Today: fiber bundles, vector bundles, principal bundles

special examples of fiber bundles, with more structure.

the 'fiber' of E at b

Def: A fiber bundle over B is a space E w/ a map $\pi: E \to B$ (continuous), satisfying (local triviality): for every $b \in B$, denoting $E_b := \pi^{-1}(b)$, $\exists$ open $U \ni p$ in B and a map $E|_U := \pi^{-1}(U) \xrightarrow{t} E_b$ such that the map

$$E|_U \xrightarrow{(\pi, t) = \varphi} U \times E_b$$

is a homeomorphism. (note φ fits into a comm. diagram

$$\begin{tikzcd} E|_U \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & & U \times E_b \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$)

proj. to first factor.

Note: any two fibers of a fiber bundle in the same connected component of B must be homeomorphic. We'll often just restrict to a connected B or assume all fibers homeo.

Example: (1) F any space, form

$$\begin{tikzcd} X \times F \arrow[d, "\pi_X"] \\ X \end{tikzcd}$$

trivial fiber bundle w/ fiber F.

(2) covering space

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"] \\ X \end{tikzcd}$$

is a fiber bundle w/ discrete fibers.

(3) (non-discrete, non-trivial example):

$S^3 \subset \mathbb{C}^2$ unit sphere, & consider $\pi: S^3 \longrightarrow \mathbb{CP}^1 = S^2$

$v \longmapsto \{\text{complex line in } \mathbb{C}^2 \text{ through } 0 \text{ \& } v\}$

'Hopf fibration'

(concretely, $\underbrace{S^3 \hookrightarrow \mathbb{C}^2 \setminus 0 \xrightarrow{\text{quotient}} \mathbb{CP}^1}_{\pi}$)

This gives a fiber bundle over S^2 whose fibers are all (S^1)'s. (b/c $\mathrm{span}_{\mathbb{C}}(\underset{S^3}{v}) = \mathrm{span}_{\mathbb{C}}(\underbrace{e^{i\theta} v}_{S^3})$).

This is not a trivial fiber bundle (i.e. not isomorphic to one): $S^3 \neq S^2 \times S^1$ (e.g., H_1's are different)

(4) $V_k(\mathbb{R}^n)$ Stiefel manifold

exercise from 535a: show this is a smooth manifold.

$= \{\text{orthogonal } k\text{-frames in } \mathbb{R}^n\} = \{A \in \mathrm{Mat}(n \times k) \mid AA^T = \mathrm{Id}_k\}.$

This is a compact manifold. (How to see this? To start, observe $O(n)$ acts on $V_k(\mathbb{R}^n)$ by composition: transitive action, & isotropy group of basepoint $\{e_1, \dots, e_k\}$ is $I_k \times O(n-k)$. $\underset{\substack{\text{using this,} \\ \text{can show}}}{\Longrightarrow}$ $V_k(\mathbb{R}^n) = O(n) \big/ I_k \times O(n-k)$

compact

($\Rightarrow$ Hausdorff, cpct).

- Get a fiber bundle (fibration) $O(n) \longrightarrow V_k(\mathbb{R}^n)$ with fiber $O(n-k)$. (why? locally trivial?) e.g., why

  e.g., $V_1(\mathbb{R}^n) = S^{n-1}$, so in particular $O(n) \to S^{n-1}$ w/ fiber $O(n-1)$.

- Forget last $(k-1)$ vectors: $V_k(\mathbb{R}^n) \longrightarrow V_1(\mathbb{R}^n) = S^{n-1}$. with fiber at $v \in S^{n-1}$ the collection of $(k-1)$ tuples of orthogonal frames that are orthogonal to v, i.e., $(k-1)$-orthogonal frames of $T_v S^{n-1}$.

tangent space = vectors $\perp v$

The basic results that allow for us to show examples in (4) are fiber bundles (& many other examples) are:

<u>Thm</u>: (Ehresmann): Say E, B smooth manifolds, $\pi : E \to B$. smooth map. If π is
- <u>proper</u> (i.e., $\pi^{-1}(\text{cpct.})$ is cpct)
- <u>submersion</u> (means $d\pi_x : T_x E \to T_x B$ surjective for all x).

then $\pi : E \to B$ is a <u>fiber bundle</u>.

