x_k, \ldots, x_1, \mathbf{a}, y_l, \ldots, y_1) := \\ &\qquad \sum (-1)^{\blacktriangle} \mu_{\mathcal{B}}^{k+1+i|1|s}(x_k, \ldots, x_1, \mathbf{a}, y_l, \ldots, y_{i+s+1}, \phi(y_{i+s}, \ldots, y_i), y_i, \ldots, y_1) \end{aligned} \tag{2.153} $$

with sign

$$ \blacktriangle := |\phi| \cdot (\sum_{j=1}^{i} ||y_j||). \tag{2.154} $$

Proposition 2.5. Ψ *is a quasi-isomorphism when* $\mathcal{A}$ *is homologically unital.*

Proof. As with previous such arguments, the quasi-isomorphism is a consequence of a length filtration argument, reducing on page 1 to the case of this quasi-isomoprhism holding for ordinary unital algebras and their bimodules. At that level, one can either explicitly construct a contracting homotopy for the cone of Ψ or make a more abstract comparison argument (the complexes in question arise from two different projective resolutions of $\mathcal{A}$, but Hochschild cohomology, an Ext group, is independent of such choices). See e.g., [**S7**, Lemma 2.9] for a slightly more detailed sketch. $\square$

Remark 2.12. *Our emphasis on multiple chain-level models for Hochschild invariants, with explicit quasi-isomorphisms between them, is contrary to the "derived" perspective that the resulting invariants are abstractly independent of choices. However, the geometric open-closed maps constructed in Section 4 depend directly on choices of chain complexes. As one consequence of the discussion here, we will obtain the (unsurprising) result that two variants of the geometric open-closed maps that with different source Hochschild chain complexes are explicitly quasi-isomorphic. This is expected but not a priori obvious from definitions.*

3. Symplectic cohomology and wrapped Floer cohomology

3.1. Liouville manifolds. Our basic object of study will be a **Liouville manifold**, a manifold M^{2n} equipped with a one form θ called the **Liouville form**, such that $d\theta = \omega$ is a symplectic form. The **Liouville vector field** Z is defined to be the symplectic dual to θ (so $i_Z\omega = \theta$). We further require M to have a **cylindrical** (or **conical**) **convex end**. That is, away from a compact region $\bar{M}$, M has the structure of the semi-infinite symplectization of a contact manifold

$$M = \bar{M} \cup_{\partial \bar{M}} \partial \bar{M} \times [1, +\infty)_r, \tag{3.1}$$

such that the flow Z is transverse to $\partial \bar{M} \times \{1\}$ and acts on the cylindrical region by translation proportional to r, the symplectization coordinate:

$$Z = r\partial_r. \tag{3.2}$$

The flow of the vector field Z is called the **Liouville flow** and denoted

$$\psi^\rho, \tag{3.3}$$

where the time flowed is $\log(\rho)$. We henceforth fix a representation of M of the form (3.1).

Remark 3.1. *One could have instead begun with a* **Liouville domain***, an exact compact symplectic manifold $\bar{M}$ with contact boundary $\partial\bar{M}$, such that the Liouville vector field Z is outward pointing along $\partial\bar{M}$. One then integrates the flow of Z in a small neighborhood of the boundary to obtain a collar neighborhood $\partial\bar{M} \times (1-\epsilon, 1]$ and then attaches the infinite cone* (3.1) *to get a Liouville manifold. This process is known as* **completion***.*

On the boundary of the compact region $\bar{M}$

$$\partial \bar{M} := \partial \bar{M} \times \{1\}, \tag{3.4}$$

$$\bar{\theta} := \theta|_{\partial \bar{M}} \text{ is a contact form.} \tag{3.5}$$

On the **conical end**

$$\partial \bar{M} \times [1, +\infty), \tag{3.6}$$

the Liouville form is given by rescaling the contact form

$$\theta = r\bar{\theta}. \tag{3.7}$$

Moreover, there is an associated **Reeb vector field** on $\partial\bar{M}$

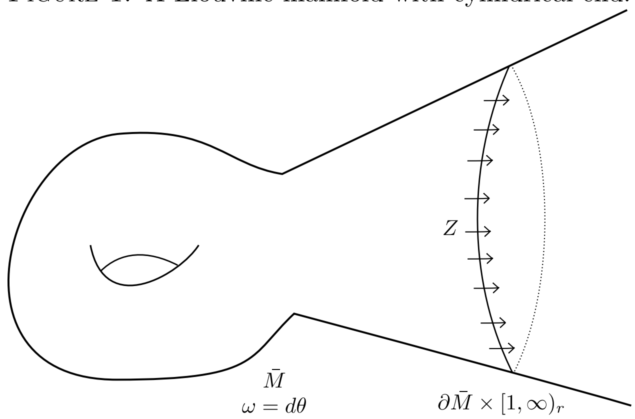
$$R \tag{3.8}$$

defined in the usual fashion by the requirements that

$$\begin{cases} d\bar{\theta}(R, \cdot) = 0. \\ \bar{\theta}(R) = 1. \end{cases} \tag{3.9}$$

Via the product identification (3.1), we view R as a vector field defined on the entire conical end.

Figure 1. A Liouville manifold with cylindrical end.



Now, we consider a finite collection ob $\mathcal{W}$ of exact properly embedded Lagrangian submanifolds in M, such that for each $L \in$ ob $\mathcal{W}$,

$$\theta \text{ vanishes on } L \cap \partial\bar{M} \times [1, +\infty). \tag{3.10}$$

Namely, the intersection ∂L of L with $\partial\bar{M}$ is Legendrian, and L is obtained by attaching an infinite cylindrical end $\partial L \times [1, +\infty)$ to $L^{in} = L \cap \bar{M}$. In addition, for each $L \in$ ob $\mathcal{W}$,

$$\text{choose and fix a primitive } f_L : L \to \mathbb{R} \text{ for } \theta|_L. \tag{3.11}$$

By the above condition, f_L is locally constant on the cylindrical end of L.

To fix an integer grading on symplectic cohomology and $\mathcal{W}$, we require each $L \in$ ob $\mathcal{W}$ to be spin, and have vanishing relative first Chern class $2c_1(M, L) \in H^2(M, L)$. We additionally need to

$$\text{Fix a spin structure (and orientation) on each } L. \tag{3.12}$$

$$\text{Fix a trivialization of } (\Lambda^n_{\mathbb{C}} T^*M)^{\otimes 2}, \text{ and a grading on each } L. \tag{3.13}$$

We will implicitly fix all of this data whenever referring to a given Lagrangian.

We restrict to a class of Hamiltonians

$$\mathcal{H}(M) \subset C^\infty(M, \mathbb{R}), \tag{3.14}$$

functions H that, away from some compact subset of M satisfy

$$H(r, y) = r^2. \tag{3.15}$$

Consider a class of almost-complex structures $\mathcal{J}_1(M)$ that are **rescaled contact type** on the conical end, meaning that

$$\frac{1}{r}\theta \circ J = dr. \tag{3.16}$$

This implies in particular that J intertwines the Reeb and r directions:

$$\begin{aligned} J(\partial_r) &= R \\ J(R) &= -\partial_r \end{aligned} \tag{3.17}$$

Remark 3.2. *Our class of complex structures differs from those used by Abouzaid* [**A1**] *and Abouzaid-Seidel* [**AS**], *who consider almost complex structures satisfying $\theta \circ J = dr$. The difference is backwards compatible with the operations constructed in* [**A1**], *as can be seen by using interpolating complex structures (those operations satisfy a maximum principle for any surface-dependent complex structure with $f(r)\theta \circ J = dr$, for $f(r)$ a positive function). However, the class we used has slightly better compactness properties for operations involving multiple orbit inputs, as we prove in §B.*

We assume that θ has been chosen generically so that

$$\begin{aligned}&\text{all Reeb orbits of } \bar{\theta} \text{ are non-degenerate, and}\\&\text{all Reeb chords between Lagrangians in ob } \mathcal{W} \text{ are non-degenerate.}\end{aligned} \tag{3.18}$$

3.2. Wrapped Floer cohomology. Fixing a choice of $H \in \mathcal{H}(M)$ define

$$\chi(L_0, L_1)$$

to be the set of time 1 Hamiltonian flows of H between L_0 and L_1. Given the data specified in the previous section, the **Maslov index** defines an absolute grading on $\chi(L_0, L_1)$, which we will denote by

$$\deg : \chi(L_0, L_1) \to \mathbb{Z}. \tag{3.19}$$

Then, given a family $J_t \in \mathcal{J}_1(M)$ parametrized by $t \in [0,1]$, define the **wrapped Floer co-chain complex** over $\mathbb{K}$ to be, as a graded vector space,

$$CW^i(L_0, L_1, H, J_t) = \bigoplus_{x \in \chi(L_0,L_1), deg(x) = i} |o_x|_{\mathbb{K}}. \tag{3.20}$$

Here, the **orientation line** o_x, a one-dimensional real vector space associated to x defined as the determinant line of a certain linearization of Floer's equation, is defined in Appendix A. Also, for any one-dimensional real vector space V (such as o_x) and any field $\mathbb{K}$, the $\mathbb{K}$-*normalization*

$$|V|_{\mathbb{K}} \tag{3.21}$$

is $\mathbb{K}$-vector space generated by the two orientations on V, modulo the relation that the sum of the orientations is zero [**S5**, §12e] (if $\mathbb{K} = \mathbb{Z}_2$, note that $|o_x|_{\mathbb{K}} \cong \mathbb{Z}_2$ canonically, so (3.20) is literally generated by a copy of $\mathbb{K}$ for each chord).

Now, consider maps

$$u : (-\infty, \infty) \times [0,1] \to M \tag{3.22}$$

converging exponentially at each end to time-1 chords of H, satisfying boundary conditions

$$\begin{aligned}u(s,0) &\in L_0\\u(s,1) &\in L_1\end{aligned}$$

and satisfying Floer's equation

$$(du - X \otimes dt)^{0,1} = 0. \tag{3.23}$$

Above, X is the Hamiltonian vector field of H and we think of the strip

$$Z = (-\infty, \infty) \times [0,1] \tag{3.24}$$

as equipped with coordinates s, t and the canonical complex structure j ($j(\partial_s) = \partial_t$). With this prescription one can rewrite the above equation in coordinates in the more familiar form

$$\partial_s u = -J_t(\partial_t u - X). \tag{3.25}$$

Given time 1 chords $x_0, x_1 \in \chi(L_0, L_1)$, denote by

$$\tilde{\mathcal{R}}^1(x_0; x_1) \tag{3.26}$$

the set of maps u converging to x_0 when $s \to -\infty$ and x_1 when $s \to +\infty$. As a component of the zero-locus of an elliptic operator on the space of smooth functions from Z into M, this set carries a natural topology. Moreover, the natural $\mathbb{R}$ action on $\tilde{\mathcal{R}}^1(x_0; x_1)$, coming from translation in the s direction, is continuous with respect to this topology. Following standard arguments, we conclude:

Lemma 3.1. *For generic J_t, the moduli space $\tilde{\mathcal{R}}^1(x_0; x_1)$ is a compact manifold of dimension* $\deg(x_0) - \deg(x_1)$. *The action of $\mathbb{R}$ is smooth and free unless* $\deg(x_0) = \deg(x_1)$.

Proof. See [**A1**, Lemma 2.3]. □

Definition 3.1. *Define*

$$\mathcal{R}(x_0; x_1) \tag{3.27}$$

to be the quotient of $\tilde{\mathcal{R}}^1(x_0; x_1)$ by the $\mathbb{R}$ action whenever it is free, and the empty set when the $\mathbb{R}$ action is not free.

Also following now-standard arguments, one may construct a bordification $\overline{\mathcal{R}}(x_0; x_1)$ by adding **broken strips**

$$\overline{\mathcal{R}}(x_0; x_1) = \coprod \mathcal{R}(x_0; y_1) \times \mathcal{R}(y_1; y_2) \times \cdots \times \mathcal{R}(y_k; x_1) \tag{3.28}$$

Lemma 3.2. *For generic J_t, the moduli space $\overline{\mathcal{R}}(x_0, x_1)$ is a compact manifold with boundary of dimension* $\deg(x_0) - \deg(x_1) - 1$. *The boundary is covered by the closure of the images of natural inclusions*

$$\mathcal{R}(x_0; y) \times \mathcal{R}(y; x_1) \to \overline{\mathcal{R}}(x_0; x_1). \tag{3.29}$$

Proof. See [**A1**, Lemma 2.4]. □

Lemma 3.3. *Moreover, for each x_1, the $\overline{\mathcal{R}}(x_0; x_1)$ is empty for all but finitely many x_0.*

Proof. A proof of this is given in [**A1**, Lemma 2.5] but it is not quite applicable as it involves a general compactness result proven for complex structures J satisfying $\theta \circ J = dr$, see [**A1**, Lemma B.1-2]. In fact, the arguments from this general compactness result directly carry over for our J_t but we can alternately apply Theorem B.1. □

Now, for regular $u \in \mathcal{R}(x_0; x_1)$, if $\deg(x_0) = \deg(x_1) + 1$, the orientation on $\mathcal{R}(x_0; x_1)$ gives, by Lemma A.1 and Remark A.2, an isomorphism

$$\mu_u : o_{x_1} \longrightarrow o_{x_0}. \tag{3.30}$$

which in particular induces a morphism on the level of $\mathbb{K}$ vector spaces $|o_{x_1}|_{\mathbb{K}} \to |o_{x_0}|_{\mathbb{K}}$, also denoted μ_u.

Thus we can define a differential

$$\begin{aligned} d : CW^*(L_0, L_1; H, J_t) &\longrightarrow CW^*(L_0, L_1; H, J_t) \\ d([x_1]) &= \sum_{x_0; \deg(x_0) = \deg(x_1)+1} \sum_{u \in \mathcal{R}(x_0; x_1)} (-1)^{\deg(x_1)} \mu_u([x_1]). \end{aligned} \tag{3.31}$$

Lemma 3.4.

$$d^2 = 0.$$

Call the resulting group $HW^*(L_0, L_1)$.

3.3. Symplectic cohomology. To define symplectic cohomology, we break the S^1 symmetry that occurs for non-trivial time 1 orbits of our autonomous Hamiltonian H. Choose $F : S^1 \times M \to \mathbb{R}$ a smooth non-negative function, with

- F and $\theta(X_F)$ uniformly bounded in absolute value, and
- all time-1 periodic orbits of X_{S^1}, the (time-dependent) Hamiltonian vector field corresponding to $H_{S^1}(t, m) = H(m) + F(t, m)$, are non-degenerate. This is possible for generic choices of F [**A1**].

Fixing such a choice, define

$$\mathcal{O}$$

to be the set of (time-1) periodic orbits of H_{S^1}. Given an element $y \in \mathcal{O}$, define the *degree* of y to be

$$\deg(y) := n - CZ(y) \tag{3.32}$$

where CZ is the Conley-Zehnder index of y. Now, define the **symplectic co-chain complex** over $\mathbb{K}$ to be

$$CH^i(M; H, F, J_t) = \bigoplus_{y \in \mathcal{O}, deg(y) = i} |o_y|_{\mathbb{K}}, \tag{3.33}$$

where the **orientation line** o_y is again defined using the determinant line of a linearization of Floer's equation in Appendix A, and $|o_y|_{\mathbb{K}}$ denotes the $\mathbb{K}$-normalization of o_y as in (3.21).

Given an S^1 dependent family $J_t \in \mathcal{J}_1(M)$, consider maps

$$u : (-\infty, \infty) \times S^1 \to M \tag{3.34}$$

converging exponentially at each end to a time-1 periodic orbit of H_{S^1} and satisfying Floer's equation

$$(du - X_{S^1} \otimes dt)^{0,1} = 0. \tag{3.35}$$

Here, as above the cylinder $A = (-\infty, \infty) \times [0, 1]$ is equipped with coordinates s, t and a complex structure j with $j(\partial s) = \partial_t$. As before, this means the above equation in coordinates is the usual

$$\partial_s u = -J_t(\partial_t u - X). \tag{3.36}$$

Given time 1 orbits $y_0, y_1 \in \mathcal{O}$, denote by $\tilde{\mathcal{M}}(y_0; y_1)$ the set of maps u converging to y_0 when $s \to -\infty$ and y_1 when $s \to +\infty$. In analogy with the maps defining wrapped Floer cohomology, this set is equipped with a topology and a natural $\mathbb{R}$ action coming from translation in the s direction. We can similarly conclude that for generic J_t, the moduli space is smooth of dimension $\deg(y_0) - \deg(y_1)$ with free $\mathbb{R}$ action unless it is of dimension 0.

Definition 3.2. *Define*

$$\mathcal{M}(y_0; y_1) \tag{3.37}$$

to be the quotient of $\tilde{\mathcal{M}}(y_0; y_1)$ by the $\mathbb{R}$ action whenever it is free, and the empty set when the $\mathbb{R}$ action is not free.

Construct the analogous bordification $\overline{\mathcal{M}}(y_0; y_1)$ by adding **broken cylinders**

$$\overline{\mathcal{M}}(y_0; y_1) = \coprod \mathcal{M}(y_0; x_1) \times \mathcal{M}(x_1; x_2) \times \cdots \times \mathcal{M}(x_k; y_1) \tag{3.38}$$

Lemma 3.5. *For generic J_t, the moduli space $\overline{\mathcal{M}}(y_0, y_1)$ is a compact manifold with boundary of dimension* $\deg(y_0) - \deg(y_1) - 1$. *The boundary is covered by the closure of the images of natural inclusions*

$$\mathcal{M}(y_0; y) \times \mathcal{M}(y; y_1) \to \overline{\mathcal{M}}(y_0; y_1). \tag{3.39}$$

Moreover, for each y_1, $\overline{\mathcal{M}}(y_0; y_1)$ is empty for all but finitely many choices of y_0.

Proof. Perturbed hamiltonians of the form $H + F_t$ cease to satisfy a maximum principle, by some bounded error term. For *rescaled contact-type complex structures*, we show in Theorem B.1 that it is still possible to ensure that solutions with fixed asymptotics stay within a compact set, and a corresponding finiteness result. □

For a regular $u \in \mathcal{M}(y_0; y_1)$ with $\deg(y_0) = \deg(y_1) + 1$, Lemma A.1 and Remark A.2 give us an isomorphism of orientation lines

$$\mu_u : o_{y_1} \longrightarrow o_{y_0}. \tag{3.40}$$

Thus we can define a differential

$$d : CH^*(M; H, F_t, J_t) \longrightarrow CH^*(M; H, F_t, J_t) \tag{3.41}$$

$$d([y_1]) = \sum_{y_0; \deg(y_0) = \deg(y_1)+1} \sum_{u \in \mathcal{M}(y_0; y_1)} (-1)^{\deg(y_1)} \mu_u([y_1]). \tag{3.42}$$

Lemma 3.6.

$$d^2 = 0.$$

Call the resulting group $SH^*(M)$.

Remark 3.3. *Our grading conventions for symplectic cohomology follow Seidel* [**S4**], *Abouzaid* [**A1**], *and Ritter* [**R**]. *These conventions are essentially determined by the fact that the identity element lives in degree zero, and the product map is also a degree zero operation, making $SH^*(M)$ a graded ring. See the sections that follow for more details.*

4. Open-closed operations

We recall definitions of abstract moduli spaces of genus 0 bordered Riemann surfaces with interior and boundary marked points, which we will call **genus-0 open-closed strings**. Then, we define Floer data for such spaces, and use these Floer data to construct chain-level open-closed operations in the wrapped setting. In the next section, we will specialize to examples such as discs, spheres, and discs with interior and boundary punctures to obtain A_∞ structure maps, TFT operations, and various open-closed operations.

4.1. Abstract moduli spaces.

DEFINITION 4.1. *A* **genus-0 open-closed string** *of type h with $n, \vec{m} = (m^1, \ldots, m^h)$ marked points Σ is a sphere with h disjoint discs removed, with n interior marked points and m^i boundary marked points on the ith boundary component $\partial^i\Sigma$. Fix some subset $\mathbf{I} \subset \{1, \ldots, n\}$ and a vector of subsets $\vec{\mathbf{K}} = (K^1, \ldots, K^h)$ with $K^i \subset \{1, \ldots, m^i\}$. Σ has* **sign-type** $(\mathbf{I}, \vec{\mathbf{K}})$ *if*

- *interior marked points p_i, with $i \in I$ are negative,*
- *boundary marked points $z_{j,k} \in \partial^j\Sigma$, $k \in K^j$ are negative, and*
- *all other marked points are positive.*

Also, a genus-0 open-closed string comes equipped with the data of

- *a choice of* **normal vector** *or* **asymptotic marker** *at each interior marked point.*

For our applications, we explicitly restrict to considering at most one negative interior marked point or at most two negative boundary marked points, i.e. the cases

$$\begin{cases} |I| = 1 \text{ and } \sum |K^i| = 0 \\ |I| = 0 \text{ and } \sum |K^i| = 1 \text{ or } 2. \end{cases} \tag{4.1}$$

DEFINITION 4.2. *The* **(non-compactified) moduli space of genus-0 open-closed strings** *of type h with $n, \vec{m}$ marked points and sign-type $(\mathbf{I}, \vec{\mathbf{K}})$ is denoted $\mathcal{N}^{\mathbf{I},\vec{\mathbf{K}}}_{h,n,\vec{m}}$.*

Denote by $\overline{\mathcal{N}}^{\mathbf{I},\vec{\mathbf{K}}}_{h,n,\vec{m}}$ the Deligne-Mumford compactification of this space, a real blow-up of the space described by Liu [**L**] (see e.g., [**KSV**]). Note that for $\mathbf{I}$ and $\vec{\mathbf{K}}$ as in (4.1), the lower-dimensional strata of $\overline{\mathcal{N}}^{\mathbf{I},\vec{\mathbf{K}}}_{h,n,\vec{m}}$ consist of nodal bordered surfaces, with each component genus 0 and also satisfying (4.1).

Fix a collection of strip-like and cylindrical ends near every marked boundary and interior point of a stable open-closed string, with the cylindrical ends chosen to have $1 \in S^1$ asymptotic to our chosen marker. Then, at a nodal surface consisting of k interior nodes and l boundary nodes, there is a chart

$$[0,1)^{k+l} \to \overline{\mathcal{N}}^{\mathbf{I},\vec{\mathbf{K}}}_{h,n,\vec{m}} \tag{4.2}$$

given by assigning to the coordinate $(\rho_1, \ldots, \rho_k, \eta_1, \ldots, \eta_l)$ the glued surface where the ith interior node and jth boundary nodes have been glued with gluing parameters ρ_i and η_j respectively, in a manner so that asymptotic markers line up for interior gluings. We see in this way that $\overline{\mathcal{N}}^{\mathbf{I},\vec{\mathbf{K}}}_{h,n,\vec{m}}$ inherits the structure of a manifold with corners. Moreover, from the corner charts described above every open-string $S \in \overline{\mathcal{N}}^{\mathbf{I},\vec{\mathbf{K}}}_{h,n,\vec{m}}$ inherits a **thick-thin decomposition**, where the *thin parts* of the surface S are by definition the finite cylinders and strips in S that are inherited from the gluing parameters if S lies in one of the above such charts. If S does not lie in such a chart, then S has no thin parts.

REMARK 4.1. *In all the moduli spaces we will actually consider, the marked direction is always determined uniquely (and somewhat arbitrarily) by requiring it to point towards one particular distinguished boundary point. This works because*

- *it is consistent with Deligne-Mumford compactifications: when disc components break off and separate the interior puncture from the preferred boundary puncture, the new preferred boundary puncture is the node connecting the components;*
- *it is consistent with the choices of cylindrical ends made at interior punctures created when later we glue pairs of discs across Δ labels.*

Given this, we will always omit these asymptotic markers from the general discussion.

REMARK 4.2. *By considering moduli spaces where these asymptotic markers vary in S^1 families, one can endow symplectic cohomology and open-closed maps with a larger set of operations, e.g. the BV operator. We will not do so here. For some additional details on such operations, see* [**S4**] *or* [**SS**].

4.2. Floer data. First, we note, following an observation of [?Fukaya:2009aa] which was systematically developed in [**A1**], that pullback of solutions to (3.23) by the Liouville flow for time $\log(\rho)$ defines a canonical isomorphism

$$CW^*(L_0, L_1; H, J_t) \simeq CW^*\left(\psi^\rho L_0, \psi^\rho L_1; \frac{H}{\rho} \circ \psi^\rho, (\psi^\rho)^* J_t\right) \tag{4.3}$$

We have two main observations which will help us define operations on the complexes $CW^*(L_0, L_1)$ and $SH^*(M)$:

LEMMA 4.1. *The function $\frac{H}{\rho^2} \circ \psi^\rho$ lies in $\mathcal{H}(M)$.*

PROOF. The Liouville flow is given on the collar by

$$\psi^\rho(r, y) = (\rho \cdot r, y) \tag{4.4}$$

so $r^2 \circ \psi^\rho = \rho^2 r^2$. □

Note however that $(\psi^\rho)^* J_t \notin \mathcal{J}_1(M)$. In fact, a computation shows that

$$\frac{\rho}{r}\theta \circ (\psi^\rho)^* J_t = dr. \tag{4.5}$$

Motivated by this,

DEFINITION 4.3. *Define $\mathcal{J}_c(M)$ to be the space of almost-complex structures J that are* **c-rescaled contact type**, *i.e.*

$$\frac{c}{r}\theta \circ J = dr. \tag{4.6}$$

Also, define $\mathcal{J}(M)$ to be the space of almost-complex structures J that are c-rescaled contact type for some c.

To simplify terminology later, we will introduce some new notation for Floer data.

DEFINITION 4.4. *A* **collection of strip and cylinder data** *for a surface S with some boundary and interior marked pointed removed is a choice of*

- **strip-like ends** $\epsilon^k_\pm : Z_\pm \to S$,
- **finite strips** $\epsilon^l : [a^l, b^l] \times [0,1] \to S$,
- **cylindrical ends** $\delta^j_\pm : A_\pm \times S^1 \to S$, *and*
- **finite cylinders** $\delta^r : [a_r, b_r] \times S^1, \to S$

all with disjoint image in S. Such a collection is said to be **weighted** *if each cylinder and strip above comes equipped with a choice of positive real number, called a weight. Label these weights as follows:*

- *$w^\pm_{S,k}$ is the weight associated to the strip-like end $\epsilon^k_\pm$,*
- *$w_{S,l}$ is associated to the finite strip ϵ^l,*
- *$v^\pm_{S,j}$ is associated to the cylindrical end $\delta^j_\pm$, and*
- *$v_{S,r}$ is associated to the finite cylinder δ^r.*

Finally, such a collection is said to be **δ-bounded** *if*

- *the length of each finite cylinder $(b_r - a_r)$ is larger than 3δ.*

DEFINITION 4.5. *Let $\mathfrak{S}$ be a δ-bounded collection of strip and cylinder data for S. The* **associated δ-collar** *of S is the following collection of finite cylinders:*

- *the restriction $\tilde{\delta}^j_+$ of each positive cylindrical end $\delta^j_+ : [0, \infty) \times S^1 \to S$ to the domain $[0, \delta] \times S^1$,*
- *the restriction $\tilde{\delta}^j_-$ of each negative cylindrical end $\delta^j_- : (-\infty, 0] \times S^1 \to S$ to the domain $[-\delta, 0] \times S^1$, and*

- *the restrictions $\tilde{\delta}^r_{in}$ and $\tilde{\delta}^r_{out}$ of each finite cylinder $\delta^r : [a_r, b_r] \times S^1 \to S$ to the domains $[a_r, a_r + \delta] \times S^1$ and $[b_r - \delta, b_r] \times S^1$ respectively.*

We will often refer to this as the **associated collar** *if δ is implicit.*

Let $(S, \mathfrak{S})$ be a surface S with a δ-bounded collection of weighted strip and cylinder data $\mathfrak{S}$.

Definition 4.6. *A one-form α_S on S is said to be* **compatible with the weighted strip and cylinder data $\mathfrak{S}$** *if, for each finite or semi-infinite cylinder or strip κ of S, with associated weight ν_κ,*

$$\kappa^* \alpha_S = \nu_\kappa dt. \tag{4.7}$$

Above, t is the coordinate of the second component of the associated strip or cylinder.

Definition 4.7. *Fix a Hamiltonian $H \in \mathcal{H}(M)$. An S-dependent Hamiltonian $H_S : S \to \mathcal{H}(M)$ is said to be* **H-compatible with the weighted strip and cylinder data $\mathfrak{S}$** *if, for each cylinder or strip κ with associated weight ν_κ,*

$$\kappa^* H_S = \frac{H \circ \psi^{\nu_\kappa}}{\nu_\kappa^2}. \tag{4.8}$$

Definition 4.8. *An* **$\mathfrak{S}$-adapted rescaling function** *is a map $a_S : S \to [1, \infty)$ that is constant on each cylinder and strip of $\mathfrak{S}$, equal to the associated weight of that cylinder or strip.*

Definition 4.9. *Fix a time-dependent almost-complex structure $J_t : S^1 \to \in \mathcal{J}_1(M)$, and an adapted rescaling function a_S. An* **$(\mathfrak{S}, a_S, J_t)$-adapted complex structure** *is a map $J_S : S \to \mathcal{J}(M)$ such that*

- *at each point $p \in S$, $J_p \in \mathcal{J}_{a_S(p)}(M)$,*
- *at each cylinder or strip κ with associated weight ν_κ,*

$$\kappa^* J_S = (\psi^{\nu_\kappa})^* J_t. \tag{4.9}$$

  Here, if κ is a strip, we mean the $[0,1]$ dependent complex structure given by pulling back J_t by the projection map $[0,1] \to S^1 = \mathbb{R}/\mathbb{Z}$.

We will need to introduce Hamiltonian perturbation terms supported on the cylinders of $(S, \mathfrak{S})$, in order to break the S^1 symmetry of orbits. Since we will be gluing nodal cylindrical punctures together, these perturbation terms need to possibly have support on the thin-parts of our gluing as well. The next definition will give us very explicit control over these perturbation terms. Let $F_T : S^1 \to C^\infty(E)$ be a time-dependent function that is absolutely bounded, with all derivatives absolutely bounded. Also, let $\phi_\epsilon(s) : [0,1] \to [0,1]$ be a smooth function that is 0 in an ϵ-neighborhood of 0, 1 in an ϵ-neighborhood of 1, with all derivatives bounded.

Definition 4.10. *For $(S, \mathfrak{S})$ as above, an* **S^1-perturbation adapted to** (F_T, ϕ_ϵ) *is a function $F_S : S \to C^\infty(E)$ satisfying the following properties:*

- *F_S is locally constant on the complement of the images of all cylinders,*
- *on each cylindrical end $\kappa^\pm$ with associated weight ν_κ, outside the associated collar,*

$$(\kappa^\pm)^* F_S = \frac{F_T \circ \psi^{\nu_\kappa}}{\nu_\kappa^2} + C_\kappa, \tag{4.10}$$

  where C_κ is a constant depending on the cylinder $\kappa^\pm$.
- *on each finite cylinder κ^r, outside the associated collar,*

$$(\kappa^r)^* F_S = m_\kappa \frac{F_T \circ \psi^{\nu_\kappa}}{\nu_\kappa^2} + C_\kappa, \tag{4.11}$$

  where C_κ and m_κ are constants depending on the cylinder κ^r.
- *On each associated δ-collar, $\kappa : [0, \delta] \times S^1 \to S$,*

$$\kappa^* F_S = (\kappa^* F_S)|_{0 \times S^1} + \phi_\epsilon(s/\delta)((\kappa^* F_S)|_{\delta \times S^1} - (\kappa^* F_S)|_{0 \times S^1}) \tag{4.12}$$

- *F_S is weakly monotonic on each cylinder κ, i.e.*

$$\partial_s \kappa^* F_S \leq 0. \tag{4.13}$$

Putting all of these together, we can make the following definition:

Definition 4.11. *A* **Floer datum $\mathbf{F}_S$** *on a stable genus zero open-closed string string S consists of the following choices on each component:*

(1) *A collection of* **weighted strip and cylinder data** $\mathfrak{S}$ *that is δ-bounded;*
(2) **sub-closed 1-form***: a one-form α_S with*
$$d\alpha_S \leq 0,$$
compatible with the weighted strip and cylinder data;
(3) *A* **primary Hamiltonian** $H_S : S \to \mathcal{H}(M)$ *that is H-compatible with the weighted strip and cylinder data $\mathfrak{S}$ for some fixed H;*
(4) *An* **$\mathfrak{S}$-adapted rescaling function** a_S*;*
(5) *An* **almost-complex structure** J_S *that is $(\mathfrak{S}, a_S, J_t)$-adapted for some J_t.*
(6) *An* **S^1-perturbation F_S adapted to** (F_T, ϕ_ϵ) *for some F_T, ϕ_ϵ as above.*

There is a notion of equivalence of Floer data, weaker than strict equality, which will imply by the rescaling correspondence (4.3) that the resulting operations are identical.

Definition 4.12. *Say that Floer data D_S^1 and D_S^2 are* **conformally equivalent** *if there exist constants C, K, K' such that*

$$\begin{aligned} a_S^2 &= C \cdot a_S^1, \\ \alpha_S^2 &= C \cdot \alpha_S^1, \\ J_S^2 &= (\psi^C)^* J_S^1, \\ H_S^2 &= \frac{H_S^1 \circ \psi^C}{C^2} + K, \text{ and} \\ F_S^2 &= \frac{F_S^1 \circ \psi^C}{C^2} + K'. \end{aligned} \tag{4.14}$$

In other words, the Floer D_S^2 is a rescaling by Liouville flow of the Floer data D_S^1, up to a constant ambiguity in the Hamiltonian terms.

Definition 4.13. *A* **universal and consistent choice of Floer data** *for genus 0 open-closed strings is a choice $\mathbf{D_S}$ of Floer data for every h, n, $\vec{m}$, $\mathbf{I}$, $\vec{\mathbf{K}}$ and every representative S of $\overline{\mathcal{N}}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}}$, varying smoothly over $\mathcal{N}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}}$, whose restriction to a boundary stratum is conformally equivalent ot the product of Floer data coming from lower dimensional moduli spaces. Moreover, with regards to the coordinates, Floer data agree to infinite order at the boundary stratum with the Floer data obtained by gluing.*

Remark 4.3. *By varying smoothly, we mean that the data of H_S, F_S, a_S, α_S, and J_S, along with the cylindrical and strip-like ends vary smoothly. Over given charts of our moduli space, finite cylinders and strips need to vary smoothly as well, but they may be different across charts (for example, some charts that stay away from lower-dimensional strata may have no finite cylinder or strip-like regions).*

All of the choices involved in the definition of a Floer datum above are contractible, so one can inductively over strata prove that

Lemma 4.2. *The restriction map from the space of universal and consistent Floer data to the space of Floer data for a fixed surface S is surjective.*

Definition 4.14. *Let $\mathbf{L}$ be a set of Lagrangians. A* **Lagrangian labeling from L** *for a genus-0 open-closed string $S \in \mathcal{N}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}}$ is a choice, for each $j = 1, \ldots, h$ and for each connected component $\partial_i^j S$ of the jth boundary disc, of a Lagrangian $L_i^j \in \mathbf{L}$. The* **space of genus-0 open-closed strings with a fixed labeling** $\vec{L} = \{\{L_i^j\}_i\}_j$ *is denoted $(\overline{\mathcal{N}}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}})_{\vec{L}}$. The* **space of all labeled open-closed strings** *is denoted $(\overline{\mathcal{N}}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}})_{\mathbf{L}}$.*

Clearly, $(\overline{\mathcal{N}}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}})_{\mathbf{L}}$ is a disconnected cover of $\overline{\mathcal{N}}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}}$. There is a notion of a **labeled Floer datum**, namely a Floer datum for the space of open-closed strings equipped with labels $(\overline{\mathcal{N}}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}})_{\mathbf{L}}$. This is simply a choice of Floer data as above in a manner also depending coherently on the particular Lagrangian labels. We will use this notion in later sections, along with the following definition.

DEFINITION 4.15. *Let $\mathbf{D}_S$ be a Floer datum on a surface S. The* **induced labeled Floer datum** *on a labeled surface $S_{\vec{L}}$ is the Floer datum $\mathbf{D}_S$ coming from forgetting the labels.*

4.3. Chain-level operations. Now, fix a compact oriented submanifold with corners of dimension d,

$$\overline{\mathcal{Q}}^d \hookrightarrow \overline{\mathcal{N}}^{\mathbf{I},\vec{\mathbf{K}}}_{h,n,\vec{m}}. \tag{4.15}$$

Fix a Lagrangian labeling

$$\{\{L_0^1, \ldots, L_{m_1}^1\}, \{L_0^2, \ldots, L_{m_2}^2\}, \ldots, \{L_0^h, \ldots, L_{m_h}^h\}\}. \tag{4.16}$$

Also, fix chords

$$\vec{x} = \{\{x_1^1, \ldots, x_{m_1}^1\}, \ldots, \{x_1^h, \ldots, x_{m_h}^h\}\} \tag{4.17}$$

and orbits $\vec{y} = \{y_1, \ldots, y_n\}$ with

$$x_i^j \in \begin{cases} \chi(L_{i+1}^j, L_i^j) & i \in K^j \\ \chi(L_i^j, L_{i+1}^j) & \text{otherwise.} \end{cases} \tag{4.18}$$

Above, the index i in L_i^j is counted mod m_j. The **outputs** $\vec{x}_{out}$, $\vec{y}_{out}$ are by definition those x_i^j and y_s for which $i \in K^j$ and $s \in \mathbf{I}$, corresponding to negative marked points. The **inputs** $\vec{x}_{in}$, $\vec{y}_{in}$ are the remaining chords and orbits from $\vec{x}$, $\vec{y}$. Fixing a chosen universal and consistent Floer datum, denote $\epsilon_{\pm}^{i,j}$ and $\delta_{\pm}^l$ the strip-like and cylindrical ends corresponding to x_i^j and y_l respectively.

Define

$$\overline{\mathcal{Q}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in}) \tag{4.19}$$

to be the space of maps

$$\{u : S \longrightarrow M : \ S \in \overline{\mathcal{Q}}^d\} \tag{4.20}$$

satisfying the inhomogenous Cauchy-Riemann equation with respect to the complex structure J_S:

$$(du - X_S \otimes \alpha_S)^{0,1} = 0 \tag{4.21}$$

and asymptotic and boundary conditions:

$$\begin{cases} \lim_{s \to \pm\infty} u \circ \epsilon_{\pm}^{i,j}(s, \cdot) = x_i^j, \\ \lim_{s \to \pm\infty} u \circ \delta_{\pm}^l(s, \cdot) = y_l, \\ \quad u(z) \in \psi^{a_S(z)} L_i^j, \qquad\qquad z \in \partial_i^j S. \end{cases} \tag{4.22}$$

Above, X_S is the (surface-dependent) Hamiltonian vector field corresponding to $H_S + F_S$.

LEMMA 4.3. *The moduli spaces $\overline{\mathcal{Q}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ are compact and there are only finitely many collections $\vec{x}_{out}$, $\vec{y}_{out}$ for which they are non-empty given input $\vec{x}_{in}$, $\vec{y}_{in}$. For a generic universal and conformally consistent Floer data they form manifolds of dimension*

$$\begin{aligned} \dim \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in}) := &\sum_{x_- \in \vec{x}_{out}} \deg(x_-) + \sum_{y_- \in \vec{y}_{out}} \deg(y_-) + \\ &(2 - h - |\vec{x}_{out}| - 2|\vec{y}_{out}|)n + d - \sum_{x_+ \in \vec{x}_{in}} \deg(x_+) - \sum_{y_+ \in \vec{y}_{in}} \deg(y_+). \end{aligned} \tag{4.23}$$

PROOF. The dimension calculation follows from a computation of the index of the associated linearized Fredholm operator. Via a gluing theorem for indices [**S5**, (11c)] [**Sc**, Thm. 3.2.12], there is a contribution coming from the index of the linearized Cauchy Riemann operator on the compactified surface $\hat{S}$, equal to $n\chi(\hat{S})$, where $\chi(\hat{S}) = (2 - h)$ is the Euler characteristic of a genus-0 open-closed string of type h. The other contributions come from the tangent space of $\mathcal{Q}$ (contributing d), and spectral-flow type calculations on the striplike and cylindrical ends. This calculation is essentially a fusion of [**S5**, Proposition 11.13] and [**R**, Lemma 10].

