Monotone Manifolds & their Lagrangians

week 6, Wednesday

- $X^{2n,\omega}$ monotone if

$$[\omega] = \tau c_1(X), \quad \tau > 0$$

- $L \subseteq X$ monotone if

$$[\omega] = \lambda [\mu], \qquad \lambda > 0, \text{ where}$$

$$[\omega] : \pi_2(X, L) \longrightarrow \mathbb{R} \quad \text{sympl. area}$$

$$[\mu] : \pi_2(X, L) \longrightarrow \mathbb{R}$$

Note: $[\omega](\underset{\pi_2(X,L)}{u} \# \underset{\pi_2(X)}{v}) = [\omega](u) + [\omega](v)$

$$[\mu](u \# v) = [\mu](u) + 2c_1(X)[v].$$

$\Rightarrow$ X is monotone w/ $\tau = 2\lambda$.

L monotone. The minimal Maslov number of L, $N_L := \{\min \mu(\alpha) \mid \alpha \in \pi_2(M, L), \mu(\alpha) > 0\}$

(or over all α supporting a hol. disk).

Assumption: $N_L \geq 2$.

(always if L orientable).

Consequences: 1) If $u : S^2 \to M$ J-hol-sphere, then $\int u^*\omega > 0 \Rightarrow c_1(X)[u] > 0$.

$\Rightarrow$ Any J-hol. sphere is a branched cover of a simple J-hol. sphere (McDuff)

$\Rightarrow$ Chern 1 spheres are simple, hence transversely cut out (by standard transversality arguments).

~~the virtual dimension of the $M(A) := \{u : S^2 \to X, \text{J-hol}\}$~~

$\Rightarrow$ "Chern 1 J-hol. spheres ~~the fiber~~ sweep out a ~~codim. 4~~ high codim ~~subspace~~ subset of X.

Namely: $M_1(A) := \{u : (S^2, x_0) \xrightarrow{J\text{-hol.}} X \mid [u] = A\}/\text{reparam.}$ has

when $c_1(A) = 1$, dimension $2n + 2c_1^{(X)}[A] + \underset{\uparrow \text{ marked pt. position}}{2} - \overset{\text{reparametrizations}}{6} = 2n - 2$,

Have $\mathrm{ev} : M_1(A) \longrightarrow X$

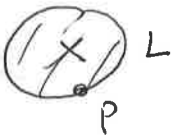
$u \longmapsto u(x_0)$. "the space of all points of all J-hol. spheres" is codim. 2.

2) Can count (at least Maslov 2) disks:

[Lazzarini, Kwon-Oh]: Maslov 2 hol. discs are simple; hence ~~can be transversely~~ are transversely cut out.

Given a class $\beta \in \pi_2(X, L)$, w/ $\mu(\beta) = 2$,

let $\underset{(;J)}{R^\circ(L;\beta)} := \{u : (D^2, \partial D^2, \underset{\overset{\uparrow}{\partial D^2}}{p}) \longrightarrow L \text{ J-hol}\} / Aut(D^2, \partial D^2, p)$



compact: Gromov compactness $\Rightarrow$ there are no bubbles; (why?).

By above, for generic J this can be made a smooth manifold of dimension

$$n + \mu(\beta) + \underset{\overset{\uparrow}{\text{marked point } p}}{1} - 2$$

Moreover, any two choices of generic J give cobordant manifolds. (why? in a one parameter family, monotonicity $\Rightarrow$ no bubbles come off, b/c they take sympl. area, hence Maslov index)

There's a map $ev : R^\circ(L;\beta) \longrightarrow L$, $u \longmapsto u(p)$.

Define $m_0(L) \in \overset{\Lambda (= \mathbb{C}}{\mathbb{C}} \cong H^0(L)$ by

$$\sum_{\substack{\beta \in \pi_2(-) \\ \mu(\beta)=2}} T^{\omega(\beta)} \cdot ev_*[R^\circ(L;\beta)] = m_0(L) \cdot [L].$$

$\Lambda = \mathbb{C}$, (note: can take $T = 1$! b/c $\mu = 2 \Rightarrow \omega$ fixed $\Rightarrow$ Gromov compactness!

More directly, $m_0(L) := \sum_\beta \# T^{\omega(\beta)} R^\circ_q(L;\beta) := \{u - \text{———} \mid u(p) = q\}$

where $q \in L$ is generic.