Using this, can prove:

<u>Prop</u>: G Lie group, and $K \subseteq H \subseteq G$ closed subgroups (so K, H also Lie groups) then the projection map

$$\begin{aligned} G/K &\longrightarrow G/H \\ g+K &\longmapsto g+H. \end{aligned}$$

is a fiber bundle with fibers isomorphic to H/K.

can apply this general result to get examples in (4), and many others.

<u>E.g.,</u>

(5) Grassmannians.

$G_k(\mathbb{R}^n)$ (or $Gr_{\mathbb{R}}(k,n)$) $:= \{ V \subset \mathbb{R}^n \mid V \text{ a real linear } k\text{-dim'l subspace} \}$.

subspace

$G_1(\mathbb{R}^{n+1}) := \mathbb{RP}^n$.

(There's also a complex version:
$G_k(\mathbb{C}^n) := \{V \subset \mathbb{C}^n \mid V \text{ a cplx-linear } k\text{-dim'l subspace}\}$.
w/ $G_1(\mathbb{C}^{n+1}) := \mathbb{CP}^n$., w/ same construction.)

can explicitly construct as

$$G_k(\mathbb{R}^n) = \underbrace{\{\text{linearly independent } k\text{-tuples in } \mathbb{R}^n\}}_{\text{open subset of } (\mathbb{R}^n)^k} / GL(k,\mathbb{R})$$

open subset of $(\mathbb{R}^n)^k$.

applying $GL(k,\mathbb{R})$ to a tuple gives same span.

equipped w/ quotient topology,

$k\times n$ matrices A w/ $AA^T = Id_k$.

can also construct as

$$= \{\text{orthonormal } k\text{-tuples in } \mathbb{R}^n\} / O(k)$$

$$= \{\text{orthonormal } n\text{-tuples in } \mathbb{R}^n\} / O(k)\times O(n-k)$$

$$= O(n) / O(k)\times O(n-k)$$

can check again that $G_k(\mathbb{R}^n)$ is a cpct, Hausdorff manifold.

The Prop above implies: $V_k(\mathbb{R}^n) \longrightarrow G_k(\mathbb{R}^n)$ is a fiber bundle w/ fibers $O(k)$.

$\{v_1, \dots, v_k\} \longmapsto \operatorname{span}(v_1, \dots, v_k)$

As we'll see, many of the above examples have the structure of principal bundles.

Vector bundles

a type of fiber bundle where all fibers are vector spaces (& trivializations are compat. w/ this structure).

X a space.

Def: A real vector bundle over X is

(i) a space E

(ii) a continuous $\pi: E \to X$

(iii) a real vector space structure on each $E_x := \pi^{-1}(x)$, $x \in X$.

satisfying (local triviality):

for every $x_0 \in X$, $\exists$ a nbhd $U \ni x_0$ in X and a homeo. for some n

$$\begin{tikzcd} E|_U = \pi^{-1}(U) \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & \circlearrowleft & U \times \mathbb{R}^n \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

π_U (proj. to first factor)

s.t. $\varphi|_{E_x} : E_x \xrightarrow[(\text{by } \circlearrowleft)]{} \{x\} \times \mathbb{R}^n \cong \mathbb{R}^n$ is a real linear isomorphism, for each $x \in U$.

Similarly, have notion of a complex vector bundle: (replace real by complex & $\mathbb{R}^n$ by $\mathbb{C}^n$).

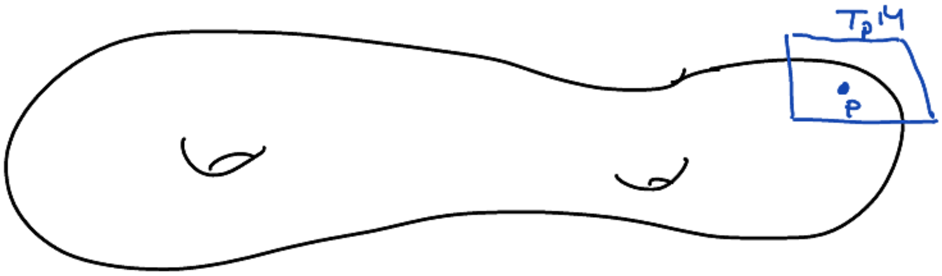
Examples:

(i) $X \times \mathbb{R}^n =: \underline{\mathbb{R}}^n$ equipped w/ $\underset{\pi_X}{\overset{\shortparallel}{\pi}} : X \times \mathbb{R}^n \longrightarrow X$ (projection to X)

trivial vector bundle.

