~~Uncountable sets of measure zero exist.~~

~~$\overline{X}^c = M$~~

Week 8, Monday:

Last time: $\mathcal{C}$ A_∞ cat. $\rightsquigarrow$ $tw\,\mathcal{C}$, $tw^\pi\mathcal{C}$ — pre-triangulated & split-closed pre-Δ hull.

$\hat{\mathcal{C}} \cong tw\,\mathcal{C}$, $perf(\mathcal{C}) \cong tw^\pi \mathcal{C}$

$\mathcal{A} \subseteq \mathcal{C}$ split-generates $\mathcal{C}$ iff $perf\,\mathcal{A} \xhookrightarrow{\simeq} perf(\mathcal{C})$.

An imprecise analogy:

categories $\mathcal{C}$ $\rightsquigarrow$ subsets $X \subseteq \mathbb{R}^n$

$\downarrow tw^\pi$ $\qquad$ $\downarrow$ span.

split-closed pre-Δ cat $\rightsquigarrow$ vector spaces $V \subseteq \mathbb{R}^n$

For example: Given $\mathcal{A} \subseteq \mathcal{C}$, can form A_∞ quotient category $\mathcal{C}/\mathcal{A}$. (Drinfeld - dg case, Lyubashenko-Mazyuk), & have $tw^\pi \mathcal{C}/(\mathcal{A}) \cong tw^\pi\mathcal{C}/tw^\pi\mathcal{A}$; moreover $\mathcal{C}/\mathcal{A} \simeq \{0\}$ iff $\mathcal{A}$ split-generates $\mathcal{C}$.

The quotient satisfies a universal property:

$\exists$ a functor $\mathcal{C} \xrightarrow{j} \mathcal{C}/\mathcal{A}$ which is "initial among functors out of $\mathcal{C}$ sending $\mathcal{A} \to \{0\}$, meaning $\forall\, \mathcal{D}$ w/

$$\begin{tikzcd} & \mathcal{C}/\mathcal{A} \\ \mathcal{C} \arrow[r] \arrow[ur] & \mathcal{D} \arrow[u, "\exists\text{ htpy uniquely}"'] \\ \mathcal{A} \arrow[u, phantom, "\cup"] \arrow[r] & \{0\} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Moreover, $\exists$ equiv. $\text{nu-}fun(\mathcal{C}/\mathcal{A}, \mathcal{D}) \simeq \text{nu-}fun_{\mathcal{A}}(\mathcal{C}, \mathcal{D})$.

Explicit construction: take a cell.

Using this, ~~another~~ operation one can perform is localization: given $W \subseteq$ ~~$H^0(hom(X,Y))$~~ $H^0(\mathcal{C})$ a subset of morphism spaces for each pair ~~X,Y~~ of objects,

can define localization: $\mathcal{C}[W^{-1}]$:= to be the universal A_∞ cat in which chain level ~~elts~~ cycles representing W are quasi-isomorphisms.

Again comes w/ functor $j: \mathcal{C} \to \mathcal{C}[W^{-1}]$ s.t. if

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\text{sends } W \to \text{isomorphisms}"] \arrow[d, "j"] & \mathcal{D} \\ \mathcal{C}[W^{-1}] \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Cor: If $\mathcal{C}$ pre-tri., can construct $\mathcal{C}[W^{-1}] := \mathcal{C}/$ sub cat. gen. by cones of morphisms in W

def $Cones(W) \subseteq \mathcal{C}$:= full subcat of objects $\{Cone(f) \mid f \in W\}$.

& $\mathcal{C}[W^{-1}] := \mathcal{C}/Cones(W)$.

otherwise define $\mathcal{C}[W^{-1}]$ to be the image of $\mathcal{C} \hookrightarrow \mathrm{tw}^\pi \mathcal{C} \to \mathrm{tw}^\pi \mathcal{C} / \mathrm{Cones}(W)$.