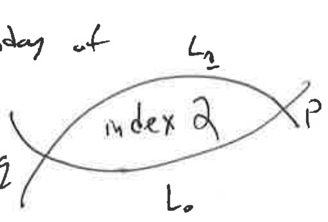
Given a local system $\overset{\mathcal{E}}{\underset{L}{\downarrow}}$ Λ,

$$m_0(L, \mathcal{E}) := \sum_\beta T^{\omega(\beta)} \cdot \# R^\circ_q(L;\beta) \cdot \underbrace{hol\, \mathcal{E}\rho(\partial\beta)} \in \Lambda^* \overset{e.g.}{=} \mathbb{C}^\times.$$

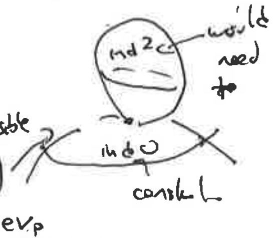
Then, if L_0, L_1 monotone Lagns (say w/ flat local systems),

Prop: $\mu^1 : CF^*(L_0, L_1) \circlearrowleft$ satisfies $(\mu^1)^2 = (m_0(L_0) - m_0(L_1)) \cdot Id.$

Sketch: Look at the boundary of



Gromov compactness $\rightsquigarrow$ sphere bubble? only possible (by monotonicity)



but

$\xrightarrow{ev_p} X$ generally misses constant strips/pts

Hence, $(\mu^1)^2 \neq 0$ unless $m_0(L_0) = m_1(L_1)$

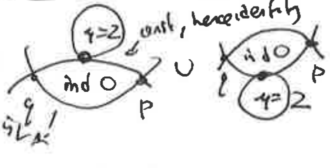
So, the unobstructed Fukaya category of X

the subcat of the Fukaya cat. where $hom(X_0, X_1) = CF^*(X_0, X_1)$ s.t. if $(\mu^1)^2 = 0$, 0 otherwise,

$\rightarrow$ Floer breaking

$(\mu^1)^2$

disc bubbles: only possible! there do exist $\mu=2$ discs though a generic pt $q \in L$! # such is $m_0(L_i)$



splits as $\bigoplus_{\lambda \in \Lambda} \mathcal{F}_\lambda(X)$

$$\mathcal{F}_\lambda := \{ \underset{\text{Lag'n brane}}{L} \overset{\left(\text{or } \overset{\mathcal{E}}{\underset{L}{\downarrow}}\right)}{} \mid m_0(L) = \lambda \}$$

<u>Prop</u>: (unjustified today): If X monotone,

$\mathcal{F}_\lambda \cong 0$ unless λ is an eigenvalue of $(\underset{\text{quantum product}}{c_1(X) \star —}) : \underset{\text{quantum cohomology,}}{QH^\bullet(X)} \circlearrowleft$

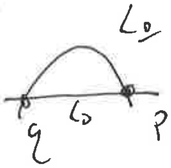
as a vec-space ordinary $H^\bullet(X)$, but product defined by J-hol. spheres.

A Morse-Bott approach to $CF^\bullet(L,L)$: it's a little inconvenient that as we defined it, $m_0(L) \notin CF^\bullet(L,L)$, but rather $m_0(L) \in H^*(L)$.

↖ recall this ~~[illegible]~~ is basically $\mathcal{F}_H$ gen. by $\phi_H L \pitchfork L$, some H.

There's are various equivalent Morse-Bott methods ~~of~~ for computing the A_∞ alg. structure on $CF^\bullet(L,L)$ using ~~[illegible]~~ $C^\bullet_{sing}(L)$ or $CM^\bullet(L,f)$ as the starting point (imagine ~~ϕ_H is modeled~~ H is modeled on εdf ~~for~~ near L in a T^*L nbhd), "take limit as $\varepsilon \to 0$).

Ex: monotone $S^1 \subseteq \mathbb{CP}^1$



p

$$\partial p = ~~T^2 q~~ - q + T^A$$

(small trajectory) area(big trajectory)

$$\partial q = 0 \quad ~~q \circ q~~$$

$$\partial p = T^{A_1} q - T^{A_2} q.$$

if the holonomy around $S^1 = z \in \mathbb{C}^*$, have

$$\partial p = \left(z \pm \frac{1}{z}\right) q.$$

This is only zero when $z = \pm 1$.

<u>Morse-Bott Floer theory</u> (FOOO):

- $CF^*(L,L) := C_*(L;\Lambda)$ "singular chains" on L (in suitable sense).

μ^k operator: ~~instead of a strip-like end around each marked pt~~ instead of removing each marked pt & putting strip-like end

, $\simeq$

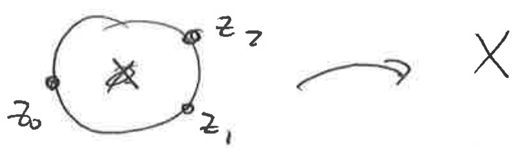
put a boundary marked pt

& require $u(z_i) \in$ chain.