The proof of transversality for generic perturbation data is a standard application of Sard-Smale, following identical arguments in [**S5**, (9k)] or alternatively [**FHS**]. The usual proof Gromov compactness also applies, assuming that solutions to Floer's equation with given asymptotic boundary conditions are a priori bounded in the non-compact target M. This is the content of Theorem B.1. $\square$

When $\mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ has dimension zero, we conclude that its elements are rigid. For any such element $u \in \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$, we obtain an isomorphism of orientation lines, by Lemma A.1

$$\mathcal{Q}_u : \bigotimes_{x \in \vec{x}_{in}} o_x \otimes \bigotimes_{y \in \vec{y}_{in}} o_y \longrightarrow \bigotimes_{x \in \vec{x}_{out}} o_x \otimes \bigotimes_{y \in \vec{y}_{out}} o_y. \tag{4.24}$$

Thus, we can define a map

$$\begin{aligned} \mathbf{F}_{\overline{\mathcal{Q}}^d} : \bigotimes_{(i,j);1 \le i \le m_j; i \notin K^j} CW^*(L_i^j, L_{i+1}^j) \otimes \bigotimes_{1 \le k \le n; k \notin \mathbf{I}} CH^*(M) \longrightarrow \\ \bigotimes_{(i,j);1 \le i \le m_j; i \in K^j} CW^*(L_{i+1}^j, L_i^j) \otimes \bigotimes_{1 \le k \le n; k \in \mathbf{I}} CH^*(M) \end{aligned} \tag{4.25}$$

given by:

$$\begin{aligned} &\mathbf{F}_{\overline{\mathcal{Q}}^d}([y_t], \ldots, [y_1], [x_s], \ldots, [x_1]) := \\ &\sum_{\dim \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \{x_1, \ldots, x_s\}, \{y_1, \ldots, y_t\}) = 0} \;\; \sum_{u \in \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \{x_1, \ldots, x_s\}, \{y_1, \ldots, y_t\})} \mathcal{Q}_u([x_s], \ldots, [x_1], [y_t], \ldots, [y_1]). \end{aligned} \tag{4.26}$$

This construction naturally associates, to any submanifold $\mathcal{Q}^d \in \mathcal{N}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}}$, a chain-level map $\mathbf{F}_{\mathcal{Q}^d}$, depending on a sufficiently generic choice of Floer data for open-closed strings. We need to modify this construction by signs depending on the relative positions and degrees of the inputs.

Definition 4.16. *Given such a submanifold $\mathcal{Q}$, a* **sign twisting datum** $\vec{t}$ *for $\mathcal{Q}$ is a vector of integers, one for each input boundary or interior marked point on an element of $\mathcal{Q}$.*

To a pair $(\mathcal{Q}, \vec{t})$ one can associate a twisted operation

$$(-1)^{\vec{t}} \mathbf{F}_{\overline{\mathcal{Q}}^d}, \tag{4.27}$$

defined as follows. If $\{\vec{x}, \vec{y}\} = \{x_1, \ldots, x_s, y_1, \ldots, y_t\}$ is a set of asymptotic inputs, the **vector of degrees** is denoted

$$\vec{\deg}(\vec{x}, \vec{y}) := \{\{\deg(x_1), \ldots, \deg(x_s)\}, \{\deg(y_1), \ldots, \deg(y_t)\}\}. \tag{4.28}$$

The corresponding sign twisting datum $\vec{t}$ is of the form

$$\vec{t} := \{\{v_1, \ldots, v_s\}; \{w_1, \ldots, w_t\}\}. \tag{4.29}$$

Then, the operation (4.27) is defined to be

$$\begin{aligned} &(-1)^{\vec{t}} \mathbf{F}_{\overline{\mathcal{Q}}^d}([y_t], \ldots, [y_1], [x_s], \ldots, [x_1]) := \\ &\sum_{\dim \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}, \vec{y}) = 0} \;\; \sum_{u \in \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out};, \vec{x}, \vec{y})} (-1)^{\vec{t} \cdot \vec{\deg}(\vec{x}, \vec{y})} \mathcal{Q}_u([x_1], \ldots, [x_s], [y_1], \ldots, [y_t]). \end{aligned} \tag{4.30}$$

The zero vector $\vec{t} = (0, \ldots, 0)$ recovers the original operation $\mathbf{F}_{\overline{\mathcal{Q}}^d}$.

Now, suppose instead that we are given a submanifold $\mathcal{Q}_{\vec{L}}^d$ of the labeled space $(\overline{\mathcal{N}}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}})_{\vec{L}}$. Then, we obtain a chain-level operation

$$(-1)^{\vec{t}} \mathbf{F}_{\mathcal{Q}_{\vec{L}}^d} \tag{4.31}$$

that *is only defined for the fixed labeling $\vec{L}$*. In the sections that follow, we will use this definition to construct associated chain-level map for specific families $\{\mathcal{Q}_{\vec{L}}^d\}_d$.

Remark 4.4. *Strictly speaking, when there are two boundary outputs on the same component, one only obtains the isomorphism of orientation lines (4.24) after choosing orientations of Lagrangians along that boundary component. Since we are working with oriented Lagrangians, we are implicitly making such choices. See Appendix A for more details.*

In a different direction, we will make repeated use of the following standard *codimension 1 boundary principle* for Floer-theoretic operations: suppose that the boundary $\partial\overline{\mathcal{Q}}^d$ is covered by the images of natural inclusions of $(d-1)$-dimensional orientable submanifolds

$$\mathcal{T}_i \hookrightarrow \partial\overline{\mathcal{Q}}^d,\ i = 1, \ldots, k. \tag{4.32}$$

Then, standard results tell us that

Lemma 4.4. *In the situation above, the Gromov bordification of the moduli space of maps* $\overline{\mathcal{Q}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ *has codimension 1 boundary covered by the images of natural inclusions of the following spaces:*

$$\mathcal{T}_i(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in}),\ i = 1, \ldots, k \tag{4.33}$$

$$\overline{\mathcal{R}}(\tilde{x}; x_a) \times \overline{\mathcal{Q}}^d(\vec{x}_{out}, \vec{y}_{out}; \tilde{\vec{x}}_{in}, \vec{y}_{in}) \tag{4.34}$$

$$\overline{\mathcal{M}}(\tilde{y}; y_b) \times \overline{\mathcal{Q}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \tilde{\vec{y}}_{in}) \tag{4.35}$$

$$\overline{\mathcal{Q}}^d(\tilde{\vec{x}}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in}) \times \overline{\mathcal{R}}(x_c; \tilde{x}) \tag{4.36}$$

$$\overline{\mathcal{Q}}^d(\vec{x}_{out}, \tilde{\vec{y}}_{out}; \vec{x}_{in}, \vec{y}_{in}) \times \overline{\mathcal{M}}(y_d; \tilde{y}). \tag{4.37}$$

Here,

- *in (4.34), $x_a \in \vec{x}_{in}$ and $\tilde{\vec{x}}_{in}$ is $\vec{x}_{out}$ with the element x_a replaced by $\tilde{x}$;*
- *in (4.35), $y_b \in \vec{y}_{in}$ and $\tilde{\vec{y}}_{in}$ is $\vec{y}_{in}$ with the element y_b replaced by $\tilde{y}$;*
- *in (4.36), $x_c \in \vec{x}_{out}$ and $\tilde{\vec{x}}_{out}$ is $\vec{x}_{out}$ with the element x_c replaced by $\tilde{x}$; and*
- *in (4.37), $y_d \in \vec{y}_{out}$ and $\tilde{\vec{y}}_{out}$ is $\vec{y}_{out}$ with the element y_d replaced by $\tilde{y}$.*

The strata (4.34) - (4.37) range over all $\tilde{x}$, $\tilde{y}$ and all possible choices of x_a, y_b, x_c, x_d.

In words, this Lemma says that the boundary of space of maps from $\mathcal{Q}$ is covered by maps from the various $\mathcal{T}_i$ plus all possible semi-stable strip or cylinder breakings.

The manifolds $\mathcal{T}_i$, which may live on the boundary strata of $\overline{\mathcal{N}}^{\mathbf{I},\vec{\mathbf{K}}}_{h,n,\vec{m}}$, inherit orientations and Floer data from the choice of $\mathcal{Q}^d$, via the convention of orienting relative to the normal vector pointing towards the boundary. Thus, there are associated signed operations

$$(-1)^{\vec{t}}\mathbf{F}_{\mathcal{T}_i}. \tag{4.38}$$

By looking at the boundary of one-dimensional elements of the moduli space $\mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$, one concludes that

Corollary 4.1. *In the situation described above, for any $\vec{t}$,*

$$\begin{aligned}&\sum_{i=1}^{k}(-1)^{\vec{t}}\mathbf{F}_{\mathcal{T}_i} + \sum_{i=1}^{s+t}(-1)^{*}(-1)^{\vec{t}}\mathbf{F}_{\overline{\mathcal{Q}}^d} \circ (id \otimes \cdots \otimes id \otimes \mu^1 \otimes id \otimes \cdots id)\\ &\quad + \sum_{j}(-1)^{\dagger}(id \otimes \cdots id \otimes \mu^1 \otimes id \otimes \cdots id) \circ \left((-1)^{\vec{t}}\mathbf{F}_{\overline{\mathcal{Q}}^d}\right) = 0,\end{aligned} \tag{4.39}$$

where we have used μ^1 to indicate the both the differential on wrapped Floer homology or symplectic cohomology depending on the input. The signs $(-1)^*$ and $(-1)^\dagger$ will be calculated in Appendix A.

In order to obtain equations such as the A_∞ equations, bimodule equations, various morphisms are chain homotopies, etc. with the correct (Koszul) signs, one needs to compare signs between the operators $(-1)^{\vec{t}}\mathbf{F}_{\mathcal{T}_i}$ and the composition of operators arising from $\mathcal{T}_i$ viewed as a families of potentially nodal surface using the consistency condition imposed on our Floer data. This, plus appropriate choices of sign twisting data for these strata, will yield all of the relevant signs. The relevant calculations are performed in Appendix A.

4.4. The product in symplectic cohomology. Symplectic cohomology is known to admit a range of TQFT-like operations, coming from surfaces with I incoming and J outgoing ends, for $J > 0$, see e.g. [**R**]. In this section, we will recall the case of a genus-0 surface with $I = 2$, $J = 1$, which gives the *pair of pants product.* More generally, one consequence of later sections is the construction of operations associated to certain *parametrized families* of such surfaces for $I > 2$.

Denote by $\mathcal{S}_2$ the configuration space of spheres with two positive and one negative punctures, with asympotic markers pointing in the tangent direction to the unique great circle containing all three points.

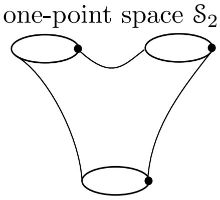
Definition 4.17. *A* **Floer datum** D_T *on a stable sphere* $T \in \mathcal{S}_{2,1}$ *consists of a Floer datum of* T *thought of as an open-closed string.*

From the previous section, considering the space $\mathcal{S}_{2,1}$ as a maximal submanifold of itself defines an operation of degree zero

$$\mathcal{F}_2 : CH^*(M)^{\otimes 2} \longrightarrow CH^*(M). \tag{4.40}$$

This is known as the *pair of pants product.*

Figure 2. A representative of the one-point space $\mathcal{S}_{2,1}$ giving the pair of pants product.



Remark 4.5. *For arbitrary families of spheres with more than two inputs, there ceases to be a preferred direction in which to point the asymptotic markers, a situation which will not be considered in our work. However, if one mandates that all marked points lie on a single great circle, then one recovers the required preferred direction. The end result, the Massey products on* $SH^*(M)$*, will be constructed as a special case of the discussion in Sections 5 and 6.*

4.5. A_∞ **structure maps and the wrapped Fukaya category.** Here we define the higher structure maps μ^k on $\mathcal{W}$ (including the product μ^2). We will recall and apply with some detail our construction of Floer-theoretic operations from Section 4.3, though the reader is warned that subsequent constructions will be more terse.

Define

$$\mathcal{R}^d \tag{4.41}$$

to be the (Stasheff) moduli space of discs with one negative marked point z_0^- and d positive marked points $z_1^+, \dots, z_d^+$ removed from the boundary, labeled in *counterclockwise* order from z_0^-. $\mathcal{R}^d$ is a special case of our general construction of open-closed strings. Denote by $\overline{\mathcal{R}}^d$ its natural (Deligne-Mumford) compactification, consisting of trees of stable discs with a a total of d exterior positive marked points and 1 exterior negative marked point, modulo compatible reparametrization of each disc in the tree. Recall from the discussion in Section 4 that $\overline{\mathcal{R}}^d$ inherits the structure of a manifold with corners, coming from standard gluing charts

$$(0, +\infty]^k \times \sigma \to \overline{\mathcal{R}}^d. \tag{4.42}$$

near (nodal) strata of codimension k.

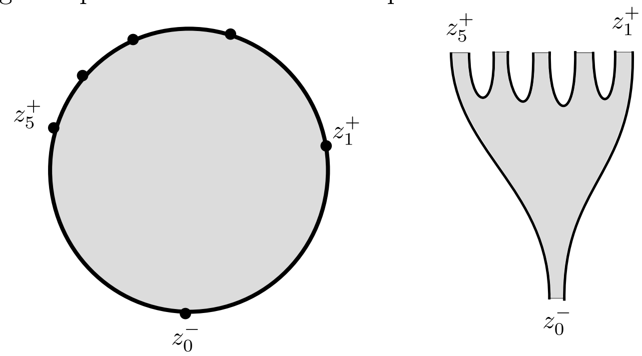
Now, in the terminology of Definition 4.11, pick a universal and consistent choice of Floer data $\mathbf{D}_\mu$ for the spaces $\mathcal{R}^d$, $d > 2$. Also, fix an orientation of the space $\mathcal{R}^d$, discussed in Appendix A.3.1.

Definition 4.18. *The* d**th order** A_∞ **operation** *is by definition the operation*

$$\mu^d := (-1)^{\vec{t}} \mathbf{F}_{\overline{\mathcal{R}}^d}, \tag{4.43}$$

in the sense of (4.27), where $\vec{t}$ *is the sign twisting datum given by* $(1, 2, \dots, d)$.

FIGURE 3. Two drawings of a representative of an element of the moduli space $\mathcal{R}^5$. The drawing on the right emphasizes the choices of strip-like ends.



We step through this construction for clarity. Let $L_0, \ldots, L_d$ be objects of $\mathcal{W}$, and consider a sequence of chords $\vec{x} = \{x_k \in \chi(L_{k-1}, L_k)\}$ as well as another chord $x_0 \in \chi(L_0, L_d)$. Given a fixed universal and consistent Floer data $\mathbf{D}_\mu$, write $\mathcal{R}^d(x_0; \vec{x})$ for the space of maps

$$u : S \to M$$

with source an arbitrary element $S \in \mathcal{R}^d$, with marked points $(z^0, \ldots, z^d)$ satisfying the boundary asymptotic conditions

$$\begin{cases} u(z) \in \psi^{a_S(z)} L_k & \text{if } z \in \partial S \text{ lies between } z^k \text{ and } z^{k+1} \\ \lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = \psi^{a_S(z)} x_k \end{cases} \tag{4.44}$$

and differential equation

$$(du - X_S \otimes \alpha_S)^{0,1} = 0 \tag{4.45}$$

with respect to the complex structure J_S and total Hamiltonian $H_S + F_S$. Using the consistency of our Floer data and the codimension one boundary of the abstract moduli spaces $\overline{\mathcal{R}}^d$, Lemma 4.4 implies that the Gromov bordification $\overline{\mathcal{R}}^d(x_0; \vec{x})$ is obtained by adding the images of the natural inclusions

$$\overline{\mathcal{R}}^{d_1}(x_0; \vec{x}_1) \times \overline{\mathcal{R}}^{d_2}(y; \vec{x}_2) \to \overline{\mathcal{R}}^d(x_0; \vec{x}) \tag{4.46}$$

where y agrees with one of the elements of $\vec{x}_1$ and $\vec{x}$ is obtained by removing y from $\vec{x}_1$ and replacing it with the sequence $\vec{x}_2$. Here, we let d_1 range from 1 to d, with $d_2 = d - d_1 + 1$, with the stipulation that $d_1 =$ or $d_2 = 1$ is the semistable case:

$$\overline{\mathcal{R}}^1(x_0; x_1) := \overline{\mathcal{R}}(x_0; x_1) \tag{4.47}$$

Thanks to Lemma 4.3, for generically chosen Floer data $\mathbf{D}_\mu$

COROLLARY 4.2. *The moduli spaces* $\overline{\mathcal{R}}^d(x_0; \vec{x})$ *are smooth compact manifolds of dimension*

$$\deg(x_0) + d - 2 - \sum_{1 \le k \le d} \deg(x_k).$$

In particular, if $\deg(x_0) = 2 - d + \sum_1^d \deg(x_k)$, then the elements of $\overline{\mathcal{R}}^d(x_0; \vec{x})$ are rigid, and for any such rigid $u \in \overline{\mathcal{R}}^d(x_0; \vec{x})$, we obtain by Lemma A.1, an isomorphism

$$\mathcal{R}_u^d : o_{x_d} \otimes \cdots \otimes o_{x_1} \longrightarrow o_{x_0}. \tag{4.48}$$

Thus, taking into account the sign twisting $\vec{t}$, we define the operation

$$\mu^d : CW^*(L_{d-1}, L_d) \otimes \cdots \otimes CW^*(L_0, L_1) \longrightarrow CW^*(L_0, L_d) \tag{4.49}$$

as a sum

$$\mu^d([x_d],\dots,[x_1]) := \sum_{\deg(x_0)=2-d+\sum\deg(x_k)} \sum_{u\in\overline{\mathcal{R}}^d(x_0;\vec{x})} (-1)^{\bigstar_d}\mathcal{R}^d_u([x_d],\dots,[x_1]) \tag{4.50}$$

where

$$\bigstar_d = \vec{t}\cdot\vec{\deg}(\vec{x}) := \sum_{i=1}^d i\cdot\deg(x_i). \tag{4.51}$$

By looking at the codimension 1 boundary of 1-dimensional families of such maps, and performing a tedious sign comparison via the algorithm outlined in Appendix A (this case is explicitly written out in [**S5**, Prop. 12.3]), one concludes that

LEMMA 4.5. *The maps μ^d satisfy the A_∞ relations.*

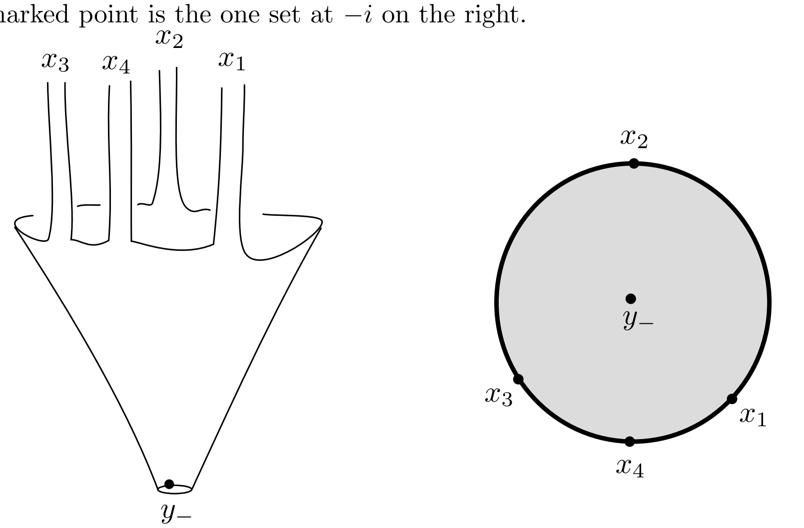
The sign twisting datum used here will reappear with variations later, so it is convenient to fix notation.

DEFINITION 4.19. *The* **incremental sign twisting datum of length** *d, denoted $\vec{t}_d$, is the vector $(1,2,\dots,d-1,d)$.*

4.6. Open-closed and closed-open maps. In this section, we take a digression and review two examples of open-closed maps between symplectic cohomology and Hochschild invariants of the Fukaya category. In Section 2.7 we defined two quasi-isomorphic complexes that one could use to define each Hochschild invariant (there are of course, many others); in this section, we will define geometric operations associated to each complex.

As in [**A1**], define $\mathcal{R}^1_d$ to be the abstract moduli space of discs with d boundary positive punctures $z_1,\dots,z_d$ labeled in counterclockwise order and 1 interior negative puncture z_{out}, with the last positive puncture x_d marked as distinguished (implicitly z_{out} is also equipped with an asymptotic marker, constrained to point at the distinguished point x_d). Its Deligne-Mumford compactification inherits the structure of a manifold with corners via the inclusion $\overline{\mathcal{R}}^1_d \hookrightarrow \overline{\mathcal{N}}^{\mathbf{I},\vec{\mathbf{K}}}_{h,n,\vec{m}}$ where $h=1$, $n=1$, $\vec{m}=(d)$, $\mathbf{I}=\{1\}$, $\vec{\mathbf{K}}=(\{\})$.

FIGURE 4. Two drawings of representative of an element of the moduli space $\mathcal{R}^1_4$. The drawing on the left emphasizes the choices of strip-like and cylindrical ends. The distinguished boundary marked point is the one set at $-i$ on the right.



In the manner of (4.19), using our fixed generic universal and consistent Floer data and an orientation for $\mathcal{R}^1_d$ fixed in Appendix A.3.2, we obtain, for every Lagrangian labeling $L_1,\dots,L_d$, and asymptotic conditions $\{x_1,\dots,x_d,y_{out}\}$ moduli spaces

$$\overline{\mathcal{R}^1}_d(y_{out};\{x_i\}) \tag{4.52}$$

which are compact smooth manifolds of dimension

$$\deg(y_{out}) - n + d - 1 - \sum_{k=0}^{d-1} \deg(x_k). \tag{4.53}$$

Then, fixing sign twisting datum

$$\vec{t}_{\mathcal{OC},d} := (1, 2, \dots, d-1, d+1) = \vec{t}_d + (0, \dots, 0, 1), \tag{4.54}$$

we obtain associated operations

$$\begin{aligned} \mathcal{OC}_d :=& (-1)^{\vec{t}_{\mathcal{OC},d}} \mathbf{F}_{\mathcal{R}^1_d} : \\ & \hom_{\mathcal{W}}(L_d, L_0) \otimes \hom_{\mathcal{W}}(L_{d-1}, L_d) \otimes \cdots \otimes \hom_{\mathcal{W}}(L_0, L_1) \longrightarrow CH^*(M); \end{aligned} \tag{4.55}$$

in other words, operations

$$\mathcal{OC}_d : (\mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes d-1})^{diag} \longrightarrow CH^*(M) \tag{4.56}$$

of degree $n - d + 1$. The composite map

$$\mathcal{OC} := \sum_d \mathcal{OC}_d \tag{4.57}$$

therefore gives a map

$$\mathcal{OC} : \mathrm{CC}_*(\mathcal{W}, \mathcal{W}) \longrightarrow CH^*(M) \tag{4.58}$$

of degree n (using the grading conventions for Hochschild homology (2.140). Recall that the codimension-1 boundary of the Deligne-Mumford compactification $\overline{\mathcal{R}}^1_d$ is covered by the following strata:

$$\overline{\mathcal{R}^m} \times_i \overline{\mathcal{R}}^1_{d-m+1} \quad 1 \leq i < d - m + 1 \tag{4.59}$$

$$\overline{\mathcal{R}^m} \times_{d-m+1} \overline{\mathcal{R}}^1_{d-m+1} \quad 1 \leq j \leq m \tag{4.60}$$

where the notation $\times_j$ means that the output of the first component is identified with the jth boundary input of the second. In the second type of stratum (4.60), the jth copy correspond to the stratum in which the jth input point on $\mathcal{R}^m$ becomes the distinguished boundary marked point on $\mathcal{R}^1_d$ after gluing.

The consistency condition imposed on Floer data implies that the Gromov bordification $\overline{\mathcal{R}}^1_d(y_0, \vec{x})$ is obtained by adding the images of natural inclusions of moduli spaces of maps coming from the boundary strata (4.59)-(4.60) along with the following semi-stable breakings

$$\overline{\mathcal{R}}^1_d(y_1, \vec{x}) \times \overline{\mathcal{M}}(y_{out}; y_1) \to \partial \overline{\mathcal{R}}^1_d(y_{out}; \vec{x}) \tag{4.61}$$

$$\overline{\mathcal{R}}^1(x_1; x) \times \overline{\mathcal{R}}^1_d(y_{out}; \tilde{\vec{x}}) \to \partial \overline{\mathcal{R}}^1_d(y_{out}; \vec{x}) \tag{4.62}$$

where in the second type of stratum, x is one of the elements of $\vec{x}$ and $\tilde{\vec{x}}$ is the sequence obtained by replacing x in $\vec{x}$ by x_1. Along with a sign comparison, this implies

Proposition 4.1. [**A1**, Lemma 5.4]] $\mathcal{OC}$ *is a chain map.*

In a similar fashion, define $\mathcal{R}^{1,1}_d$ to be the moduli space of discs with

- $d + 1$ boundary marked points removed, 1 of which is negative and labeled z_0^-, and d of which are positive and labeled $(z_1, \dots, z_d)$ in counterclockwise order from z_0^-; and
- one interior positive marked point y_{in} removed (again, with an asymptotic marker which is constrained to point at the distinguished point z_0^-).

Its Deligne-Mumford compactification inherits the structure of a manifold with corners via the inclusion $\overline{\mathcal{R}}^{1,1}_d \hookrightarrow \overline{\mathcal{N}}^{\mathbf{I},\vec{\mathbf{K}}}_{h,n,\vec{m}}$, where $h = 1$, $n = 1$, $\vec{m} = (d + 1)$, $\mathbf{I} = \{\}$, $\vec{\mathbf{K}} = (\{1\})$. Thus, let us fix a universal and conformally consistent choice of Floer datum $\mathbf{D}_{\mathcal{CO}}$ on $\mathcal{R}^{1,1}_d$ for every $d \geq 1$. Given a Lagrangian labeling and a set of compatible asymptotic conditions, along with an orientation of $\mathcal{R}_d^{1,1}$ discussed in Appendix A.3.3, we obtain the moduli space $\mathcal{R}_d^{1,1}(x_{out}; y_{in}, \vec{x})$ as in (4.19), which are compact smooth manifolds of dimension

$$\deg(x_{out}) + d - \deg(y_{in}) - \sum_{k=1}^{d} \deg(x_k). \tag{4.63}$$

Fix sign twisting datum

$$\vec{t}_{\mathcal{CO},d} = (0, 1, 2, \dots, d) \tag{4.64}$$

with respect to the ordering of inputs $(y_{in}, x_1, \dots, x_d)$. Then, define

$$\mathcal{CO}_d : CH^*(M) \longrightarrow \hom_{Vect}(\mathcal{W}^{\otimes d}, \mathcal{W}) \tag{4.65}$$

as

$$\mathcal{CO}_d(y_{in})(x_d, \dots, x_1) := (-1)^{\vec{t}_{\mathcal{CO},d}} \mathbf{F}_{\overline{\mathcal{R}}_d^{1,1}}(y_{in}, x_d, \dots, x_1) \tag{4.66}$$

The composite map

$$\mathcal{CO} = \sum_d \mathcal{CO}_d \tag{4.67}$$

gives a map $CH^*(M) \to \mathrm{CC}^*(\mathcal{W}, \mathcal{W})$.

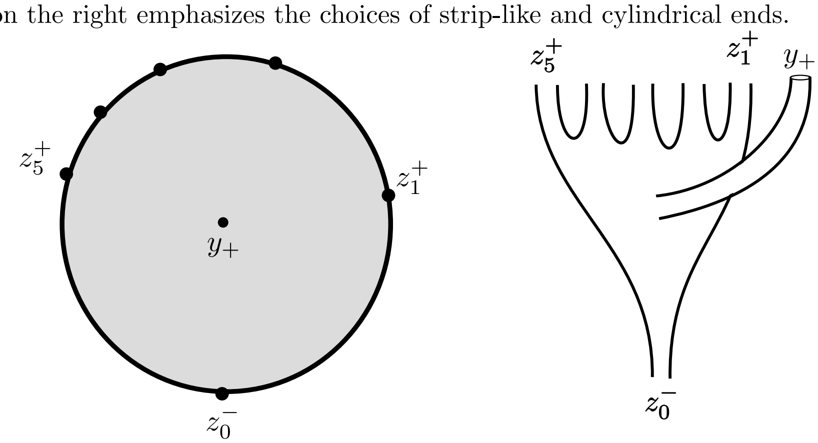
The codimension-1 boundary of the Deligne-Mumford compactification $\overline{\mathcal{R}}_d^{1,1}$ is covered by the natural images of the following products:

$$\overline{\mathcal{R}}^m \times_i \overline{\mathcal{R}}_{d-m+1}^{1,1} \quad 1 < m < d-1,\ 1 \leq i < d-m+1 \tag{4.68}$$

$$\overline{\mathcal{R}}_{d-(k+l+1)+1}^{1,1} \times_{k+1} \overline{\mathcal{R}}^{k+l+1} \quad 0 < k+l < d-1. \tag{4.69}$$

using the notation $\times_j$ as in the last section to indicate gluing the distinguished output of the first component to the jth (boundary) input of the second component.

FIGURE 5. Two drawings of representative of an element of the moduli space $\mathcal{R}_5^{1,1}$. The drawing on the right emphasizes the choices of strip-like and cylindrical ends.



The consistency condition implies that the Gromov bordification $\overline{\mathcal{R}}_d^{1,1}(x_{out}; y_{in}, \vec{x})$ is obtained by adding images of the natural inclusions

$$\mathcal{M}(y_1; y_{in}) \times \mathcal{R}_d^{1,1}(x_{out}; y_1, \vec{x}) \to \partial\overline{\mathcal{R}}_d^{1,1}(x_{out}; y_{in}, \vec{x}) \tag{4.70}$$

$$\mathcal{R}_{d_2}^{1,1}(x^a; y_{in}, \vec{x}^2) \times \mathcal{R}^{d_1}(x_{out}, \vec{x}^1) \to \partial\overline{\mathcal{R}}_d^{1,1}(x_{out}; y_{in}, \vec{x}) \tag{4.71}$$

$$\mathcal{R}^{d_2}(x^a; \vec{x}^2) \times \mathcal{R}_{d_1}^{1,1}(x_{out}; y_{in}, \vec{x}^1) \to \partial\overline{\mathcal{R}}_d^{1,1}(x_{out}; y_{in}, \vec{x}) \tag{4.72}$$

where $d_1 + d_2 - 1 = d$, $\vec{x}^2$ is any consecutive sub-vector of size d_2 and $\vec{x}^1$ is obtained by replacing $\vec{x}^2$ in $\vec{x}$ by x^a. Using the above result about Gromov bordification, we see that:

PROPOSITION 4.2. *$\mathcal{CO}$ is a chain map.*

PROOF. For the simplest portion of this map $\mathcal{CO}_0$, as noted in [**A1**, §5.2], the argument that $\mathcal{CO}_0$ is a chain map with the correct signs is essentially identical to the proof that various differentials square to zero. The higher cases follow from Koszul sign reordering essentially as in [**S5**, Prop. 12.3], using the algorithm discussed in Appendix A. $\square$

Since the two-pointed Hochschild chain and co-chain complexes

$$_2\mathrm{CC}_*(\mathcal{W}, \mathcal{W}),\ _2\mathrm{CC}^*(\mathcal{W}, \mathcal{W})$$

arise naturally from a bimodule perspective, we now define variants of the chain-level map $\mathcal{OC}$ and $\mathcal{CO}$ between $CH^*(M)$ and the respective two-pointed complexes:

$$_2\mathcal{OC} : {}_2\mathrm{CC}_*(\mathcal{W}, \mathcal{W}) \longrightarrow CH^*(M) \tag{4.73}$$

$$_2\mathcal{CO} : CH^*(M) \longrightarrow {}_2\mathrm{CC}^*(\mathcal{W}, \mathcal{W}) \tag{4.74}$$

As one example of the type of argument allowed by our construction of open-closed operations, we prove that the resulting maps are in fact quasi-isomorphic to $\mathcal{OC}$ and $\mathcal{CO}$.

DEFINITION 4.20. *The* **two-pointed open-closed moduli space** *with (k, l) marked points*

$$\mathcal{R}^1_{k,l} \tag{4.75}$$

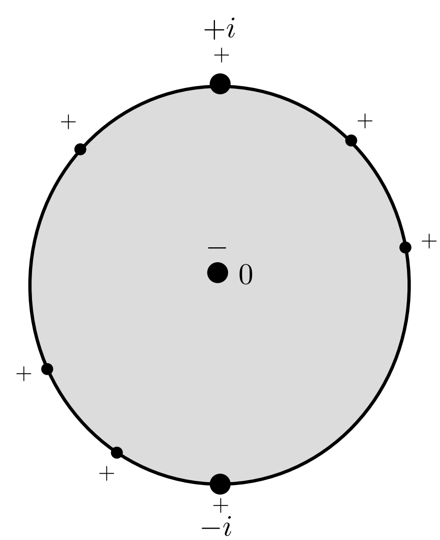
is the space of discs with one interior negative puncture labeled y_{out}, and $k+l+2$ boundary punctures, labeled in counterclockwise order $z_1, z_2, \ldots, z_k, z_L, z'_1, z'_2, \ldots, z'_l, z'_R$, such that:

(4.76) *up to automorphism, z_L, z'_R, and y_{out} are constrained to lie at $-i$, i and 0 respectively.*

Call z_L and z'_R the **special inputs** *of any such disc.*

REMARK 4.6. *The moduli space $\mathcal{R}^1_{k,l}$ is a codimension one submanifold of $\mathcal{R}^1_{k+l+2}$, and thus has dimension $k + l$.*

FIGURE 6. A representative of an element of the moduli space $\mathcal{R}^1_{3,2}$ with special points at 0 (output), $-i$, and i.



The boundary strata of Deligne-Mumford compactification $\overline{\mathcal{R}}^1_{k,l}$ is covered by the images of the natural inclusions of the following products:

$$\overline{\mathcal{R}}^{k'} \times_{n+1} \overline{\mathcal{R}}^1_{k-k'+1,l}, \quad 0 \le n < k - k' + 1 \tag{4.77}$$

$$\overline{\mathcal{R}}^{l'} \times_{(n+1)'} \overline{\mathcal{R}}^1_{k,l-l'+1}, \quad 0 \le n' < l - l' + 1 \tag{4.78}$$

$$\overline{\mathcal{R}}^{k'+l'+1} \times_0 \overline{\mathcal{R}}^1_{k-k',l-l'} \tag{4.79}$$

$$\overline{\mathcal{R}}^{l'+k'+1} \times_{0'} \overline{\mathcal{R}}^1_{k-k',l-l'}. \tag{4.80}$$

Here the notation $\times_j$ indicates that one glues the distinguished output of the first factor to the input z_j, and the notation $\times_{j'}$ indicates that one glues the distinguished output of the first factor to the input z'_j. Moreover, in (4.79), after gluing the output of the first disc to to the first special point z_0, the $k'+1$st input becomes the new special point z_0. Similarly in (4.80), after gluing the output of the first stable disc to the second special point z'_0, the $l'+1$st input becomes the new special point z'_0. Thinking of $\mathcal{R}^1_{k,l}$ as a submanifold of open-closed strings, we obtain, given a compatible Lagrangian labeling $\{L_0, \ldots, L_k, L'_0, \ldots, L'_l\}$ asymptotic input chords $\{x_0, x_1, \ldots, x_k, x'_0, x'_1, \ldots, x'_l\}$ and output orbit y, Floer theoretic moduli spaces

$$\overline{\mathcal{R}}^1_{k,l}(y; x_0, x_1, \ldots, x_k, x'_0, x'_1, \ldots, x'_l) \tag{4.81}$$

of dimension

$$k + l - n + \deg(y) - \deg(x_0) - \deg(x'_0) - \sum_{i=1}^{k} \deg(x_i) - \sum_{j=1}^{l} \deg(x'_j). \tag{4.82}$$

Here, L_k, L'_0 are adjacent to the second special point z'_0 and L'_l, L_0 are adjacent to z_0 (with corresponding inputs x'_0, x_0). Using the sign twisting datum

$$\vec{t}_{{}_2\mathcal{OC}_{k,l}} = (1, 2, \ldots, k+1, k+3, k+4, \ldots, k+2+l) \tag{4.83}$$

with respect to the ordering of inputs $(z_0, \ldots, z_k, z'_0, \ldots, z'_l)$, define associated Floer-theoretic operations

$${}_2\mathcal{OC}_{k,l} := (-1)^{\vec{t}_{{}_2\mathcal{OC}}} \mathbf{F}_{\overline{\mathcal{R}}^1_{k,l}} : (\mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes l} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes k})^{diag} \longrightarrow CH^*(M). \tag{4.84}$$

The two-pointed open-closed map is defined to be the sum of these operations:

$${}_2\mathcal{OC} = \sum_{k,l} {}_2\mathcal{OC}_{k,l} : (\mathcal{W}_\Delta \otimes T\mathcal{W} \otimes \mathcal{W}_\Delta \otimes T\mathcal{W})^{diag} \longrightarrow CH^*(M). \tag{4.85}$$

With respect to the grading on the 2-pointed Hochschild complex, ${}_2\mathcal{OC}$ is once more a map of degree n. By analyzing the boundary of the one-dimensional components of $\overline{\mathcal{R}}^1_{k,l}$, seeing that the relevant boundary behavior is governed by the codimension-1 boundary of the abstract moduli space $\overline{\mathcal{R}}^1_{k,l}$ described from (4.77)-(4.80) and strip-breaking, and performing a sign verification in Appendix A, we conclude that

Corollary 4.3. *The map ${}_2\mathcal{OC} : {}_2\mathrm{CC}_*(\mathcal{W}, \mathcal{W}) \longrightarrow CH^*(M)$ is a chain map.*

Definition 4.21. *The* **two-pointed closed-open moduli space** *with (r, s) marked points*

$$\mathcal{R}^{1,1}_{r,s} \tag{4.86}$$

is the space of discs with one interior positive puncture labeled y_{in}, one negative boundary puncture z_{out}, and $r+s+1$ positive boundary punctures, labeled in clockwise order from z_{out} as $z_1, \ldots, z_r, z_{fixed}, z'_1, \ldots, z'_s$, subject to the following constraint:

(4.87) *up to automorphism, z_{out}, z_{fixed}, and y_{in} lie at $-i$, i and 0 respectively.*

The boundary strata of the Deligne-Mumford compactification $\overline{\mathcal{R}}^{1,1}_{r,s}$ is covered by the natural inclusions of the following products:

$$\overline{\mathcal{R}}^{r'} \times_{n+1} \overline{\mathcal{R}}^{1,1}_{r-r'+1,s}, \quad 0 \leq n < r - r' + 1 \tag{4.88}$$

$$\overline{\mathcal{R}}^{s'} \times_{(m+1)'} \overline{\mathcal{R}}^{1,1}_{r,s-s'+1}, \quad 0 \leq m < s - s' + 1 \tag{4.89}$$

$$\overline{\mathcal{R}}^{r'+s'+1} \times_0 \overline{\mathcal{R}}^{1,1}_{r-r',s-s'} \tag{4.90}$$

$$\overline{\mathcal{R}}^{1,1}_{r-a',s-b'} \times_{a'+1} \overline{\mathcal{R}}^{a'+b'+1}. \tag{4.91}$$

Here in (4.90), the output of the stable disc is glued to the special input z_{fixed} with the $r' + 1$st point becoming the new distinguished z_{fixed}. Similarly, in (4.91), the output of the two-pointed closed-open disc z_{out} is glued to the $a + 1$st input of the stable disc.