~~symplectic~~ Landau-Ginzburg (LG) models:

In mirror symmetry & physics, an LG model is a pair $(\check{X}, W)$ — W hol. fn. $W: \check{X} \to \mathbb{C}$; $\check{X}$ non-cpct, Kähler, $c_1(\check{X}) = 0$

HMS away from the Calabi-Yau case predicts; for instance when X is cpct, Fano,

$$D^\pi \mathrm{Fuk}(X) \underset{\text{"}H^0(\mathrm{tw}^\pi \mathcal{F}(X))\text{"}}{\cong} MF(\check{X}, W)$$ (we've seen glimpses of this for $X = \mathbb{P}^n$, $\check{X} = (\mathbb{C}^*)^n$, $W = z_1 + z_2 + \dots + z_n + \frac{1}{z_1 \cdots z_n}$

$$D^b \mathrm{Coh}(X) \simeq D^\pi \mathcal{F}(\check{X}, W)$$ ← today.

Goal: define a category $\mathcal{F}(\check{X}, w)$ assoc. to such a pair & see how its ~~[illegible]~~ structure existence induces relations in $\check{X}$ & $W^{-1}(pt.)$.

Sympl. setup: (E^{2n}, ω)

← non-compact sympl., w/ technical hypotheses (exact, monotone, ...)

$W: E \to \mathbb{C}$ "symplectic fibration with singularities," meaning

- away from $K_{cpct} \subseteq \mathbb{C}$, $W: E \setminus W^{-1}(K) \to \mathbb{C}$ ~~generic~~ sympl. fibration

(meaning ~~[illegible]~~ each $(M_p \overset{\text{"}}{:} W^{-1}(p), \omega|_{M_p})$ sympl.,

loc. trivial w/ transition fns. symplectic.

Note: Symplectic fibrations carry canonical connections

(at a point $x \in M_p$, have canon. splitting $T_x E = T_x M_p \oplus (T_x M_p)^{\perp_\omega}$ ← sympl. complement

$\Rightarrow$ ~~parallel to~~ get symplectic parallel transport maps (as long as flow doesn't escape to ∞ in the fiber!

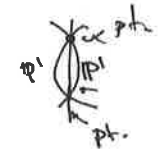
(Sometimes ~~more~~ convenient, work w/ $\bar{E}$: symplectic manifold w/ convex boundary $\partial\bar{E}$: means near $\bar{E}$ $\exists$ a primitive λ of ω s.t. $Z = $ ~~[illegible]~~ ω dual of λ points outward along $\partial\bar{E}$: ~~[illegible]~~.

(local model at $z \in Z_i$: $\mathbb{C}^n \xrightarrow{W} \mathbb{C}$
$\vec{z} \longmapsto \sum_1^{k_i} z_j^2$.)

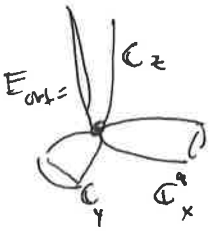
(3) More genl "stratified singularities"

Ex: ~~$\mathbb{C}$~~ $(\mathbb{C}^3, \hat{W} = -xyz + x^5 + y^5 + z^5)$.

compute $\hat{E}_{crit}$ =

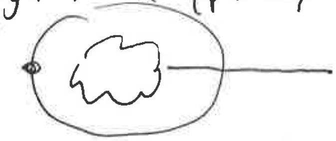


or $(\mathbb{C}^3, W = xyz)$

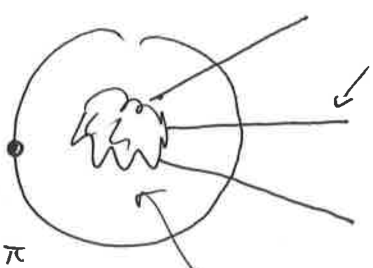


(this example is a little more delicate b/c critical locus is not compact.)

<u>Def</u>: (Given a pair (E, W), an <u>admissible Lag'n</u> is a (possibly non-compact) $L \subseteq E$ such that $W(L)$ is contained in



some ~~radial~~ radial rays of the form $\theta = $ const;



(equiv.

