Math 257B week 2 Day 2

Left to do: recast Lag'n Floer homology in terms of Floer's equation, stable maps remark

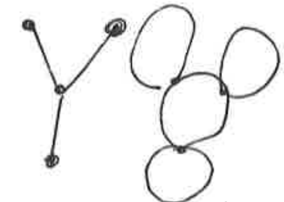
- continuation maps
- ~~88~~ Floer = Morse in exact case..

In general, Oh spectral sequence.

Signs: maybe not this time. Or maybe just a brief mention.

~~A more flexible variant on Floer hom~~

Rmk (last time): A J-hol. map $u: \Sigma_\infty \longrightarrow X$ ('nodal', w/ marked points, bordered) ~~is~~ called a prestable map, stable if its automorphism group is finite.



(Ex: an automorphism of the tree underlying the nodal configuration + automorphisms of surfaces fixing our nodes)

$\Rightarrow$ each component of Σ_∞ is either

- non-constant (Ex: non-const is stable, b/c the $\mathbb{R}$ action on strip gives different u's; not an automorphism)
- constant, but then underlying domain is stable: (w/ fixed marked pts & nodal pts.)

(If u is constant, stability of $u \iff$ stability of Σ_∞).

Continuation maps

Helpful to have a more flexible way of constructing Floer homology groups, via Floer trajectories instead of J-curves

Define, for $H: [0,1]\times X \longrightarrow \mathbb{R}$, J, a Floer chain complex

$$CF^*(L_0; L_1; \underset{\uparrow \text{ new}}{H}, J) \quad \text{whenever} \quad \phi^1_H(L_0) \pitchfork L_1.$$

generators: $\Lambda\langle$ time 1 orbits of X_H from $L_0 \to L_1\rangle$

Ex: $\mathcal{X}_{L_0,L_1} := \{\gamma: [0,1] \to X,\ \gamma(i) \in L_i\ \ i\in\{0,1\}$
$\qquad \dot\gamma(t) = X_{H_t}\}$

$\Lambda\langle \mathcal{X}_{L_0,L_1}\rangle$.

Differential: Now, for ~~[illegible]~~ $x^+, x^- \in \mathcal{X}_{L_0,L_1}$, $\beta \in \pi_2$.

$$\underset{\underset{\pi_2(-)}{\cap}}{\mathcal{M}(x^+, x^-, \beta)} = \left\{\begin{array}{l} u: \mathbb{R}\times[0,1] \longrightarrow X \quad \lim_{s\to\pm\infty} u(s,t) = x^\pm. \\ u(\mathbb{R}\times\{i\}) \subseteq L_i. \\ \partial_s u + J(\partial_t u - X_H) = 0 \iff (du - X\otimes dt)^{(0,1)} = 0 \quad \text{new equation} \\ E(u) = \int_\Sigma |\partial_s u - X_H|^2 < \infty = \int_\Sigma (u^*\omega - dH\wedge dt) \quad \text{(new energy identity)} \end{array}\right\},$$

where, for a map $u: \Sigma \to X$, $\alpha \in \Omega^1(\Sigma, u^*TX)$ & $\boxed{\alpha^{0,1} = \frac{1}{2}(\alpha + J\alpha j)}$

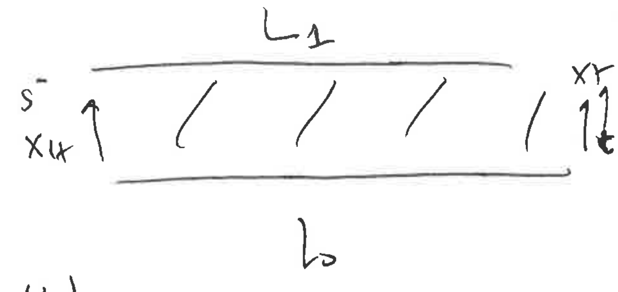
These are manifolds for generic J (same reason as before); look at $\mathcal{M}(x^+, x^-, \beta, H, J)/\mathbb{R}$ whenever $\mathrm{ind}(\beta) \geq 1$, & Gromov compactify:

Define: $$d(x^+) = \sum_{\substack{x^-,\ \beta \in \pi_2(x^+, x^-) \\ \mathrm{ind}(\beta)}} T^{E(\beta)} \#\left(\mathcal{M}(x^+, x^-, \beta, H, J)/\mathbb{R}\right) \cdot x^-.$$

Usual argument establishes $d^2 = 0$.