Ex:

μ^2 ~~counts~~ ; consider ~~$\mathcal{M}_{0,3}$~~ $\overline{\mathcal{R}^2}(X; L, J, \beta)$

$\shortparallel$



entire $\partial u \longrightarrow L$

chains in L

$$\mu^2(\overset{\downarrow}{c_2}, c_1) = \sum_{\beta \in \pi_2(X, L)} (ev_0)_* \left([~~\mathcal{M}_{0,3}~~ \overline{\mathcal{R}}^2(X, L; J, \beta)] \cap ev_1^* c_1 \cap ev_2^* (c_2) \right) T^{\omega(\beta)}$$

where $ev_i : \overline{\mathcal{R}}^2(X, L; J, \beta) \to L$
$u \longmapsto u(z_i)$

constant discs contribute by intersection product,

except w/ μ^1: don't count constant  . just take ∂c as chain.

{More generally, if L_0, L_1 have "clean intersection",

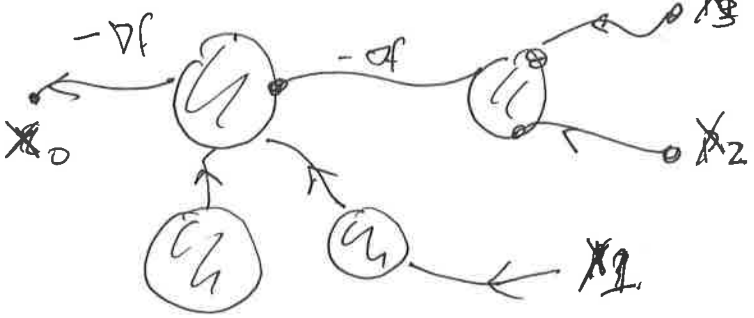
want $CF^*(L_0, L_1) := C_*(L_0 \cap L_1; \ )$; but unclear (analytic foundations still)
)

or

Cornea-Lalonde cluster homology "clusters" "pearl complex":

Fix a $f: L \to \mathbb{R}$ Morse fcn, define $CF^*(L, L) := \Lambda \otimes CM^*(f)$
$\shortparallel$
$\bigoplus_{x \in crit f} \mathbb{K}\langle x \rangle$.

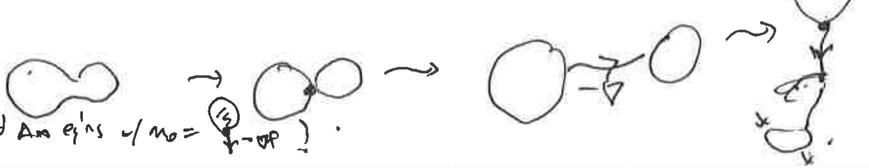
μ^k counts "clusters" of J-hol. discs + gradient flowlines



$\swarrow$ this contributes $T^{\text{area}(\text{discs})} \cdot x_0$ in $\mu^3(x_3, x_2, x_1)$!

bubbling of discs is no longer a boundary;

$\rightsquigarrow$ broken morse trajectories are boundaries ($\Rightarrow$ curved A_∞ eq'ns w/ $m_0 = \,$?)



General structures:

Say ~~[illegible]~~ L in $\mathcal{F}(X)$ is weakly unobstructed if , ~~[illegible]~~ using one of these two models,

$$m^0_L = \lambda_L \cdot \text{ ~~multiple~~ of a strict unit.}$$

$$\Rightarrow (m^1)^2_L = 0$$

for weakly unobstructed

$$\& \ (m^1)^2_{K,L} = 0 \text{ if } \cancel{m^0_K = m^0_L} \ \lambda_K$$

$$\mathcal{F}(X)^{w.o.} = \bigoplus_{\lambda \in \Lambda} \mathcal{F}_\lambda(X) \text{ a genuine } A_\infty \text{ cat.}$$

^ weakly unobstructed objects w/ "charge λ"

<u>Example</u>: $T^n \subseteq \mathbb{P}^n$ clifford torus.

define $T^{n+1} = \left(S^1\left(\frac{1}{\sqrt{n+1}}\right)\right)^{n+1} \hookrightarrow S^{2n+1}(1) \subset \mathbb{C}^{n+1}$

^ radius. ^ radius

~~[illegible]~~ and define $T^n := T^{n+1}/\text{diagonal } S^1 \subseteq S^{2n+1}/S^1 \cong \mathbb{CP}^n$.

Prop: [~~Cho~~ Oh]: T^n_{cl} is monotone w.r.t. the standard sympl. str. on $\mathbb{P}^n$, w/ minimal Maslov 2.

(follows from explicit analysis of ω & μ on $\pi_2(X, T^n)$

using SES

$$0 \to \pi_2(\mathbb{P}^n) \to \pi_2(\mathbb{P}^n, T^n) \to \pi_1(T^n) \to 0 \quad).$$

Ex: T^2 in $\mathbb{CP}^2$ ~~schematically, T^2 [illegible]~~ Presenting $\mathbb{C}^2 = \mathbb{CP}^2 \setminus H_\infty$,

we have $T^2 \subseteq \mathbb{C}^2$ is one of the standard tori.