Thinking of $\mathcal{R}^{1,1}_{r,s}$ as a submanifold of open-closed strings, we obtain, given a compatible Lagrangian labeling

$$\{L_0, \ldots, L_r, L'_0, \ldots, L'_s\} \tag{4.92}$$

and input chords $\{x_1, \ldots, x_r, x_{fixed}, x'_1, \ldots, x'_s\}$ input orbit y, and output chord x_{out}, a moduli space

$$\overline{\mathcal{R}}^{1,1}_{r,s}(x_{out}; y, x_1, \ldots, x_r, x_{fixed}, x'_1, \ldots, x'_s) \tag{4.93}$$

of dimension

$$r + s + \deg(x_{out}) - \deg(y) - \deg(x_{fixed}) - \sum_{i=1}^{r} \deg(x_i) - \sum_{j=1}^{s} \deg(x'_j). \tag{4.94}$$

Here, L_r, L'_0 are adjacent to the second special output z_{out} and L'_s, L_0 are adjacent to z_{fixed} (with corresponding asymptotic conditions x_{out}, x_{fixed}). We also obtain associated Floer-theoretic operations

$$\mathbf{F}_{\overline{\mathcal{R}}^{1,1}_{r,s}} : CH^*(M) \otimes (\mathcal{W}^{\otimes s} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes r}) \longrightarrow \mathcal{W}_\Delta. \tag{4.95}$$

Now, define

$$_2\mathcal{CO}_{r,s} : CH^*(M) \longrightarrow \hom_{Vect}(\mathcal{W}^{\otimes s} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes r}, \mathcal{W}_\Delta) \tag{4.96}$$

as

$$_2\mathcal{CO}_{r,s}(y)(y_s, \ldots, y_1, \mathbf{b}, x_r, \ldots, x_1) := (-1)^{\vec{t}_{_2\mathcal{CO}_{r,s}}} \mathbf{F}_{\overline{\mathcal{R}}^{1,1}_{r,s}}(y, y_s, \ldots, x_1, \mathbf{b}, x_r, \ldots, x_1) \tag{4.97}$$

where $\vec{t}_{_2\mathcal{CO}_{r,s}}$ is the sign twisting datum

$$\vec{t}_{_2\mathcal{CO}_{r,s}} := (-1, 0, \ldots, r-1, r+1, r+2, \ldots, r+s+1) \tag{4.98}$$

with respect to the input ordering $(y_{in}, z_1, \ldots, z_r, z_{fixed}, z'_1, \ldots, z'_s)$. Define the two-pointed closed-open map to be the sum of these operations

$$_2\mathcal{CO} = \sum_{r,s} {}_2\mathcal{CO}_{r,s} : CH^*(M) \longrightarrow \hom_{\mathcal{W}\text{-}\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{W}_\Delta) \tag{4.99}$$

With respect to the grading on the 2-pointed Hochschild co-chain complex, $_2\mathcal{CO}$ is once more a map of degree 0. An analysis of the boundary of the one-dimensional components of $\overline{\mathcal{R}}^{1,1}_{k,l}$ coming from strip-breaking and the codimension-1 boundary of the abstract moduli space $\overline{\mathcal{R}}^{1,1}_{k,l}$ described in (4.88)-(4.91), along with an application of the sign verification algorithm discussed in Appendix A, we conclude that

Corollary 4.4. *The map $_2\mathcal{CO} : CH^*(M) \longrightarrow {}_2\mathrm{CC}^*(\mathcal{W}, \mathcal{W})$ is a chain map.*

We remark that the quasi-isomorphisms of chain complexes

$$\mathbf{\Phi} : {}_2\mathrm{CC}_*(\mathcal{W}, \mathcal{W}) \xrightarrow{\sim} \mathrm{CC}_*(\mathcal{W}, \mathcal{W})$$

$$\mathbf{\Psi} : \mathrm{CC}^*(\mathcal{W}, \mathcal{W}) \xrightarrow{\sim} {}_2\mathrm{CC}^*(\mathcal{W}, \mathcal{W})$$

defined in (2.148) and (2.153) induce identifications of the two-pointed open-closed maps with the usual open-closed maps. The precise statement is:

Proposition 4.3. *There are homotopy-commutative diagrams*

$$\begin{tikzcd} {}_2\mathrm{CC}_*(\mathcal{W},\mathcal{W}) \arrow[d, "\mathbf{\Phi}"] \arrow[dr, "{}_2\mathcal{OC}"] & \\ \mathrm{CC}_*(\mathcal{W},\mathcal{W}) \arrow[r, "\mathcal{OC}"] & SH^*(M) \end{tikzcd} \tag{4.100}$$

and

$$\begin{tikzcd} SH^*(M) \arrow[d, "\mathcal{CO}"] \arrow[dr, "{}_2\mathcal{CO}"] & \\ \mathrm{CC}^*(\mathcal{W},\mathcal{W}) \arrow[r, "\mathbf{\Psi}"] & {}_2\mathrm{CC}^*(\mathcal{W},\mathcal{W}) \end{tikzcd} . \tag{4.101}$$

Corollary 4.5. *The maps $({}_2\mathcal{OC}, {}_2\mathcal{CO})$ are equal in homology to the maps $(\mathcal{OC}, \mathcal{CO})$.*

The homotopies (4.100) and (4.101) are controlled by the following moduli spaces.

Definition 4.22. *The moduli space*

$$\mathcal{S}^1_{k,l} \tag{4.102}$$

is the space of discs with one interior negative puncture labeled y_{out} and $k+l+2$ positive boundary punctures, labeled in clockwise order $z_0, z_1, \dots, z_k, z_0', z_1', \dots, z_l'$, such that:

$$\text{up to automorphism, } z_0,\ z_0',\ \text{and } y_{out} \text{ are constrained to lie at } -i,\ e^{-i\frac{\pi}{2}(1-2t)} \text{ and } 0 \text{ respectively, for some } t \in (0,1). \tag{4.103}$$

The space $\mathcal{S}^1_{k,l}$ fibers over the open interval $(0,1)$, by the value of t above. Compactifying, we see that $\overline{\mathcal{S}}^1_{k,l}$ submerses over $[0,1]$ and its codimension 1 boundary strata are covered by the images of the natural inclusions of the following products (some corresponding to the limits $t = 0, 1$ and some occuring over the entire interval):

$$\overline{\mathcal{R}}^{k+2+l'+l''} \times_{l-l'-l''+1} \overline{\mathcal{R}}^1_{l-l'-l''+1} \ (t=0) \tag{4.104}$$

$$\overline{\mathcal{R}}^1_{k,l} \ (t=1) \tag{4.105}$$

$$\overline{\mathcal{R}}^{k'} \times_{n+1} \overline{\mathcal{S}}^1_{k-k'+1,l} \quad 0 \leq n < k-k'+1 \tag{4.106}$$

$$\overline{\mathcal{R}}^{l'} \times_{(m+1)'} \overline{\mathcal{S}}^1_{k,l-l'+1} \quad 0 \leq m < l-l'+1 \tag{4.107}$$

$$\overline{\mathcal{R}}^{k'+l'+1} \times_0 \overline{\mathcal{S}}^1_{k-k',l-l'} \tag{4.108}$$

$$\overline{\mathcal{R}}^{l'+k'+1} \times_0' \overline{\mathcal{S}}^1_{k-k',l-l'} \tag{4.109}$$

where

- in (4.104), the $k+1$st and $k+l'+2$nd marked points of the stable disc become the special points z_0 and z_0' after gluing; and
- the products (4.106)-(4.106) are as in (4.77)-(4.80).

Fixing sign twisting datum

$$\vec{t}_{{}_2\mathcal{OC}\to\mathcal{OC},k,l} := (1, \dots, k+1, k+3, \dots, k+l+2), \tag{4.110}$$

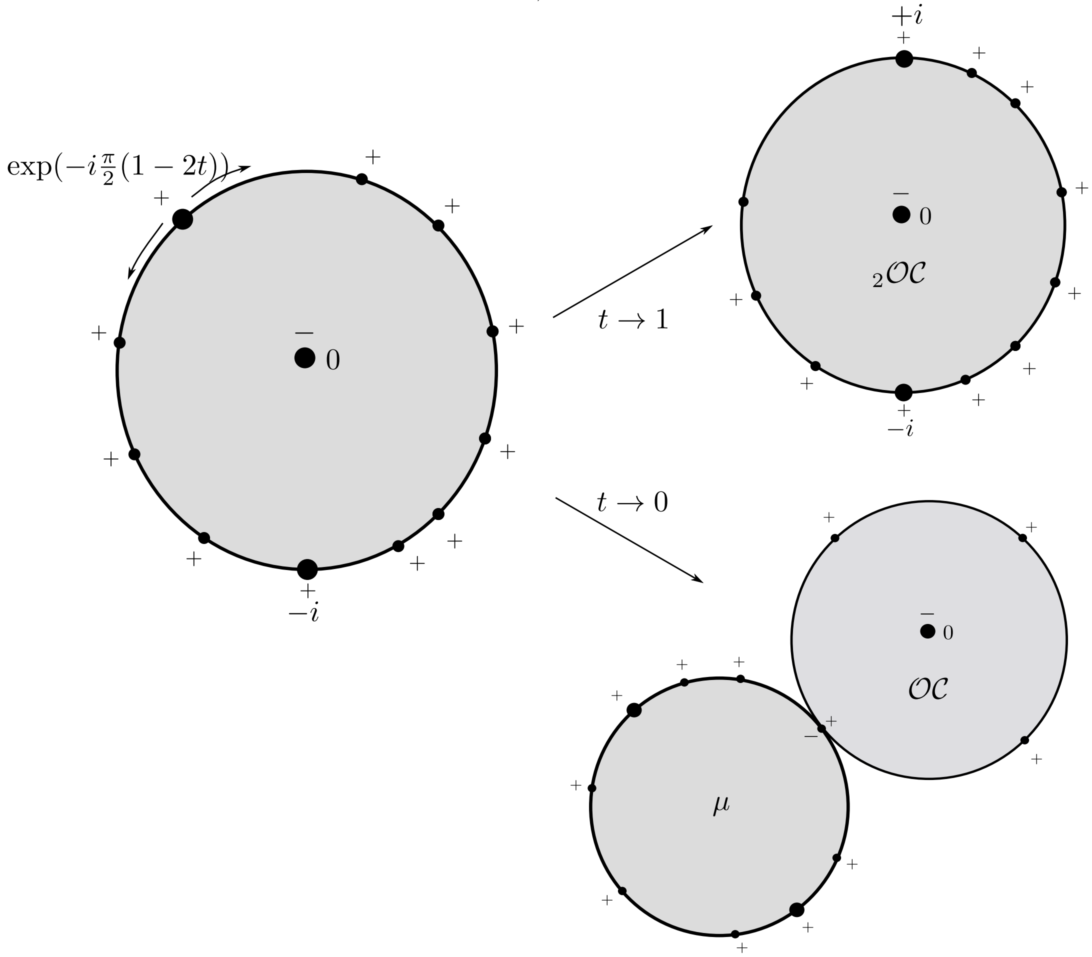
we obtain an associated operation

$$\mathcal{H} := \bigoplus_{k,l} (-1)^{\vec{t}_{{}_2\mathcal{OC}\to\mathcal{OC},k,l}} \mathbf{F}_{\overline{\mathcal{S}}^1_{k,l}} : {}_2\mathrm{CC}_*(\mathcal{W},\mathcal{W}) \longrightarrow CH^*(M) \tag{4.111}$$

of degree $n-1$. By analyzing the boundary of the 1-dimensional Floer moduli spaces associated to $\mathcal{S}^1_{k,l}$ coming from (4.104)-(4.109) and strip-breaking, as well as verifying signs (see Appendix A), we see that

$$d_{CH} \circ \mathcal{H} \pm \mathcal{H} \circ d_{{}_2\mathrm{CC}} = \mathcal{OC} \circ \mathbf{\Phi} - {}_2\mathcal{OC}, \tag{4.112}$$

FIGURE 7. A schematic of $\mathcal{S}^1_{k,l}$ and its $t = 0, 1$ degenerations.



verifying the first homotopy commutative diagram.

DEFINITION 4.23. *The moduli space*

$$\mathcal{S}^{1,1}_{r,s} \tag{4.113}$$

is the space of discs with one interior positive puncture labeled y_{in}, one negative boundary puncture z_{out}, and $r+s+1$ positive boundary punctures, labeled in clockwise order from z_{out} as $z_1, \ldots, z_r, z_{fixed}, z'_1, \ldots, z'_s$, subject to the following constraint:

$$\begin{gathered}\textit{up to automorphism, } z_{out},\ z_{fixed}, \textit{ and } y_{in} \textit{ lie at } -i,\ e^{i(-\pi/2+\pi\cdot t)} \textit{ and } 0 \textit{ respectively,}\\ \textit{for some } t \in (0,1).\end{gathered} \tag{4.114}$$

The space $\mathcal{S}^{1,1}_{r,s}$ again fibers over the open interval $(0, 1)$, by the value of t above. Compactifying, we see that the codimension 1 boundary strata of the space $\overline{\mathcal{S}}^{1,1}_{r,s}$ submerses over $[0, 1]$ and is covered by the images of the natural inclusions of the following products (some corresponding to the limits $t = 0, 1$ and some occuring over the entire interval):

$$\overline{\mathcal{R}}^{1,1}_{s-s'-s''+1} \times_{r+2+s'} \overline{\mathcal{R}}^{r+2+s'+s''} \ (t=0) \tag{4.115}$$

$$\overline{\mathcal{R}}^{1,1}_{r,s} \ (t=1) \tag{4.116}$$

$$\overline{\mathcal{R}}^{r'} \times_{n+1} \overline{\mathcal{S}}^{1,1}_{r-r'+1,s} \quad 0 \leq n < r-r'+1 \tag{4.117}$$

$$\overline{\mathcal{R}}^{s'} \times_{(m+1)'} \overline{\mathcal{S}}^{1,1}_{r,s-s'+1} \quad 0 \leq m < s-s'+1 \tag{4.118}$$

$$\overline{\mathcal{R}}^{r'+s'+1} \times_0 \overline{\mathcal{S}}^{1,1}_{r-r',s-s'} \tag{4.119}$$

$$\overline{\mathcal{S}}^{1,1}_{r-a',s-b'} \times_{a'+1} \overline{\mathcal{R}}^{a'+b'+1} \tag{4.120}$$

Using sign twisting datum

$$\vec{t}_{{}_2\mathcal{CO}\to\mathcal{CO},r,s} = (-1,0,\ldots,r-1,r+1,r+2,\ldots,r+s+1), \tag{4.121}$$

the associated operation

$$\mathcal{G} := \bigoplus_{r,s} (-1)^{\vec{t}_{{}_2\mathcal{CO}\to\mathcal{CO},r,s}} \mathbf{F}_{\overline{\mathcal{S}}^{1,1}_{r,s}} : CH^*(M) \longrightarrow {}_2\mathrm{CC}^*(\mathcal{W},\mathcal{W}) \tag{4.122}$$

has degree -1. By analyzing the boundary of the 1-dimensional Floer moduli spaces associated to $\mathcal{S}^{1,1}_{r,s}$ coming from (4.104)-(4.109) and strip-breaking, and verifying signs (Appendix A), we see that

$$\mathcal{G} \circ d_{CH} \pm d_{{}_2\mathrm{CC}^*} \circ \mathcal{G} = \mathbf{\Psi} \circ \mathcal{CO} - {}_2\mathcal{CO}, \tag{4.123}$$

verifying the second homotopy commutative diagram.

4.7. A generalized coproduct. For any pair of objects $K, L \in \mathrm{ob}\ \mathcal{W}$, by counting a certain family of discs with two outputs, we define a chain map of degree $+n$:

$$cop_{K,L} : \hom^*_{\mathcal{W}}(K,L) \to \hom^{*+n}_{\mathcal{W}-\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{Y}^l_K \otimes \mathcal{Y}^r_L). \tag{4.124}$$

We recall that Abouzaid [**A1**] constructed for any object K, an A_∞ coproduct of degree $+n$ $\Delta_K : \mathcal{W}_\Delta \to \mathcal{Y}^l_K \otimes \mathcal{Y}^r_K$. Our construction slightly generalizes [**A1**] in the sense that, given a chain level representative of the unit $e_K \in \hom_{\mathcal{W}}(K,K)$, $cop_{K,K}(e_K)$ is chain-homotopic to Δ_K (we do not prove this, but the proof is a rather straightforward gluing argument).

Definition 4.24. *The* **moduli space of discs with two negative punctures, two positive punctures, and** $(0,0;s,t)$ **positive marked points**

$$\mathcal{R}^{0,0;s,t}_2 \tag{4.125}$$

is the abstract moduli space of discs with

- *two distinguished negative marked points z^1_-, z^2_-,*
- *two distinguished positive marked points z^1_+, z^2_+, one removed from each boundary component cut out by z^1_- and z^2_-,*
- *s positive marked points $c_1,\ldots,c_s$ between z^2_- and z^2_+; and*
- *t positive marked points $d_1,\ldots,d_t$ between z^2_+ and z^1_-.*

Moreover, the distinguished points z^1_-, z^2_-, z^1_+, and z^2_+ are constrained to lie (after automorphism) at 1, -1, i and $-i$ respectively. Namely, we fix the cross-ratios of these 4 points.

The boundary strata of the Deligne-Mumford compactification

$$\overline{\mathcal{R}}^{0,0;s,t}_2 \tag{4.126}$$

is covered by the images of natural inclusions of the following products:

$$\overline{\mathcal{R}}^{s'+1+t'} \times_{2+} \overline{\mathcal{R}}_2^{0,0,s-s',t-t'} \tag{4.127}$$

$$\overline{\mathcal{R}}^{s'} \times_{n+1}^{c} \overline{\mathcal{R}}_2^{0,0;s-s'+1,t}, \quad 0 \leq n < s-s'+1 \tag{4.128}$$

$$\overline{\mathcal{R}}^{t'} \times_{n+1}^{d} \overline{\mathcal{R}}_2^{0,0;s,t-t'+1}, \quad 0 \leq n < t-t'+1 \tag{4.129}$$

$$\overline{\mathcal{R}}_2^{0,0;s',t-t'} \times_{(1,1)} \overline{\mathcal{R}}^{1+t'} \tag{4.130}$$

$$\overline{\mathcal{R}}_2^{0,0;s-s',t} \times_{(2,s'+1)} \overline{\mathcal{R}}^{1+s'}. \tag{4.131}$$

Above, the notation $\times_{j+}$ in (4.127) indicates that the output of the first component is glued to the special input z_j^+ of the second component, $\times_j^c$ and $\times_j^d$ indicate gluing to the input c_j and d_j respectively, and $\times_{(i,j)}$ in (4.130) and (4.131) indicate gluing the ith output of the first component to the jth input of the second. Also, in (4.127), the $s'+1$st input points of the first component becomes the special point z_+^2 after gluing.

Definition 4.25. *A* **Floer datum** *for a disc S with two positive, two negative, and $(0,0;s,t)$ positive boundary marked points is a Floer datum of S thought of as an open-closed string.*

Fix a sequence of Lagrangians

$$K, L, C_0, \ldots, C_s, D_0, \ldots, D_t, \tag{4.132}$$

corresponding to a labeling of the boundary of an element of $\mathcal{R}_2^{0,0;s,t}$ by specifying that z_+^1 be the intersection point between K and L, z_+^2 be the intersection point between C_s and D_0, c_i be the intersection point between C_{i-1} and C_i, and so on for d_i. In the manner described in (4.27), the space $\overline{\mathcal{R}}_2^{0,0;s,t}$, along with the sign twisting datum

$$\vec{t}_{cop_{s,t}} := (0,1,2,\ldots,s,s,s+1,\ldots,s+t) \tag{4.133}$$

corresponding to inputs $(z_1^+, c_1, \ldots, c_s, z_2^+, d_1, \ldots, d_t)$, determines an operation

$$\begin{aligned} cop_{t,s}^{K,L} : \mathbf{hom}(K,L) \otimes \big(&\hom(D_{t-1},D_t) \otimes \cdots \otimes \hom(D_0,D_1) \\ &\otimes \mathbf{hom}(C_s,D_0) \otimes \hom(C_{s-1},C_s) \otimes \cdots \otimes \hom(C_0,C_1)\big) \\ &\longrightarrow \mathbf{hom}(K,D_t) \otimes \mathbf{hom}(C_0,L). \end{aligned} \tag{4.134}$$

Definition 4.26. *The* **generalized coproduct** *is defined by*

$$cop_{K,L} : \hom_{\mathcal{W}}(K,L) \longrightarrow \hom^{*+n}(\mathcal{W}_\Delta, \mathcal{Y}_K^l \otimes_{\mathbb{K}} \mathcal{Y}_L^r) \tag{4.135}$$

$$\mathbf{a} \longmapsto \phi_{\mathbf{a}} \tag{4.136}$$

where $\phi_{\mathbf{a}}$ is the morphism whose $t|1|s$ term is

$$\phi_{\mathbf{a}}^{t|1|s}(d_t,\ldots,d_1,\mathbf{b},c_s,\ldots,c_1) := cop_{t,s}^{K,L}(\mathbf{a},d_t,\ldots,d_1,\mathbf{b},c_s,\ldots,c_1). \tag{4.137}$$

The Gromov bordification $\overline{\mathcal{R}}_2^{0,0;s,t}(\vec{x}_{in},\vec{x}_{out})$ has boundary covered by the images of the Gromov bordifications of spaces of maps from the nodal domains (4.127) - (4.131), along with standard strip breaking, which put together implies that:

Proposition 4.4. *$cop_{K,L}$ is a chain map of degree n. This implies e.g., that for closed element $a \in \hom_w(K,L)$, $cop_{K,L}(a)$ is a closed morphism of bimodules of degree $n+|a|$.*

We will omit further details about the proof, as it is not necessary for this paper, and will anyway appear as a special case of this discussion of the *Calabi-Yau morphism* in the sequel [**G**].

5. Operations from glued pairs of discs

In this section, we define a broad class of abstract moduli spaces and their associated Floer theoretic operations, corresponding to a pair of discs glued together along some boundary components. This class will arise when defining operations in the product $M^- \times M$, and in setting up the theory of quilts.

5.1. Connect sums. Recall first the notion of a boundary connect sum between two Riemann surfaces with boundary, a notion already implicit in the constructions of boundary and corner charts for Deligne-Mumford compactifications of moduli spaces.

Definition 5.1. *Let Σ_1, Σ_2 be two Riemann surfaces with boundary, with marked points $z_1 \in \partial\Sigma_1$, $z_2 \in \partial\Sigma_2$ removed. Let $\epsilon_1 : Z_+ \to \Sigma_1$ be a positive strip-like end for z_1 and $\epsilon_2 : Z_- :\to \Sigma_2$ a negative strip-like end for z_2 and let $\lambda = \frac{\log \rho}{1+\log \rho} \in (0,1)$ (correspondingly $\rho \in [1,\infty)$). The* **λ-connect sum**

$$\Sigma_1 \#^{\lambda}_{(\epsilon_1,z_1),(\epsilon_2,z_2)} \Sigma_2 \tag{5.1}$$

is

$$(\Sigma_1 - \epsilon_1([\rho,\infty) \times [0,1])) \cup_{\varphi_\rho} (\Sigma_2 - \epsilon_2((-\infty,-\rho] \times [0,1])) \tag{5.2}$$

where

$$\varphi_\rho : \epsilon_1((0,\rho) \times [0,1]) \to \epsilon_2((-\rho,0) \times [0,1]). \tag{5.3}$$

is the composition

$$\epsilon_1((0,\rho) \times [0,1]) \xrightarrow{\epsilon_1^{-1}} (0,\rho) \times [0,1] \xrightarrow{(\cdot-\rho, id)} (-\rho,0) \times [0,1] \xrightarrow{\epsilon_2} \epsilon_2((-\rho,0) \times [0,1]). \tag{5.4}$$

We will often write $\Sigma_1 \#^{\lambda}_{z_1,z_2} \Sigma_2$ when the choice of strip-like ends is implicit.

Definition 5.2. *In the notation of above, the* **associated thin part** *of a λ-connect sum $\Sigma_1 \#^{\lambda}_{z_1,z_2} \Sigma_2$ is the finite strip parametrization*

$$\epsilon_1 : [0,\rho] \times [0,1] \to \Sigma_1 \#^{\lambda}_{z_1,z_2} \Sigma_2. \tag{5.5}$$

The **associated thick parts** *of this connect sum are the regions $\Sigma_1 - \epsilon_1([0,\infty) \times [0,1])$, $\Sigma_2 - \epsilon_2([-\infty,0] \times [0,1])$ respectively, thought of as living in the connect sum.*

The notion of λ connect sum extends continuously to the nodal case $\lambda = 1$.

5.2. Pairs of discs.

Definition 5.3. *The* **moduli space of pairs of discs with (k,l) marked points***, denoted*

$$\mathcal{R}_{k,l} \tag{5.6}$$

is the moduli space of pairs of discs with k and l positive marked points and one negative marked point each in the same position, modulo simultaneous automorphisms.

Remark 5.1. *This definition is* **not** *identical to the product of associahedra $\mathcal{R}^k \times \mathcal{R}^l$. The latter space is a further quotient of the former space by automorphisms of the right or left disc, at least when both k and l are in the stable range. Operations at the level of the moduli space $\mathcal{R}_{k,l}$ will arise via quilted strips and homotopy units.*

Remark 5.2. *To construct moduli spaces, we require a pair of discs with (k,l) marked points to be* **stable***: one of k or l must be at least 2.*

The Stasheff associahedron embeds in $\mathcal{R}_{k,l}$ in several ways. There is the **diagonal embedding**

$$\mathcal{R}^d \xhookrightarrow{\Delta_d} \mathcal{R}_{d,d}. \tag{5.7}$$

which is self-explanatory. When $l = 1$ and $k \geq 2$, there is a one-sided embedding

$$\mathcal{R}^k \xhookrightarrow{\mathcal{I}_k} \mathcal{R}_{k,1}, \tag{5.8}$$

where $\mathcal{I}_k = (id, For_{k-1})$ is the pair of maps corresponding to inclusion and forgetting the first $k-1$ boundary marked points respectively. Since the right factor in the image has only one incoming marked point, we call $\mathcal{I}_k$ the **right semi-stable embedding**. Similarly when $k = 1$ and $l \geq 2$, forgetting $l-1$ marked points and inclusion gives us the **left semi-stable embedding**

$$\mathcal{R}^k \xhookrightarrow{\mathcal{J}_k} \mathcal{R}_{1,k}. \tag{5.9}$$

When $l = 0$ or $k = 0$, there are also equivalences

$$\begin{aligned} \mathcal{R}^k &\xhookrightarrow{\sim} \mathcal{R}_{k,0} \\ \mathcal{R}^l &\xhookrightarrow{\sim} \mathcal{R}_{0,l}, \end{aligned} \tag{5.10}$$

which we call the **right** and **left ghost embeddings** respectively, corresponding to the fact that the right or left component of $\mathcal{R}_{k,l}$ is a ghost disc. In fact, we will never consider operations with either k or l equal to 0 on their own, but they arise as boundaries in the compactifications of $\overline{\mathcal{R}}_{k,l}$ $(k, l > 0)$, where the equivalences (5.10) help us explain appearances of ordinary associahedra. Henceforth, we restrict to $k, l \geq 1$ and one of $k, l \geq 2$.

The open moduli space $\mathcal{R}_{k,l}$ admits a stratification by *coincident points* between factors, which we will find useful to explicitly describe.

Definition 5.4. *A* (k,l)**-point identification** $\mathfrak{P}$ *is a sequence of tuples*

$$\{(i_1, j_1), \ldots, (i_s, j_s)\} \subset \{1, \ldots, k\} \times \{1, \ldots, l\} \tag{5.11}$$

which are strictly increasing, i.e.

$$\begin{aligned} i_r &< i_{r+1} \\ j_r &< j_{r+1} \end{aligned} \tag{5.12}$$

The **number of coincidences** *of* $\mathfrak{P}$ *is the size* $|\mathfrak{P}|$.

Definition 5.5. *Take a representative* (S_1, S_2) *of a point in* $\mathcal{R}_{k,l}$*. A boundary input marked point* p_1 *on* S_1 *is said to* **coincide** *with a boundary marked input marked point* p_2 *on* S_2 *if they are at the same position when* S_1 *is superimposed upon* S_2*. This notion is independent of the representative* (S_1, S_2)*, as we act by simultaneous automorphism.*

We remark by definition $\mathcal{R}_{k,l}$ is the space of pairs of discs whose output marked points coincide.

Definition 5.6. *The space of* $\mathfrak{P}$**-coincident pairs of discs with** (k,l) **marked points**

$$_{\mathfrak{P}}\mathcal{R}_{k,l} \tag{5.13}$$

is the subspace of $\mathcal{R}_{k,l}$ *where pairs of input marked points on each factor specified by* $\mathfrak{P}$ *are required to coincide, and no other input marked points are allowed to coincide. Here the indices in* $\mathfrak{P}$ *coincide with the* **counter-clockwise ordering** *of input marked points on each factor.*

Example 5.1. When $\mathfrak{P} = \emptyset$, $_{\mathfrak{P}}\mathcal{R}_{k,l}$ is the space of pairs of discs where none of the inputs are allowed to coincide. This is a disconnected space, with connected components determined by the relative ordering of the k inputs on the first disc with the l inputs on the second disc. The number of connected components is exactly the number of (k,l) **shuffles**, i.e. re-orderings of the sequence

$$\{a_1, \ldots, a_k, b_1, \ldots, b_l\} \tag{5.14}$$

that preserve relative ordering of the a_i, and the relative ordering of the b_j. Given a fixed (k,l) shuffle, the subspace of pairs of discs with appropriate relatively ordered inputs is a copy of the $k+l$ (open) associahedron $\mathcal{R}^{k+l}$. Equivalently, there is one copy of $\mathcal{R}^{k+l}$ for each (k,l) *two-coloring* of the combined set of points, that is a collection of subsets

$$I, J \subset [k+l],\ |I| = k,\ |J| = l,\ I \cup J = [k+l]. \tag{5.15}$$

where $[k+l] := \{1, \ldots, k+l\}$.

Example 5.2. When $k = l$ and $|\mathfrak{P}| = k$ is maximal, the associated space $_{\mathfrak{P}}\mathcal{R}_{k,l}$ is just the diagonal associahedron $\Delta_k(\mathcal{R}^k)$.

We often group these spaces $_{\mathfrak{P}}\mathcal{R}_{k,l}$ by the number of coincident points.

Definition 5.7. *The* **space of pairs of discs with** (k,l) **marked points and** i **coincident points** *is defined to be*

$$_{i}\mathcal{R}_{k,l} := \coprod_{|\mathfrak{P}|=i} {}_{\mathfrak{P}}\mathcal{R}_{k,l}. \tag{5.16}$$

The closure of a stratum ${}_i\mathcal{R}_{k,l}$ in $\mathcal{R}_{k,l}$ is $\coprod_{j\geq i} {}_i\mathcal{R}_{k,l}$. Moreover, each stratum ${}_i\mathcal{R}_{k,l}$ can be explicitly described as a union of associahedra.

Definition 5.8. *Fix disjoint subsets I,J,K of $[d] = \{1,\ldots,d\}$ such that*

$$I \cup J \cup K = [d]. \tag{5.17}$$

The **space of (I,J,K) tricolored discs with d inputs**

$${}_{I,J,K}\mathcal{R}^d \tag{5.18}$$

is exactly the ordinary associahedron, with inputs labeled by the elements $\{L,R,LR\}$ according to whether they are in the set I,J,or K. The **space of (i,j,k) tricolored discs with d inputs**, *where $i+j+k=d$, is the disjoint union over all possible tricolorings of cardinality i,j,k:*

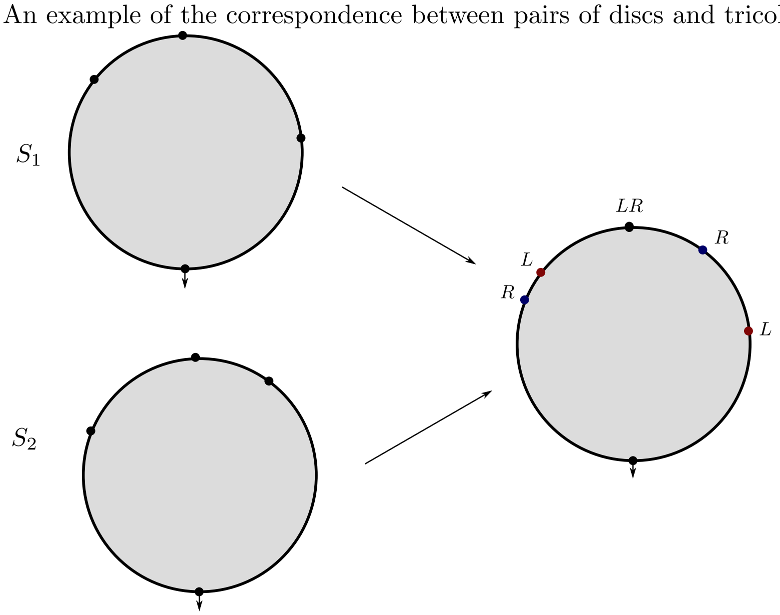
$${}_{i,j,k}\mathcal{R}^d := \coprod_{|I|=i,|J|=j,|K|=k} {}_{I,J,K}\mathcal{R}^d. \tag{5.19}$$

There is a canonical identification

$${}_i\mathcal{R}_{k,l} \simeq {}_{k-i,l-i,i}\mathcal{R}^{k+l-i} \tag{5.20}$$

given as follows: To a pair of discs (S_1,S_2) with i coincidences, consider the *overlay* (superimposition) of S_1 and S_2 along with their marked points. This is a disc with $k+l-i$ input marked points and one output. Color a marked point L if the marked point came only from S_1, R if the marked point came only from S_2, and LR if the marked point came from both factors. Similarly, given a tricolored disc, one can reconstruct a pair of discs with i coincidences by reversing the above procedure.

Figure 8. An example of the correspondence between pairs of discs and tricolored discs.



The disjoint union of spaces

$$\coprod_i {}_{k-i,l-i,i}\mathcal{R}^{k+l-i} \tag{5.21}$$

is set theoretically equal to $\mathcal{R}_{k,l}$, but has forgotten some of the topology. Namely, points colored L and R are not allowed to coincide, and coincident points (those colored LR) are not allowed to separate arbitrarily.

We now construct a model for the Deligne-Mumford compactification

$$\overline{\mathcal{R}}_{k,l}. \tag{5.22}$$

The main idea in our construction is to recover this compactification from the Deligne-Mumford compactifications of the spaces ${}_{k-i,l-i,i}\overline{\mathcal{R}}^{k+l-i}$ by reconstructing the topology with which points colored L and R are allowed to coincide, and points colored LR are allowed to separate. Note that the compactification of tricolored spaces

$$(5.23)\qquad {}_{I,J,K}\overline{\mathcal{R}}^{d}$$

is exactly the usual Deligne-Mumford compactification, where boundary marked points on components of nodal discs are colored in a manner induced by the gluing charts (4.42). Internal positive marked points are colored in the following induced fashion: If the subtree of nodal discs lying above a given positive marked point is a tree of discs with all L or all R labels, then color this marked point L or R respectively. If the subtree contains two out of the three colors (R, L, RL) then color the input LR.

Let

$$(5.24)\qquad D_{LR}^{+1}$$

be a representative of the one-point moduli space ${}_{\{1\},\{2\},\emptyset}\mathcal{R}^2$, i.e. a disc with inputs labeled L, R in clockwise order. Similarly, let

$$(5.25)\qquad D_{LR}^{-1}$$

be a representative of ${}_{\{2\},\{1\},\emptyset}\mathcal{R}^2$, i.e. a disc with inputs labeled R, L in clockwise order. Fix a choice of strip-like ends on $D_{LR}^{\pm 1}$, and let z_{LR} denote the output of each of these discs. Also, suppose we have fixed a universal and consistent choice of strip like ends on the various ${}_{k-i,l-i,i}\overline{\mathcal{R}}^{k+l-i}$.

Now, take a (potentially nodal) representative S of a point of ${}_{k-i,l-i,i}\overline{\mathcal{R}}^{k+l-i}$. Let

$$(5.26)\qquad \vec{p} = p_{j_1}, \ldots p_{j_s},\ \ s \leq i$$

be a subset of the points colored LR, and $\epsilon_{j_1}, \ldots, \epsilon_{j_s}$ the associated strip-like ends. Given a vector

$$(5.27)\qquad \vec{v} = (v_1, \ldots, v_{j_s}) \in [(-\epsilon, \epsilon)^*]^i,$$

where the $*$ means none of the v_r are allowed to be zero, define an element

$$(5.28)\qquad \mathbf{\Pi}_{\vec{v}}^{\vec{p}}(S) \in {}_{k-i+s,l-i+s,i-s}\overline{\mathcal{R}}^{k+l-i+s}$$

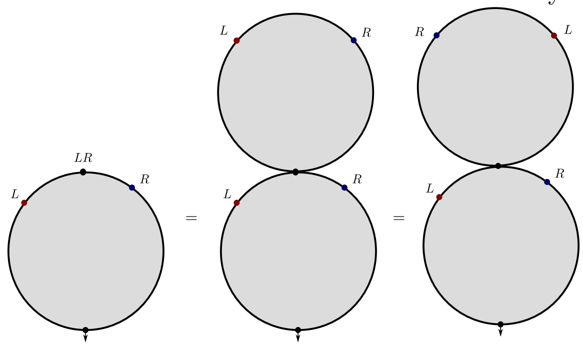
by the iterated connect sum

$$(5.29)\qquad \mathbf{\Pi}_{\vec{v}}^{\vec{p}}(S) := S \#_{p_{j_1}, z_{LR}}^{1-|v_1|} D_{LR}^{\mathrm{sign}(v_1)} \# \cdots \#_{p_{j_k}, z_{LR}}^{1-|v_j|} D_{LR}^{\mathrm{sign}(v_j)}.$$

Here $\#^{1-|v_r|}$ is the operation of connect sum with gluing parameter $1 - |v_r|$ in the notation of §5.1, which as $|v_R|$ approaches zero is very close to nodal. Also, $\mathrm{sign}(v_r)$ is $+1$ if v_r is positive and -1 if v_r is negative. In other words, at input point p_{j_r} on S, we are taking a connect sum with the disc $D_{LR}^{\mathrm{sign}(v_r)}$ at the point z_{LR}, i.e. gluing in two points labeled L and R in clockwise or counterclockwise order depending on the sign of v_r.

Example 5.3. It is useful before proceeding to describe the map $\Pi_{\vec{v}}^{\vec{p}}(S)$ in a simple example. Suppose we are in ${}_{1,1,1}\mathcal{R}^3$, the moduli space of discs with three inputs, one with each color. Pick a non-nodal representative of an element of this space, without loss of generality one in which the points are colored L, LR and R in clockwise order from the output. Let p_2 be the point colored LR with associated vector $\vec{v} = (v)$, and let us examine the representative of $\Pi_{(v)}^{p_2}(S)$ for different values of $v \in (-\epsilon, 0) \cup (0, \epsilon)$. For v positive, $\Pi_{(v)}^{p_2}(S)$ corresponds to resolving the LR point by two points, one labeled L and one labeled R, with the L point to the left of the R point, which lives in ${}_{\{1,2\},\{3,4\},\emptyset}\mathcal{R}^4$. For v negative, $\Pi_{(v)}^{p_2}(S)$ resolves the LR point in the opposite direction, giving an element of a different associahedron ${}_{\{1,3\},\{2,4\},\emptyset}\mathcal{R}^4$. As v approaches zero from either direction, the newly created L and R points come together and bubble off, giving a nodal element in each of these respective associahedra $\Pi_{(0+)}^{p_2}(S)$ and $\Pi_{(0-)}^{p_2}(S)$ with bubble component a D_{LR}^{+1} or D_{LR}^{-1} respectively. To partially recover the topology of $\mathcal{R}_{2,2}$, we would like to identify the points $\Pi_{0-}^{p_2}(S)$ and $\Pi_{0+}^{p_2}(S)$, in a manner preserving the manifold structure near the identification. See Figure 9.