<u>Note:</u> Given such a solution u: gauge transform:

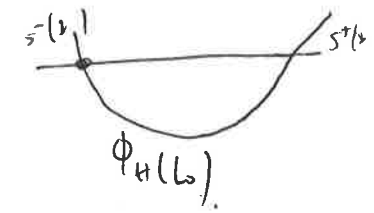
define $$\tilde{u}(s,t) = \phi_H^{(1-t)} u(s,t).$$



is a solution to

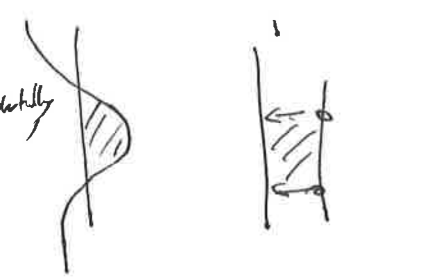
$\bar{\partial}_{\tilde{J}} u = 0$ with boundary on $\phi_H(L_0)$ and L_1

$\tilde{J} = (\phi_H^{1-t})_* J$. asymptotic to $s^+(1)$ and $s^-(1)$.



So have a canonical equiv:
$$CF^*(L_0, L_1; H, J) \cong CF^*(\phi_H(L_0), L_1; (\phi_H^{1-t})_* J).$$

↑ time dependent, potentially but ok.

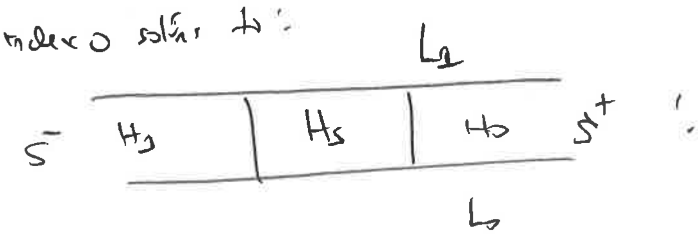


<u>continuation maps</u>:

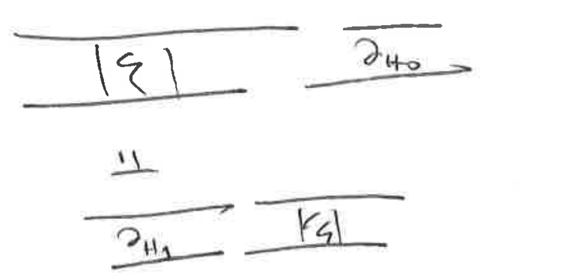
$$\psi: CF^*(L_0, L_1; H_0, J) \to CF^*(L_0, L_1; H_1, J)$$

(J_0 similar).

Count index 0 solutions to:



Chain map? Follows from Gromov compactness & gluing, applied to see that ∂(index 1 moduli spaces) is

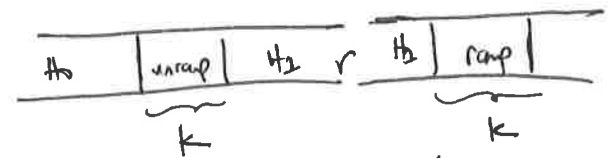


<u>Note</u>: such a u can have <u>negative energy</u>!

Namely $E(u) = \int u^*\omega - d(u^* H dt)$ ~~this~~ now equals $\int |\partial_t u - X_{H_{s,t}}|^2$

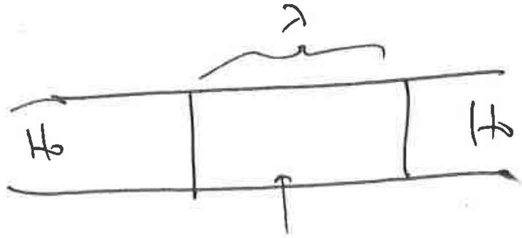
$- \int \partial_s H \, ds\, dt \ldots$

(H has an s derivative.) Not problematic (bounded); but pick up neg. powers of T (need $\mathbb{F}$).