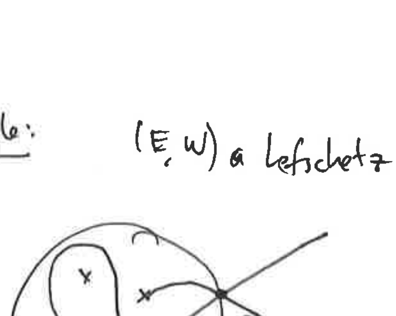
horiz. pullel to $\mathbb{R}_+$)

compact Lag'ns allowed; ~~in this case~~

$\rightsquigarrow$ L induces a subset $D_L \subseteq (-\pi, \pi)$ of L's "directions near ∞."

(L compact: $D_L = \emptyset$.).

<u>key: Example</u>: (E, W) a Lefschetz fibration, $\gamma: [0, \infty) \longrightarrow \mathbb{C}$ a path w/ $\gamma(0) \in Z_i \in$ crit W asymptotically a radial ray

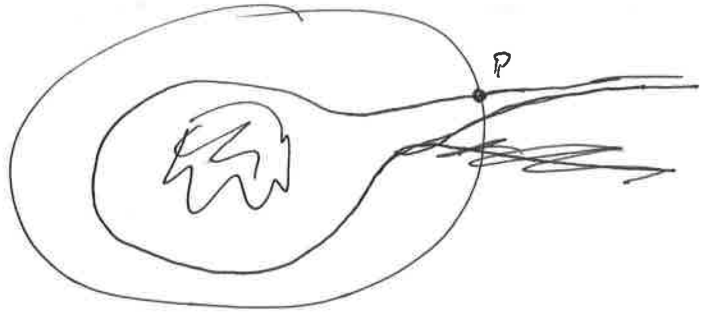


$\rightsquigarrow$ ~~gen.~~ Lefschetz thimble $\Delta_\gamma := $ pts. in $W^{-1}(\text{im } \gamma)$ which parallel transport along γ to ~~…~~ Z_i
$\cong \mathbb{R}^n$.

(E, W) Lefschetz-Bott, γ as above.
Given a Lag'n $L \subseteq Z_i$ a crit. submfld,
get. gen. thimble $\Delta_\gamma^L \subseteq E := $ "pts. — parallel ~~#~~ trans. along γ to $L \subseteq Z_i$."

Examples: "Lag'ns can have multiple ends"; $M = w^{-1}(p)$ ref. fiber near ∞.

- Draw $\gamma : (0,\infty) \longrightarrow \mathbb{C}$ positively oriented path through p around K_{cpct}



Given $L \subseteq M_p$ a Lag'n in M_p

define $U_\gamma^L :=$ points in $w^{-1}(\text{im } \gamma)$

which $\|$ transport to L along γ.

~~∞ functor $U_\gamma : \mathcal{F}(M) \to \mathcal{F}(E,W)$.~~

"[Orlov] Lagrangian"

The Fukaya category of an LG model

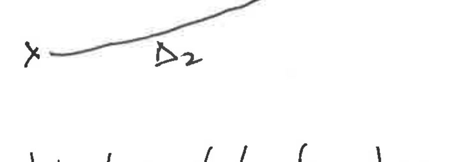
Floer cohomology of admissible Lag'ns: $H^* \hom(K,L)$

Given K, L admissible, want to define $\overset{\shortparallel}{H}om_{\mathcal{F}(E,w)}(K,L) = HF^*(K,L)$

Problems: • (K,L) not transverse at ∞

- Is HF^* defined for non-compact Lag'ns?
- Lack of Ham. isotopy invariance near ∞!

Ex:



Solution: there is a distinguished choice of direction at ∞ in $\mathbb{C}$, "counterclockwise".