Figure 9. The strata we would like to identify.



The above example illustrates the following properties: $\Pi_{\vec{v}}$ varies smoothly in S and the parameters $\vec{v} \in [(-\epsilon, \epsilon)^*]^i$, and there are well defined, but different, nodal limits of the curve $\Pi_{\vec{v}}(S)$ as components v_r approach 0 from the left or right, corresponding to gluing on a nodal $D^{\pm 1}_{LR}$ respectively at input p_{j_r}. Indicate these different limits by values 0_+ and 0_- respectively. Then $\Pi_{\vec{v}}(S)$ extends to a map

$$\bar{\Pi}_{\vec{v}} \tag{5.30}$$

defined over domain

$$\vec{v} \in [(-\epsilon, 0_-] \cup [0_+, \epsilon)]^i . \tag{5.31}$$

Define

$${}_{k-i,l-i,i}(\overline{\mathcal{R}}^{k+l-i})^* \tag{5.32}$$

to be the locus of the compactifications ${}_{I,J,K}\overline{\mathcal{R}}^{k+l-i}$ with $|I| = k-i$, $|J| = l-i$, $|K| = i$ where there are no leaf bubbles with one L and one R. Put another way, remove the images of $\bar{\Pi}^{\vec{p}}_{(\cdots,0_+,\cdots)}$, $\bar{\Pi}^{\vec{p}}_{(\cdots,0_-,\cdots)}$ for all relevant domains of definition of Π. Now, define the manifold structure on $\overline{\mathcal{R}}_{k,l}$ as follows. To simplify notation, denote

$$\mathbb{I}_\epsilon := (-\epsilon, \epsilon). \tag{5.33}$$

Charts consist of

$$\mathcal{U} \times \mathbb{I}^i_\epsilon,\ \mathcal{U} \subset {}_{k-i,l-i,i}(\overline{\mathcal{R}}^{k+l-i})^*. \tag{5.34}$$

where ϵ may depend on $\mathcal{U}$.

For every such $\mathcal{U}$ above take any subset of indices $\vec{j} = \{j_1, \ldots, j_s\}$ of the i points colored LR, indexed for now from 1 to i. Let $\vec{p} = \{p_{j_1}, \ldots, p_{j_s}\}$ be the associated points. Given any such subset $\vec{j}$ of $\{1, \ldots, i\}$, define

$$\mathcal{P}_{\vec{j}} : \mathbb{I}^i_\epsilon \longrightarrow \mathbb{I}^{|\vec{j}|}_\epsilon \tag{5.35}$$

to be the projection onto the coordinates with indices in $\vec{j}$ and

$$\mathcal{P}_{\vec{j}^c} : \mathbb{I}^i_\epsilon \longrightarrow \mathbb{I}^{i-|\vec{j}|}_\epsilon \tag{5.36}$$

to be the projection onto the complementary coordinates.

Then shrinking $\mathcal{U}$ and ϵ if necessary, perform a smooth identification of the restriction of the basic chart (5.34), where the coordinates for indices in $\vec{j}$ are all non-zero

$$\mathcal{U} \times \mathbb{I}^i_\epsilon|_{\mathcal{P}^{-1}_{\vec{j}}((\mathbb{I}^*_\epsilon)^{|\vec{j}|})} \tag{5.37}$$

onto its image under the smooth map

$$(S, \vec{v}) \longmapsto (\Pi^{\vec{p}}_{\mathcal{P}_{\vec{j}}(\vec{v})}(S), \mathcal{P}_{\vec{j}^c}(\vec{v})). \tag{5.38}$$

All such identifications are manifestly compatible with each other, and along with the identifications of charts within each ${}_{k-i,l-i,i}\mathcal{R}^{k+l-i}$, give $\overline{\mathcal{R}}_{k,l}$ the structure of a smooth manifold with corners of dimension

$k+l-2$. The result is moreover compact, as topologically it is the quotient of the compact space ${}_{k,l,0}\overline{\mathcal{R}}^{k+l}$ by identifications between otherwise identical nodal surfaces containing $D^{\pm}_{LR}$ components.

EXAMPLE 5.4. Let us examine the resulting manifold structure on $\overline{\mathcal{R}}_{2,2}$, in a neighborhood of the one point coincidence discussed in Example 5.3. Consider once more the element q of ${}_{1,1,1}\mathcal{R}^3$ we discussed there, which is represented by some disc with inputs colored L, LR, and R in clockwise order. Then, in a neighborhood U_q of q, we have a chart

$$U_q \times (-\epsilon, \epsilon) \tag{5.39}$$

for some small value of ϵ. There are two distinguished smooth identifications of subsets of (5.39). In the first, one resolves the LR point p_2 by an L followed by R

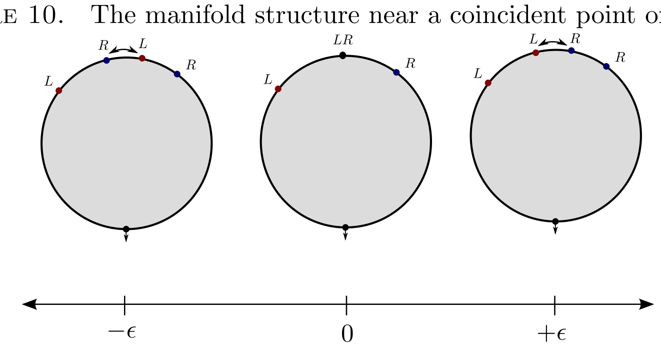
$$\begin{aligned} U_q \times (0,\epsilon) &\longrightarrow {}_{2,2,0}\mathcal{R}^4 \\ (S,v) &\longmapsto \Pi^{p_2}_{(v)}(S) := S\#^{1-v}_{p_2} D^{+1}_{LR} \end{aligned} \tag{5.40}$$

and in the second, one resolves p_2 by an R followed by an L

$$\begin{aligned} U_q \times (-\epsilon,0) &\longrightarrow {}_{2,2,0}\mathcal{R}^4 \\ (S,v) &\longmapsto \Pi^{p_2}_{(v)}(S) := S\#^{1-(-v)}_{p_2} D^{-1}_{LR}. \end{aligned} \tag{5.41}$$

These identifications, along with the existing manifold structure on $({}_{2,2,0}\overline{\mathcal{R}}^4)^*$ determine the manifold structure in a neighborhood of p_2. See Figure 10 for an illustration of the manifold structure near p_2.

FIGURE 10. The manifold structure near a coincident point on $\mathcal{R}_{2,2}$.



This definition of the compactifiction $\overline{\mathcal{R}}_{k,l}$ remembers the structure of the left and right components. Namely, a (potentially nodal) element S in $\overline{\mathcal{R}}_{k,l}$ can be thought of as an element of ${}_{k-i,l-i,i}\overline{\mathcal{R}}^{k+l-i}$ for some i, with no $D^{\pm}_{LR}$ leaf bubbles. Define the associated **unreduced left disc** $\tilde{S}_1$ to be obtained from S by deleting all points colored R, and forgetting the L, LR colorings. Similarly, define the associated **unreduced right disc** $\tilde{S}_2$ to be obtained by deleting all points colored L and forgetting the R, RL colorings. The associated **reduced discs** (S_1, S_2) are given by stabilizing $\tilde{S}_1$ and $\tilde{S}_2$, and there are well defined inputs on these stabilizations corresponding to the inputs colored (L, LR) and (LR, R) on S respectively.

In this way, we obtain a **projection map**

$$\pi_{reduce} : \overline{\mathcal{R}}_{k,l} \longrightarrow \bar{\mathcal{R}}^k \times \bar{\mathcal{R}}^l \tag{5.42}$$

Since the manifolds ${}_{k-i,l-i,i}\mathcal{R}^{k+l-i}$ are codimension i in the space $\mathcal{R}_{k,l}$, we see that the codimension 1 boundary of the compactification $\overline{\mathcal{R}}_{k,l}$ is contained in the image of the codimension 1 boundary of the top stratum ${}_{k,l,0}\overline{\mathcal{R}}^{k+l}$. The chart identifications (5.38) show that points where an L and R would have bubbled off in codimension 1 now cease to be boundary points; thus any codimension 1 bubble must contain at least two L/LR points or two R/LR points.

The result of this discussion is as follows: we see that the codimension one boundary of the Deligne-Mumford compactification (5.22) is covered by the images of the natural inclusions of the following products:

$$\overline{\mathcal{R}}_{k',l'} \times {}_1\overline{\mathcal{R}}_{k-k'+1,l-l'+1},\ k', l', k-k'+1, l-l'+1 \geq 2 \tag{5.43}$$

$$\overline{\mathcal{R}}_{k',1} \times {}_1\overline{\mathcal{R}}_{k-k'+1,l},\ k', k-k'+1 \geq 2 \tag{5.44}$$

$$\overline{\mathcal{R}}_{1,l'} \times {}_1\overline{\mathcal{R}}_{k,l-l'+1},\ l', l-l'+1 \geq 2 \tag{5.45}$$

$$\overline{\mathcal{R}}_{k',0} \times \overline{\mathcal{R}}_{k-k'+1,l},\ k', k-k'+1 \geq 2 \tag{5.46}$$

$$\overline{\mathcal{R}}_{0,l'} \times \overline{\mathcal{R}}_{k,l-l'+1},\ l', l-l'+1 \geq 2 \tag{5.47}$$

The appearance of 1-coincident spaces ${}_1\mathcal{R}_{k-k'+1,l-l'+1}$ in (5.43)-(5.45) has a simple explanation. For simultaneous bubbling of L and R's to occur, the bubble point must be coincident, i.e. colored LR.

Strictly speaking, the compactification we have described is somewhat larger than we would ideally like; the strata of most interest to us are (5.46) and (5.47). However, we will be able to make arguments showing that operations coming from the other strata must all be zero.

Remark 5.3. *We could further reduce the compactification of $\mathcal{R}_{k,l}$ by collapsing strata whose subtrees are monochromatic L or R except for a single off-color point. The construction would make points colored R point view entire subtrees colored L as invisible and vice versa. Having made this construction, one can then check that the resulting operations we construct will not change. We have thus opted for a simpler construction at the expense of having a larger compactification.*

5.3. Sequential point identifications. We study a particular classes of submanifolds of pairs of discs, and examine its compactifications carefully. This compactification is the one that will arise when defining quilts and Floer theoretic operations in the product.

Definition 5.9. *A point identification $\mathfrak{P}$ is said to be* **sequential** *if it is of the form*

$$\mathfrak{S} = \{(i_1, j_1), (i_1+1, j_1+1), \ldots, (i_1+s, j_1+s)\}. \tag{5.48}$$

It is further said to be **initial** *if $(i_1, j_1) = (1, 1)$.*

Definition 5.10. *A* **cyclic sequential point identification** *of type (r, s) is one of the form*

$$\mathfrak{S} = \{(1,1), (2,2), \ldots, (r,r), (k-s, l-s), (k-s+1, l-s+1), \ldots, (k, l)\}. \tag{5.49}$$

In other words, it is a sequential point identification where we need to take indices mod (k, l).

Proposition 5.1. *Let $\mathfrak{P}$ be a (k, l) initial sequential point identification of length s. Then the codimension-1 boundary of the compactification of $\mathfrak{P}$-identified pairs of discs*

$$_{\mathfrak{P}}\overline{\mathcal{R}}_{k,l} \tag{5.50}$$

is covered by the natural inclusions of the following products:

$$_{\mathfrak{P}_{max}}\overline{\mathcal{R}}_{d,d} \times {}_{\mathfrak{P}'}\overline{\mathcal{R}}_{k-d+1,l-d+1} \tag{5.51}$$

$$_{\mathfrak{P}'}\overline{\mathcal{R}}_{k',l'} \times {}_{\mathfrak{P}''}\overline{\mathcal{R}}_{k-k'+1,l-l'+1} \tag{5.52}$$

$$\overline{\mathcal{R}}_{k',l'} \times {}_{\mathfrak{P}\cup(s,t)}\overline{\mathcal{R}}_{k-k'+1,l-l'+1} \tag{5.53}$$

$$\overline{\mathcal{R}}_{k',1} \times {}_{\mathfrak{P}\cup(s,t)}\overline{\mathcal{R}}_{k-k'+1,l} \tag{5.54}$$

$$\overline{\mathcal{R}}_{1,l'} \times {}_{\mathfrak{P}\cup(s,t)}\overline{\mathcal{R}}_{k,l-l'+1} \tag{5.55}$$

$$\overline{\mathcal{R}}_{0,l'} \times {}_{\mathfrak{P}}\overline{\mathcal{R}}_{k,l-l'+1} \tag{5.56}$$

$$\overline{\mathcal{R}}_{k',0} \times {}_{\mathfrak{P}}\overline{\mathcal{R}}_{k-k'+1,l}. \tag{5.57}$$

Proof. We will only say a few words about the Proposition. Note that under the point-coincidence stratification, the top stratum of the non-compactified space ${}_{\mathfrak{P}}\mathcal{R}_{k,l}$, in which there are no other coincidences, correspond to all tricolorings of discs with $k+l-i$ marked points with LR colorings specified by $\mathfrak{P}$ and L, R colorings arbitrary

$$_{\mathfrak{P}}\mathcal{R}_{k,l} = \coprod_{I \cup J \cup \mathfrak{P} = [k+l-i]} {}_{I,J,\mathfrak{P}}\mathcal{R}^{k+l-|\mathfrak{P}|}. \tag{5.58}$$

Thus, we can determine the codimension-one boundary of the compactification by looking at the boundary components of the compactifications ${}_{I,J,\mathfrak{p}}\overline{\mathcal{R}}_{k,l}$ which survive our chart maps (5.38).

The possible strata that arise fall into three different cases: bubbling occurs entirely within the coincident points (5.51), bubbling overlaps somewhat with the coincident points (5.52), and bubbling stays entirely away from the coincident points (5.53) - (5.57). Note once more that when L and R points simultaneously bubble, there is an additionl coincident point created, hence the need to add various (s,t) to the coincident set in (5.53)-(5.55). $\square$

REMARK 5.4. *For the diagonal associahedron $\Delta_d(\mathcal{R}^d)$, thought of as the space ${}_{\mathfrak{p}_{max}}\mathcal{R}_{d,d}$ with $\mathfrak{p}_{max} = \{(1,1),(2,2),\ldots,(d,d)\}$, one can check that the compactification constructed in the previous subsection coincides precisely is just the usual Deligne-Mumford compactification ${}_{0,0,d}\overline{\mathcal{R}}^d$. In particular Δ_d extends to a diffeomorphism $\overline{\mathcal{R}}^d \cong {}_{\mathfrak{p}_{max}}\overline{\mathcal{R}}_{d,d}$, and equivalently the only codimension-1 stratum appearing above is* (5.51).

5.4. Gluing discs. We now make precise the notion of *gluing* pairs of discs along some identified boundary components, a construction that will arise from incorporating the diagonal Lagrangian Δ as an admissible Lagrangian in the product $M^- \times M$. We begin by discussing the combinatorial type of a **boundary identifications** of a pair of discs.

DEFINITION 5.11. *A (k,l)* **boundary identification** *is a subset $\mathfrak{S}$ of the set of pairs $\{0,\ldots,k\} \times \{0,\ldots,l\}$ satisfying the following conditions:*

- *$(0,0)$ and (k,l) are the only admissible pairs in $\mathfrak{S}$ containing extrema.*
- *(***monotonicity***) $\mathfrak{S}$ can be written as $\{(i_1,j_1),\ldots,(i_s,j_s)\}$ with $i_r < i_{r+1}$ and $j_r < j_{r+1}$.*

DEFINITION 5.12. *Let S and T be unit discs in $\mathbb{C}$ with k and l incoming boundary marked points respectively, and one outgoing boundary point each. Assume further that the outgoing boundary points of S and T are in the same position. Label the boundary components of S*

$$\{\partial^0 S,\ldots,\partial^k S\} \tag{5.59}$$

in **counterclockwise** *order from the outgoing point, and label the components of T*

$$\{\partial^0 T,\ldots,\partial^l T\} \tag{5.60}$$

in **counterclockwise** *order from the outgoing point. Let $\mathfrak{S}$ be a (k,l) boundary identification. S and T are said to be* $\mathfrak{S}$**-compatible** *if*

- *the outgoing points of S and T are at the same position.*
- *The identity map induces a one-to-one identification of $\partial^x S$ with $\partial^y T$ for each $(x,y) \in \mathfrak{S}$.*

The notion of $\mathfrak{S}$-compatibility is manifestly invariant under simultaneous automorphism of the pair (S,T). $\mathfrak{S}$-compatibility of a pair (S,T) also implies certain point coincidences, in the sense of the previous section.

DEFINITION 5.13. *Let $\mathfrak{S}$ be a (k,l) boundary identification. The* **associated** (k,l) **point identification**

$$p(\mathfrak{S}) \tag{5.61}$$

is defined as follows:

$$p(\mathfrak{S}) := \{(i,j) | (i,j) \in \mathfrak{S} \text{ or } (i-1,j-1) \in \mathfrak{S}\}. \tag{5.62}$$

Moreover, if S and T have coincident points consistent with the induced point-identification $p(\mathfrak{S})$, then S and T are also $\mathfrak{S}$ compatible. Hence, we can make the following definition:

DEFINITION 5.14. *A (k,l) boundary identification $\mathfrak{S}$ is said to be* **compatible with a** (k,l) **point identification** $\mathfrak{T}$ *if*

$$p(\mathfrak{S}) \subseteq \mathfrak{T} \tag{5.63}$$

where $p(\mathfrak{S})$ is the associated point identification.

Now, let $\mathfrak{S}$ be a (k,l) boundary identification with compatible point identification $\mathfrak{T}$. Given any pair of discs with $\mathfrak{T}$ point coincidences, there is an associated tricolored disc in the manner described in the previous section. We see that a boundary identification can be thought of as a binary {“identified”, “not identified”} labeling of the boundary components between $\mathfrak{T}$-coincident points, which were colored LR. In this manner, a boundary identification $\mathfrak{S}$ induces a boundary identification on nodal elements in the compactification

$$ {}_{\mathfrak{T}}\overline{\mathcal{R}}_{k,l}; \tag{5.64}$$

We can see this as follows. On any nodal component of this space, there are induced point coincidences coming from gluing maps. Label a boundary component between two coincident points as "identified," if, after gluing, the boundary component corresponds to one labeled "identified."

Thus, in the same manner that we have already spoken about boundary-labeled moduli spaces, we can define the **moduli space of $\mathfrak{S}$ identified pairs of discs with $\mathfrak{T}$ point identifications and (k,l) marked points**

$$ {}_{\mathfrak{T},\mathfrak{S}}\overline{\mathcal{R}}_{k,l} \tag{5.65}$$

to be exactly ${}_{\mathfrak{T}}\overline{\mathcal{R}}_{k,l}$ with the additional boundary labelings that we described above.

Our reason for defining boundary identification is so that we can speak more easily about gluings.

Definition 5.15. *Let S and T be compatible with a (k,l) boundary identification datum $\mathfrak{S}$. The* **$\mathfrak{S}$-gluing**

$$ \pi_{\mathfrak{S}} := S \coprod_{\mathfrak{S}} T \tag{5.66}$$

is the genus 0 open-closed string defined as follows: view $-S$, i.e. S with the **opposite complex structure** *as being the south half of a sphere bounding the equator via the complex doubling procedure, one of the methods of constructing the moduli of bordered surfaces* [**L**, §3.1]. *Similarly, view T as the north half of the sphere. Then*

$$ (S \coprod_{\mathfrak{S}} T) := (-S) \coprod T / \sim \tag{5.67}$$

where $\sim$ identifies $\partial^x(-S)$ to $\partial^y T$ ($\partial^x(-S)$ is the same boundary component of S as before, now with the reverse orientation) under the identification coming from inclusion into the sphere if and only if $(x,y) \in \mathfrak{S}$. Boundary marked points are identified as follows: Let z^i_{-S} be the boundary marked point between $\partial^{i-1}(-S)$ and $\partial^i(-S)$, z^0_{-S} the outgoing marked point, and z^j_T similar. Then:

- *if $(x-1,y-1),(x,y) \in \mathfrak{S}$, then $z^x_{-S} \sim z^y_T$ becomes a single interior marked point.*
- *if $(x-1,y-1) \in \mathfrak{S}$ but (x,y) is not, then $z^x_{-S} \sim z^y_T$ becomes a single boundary marked point, between $\partial^y T$ and $\partial^x(-S)$*
- *if $(x,y) \in \mathfrak{S}$ but $(x-1,y-1)$ is not, then $z^x_{-S} \sim z^y_T$ becomes a single boundary marked point, between $\partial^x(-S)$ and $\partial^y T$*
- *otherwise, z^x_{-S} and z^y_T are kept distinct, becoming two boundary marked points.*

By $\mathfrak{S}$-compatibility, S and T can be viewed as the south and north halves of a sphere in a manner preserving the alignment of outgoing marked points and boundary components specified by $\mathfrak{S}$, so the above definition is sensible.

One can read off the characteristics of the resulting bordered surface from k, l, and $\mathfrak{S}$, which we leave as an exercise. Denote the resulting number of boundary components of the open-closed string

$$ h(k,l,\mathfrak{S}) \tag{5.68}$$

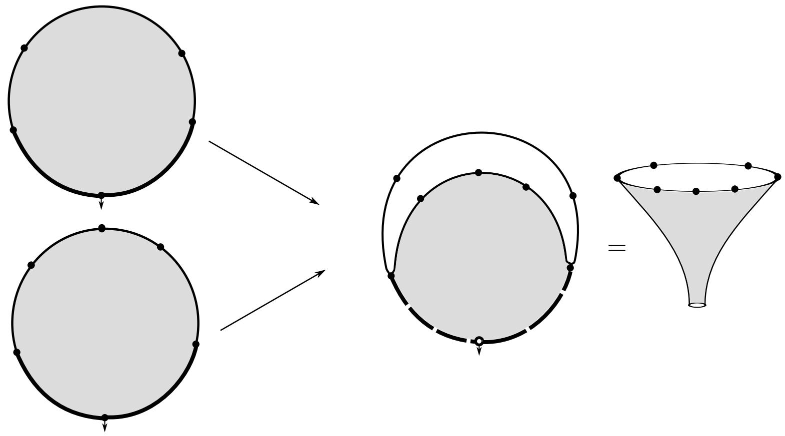
and the number of connected components (which is 1 unless $\mathfrak{S} = \emptyset$, in which case it is 2)

$$ c(k,l,\mathfrak{S}). \tag{5.69}$$

Proposition 5.2. *Let $\mathfrak{S}$ be a boundary identification, with compatible point identification $\mathfrak{T}$. Then, the gluing operation $S \coprod_{\mathfrak{S}} T$ extends to an operation on the Deligne-Mumford compactifications ${}_{\mathfrak{T}}\overline{\mathcal{R}}_{k,l}$.*

Proof. Nodal components of an element P in the Deligne-Mumford compactification ${}_{\mathfrak{T}}\overline{\mathcal{R}}_{k,l}$ can be thought of as nodal tri-colored discs, with induced boundary identifications in a manner we have already described. As we have earlier indicated, the L/R forgetful maps applied to P give us **unreduced left and right disc trees** $(\tilde{S}, \tilde{T})$. Boundary identifications descend to these trees, because by definition they were

FIGURE 11. An example of the gluing $\pi_{\mathfrak{S}}$ associated to a $\{(1,1),(k,l)\}$ boundary identification.



labelings between points colored LR. We now perform the above procedure component-wise on this pair to obtain a nodal open-closed string $\tilde{S}\coprod_{\mathfrak{S}}(\tilde{T})$, which potentially has semi-stable/unstable components. Finally, define

$$\pi_{\mathfrak{S}}(P) := S\coprod_{\mathfrak{S}} T \tag{5.70}$$

to be the nodal open-closed string obtained by stabilizing $\tilde{S}\coprod_{\mathfrak{S}}\tilde{T}$. □

In the case when the boundary identification $\mathfrak{S}$ is empty and both $k,l \geq 2$, the gluing operation $\pi_{\mathfrak{S}}$ reduces to the projection we defined earlier, where we conjugate the first factor:

$$\pi_{\emptyset} := \pi_{reduce} \circ (-1 \times id). \tag{5.71}$$

In the semi-stable case e.g., $k=1, l \geq 2$, $\pi_{\mathfrak{S}}(P)$ is defined simply to be the disjoint union of a conjugaged (semi-stable) strip $S := \Sigma_1$ (collapsing any possible higher semi-stable or unstable bubbles that may occur in the unreduced left tree $\tilde{S}$ but keeping the root semi-stable disc) with the stabilized right factor T. The case ($l=1$, $k \geq 2$ is similar.

5.5. Floer data and operations.

DEFINITION 5.16. *A* **Floer datum** *for a glued pair of discs $(P,\mathfrak{S})$ is a Floer datum for the resulting open-closed string $\pi_{\mathfrak{S}}(P)$, in the sense of Definition 4.11.*

Now, let us assume that our point identification $\mathfrak{T}$ was sequential or cyclic sequential.

DEFINITION 5.17. *A* **universal and consistent choice of Floer data for glued pairs of discs $\mathbf{D}_{glued}$** *is a choice $D_{P,\mathfrak{S}}$ of Floer data in the sense of Definition 5.16 for every k, l, (k,l) boundary identification $\mathfrak{S}$ and compatible sequential point identification $\mathfrak{T}$, and every representative ${}_{\mathfrak{T},\mathfrak{S}}\overline{\mathcal{R}}_{k,l}$, varying smoothly over ${}_{\mathfrak{T},\mathfrak{S}}\overline{\mathcal{R}}_{k,l}$, whose restriction to a boundary stratum is conformally equivalent to the product of Floer data coming from lower dimensional moduli spaces. Moreover, with regards to the coordinates, Floer data agree to infinite order at the boundary stratum with the Floer data obtained by gluing. Finally, we require that this choice of Floer datum satisfy the following conditions:*

(5.72) *The Floer datum only depends on the open-closed string $\pi_{\mathfrak{S}}(P)$; and*

(5.73) *The Floer datum agrees with our previously chosen Floer datum on $\pi_{\mathfrak{S}}(P)$; and*

(5.74) *When $\mathfrak{S} = \emptyset$ and $k=1$ or $l=1$, this datum agrees with the union of the Floer data $\mathbf{D}_{\mu}$ for the A_{∞} operations on the stable factor with the translation invariant Floer datum on the other.*

Definition 5.18. *A* **Lagrangian labeling from L** *for a glued pair of discs* $(P,\mathfrak{S})$ *is a Lagrangian labeling from* $\mathbf{L}$ *for the gluing* $\pi_{\mathfrak{S}}(P) = S\coprod_{\mathfrak{S}} T$*, thought of as a (possibly disconnected) open-closed string. Given a fixed labeling* $\vec{L}$*, denote by*

$$({}_{\mathfrak{T},\mathfrak{S}}\overline{\mathcal{R}}_{k,l})_{\vec{L}} \tag{5.75}$$

the space of labeled $\mathfrak{S}$*-identified pairs of discs with* $\mathfrak{T}$ *point coincidences.*

Now, fix a compact oriented submanifold with corners of dimension d,

$$\overline{\mathcal{L}}^d \hookrightarrow {}_{\mathfrak{T},\mathfrak{S}}\overline{\mathcal{R}}_{k,l} \tag{5.76}$$

Fix a Lagrangian labeling

$$\vec{L} = \{\{L_0^1,\ldots,L_{m_1}^1\},\{L_0^2,\ldots,L_{m_2}^2\},\ldots,\{L_0^h,\ldots,L_{m_h}^h\}\}. \tag{5.77}$$

Also, fix chords

$$\vec{x} = \{\{x_1^1,\ldots,x_{m_1}^1\},\ldots,\{x_1^h,\ldots,x_{m_h}^h\}\} \tag{5.78}$$

and orbits $\vec{y} = \{y_1,\ldots,y_n\}$ with

$$x_i^j \in \begin{cases} \chi(L_{i+1}^j, L_i^j) & i\in K^j \\ \chi(L_i^j, L_{i+1}^j) & \text{otherwise.}\end{cases} \tag{5.79}$$

Above, the index i in L_i^j is counted mod m_j. Collectively, the $\vec{x},\vec{y}$ are called a set of **asymptotic conditions** for the labeled moduli space $\mathcal{L}^d_{\vec{L}}$. The **outputs** $\vec{x}_{out}$, $\vec{y}_{out}$ are by definition those x_i^j and y_s for which $i\in K^j$ and $s\in\mathbf{I}$, corresponding to negative marked points. The **inputs** $\vec{x}_{in}$, $\vec{y}_{in}$ are the remaining chords and orbits from $\vec{x}$, $\vec{y}$. Fixing a chosen universal and consistent Floer datum, denote $\epsilon_{\pm}^{i,j}$ and $\delta_{\pm}^l$ the strip-like and cylindrical ends corresponding to x_i^j and y_l respectively.

Finally, define

$$\overline{\mathcal{L}}^d(\vec{x}_{out},\vec{y}_{out};\vec{x}_{in},\vec{y}_{in}) \tag{5.80}$$

to be the space of maps

$$\{u:\pi_{\mathfrak{S}}(P)\longrightarrow M:\ P\in\overline{\mathcal{L}}^d\} \tag{5.81}$$

satisfying, at each element P, *Floer's equation for* $(\mathbf{D}_{\mathfrak{S}})_P$ with boundary and asymptotic conditions

$$\begin{cases} \lim_{s\to\pm\infty} u\circ\epsilon_{\pm}^{i,j}(s,\cdot) = x_i^j, \\ \lim_{s\to\pm\infty} u\circ\delta_{\pm}^{l}(s,\cdot) = y_l, \\ \quad u(z)\in\psi^{a_S(z)}L_i^j, & z\in\partial_i^j S. \end{cases} \tag{5.82}$$

We have the usual transversality and compactness results:

Lemma 5.1. *The moduli spaces* $\overline{\mathcal{L}}^d(\vec{x}_{out},\vec{y}_{out};\vec{x}_{in},\vec{y}_{in})$ *are compact and there are only finitely many collections* $\vec{x}_{out}$*,* $\vec{y}_{out}$ *for which they are non-emtpy given input* $\vec{x}_{in}$*,* $\vec{y}_{in}$*. For a generic universal and conformally consistent Floer data they form manifolds of dimension*

$$\begin{aligned}\dim\mathcal{L}^d(\vec{x}_{out},\vec{y}_{out};\vec{x}_{in},\vec{y}_{in}) &:= \sum_{x_-\in\vec{x}_{out}}\deg(x_-) + \sum_{y_-\in\vec{y}_{out}}\deg(y_-) \\ &+(2c(k,l,(S)-h(k,l,\mathfrak{S})-|\vec{x}_{out}|-2|\vec{y}_{out}|)n + d - \sum_{x_+\in\vec{x}_{in}}\deg(x_+) - \sum_{y_+\in\vec{y}_{in}}\deg(y_+).\end{aligned} \tag{5.83}$$

Proof. The index computation follows from the arguments outlined in the proof of Lemma 4.3. The proof of transversality for generic perturbation data is once more an application of Sard-Smale, following arguments in [**S5**, (9k)] or alternatively [**FHS**]. These arguments show that the extended linearized operator for Floer's equation, in which one allows deformations of the almost complex structure and one-form, is surjective. As their arguments are on the level of stabilized moduli spaces, they imply that transversality can be achieved in our situation by taking perturbations of Floer data that are constant along the fibers of the projection map $\pi_{\mathfrak{S}}$. In other words, the class of Floer data satisfying (5.72) is large enough to achieve transversality.

The usual Gromov compactness applies once Theorem B.1 is applied to obtain a priori bounds on maps satisfying Floer's equation with fixed asymptotics. □

When the dimension of $\mathcal{L}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ is 0, we conclude that its elements are rigid. In particular, any such element $u \in \mathcal{L}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ gives (by Lemma A.1) an isomorphism of orientation lines

$$\mathcal{L}_u : \bigotimes_{x \in \vec{x}_{in}} o_x \otimes \bigotimes_{y \in \vec{y}_{in}} o_y \longrightarrow \bigotimes_{x \in \vec{x}_{out}} o_x \otimes \bigotimes_{y \in \vec{y}_{out}} o_y. \tag{5.84}$$

Using this we define a map

$$\begin{aligned} \mathbf{G}_{\overline{\mathcal{L}}^d} : \bigotimes_{(i,j); 1 \leq i \leq m_j; i \notin K^j} CW^*(L_i^j, L_{i+1}^j) \otimes \bigotimes_{1 \leq k \leq n; k \notin \mathbf{I}} CH^*(M) \longrightarrow \\ \bigotimes_{(i,j); 1 \leq i \leq m_j; i \in K^j} CW^*(L_{i+1}^j, L_i^j) \otimes \bigotimes_{1 \leq k \leq n; k \in \mathbf{I}} CH^*(M) \end{aligned} \tag{5.85}$$

given by, as usual (abbreviating $\vec{x}_{in} = \{x_1, \ldots, x_s\}$, $\vec{y}_{in} = \{y_1, \ldots, y_t\}$)

$$\begin{aligned} &\mathbf{G}_{\overline{\mathcal{L}}^d}([y_t], \ldots, [y_1], [x_s], \ldots, [x_1]) := \\ &\sum_{\dim \mathcal{L}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in}) = 0} \; \sum_{u \in \mathcal{L}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})} \mathcal{L}_u([x_1], \ldots, [x_s], [y_1], \ldots, [y_t]). \end{aligned} \tag{5.86}$$

This construction naturally associates, to any submanifold $\mathcal{L}^d \in {}_{\mathfrak{S},\mathfrak{T}}\mathcal{R}_{k,l}$, a map $\mathbf{G}_{\mathcal{L}^d}$, depending on a sufficiently generic choice of Floer data for glued pairs of discs. In a similar fashion, this can be done for a submanifold of the labeled space

$$\mathcal{L}^d_{\vec{L}} \subset ({}_{\mathfrak{S},\mathfrak{T}}\mathcal{R}_{k,l})_{\vec{L}}, \tag{5.87}$$

in which case the result is an operation defined only for a specific labeling,

$$\mathbf{G}_{\overline{\mathcal{L}}^d_{\vec{L}}}, \tag{5.88}$$

This operation can also be constructed with a sign twisting datum to create an operation

$$(-1)^{\vec{t}} \mathbf{G}_{\overline{\mathcal{L}}^d} \tag{5.89}$$

in an identical fashion to (4.30).

5.6. Examples. As a first example, consider the case $\mathfrak{S} = \emptyset$ and $\mathcal{L}$ equal to the full $\overline{\mathcal{R}}_{k,l}$.

Proposition 5.3. *The operation associated to $\mathcal{L} = \overline{\mathcal{R}}_{k,l}$ with arbitrary Lagrangian labeling (and any sign twist) is 0 if both k and l are ≥ 1 and one of (k,l) is ≥ 2.*

Proof. Let u be a rigid element in the associated moduli space $\overline{\mathcal{R}}_{k,l}(\vec{x}_{in}; \vec{x}_{out})$; since we are in the transverse situation, we can assume the domain of u is a point in the interior $p \in \mathcal{R}_{k,l}$. On the interior, the projection map

$$\pi_\emptyset : \mathcal{R}_{k,l} \longrightarrow \mathcal{R}^k \times \mathcal{R}^l \tag{5.90}$$

has fibers of dimension at least 1, parametrized by automorphisms of one factor relative to the other. (when $k = 1$, we implicitly replace $\mathcal{R}^k$ by a point, and same for l—stabilization in this case completely collapses the left or right component). Since our Floer data was chosen to only depend on $\pi_\emptyset(p)$, we conclude that any map from an element of the fiber $\pi_\emptyset^{-1}(\pi_\emptyset(p))$ also satisfies Floer's equation; hence u cannot be rigid. □

Proposition 5.4. *The operation associated to the compactification of the inclusion $\mathcal{I}_k$ in (5.8) is $(\mu^k_{\mathcal{W}})^{op} \otimes id$. Similarly the operation associated to $\mathcal{J}_l$ as in (5.9) is $id \otimes \mu^l_{\mathcal{W}}$.*

Proof. A rigid element u of the moduli space associated to $\overline{\mathrm{im}(\mathcal{I}_k)}$ has, without loss of generality, domain in the interior $\mathrm{im}(\mathcal{I}_k)$. We note first that the projection map

$$\pi_\emptyset : \mathrm{im}(\mathcal{I}_k) \to \mathcal{R}^k \times \{*\} \to \mathcal{R}^k \tag{5.91}$$

is is compatible with Floer data, in particular that the Floer datum on the left factor coincides with the one chosen for the A_∞ structure $\mathbf{D}_\mu$. Next we note that if the right factor of an element of the moduli space

$$\overline{\mathcal{I}_k(\mathcal{R}^k)}((x_0, x_0'); ((x_1, \dots, x_k), x_0')) \tag{5.92}$$

is not a constant map, then elements of this moduli space cannot be rigid, as one can act by translations on the right factor by the condition (5.74). Hence, we obtain an isomorphism of 0-dimensional moduli spaces

$$\overline{\mathcal{I}_k(\mathcal{R}^k)}((x_0, x_0'); ((x_1, \dots, x_k), x_0')) \simeq \mathcal{R}^k(x_0; x_k, \dots, x_1) \times \{*\}, \tag{5.93}$$

where now $\{*\}$ denotes the constant solution to Floer's equation, implying the result for $\mathcal{I}_k$. The result for $\mathcal{J}_l$ is analogous. □

More generally, one can look at the submanifold $\mathcal{Q}_2^{k,l} \subseteq \mathcal{R}_{k,l}$ of pairs of discs where the negative marked points are required to coincide and the marked points immediately counterclockwise are required to coincide.

Proposition 5.5. *The associated operation is 0 unless $k = 1$ or $l = 1$, in which case the previous proposition applies.*

Proof. In this case, the projection

$$\pi_\emptyset : \mathcal{Q}_2^{k,l} \longrightarrow \mathcal{R}^k \times \mathcal{R}^l \tag{5.94}$$

has one dimensional fibers, parametrized by automorphism of one factor relatve another. We conclude in the manner of the previous two propositions that elements of the associated moduli spaces can never be rigid. □

We can also look at the submanifold

$$_{\emptyset,\mathfrak{T}}\mathcal{R}_{k,l} \tag{5.95}$$

of pairs of discs corresponding to the point identification

$$\mathfrak{T} = \{(1,1), (2,2)\}, \tag{5.96}$$

i.e. discs where the negative point, and first two positive points, are required to coincide. This submanifold can be thought of as the image of an open embedding from the pair of associahedra

$$\mathcal{R}^r \times \mathcal{R}^s \xrightarrow{Q} \mathcal{R}_{r,s} \tag{5.97}$$

which can be described as follows: Take the representative of each disc on the left $(-S_1, S_2)$ for which the negative marked point and the two marked points immediately counterclockwise of (S_1, S_2) have been mapped to $-i$, 1, and i respectively. Define

$$Q([S_1], [S_2]) := [(S_1, S_2)]. \tag{5.98}$$

In other words, associate to a pair of discs mod automorphism the pair mod simultaneous automorphism in which the output and the first two inputs are required to coincide. The embedding Q not quite extend to an embedding of $\overline{\mathcal{R}^r} \times \overline{\mathcal{R}^s}$ because, among other phenomena, the three chosen points on each disc will come together simultaneously in codimension 1. We can still study the operation determined by the compactification of the embedding.