(ii) M any smooth (C^∞) manifold, then its tangent bundle $TM \xrightarrow{\pi} M$ (fiber at $p \in M$ is T_pM tangent space) is a vector bundle.

e.g., if $M \subset \mathbb{R}^N$



(so are T^*M, $\Lambda^k T^*M$, etc.)

(iii) Tautological vector bundles on Grassmannians

Define $E_{taut} \xrightarrow{\pi} Gr_k(\mathbb{R}^n)$ by:

(similarly $E_{taut} \longrightarrow Gr_k(\mathbb{C}^n)$ tautological complex vec. bundle)

$E_{taut} \subseteq Gr_k(\mathbb{R}^n) \times \mathbb{R}^n$

$\shortparallel$

$\{ (X, v) \mid X \in Gr_k(\mathbb{R}^n),\ v \in X \}$

and $\pi(X, v) := X$.

the point / the subspace of $\mathbb{R}^n$

Observe: $(E_{taut})_X := \pi^{-1}(X) = \{X\} \times X \cong X$, i.e., has a linear structure.

Local triviality?

Choose a surjection $\alpha : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^k$ (linear).

true whenever $X \cap \overbrace{\ker(\alpha)}^{(n-k)\text{-dim'l}} = \{0\}$.

Define $U_\alpha := \{ X \in Gr_k(\mathbb{R}^n) \mid \alpha|_X : X \to \mathbb{R}^k$ is an isomorphism $\}$

(open dense subset, and $\{U_\alpha\}_{\alpha \in Surj(\mathbb{R}^n, \mathbb{R}^k)}$ cover $Gr_k(\mathbb{R}^n)$)

On U_α have a trivialization

$$\begin{aligned} E|_{U_\alpha} &\xrightarrow{\varphi_\alpha} U_\alpha \times \mathbb{R}^k \\ (X, v) &\longmapsto (x, \alpha|_X(v)). \end{aligned}$$

$(x \in U_\alpha, v \in X)$

check (exercise):
- homeomorphism, compat w/ projections.
- linear in each fiber.

<u>Def</u>: The <u>rank</u> of $E \xrightarrow{\pi} X$ is $\dim_{\mathbb{R} \text{ or } \mathbb{C}}(E_x)$, provided this number is constant in X.
(over $\mathbb{R}$ or $\mathbb{C}$)
any x

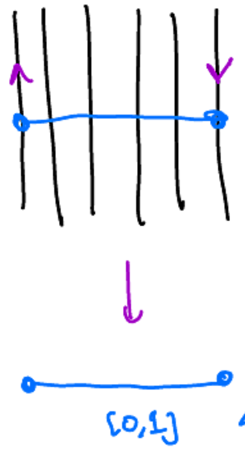
(know it has to be locally constant b/c local triviality, we'll usually assume global constancy so we can talk about rank)

(real or complex) <u>line bundle</u>: vector bundle of (real or complex) rank $= 1$.

e.g., tautological bundles over $Gr_k(\mathbb{R}^n)$ $Gr_k(\mathbb{C}^n)$ when $k=1$ give:

- $L_{taut} \to \mathbb{CP}^n$ tautological (complex) line bundle
- $L_{taut} \to \mathbb{RP}^n$ tautological (real) line bundle

<u>subexample/exercise</u>: Look at $L_{taut} \to \mathbb{RP}^1 \cong S^1$ & verify L_{taut} is $\cong$ <u>Möbius bundle</u>:



$[0,1] \times \mathbb{R} \big/ (0,v) \sim (1,-v)$

$[0,1] \big/ 0 \sim 1.$

& verify L_{taut} is <u>not trivial</u>.