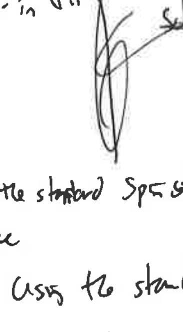
Thm: [Cho]: ~~[illegible]~~ using the standard Spin str. on T^n to get a Lag. brane, $\exists$ exactly ~~3~~ $n+1$ rk. 1 local systems ε for which

$$HF((T^n_{cl}, \varepsilon), [illegible]) \neq 0.$$

(& all have distinct m^0 values!).

~~[illegible]~~ using the standard integrable J, ~~the hol. discs~~ ~~can be descr.~~ there are 3 families of Maslov index 2 discs

A "toric picture of $\mathbb{CP}^2$"



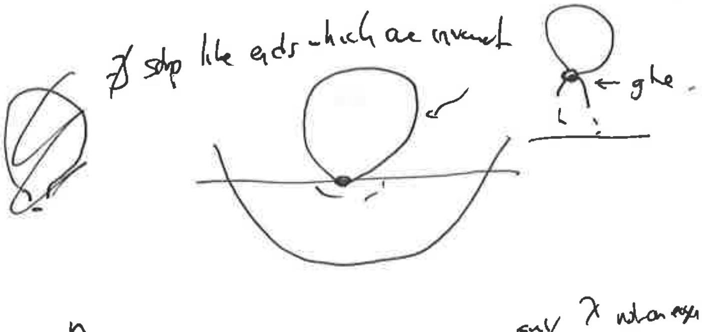
using $[x:y:z] \longmapsto$

There's a map

$$\mathcal{R}^0(L;\overset{\text{dim class}}{\beta}) \longrightarrow L \quad \mu(\beta)=2.$$

$$\mathrm{ev}: \mathcal{R}^0(L;\beta) \longrightarrow L$$

$$u \longmapsto u(p).$$



Define $\mathcal{Y}^0 \in H^0(L) :=$

$$\mathrm{ev}_*[\mathcal{R}^0(L;\beta)]..$$

(Maybe today assume $N_L \geq 3$. $\mathbb{RP}^n$, $\mathbb{CP}^n$ $n \geq 2$)

Given a pair $(L, ~~\beta~~ \overset{(\mathcal{E},\nabla)}{\underset{L}{\downarrow}}$ local system)

Define

$$= m_L := \sum_{\substack{\beta \\ \mu(\beta)=2 \\ u \in \mathcal{R}^0(L,\beta)}} \mathrm{sign}(u) \cdot \mathrm{hol}_{\mathcal{E}}([\partial u]). \quad \in \mathbb{C}.$$

say λ not an eigenvalue of $c_1 \star$

Then $(c_1 - \lambda I)\star$ invertible

$$\mathcal{CO}^0(c_1) = m_0(L)[L],$$

$$\Rightarrow (m_0(L) - \lambda I)\overset{?}{\star} \text{ invertible,}$$

not implies $\lambda \neq m_0(L)$.

(A): $(\mathcal{Y}^1)^2 = m_0(L_1) - m_0(L_0)$

(B) A Morse-Bott approach to HF.

(C) Discs on Clifford torus,

& local systems w/ non-vanishing HF^*(

$$\dim \mathcal{M}_0(A) = 2n-4.$$

Claim: $(\mathcal{Y}^1_{\mathcal{C}})^2$

X

$\mathcal{M}(A) \to X$? how? subsets of

(A): Show: $(\mathcal{Y}^1)^2 = m_0(L_1) - m_0(L_0)$..

$$\frac{1}{2}|x|^2,\ \frac{1}{2}|y|^2$$

$$\frac{1}{1-|z|^2} \quad \frac{1-|z|^2}{|x|^2-|z|^2} \quad \frac{1}{\frac{1}{2}|y|^2-|x|^2}$$

so if $L_1 = L_0$, or otherwise

Say $\mathcal{E}$

$$m_0(L_1) = m_0(L_0)$$

isotropic decomp. of Fukaya category

Prop: (unjustified today): X monotone; $\mathcal{F}_\lambda \cong 0$ unless

~~$\Rightarrow (L,\lambda) \neq 0$~~ λ is an eigenvalue of $(c_1(X) \star -) : QH^*(X)$ ↻

quantum cohomology,

a deformation of cohomology product using J-hol. spheres,

(B) A model of Floer cohomology from $H^*(L)$. (Morse-Bott)

(C) ~~Example~~: The ~~Spec~~, disc potential,

and

$$\left\{ 1 - \frac{1}{z^2} = 0 \right. \quad z = \pm 1. \qquad z \times \frac{1}{z} = 0$$