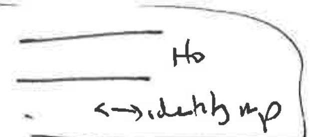
$\psi_{H_2\to H_0}\circ\psi_{H_0\to H_2}$ counts S^-  S^+

~~Consider the one~~ 'homotopy of homotopies argument':

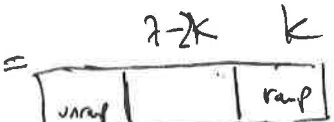
Let $\mathcal{M}_{\lambda,\{H_s\}}$ denote the space of pairs $\{\lambda, u$ a sol'n to



Gromov compactifying, the boundary of ind 1 has 4 types of :

- ~~strip~~ breaking (energy escape to $\pm\infty$)
- $\lambda \to 0$: counts index 0  identity map
- $\lambda \to \infty$ . . .

$\Rightarrow \psi_{H_2 H_0}\psi_{H_0 H_2} - id$.

$= \begin{cases} H_0 & \text{if } \lambda \leq 1 \\ = \end{cases}$ 

Define $HF^*(L,L) = HF^*(L,L;H,J)$ some H $= HF^*(\phi_H L, L)$.

Example: $M = T^*Q$ cotangent bundle, $\omega_{can} = dp\wedge dq = d\lambda_{can}$, $\lambda_{can} = p\,dq$.

$L_0 = Q$, zero section. Note that L is exact: $p\,dq|_L = 0$ (more generally, it would suffice if $p\,dq|_L = df$!)

in particular, by Stokes' thm, $\pi_2(M,L) = 0$ and $\pi_2(M) = 0$.

$\Rightarrow$ Floer theory is well-defined as long as Gromov compactness holds (which requires strips to map to an a priori compact target, not T^*Q).

"a priori C^0 estimate."

Claim: Such an estimate holds.* (by a 'maximum principle', 'monotonicity', —).

Assuming this, have, for any H, w/ $\phi_H L \pitchfork L$, as T^*Q "convex"

Thm [Floer]: $HF^*(L,L) \cong H^*(L)$.

By invariance, etc., sufficient to make the computation for a nice H, J.

- Pick metric g on L $\rightsquigarrow$ induces a splitting $TT^*L \cong T_{vert}T^*L \oplus T_{horiz}T^*L$.

& an almost cx. structure, ex: on zero section

$$TT^*L|_0 = T^*L \oplus TL, \ \&$$

J is the natural pairing induced by g.

- Pick Morse fcn. f on L s.t. (f,g) is 'Morse Smale':

induces Hamiltonian $H: T^*L \to \mathbb{R}$ (though cut off near ∞ to make cptly support)

$$\begin{array}{ccc} T^*L & \longrightarrow & \mathbb{R} \\ \downarrow \pi & \nearrow f & \\ L & & \end{array}$$

Notes "$H(q,p) = f(q)$"

$dH = f'(q)\,dq$

$X_H = f'(q)\,\partial_p$

$JX_H = \nabla f(q)\,\partial_q$..

[F'88]

Thm: If f C^2 small, there is a bijection

$\left\{ \begin{array}{l} \gamma: \mathbb{R} \to L \text{ flowlines for } (f,g) \\ \dot\gamma(s) = \nabla f_{\gamma(s)} \\ \lim_{s\to\pm\infty} \gamma(s) = x^\pm \end{array} \right\}$ $\longleftrightarrow$ $\left\{ \begin{array}{l} \text{sol'ns to } \partial_s u + J(\partial_t u - X) = 0 \\ \text{between } x^+ \ \& \ x^- \end{array} \right\}$

$\longleftrightarrow$

$\updownarrow$

J-curves between L & $\phi_H^1(L)$

[Floer]

Thm: In general, if $\pi_2(M,L) = \pi_2(M) = 0$,

$HF^*(L,L) \cong H^*(L)$ too

Idea: ~~Near every~~ ~~For~~ ~~given H~~ modeled on Morse. Near every $L \ni U \cong T^*L$.

For given H, show (a)