Def: An <u>isomorphism of vector bundles</u> $E \xrightarrow{\pi_E} X$, $F \xrightarrow{\pi_F} X$ is a homeomorphism,

compat. w/ projections:
$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & \circlearrowleft & F \arrow[dl, "\pi_F"] \\ & X & \end{tikzcd}$$
, such that $\varphi|_{E_x} : E_x \to F_x$ is a linear isomorphism for each $x \in X$.

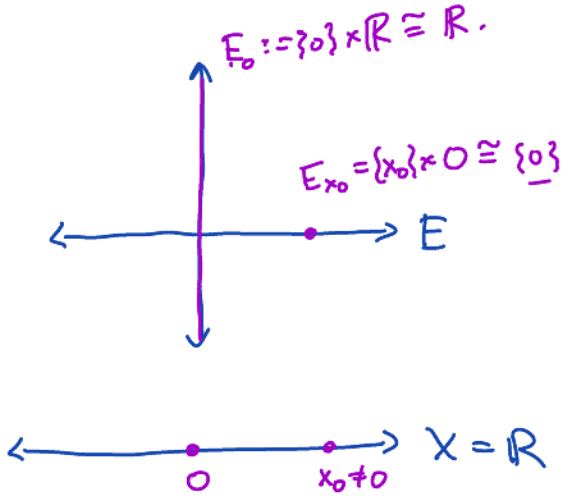
<u>Automorphisms</u> are self-isomorphisms.

E.g., $\mathrm{Aut}(\underline{\mathbb{R}}^k) = \mathrm{Maps}(X, GL(k,\mathbb{R}))$.
vec. bdle over X

—

Non-example of a vector bundle (also not a fiber bundle):

$$\begin{array}{ccc} (x,y) & & E = \{xy=0\} \subseteq \mathbb{R}^2 = x\text{-axis} \cup y\text{-axis} \\ \downarrow & & \downarrow \pi_x \\ x & & \mathbb{R} \end{array}$$



(this example can be viewed as, in a suitable sense, a <u>sheaf</u>).

<u>Principal bundles</u>

G a topological group, X a space.

Def: A <u>principal G-bundle</u> (or a principal bundle w/ <u>structure group</u> G) over X is a fiber bundle $\pi: P \to X$, along with a right action of G $P \times G \to P$, such that $\pi: P \to X$ is the quotient by this action, and (implies G preserves each P_x & hence $P|_U$.)

(<u>local triviality</u>) $\exists$ an open cover $\mathcal{U}$ of X s.t. for every $U \in \mathcal{U}$, $\exists$ a trivialization ("local trivialization along U")

$$\begin{tikzcd} P|_U \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & \circlearrowleft & U \times G \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

G acts on $P|_U$ and on $U \times G$; π: map from above; π_U ← proj. to U

which is G-equivariant; i.e., if $\varphi(p) = (z, g_0)$ then $\varphi(pg) = (z, g_0 g)$.

note: G acts freely on P, & each $P_x \cong G$, as spaces w/ G action (but no canon. group str. on P_x)

Obs: If $\pi: E \to X$ is a vector bundle of rank k, $\exists$ an associated principal $GL(k,\mathbb{R})$ bundle $\tilde{\pi}: P \to X$, defined as $P = \{(x, v_1, \dots, v_k) \mid x \in X, (v_1, \dots, v_k) \text{ basis for } E_x\}$.

"frame bundle", Frame(E).

$GL(k,\mathbb{R})$ acts on P by "change of basis" action, local triviality follows from local triviality of $E \to X$.

It turns out one can naturally go back from Frame(E) to E, as a special case of a more general construction that associates

$$\left(P := \text{principal } G \text{ bundle}, \underset{\text{representation}}{G \to GL(V)}\right) \longmapsto P \times_G V \text{ associated vector bundle.}$$

Applying this to

$$\left(\text{Frame}(E), GL(k) \xrightarrow{id} GL(k)\right) \quad \text{produces } E.$$

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Operations on principal bundles:

$P \xrightarrow{\pi} X$ principal G-bundle, F any top. space w/ a left G action $G \times F \to F$

$\left(G \to \underset{\text{map of top. groups}}{\text{Homeo}(F)}\right)$

$\rightsquigarrow$ can form the associated fiber bundle

$$P \times_G F := P \times F / \sim \quad \text{where } (zg, f) \sim (z, gf). \ \forall g, z, f.$$

$$\pi: P \times_G F \to X \quad \text{defined by} \quad \pi([z,f]) := \pi(z) \quad (\text{check well-defined});$$

fibers non-canonically isomorphic to F, & locally trivial (check: uses local triviality of P).