(e.g. ∂_θ, or w.r.t. ω_{std} on $\mathbb{C}$, the flow near ∞ assoc. to $h\theta = r$.
$\overset{\shortparallel}{dx \wedge dy}$

& define $\mathrm{Hom}(L_0, L_1) := HF^*(\phi_\varepsilon L_0, L_1) := H^*(\mathbb{K}\langle \phi_\varepsilon L_0 \cap L_1 \rangle, \mu^1)$

$\uparrow$ "time ε ccw bend"

time ~~H~~ flow ~~any Ham~~

the pull back

Remarks: Use J s.t. $W : (E, J) \to (\mathbb{C}, j)$ is j-holomorphic near ∞

$\Rightarrow$ any disc $u: D^2 \to E$ w/ boundary on K, L disjoint Lagns near ∞, ~~and~~ which are radial

$W(u): D^2 \to \mathbb{C}$ achieves its maximum at punctures

disc w/ bndry in radial rays

$\Rightarrow$ cst.

λ $r^2 d\theta \Big|_L = 0$. strictly

so can fix. $f = 0$ near ∞

Then a sympl. LG model is a map $W : E^{2n} \to \mathbb{C}$

s.t. outside $D^2 \subseteq \mathbb{C}$, W presents E^{2n} as a symplectic mapping torus:

$\exists\, (M, \partial M, \lambda_M)$ defined near ∂M, convex)

& $\varphi : M \to M$ preserving λ_M, ω_M, s.t.

$$\begin{array}{ccc} E \setminus W^{-1}(D^2) & \cong & \dfrac{M \times [0,2\pi]_\theta \times [1,\infty)_r}{(x,0,r) \sim (\varphi(x), 2\pi, r)} \\ \downarrow W & & \downarrow \, re^{i\theta} \\ \mathbb{C} \setminus D^2 & = & \mathbb{C} \setminus D^2 \end{array}$$

Examples:

(1) W is a Lefschetz ~~(Bott)~~ fibration:

if $W : E \to \mathbb{C}$ ~~~~

$\mathrm{crit}\, W = \{p_i\}$

$E^{crit} = \coprod \{z_i\}$ ← ~~smooth symplectic submanifolds of codim $2k_i$~~

↖ crit. locus above $W^{-1}(p_i)$.

(WLOG assume crit. values are separate)

w/ W "hol. Morse" i.e. non-deg. ~~~~ at z_i

(means the local model at a point $z_i \in E^{crit}$ is the pt. 0 in

$$\mathbb{C}^n \xrightarrow{W_{loc}} \mathbb{C}$$
$$(z_1, \ldots, z_n) \longmapsto \sum_{i=1}^n z_i^2 .$$

(2) Lefschetz–Bott fibrations

$\mathrm{crit}\, W = \{p_i\}$

$E^{crit} = \coprod Z_i$ ← smooth sympl. submfld codim $2k_i$

↖ crit. locus above $W^{-1}(p_i)$.

w/ W "hol. Morse" in normal directions.

Example in nature: Lefschetz pencils.

$\bar{X}$ $(n+1)$-dim'l sm. proj. variety, $\mathcal{L} \to \bar{X}$ ample line bdle.

Take $s_0, s_\infty \in H^0(\mathcal{L})$ lin. indep.

$\leadsto \bar{p} = s_0/s_\infty$ rat'l fxn,

fibers are $M_y := \{s_0(x) = y s_\infty(x)\}$

$B :=$ base locus $= s_0^{-1}(0) \cap s_\infty^{-1}(0)$.

Lefschetz pencil: B is smooth, $\bar{p}$ non-deg. crit pts, $s_\infty^{-1}(0)$ smooth

⇓

$p : = X = \bar{X} \setminus s_\infty^{-1}(0) \to \mathbb{C}$.

metric on $\mathcal{L}$ → Kähler form on X

→ ~~~~ potential ϕ w/ $-\log|s_\infty|$.

$\phi^{-1}((-\infty, c))$ convex w/ boundary, $c \gg 0$.