Proposition 5.6. *The operation associated to the compactification $\overline{Q(\mathcal{R}^r \times \mathcal{R}^s)}$ is $(\mu^r)^{op} \otimes \mu^s$.*

Proof. The map Q is a left and right inverse to the projection map

$$\pi_\emptyset :_{\emptyset,\{(1,1),(2,2)\}} \mathcal{R}_{k,l} \longrightarrow \mathcal{R}^k \times \mathcal{R}^l \tag{5.99}$$

so $\pi_\emptyset$ is an isomorphism, up to direction reversal of first factor. We conclude that there is an identification of dimension zero moduli spaces

$$\overline{\mathrm{im}(Q)}(x^1_{out}, x^2_{out}; \vec{x}^1_{in}, \vec{x}^2_{in}) \simeq \overline{\mathcal{R}}^k(x^1_{out}; (\vec{x}^1_{in})^{op}) \times \overline{\mathcal{R}}^l(x^2_{out}; \vec{x}^2_{in}). \tag{5.100}$$

where the *op* superscript indicates an order reversal. □

Now, consider the case of a single gluing adjacent to the outgoing marked points, i.e. $\mathfrak{S} = \{(1,1)\}$ or $\mathfrak{S} = \{(k,l)\}$ with the induced point identification.

Proposition 5.7. *The resulting operation in either case is μ^{k+l+1}.*

Proof. We will without loss of generality do $\mathfrak{S} = \{(1,1)\}$; the associated point identification is also $p(\mathfrak{S}) = \{(1,1)\}$. The gluing morphism is of the form

$$\pi_{\mathfrak{S}} :_{\mathfrak{S},p(\mathfrak{S})} \mathcal{R}_{k,l} \longrightarrow \mathcal{R}^{k+l+1}. \tag{5.101}$$

if k or l is ≥ 1, the unreduced gluing is automatically stable, implying that (5.101) is an isomorphism. We obtain a corresponding identification of moduli spaces. $\square$

Our next example is the case $\mathfrak{S} = \{(1,1),(k,l)\}$ with the induced point identification.

Proposition 5.8. *The resulting operation is exactly ${}_2\mathcal{OC}^{k-2,l-2}$.*

Proof. The surface obtained by gluing the $(1,1)$ and (k,l) boundary components together in $\mathcal{R}_{k,l}$ is stable, and has one interior output marked point. There are also $k+l$ boundary marked points, two of which are special. In cyclic order on the boundary, there is the identified point p_1 coming from the $(1,1)$ boundary points, the $k-2$ non-identified points from the left disc, the identified point p_2 coming from the (k,l) boundary points, and the $l-2$ non-identified points from the right disc. Moreover, the identified points p_1, p_2, and the interior boundary point are required to, up to equivalence, lie at the points $-i$, 0, and i respectively. We conclude that the projection is an isomorphism onto

$$\pi_{\mathfrak{S}} : {}_{\mathfrak{S},p(\mathfrak{S})}\mathcal{R}_{k,l} \longrightarrow \mathcal{R}^1_{k-2,l-2}, \tag{5.102}$$

the moduli space controlling ${}_2\mathcal{OC}^{k-2,l-2}$.

See also Figure 11 for an image of this situation. $\square$

Our final example is the case $\mathfrak{S} = \{(1,1),(2,2)\}$, with the induced point identification.

Proposition 5.9. *The resulting operation is exactly ${}_2\mathcal{CO}^{k-2,l-2}$.*

Proof. The surface obtained by gluing the $(1,1)$ and $(2,2)$ boundary components together in $\mathcal{R}_{k,l}$ is stable, has one interior input marked point, and in counterclockwise order on the boundary has one output boundary marked point (which was adjancent to the $(1,1)$ gluing), $l-2$ additional boundary inputs, one special boundary input (which was adjacent to the $(2,2)$ gluing), and $k-2$ additional boundary inputs. Moreover, the output boundary marked point, the interior input, and the special boundary input are required to, up to equivalence, lie at the points $-i$, 0, and i respectively. We conclude that the projection is an isomorphism onto

$$\pi_{\mathfrak{S}} : {}_{\mathfrak{S},p(\mathfrak{S})}\mathcal{R}_{k,l} \longrightarrow \mathcal{R}^{1,1}_{k-2,l-2}, \tag{5.103}$$

the moduli space controlling ${}_2\mathcal{CO}^{k-2,l-2}$. $\square$

6. Floer theory in the product

The Liouville manifold $M^2 := M^- \times M$ carries a natural symplectic form, $(-\omega_M, \omega_M)$ for which the diagonal is a Lagrangian submanifold. Let

$$\pi_i : M^2 \to M, \ i = 1,2$$

be the projection to the ith component. As observed by Oancea [**O1**], there is a natural cylindrical end on $M^- \times M$, with coordinate given by $r_1 + r_2$, where $r_i = \pi_i^* r$ is the coordinate on the i-th factor. Thus one could define symplectic homology and wrapped Floer theory by considering Hamiltonians of the form $(r_1+r_2)^2$ at infinity. To obtain the comparisons that we desire, we must consider Floer theory for split Hamiltonians of the form $\pi_1^* H + \pi_2^* H$, for $H \in \mathcal{H}(M)$. There are immediately some technical difficulties: split Hamiltonians are *not* admissible in the above sense, and in general will admit some additional chords near infinity. Using methods similar to [**O1**], one could prove that these orbits and chords have sufficiently negative action, do not contribute to the homology, and thus such split Hamiltonians are a posteriori admissible (thereby proving a Künneth theorem for wrapped Floer homology).

We bypass this issue and instead define all Floer-theoretic operations on the product for split Hamiltonians and almost complex structures. In this case, with suitably chosen Floer data, compactness and transversality follow, via *unfolding,* from compactness and transversality of certain open-closed moduli spaces of maps into M constructed from glued pairs of discs. The end result will be a model

$$\mathcal{W}^2 \tag{6.1}$$

for the wrapped Fukaya category of split Lagrangians and the diagonal in M^2, using only maps and morphisms in M.

REMARK 6.1. *There is another technical difficulty with considering Hamiltonians of the form $(r_1+r_2)^2$ at infinity: Products of admissible Lagrangians $L_i \times L_j$ are no longer a priori admissible in the product. Namely, it is not guaranteed (and highly unlikely) that the primitive $\pi_1^* f_{L_i} + \pi_2^* f_{L_j}$ is constant as $(r_1+r_2) \to \infty$. The usual method of proving that the relevant moduli spaces are compact does not work in this situation, and a more refined argument is needed.*

6.1. Floer homology with split Hamiltonians. First, let us examine Floer homology groups in M^2 for a class of split Hamiltonians $\tilde{\mathcal{H}}(M \times M)$ of the form

$$\pi_1^* H + \pi_2^* H, \ H \in \mathcal{H}(M). \tag{6.2}$$

For Floer homology between split Lagrangians we immediately obtain a Künneth decomposition.

LEMMA 6.1. *For H and J generic, there is an identification of (relatively graded) complexes over $\mathbb{Z}_2$*

$$CW^*(L_1 \times L_2, L_1' \times L_2'; \pi_1^* H + \pi_2^* H, (-J, J)) = CW^*(L_1', L_1, H, J) \otimes CW^*(L_2, L_2'; H, J) \tag{6.3}$$

where the differential on the right hand side is $\delta_{L_1', L_1} \otimes 1 + 1 \otimes \delta_{L_2, L_2'}$.

PROOF. If X is the Hamiltonian vector field corresponding to H, note that the complex on the left-hand side of 6.3 is generated by time 1 flows of the vector field $(-X, X)$, so there is a one-to-one correspondence of generators. By examining equations we see that there is a one-to-one correspondence of strips

$$\mathcal{R}^1((x_0, x_0'), (x_1, x_1')) = \mathcal{R}^1(x_0; x_1) \times \mathcal{R}^1(x_0'; x_1') \tag{6.4}$$

In particular, the dimension of $\mathcal{R}^1((x_0, x_0'), (x_1, x_1'))$ is

$$\deg x_0 - \deg x_1 + \deg x_0' - \deg x_1' = \dim \mathcal{R}^1(x_0; x_1) + \dim \mathcal{R}^1(x_0'; x_1')$$

This implies that the one-dimensional component of the moduli space is the union of

- the one-dimensional component of $\mathcal{R}^1(x_0; x_1)$ times the zero-dimensional component of $\mathcal{R}^1(x_0'; x_1')$.
- the one-dimensional component of $\mathcal{R}^1(x_0'; x_1')$ times the zero-dimensional component of $\mathcal{R}^1(x_0; x_1)$.

But the zero-dimensional components of the above moduli spaces must be constant maps. If they were not, $\mathbb{R}$-translation of strips would imply that solutions come in 1-parameter families, a contradiction. This implies the Lemma. ☐

Floer trajectories with the diagonal Lagrangian Δ unfold and can be compared to symplectic cohomology trajectories, for appropriate Hamiltonians.

LEMMA 6.2. *For H_t and J generic,*

$$CW^*(\Delta, \Delta, \pi_1^* \frac{1}{2} H_{1-t/2} + \pi_2^* \frac{1}{2} H_{t/2}, (-J, J)) = CH^*(M, H_t, J) \tag{6.5}$$

as relatively graded chain complexes and over $\mathbb{Z}_2$.

PROOF. Denote $\hat{H}_t = \pi_1^* \frac{1}{2} H_{1-t/2} + \pi_2^* \frac{1}{2} H_{t/2}$. The correspondence between generators is as follows: Given a time 1 orbit x of H_t, we construct a time 1 chord from Δ to Δ of $\hat{H}_t$

$$\hat{x}(t) = (x(1-t/2), x(t/2)). \tag{6.6}$$

Conversely, given a time 1 chord $\hat{x} = (x_1, x_2)$ of $\hat{H}_t$, the corresponding orbit of H_t is given by:

$$x(t) = \begin{cases} x_2(2t) & t \leq 1/2 \\ x_1(2(1-t)) & 1/2 \leq t \leq 1 \end{cases} \tag{6.7}$$

Let us now identify the moduli spaces counted by either differential. First, suppose we have a map $u : \mathbb{R}\times\mathbb{R}/2\mathbb{Z} \to M$ satisfying:

$$\begin{cases} \partial_s u + J_t(\partial_t u - X_t) = 0 \\ \lim_{s\to\pm\infty} u(s,\cdot) = x_\pm \end{cases}$$

where X_t is the Hamiltonian vector field corresponding to H_t. Then, note that the map

$$\hat{u}(s,t) := (u_1(s, 1 - t/2), u_2(s, t/2)) \tag{6.8}$$

satisfies the following equation with asymptotics:

$$\begin{cases} \partial_s \hat{u} = -(-J_{1-t/2}, J_{1-t/2})\left(\partial_t \hat{u} - (-X_{1-t/2}, X_{t/2})\right) \\ \lim_{s\to\pm\infty} \hat{u}(s,\cdot) = \hat{x}_\pm, \end{cases} \tag{6.9}$$

Conversely, suppose we have a map $\hat{u} : \mathbb{R}\times[0,1] \to M^- \times M$ satisfying

$$\partial_s u + (-J_{1-t/2}, J_{1-t/2})(\partial_t u - (-X_{1-t/2}, X_{t/2})) = 0 \tag{6.10}$$

$$\lim_{s\to\pm\infty} u(s,\cdot) = \hat{x}_\pm \tag{6.11}$$

Let $u_i = \pi_i \circ \hat{u}$, and define

$$u(s,t) = \begin{cases} u_2(s, 2t) & 0 \le t \le 1/2 \\ u_1(s, 2(1-t)) & 1/2 \le t \le 1. \end{cases} \tag{6.12}$$

Because $\hat{u}(s,0)$ and $\hat{u}(s,1)$ lie on Δ, $u(s,t)$ is continuous across the *seams* $t = 0,\ 1/2$. It is clear that $\partial_s u$ is continuous along $t = 0,\ 1/2$. Thus, as u solves $\partial_s u + J_t(\partial_t u - X_t) = 0$ on both sides of $t = 0$ and $t = 1$, we see that $\partial_t u = J_t(\partial_s u) + X_t$ is continuous, so u is at least C^1 across the seams. Now inductively use the fact that ∂_s^k is continuous for all k along with applications of ∂_s and ∂_t to Floer's equation, to conclude that all other mixed partials are continuous. Therefore u is C^∞ across the seams. ☐

The cases of $HW^*(\Delta, L_i \times L_j)$ and $HW^*(L_i \times L_j, \Delta)$ are analogous, so we simply state them.

PROPOSITION 6.1. *As relatively graded chain complexes over $\mathbb{Z}_2$:*

$$CW^*(L_i \times L_j, \Delta, \frac{1}{2}(\pi_1^* H + \pi_2^* H), (-J, J)) = CW^*(L_j, L_i, H, J) \tag{6.13}$$

$$CW^*(\Delta, L_i \times L_j, \frac{1}{2}(\pi_1^* H + \pi_2^* H), (-J, J)) = CW^*(L_i, L_j, H, J). \tag{6.14}$$

(6.15)

In the setting of the ordinary Fukaya category, the analogue of Lemma 6.2 is the well-known correspondence between $HF^*(\Delta,\Delta)$ with the ordinary Hamiltonian Floer homology, or quantum cohomology, of the target manifold (note that to have such a correspondence over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$, one would need to equip $M \cong \Delta$ with a Spin structure or *relative Spin structure* with respect to a background class). Instead of continuing this correspondence for higher operations and, e.g. *unfolding* Floer data for discs mapping into M^2, we will take the above correspondence as a starting point for a definition of the category $\mathcal{W}^2$ using operations and Floer data in M. Define the **objects** of $\mathcal{W}^2$ as

$$\mathrm{ob}\ \mathcal{W}^2 := \{L_i \times L_j | L_i, L_j \in \mathrm{ob}\ \mathcal{W}\} \cup \{\Delta\}. \tag{6.16}$$

For objects X_k, X_l, define the *generators of the hom complexes*

$$\chi_{M^2}(X_k, X_l) := \begin{cases} \chi(L_j, L_i, H) \times \chi(L_i', L_j', H) & X_k = L_i \times L_i',\ X_l = L_j \times L_j' \\ \chi(L_j, L_i) & X_k = L_i \times L_j,\ X_l = \Delta \\ \chi(L_i, L_j) & X_k = \Delta,\ X_l = L_i \times L_j \\ \mathcal{O} & X_k = X_l = \Delta \end{cases} \tag{6.17}$$

We go back and forth between generators in $M^- \times M$ and those in M via the following corrrespondences:

$$\{x \in \chi_{M^2}(L_0 \times L_1, L_0' \times L_1')\} \longleftrightarrow \{\hat{x} = (x_1, x_2) \in \chi(L_0', L_0) \times \chi(L_1, L_1')\} \tag{6.18}$$

$$\{z \in \chi_{M^2}(L_0 \times L_1, \Delta)\} \longleftrightarrow \{\hat{z} \in \chi(L_1, L_0)\} \tag{6.19}$$

$$\{w \in \chi_{M^2}(\Delta, L_0 \times L_1)\} \longleftrightarrow \{\hat{w} \in \chi(L_0, L_1)\} \tag{6.20}$$

$$\{y \in \chi_{M^2}(\Delta, \Delta)\} \longleftrightarrow \{\hat{y} \in \mathcal{O}\} \tag{6.21}$$

(6.22)

We observe that naively attempting to inherit the absolute grading from M under these identifications will result in the A_∞ operations constructed having the wrong degree.

Proposition 6.2. *Fix a choice of gradings in M. Then, choose gradings for M^2 in the following manner: given correspondences of hom generators* (6.18)-(6.21)*: assign gradings to generators (and hence, associated orientation lines) as follows:*

$$\deg x = \deg \hat{x} = \deg x_1 + \deg x_2 \tag{6.23}$$

$$\deg z = \deg \hat{z} \tag{6.24}$$

$$\deg w = \deg \hat{w} + n \tag{6.25}$$

$$\deg y = \deg \hat{y}. \tag{6.26}$$

For this choice, the operations $\mu^d_{\mathcal{W}^2}$ constructed in the next section are of degree $2 - d$.

We defer a proof to the end of §6.2. Hence, as graded vector spaces, define morphism spaces:

$$\hom^i_{\mathcal{W}^2}(X_k, X_l) := \bigoplus_{x \in \chi^i_{M^2}(X_k, X_l)} \begin{cases} |o_{x_1}|_{\mathbb{K}} \otimes_{\mathbb{K}} |o_{x_2}| & X_k = L_i \times L_i',\ X_l = L_j \times L_j' \\ |o_{\hat{x}}|_{\mathbb{K}} & X_k = L_i \times L_j,\ X_l = \Delta \\ |o_{\hat{x}}|_{\mathbb{K}} & X_k = \Delta,\ X_l = L_i \times L_j \\ |o_y|_{\mathbb{K}} & X_k = X_l = \Delta \end{cases} \tag{6.27}$$

We define the differentials $\mu^1_{\mathcal{W}^2}$ to be the usual signed differentials coming from the basic correspondences in the Lemmas above, with slight sign twists to account for Koszul sign rules (in the case of morphisms between product Lagrangians) and the degree shift of n (for morphisms from the diagonal to a product):

$$(\mu^1_{\mathcal{W}^2})_{X_k, X_l}(x) = \begin{cases} (-1)^{|x_2|}\mu^1_{\mathcal{W}}(x_1) \otimes x_2 + x_1 \otimes \mu^1(x_2) & X_k = L_i \times L_i',\ X_l = L_j \times L_j', \hat{x} = (x_1, x_2) \\ \mu^1_{\mathcal{W}}(\hat{x}) & X_k = L_i \times L_j,\ X_l = \Delta \\ (-1)^n \mu^1_{\mathcal{W}}(\hat{x}) & X_k = \Delta,\ X_l = L_i \times L_j \\ d(\hat{y}) & X_k = X_l = \Delta \end{cases} \tag{6.28}$$

Lemma 6.3. $(\mu^1_{\mathcal{W}^2})^2 = 0$.

6.2. The A_∞ category. To complete the construction of $\mathcal{W}^2$, we construct higher A_∞ operations $\mu^d_{\mathcal{W}^2}$, $d \geq 2$. First, suppose we have *fixed a universal and conformally consistent Floer datum for pairs of glued discs and genus-0 open closed strings.* Consider the space of labeled associahedra

$$\mathcal{R}^d_{\mathbf{L}^2} \tag{6.29}$$

with label set the relevant Lagrangians in M^2:

$$\mathbf{L}^2 = \{\Delta\} \cup \{L_i \times L_j | L_i, L_j \in \text{ob } \mathcal{W}\}. \tag{6.30}$$

Consider first the case where all Lagrangians are split. Discs in $M^- \times M$ solving the inhomogenous Cauchy-Riemann equation with respect to a split Hamiltonian in $\tilde{\mathcal{H}}(M^- \times M)$ split almost complex structure $(-J, J)$ and split Lagrangian boundary conditions are exactly pairs of discs u_1, u_2 with the *same conformal structure* (up to conjugation) solving the inhomogenous Cauchy-Riemann equation with respect to ω, J and respective Lagrangian boundary conditions. The relevant moduli space of abstract discs is the *diagonal associahedron*

$$\overline{\mathcal{R}}^d \xhookrightarrow{\Delta_d}_{\emptyset, \mathfrak{T}_{max}} \overline{\mathcal{R}}_{d,d}. \tag{6.31}$$

For labeling sequences $\vec{L}^2$ from $\mathbf{L}^2$ not containing Δ, we can think of Δ_d as an embedding of *labeled moduli spaces*

$$(\Delta_d)_{\vec{L}^2} : (\mathcal{R}^d)_{\vec{L}^2} \longrightarrow (\mathcal{R}_{d,d})_{\mathbf{L}} \tag{6.32}$$

in the obvious manner: if a boundary component of $S \in \overline{\mathcal{R}}^d_{\mathbf{L}^2}$ was labeled $L_i \times L_j$, applying Δ_d, label the respective component of the first factor L_i and the second component L_j.

Definition 6.1. *Define the operation $\mu^d_{\mathcal{W}^2}$, for sequences of Lagrangians $\vec{L}^2$ in $\mathbf{L}^2$ not containing Δ, to be the operation controlled by the image of $(\Delta_d)_{\vec{L}^2}$ as in Equation* (5.88)*, with sign twisting datum given by the sequential sign twisting datum*

$$\vec{t}_d = (1, \ldots, d) \tag{6.33}$$

with respect to the vector of degrees in $\mathcal{W}^2$, $(\deg(x_1), \ldots, \deg(x_d))$*, which are compared to degrees in $\mathcal{W}$ by* (6.23)*.*

Now, let us give a more general construction of the operations, for cases including Δ. Let S be a disc in $\mathcal{R}^d$ with labels $\vec{L}^2$ from $\mathbf{L}^2$, with at least one label equal to Δ. Let

$$D(\vec{L}^2) \tag{6.34}$$

be the set of indices of boundary components of S labeled Δ. Then, let

$$\mathfrak{T}_{max} = \{(1,1), (2,2), \ldots, (d,d)\} \tag{6.35}$$

be the maximal boundary identification data and let

$$\mathfrak{S}(\mathbf{L}^2) = \{(i,i) | i \in D(\vec{L}^2)\} \tag{6.36}$$

be the set of boundary components determined by the positions of Δ. Finally, define

$$\mathbf{\Phi}_{\vec{L}^2}(\mathcal{R}^d) :=_{\mathfrak{S}(\mathbf{L}^2), \mathfrak{T}_{max}} \mathcal{R}_{d,d} \tag{6.37}$$

Label the boundary components of the resulting pair of discs as follows: if $\partial_k S$ was labeled $L_i \times L_j$, then in $\mathbf{\Phi}_{\vec{L}}(S)$, the left image of $\partial_k S$ will be labeled L_i and the right of $\partial_k S$ will be labeled L_j. If $\partial_k S$ was labeled Δ, then it will become part of a boundary identification and disappear under gluing so there is nothing to label.

Definition 6.2. *Define the operation*

$$\mu^d_{\mathcal{W}^2}, \tag{6.38}$$

for sequences of Lagrangians $\vec{L}^2$ in $\mathbf{L}^2$, to be the operation controlled by the image of $\mathbf{\Phi}_{\vec{L}^2}$ as in (5.88)*, with additional sign twists:*

$$\mu^d_{\mathcal{W}^2}(x_d, \ldots, x_1) := (-1)^{f(\vec{L}^2)} \cdot (-1)^{\vec{\deg}(\vec{x}) \cdot \vec{t}_d} \mathbf{F}_{\mathbf{\Phi}_{\vec{L}^2}(\mathcal{R}^d)} \tag{6.39}$$

where $\vec{t}_d$ as usual denotes the the sequential sign twisting datum

$$\vec{t}_d = (1, \ldots, d) \tag{6.40}$$

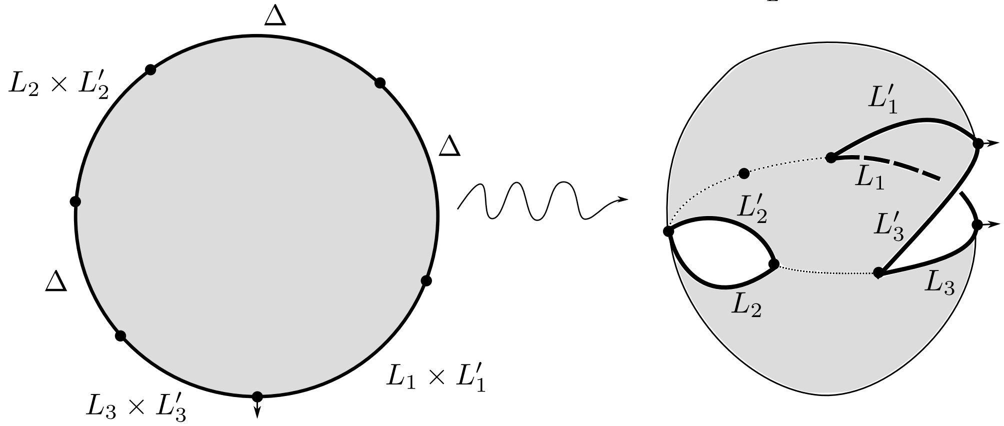
which is applied to the vector of degrees in $\mathcal{W}^2$, $(\deg(x_1), \ldots, \deg(x_d))$ *(which are compared to degrees in $\mathcal{W}$ by* (6.23)-(6.26)*). There is one additional sign*

$$f(\vec{L}^2) = \begin{cases} \frac{n(n+1)}{2} & \Delta \text{ is in the middle of } \vec{L}^2 \\ 0 & \text{otherwise} \end{cases} \tag{6.41}$$

Proposition 6.3. *The operations $\mu^d_{\mathcal{W}^2}$ as constructed satisfy the A_∞ equations.*

Proof. The unfolding maps $\mathbf{\Phi}_{\vec{L}^2}$ are embeddings of associahedra, which (along with the usual strip and cylinder breaking occuring in compactifications of spaces of maps) immediately implies the A_∞ equations up to sign. □

FIGURE 12. An example of the labeled gluing $\mathbf{\Phi}_{\vec{L}^2}$.



REMARK 6.2 (Massey products on $SH^*(M)$). *As a special case of the above discussion, we obtain an A_∞ algebra structure*

$$\mu^d : CH^*(M)^{\otimes d} = \hom_{\mathcal{W}^2}(\Delta, \Delta)^{\otimes d} \to \hom_{\mathcal{W}^2}(\Delta, \Delta) = CH^*(M), \tag{6.42}$$

with μ^1 the ordinary differential. For $d \geq 2$, μ^d counts (with a suitable sign twist) solutions to Floer's equation on the glued surface $S \coprod_{\mathfrak{S}_{max}} T$ for pairs of discs $(S,T) \in \overline{\mathcal{R}}_{d,d}$ lying in the diagonal associahedron, with $\mathfrak{S}_{max} = \{(0,0),(1,1),\ldots,(d,d)\}$ the boundary identification identifying every boundary of S with the corresponding boundary of T. The gluing map identifies this space of domains (${}_{\mathfrak{T}_{max},\mathfrak{S}_{max}}\overline{\mathcal{R}}_{d,d}$ in the language of §5) with the space of spheres with d input punctures and 1 output puncture, such that all of the punctures are constrained to lie on an equator circle (this is an empty constraint for $d = 2$). The asymptotic markers, which have been mostly omitted from our discussion, are all constrained to point in the same direction along this equator (except for the output asymptotic marker, which points in the opposite direction).

PROOF OF PROP. 6.2. There are two proofs of this fact. In the first, we can treat the numbers $\deg(x)$, $\deg(y)$ as black boxes and verify that the degree assignment given above makes the A_∞ operations $\mathcal{W}^2$ have correct degree $2-d$ for any sequence of labeled Lagrangians. This could be done as follows: Take an arbitrary labeling $\vec{L}^2$ of Lagrangians, some of which are Δ and some of which are $L_i \times L_j$'s, and calculate the number of boundary components, number of boundary outputs, and number of interior outputs of the resulting open-closed string $\pi_{\mathfrak{S}}(\mathbf{\Phi}_{\vec{L}^2}(S))$, thus arriving at the dimension (and therefore degree) of the operation controlled by $\mathbf{\Phi}_{\vec{L}^2}$. The main observation here is that, inductively, any sequence of consecutive labels of Δ that do not appear at the end of the sequence shift the index by n, either by gluing a pair of discs together for the first time, by adding an additional boundary component or or by turning a boundary output into an interior output (note that interior outputs are only formed if there are Δ labels on both ends, an edge case). Correspondingly, any such sequence contributes a term of the form $\hom(\Delta, L_i \times L_j)$. Thus in terms of the grading given above, the operation continues to have degree $2 - d$.

Alternatively, we give a conceptual argument, assuming that M is a compact manifold. Suppose we had chosen a grading for Δ such that $\hom(L_1 \times L_2, \Delta) \simeq \hom(L_1, L_2)$ as graded complexes. Then, by Poincaré duality on M^2 and on M, we must have that

$$\hom(\Delta, L_1 \times L_2) \simeq \hom(L_1 \times L_2, \Delta)^\vee[2n] \simeq \hom(L_2, L_1)^\vee[2n] \simeq \hom(L_1, L_2)[n]. \tag{6.43}$$

Of course, Poincaré duality fails in our situation, but this argument gives a reasonable sanity check regarding gradings. □

7. From the product to bimodules

7.1. Moduli spaces of quilted strips. The next three definitions are due to Ma'u [**M1**]:

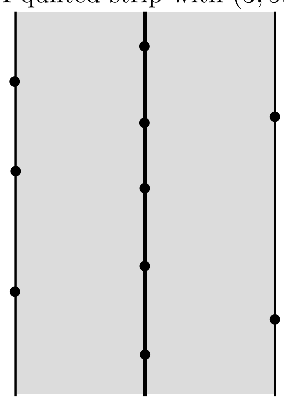
DEFINITION 7.1. *Fix $-\infty < x_1 < x_2 < x_3 < \infty$. A* **3-quilted line** *consists of the three parallel lines l_1, l_2, l_3, each of which is a vertical line $\{x_j + i\mathbb{R}\}$ considered as a subset of $[x_1, x_3] \times (-\infty, \infty) \subset \mathbb{C}$.*

Definition 7.2. *Let* $\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{Z}^3_{\geq 0}$. *A* **3-quilted line with r markings** *consists of the data* $(Q, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3)$*, where* Q *is a 3-quilted line, and each vector* $\mathbf{z}_i = (z_i^1, \dots, z_i^{r_i})$ *is an upwardly ordered configuration of points in* l_i*, i.e.* $\mathrm{Re}(z_i^k) = l_i$ *and* $-\infty < \mathrm{Im}(z_i^1) < \mathrm{Im}(z_i^2) < \cdots < \mathrm{Im}(z_i^{r_i}) < \infty$.

There is a free and proper $\mathbb{R}$ action on such quilted lines with markings, given by simultaneous translation in the $\mathbb{R}$ direction.

Definition 7.3. *The* **moduli space of 3-quilted, r-marked strips** $Q(3, \mathbf{r})$ *is the set of such 3-quilted lines with* $\mathbf{r}$ *markings, modulo translation.*

Figure 13. A quilted strip with $(3, 5, 2)$ markings.



Remark 7.1. *As the notation suggests, Ma'u gives a more general definition of a space* $Q(n+1, \mathbf{r})$ *of* $n+1$*-quilted lines with* $\mathbf{r}$*-markings, which are used to define a more general theory of* n*-modules (bimodules are the case* $n = 2$*). We will not need this theory for our current work.*

Ma'u gives a description of the Deligne-Mumford compactification

$$\overline{Q(3, \mathbf{r})}$$

of the above moduli space, the **moduli space of stable, nodal 3-quilted lines with r markings**. Strata consist of multi-level broken 3-quilted lines with stable discs glued to marked points on each of the three lines at any level. The manifold with corners structure near these strata comes from gluing charts, which are similar to ones we have already written down. We refer the reader to [**M1**, §2] for more details on this moduli space, but in codimension 1

Proposition 7.1 ([**M1**]). *The boundary* $\partial\overline{Q(3, \mathbf{r} = (r_1, r_2, r_3))}$ *is covered by the images of the codimension 1 inclusions*

$$\begin{aligned} \overline{Q(3, (a, b, c))} \times \overline{Q(3, (r_1 - a, r_2 - b, r_3 - c))} &\to \partial\overline{Q(3, \mathbf{r})} \\ \overline{Q(3, (a+1, r_2, r_3))} \times \overline{\mathcal{R}^{r_1 - a}} &\to \partial\overline{Q(3, \mathbf{r})} \\ \overline{Q(3, (r_1, b+1, r_3))} \times \overline{\mathcal{R}^{r_2 - b}} &\to \partial\overline{Q(3, \mathbf{r})} \\ \overline{Q(3, (r_1, r_2, c+1))} \times \overline{\mathcal{R}^{r_3 - c}} &\to \partial\overline{Q(3, \mathbf{r})}. \end{aligned} \tag{7.1}$$

In fact, we do not need to make use of use Proposition 7.1, which we have presented for motivation. Instead, we will use the open space $Q(3, \mathbf{r})$ to construct operations controlled by various glued pairs of discs. The codimension 1 compactification of the resulting moduli spaces we consider will not quite be (7.1), but will only differ by some strata whose associated operations are zero.

Definition 7.4. *Let* $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$ *be sets of Lagrangians in* M, M^2, *and* M *respectively. A* **Lagrangian labeling from** $(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3)$ *for a 3-quilted line with* $\mathbf{r}$ *markings* $(Q, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3)$ *consists of, for each* i*, an assignment of a label in* $\mathbf{L}_i$ *to each of the* $r_i + 1$ *components of the punctured line* $l_i - \mathbf{z}_i$. *The* **space of 3-quilted lines** *with* **r-markings** *and* $(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3)$ **labels** *is denoted* $Q(3, \mathbf{r})_{(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3)}$.

7.2. Unfolding labeled quilted strips. Fix the label set $\hat{\mathbf{L}} = (\mathbf{L}, \mathbf{L}^2, \mathbf{L})$. Let S be a stable labeled, 3-quilted strip with $\mathbf{r}$-markings, $S \in Q(3, \mathbf{r})_{\hat{\mathbf{L}}}$, labeled by $\hat{\vec{L}} = (\vec{L}_0, \vec{L}^2, \vec{L}_1)$. We associate to S a pair of glued discs

$$\boldsymbol{\Psi}_{\hat{\vec{L}}}(S). \tag{7.2}$$

in a manner analogous to the construction of $\boldsymbol{\Psi}$ in Section 6.2. From a 3-quilted line with marked points S, consider the substrips $-S_1$ and S_2, where S_i is $(i = 1, 2)$ given by the regions in between and including lines l_i and l_{i+1} ($-S_1$ denotes the reflection of S_1 across a vertical axis).

$-S_1$ and S_2 are conformally discs with boundary marked points $\mathbf{z}_i \cup \mathbf{z}_{i+1} \cup \{a_i^{\pm}\}$, where $a_i^{\pm}$ are the marked points corresponding in the strip-picture to $\pm\infty$. Denote the connected components of the line $l_2 - \mathbf{z}_2$ by $\partial_j^2 S$, $j = 1, \ldots, r_2 + 1$, and the images of these boundary components in S_i by $\partial_j^2 S_i$. Pick some conformal map ϕ from the strip to a disc with marked points at $\pm\infty$ sent to ± 1. Apply this same conformal map to $-S_1$ and S_2 and call the results $-\tilde{S}_1$, $\tilde{S}_2$. By construction $\tilde{S}_1$ and $\tilde{S}_2$ have $r_2 + 1$ coincident points

$$\mathfrak{T} = \{(1, 1), \ldots, (r_2 + 1, r_2 + 1)\} \tag{7.3}$$

coming from the marked points on the strip l_2 and the point at $+\infty$.

Now, define the boundary identification

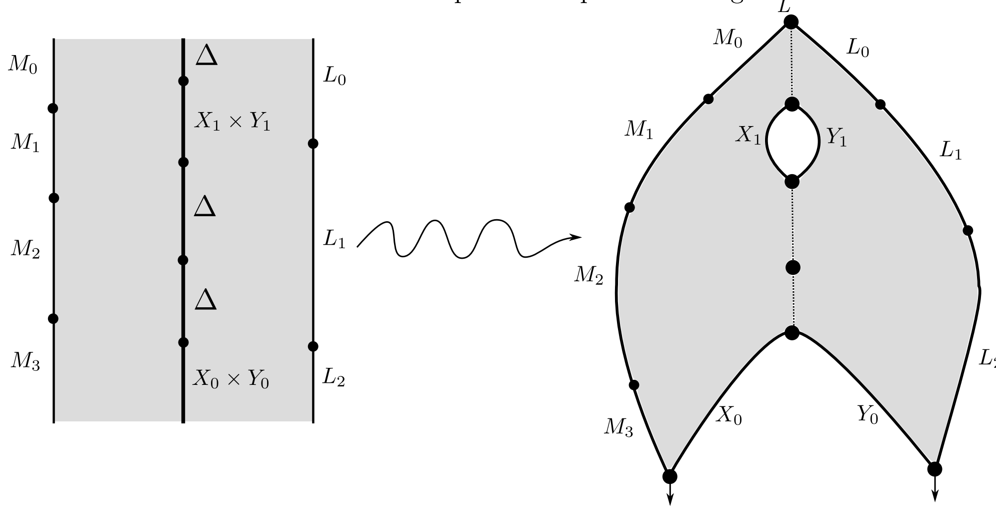
$$\mathfrak{S}(\hat{\vec{L}}) := \{(i, i) | 1 \le i \le r_2 + 1,\ \partial_i^2 S \text{ is labeled } \Delta\}. \tag{7.4}$$

Thus, we can define

$$\boldsymbol{\Psi}_{\hat{\vec{L}}}(Q(3, (r_1, r_2, r_3))) := {}_{\mathfrak{S}(\hat{\vec{L}}), \mathfrak{T}}\overline{\mathcal{R}}_{r_2 + r_1 + 1, r_2 + r_3 + 1}. \tag{7.5}$$

The resulting space is labeled as follows: The connected components of $l_i - z_i$ for $i = 1, 3$ in the image of $\boldsymbol{\Psi}$ retain the same labeling. If $\partial_j^2 S$ was labeled by some $L_s \times L_t$, then label the image of $\partial_j^2 S_1$ by L_s and the image of $\partial_j^2 S_2$ by L_t.

FIGURE 14. An example of the quilt unfolding $\boldsymbol{\Psi}_{\hat{\vec{L}}}$.



7.3. The A_∞ functor. Using the above embeddings of labeled moduli spaces, we construct an A_∞ functor

$$\mathbf{M} : \mathcal{W}^2 \longrightarrow \mathcal{W}\text{–mod–}\mathcal{W}. \tag{7.6}$$

On an object $X \in \mathcal{W}^2$, the bimodule $\mathbf{M}(X)$ is specified by the following data:

- for pairs of objects $A, B \in \operatorname{ob} \mathcal{W}$, $\mathbf{M}(X)(A,B)$ is generated as a graded vector space by $\chi_{M^2}(A \times B, X)$, which we recall to be:

$$\begin{cases} \chi(L_1, A) \times \chi(B, L_2) & X = L_1 \times L_2 \\ \chi(B, A) & X = \Delta \end{cases} \tag{7.7}$$

- differential

$$\mu_{\mathbf{M}}^{0|1|0} : \mathbf{M}(X)(L, L') \longrightarrow \mathbf{M}(X)(L, L') \tag{7.8}$$

which is exactly the differential $\mu^1_{\mathcal{W}^2}$ on $\hom_{\mathcal{W}^2}(L \times L', X)$, counting pairs of strips modulo simultaneous automorphisms.

- for objects $(A_0, \ldots, A_r, B_0, \ldots, B_s)$, higher bimodule structure maps

$$\begin{aligned} \mu_{\mathbf{M}}^{r|1|s} : &\hom_{\mathcal{W}}(A_{r-1}, A_r) \times \cdots \hom_{\mathcal{W}}(A_0, A_1) \times \mathbf{M}(X)(A_0, B_0) \times \\ &\times \hom_{\mathcal{W}}(B_1, B_0) \times \cdots \times \hom_{\mathcal{W}}(B_s, B_{s-1}) \longrightarrow \mathbf{M}(X)(A_r, B_s). \end{aligned} \tag{7.9}$$

These maps are the ones determined by the moduli space

$$\overline{\mathbf{\Psi}_{\hat{\vec{L}}}(Q(3,(r,0,s))_{\hat{\vec{L}}})} \tag{7.10}$$

in the sense of equations (4.31) and (5.88), where

$$\hat{\vec{L}} = ((A_0, \ldots, A_r), (X), (B_0, \ldots, B_s)), \tag{7.11}$$

using existing choices of Floer data and sign twisting datum

$$\vec{t} = (1, 2, \ldots, s, s, s+1, \ldots, s+r) \tag{7.12}$$

with respect to the reverse ordering of inputs in (7.9) (considering $\mathbf{M}(X)(A_0, B_0)$ as a single input).