If the action has 'more structure', the associated fiber bundle will have more structure too.

e.g.,
- If $F = V$ a vector space (over $\mathbb{R}$ or $\mathbb{C}$) and $G \times V \to V$ is a linear action (meaning $G \to GL(V) \subset \text{Homeo}(V)$),

then $P \times_G V$ is a vector bundle of rank $= \dim(V)$, w/ fibers all (non-canonically) isomorphic to V.

- If have a map of top. groups $G \to H$ (e.g., continuous group hom.), induces an action $G \times H \to H$.

then $P \times_G H$ is a principal H-bundle.

Let's give some examples of this construction.

Note: Have the tautological action $GL(\mathbb{R}^k) \times \mathbb{R}^k \to \mathbb{R}^k$ $\quad (GL(\mathbb{R}^k) \xrightarrow{id} GL(\mathbb{R}^k))$.
$(T, v) \mapsto T(v)$

using this action

Claim: If $\begin{matrix} E \\ \pi\downarrow \\ X \end{matrix}$ any vector bundle $\rightsquigarrow$ $\begin{matrix} \mathrm{Frame}(E) \\ \downarrow \\ X \end{matrix}$ principal $GL(\mathbb{R}^k)$ bundle $\rightsquigarrow$ $\mathrm{Frame}(E) \times_{GL(\mathbb{R}^k)} \mathbb{R}^k$

Then $\mathrm{Frame}(E) \times_{GL(\mathbb{R}^k)} \mathbb{R}^k \cong E$.

In fact, (exercise): The following are inverse operations

$$(*) \quad \left\{ \begin{matrix} \text{Vector bundles of} \\ \text{rank } k \\ \text{on } X \end{matrix} \right\} \underset{(-)\times_{GL(\mathbb{R}^k)} \mathbb{R}^k}{\overset{\text{Frame}}{\rightleftarrows}} \left\{ \begin{matrix} \text{Principal } GL(k,\mathbb{R}) \\ \text{bundles on } X \end{matrix} \right\}$$

tautological action.

In particular by applying $(*)$, given a representation $GL(\mathbb{R}^k) \to GL(\mathbb{R}^m)$, we get an associated operation {rank k vector bundles} - - - -> {rank m vector bundles}

$\downarrow$ $\qquad\qquad\qquad\qquad \uparrow$

Principal $GL(k)$-bundle $\xrightarrow[\text{bundle}]{\text{assoc.}}$ Principal $GL(m)$-bundle

Ex: (1) $\underset{GL(k,\mathbb{R})}{\overset{GL(\mathbb{R}^k)}{\shortparallel}}$ acts on $\mathbb{R}$ by $GL(k,\mathbb{R}) \to GL(1,\mathbb{R}) = \mathbb{R} \setminus 0$
$A \longmapsto \det(A)$

$\rightsquigarrow$ get for any rank k $E \to X$ an associated line bundle
$\det(E) \to X$. $\quad$ (note: this coincides w/ "$\begin{matrix} \Lambda^{top} E \\ \downarrow \\ X \end{matrix}$").

(2) Consider $GL(k,\mathbb{R})$ acting on $\mathbb{R}^k$ via
$$(A, \vec{v}) \longmapsto (A^{-1})^T \vec{v}.$$

The associated vector bundle (starting from E) is called the dual vector bundle $E^\vee$.
(similar constructions work over $\mathbb{C}$)

<u>Other operations on vector bundles</u>: (over $\mathbb{R}$ or $\mathbb{C}$)

- <u>Pullback</u>: Given a vector bundle $\begin{matrix} E \\ \downarrow\pi \\ Y \end{matrix}$ and a continuous map $f: X \to Y$, get a vector bundle $\begin{matrix} f^*E \\ \downarrow f^*\pi \\ X \end{matrix}$, along with a map (lying over f)

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(linear on each fiber)

by definition, $f^*E := \{(x,e) \mid f(x) = \pi(e)\} \subseteq X \times E$, and $(f^*\pi)(x,e) = x$.

$$\left(\text{"} X \times_Y E \text{" or "} X \times_{(f,\pi)} E \text{"}\right)$$

Note: $(f^*E)_x := E_{f(x)}$. (a vector space).

