Last time:

Pontryagin classes of real vector bundles

$E \to X$ real vec. bundle of rank k.

Form $E \otimes_{\mathbb{R}} \mathbb{C} \to X$ (fiberwise) complexification, complex rank k vec. bundle w/ an iso.

$$E \otimes_{\mathbb{R}} \mathbb{C} \underset{\text{conjugation}}{\cong} \overline{E \otimes_{\mathbb{R}} \mathbb{C}} \underset{\substack{\text{using} \\ \text{Hermitian metric} \\ \text{as in last time}}}{\cong} (E \otimes_{\mathbb{R}} \mathbb{C})^* . \quad (\star)$$

Taking Chern classes $c_i(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{2i}(X;\mathbb{Z})$, and $(\star)$ implies:

$$\underline{c_i(E \otimes_{\mathbb{R}} \mathbb{C})} = c_i((E \otimes_{\mathbb{R}} \mathbb{C})^*) \underset{\text{last time}}{=} (-1)^i \underline{c_i(E \otimes_{\mathbb{R}} \mathbb{C})}.$$

If i is odd, this tells us that $2c_i(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$ in $H^{4k+2}(X;\mathbb{Z})$.
$i = 2k+1$

Def: $E \to X$ real vec. bundle of rank n, define its k^{th} Pontryagin class by

$$p_k(E) := (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}(X;\mathbb{Z}).$$

By definition, $p_k(E) = 0$ if $2k > \text{rank}(E)$.

Whitney sum formula

E, E' vector bundles,

then $p_k(E \oplus E') := (-1)^k c_{2k}(E \oplus E')$

$$\underset{\substack{\text{(Whitney} \\ \text{sum for} \\ \text{Chern classes)}}}{=} (-1)^k \underbrace{\sum_{\substack{i+j=2k \\ i \geq 0 \\ j \geq 0}} c_i(E \otimes_{\mathbb{R}} \mathbb{C}) \smile c_j(E' \otimes \mathbb{C})} \qquad (c_0 = 1)$$

terms where both i,j even — terms where one of i,j odd

$$\sum_{\substack{r+s=k \\ r \geq 0, s \geq 0}} (-1)^{k=r+s} c_{2r}(E \otimes_{\mathbb{R}} \mathbb{C}) \smile c_{2s}(E' \otimes \mathbb{C}) \quad + \quad (2\text{-torsion terms})$$

$$= \sum_{\substack{r+s=k \\ r\geq 0 \\ s \geq 0}} p_r(E) \cup p_s(E') \quad + \quad (2\text{ torsion terms})$$

So, denoting $p(E) = 1 + p_1(E) + p_2(E) + \cdots$ total Pontryagin class

$$\Rightarrow \quad p(E \oplus E') = p(E)\,p(E') + (2\text{-torsion terms})$$

Special case: (towards $p(T\mathbb{C}P^n)$)

Say $F \to X$ complex rank n vec. bdle $\rightsquigarrow$ $F_{\mathbb{R}} \to X$ underlying real rank $2n$ vec. bdle.

By a fiberwise version of Lemma from last time,

$$F_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong F \oplus \overline{F} \underset{\substack{\text{Hermitian} \\ \text{metric}}}{\cong} F \oplus F^*.$$

so $$p_k(F_{\mathbb{R}}) = (-1)^k c_{2k}(F_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^k c_{2k}(F \oplus F^*)$$

$$\underset{\substack{\text{Whitney} \\ \text{sum formula} \\ \text{for Chern classes}}}{=} (-1)^k \sum_{\substack{i+j=2k \\ i,j\geq 0}} c_i(F) \cup \underbrace{c_j(F^*)}_{(-1)^j c_j(F)}$$

$$= (-1)^k \sum (-1)^j c_i(F) \cup c_j(F)$$

$$= \Big((-1)^k (1 + c_1(F) + c_2(F) + \cdots)(1 - c_1(F) + c_2(F) - \cdots)\Big)_{2k}$$

Ex: $p_k(\mathbb{C}P^n) = ?$

$L := L_{taut}.$

know that as complex vector bundles, $T\mathbb{CP}^n \oplus \underline{\mathbb{C}} \cong \underbrace{L^* \oplus \cdots \oplus L^*}_{n+1}$ (*)

$$\Rightarrow c(T\mathbb{CP}^n) = \underbrace{(1+h)}_{c(L^*)}{}^{n+1} \text{ in } H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}$$

complex conjugating (*) $\Rightarrow$

$$\overline{T\mathbb{CP}^n \oplus \underline{\mathbb{C}}} \cong \overline{\underbrace{L^* \oplus \cdots \oplus L^*}_{n+1}} \cong \underbrace{L \oplus \cdots \oplus L}_{n+1}$$

$$\overline{T\mathbb{CP}^n \oplus \underline{\mathbb{C}}} = \overline{T\mathbb{CP}^n} \oplus \underline{\mathbb{C}}$$