The consistency condition imposed on the choice of Floer data pairs of glued discs and the codimension-1 strata (7.1) imply that

Proposition 7.2. *$\mathbf{M}(X)$ is an A_∞ bimodule.*

Proof. We look at the boundary of the associated one-dimensional moduli spaces. The resulting pair of glued discs has a sequential point identification $\mathfrak{T} = \{(1,1),(2,2)\}$. We have already examined the boundary strata of ${}_{\mathfrak{S}(\hat{\vec{L}}),\mathfrak{T}}\mathcal{R}_{r+1+1,r_3+1}$ in Proposition 5.1. The composed operations corresponding to stratas (5.53) - (5.55) vanish by Proposition 5.3. The strata (5.52), (5.56), and (5.57) correspond exactly to equations of the form

$$\begin{gathered} \mu_{\mathbf{M}(X)}(\cdots \mu_{\mathbf{M}(X)}(\cdots, \mathbf{b}, \cdots), \cdots), \\ \mu_{\mathbf{M}(X)}(\cdots, \mathbf{b}, \cdots \mu_{\mathcal{W}}(\cdots), \cdots), \\ \mu_{\mathbf{M}(X)}(\cdots, \mu_{\mathcal{W}}(\cdots), \cdots, \mathbf{b}, \cdots, \cdots) \end{gathered} \tag{7.13}$$

respectively, which together comprise the terms of the A_∞ bimodule relations. There are final terms coming from strip-breaking, corresponding to allowing ourselves to apply μ^1 or $\mu^{0|1|0}$ before or after applying $\mu_{\mathbf{M}(X)}$. Verification of signs is as in Appendix A. □

Given objects $X_0, \ldots, X_d \in \mathcal{W}^2$, the higher terms of the functor are maps

$$\mathbf{M}^d : \hom_{\mathcal{W}}(X_{d-1}, X_d) \otimes \cdots \hom_{\mathcal{W}}(X_0, X_1) \longrightarrow \hom_{\mathcal{W}\text{–mod–}\mathcal{W}}(\mathbf{M}(X_0), \mathbf{M}(X_d)) \tag{7.14}$$

sending

$$x_d \otimes \cdots \otimes x_1 \longmapsto \mathbf{m}_{(x_d, \ldots, x_1)} \in \hom_{\mathcal{W}\text{–mod–}\mathcal{W}}(\mathbf{M}(X_0), \mathbf{M}(X_d)). \tag{7.15}$$

The bimodule homomorphism $\mathbf{m}_{(x_d, \ldots, x_1)}$ consists of, for objects $(A_0, \ldots, A_r, B_0, \ldots, B_s)$ in $\mathcal{W}$, maps:

$$\begin{aligned} \mathbf{m}_{(x_d, \ldots, x_0)}^{r|1|s} : &\hom_{\mathcal{W}}(A_{r-1}, A_r) \times \cdots \hom_{\mathcal{W}}(A_0, A_1) \times \mathbf{M}(X_0)(A_0, B_0) \times \\ &\times \hom_{\mathcal{W}}(B_1, B_0) \times \cdots \times \hom_{\mathcal{W}}(B_s, B_{s-1}) \longrightarrow \mathbf{M}(X_d)(A_r, B_s) \end{aligned} \tag{7.16}$$

Letting $\hat{\vec{L}} = ((A_0, \ldots, A_r), (X_0, \ldots, X_d), (B_0, \ldots, B_s))$, we define the above operation to be the one controlled in the sense of Equations (4.31) and (5.88) by the unfolded image

$$\overline{\mathbf{\Psi}_{\hat{\vec{L}}}(Q(3,(r,d,s))_{\hat{\vec{L}}})} \tag{7.17}$$

with sign twisting datum

$$\vec{t} = (1, \ldots, d, 1, \ldots, r, r, r+1, \ldots, r+s) \tag{7.18}$$

with respect to the ordering of inputs given by $x_1, \ldots, x_d$ followed by the reverse of the order of inputs in (7.16) (as before, this means that we twist the image of the inputs after unfolding by these quantities). The consistency condition for Floer data for open-closed strings and pairs of discs, along with the codimension 1 boundary of quilted strips (7.1) imply

Proposition 7.3. *The data $\mathbf{M}^d$ as defined above gives an A_∞ functor*

$$\mathbf{M} : \mathcal{W}^2 \longrightarrow \mathcal{W}\text{–mod–}\mathcal{W}. \tag{7.19}$$

Proof. We need to verify the A_∞ functor equation, which (as $\mathcal{W}$–mod–$\mathcal{W}$ is a dg category), takes the form:

$$\mu^1_{\mathcal{W}\text{–}\mathcal{W}} \circ \mathbf{M}^d + \sum_{i_1+i_2=d} (\mathbf{M}^{i_1} \text{``} \circ \text{''} \mathbf{M}^{i_2}) = \mathbf{M} \circ \hat{\mu}_{\mathcal{W}}. \tag{7.20}$$

We examine the boundary strata ${}_{\mathfrak{S}(\hat{\vec{L}}),\mathfrak{T}}\mathcal{R}_{r+d+1,d+r_3+1}$ computed in Proposition 5.1, for

$$\mathfrak{T} = \{(1,1), \ldots, (d+1, d+1)\}.$$

The first term, $\mu^1_{\mathcal{W}\text{–mod–}\mathcal{W}}(\mathbf{M}^d) = \mu_{\mathbf{M}(X_d)} \circ \hat{\mathbf{M}}^d \mp \mathbf{M}^d \circ \hat{\mu}_{\mathbf{M}(X_0)}$, matches up exactly with the strata (5.56), (5.57), and (5.52) (in the case that one of $\mathcal{P}$' or $\mathcal{P}''$ has size $d+1$). The cases of (5.52) in which neither $\mathcal{P}'$ or $\mathcal{P}''$ are maximal give exactly $(\mathbf{M}^{i_1} \text{``} \circ \text{''} \mathbf{M}^{i_2})$, and $\mathbf{M} \circ \mu_{\hat{\mathcal{W}}}$ is given by (5.51). Finally, there is strip-breaking of the geometric moduli spaces, giving the $\mu^1_{\mathcal{W}}$ portions of the equations, and the remaining boundary strata (5.53) - (5.55) vanish by Proposition 5.3. Once more, details on how to fill in the sign verification are discussed in Appendix A. □

8. Calculations

First, we examine the functor $\mathbf{M}$ on split Lagrangians.

Proposition 8.1. *The bimodule $\mathbf{M}(L_i \times L_j)$ is exactly the tensor product of Yoneda modules*

$$\mathcal{Y}^l_{L_i} \otimes_k \mathcal{Y}^r_{L_j}. \tag{8.1}$$

Proof. For objects (A, B) in $\mathcal{W}$, $\mathbf{M}(L_i \times L_j)(A, B)$ and $\mathcal{Y}^l_{L_i}(A) \otimes \mathcal{Y}^r_{L_j}(B)$ are identical as chain complexes. The bimodule maps $\mu^{r|1|s}_{\mathbf{M}(L_i \times L_j)}$ are zero if $r, s > 0$, by Proposition 5.3. If $r = 0$ or $s = 0$, by Proposition 5.4, the operations are:

$$\begin{aligned} \mu^{0|1|s}_{\mathbf{M}(L_i \times L_j)} &= id \otimes \mu^s \\ \mu^{r|1|0}_{\mathbf{M}(L_i \times L_j)} &= \mu^r \otimes id, \end{aligned} \tag{8.2}$$

concluding the proof. □

Proposition 8.2. *$\mathbf{M}$ is full and faithful on the subcategory generated by objects of the form $L_i \times L_j$.*

Proof. The first-order map

$$\mathbf{M}^1 : \hom_{\mathcal{W}^2}(L_i \times L_j, L_i' \times L_j') \to \hom_{\mathcal{W}\text{–mod–}\mathcal{W}}(\mathbf{M}(L_i \times L_j), \mathbf{M}(L_i' \times L_j')). \tag{8.3}$$

is the operation $(\mathbf{M}^1(\alpha \otimes \beta))^{r|1|s}$ controlled by the embeddings

$$\mathbf{\Psi}_{(\vec{L}^1,\vec{L}^2,\vec{L}^3)}(Q(3,(r,1,s))) \subset (\mathcal{R}_{r+2,s+2})_{\hat{L}'} \tag{8.4}$$

where $\vec{L}^2 = (L_i \times L_j, L_i' \times L_j')$. On the level of unlabeled surfaces, this map takes a 3-quilted line with one marked point on the interior line and associates a pair of discs S_1, S_2, with $r+2$ and $s+2$ positive marked points respectively, such that 3 of the marked points of S_1 (corresponding to $\pm\infty$ and the marked point on the interior line) are coincident with 3 of the marked points of S_2. By Proposition 5.6, the corresponding operation is $\mu^{r+2} \otimes \mu^{s+2}$.

This implies that $\mathbf{M}^1$ is exactly the first order Yoneda map, followed by the inclusion in Corollary 2.2:

$$\begin{aligned} CW^*(L_i, L_i') \otimes CW^*(L_j', L_j) &\xrightarrow{(\mathbf{Y}^l)^1 \otimes (\mathbf{Y}^r)^1} \hom_{\mathcal{W}\text{–mod}}(\mathcal{Y}^l_{L_i}, \mathcal{Y}^l_{L_i'}) \otimes \hom_{\text{mod–}\mathcal{W}}(\mathcal{Y}^r_{L_j}, \mathcal{Y}^r_{L_j'}) \\ &\hookrightarrow \hom_{\mathcal{W}\text{–mod–}\mathcal{W}}(\mathcal{Y}^l_{L_i} \otimes \mathcal{Y}^r_{L_j}, \mathcal{Y}^l_{L_i'} \otimes \mathcal{Y}^r_{L_j'}) \end{aligned} \tag{8.5}$$

Fullness follows immediately from the fullness of the Yoneda embedding and Corollary 2.2. ☐

Proposition 8.2 may be regarded as an A_∞ version of the Künneth decomposition for Floer homology. Namely, let $\mathcal{W}^L$ be the dg category of *left Yoneda modules* over $\mathcal{W}$ and $\mathcal{W}^R$ be the dg category of *right Yoneda modules* over $\mathcal{W}$. The Yoneda embedding gives equivalences $\mathcal{W}^{op} \simeq \mathcal{W}^L$ and $\mathcal{W} \simeq \mathcal{W}^R$. Now, if $\mathcal{W}^2_{split}$ denotes the category of product Lagrangians, then we have shown $\mathbf{M}$ gives an A_∞ equivalence $\mathcal{W}^2_{split} \simeq \mathcal{W}^L \otimes \mathcal{W}^R$. This provides one method of circumventing the difficulty of discussing tensor products of A_∞ algebras and categories.

Now, we examine $\mathbf{M}$ and $\mathbf{M}^1$ for the remaining object of $\mathcal{W}^2$: Δ.

Proposition 8.3. *$\mathbf{M}(\Delta)$ is the diagonal bimodule $\mathcal{W}_\Delta$.*

Proof. Consider the unfolding map $\mathbf{\Psi}$ when the middle strip is labeled Δ. The space of quilted strips $Q(3,(r_1,0,r_2))$ is sent to the associahedron $\mathcal{R}^{r_1+1+r_2}$ with a distinguished input marked point corresponding to the intersection point at $+\infty$ in the quilt. These are exactly the structure maps corresponding to the diagonal bimodule. ☐

In Section 4.6, we defined two (different, but quasi-isomorphic) geometric closed-open maps from symplectic cohomology to Hochschild cohomology of the wrapped Fukaya category. Here, we observe that the first order term $\mathbf{M}^1$ agrees on the chain level with one of these maps, and thus on homology with both.

Proposition 8.4. *There is a chain-level identification between*

$$\mathbf{M}^1 : \hom_{\mathcal{W}^2}(\Delta, \Delta) \longrightarrow \hom_{\mathcal{W}\text{–mod–}\mathcal{W}}(\mathbf{M}(\Delta), \mathbf{M}(\Delta)) \tag{8.6}$$

and the two-pointed closed-open string map (4.99),

$$_2\mathcal{CO} : CH^*(M) \longrightarrow \hom_{\mathcal{W}\text{–mod–}\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{W}_\Delta). \tag{8.7}$$

Proof. In this case, the relevant space of quilted strips is $Q(3,(r_1,1,r_2))$ with middle strip Lagrangian labels both Δ. The relevant boundary identification datum is $\mathfrak{S} = \{(1,1),(2,2)\}$. Proposition 5.9 shows that the operation corresponding to the moduli space ${}_{\mathfrak{S},p(\mathfrak{S})}\overline{\mathcal{R}}_{k,l}$ is exactly ${}_2\mathcal{CO}^{r_1,r_2}$. See also Figure 15. ☐

Proposition 8.5. *For any $A, B \in \operatorname{ob} \mathcal{W}$, there is an equality*

$$cop_{A,B} = \mathbf{M}^1_{\Delta, A\times B}. \tag{8.8}$$

Proof. The maps

$$cop_{A,B} : \hom_{\mathcal{W}}(A,B) \longrightarrow \hom_{\mathcal{W}\text{–}\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{Y}^l_A \otimes \mathcal{Y}^r_B)[n] \tag{8.9}$$

and

$$\mathbf{M}^1 : \hom_{\mathcal{W}^2}(\Delta, A\times B) := \hom_{\mathcal{W}}(A,B)[-n] \longrightarrow \hom_{\mathcal{W}\text{–}\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{Y}^l_A \otimes \mathcal{Y}^r_B) \tag{8.10}$$

have the same source and targets, so we need to verify that the spaces controlling the Floer equations are the same. The unfolding map, $\mathbf{\Psi}_{\hat{\underline{L}}}$, defined in (7.5), when applied to a quilted strip with $(r,1,s)$ marked points, with middle label sequence $(\Delta, A\times B)$, produces a surface with two output marked points, two distinguished input marked points, a distinguished input marked point between them corresponding to $\hom(A,B)$, and then $r+1+s$ input marked points (the $r+1$st of which is distinguished) around the two outputs. This is exactly the definition of the operation given by $cop^{A,B}_{r,s}$. See also Figure 16 for a picture of this unfolding. ☐

FIGURE 15. The unfolding of $\mathbf{M}^1 : \hom_{\mathcal{W}^2}(\Delta, \Delta) \to \hom_{\mathcal{W}\text{–mod–}\mathcal{W}}(\mathbf{M}(\Delta), \mathbf{M}(\Delta))$ to give the glued pair of discs corresponding to ${}_2\mathcal{CO}$.

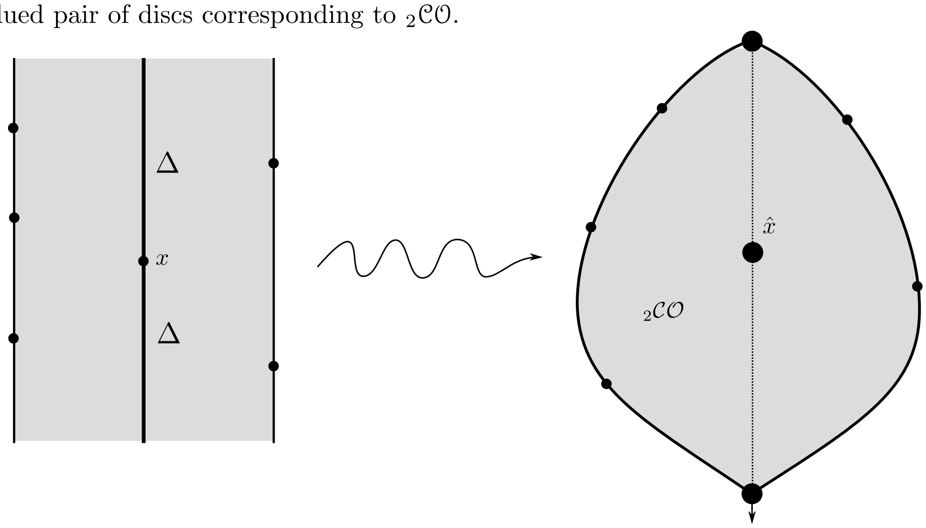
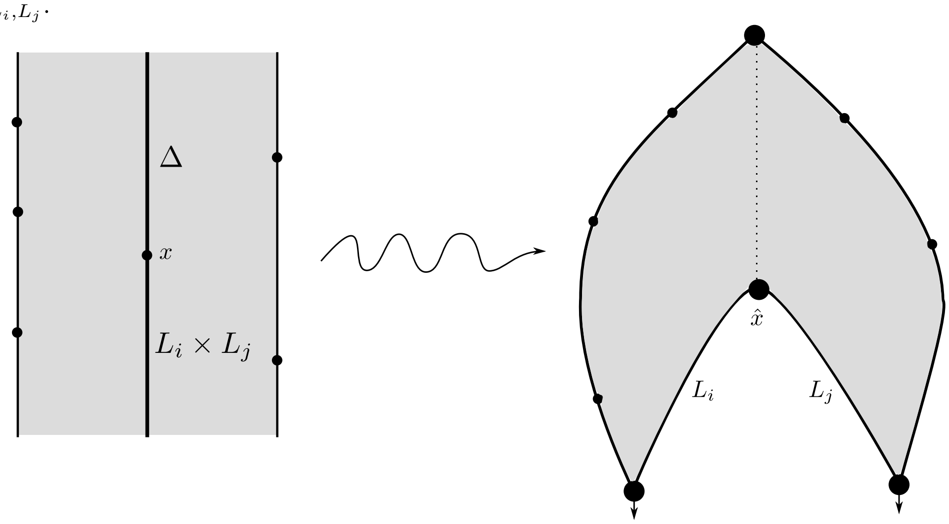


FIGURE 16. The equality between the quilted strip controlling $\mathbf{M}^1_{\Delta, L_i \times L_j}$ and the map cop_{L_i, L_j}.



PROPOSITION 8.6. *For any $A, B \in \mathrm{ob}\, \mathcal{W}$, the map*

$$\mathbf{M}^1_{A\times B,\Delta} : \hom_{\mathcal{W}}(B, A) = \hom_{\mathcal{W}^2}(A \times B, \Delta) \to \hom_{\mathcal{W}\text{–}\mathcal{W}}(\mathcal{Y}^l_A \otimes \mathcal{Y}^r_B, \mathcal{W}_\Delta) \tag{8.11}$$

is homotopic to $\lambda_{\mathcal{W}_\Delta, A, B}$ defined in (2.116).

SKETCH. The unfolding map, $\mathbf{\Psi}_{\hat{L}}$ defined in (7.5), when applied to a quilted strip with $(r, 1, s)$ marked points with middle label sequence $(A \times B, \Delta)$ produces a surface with one outgoing marked point z_{out} and $r + s + 3$ incoming marked points $z_1, \ldots, z_r, d_1, d_2, d_3, z'_1, \ldots, z'_s$, with d_2 corresponding to $\hom(B, A)$ and the middle three marked points d_1, d_2, d_3 (in particular, the cross ratio of z_{out}, d_1, d_2, d_3 are fixed, so this is a codimension 1 submanifold of $\overline{\mathcal{R}}^{r+s+3}$). As in Proposition 4.3, we now consider the operation induced by the family of discs where this cross ratio is allowed to vary in an open interval; for concreteness we look at the family of discs where up to automorphism d_1, d_2, d_3 and z_{out} are fixed at -1, i, $e^{i(\pi/2-t)}$, and $-i$ respectively, for $t \in [0, 1)$ (for $t = 0$, this is just the unfolded moduli space associated to $\mathbf{M}^1_{A\times B,\Delta}$. The compactified family fibers over $[0, 1]$ with fiber over 1 given by products of associahedra where the marked points d_2 and d_3, along with $z'_1, \ldots, z'_k$, for some $k \leq s$, have bubbled off. The operation induced by the fiber over 1 is therefore visibly $\lambda_{\mathcal{W}_\Delta, A, B}$ (up to a sign verficiation), and the operation induced by the total family over $[0,1]$ produces the chain homotopy between $\mathbf{M}^1_{A \times B, \Delta}$ and $\lambda_{\mathcal{W}_\Delta, A, B}$.

See Figure 17 for an image of the unfolding and the homotopy. $\square$

An immediate application of Proposition 2.4 implies:

COROLLARY 8.1. *$\mathbf{M}^1_{A \times B, \Delta}$ is always a homology isomoprhism.*

9. Further directions

9.1. Symplectic cohomology as Hochschild (co)homology. In the sequel to this article, we will use the following corollary as an essential ingredient in the proof that under certain hypotheses, the closed-open and open-closed maps are isomorphisms.

COROLLARY 9.1. *If, in the category $\mathcal{W}^2$, Δ is split-generated by product Lagrangians, then the closed open map and generalized coproducts*

$$
\begin{aligned}
cop_{A,B} : \hom^*_{\mathcal{W}}(A, B) &\longrightarrow \hom^{*+n}_{\mathcal{W}\text{–}\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{Y}^l_A \otimes \mathcal{Y}^r_B) \\
\mathcal{CO} : SH^*(M) &\longrightarrow \mathrm{HH}^*(\mathcal{W}, \mathcal{W})
\end{aligned}
\tag{9.1}
$$

are isomorphisms.

PROOF. It is a general fact, which does not have a clean proof in the literature that if $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is an A_∞ functor which is full and faithful on a subcategory $\mathcal{X} \subseteq \mathcal{C}$ which split generates $\mathcal{C}$, Then $\mathcal{F}$ is full and faithful on $\mathcal{C}$. The case that $\mathcal{X}$ *generates* $\mathcal{C}$ can be found in [**S5**, Lemma 3.25]. Parts of the argument for the split-generation case are in [**S2**, Lemma 2.5].

At any rate, we conclude by Proposition 8.4 that if the hypotheses hold, then ${}_2\mathcal{CO} = \mathbf{M}^1 : SH^*(M) = \hom_{\mathcal{W}^2}(\Delta, \Delta) \to \hom_{\mathcal{W}\text{–mod–}\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{W}_\Delta)$ is a homology isomorphism. But Corollary 4.5 implies that $[\mathcal{CO}] = [{}_2\mathcal{CO}]$. Similarly by Proposition 8.5, $cop_{A,B} = \mathbf{M}^1 : \hom^*_{\mathcal{W}}(A, B) = \hom_{\mathcal{W}^2}(\Delta, A \times B) \to \hom^{*+n}_{\mathcal{W}\text{–}\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{Y}^l_A \otimes \mathcal{Y}^r_B)$ is a homology isomorphism. $\square$

9.2. Other Lagrangians in other products. Our construction of the functor $\mathbf{M}$ restricted the category $\mathcal{W}^2$ to contain only product Lagrangians and the diagonal in $M^- \times M$. It is only a notational matter to extend this, as discussed in the introduction, to a functor from product Lagrangians in $M^- \times N$ to the category $\mathcal{W}(M)$–mod–$\mathcal{W}(N)$. With only slightly more effort, one should be able to include graphs of *exact symplectomorphisms* $\phi : M \to N$ as well, including the case $\phi \neq id : M \to M$. One would need to enlarge the class of open-closed operations to contain maps from bordered surfaces with slits across which the components of the map is glued by a given ϕ. Such operations can equivalently be viewed as counts of sections of a suitable symplectic fibration over the domain (as is usual in fixed point Floer homology); from this point, the "seam unfolding" construction would proceed as discussed in this paper. The end result would contain, to first order, closed-open maps of the form

$$
HW^*(\Delta, \Gamma_\phi) := SH^*(\phi) \to \mathrm{HH}^*(\mathcal{W}, \Gamma_\phi),
\tag{9.2}
$$

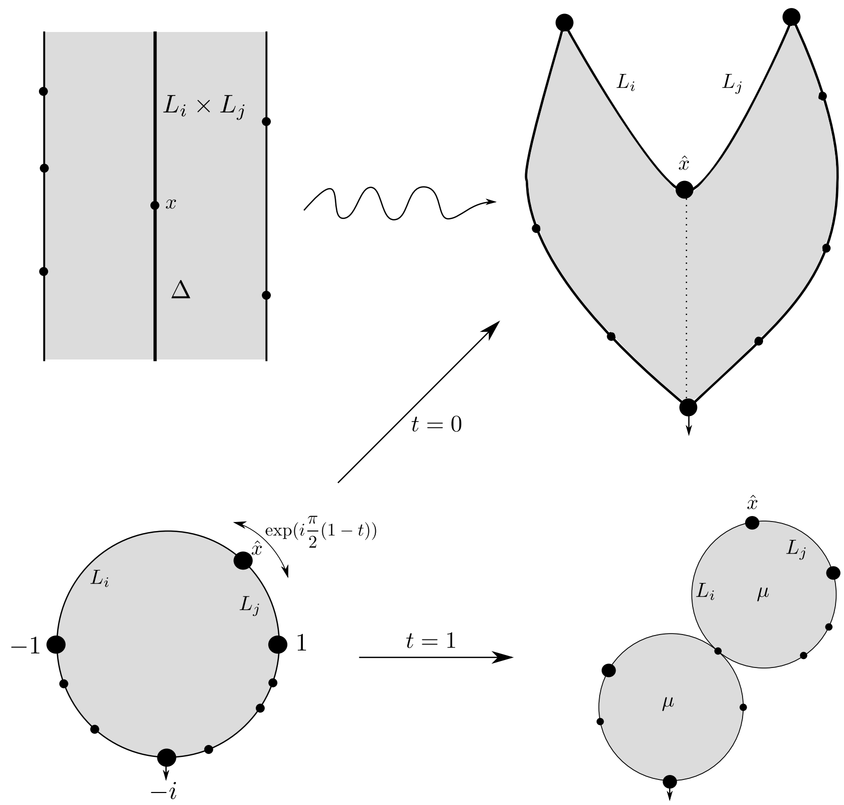
which are isomorphisms whenever product Lagrangians split-generate $\mathcal{W}^2$.

In a different direction, the functor $\mathbf{M}$ restricted to product Lagrangians, along with additional homological algebra, actually produces a functor from any other admissible object of $\mathcal{W}^2$ to bimodules. This should include, e.g., the case of Lagrangians that are split near ∞, exact Lagrangians with vanishing primitive, and graphs of exact symplectomorphisms. Whenever $\mathcal{W}^2_{split}$ embeds into the wrapped Fukaya category of $M^- \times M$ (whose objects are admissible with respect to $-\lambda_1 + \lambda_2$), this includes the entire wrapped Fukaya category of $M^- \times M$. See the remark below for more details.

REMARK 9.1. *Let $\tilde{\mathcal{W}}^2$ be any collection of admissible objects of the Fukaya category of $M^- \times M$ containing a subcategory quasi-isomorphic to the category $\mathcal{W}^2_{split}$ of product Lagrangians with split-Hamiltonian data, and let*

$$
i : \mathcal{W}^2_{split} \hookrightarrow \tilde{\mathcal{W}}^2
\tag{9.3}
$$

FIGURE 17. The homotopy between the quilted strip controlling $\mathbf{M}^1_{L_i \times L_j \times \Delta}$ and the map $\lambda_{\mathcal{W}_\Delta, L_i, L_j}$.



be the inclusion of this full subcategory. We claim that the functor

$$\mathbf{M} : \mathcal{W}^2_{split} \to \mathcal{W}\text{–mod–}\mathcal{W} \tag{9.4}$$

extends via homological algebra to an A_∞ *functor*

$$\tilde{\mathbf{M}} : \tilde{\mathcal{W}}^2 \to \mathcal{W}\text{–mod–}\mathcal{W}. \tag{9.5}$$

in the following fashion. The Yoneda embedding gives a functorial way to associate to any Lagrangian in $\tilde{\mathcal{W}}^2$ *a module over* $\mathcal{W}^2_{split}$*:*

$$i^* \circ \mathbf{Y}^R : \tilde{\mathcal{W}}^2 \to \mathcal{W}^2_{split}\text{–mod}, \tag{9.6}$$

(we write $i^* \circ \mathbf{Y}^R$ *to indicate the version of the Yoneda embedding in which we only test against objects in* $\mathcal{W}^2_{split}$*). Thus, our real claim is that we can naturally map* $\mathcal{W}^2_{split}$–mod *to* $\mathcal{W}$–mod–$\mathcal{W}$*, the category of* $\mathcal{W}-\mathcal{W}$ *bimodules.*

By Proposition 8.2, $\mathbf{M}$ *induces quasi-equivalence*

$$\mathbf{M} : \mathcal{W}^2_{split} \xrightarrow{\sim} \mathcal{W}^L \otimes \mathcal{W}^R \subset \mathcal{W}\text{–mod–}\mathcal{W}, \tag{9.7}$$

where $\mathcal{W}^L \otimes \mathcal{W}^R$ *is the tensor product of the dg categories of left Yoneda modules over* $\mathcal{W}$ *and right Yoneda modules over* $\mathcal{W}$ *respectively. By general abstract nonsense,* $\mathbf{M}$ *induces a* pushforward *functor on categories of modules, which is an equivalence if* $\mathbf{M}$ *is:*

$$\mathbf{M}_* : \mathcal{W}^2_{split}\text{–mod} \xrightarrow{\sim} (\mathcal{W}^L \otimes \mathcal{W}^R)\text{–mod}, \tag{9.8}$$

($\mathbf{M}_*$ *is defined by tensor product with the graph bimodule of* $\mathbf{M}$*, which is the one-sided pullback of the diagonal bimodule* $(\mathcal{W}^L \otimes \mathcal{W}^R)_\Delta$ *by* $\mathbf{M}$*).*

Next, an A_∞ *module* $\mathcal{M}$ *over the dg category* $\mathcal{A} = (\mathcal{W}^L \otimes \mathcal{W}^R)$ *(arising as the tensor product of dg categories) is quasi-isomorphic to a dg module over the category, via a functorial construction* $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_\Delta$*. In turn, a dg module over a tensor product of dg categories is equivalent to the data of a dg bimodule, and in particular gives an* A_∞ *bimodule. Moreover, the categories of dg modules over* $\mathcal{C} \otimes \mathcal{D}$ *and* $\mathcal{C}^{op} - \mathcal{D}$ *bimodules, thought of as subcategories of* A_∞ *modules and bimodules respectively, are actually equivalent. The correspondence for objects is basically obvious but checking the categories are equivalent requires the use of* Eilenberg-Zilber maps *to compare morphism spaces of the form* $\mathcal{B}_0 \otimes T(\mathcal{C} \otimes \mathcal{D}) \to \mathcal{B}_1$ *to* $T\mathcal{C}^{op} \otimes \mathcal{B}_0 \otimes T\mathcal{D} \to \mathcal{B}_1$*).*

Putting these observations together we obtain a strictification functor

$$s : (\mathcal{W}^L \otimes \mathcal{W}^R)\text{–mod} \xrightarrow{\sim} (\mathcal{W}^L)^{op}\text{–mod–}\mathcal{W}^R \tag{9.9}$$

Now, we note that each of $(\mathcal{W}^L)^{op}$ *and* $\mathcal{W}^R$ *is quasi-isomorphic to* $\mathcal{W}$ *via the Yoneda embedding, so* $(\mathcal{W}^L)^{op}-\mathcal{W}^R$ *bimodules should be basically the same as* $\mathcal{W}-\mathcal{W}$ *bimodules. Indeed, pull-back via the pair of Yoneda embeddings*

$$\mathbf{Y}_L : \mathcal{W}^{op} \xrightarrow{\sim} \mathcal{W}^L \tag{9.10}$$

$$\mathbf{Y}_R : \mathcal{W} \xrightarrow{\sim} \mathcal{W}^R \tag{9.11}$$

induces an equivalence of bimodule categories

$$(\mathbf{Y}_L \otimes \mathbf{Y}_R)^* : (\mathcal{W}^L)^{op}\text{–mod–}\mathcal{W}^R \xrightarrow{\sim} \mathcal{W}\text{–mod–}\mathcal{W}. \tag{9.12}$$

Putting this all together, we obtain a functor $\mathbf{F} : \tilde{\mathcal{W}}^2 \to \mathcal{W}$–mod–$\mathcal{W}$ *as* $(\mathbf{Y}_L \otimes \mathbf{Y}_R)^* \circ s \circ \mathbf{M}_* \circ i^* \circ \mathbf{Y}^R$*. We emphasize that each of the functors in the composition except possibly the first was a quasi-equivalence, so* $\mathbf{F}$*, like our geometric construction of* $\mathbf{M}$*, will be full and faithful if and only if* $i^* \circ \mathbf{Y}^R$ *is (for instance, on any object split-generated by product Lagrangians).*

A version of this approach for the infinitesimal Fukaya category [**NZ**] *seems to be implicitly used in work of Nadler* [**N**]*, assuming the existence of* $\mathbf{M}$ *for product Lagrangians (or really just some version of* $\mathbf{M}_*$*). There is one slight disadvantage of the approach in this Remark: in light of all of the homological algebra involved, it is difficult to calculate; for instance, it is somewhat non-trivial to see that* Δ *is sent to the diagonal bimodule. However, it is easy to see that for any* $\mathcal{L} \in \text{ob}\, \mathcal{W}^2$*, there is a quasi-isomorphism* $\mathbf{F}(\mathcal{L})(B, A) \simeq \hom_{\tilde{\mathcal{W}}^2}(A \times B, \mathcal{L})$ *of chain complexes.*

Appendix A. Orientations and signs

In this appendix, we recall the ingredients necessary to orient various moduli spaces of maps, thereby obtaining operations defined over the integers (or over a field of arbitrary characteristic). The relevant theory was first developed in [**FH1**] and adapted to the Lagrangian case in [**FOOO**, §8]. We will proceed as follows:

- In Section A.1, we associate, to every time-1 chord $x \in \chi(L_i, L_j)$ or orbit $y \in \mathcal{O}$, real one-dimensional vector spaces called **orientation lines**

$$o_x, o_y, \tag{A.1}$$

coming from the linearization of Floer's equation on any strip-like or cylindrical end. Then we recall from general theory how orientation lines, orientations of our Lagrangians and orientations of abstract moduli spaces determine canonical orientations of moduli spaces of maps. In the semi-stable case, this orientation is canonical up to a choice of trivialization of the natural $\mathbb{R}$ action.
- In Section A.2, we give a recipe for computing the sign of terms in the expression arising from the codimension 1 boundary components of a moduli space of maps.
- In Section A.3, we choose orientations for the top strata of various abstract moduli spaces of open-closed strings/glued pairs of discs.
- Finally, in Section A.4, we will use all of the ingredients discussed to carefully verify the signs arising in a single case.

We will draw heavily from the discussion in [**S5**, (11l)] (which discusses surfaces without interior punctures), along with the extension to general open-closed strings in [**A1**, §C]. Our notation will primarily follow [**A1**, §C].

A.1. Orientation Lines and Moduli Spaces of Maps. Given $y \in \mathcal{O}$, there is a unique homotopy class of trivializations of the pullback of TM to S^1 that is compatible with our chosen trivialization of $\Lambda^n_{\mathbb{C}} TM$. The linearization of Floer's equation (3.36) on a cylindrical end $[1, \infty) \times S^1$ with respect to such a trivialization exponentially converges to an operator of the form

$$Y \longmapsto \partial_s Y - J_t \partial_t Y - A(+\infty, t) Y, \tag{A.2}$$

where $J_t - A(+\infty, t)$ is a self-adjoint operator (see [**FH1**] for more details). Thus, to define an orientation line, we once and for all fix a local operator

$$\mathrm{D}_y : H^1(\mathbb{C}, \mathbb{C}^n) \longrightarrow L^2(\mathbb{C}, \mathbb{C}^n) \tag{A.3}$$

extending the asymptotics (A.2) in the following fashion. Endow $\mathbb{C}$ with a negative strip-like end around ∞ of the form

$$\begin{aligned} \epsilon : (-\infty, 0] \times \mathbb{R}/\mathbb{Z} &\longrightarrow \mathbb{C} \\ s, t &\longmapsto \exp(-2\pi(s + it)) \end{aligned} \tag{A.4}$$

and consider extensions of J_t and $A(-\infty, t)$ to families $J_{\mathbb{C}}$ of complex structures and $A_{\mathbb{C}}$ of endomorphisms of $\mathbb{C}^n$. Using these families, we define the operator D_y to be as in (A.2) using the extended families $J_{\mathbb{C}}$, $A_{\mathbb{C}}$.

Definition A.1. *The* **orientation line** o_y *is the determinant line* $\det(\mathrm{D}_y)$.

In a similar fashion, given $x \in \chi(L_i, L_j)$, after applying a canonical up to homotopy trivialization of the pullback of TM to $[0, 1]$, x can be thought of as a path between two Lagrangian subspaces Λ_i and Λ_j of $\mathbb{C}^n$, and the linearized operator corresponding to Floer's equation (3.25) asymptotically takes the same form as (A.2). Now, choose a negative striplike end around ∞ in the upper half plane

$$\begin{aligned} \epsilon : (-\infty, 0] \times [0, 1] &\longrightarrow \mathcal{H} \\ s, t &\longmapsto \exp(\pi i - \pi(s + it)). \end{aligned} \tag{A.5}$$

Also choose some family of Lagrangian subspaces F_z, $z \in \mathbb{R} \subset \mathcal{H}$ such that $F_{\epsilon(s \times \{0\})} = \Lambda_i$, $F_{\epsilon(s \times \{1\})} = \Lambda_j$, and choose extensions of A and J to all of $\mathcal{H}$ as before. One thus obtains an operator

$$\mathrm{D}_x : H^1(\mathbb{H}, \mathbb{C}^n, F) \longrightarrow L^2(\mathbb{H}, \mathbb{C}^n) \tag{A.6}$$

Definition A.2. *The* **orientation line** o_x *is the determinant line* $\det(\mathrm{D}_x)$.

Remark A.1. *We have omitted from discussion the grading structure, but we should remark that in reality, when working with graded Lagrangians, trivialization gives us* **graded Lagrangian spaces** $\Lambda_i^{\#}$, $\Lambda_j^{\#}$ *of* $\mathbb{C}^n$ *(thought of as living in the universal cover of the Lagrangian Grassmannian of* $\mathbb{C}^n$*). Instead of giving such a discussion now, we simply note that the family F of Lagrangian subspaces chosen above must lift to a family of graded Lagrangian subspaces interpolating between the lifts* $\Lambda_i^{\#}$ *and* $\Lambda_j^{\#}$ *(this determines F up to homotopy rel endpoints).*

The reader is referred to [**S5**, (11g)] for a more explicit spectral flow description of these determinant lines and indices.

By definition, orientation lines are naturally graded by the indices of the operators we have constructed above, meaning that the natural isomorphism

$$o_{x_1} \otimes o_{x_2} \xrightarrow{\sim} o_{x_2} \otimes o_{x_1} \tag{A.7}$$

introduces a Koszul-type sign

$$v \otimes w \mapsto (-1)^{|x_1|\cdot|x_2|} w \otimes v. \tag{A.8}$$

where $|x|$ is the degree of the chord (or orbit) x. Also, there is a natural pairing

$$o_x^\vee \otimes o_x \xrightarrow{\sim} \mathbb{R}. \tag{A.9}$$

Given a vector $\vec{x} = (x_1, \ldots, x_n)$ of chords or orbits, abbreviate the tensor product of orientation lines in $\vec{x}$ as

$$o_{\vec{x}} := \bigotimes_{i=1}^{n} o_{x_i} \tag{A.10}$$

The application of orientation lines to our setup is this: Standard gluing theory tells us that given a regular point u of some moduli space of maps with asymptotic conditions, orientation lines for the asymptotic conditions and an orientation for the abstract domain moduli space canonically determine an orientation of the tangent space at u.

To elaborate, let $\mathcal{M}$ be some abstract moduli space of Riemann surfaces Σ with boundary S. Denote by $(\bar{\Sigma}, \bar{S})$ the surface obtained by compactifying, i.e. filling in the boundary and interior punctures. Given a collection of asymptotic conditions $(\vec{x}_{out}, \vec{y}_{out}, \vec{x}_{in}, \vec{y}_{in})$ one can form the moduli space of maps

$$\mathcal{M}(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in}) \tag{A.11}$$

as described, for example, in Section 4. Suppose we have chosen an orientation of $\mathcal{M}$ and orientation lines o_x, o_y (and implicitly also an ordering of our inputs and outputs). Then:

Lemma A.1. *Let $\bar{C}_1, \ldots, \bar{C}_k$ be the components of the boundary of $\bar{S}$, and e_j denote the number of negative ends of $\bar{C}_j$. If we fix a marked point $z_j \in C_j$ mapping to a Lagrangian L_j then, assuming the moduli space $\mathcal{M}(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ is regular at a point u, we have a canonical isomorphism*

$$\lambda(\mathcal{M}(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})) \cong \lambda(\mathcal{M}) \otimes \bigotimes_j \lambda(T|_{u(z_j)} L_j)^{\otimes 1-e_j} \otimes o_{\vec{x}_{out}} \otimes o_{\vec{y}_{out}} \otimes o_{\vec{x}_{in}}^\vee \otimes o_{\vec{y}_{in}}^\vee, \tag{A.12}$$

where λ denotes top exterior power.