Locally trivial? (exercise).

Note: • We can also pull back principal bundles via the same construction (replace E w/ P), & the G action is inherited from G action on $X \times P \supseteq X \times_Y P = f^*P$.

• special case: $X \overset{i}{\subset} Y$ inclusion of subset, then $i^*E = E|_X$ $(= \pi^{-1}(X))$.

• Cartesian product of vector bundles (or principal bundles)

if
$$\begin{array}{ccccc} \overset{\text{rank } m}{E} & & \overset{\text{rank } n}{F} \\ \downarrow \pi_E & & \downarrow \pi_F \\ X & & Y \end{array} \quad \left(\text{resp.} \quad \begin{array}{cc} P^{\circlearrowleft G} & Q^{\circlearrowleft H} \\ \downarrow \pi_P & \downarrow \pi_Q \\ X & Y \end{array}\right), \text{ then}$$

$$\begin{array}{c} E \times F \\ \downarrow (\pi_E, \pi_F) \\ X \times Y \end{array} \quad \left(\text{resp.} \quad \begin{array}{c} P \times Q \\ \downarrow (\pi_P, \pi_Q) \\ X \times Y \end{array}\right)$$

is a vector bundle of rank $m+n$. (resp. principal $G \times H$ bundle).

(exercise)

• "fiberwise direct sum" of vector bundles (Whitney sum):

Given $\begin{array}{c} E \\ \downarrow \pi_E \\ X \end{array}$, $\begin{array}{c} F \\ \downarrow \pi_F \\ X \end{array}$, first take $\begin{array}{c} E \times F \\ \downarrow (\pi_E, \pi_F) \\ X \times X \end{array}$, then

define $E \oplus F := \Delta^*(E \times F)$, where $\Delta : X \to X \times X$, $x \mapsto (x,x)$ diagonal embedding.

check: $(E \oplus F)_x := E_x \oplus F_x$.

• We can similarly define "fiberwise" operations $E \otimes F$, $\mathrm{Hom}_{\mathbb{R}}(E,F)$; easiest way to see this is as follows:

or $\mathrm{Hom}_{\mathbb{C}}$ if $\mathbb{C}$-vec. bdles

starting with $\overset{\text{rank } m}{E}, \overset{\text{rank } n}{F}$ ($\pi_E : E \to X$, $\pi_F : F \to X$), let P, Q ($\pi_P : P \to X$, $\pi_Q : Q \to X$) be associated frame bundles over X.

P has structure group $G := GL(m, \mathbb{R})$

Q " " $H := GL(n, \mathbb{R})$.

Form $\Delta^*(P \times Q) := P \times_X Q$ · a principal $G \times H$ bundle over X.

Observe that $G \times H = GL(m,\mathbb{R}) \times GL(n,\mathbb{R})$ acts naturally on

- $\mathbb{R}^m \oplus \mathbb{R}^n$ by $(g,h)(v \oplus w) = gv \oplus hw$
- $\mathbb{R}^m \otimes \mathbb{R}^n$ by $(g,h)(v \otimes w)$ is $gv \otimes hw$
- $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n)$ $\quad (g,h)(T) = h \circ T \circ (g^{-1})^T$

We call the associated bundles $E \oplus F$, $E \otimes F$, $\mathrm{Hom}_{\mathbb{R}}(E,F)$ resp. $\mathrm{Hom}_{\mathbb{C}}(-,-)$.
check: agrees w/ def'n above. The fiber at each $x \in X$ is $E_x \oplus F_x$, $E_x \otimes F_x$, $\mathrm{Hom}_{\mathbb{R}}(E_x, F_x)$ respectively.

- the dual bundle can be realized as $E^* = \mathrm{Hom}_{\mathbb{R}}(E, \underline{\mathbb{R}})$.