So, $c(\overline{T\mathbb{CP}^n}) = c(L)^{n+1} = (1-h)^{n+1}$ in

Now, $p_k(\mathbb{CP}^n) := p_k(T\mathbb{CP}^n) = (-1)^k c_{2k}(T\mathbb{CP}^n \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^k c_{2k}(T\mathbb{CP}^n \oplus \overline{T\mathbb{CP}^n})$

$$= (-1)^k \cdot (\deg 2k \text{ part of } (1+h)^{n+1}(1-h)^{n+1}).$$

So $$\boxed{p(\mathbb{CP}^n)} = \sum_{k\geq 0} (-1)^k \left((1+h)^{n+1}(1-h)^{n+1}\right)_{\deg 2k \text{ part}}$$

$$= \sum_{k\geq 0} (-1)^k \Big(\underbrace{(1-h^2)^{n+1}}_{\substack{\text{degree } 2k \text{ part of this is} \\ (-1)^k \cdot \deg 2k \text{ part of } (1+h^2)^{n+1}}} \Big)_{\deg 2k \text{ part}}$$

$$\boxed{= (1+h^2)^{n+1}} \quad \left(\text{in } H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}\right)$$

Special case: $n = 2m$ is even; we get $p(\mathbb{C}P^{2m}) = (1+h^2)^{2m+1}$

In particular, $p_m(\mathbb{C}P^{2m}) := p_m(T\mathbb{C}P^{2m}) = \binom{2m+1}{m} h^{2m}$

$$h^{2m} \in H^{4m}(\mathbb{C}P^{2m}; \mathbb{Z}) = \mathbb{Z}\langle h^{2m} \rangle$$

Pairing with the fundamental class $[\mathbb{C}P^{2m}]$ (complex oriented)

sends $h^{2m} \longrightarrow +1$, hence.

$$\underbrace{\langle p_m(\mathbb{C}P^{2m}), [\mathbb{C}P^{2m}] \rangle}_{\text{called the Pontryagin number } p_m[\mathbb{C}P^{2m}]} = \binom{2m+1}{m}.$$

More generally, have numerical invariants of manifolds from char. classes:

e.g., • $\boxed{\text{Stiefel-Whitney numbers}}$

$$w_I[X] := \langle \underbrace{\prod w_i^{n_i}(TX)}_{\text{lives in } H^{\dim X}(X; \mathbb{Z}/2)}, [X] \rangle \in \mathbb{Z}/2$$

cpct. manifold of dim $\dim X$

$I = \{n_i \geq 0\}$ s.t. $\sum i n_i = \dim X$

• $\boxed{\text{Pontryagin numbers:}}$

X cpct oriented, & $I = \{n_i \geq 0\}$ w/ $\sum 4 i n_i = \dim X$

(so $X = 4k$-dim'l)

$$\Rightarrow \; p_I[X] := \prod p_i^{n_i}[X] := \langle \prod p_i(TX)^{n_i}, [X] \rangle \in \mathbb{Z}.$$

($\prod p_i(TX)^{n_i} \in H^{\dim X}(X; \mathbb{Z})$ by hypothesis)

It turns out that

- Stiefel-Whitney numbers are invariants of X up to cobordism

  (so if $X \underset{\text{cob.}}{\sim} X'$ i.e., $\exists\ W^{\dim X+1}$ w/ $\partial W = X \amalg X'$,

  then $\prod w_i^{n_i}[X] = \prod w_i^{n_i}[X']$.

- Pontryagin numbers are invariants of X up to oriented cobordism

  ($X \underset{\substack{\text{oriented}\\ \text{cob.}}}{\sim} X'$ if $\exists$ oriented $W^{\dim+1}$ s.t.

  $\partial W \cong X \amalg \overline{X'}$ as oriented manifolds)

Cor: $\mathbb{CP}^{2m}$ is not the oriented boundary of any cpct. $(4m+1)$-dim'l oriented manifold. $\left(\iff \mathbb{CP}^{2m} \underset{\substack{\text{or.}\\ \text{cob.}}}{\not\sim} \emptyset\right)$

as pontryagin #'s of $\mathbb{CP}^{2m}$ non-zero.

(In contrast: $\mathbb{CP}^1 = S^2 = \partial B^3$).

Want to compute cohomology of $\underset{\substack{\shortparallel \\ G_k(\mathbb{C}^\infty)}}{\overset{\mathbb{Z}\text{ coeffs.}}{BU(k)}}$ resp. $\underset{\substack{\shortparallel \\ G_k(\mathbb{R}^\infty)}}{\overset{\mathbb{Z}/2\text{ coeffs.}}{BO(k)}}$.

(why? any char. class of cplx resp. real rank k vec. bdles is by naturality the pullback of a class in one of these two spaces. So if we know these coh. rings, know all char. classes).

We'll focus on $BU(k)$ ($BO(k)$ is parallel)

Start w/

$$\begin{array}{c} E_{taut} \\ \downarrow \\ BU(k) = G_k(\mathbb{C}^\infty) \end{array}$$

Idea: use splitting principle (for E_{taut}) to embed $H^*(G_k(\mathbb{C}^\infty))$ to $H^*(\text{simpler space})$

The usual proof of splitting principle produces a space $Z = \mathbb{F}(E_{taut}) = \mathbb{F}_k(\mathbb{C}^\infty)$

fiberwise flags in E_{taut}.