Proof. The version of this Lemma in the absence of interior punctures can be found in [**S5**, Prop. 11.13]. The minor generalization of including interior punctures is discussed in [**A1**, Lem. C.4]. ☐

In particular, given fixed orientation of $\mathcal{M}$, when the moduli space $\mathcal{M}(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ is rigid, we obtain, at any regular point $u \in \mathcal{M}(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$, an isomorphism

$$\mathcal{M}_u : o_{\vec{x}_{in}} \otimes o_{\vec{y}_{in}} \longrightarrow \bigotimes_j \left(\lambda(T|_{u(z_j)} L_j)^{\otimes 1-e_j} \right) \otimes o_{\vec{x}_{out}} \otimes o_{\vec{y}_{out}}. \tag{A.13}$$

If we fix orientations for the L_j, then we obtain an isomorphism of the form (A.13) without the L_j factors. But the L_j factors will continue to have relevance in sign comparison arguments.

Remark A.2 (The semistable case). *The moduli spaces $\mathcal{M}(y_0; y_1)$, $\mathcal{R}(x_0; x_1)$ arise as a further quotient of the non-rigid elements of $\tilde{\mathcal{M}}(y_0; y_1)$ and $\tilde{R}^1(x_0; x_1)$ by the natural $\mathbb{R}$ actions. Thus, at rigid points $u \in \mathcal{M}(y_0; y_1)$, $v \in \mathcal{R}(x_0; x_1)$ one obtains trivializations of $\lambda(\tilde{\mathcal{M}}(y_0; y_1))$, $\lambda(\tilde{\mathcal{R}}^1(x_0; x_1))$ and hence isomorphisms*

$$\begin{gathered} o_{y_1} \longrightarrow o_{y_0} \\ o_{x_1} \longrightarrow o_{x_0} \end{gathered} \tag{A.14}$$

by choosing a trivialization of the $\mathbb{R}$ actions. In both cases, following the conventions in [**S5**, (12f)] *and* [**A1**, §C.6], *choose ∂_s to be the vector field inducing the trivialization.*

A.2. Comparing Signs. Let $\overline{\mathcal{Q}}$ be some abstract compact moduli space, and suppose its codimension one boundary has a component covered by a product of lower dimensional moduli spaces (either of which may also decompose as a product).

$$\overline{\mathcal{A}} \times_{\vec{v},\vec{w}} \overline{\mathcal{B}}. \tag{A.15}$$

We should first elaborate upon the notation $\times_{\vec{v},\vec{w}}$. We suppose first that we have fixed separate orderings of the input and output boundary and interior marked points for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$; such orderings will be specified case by case.

Definition A.3. *Given vectors of the form*

$$\begin{aligned} \vec{v} &= \{(v_1^-, v_1^+), \ldots, (v_k^-, v_k^+)\} \\ \vec{w} &= \{(w_1^-, w_2^+), \ldots, (w_l^-, w_l^+)\}, \end{aligned} \tag{A.16}$$

the notation

$$\overline{\mathcal{A}} \times_{\vec{v},\vec{w}} \overline{\mathcal{B}} \tag{A.17}$$

refers to the product of abstract moduli spaces $\overline{\mathcal{A}}$ with $\overline{\mathcal{B}}$ in which

- *the v_i^-th boundary output of $\overline{\mathcal{A}}$ is (nodally) glued to the v_i^+th boundary input of $\overline{\mathcal{B}}$, for $1 \leq i \leq k$, and*
- *the w_j^-th interior output of $\overline{\mathcal{A}}$ is (nodally) glued to the w_j^+th interior input of $\overline{\mathcal{B}}$, for $1 \leq j \leq l$.*

We refer to such $(\vec{v}, \vec{w})$ as a **nodal gluing datum**.

Now, suppose we had fixed orientations for $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{Q}$. The associated space of maps

$$\overline{\mathcal{Q}}(\vec{x}_{in}, \vec{y}_{in}; \vec{x}_{out}, \vec{y}_{out}) \tag{A.18}$$

inherits an orientation from Lemma A.1 and has as a codimension-1 boundary component the product of moduli spaces

$$\mathcal{A}(\vec{x}^1_{in}, \vec{y}^1_{in}; \vec{x}^1_{out}, \vec{y}^1_{out}) \times_{\vec{v},\vec{w}} \mathcal{B}(\vec{x}^2_{in}, \vec{y}^2_{in}; \vec{x}^2_{out}, \vec{y}^2_{out}) \tag{A.19}$$

for suitable input and output vectors $\vec{x}^i_{in}, \vec{y}^i_{in}; \vec{x}^i_{out}, \vec{y}^i_{out}$. Thus, the product (A.19) inherits a boundary orientation from (A.18). However, Lemma A.1 and our chosen orientations for $\mathcal{A}$ and $\mathcal{B}$ also give (A.19) a canonical *product orientation*. The question of relevance to us is

(A.20) What is the sign difference between the product orientation and boundary orientation of (A.19)?

Actually, we will also equip $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{Q}$ with sign twisting data $\vec{t}_{\mathcal{A}}$, $\vec{t}_{\mathcal{B}}$ and $\vec{t}_{\mathcal{Q}}$ and calculate the sign difference with these twistings incorporated. But we can just add them at the end.

Abbreviate

$$o_{\vec{x}_{in}, \vec{y}_{in}; \vec{x}_{out}, \vec{y}_{out}} := o_{\vec{x}_{out}} \otimes o_{\vec{y}_{out}} \otimes o^\vee_{\vec{x}_{in}} \otimes o^\vee_{\vec{y}_{in}} \tag{A.21}$$

and

$$\vec{xy}^i := (\vec{x}^i_{in}, \vec{y}^i_{in}; \vec{x}^i_{out}, \vec{y}^i_{out}). \tag{A.22}$$

Then, (A.20) can be rephrased as: what is the sign difference in the failure of commutativity of the following diagram?

$$\begin{tikzcd} \lambda(\mathcal{Q}(\vec{xy})) \arrow[rr] \arrow[d] & & \lambda(\mathcal{Q}) \otimes \mathcal{L}_{\mathcal{Q}} \otimes o_{\vec{xy}} \arrow[d] \\ \lambda(\mathcal{B}(\vec{xy}^2)) \otimes \lambda(\mathcal{A}(\vec{xy}^1)) \arrow[r] & \lambda(\mathcal{B}) \otimes \mathcal{L}_{\mathcal{B}} \otimes o_{\vec{xy}^2} \otimes \lambda(\mathcal{A}) \otimes \mathcal{L}_{\mathcal{A}} \otimes o_{\vec{xy}^1} & \end{tikzcd} \tag{A.23}$$

Here $\mathcal{L}_{\mathcal{Q}}$, $\mathcal{L}_{\mathcal{A}}$, and $\mathcal{L}_{\mathcal{B}}$ are the powers of orientation of a fixed boundary Lagrangians, one for each boundary component of representatives of the moduli spaces, appearing in Lemma A.1; these satisfy $\mathcal{L}_{\mathcal{B}} \otimes \mathcal{L}_{\mathcal{A}} = \mathcal{L}_{\mathcal{Q}}$ (up to an even power of the top exterior power of Lagrangians, which is trivial). The top and bottom horizontal arrows are the ones given by Lemma A.1. The reversal of $\mathcal{B}$ and $\mathcal{A}$ above comes from the fact that we originally listed the boundary strata of $\mathcal{A}$, $\mathcal{B}$ in the reverse order of composition.

Proposition A.1 (Sign Comparison). *The sign difference between the product and boundary orientations is the sum of four contributions:*

- **Koszul signs** *from* **reordering** $\lambda(A)$ *past* $o_{\vec{x}\vec{y}^2}$ *and* $\mathcal{L}_{\mathcal{B}}$.
- **Koszul signs** *from* **reordering** $\mathcal{L}_{\mathcal{A}}$ *past* $o_{\vec{x}\vec{y}^2}$.
- **Koszul signs** *from* **reordering** $o_{\vec{x}\vec{y}^2} \otimes o_{\vec{x}\vec{y}^1}$ *to become* $o_{\vec{x}\vec{y}}$ *(using the natural pairings (A.9) on elements coming from the gluing* $(\vec{v}, \vec{w})$*).*
- *Comparing the* **product versus boundary orientation** *on* **abstract moduli spaces** $\mathcal{A} \times_{\vec{v},\vec{w}} \mathcal{B}$.

Since all of the operations we construct involve *sign twisting data* (Definition 4.16), we add back in said data to obtain the right signs.

Corollary A.1. *The sign of the composed term* $(-1)^{\vec{t}_1}\mathbf{F}_{\mathcal{B}} \circ_{\vec{v},\vec{w}} (-1)^{\vec{t}_2}\mathbf{F}_{\mathcal{A}}$ *in the expression arising from the codimension 1 boundary principle (Lemma 4.4) applied to* $\mathcal{Q}$ *is the sum of:*

- *all terms from Proposition A.1; and*
- *contributions from the sign twisting data* $\vec{t_1}$ *and* $\vec{t_2}$*, in the sense of (4.30).*

A.3. Abstract Moduli Spaces and their orientations. By Lemma A.1, we must choose orientations of the various abstract moduli spaces we consider in order to orient the associated operations. In this section, we do precisely that. We will also, in a sample case, compute explicitly the sign difference between the induced and chosen orientation on boundary strata.

A.3.1. $\mathcal{R}^d$. Fix a slice of $\mathcal{R}^d$ in which the first three boundary marked points z_0^-, z_1^+, and z_2^+ are fixed, and consider the positions of the remaining points $(z_3, \dots, z_d)$ with respect to the counterclockwise boundary orientation as a local chart. With respect to this chart, orient $\mathcal{R}^d$ by the top form

$$dz_3 \wedge \cdots \wedge dz_d. \tag{A.24}$$

This agrees with the conventions in [**S5**] and [**A1**], so we will not discuss this case or its signs further.

A.3.2. $\mathcal{R}_d^1$. Take a slice of $\mathcal{R}_d^1$ in which the interior point y_{out} is fixed, as is the distinguished marked point z_d. With respect to the induced coordinates $(z_1, \dots, z_d)$ induced by the positions of the remaining marked points, pick orientation form

$$-\, dz_1 \wedge \cdots \wedge dz_{d-1}. \tag{A.25}$$

This agrees with the choice made in [**A1**, C.3].

A.3.3. $\mathcal{R}_d^{1,1}$. Take a slice of $\mathcal{R}_d^{1,1}$ in which the interior point y_{in} and outgoing boundary point z_0^-, are fixed, and using the positions of the remaining coordinates $(z_1, \dots, z_d)$ as the local chart, again pick orientation form

$$-\, dz_1 \wedge \cdots \wedge dz_d. \tag{A.26}$$

A.3.4. $\mathcal{R}^1_{d_1,d_2}$. Fix a slice of $\mathcal{R}^1_{d_1,d_2}$ (see Definition 4.20) in which z_0, z_0', and y_{out} are fixed at $-i$, i, and 0 respectively, and consider the positions of the remaining points in the slice $(z_1, \dots, z_{d_1}, z_1', \dots, z_{d_2}')$ as the local chart. With respect to these coordinates, pick orientation form

$$-\, dz_1 \wedge \cdots \wedge dz_{d_1} \wedge dz_1' \wedge \cdots \wedge dz_{d_2}'. \tag{A.27}$$

Proposition A.2. *With respect to the strata listed in (4.77)-(4.80), two of the differences in sign between chosen and induced orientations on boundary strata are as follows:*

stratum	**sign difference**
(4.77)	$1 + n + k' + k'(l + k - n)$
(4.80)	$1 + l(l'+1) + k(l - l')$

Proof. We have not listed the sign differences for the strata (4.78) and (4.79), which follow from identical calculations. For (4.77), with respect to the local charts $(z_{n+3}, \dots, z_{n+k'})$ on $\mathcal{R}^{k'}$ and

$$(z_1, \dots, z_n, \tilde{z}, z_{n+k'+1}, \dots, z_k, z_1', \dots, z_l')$$

on $\mathcal{R}^1_{k-k'+1,l}$, the gluing map

$$\rho : [0,1) \times \mathcal{R}^{k'} \times \mathcal{R}^1_{k-k'+1,l} \longrightarrow \mathcal{R}^1_{k,l} \tag{A.28}$$

has the approximate form

$$\begin{aligned}t, (z_{n+3}, \dots, z_{n+k'}), &(z_1, \dots, z_n, \tilde{z}, z_{n+k'+1}, \dots, z_k, \cdots) \longmapsto \\ &(z_1, \dots, z_n, \tilde{z}, \tilde{z}+t, \tilde{z}+tz_{n+3}, \dots, \tilde{z}+tz_{n+k'}, z_{n+k'+1}, \dots).\end{aligned} \tag{A.29}$$

Thus, the pullback under ρ of the top form

$$-\, dz_1 \wedge \cdots \wedge dz_k \wedge dz'_1 \wedge \cdots \wedge dz'_l. \tag{A.30}$$

is, modulo positive rescaling,

$$dz_1 \wedge \cdots \wedge dz_n \wedge d\tilde{z} \wedge dt \wedge dz_{n+3} \wedge \cdots \wedge dz_{n+k'} \wedge dz_{n+k'+1} \wedge \cdots \wedge dz_k \wedge dz'_1 \wedge \cdots dz'_l \tag{A.31}$$

which visibly differs from the product orientation (using the outward pointing vector $-dt$)

$$(-dt) \wedge (-dz_1 \wedge dz_n \wedge d\tilde{z} \wedge dz_{n+k'+1} \wedge \cdots) \wedge (dz_{n+3} \wedge \cdots dz_{n+k'}) \tag{A.32}$$

by a sign of parity

$$n + 1 + (k' - 2) \cdot (l + k - n - k') = 1 + n + k' + k'(l + k - n). \tag{A.33}$$

For (4.80), the gluing map

$$\tilde{\rho} : [0, 1) \times \mathcal{R}^{l'+k'+1} \times \mathcal{R}^1_{k-k', l-l'} \longrightarrow \mathcal{R}^1_{k,l} \tag{A.34}$$

takes the following approximate form:

$$\begin{aligned}&t, (z_{k-k'+3}, \dots, z_k, z'_0, z'_1, \dots, z'_{l'}), (z_1, \dots, z_{k-k'}, z'_{l'+1}, \dots, z'_l) \longmapsto \\ &\quad (z_1, \dots, z_{k-k'}, i - t(z'_0 - a), i - t(z'_0 - b), i - tz_{k-k'+3}, \dots, i - tz_k, i + tz'_1, \\ &\quad\quad \dots, i + tz'_{l'}, z'_{l'+1}, \dots, z'_l)\end{aligned} \tag{A.35}$$

for some constants $b > a > 0$. Thus, the pull back of the top form (A.30) is, up to positive rescaling

$$(-1)^{k'-2} dz_1 \wedge \cdots dz_{k-k'} \wedge dt \wedge dz'_0 \wedge dz_{k-k'+3} \wedge \cdots \wedge dz_k \wedge dz'_1 \wedge \cdots dz'_{l'} \wedge \cdots \wedge dz'_l, \tag{A.36}$$

which differs from the product orientation

$$\begin{aligned}(-dt) \wedge (dz_1 \wedge \cdots \wedge dz_{k-k'} \wedge dz'_{l'+1} \wedge \cdots \wedge dz_{l'}) \\ \wedge (dz_{k-k'+3} \wedge \cdots dz_k \wedge dz'_0 \wedge \cdots \wedge dz'_{l'})\end{aligned} \tag{A.37}$$

by a sign of parity

$$1 + (k' - 2) + (k - k') + (k' - 2) + (k' + l' - 1) \cdot (l - l') = 1 + l(l' + 1) + k'(l - l') \pmod 2. \tag{A.38}$$

□

A.3.5. $\mathcal{R}^{0,0,s,t}_2$. Fixing a slice of the action for which z^1_-, z^2_-, z^1_+, z^2_+ are fixed at i, $-i$, 1 and -1 and using the positions of the remaining coordinates

$$(z^2_1, \dots, z^2_s, z^3_1, \dots, z^3_t) \tag{A.39}$$

as a chart, pick orientation form

$$-\, dz^2_1 \wedge \cdots \wedge dz^2_s \wedge dz^3_1 \wedge \cdots \wedge dz^3_t. \tag{A.40}$$

A.3.6. $Q(3, \mathbf{r})$. Strictly speaking, we embed the open locus of quilted strips into glued discs with sequential point identifications, but since this embedding is an isomorphism on the open locus, it will suffice to write down a top form on the level of $Q(3, \mathbf{r})$. There are three cases:

- If $r_2 > 0$, then picking a slice of the $\mathbb{R}$ action for which the highest marked point on the middle strip $z^{r_2}_2$ is fixed, we obtain coordinates

$$(z^1_3, \dots, z^{r_3}_3, z^{r_1}_1, \dots, z^1_1, z^{r_2-1}_2, \dots, z^1_2). \tag{A.41}$$

- If $r_2 = 0$ but $r_3 > 0$, pick a slice for which z^1_3 is fixed to obtain the chart

$$(z^2_3, \dots, z^{r_3}_3, z^{r_1}_1, \dots, z^1_1). \tag{A.42}$$

- Lastly, if $r_2 = r_3 = 0$, picking a slice for which $z^{r_1}_1$ is fixed, we obtain the chart

$$(z^{r_1-1}_1, \dots, z^1_1) \tag{A.43}$$

In all three cases, pick orientation form the top exterior power of these coordinates of the chart in the orders specified above.

A.4. Sign Verification. In this section, we use all of the ingredients above to verify the signs of equations in one case. Namely, we will (partly) show that

PROPOSITION A.3 (Corollary 4.3 with signs). ${}_2\mathcal{OC}$ *is a chain map (with the right signs).*

PROOF OF PROP. A.3. We need to establish that the boundary strata (4.77)-(4.80) along with strip-breaking and our chosen sign twisting data, determine the equation

$$ {}_2\mathcal{OC} \circ d_{{}_2\mathrm{CC}_*} - d_{CH} \circ {}_2\mathcal{OC} = 0 \tag{A.44} $$

up to an overall sign. As all the cases are analogous, we will simply show that some of the terms in

$$ {}_2\mathcal{OC} \circ d_{{}_2\mathrm{CC}_*} \tag{A.45} $$

appear with the correct sign (up to the overall sign); in particular we focus upon all of the terms in the strata (4.77), which should contribute to the terms

$$ \sum_{m,k} (-1)^{\maltese_1^n} {}_2\mathcal{OC}(\mathbf{a}, x_l, \ldots, x_1, \mathbf{b}, y_s, \ldots, y_{m+k+1}, \mu^k(y_{m+k}, \ldots, y_{m+1}), y_m, \ldots, y_1), \tag{A.46} $$

where $\maltese_1^m$ is the sign

$$ \sum_{j=1}^{m} ||y_i||. \tag{A.47} $$

So, fix a set of asymptotic inputs $(y_1, \ldots, y_s, \mathbf{b}, x_1, \ldots, x_l, \mathbf{a})$. The strata (4.77) are, for $k < s$ and $0 \leq m < s - k' + 1$,

$$ \mathcal{R}^k \times_{m+1} \mathcal{R}^1_{s-k+1,l}. \tag{A.48} $$

Abbreviating

$$ \begin{aligned} x_{i \to j} &:= \{x_i, x_{i+1}, \ldots, x_j\} \\ y_{i \to j} &:= \{y_i, y_{i+1}, \ldots, y_j\} \end{aligned} \tag{A.49} $$

the corresponding moduli spaces are

$$ \mathcal{R}^k(\tilde{y}, y_{m+1 \to m+k}) \times \mathcal{R}^1_{s-k'+1,1}(z; y_{1 \to m}, \tilde{y}, y_{m+k+1 \to s}, \mathbf{b}, x_{1 \to l}, \mathbf{a}) \tag{A.50} $$

in reverse order of composition, where z is an output orbit, and $\tilde{y}$ ranges over all possible admissible asymptotic conditions. Abbreviating $\lambda_\mu := \lambda(\mathcal{R}^k)$, and $\lambda_{2\mathcal{OC}} := \lambda(\mathcal{R}^1_{s-k+1,l})$, Lemma A.1 tells us that the natural product orientation form is isomorphic to

$$ \begin{aligned} &\big(\lambda_{2\mathcal{OC}} \otimes \lambda(L_0) \otimes o_z \otimes o^\vee_{y_{1\to m}} \otimes o^\vee_{\tilde{y}} \otimes o^\vee_{y_{m+k+1\to s}} \otimes o^\vee_{\mathbf{b}} \otimes o^\vee_{x_{1\to l}} \otimes o^\vee_{\mathbf{a}}\big) \\ &\qquad\qquad \otimes \big(\lambda_\mu \otimes o_{\tilde{y}} \otimes o^\vee_{y_{m+1\to m+k}}\big). \end{aligned} \tag{A.51} $$

where, as before, we've abbreviated $o^\vee_{y_{1\to k}} = o^\vee_{y_1} \otimes \cdots o^\vee_{y_k}$ and so on, and we've abbreviated $\lambda(L_0) := \lambda(T|_{u(z_i)} L_0)$ for one of the Lagrangian boundary conditions L_0 of the moduli space

$$ \mathcal{R}^1_{s-k'+1,1}(z; y_{1\to m}, \tilde{y}, y_{m+k+1\to s}, \mathbf{b}, x_{1\to l}, \mathbf{a}). $$

From the above description, we can immediately calculate some of the sign contributions in Proposition A.1:

- $\mathcal{R}^k$ has dimension $(k-2)$, so the sign for reordering λ_μ to be next to $\lambda_{2\mathcal{OC}}$ has parity

$$ \star_1 := (k-2)(n - l - s + k + 1 + n) = k(l+s) \pmod 2. \tag{A.52} $$

- there are no Lagrangian terms $\lambda(T|_{u(z_j)} L_j)$ in orientation form of the moduli space associated to $\mathcal{R}^k$, so the associated signs of this sort are zero,

- the sign for reordering the orientation lines $o_{\tilde{y}} \otimes o^\vee_{y_{m+1\to m+k}}$ to be immediately to the right of $o^\vee_{\tilde{y}}$ (allowing one also to pair and cancel the $o^\vee_{\tilde{y}}$, $o_{\tilde{y}}$) has parity

$$\begin{aligned}\star_2 &:= (2-k)(|\mathbf{a}| + \sum_{i=m+k+1}^{s} |y_i| + |\mathbf{b}| + \sum_{i=1}^{l} |x_i|).\\ &= k(|\mathbf{a}| + \sum_{i=m+k+1}^{s} |y_i| + |\mathbf{b}| + \sum_{i=1}^{l} |x_i|) \pmod 2.\end{aligned} \tag{A.53}$$

- the sign difference between boundary and product orientations on the moduli space $\mathcal{R}^k \times \mathcal{R}^1_{s-k+1,l}$ was computed in Proposition A.2 to have parity

$$\star_3 := 1 + m + k + k(l+s-m). \tag{A.54}$$

Finally, we can add in the sign twist contributions mentioned in Corollary A.1, corresponding to the operations ${}_2\mathcal{OC}_{l,s-k+1}$, and μ^k:

- The sign twist contribution from μ^k has parity

$$\S_2 := (1,\dots,k)\cdot(|y_{m+1}|,\dots,|y_{m+k}|) = \sum_{i=m+1}^{m+k} (i-m)|y_i|. \tag{A.55}$$

- The sign twist contribution from ${}_2\mathcal{OC}_{l,s-k+1}$ has parity

$$\begin{aligned}\S_1 := \sum_{i=1}^{m} i|y_i| + (m+1)|\tilde{y}| + \sum_{i=m+k+1}^{s} (i-k+1)|y_i| + (s-k+1)|\mathbf{b}|\\ + \sum_{i=1}^{l}(s-k+1+i)|x_i| + (s-k+1+l)|\mathbf{a}|\end{aligned} \tag{A.56}$$

  where

$$|\tilde{y}| = 2 - k + \sum_{i=m+1}^{m+k} |y_i|. \tag{A.57}$$

Combining all of these signs, we compute that

$$\S_1 + \S_2 + \star_1 + \star_2 + \star_3 = \sum_{i=1}^{m} |y_i| + m + \bigstar_{l,s} \pmod 2, \tag{A.58}$$

where

$$\bigstar_{l,s} = \sum_{i=1}^{s}(i+1)|y_i| + (s+1)|\mathbf{b}| + \sum_{i=1}^{l}(s+1+i)|x_i| + (s+1+l)|\mathbf{a}| \tag{A.59}$$

is independent of k, m, and

$$\sum_{i=1}^{m} |y_i| + m = \maltese_1^m \pmod 2 \tag{A.60}$$

as desired. This calculation extends formally to the semi-stable case $k = 1$ as well. The only extra ingredient, following Remark A.2, is an extra sign of parity 0 or 1 coming from determining whether the vector ∂_s after gluing is inward pointing (1) or outward pointing (0). In this case, the vector is outward pointing so there is no additional sign contribution. Note that when the second component is semi-stable instead, the vector will be inward pointing, contributing to e.g., the -1 coefficient in $-d_{CH} \circ {}_2\mathcal{OC}$. See [**S5**, (12f)] for more details. $\square$

Appendix B. Action, energy and compactness

The goal of this appendix is to prove a compactness result for Floer-theoretic operations controlled by bordered Riemann surfaces mapping into a Liouville manifold, under some assumptions about the almost complex structure and Hamiltonian perturbation terms. There are several existing compactness results for the wrapped Fukaya category and some open-closed maps, e.g. [**A1**, §B], which are unfortunately not directly applicable for our choices of Hamiltonian perturbations. The problems occur because we use time-dependent perturbations of a standard Hamiltonian, which we cannot guarantee will vanish at infinitely many levels of the cylindrical coordinate r (this is an essential assumption in [**A1**, §B]). Solutions to Floer's equation for such perturbed Hamiltonians will fail to satisfy a maximum principle, but if the complex structure has been carefully chosen and the time-dependent perturbations are sufficiently small, this failure can be controlled. We make use in an essential way of a delicate technique for obtaining a-priori C^0 bounds on such solutions due to Floer-Hofer and Cieliebak [**FH2**] [**C**]. This technique has also been used by Oancea [**O2**], whose work we draw upon.

Remark B.1. *Our situation is a little different from* [**FH2**] [**C**] [**O2**] *in that we need a variant of their compactness result for potentially finite cylindrical regions in a larger Riemann surface. This, and differing conventions regarding Hamiltonians (quadratic versus linear) and complex structures (contact type versus rescaled contact type) prevent us from citing any of these papers directly.*

Our setup is as follows: Let W be a Liouville manifold with cylindrical end

$$W = \bar{W} \cup_{\partial W} \partial W \times [1, \infty)_r. \tag{B.1}$$

The coordinate on the end is given by a function π_r on $W - \bar{W}$, which we extend over the interior of W to a function

$$\pi_r : W \longrightarrow [0, \infty) \tag{B.2}$$

such that

$$\bar{W} = \pi^{-1}([0, 1]). \tag{B.3}$$

Let S be a bordered surface with boundary ∂S, and equip S with a Floer datum in the sense of Definition 4.11, namely:

- a collection of **δ-bounded weighted strip and cylinder data**,
- a **sub-closed one form** α_S, compatible with the weighted strip and cylinder data
- a **primary Hamiltonian** $H_S : S \to \mathcal{H}(M)$ that is H compatible with the weighted strip and cylinder data
- an adapted **rescaling function** a_S, constant and equal to the weights on each strip and cylinder region,
- an **almost complex structure** J_S that is adapted to the weighted strip and cylinder data, the rescaling function a_S, and some fixed J_t, and
- an S^1 **perturbation** F_S adapted to (F_T, ϕ_ϵ) for some F_T, ϕ_ϵ as in the definition.

Fix a Lagrangian labeling $\vec{L}$ for the boundary components of S and a compatible choice of input and output chords and orbits corresponding to the positive and negative marked points on S

$$\vec{x}_{in}, \vec{x}_{out}, \vec{y}_{in}, \vec{y}_{out}. \tag{B.4}$$

We study maps $u : S \to W$ satisfying Floer's equation for this datum, namely

$$(du - X_S \otimes \gamma)^{0,1} = 0 \tag{B.5}$$

with asymptotic/boundary conditions

$$\begin{cases} \lim_{p \to z_k^\pm} u = y_k^\pm \\ \lim_{p \to b_j^\pm} u = x_j^\pm \\ \text{for } p \in \partial^n S, \qquad u(p) \in \psi^{\rho(p)}(L_n) \end{cases}.$$

Here X_S is the Hamiltonian vector field corresponding to the **total Hamiltonian**

$$H_S^{tot} = H_S + F_S. \tag{B.6}$$

The compactness result we need is:

THEOREM B.1. *Given such a map $u : S \to W$, there is a constant C depending only on F_t, and ϕ_ϵ, $\vec{x}_{in}$, $\vec{x}_{out}$, $\vec{y}_{in}$, $\vec{y}_{out}$ such that*

$$(\pi_r \circ u) \leq C. \tag{B.7}$$

Moreover, given any set of $\vec{y}_{in}$ $\vec{x}_{in}$, there are a finite number of collections $\vec{y}_{out}$, $\vec{x}_{out}$ for which the relevant moduli spaces are non-empty.

First, we define appropriate notions of action and energy. Suppose we have fixed a Hamiltonian H and a time-dependent perturbation F_t, and have picked a surface S with compatible Floer data. Let $x \in \chi(L_i, L_j)$ be the asymptotic condition at strip-like end ϵ^k with corresponding *weight* w_k. Moreover, suppose the perturbation term F_S is equal to the constant C_k on this strip-like end (this can be chosen to be zero if there are no interior marked points in S).

DEFINITION B.1. *The* **action** *of x is defined to be the quantity*

$$\mathcal{A}(x) := -\int_0^1 (\tilde{x}_{w_k})^*\theta + \int_0^1 w_k \cdot H^{w_k}(x(t))dt + f_{L_j}(x(1)) - f_{L_i}(x(0)) + w_k C_k. \tag{B.8}$$

where the f_{L_i} are the chosen fixed primitives of the Lagrangians L_i,

$$H^{w_k} := \frac{(\psi^{w_k})^* H}{w_k^2}, \tag{B.9}$$

and

$$\tilde{x}_{w_k} \tag{B.10}$$

is the chord x thought of as a time-1 chord of $w_k \cdot H^{w_k}$, under the rescaling correspondence (4.3).

Similarly, let $y \in \mathcal{O}$ be the asymptotic condition at cylindrical end δ^l with corresponding weight v_l.

DEFINITION B.2. *The* **action** *of y is the quantity*

$$\mathcal{A}(y) := -\int_{S^1} (\tilde{y}_{v_l})^*\theta + \int_0^1 v_l \cdot H^{v_l}(x(t))dt + \int_0^1 v_l \cdot F_t^{v_l}(t, x(t))dt \tag{B.11}$$

where H^{v_l} is as before and $F_t^{v_l}$ is defined as

$$F_t^v := \frac{(\psi^{v_l})^* F_t}{v_l^2}. \tag{B.12}$$

LEMMA B.1. *The action of a Hamiltonian chord or orbit becomes arbitrarily negative as $r \to \infty$.*

PROOF. This Lemma is a variant of one in [**A1**, §B.2]. The first observation is, by Lemma 4.1, that

$$H^{w_k} = H = r^2 \tag{B.13}$$

away from a compact subset. Also, for a Hamiltonian chord $\tilde{x}_{w_k}$

$$\begin{aligned} \tilde{x}_{w_k}^* \theta &= \theta(X_{w_k \cdot H^{w_k}})dt \\ &= w_k \cdot \omega(Z, X_{H^{w_k}})dt \\ &= w_k \cdot dH^{w_k}(Z)dt \\ &= 2w_k \cdot rdr(r\partial_r)dt \\ &= 2w_k r^2 dt. \end{aligned} \tag{B.14}$$

Thus, for a chord $x \in \chi(L_i, L_j)$ away from a compact set (so f_{L_j} and f_{L_i} are zero):

$$\mathcal{A}(x) = -\int_0^1 \tilde{x}_{w_k}^* \theta + \int_0^1 w_k H^{w_k}(x(t))dt + w_k C_k \tag{B.15}$$

$$= -\int_0^1 w_k \cdot r^2 dt + w_k C_k, \tag{B.16}$$

which satisfies the Lemma. Similarly for an orbit $y \in \mathcal{O}$,

$$\mathcal{A}(y) = -\int_0^1 \tilde{y}_{v_l}^* \theta + \int_0^1 v_l \cdot H_S(t, y(t))dt \tag{B.17}$$

$$= -\int_0^1 v_l \cdot (\theta(X_{H^{v_l}}) + \theta(X_{F^{v_l}}) - H^{v_l} - F^{v_l})dt \tag{B.18}$$

$$= -\int_0^1 v_l \cdot (H^{v_l} + \theta(X_{F^{v_l}}) - F^{v_l})dt. \tag{B.19}$$

The above expression also satisfies the Lemma, as H^{v_l} dominates $\theta(X_{F^{v_l}})$ and F^{v_l} away from a compact set. □

Following [**AS**], given a map u satisfying Floer's equation, we define two notions of energy. The **geometric energy** of u is defined as

$$E_{geo}(u) := \int_S ||du - X_S^{tot} \otimes \gamma||^2. \tag{B.20}$$

where the norm $|| \cdot ||$ comes from the complex structure J_S. Picking local coordinates $z = s + it$ for S, we see that for a solution u to Floer's equation (abbreviating $X = X_S$),

$$\begin{aligned} E_{geo}(u) &= \int \omega((du - X \otimes \gamma)(\partial_s), J_S(du - X \otimes \gamma)(\partial_s)dsdt \\ &= \int \omega(du(\partial_s) - X \otimes \gamma(\partial_s), (du - X \otimes \gamma) \circ j(\partial_s))dsdt \\ &= \int_S (\omega(\partial_s u, \partial_t u) - \omega(X \otimes \gamma(\partial_s), \partial_t u) - \omega(\partial_s u, X \otimes \gamma(\partial_t)))\, dsdt \\ &= \int_S (u^*\omega - (d_W H(\partial_s u)\gamma(\partial_t) - d_W H(\partial_t u)\gamma(\partial_s))dsdt) \\ &= \int_S u^*\omega - d_W(u^* H_S^{tot})\gamma, \end{aligned} \tag{B.21}$$

a version of the energy identity for J-holomorphic curves (here d_W denotes differentiation in the direction of W). The **topological energy** of u is defined as

$$E_{top}(u) := \int_S u^*\omega - \int_S d(u^*(H_S^{tot}) \cdot \gamma). \tag{B.22}$$

Since γ is sub-closed and H_S is positive, whenever $d_S(u^* H_S) = 0$ (that is, the Hamiltonian term is a constant function along the surface S, so $d(u^* H_S) = d_W(u^* H_S)$), we have an *energy inequality*

$$0 \le E_{geo}(u) \le E_{top}(u). \tag{B.23}$$

In general, one has a *corrected energy inequality* involving the integral of a curvature term (namely, the the derivatives of H_S^{tot} along S):

$$0 \le E_{geo}(u) \le E_{top}(u) + \int_S d_S(u^* H_S^{tot}) \wedge \gamma. \tag{B.24}$$

It follows from the choices made that the curvature is bounded on $S \times W$ and vanishes on strip-like and cylindrical ends. We therefore get an inequality

$$0 \le E_{geo}(u) \le E_{top}(u) + C, \tag{B.25}$$

for some C which is independent of any particular u (whose domain, equipped with Floer data, belongs to a given compact family). Noting that

$$\epsilon_k^*(u^* H_S \gamma) = w_k \cdot (H^{w_k} + C_k)dt \tag{B.26}$$

on strip-like ends and

$$\delta_l^*(u^* H_S \gamma) = (w_k \cdot H^{v_l} + F_T^{v_l})dt \tag{B.27}$$

on cylindrical ends, we apply Stokes' theorem to (B.22) to conclude:

$$E_{top}(u) = (\sum_{y\in\vec{y}_{out}} \mathcal{A}(y) - \sum_{y\in\vec{y}_{in}} \mathcal{A}(y)) + (\sum_{x\in\vec{x}_{out}} \mathcal{A}(x) - \sum_{x\in\vec{x}_{in}} \mathcal{A}(x)). \tag{B.28}$$

PROPOSITION B.1. *For any x_{in}, y_{in}, there are finitely many choices of x_{out}, y_{out} such that there is a solution u to the relevant Floer's equation.*

PROOF. By Lemma B.1 and (B.28), for fixed inputs $\vec{x}_{in}$, $\vec{y}_{in}$, for all but a finite selection of $\vec{x}_{out}$, $\vec{y}_{out}$, any u satisfying Floer's equation with asymptotic conditions $(\vec{x}_{in}, \vec{y}_{in}, \vec{x}_{out}, \vec{y}_{out})$ has arbitrarily negative $E_{top}(u)$, which is impossible by (B.25). □

Given a map

$$u : S \longrightarrow W \tag{B.29}$$

as above, there is a notion of *intermediate action* of u along a loop $S^1 \hookrightarrow S$.

DEFINITION B.3. *Define the* **intermediate action** *of an embedded oriented loop $\mathcal{L} : S^1 \to S$ to be:*

$$\mathcal{A}(\mathcal{L}) := -\int_{S^1} \mathcal{L}^* u^* \theta + \int_{S^1} \mathcal{L}^*(u^*(H_S \cdot \gamma)) \tag{B.30}$$

A useful fact about intermediate action is:

LEMMA B.2. *There are constants c_1, c_2 depending on the input and output chords and orbits, such that*

$$\mathcal{A}(\mathcal{L}) \in [c_1, c_2] \tag{B.31}$$

for any embedded oriented loop $\mathcal{L}$.

PROOF. As S is genus 0, any embedded loop $\mathcal{L}$ is separates S into two regions, one S_{in} that has outgoing boundary on $\mathcal{L}$ and one S_{out} that has incoming boundary on $\mathcal{L}$. We note that topological energy $E_{top}(u)$ is additive on any sub-region of S, i.e.

$$E_{top}(u) = E_{top}(u|_{S_{in}}) + E_{top}(u|_{S_{out}}), \tag{B.32}$$

and by (B.25), topological energy for any subregion is bounded above by some negative constant

$$E_{top}(u|_{S_{in}}), E_{top}(u|_{S_{out}}) \geq -C. \tag{B.33}$$

Hence

$$\begin{aligned} E_{top}(u|_{S_{out}}) &\leq E_{top}(u) + C \\ E_{top}(u|_{S_{in}}) &\leq E_{top}(u) + C. \end{aligned} \tag{B.34}$$

Now each of $E_{top}(u|_{S_{out}})$, $E_{top}(u|_{S_{in}})$ can be expressed via Stokes theorem as the positive or negative action of $\mathcal{L}$ respectively plus/minus actions of inputs and outputs on each of S_{out}/S_{in}, so we obtain upper and lower bounds for $\mathcal{A}(\mathcal{L})$ in terms of actions of inputs and outputs, and the overall constant C. □

Now, given a fixed Floer datum on S, we view S as the union of two regions with different Hamiltonian behavior. On the first region, there is a non-zero time-dependent perturbation:

DEFINITION B.4. *Define the* **cylindrical perturbed regions** *S^c of S to be the union of the images of the*

- *cylindrical ends*
$$\delta^j_{\pm} : A_{\pm} \longrightarrow S,$$
and
- *finite cylinders*
$$\delta^r : [a_r, b_r] \times S^1 \longrightarrow S$$

in the strip and cylinder data chosen for S.