Def: A *section* of a fiber bundle $\begin{array}{c} Q \\ \downarrow \pi \\ X \end{array}$ is a map $s : X \to Q$ with $\pi \circ s = \mathrm{id}_X$. ✱

denoted $\begin{array}{c} Q \\ s \uparrow \downarrow \pi \\ X \end{array}$.

✱ $\Rightarrow$ $s(x) = (x, s_x)$ where $s_x \in Q_x$
(thinking of Q set theoretically as $\coprod_{x \in X} Q_x$).

Thm: A principal bundle is trivial iff it has a section.

(rmk: in contrast, while it is true a line bundle is trivial $\iff \exists$ non-zero section, not nec. true for higher rank vec. bundles)

(E vec. bundle rank k $\rightsquigarrow$ $\mathrm{Frame}(E)$ is trivial iff $\exists$ a section $X \to \mathrm{Frame}(E)$ $\rightsquigarrow$ E is trivial iff $\exists$ a k-tuple of sections which form a frame at each point x (i.e., a basis for each fiber).)

Pf: $\Rightarrow$ ✓ b/c $\begin{array}{c} X \times G \\ \downarrow \uparrow s(x) = (x, \mathrm{id}). \\ X \end{array}$

$\Leftarrow$ Say $\exists$ s: $\begin{array}{c} P \\ s \uparrow \downarrow \pi \\ X \end{array}$. Then define

$$\begin{tikzcd} X \times G \arrow[rr, "\varphi"] \arrow[dr, "\pi_X"'] & & P \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

a map of principal bundles (i.e., φ G-equiv.), by $\varphi(x,g) = s(x) \cdot g$.

φ is automatically an iso. by next lemma. ∎

Lem: Any non-trivial morphism of G-bundles

$$\begin{tikzcd} P_0 \arrow[rr, "f"] \arrow[dr] & G & P_1 \arrow[dl] \\ & X & \end{tikzcd}$$

(i.e., G-equiv.) is an isomorphism.

Pf: Special case $\begin{array}{c} P_0 = X \times G, \\ \downarrow \pi_X \\ X \end{array}$ $\begin{array}{c} P_1 = X \times G \\ \downarrow \pi_X \\ X \end{array}$ & f: $\begin{tikzcd} P_0 \arrow[rr] \arrow[dr] & & P_1 \arrow[dl] \\ & X & \end{tikzcd}$ $\overset{\text{equivariance}}{\Rightarrow} f(x,g) = (x, g\,h(x))$ for some $h : X \to G$.

But now this map has inverse $(x,g) \longmapsto (x, g(h(x))^{-1})$.

Since a general P_0, P_1 are locally trivial, this argument applies f is an iso. in a nbhd of any x, hence everywhere. $\boxtimes$.

Inner products on vector bundles: (an inner product on V is an element of $(V\otimes V)^* \ni g$ s.t. the map $\langle -,- \rangle : V\times V \to V\otimes V \to \mathbb{R}$ satisfies ...)

An inner product on a vector bundle $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ is a section (g) of $(E\otimes E)^*$,

s.t. the associated pairing $\langle -,- \rangle_x$ on E_x defined by $\langle v, w\rangle_x := g_x(v\otimes w)$

is an inner product (pos definite, symmetric bilinear).

Can think of as a collection of $\langle -,- \rangle_x$ on each E_x "varying continuously" (in sense g is a continuous section)

$\Rightarrow$ If s, t are (continuous) sections, then
$x \longmapsto \langle s_x, t_x\rangle_x$ is continuous.

OR $\langle -,- \rangle \in$ "$\Gamma(\text{Bilinear}(E\times E, \mathbb{R}))$".

Lem: An inner product exists (at least if X is paracompact, i.e., admits partitions of unity) c.f. 535a or Hatcher

Sketch: Given a cover $\{U_\alpha\}$ over which E is loc. trivial, $\exists$ an inner product $\langle -,- \rangle_\alpha$ on each $E|_{U_\alpha}$ b/c $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ (& use $\langle -,- \rangle_{\text{Euclidean}}$ on $\mathbb{R}^k$).

Then if $\{\varphi_\alpha\}$ is a partition of 1 subordinate to $\{U_\alpha\}$, we claim
$\sum \varphi_\alpha \langle -,- \rangle_\alpha$ gives an inner product on E. (exercise). $\boxtimes$.