Recall that the Floer datum on S consists of δ-bounded cylinder data, and an S^1 perturbation adapted to (F_T, ϕ_ϵ), for some chosen $\delta \ll 1$ and $\epsilon \ll 1$, in the sense of Definitions 4.4 and 4.10. This implies that on the $\bar{\delta}$-collar (see Definition 4.5)

$$S^{\bar{\delta}} \tag{B.35}$$

of S, with

$$\bar{\delta} = \delta \cdot \epsilon, \tag{B.36}$$

the perturbation F_T is locally constant.

Definition B.5. *Define the* **unperturbed region** S^u *of* S *to be the union of the complement of the cylindrical perturbed region* $S \backslash S^c$ *with the* $\bar{\delta}$*-collar* $S^{\bar{\delta}}$*. This is the region where the perturbation term* F_T *is locally constant.*

The intersection of the two regions S^u and S^c is exactly the $\bar{\delta}$ collar $S^{\bar{\delta}}$.

We now examine the function

$$\rho = \pi_r \circ u \tag{B.37}$$

on each of the two regions of S, S^u and S^c. In the next two sections, we prove the following claims:

Proposition B.2. *A maximum principle holds for* ρ *on the unperturbed region* S^u*.*

Proposition B.3. *On the portion of the cylindrical perturbed region* S^c *outside the* $\bar{\delta}$*-collar*

$$S^c \backslash S^{\bar{\delta}} \tag{B.38}$$

there is an upper bound on ρ *in terms of the Floer data and asymptotic conditions.*

Remark B.2. *Note that we are only able to directly establish direct upper bounds for* ρ *outside of a collar of* S^c*, and are forced to rely on the maximum principle to deduce bounds for* ρ *on the collar. This has to do with the technique used to establish such bounds.*

These results, along with Proposition B.1 imply Theorem B.1.

B.1. The unperturbed region. In a portion of the unperturbed region S^u mapping to the conical end of W, the total Hamiltonian is given by H_S, a quadratic function, plus a locally constant function F_S. Thus, the Hamiltonian vector field X_S is equal to

$$X_S = 2r \cdot R, \tag{B.39}$$

where R is the Reeb flow on ∂W. In particular,

$$dr(X) = 0. \tag{B.40}$$

Recall that on the conical end, our (surface-dependent) almost complex structure J_S satisfies

$$dr \circ J = \frac{a_S}{r} \cdot \theta \tag{B.41}$$

for some positive rescaling function $a_S : S \to [1, \infty)$. Namely, setting

$$\mathcal{S} := \frac{1}{2} r^2 \tag{B.42}$$

we have that

$$d\mathcal{S} \circ J = r dr \circ J = a_S \cdot \theta. \tag{B.43}$$

Now, consider Floer's equation on the conical end

$$J \circ (du - X \otimes \gamma) = (du - X \otimes \gamma) \circ j \tag{B.44}$$

and apply $d\mathcal{S}$ to both sides, with $\xi = \mathcal{S} \circ u = \frac{1}{2}\rho^2$ to obtain

$$d^c \xi = -a_S \cdot (\theta \circ (du - X \otimes \gamma)) \tag{B.45}$$

Differentiating once more,

$$dd^c \xi = -a_S(u^* \omega - d(\theta(X) \cdot \gamma)) - da_S \wedge (\theta \circ (du - X \otimes \gamma)). \tag{B.46}$$

Substituting (B.45) into (B.46), we obtain a second-order differential equation for ξ:

$$dd^c\xi = -a_S(u^*\omega - d(\theta(X)\cdot\gamma)) + \frac{da_S}{a_S}\wedge d^c\xi \tag{B.47}$$

On the cylindrical end, we have that up to a locally constant function

$$\theta(X) = 2r^2 = 2H; \tag{B.48}$$

hence

$$\begin{aligned} u^*\omega - d(\theta(X)\cdot\gamma) &= u^*\omega - d(\theta(X))\wedge\gamma - \theta(X)d\gamma \\ &= u^*\omega - 2d(u^*H)\wedge\gamma - \theta(X)d\gamma \\ &= (u^*\omega - d(u^*H)\wedge\gamma) + \theta(X)d\gamma - d(u^*H)\wedge\gamma \\ &= (u^*\omega - d(u^*H)\wedge\gamma) - \theta(X)d\gamma - d(2\xi)\wedge\gamma. \end{aligned} \tag{B.49}$$

Thus, ξ satisfies

$$dd^c\xi + 2a_S d\xi\wedge\gamma - \frac{da_S}{a_S}\wedge d^c\xi = -a_S(u^*\omega - d(u^*H)\wedge\gamma) + a_S\theta(X)d\gamma. \tag{B.50}$$

Note that by the energy identity (B.21) and the fact that γ is sub-closed,

$$-a_S(u^*\omega - d(u^*H)\wedge\gamma) + a_S\theta(X)d\gamma \leq 0. \tag{B.51}$$

Thus, ξ satisfies an equation of the form

$$dd^c\xi + 2a_S d\xi\wedge\gamma - \frac{da_S}{a_S}\wedge d^c\xi \leq 0 \tag{B.52}$$

which in local coordinates $z = s + it$ looks like

$$\Delta\xi + v(s,t)\partial_s\xi + w(s,t)\partial_t\xi \geq 0. \tag{B.53}$$

for some functions v, w. Such equations are known to satisfy the maximum principle; see e.g. [**E**]. To finally establish Proposition B.2, we must show that maxima of ξ achieved along portions of the Lagrangian boundary $\partial S|_{S^u}$ mapping to the cylindrical end also satisfy

$$d\xi = 0 \tag{B.54}$$

hence are subject to the usual maximum principle. Pick local coordinates $z = s + it$ near a boundary point p with boundary locally modeled by $\{t = 0\}$. For a boundary maximum,

$$\partial_s\xi = 0. \tag{B.55}$$

Using (B.45) to calculate $\partial_t\xi$, we see that

$$\begin{aligned} \partial_t\xi &= d^c\xi(\partial_s) \\ &= -a_S\theta\circ(du - X\otimes\gamma)(\partial_s). \end{aligned} \tag{B.56}$$

Since γ was chosen to equal zero on the boundary of S, we have that $X\otimes\gamma(\partial_s) = 0$. Similarly, at our point p, $\partial_s u$ lies in the tangent space of an exact Lagrangian L with chosen primitive f_L vanishing on the cylindrical end. Thus $\theta(\partial_s u) = 0$. Putting these together,

$$\partial_t\xi = 0 \tag{B.57}$$

as desired.

B.2. A convexity argument for the unperturbed region. Below we present an alternate convexity argument for the unperturbed region. This section is not strictly necessary, but we have included it for its potential usefulness.

Let C be the overall bound for $\rho = \pi_r \circ u$ on the perturbed region and consider

$$\tilde{S} := \rho^{-1}([C, \infty)) \subset S^u \tag{B.58}$$

$\tilde{S}$ splits as a disjoint union of surfaces $\bar{S}$ on which the total Hamiltonian is equal to a quadratic Hamiltonian r^2 plus a constant term K (different surfaces have different constant term). On any such region $\bar{S}$, note that there is a refinement of the basic geometric/topological energy inequality (B.23) as follows:

$$\begin{aligned} E_{top}(u) &\geq E_{geo}(u) + \int_{\bar{S}} u^* H(-d\gamma) \\ &\geq E_{geo}(u) + (C^2 + K) \int_{\bar{S}} (-d\gamma) \\ &\geq (C^2 + K) \int_{\bar{S}} (-d\gamma), \end{aligned} \tag{B.59}$$

with equality if and only if $E_{geo}(u) = 0$ (we remark that one does not need to worry about (B.24) for $\tilde{S}$ as we are in a region where $d_S(H_S) = 0$). The boundary of $\bar{S}$ splits as $\partial^n \bar{S}$, the portion mapping via u to $\partial W \times \{C\}$, $\partial^l \bar{S}$, the portion with Lagrangian boundary, and $\partial^p \bar{S}$, the punctures. Suppose there are boundary punctures $\{x_i\}$ in $\partial^l \bar{S}$. We calculate,

$$E_{top}(u|_{\bar{S}}) = \int_{\partial^n \bar{S}} (u^*\theta - u^* H\gamma) + \int_{\partial^l \bar{S}} (u^*\theta - u^* H\gamma) + \sum_i \mathcal{A}(x_i). \tag{B.60}$$

Note first that θ restricted to the cylindrical end of any Lagrangian is zero, and similarly for γ; implying the second term above vanishes. Moreover,

$$\theta(X) = 2r^2 = 2(u^* H - K) \tag{B.61}$$

and

$$u^* H|_{\partial^n \bar{S}} = C^2 + K; \tag{B.62}$$

hence

$$\begin{aligned} E_{top}(u|_{\bar{S}}) &= \int_{\partial^n \bar{S}} \theta \circ (du - X \otimes \gamma) + \int_{\partial^n \bar{S}} (u^* H - 2K)\gamma + \sum_i \mathcal{A}(x_i) \\ &= \int_{\partial^n \bar{S}} \theta \circ (du - X \otimes \gamma) + \int_{\partial^n \bar{S}} (C^2 - K)\gamma + \sum_i \mathcal{A}(x_i). \end{aligned} \tag{B.63}$$

Following action arguments in [**A1**, Appendix B] and [**AS**, Lemma 7.2], we rewrite

$$\begin{aligned} \int_{\partial^n \bar{S}} \theta \circ (du - X \otimes \gamma) &= \int_{\partial_n \bar{S}} \theta \circ J(du - X \otimes \gamma) \circ (-j) \\ &= \int_{\partial^n \bar{S}} \frac{r}{a_S} dr \circ (du - X \otimes \gamma) \circ (-j) \\ &= \int_{\partial^n \bar{S}} -\frac{r}{a_S} dr \circ (du) \circ j. \end{aligned} \tag{B.64}$$

as $dr \circ X = dr \circ (2r \cdot R) = 0$ on $\bar{S}$. As in [**AS**, Lemma 7.2], note now that for a vector ξ tangent to $\partial^n \bar{S}$ with the positive boundary orientation, $j\xi$ points inward. Apply du and note that in order for $j\xi$ to point inward, $du \circ (j\xi)$ must not decrease the r-coordinate. Namely $dr \circ du \circ j(\xi) \geq 0$, and

$$\int_{\partial^n \bar{S}} \theta \circ (du - X \otimes \gamma) \leq 0. \tag{B.65}$$

Thus,

$$E_{top}(u|_{\bar{S}}) \leq \int_{\partial^n \bar{S}} (C^2 - K)\gamma + \sum_i \mathcal{A}(x_i). \tag{B.66}$$

Outside a sufficiently large compact set, actions are negative, so (increasing C if necessary)

$$
\begin{aligned}
E_{top}(u|_{\bar{S}}) &\leq \int_{\partial^n \bar{S}} (C^2 - K)\gamma \\
&\leq (C^2 - K)\Big(\int_{\bar{S}} d\gamma - \int_{\partial \bar{S}^p} \gamma\Big) \\
&\leq (C^2 - K) \int_{\bar{S}} \Big(d\gamma - \sum_i w_i\Big) \\
&\leq (C^2 + K) \int_{\bar{S}} (-d\gamma).
\end{aligned} \tag{B.67}
$$

Along with the opposite inequality (B.59), this implies that $E_{geo}(u|_{\bar{S}}) = 0$. So du must be a constant multiple of the Reeb flow, which is possible only if the image of $u|_{\bar{S}}$ is contained in a single level $\partial W \times \{C\}$.

B.3. The perturbed cylindrical regions. The starting point for this case is the following classical refinement of the maximum principle for uniformly elliptic second order linear differential operators associated with Dirichlet problems in bounded domains:

Proposition B.4 (Compare [**C**, Prop. 5.1]). *Let L be a strongly positive second-order elliptic differential operator on a domain Ω (such as $-\Delta$ on a finite cylinder $[a, b] \times S^1$). Let $\overline{\lambda}$ be the smallest eigenvalue of L. Then $\overline{\lambda}$ is positive. Moreover, for any positive $\lambda < \overline{\lambda}$, if $f : \Omega \to \mathbb{R}$ is smooth and satisfies the following properties:*

$$
\begin{cases} Lf \geq \lambda f & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega \end{cases} \tag{B.68}
$$

then $f \geq 0$ on Ω.

Proof. The proof combines a theorem of Krein-Rutman with the maximum principle, and can be found in Thorems 4.3 and 4.4 of [**Am**]. ☐

Using this Proposition we prove a variant of a result of Floer-Hofer [**FH2**, Prop. 8]:

Proposition B.5. *Let Z be a cylinder of the form $[c, d]_s \times S^1_t$, with $c \in [-\infty, \infty)$, $d \in (\infty, \infty]$. Let*

$$
g : Z \longrightarrow \mathbb{R} \tag{B.69}
$$

be a function satisfying the following two properties:

(1) *For any $\eta \ll 1$, there exists a c_η with the following property: On any η-width sub-cylinder*

$$
(s, s + \eta) \times S^1 \subseteq Z \tag{B.70}
$$

there is a loop $\{s'\} \times S^1$ satisfying

$$
\sup_t [g(s', t)] < c_\eta. \tag{B.71}
$$

(2) *For some $\lambda > 0$ and (possibly negative) constant A, g satisfies the following equation*

$$
\Delta g + \lambda g \geq A. \tag{B.72}
$$

Then for sufficiently small η, there is a constant $C(\lambda, A, \eta)$ such that

$$
g(s, t) < C \tag{B.73}
$$

everywhere except possibly outside a 2η-collar of Z.

Proof. Starting from one end, partition Z into a maximal collection of adjacent η-sized cylinders $[s, s + \eta] \times S^1$—this covers all of Z except a at most a portion of Z's collar of width at most η (when Z is finite). To each such sub-cylinder $Z_k = [s_k, s_k + \eta] \times S^1$ of Z associate a number

$$
s'_k \in (s_k, s_k + \eta) \tag{B.74}
$$

satisfying (B.71). An adjacent pair of such s'_k, s'_{k+1} satisfies

$$
s'_{k+1} - s'_k < 2\eta \tag{B.75}
$$

and

$$g(s'_k, t) < c_\eta \\ g(s'_{k+1}, t) < c_\eta. \tag{B.76}$$

We will now examine the new regions

$$Z'_k = [s'_k, s'_{k+1}] \times S^1, \tag{B.77}$$

which cover all of Z except a portion of Z's collar of width at most 2η. Let

$$\epsilon_k = \frac{1}{2}(s'_{k+1} - s'_k) \tag{B.78}$$

and consider the function

$$h(s,t) := (\lambda c_\eta + |A|)(\epsilon_k^2 - (s - s'_k - \epsilon_k)^2) + c_\eta. \tag{B.79}$$

$h(s,t)$ satisfies the following properties on $[s'_k, s'_{k+1}]$:

$$h(s,t) \geq c_\eta. \tag{B.80}$$

$$h(s,t) \leq (\lambda c_\eta + |A|)\epsilon_k^2 + c_\eta \leq (\lambda c_\eta + |A|)\eta^2 + c_\eta. \tag{B.81}$$

Moreover, for η chosen sufficiently small ($< \sqrt{\frac{1}{\lambda}}$),

$$\begin{aligned}(-\Delta - \lambda)h(s,t) &= 2(\lambda c_\eta + |A|) - \lambda h(s,t) \\ &\geq (2(\lambda c_\eta + |A|)) - \lambda[(\lambda c_\eta + |A|)\eta^2 + c_\eta] \\ &\geq |A|(2 - \lambda\eta^2) + \lambda c_\eta(2 - \lambda\eta^2 - 1) \\ &\geq |A|.\end{aligned} \tag{B.82}$$

Therefore, the function

$$-g(s,t) + h(s,t) \tag{B.83}$$

satisfies the following two properties:

$$-g(s,t) + h(s,t) \geq 0 \text{ on } \partial C'_k, \tag{B.84}$$

$$(-\Delta - \lambda)(-g(s,t) + h(s,t)) \geq 0 \text{ on } C'_k. \tag{B.85}$$

Now, the smallest eigenvalue of $-\Delta$ on Z'_k subject to the boundary condition of 0 on $\partial Z'_k$ can be explicitly calculated via Fourier series to be

$$\bar{\lambda} = \frac{\pi^2}{4\epsilon_k^2} \geq \frac{\pi^2}{4\eta^2}. \tag{B.86}$$

so for η sufficiently small, λ is smaller than $\bar{\lambda}$. Thus Proposition B.4 applies, and we conclude that on all of Z'_k:

$$-g(s,t) + h(s,t) \geq 0; \tag{B.87}$$

namely on Z'_k

$$g(s,t) \leq h(s,t) \leq (\lambda c_\eta + |A|)\eta^2 + c_\eta. \tag{B.88}$$

This final bound holds on every new cylinder Z'_k and is independent of k. Since the cylinders Z'_k cover all but a 2η collar of Z, we conclude the result. □

Returning to our main argument, let us recall that pulled back to a particular cylinder $[c,d] \times S^1$ in the cylindrical region S^c with associated **weight** v, our chosen Floer datum has the following form:

- The **sub-closed one-form** γ is actually closed and equal to vdt.
- The **main Hamiltonian** H_S is equal to $\frac{H \circ \psi^v}{v^2}$
- The **rescaling function** a_S is equal to the constant v.

- The **almost-complex structure** $J_S = (\psi^v)^* J_t$ is v**-rescaled contact type**, e.g. on the conical end,

$$dr \circ J = -\frac{v}{r} \cdot \theta \tag{B.89}$$

- The **perturbation term** F_T is **monotonic** in s, i.e.

$$\partial_s F_T \leq 0, \tag{B.90}$$

  and has norm and all derivatives bounded by constants independent of the particular cylinder.

In particular, we note that on any such region, the **total Hamiltonian**

$$H_S^{tot} := H_S + F_S \tag{B.91}$$

is monotonic in s. Also, u pulled back to any such region satisfies the usual form of Floer's equation:

$$\begin{aligned} (du - X \otimes (v \cdot dt))^{0,1} = 0, \text{ i.e.} \\ \partial_s u + J_t(\partial_t u - X_{v \cdot (H_S^{tot})}) = 0. \end{aligned} \tag{B.92}$$

Below, we will frequently suppress the weight v, building it into the total Hamiltonian $\tilde{H}_S^{tot} = v \cdot H_S^{tot}$. On a cylindrical region of S, we examine the function

$$\xi := \frac{1}{2}\rho^2 = \frac{1}{2}(\pi_r \circ u)^2 : [c, d] \times S^1 \longrightarrow \mathbb{R}. \tag{B.93}$$

Proposition B.3 can now be refined as follows:

Proposition B.6. *ξ (almost) satisfies Conditions 1 and 2 from Proposition B.5. Namely, there is a replacement $\tilde{\xi}$ satisfying Conditions 1 and 2, with*

$$\begin{aligned} \xi < \tilde{\xi} + C \\ \tilde{\xi} < \xi + C'. \end{aligned} \tag{B.94}$$

Choosing η smaller than $\frac{\bar{\delta}}{2}$, we see that on all but a $\bar{\delta}$ collar of S, $\tilde{\xi}$ and hence ξ and ρ are absolutely bounded in terms of the Floer data and asymptotic conditions.

In order to better examine intermediate actions along loops $\mathcal{L} : S^1 \hookrightarrow S^c$ in the cylindrical region, we will define define relevant function spaces for maps $x : S^1 \to W$, following [**O2**, §4]. First, define the following continuous projection to the compact region:

$$\pi_{in} : W \longrightarrow \bar{W} \tag{B.95}$$

$$p \longmapsto \begin{cases} p & p \in \bar{W} \\ \bar{p} & p = (\bar{p}, r) \in \partial\bar{W} \times [1, \infty). \end{cases} \tag{B.96}$$

Given $x : S^1 \to W$, denote by $\bar{x}$ the composition

$$\bar{x} := \pi_{in} \circ x. \tag{B.97}$$

Definition B.6. *Define the following function spaces:*

$$L^2(S^1, W) := \{x : S^1 \to W \text{ measurable} : \pi_r \circ x \in L^2(S^1, \mathbb{R})\} \tag{B.98}$$

$$H^1(S^1, W) := \{x \in L^2(S^1, W) : \dot{\bar{x}} \in L^2(x^* T\bar{W}), (\pi_r \circ x)' \in L^2(S^1, \mathbb{R})\}. \tag{B.99}$$

Here, measurability is with regards to some metric $g = \omega(\cdot, J\cdot)$, and independent of choices. Also, for a smooth map x, $\dot{\bar{x}}$ is well-defined as a distribution given an embedding of $\bar{W}$ into Euclidean space. The requirement that it be $L^2(x^*T\bar{W})$ is independent of embedding, though we declare that for some fixed embedding that $||\dot{\bar{x}}||^2_{L^2}$ be the restriction of the usual Euclidean L^2 norm. We define the norms associated to these spaces as follows:

$$||x||^2_{L^2} := ||\pi_r \circ x||^2_{L^2} \tag{B.100}$$

$$||x||^2_{H^1} := ||\pi_r \circ x||^2_{H^1} + ||\dot{\bar{x}}||^2_{L^2}. \tag{B.101}$$

Definition B.7. *Define the normed space*

$$C^0(S^1, W) \tag{B.102}$$

of continuous functions from S^1 to W as follows: The norm of a continuous map x is given by by choosing an embedding of $\bar{W}$ into Euclidean space and restricting the standard Euclidean sup norm on $\bar{x}$, plus the sup norm of $\pi_r \circ x$. This space is independent of embedding of $\bar{W}$.

Lemma B.3 (Sobolev embedding). *There is a compact embedding*

$$H^1(S^1, W) \subset C^0(S^1, W). \tag{B.103}$$

Proof. See [**O2**, Lemma 4.7]. The main point is to leverage the known compact embedding $H^1(S^1, \mathbb{R}) \subset C^0(S^1, \mathbb{R})$ to ensure that any sequence f_k bounded in $H^1(S^1, W)$ takes values in a compact set. Thus, one can apply Sobolev embedding for maps from S^1 to a compact target manifold. $\square$

By definition, we have that

$$\text{if } f \in C^0(S^1, W) \text{ then } \pi_r \circ f \text{ is bounded.} \tag{B.104}$$

Given an almost complex structure J, we recall the associated metric

$$\langle X, Y \rangle_J = \omega(X, JY). \tag{B.105}$$

For a smooth map $x : S^1 \longrightarrow W$ and a S^1 dependent complex structure J_t we use this metric to define the following L^2 norm

$$||\dot{x}||^2_{L^2} := \int_{S^1} \omega(\dot{x}(t), J_t \dot{x}(t)) dt. \tag{B.106}$$

The relation to the function spaces defined earlier is as follows:

Claim B.1. *For J_t of rescaled contact type, the L^2 norm $||\dot{x}||^2_{L^2}$ bounds $||\dot{\bar{x}}||^2_{L^2}$ and $||(\pi_r \circ x)'||^2_{L^2}$.*

Proof. Suppose first that x maps entirely to the cylindrical end of W. Then $\bar{x}$ is smooth and the norm of $\dot{\bar{x}}$ given by a choice of embedding into Euclidean space is equivalent to the one coming from $\omega(\cdot, J\cdot)$ on $\bar{x}$, for any J. Now, for any J of rescaled contact type, ∂_r and $T\partial W$ are $\langle \cdot, \cdot \rangle_J$ orthogonal, implying that

$$\begin{aligned} |\dot{x}|^2 &= |\, (\dot{\bar{x}}, (\pi_r \circ x)') \,|^2 \\ &= |\dot{\bar{x}}|^2_{x(t)} + |(\pi_r \circ x)'|^2. \end{aligned} \tag{B.107}$$

Above, the notation $|\dot{\bar{x}}|^2_{x(t)}$ refers to the fact that we are taking the norm of $\dot{\bar{x}}$ with respect to the metric at level $\pi_r \circ x$. The norm $\langle \cdot, \cdot \rangle_J$ behaves in the following manner with regards to level on the cylindrical portion $\partial W \times [1, \infty)$: For R and ∂_r,

$$\begin{aligned} \langle R, R \rangle_J &= v \\ \langle \partial_r, \partial_r \rangle_J &= \frac{1}{v}, \end{aligned} \tag{B.108}$$

independent of level r (here v is the rescaling constant). For vectors in the orthogonal complement of R, ∂_r, the norm grows linearly in r. In particular,

$$|\dot{\bar{x}}|^2_{\bar{x}(t)} \leq |\dot{\bar{x}}|^2_{x(t)}; \tag{B.109}$$

thus both $|\dot{\bar{x}}|$ and $|(\pi_r \circ x)'|$ are bounded by a constant multiple of $|\dot{x}|$. Now extend this bound to arbitrary x as follows: suppose that $\pi_r \circ x \leq 1$, e.g. $x \subset \bar{W}$. Then, since $d\pi_r$ is an operator with bounded norm on $\bar{W}$,

$$(\pi_r \circ x)^2 = (d\pi_r \circ \dot{x})^2 \leq (\text{const}) \cdot |\dot{x}|^2. \tag{B.110}$$

Similarly, when $x \subset \bar{W}$, $\dot{\bar{x}} = \dot{x}$, so $|\dot{\bar{x}}|$ is trivially bounded by $|\dot{x}|$. $\square$

Lemma B.4 (Condition 1). *For any $\eta > 0$ there is a c_η such that on any sub-cylinder $[s, s+\eta] \times S^1$ of the cylindrical region S^c,*

$$\sup_t [\pi_r \circ u(s', t)] \leq c_\eta \tag{B.111}$$

for some $s' \in [s, s+\eta]$.

Proof. Let

$$\bar{Z} = (s_0, s_0 + \eta) \times S^1 \tag{B.112}$$

be a given sub-cylinder of the cylindrical region S^c. Let

$$\mathcal{A}_{\bar{Z}}(s) \tag{B.113}$$

denote the intermediate action of the loop $\{s\} \times S^1 \subset \bar{Z}$, and let $u(s,t)$ denote the restriction of u to $\bar{Z}$. By the positivity of topological energy,

$$\mathcal{A}_{\bar{Z}}(s_0) - \mathcal{A}_{\bar{Z}}(s_0 + \eta) = E_{top}(u|_{\bar{Z}}) \leq K, \tag{B.114}$$

where K is the topological energy of u. The mean-value theorem therefore implies the existence of

$$s' \in (s_0, s_0 + \eta) \tag{B.115}$$

satisfying

$$|\partial_s(\mathcal{A}_{\bar{Z}}(s))||_{s=s'} \leq K/\eta. \tag{B.116}$$

Moreover, we know from Lemma B.2 that

$$(-\mathcal{A}_{\bar{Z}}(s')) \leq M \tag{B.117}$$

for some constant M depending only on the asymptotics of S. We claim that the equations (B.116) and (B.117) give a bound for the loop $u(s', \cdot)$ in the H^1 norm (B.99), establishing the Lemma for s' (using the Sobolev embedding (B.103)).

Recalling the special form of our Floer data on $\bar{Z}$, and abbreviating $H := H_S^{tot}$, $J := J_S$, $X = X_{H_S^{tot}}$ the derivative $\partial_s \mathcal{A}_{\bar{Z}}(s)$ can be expressed as:

$$\begin{aligned}
\partial_s \mathcal{A}_{\bar{Z}}(s) &= -\int_{\{s\}\times S^1} \omega(\partial_s u, \partial_t u) dt + \int_{\{s\}\times S^1} \partial_s(u^*(H)) dt \\
&= -\int_{s\times S^1} \omega(\partial_t u, J(\partial_t u - X)) dt + \int_{s\times S^1} (dH \circ \partial_s u + \partial_s H) dt \\
&= -\int_{s\times S^1} \omega(\partial_t u, J(\partial_t u - X)) dt + \int_{s\times S^1} \omega(X, \partial_s u) dt + \int_{s\times S^1} \partial_s H dt \\
&= -\int_{s\times S^1} \omega(\partial_t u, J(\partial_t u - X)) dt - \int_{s\times S^1} \omega(X, J_S(\partial_t u - X)) dt + \int_{s\times S^1} \partial_s H dt \\
&= -\int_{s\times S^1} \omega(\partial_t u - X, J(\partial_t u - X)) dt + \int_{s\times S^1} \partial_s H dt \\
&= -||\partial_t u(s,\cdot) - X||^2 + \int_{S^1} \partial_s H(s,\cdot) dt,
\end{aligned} \tag{B.118}$$

where above we have twiced used that u satisfies Floer's equation. Abbreviate

$$x_s(t) := u(s,t), \tag{B.119}$$

and note that since H_S^{tot} is monotonically decreasing, (B.118) and (B.116) imply that

$$||\partial_t x_{s'} - X||^2 \leq \frac{K}{\eta}. \tag{B.120}$$

For a rescaled-standard complex structure J, and a Hamiltonian vector field X_S equal to $2r \cdot R$ plus a bounded term,

$$|X_S|^2 \leq C(\omega(2r \cdot R, J(2r \cdot R)) + 1) = C(r^2 + 1), \tag{B.121}$$

i.e.

$$|X_S| \leq \tilde{C}(|r| + 1). \tag{B.122}$$

for some constants $C, \tilde{C}$. Thus by (B.120),

$$||\partial_t x_{s'}|| \leq ||\partial_t x_s - X|| + ||X|| \leq \tilde{C}(1 + ||\pi_r \circ x_{s'}||_{L^2}). \tag{B.123}$$

By Claim B.1, a bound on $||\partial_t x_{s'}||$ is as good as a bound on $||(\pi_r \circ x_{s'})'||$ and $||\dot{\bar{x}}_{s'}||$. The equation (B.123) implies that a bound for $||\pi_r \circ x_{s'}||_{L^2}$ suffices to establish the desired H^1 bound on $x_{s'}$.

We know by hypothesis that for any s the action of the loop x_s is bounded below:

$$-\mathcal{A}(x_s) = \int_{S^1} x_s^*\theta - \int x_s^* H \leq M \tag{B.124}$$

for some M. We rewrite the first term of (B.124) as

$$\begin{aligned}\int_{S^1} x_{s'}^*\theta &= \int \omega(Z, \dot{x}_{s'})dt \\ &= \int \omega(Z, (\partial_t x_{s'} - X(x_{s'})) + X(x_{s'})) \\ &= \int \langle JZ, \partial_t x_{s'} - X\rangle_J + \int_{x_{s'}(S^1)} dH(Z)dt.\end{aligned} \tag{B.125}$$

Thus by Cauchy-Schwarz

$$\int_{S^1} x_{s'}^*\theta \geq -||Z|| \cdot ||\partial_t x_{s'} - X|| + \int_{x_{s'}(S^1)} dH(Z) \tag{B.126}$$

Substituting into (B.124) we see that

$$M \geq \int_{x_{s'}(S^1)} (dH(Z) - H) - ||Z|| \cdot ||\partial_t x_{s'} - X||, \tag{B.127}$$

e.g.

$$\int_{x_{s'}(S^1)} (dH(Z) - H) \leq M + ||Z|| \cdot ||\partial_t x_{s'} - X||. \tag{B.128}$$

By (B.120), $||\partial_t x_{s'} - X||$ is bounded. Moreover, $Z = r \cdot \partial_r$ has norm equal to r times a constant. Thus,

$$\int_{x_{s'}(S^1)} (dH(Z) - H) \leq \tilde{M} + C_0||\pi_r \circ x_{s'}|| \tag{B.129}$$

for some new constant $\tilde{M}$. For $x_{s'}$ mapping entirely to the cylindrical end, we see that

$$\begin{aligned} dH(Z) - H &= r \cdot dH(\partial_r) - H \\ &= r^2 + dF_S(Z) - F_S. \end{aligned} \tag{B.130}$$

The last two terms are totally bounded by assumption, so

$$\int_{x_{s'}(S^1)} r^2 = ||\pi_r \circ x_{s'}||^2 \leq N + C_0||\pi_r \circ x_{s'}||, \tag{B.131}$$

for constants N, C_0, implying a bound for $||\pi_r \circ x_{s'}||$. We extend to the general case by noting that whenever $x_{s'}$ maps to $\bar{W}$, $|\pi_r \circ x_{s'}|$ is bounded by 1. $\square$

We have just proven that Condition 1 holds for

$$\rho = \pi_r \circ u \tag{B.132}$$

This implies that it holds for

$$\xi := \frac{1}{2}\rho^2 \tag{B.133}$$

as well as any $\tilde{\xi}$ satisfying (B.94).

Lemma B.5 (Condition 2). *On any cylindrical part of S there exists a $\tilde{\xi}$, satisfying*

$$\begin{aligned} \xi &< \tilde{\xi} + C \\ \tilde{\xi} &< \xi + C' \end{aligned} \tag{B.134}$$

such that

$$\Delta\tilde{\xi} + \lambda\tilde{\xi} \geq -A \tag{B.135}$$

for constants C, C' depending only on the asymptotic conditions of S and the Floer data.

Proof. Actually, we will prove that ξ itself satisfies (B.135) on the cylindrical end of W; we will then perform the replacement $\tilde{\xi}$ to extend the validity of (B.135) to the compact region $\bar{W}$.

Letting

$$\mathcal{S} = \frac{1}{2}r^2, \tag{B.136}$$

and $\xi = \mathcal{S} \circ u$ we calculate the Laplacian $\Delta\xi$ on the cylindrical end of W. Begin with Floer's equation

$$J \circ (du - X \otimes dt) = (du - X \otimes dt) \circ j, \tag{B.137}$$

apply $d\mathcal{S}$ to both sides. Since J is v-rescaled-contact type

$$d\mathcal{S} \circ J = rdr \circ J = -vr\bar{\theta} = -v\theta, \tag{B.138}$$

we have that

$$d^c\xi = d\xi \circ j = v(-u^*\theta + \theta(X)dt) + d\mathcal{S}(X)(dt \circ j) \tag{B.139}$$

Differentiating once more, we see that

$$dd^c\xi = v(-u^*\omega + (\partial_s\theta(X))\ ds \wedge dt) - \partial_t(d\mathcal{S}(X))\ ds \wedge dt. \tag{B.140}$$

Since $dd^c\xi = -\Delta\xi\ ds \wedge dt$, we see that

$$\Delta\xi = v(\omega(\partial_s u, \partial_t u) - \partial_s(\theta(X))) + \partial_t(d\mathcal{S}(X)). \tag{B.141}$$

Now, as in [**CFH**] and [**O2**], rewrite $\omega(\partial_s u, \partial_t u)$ as:

$$\begin{aligned} \omega(\partial_s u, \partial_t u) &= \frac{1}{2}\omega(\partial_s u, \partial_t u) + \frac{1}{2}\omega(\partial_s u, \partial_t u) \\ &= \frac{1}{2}\left(\omega(\partial_s u, J\partial_s u + X) + \omega(-J\partial_t u + JX, \partial_t u)\right) \\ &= \frac{1}{2}\left(|\partial_s u|^2 + |\partial_t u|^2 + \omega(\partial_s u, X) - \omega(\partial_t u, JX)\right) \end{aligned} \tag{B.142}$$

The other terms in (B.141) can be expanded as follows, where $\bar{\theta}$ is the contact form on ∂W (as in (3.5)):

$$\partial_t(d\mathcal{S}(X)) = d\mathcal{S} \circ \partial_t X + d\mathcal{S} \circ \nabla_{u_t} X, \tag{B.143}$$

and

$$\begin{aligned} \partial_s(\theta(X)) &= \partial_s(r\bar{\theta}(X)) \\ &= (dr \circ u_s)\bar{\theta}(X) + r\bar{\theta}(\partial_s X) + r\bar{\theta}(\nabla_{u_s} X) \end{aligned} \tag{B.144}$$

Putting these together, we have that

$$\begin{aligned} \Delta\xi = &\frac{v}{2}|\partial_s u|^2 + \frac{v}{2}|\partial_t u|^2 + \frac{v}{2}\omega(\partial_s u, X) + \frac{v}{2}\omega(\partial_t u, JX) + d\mathcal{S} \circ \partial_t X \\ &+ d\mathcal{S} \circ \nabla_{u_t} X + v(dr \circ u_s)\bar{\theta}(X) + vr\bar{\theta}(\partial_s X) + vr\bar{\theta}(\nabla_{u_s} X). \end{aligned} \tag{B.145}$$

When J is of v-rescaled contact type, we recall that as linear operators

$$\bar{\theta} \text{ has constant norm; and} \tag{B.146}$$

$$dr = d(\sqrt{2\mathcal{S}}) = \frac{d\mathcal{S}}{\sqrt{2\mathcal{S}}} \text{ has constant norm, hence} \tag{B.147}$$

$$d\mathcal{S} \text{ has norm } O(\sqrt{\mathcal{S}}). \tag{B.148}$$

Moreover, we have the following inequalities:

$$\begin{aligned} |X| &\le C(1 + \sqrt{\mathcal{S}}), \\ |\nabla_Y X| &\le C|Y|, \text{ for any vector field } Y, \\ |\partial_s X| &\le C(1 + \sqrt{\mathcal{S}}) \end{aligned} \tag{B.149}$$

for some (possibly different) constants C depending on the rescaling constant v and the time-dependent perturbation term of our total Hamiltonian. We use these inequalities to estimate the terms in (B.145):

$$|\omega(\partial_s u, X)| \leq |\partial_s u||X| \leq C(1+\sqrt{\xi})|\partial_s u| \tag{B.150}$$

$$|\omega(\partial_t u, JX)| \leq |\partial_t u||X| \leq C(1+\sqrt{\xi})|\partial_t u| \tag{B.151}$$

$$|d\mathcal{S} \circ \partial_t X| \leq C(1+\sqrt{\xi})^2 \tag{B.152}$$

$$|d\mathcal{S} \circ \nabla_{u_t} X| \leq C(1+\sqrt{\xi})|\partial_t u| \tag{B.153}$$

$$|(dr \circ u_s)\bar{\theta}(X)| \leq C(1+\sqrt{\xi})|\partial_s u| \tag{B.154}$$

$$|r\bar{\theta}(\partial_s X)| \leq |r||\partial_s X| \leq C(1+\sqrt{\xi})^2 \tag{B.155}$$

$$|r\bar{\theta}\nabla_{u_s} X| \leq |r||\partial_s u| \leq C(1+\sqrt{\xi})|\partial_s u| \tag{B.156}$$

again for potentially different constants C depending on those in (B.149). Putting these together, there exists constants c_1, c_2 and c_3 such that ξ satisfies an equation of the following form:

$$\begin{aligned}\Delta\xi \geq& \frac{1}{2}|\partial_s u|^2 + \frac{1}{2}|\partial_t u|^2 - c_1(1+\sqrt{\xi})|\partial_s u| \\ &- c_2(1+\sqrt{\xi})|\partial_t u| - c_3(1+\sqrt{\xi})^2.\end{aligned} \tag{B.157}$$

which implies that

$$\Delta\xi + \lambda\xi \geq -A \tag{B.158}$$

for obvious constants λ, A depending on c_1, c_2, c_3. This holds on the cylindrical end $r \geq 1$, where the estimates above apply.

We extend as follows, directly following an argument in [**O2**, Thm. 4.6]. Let

$$\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}+ \tag{B.159}$$

be a smooth function such that $\varphi(r) = 0$ for $r \leq 1$, $\varphi'(r) = 1$ for $r \geq 2$, and $\varphi''(r) > 0$ for $1 \leq r \leq 2$. Clearly

$$\begin{aligned} r &< \varphi(r) + C \\ \varphi(r) &< r + C' \end{aligned} \tag{B.160}$$

Thus, the modification

$$\tilde{\xi} := \varphi \circ \xi \tag{B.161}$$

satisfies (B.94) as required. Moreover, note that

$$\begin{aligned}\Delta\tilde{\xi} &= \partial_s(\varphi'(\xi(s,t)) \cdot \partial_s\xi) + \partial_t(\varphi'(\xi(s,t)) \cdot \partial_t\xi) \\ &= \varphi''(\xi(s,t))(|\partial_s\xi|^2 + |\partial_t\xi|^2) + \varphi'(\xi(s,t)) \cdot (\Delta\xi) \\ &\geq \varphi'(\xi(s,t))(-A - \lambda\xi) \geq (-A - \lambda\xi) \geq (-A - \lambda\tilde{\xi})\end{aligned} \tag{B.162}$$

as desired. □

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