- [Hatcher Ch. 4] proceeds by using this space & computing $H^*(\mathbb{F}(E_{taut}))$ by applying Leray-Hirsch to various fibrations, e.g.,

$$\mathbb{F}_k(\mathbb{C}^\infty) \longrightarrow \mathbb{CP}^\infty \quad \text{w/ fiber } \mathbb{F}_{k-1}$$
$$(L_1, \dots, L_k) \longmapsto L_1$$

- We'll take a shortcut, appealing to a different (simple) splitting map: ([Husemoller, Fibre Bundles])

Consider $X = \underbrace{\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty}_{k \text{ times}}$. On X we have the rank k vector bundle

$$E := L^1_{taut} \times \cdots \times L^k_{taut}$$

equivalently, $E = \bigoplus_{i=1}^{k} \pi_i^* L_{taut}$; $\pi_i : X \to \mathbb{CP}^\infty$ ith projection.

Since $BU(k)$ classifies rank k vector bundles, $\exists!$ (up to homotopy)

$$f_k : X \to BU(k) \quad \text{w/} \quad f_k^* E_{taut} = E = \bigoplus_{i=1}^{k} \pi_i^* L_{taut}.$$

Prop: f_k is a splitting map for E_{taut}, i.e., , and f_k^* is injective.

Pf: Let $s: Z \to BU(k)$ be any splitting map for E_{taut} ($\exists$ by splitting principle), i.e., $s^* E_{taut} = L_1 \oplus \cdots \oplus L_k$ for $L_i \to Z$; & s^* is injective.

Each L_i is a complex line bundle, so is classified by a map $g_i : Z \to \mathbb{CP}^\infty$.
(i.e., $g_i^* L_{taut} = L$).

$\rightsquigarrow$ get $g = (g_1, \dots, g_k) : Z \longrightarrow (\mathbb{CP}^\infty)^k$, and observe that

$$g^*\left(E = \bigoplus_{i=1}^{k} \pi_i^* L_{taut}\right) = \bigoplus_{i=1}^{k} g^* \pi_i^* L_{taut}$$

$$= \bigoplus_{i=1}^{k} g_i^* L_{taut} = \bigoplus_{i=1}^{k} L_i = s^* E_{taut}$$

In particular, $f_k \circ g : Z \xrightarrow{g} (\mathbb{CP}^\infty)^k \xrightarrow{f_k} BU(k)$

classifies $s^* E_{taut}$, because $(f_k \circ g)^* E_{taut}$

$$= g^* f_k^* E_{taut}$$

$$= g^*\left(E = \bigoplus_{i=1}^{k} \pi_i^* L_{taut}\right)$$

$$= \bigoplus_{i=1}^{k} L_i = s^* E_{taut}$$

i.e., $(f_k \circ g)^* E_{taut} \cong s^* E_{taut}$

Since classifying maps are unique up to homotopy

$\Rightarrow f_k \circ g \simeq s.$

$\Rightarrow s^* = g^* f_k^*$. But s^* is injective $\Rightarrow f_k^*$ is injective. $\boxed{E}$

Using this:

Thm: Let $c_i := c_i(E_{taut}) \in H^{2i}(BU(k);\mathbb{Z})$. Then, the classes c_i are algebraically independent for $i = 1, \dots, k$ & moreover

$$H^\bullet(BU(k);\mathbb{Z}) \underset{\text{as rings}}{\cong} \mathbb{Z}[c_1, \dots, c_k] \qquad (|c_j| = 2j).$$

Cor: Each char. class $\phi : \mathrm{Vect}^k_{\mathbb{C}}(-) \to H^\bullet(-;\mathbb{Z})$ must have the form $E \longmapsto q(c_1(E), \dots, c_k(E))$ where q is a poly. uniquely determined by the class (e.g., $q = \phi(E_{taut}) \in H^\bullet(BU(k);\mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_k]$)

Betti #: rank $H^i = b_i$

Cor: $b_{2k+1}(BU(n)) = 0$, and $b_{2k}(BU(n)) = \mathrm{rk}\, H^{2k}(BU(n))$

$= \dim$ (deg $2k$ part of $\mathbb{Z}[c_1, \dots, c_n]$, $|c_i| = 2i$).

$=$ # of monomials $c_1^{r_1} \cdots c_n^{r_n}$ of degree $2k = 2(r_1 + 2r_2 + 3r_3 + \cdots + nr_n)$

$=$ # of n-tuples $(r_1, \dots, r_n)$ with $k = r_1 + 2r_2 + \cdots + nr_n$.

$=$ # of unordered partitions of k into at most n integers.

$\Longleftrightarrow \{k_1, \dots, k_n\}$ w/ $k_1 \le k_2 \le \cdots \le k_n$ w/ $\sum k_i = k$.

via

$$(r_1, \dots, r_n) \longleftrightarrow \overbrace{r_n}^{k_1} \le \overbrace{r_n + r_{n-1}}^{k_2} \le \overbrace{r_n + r_{n-1} + r_{n-2}}^{k_3} \le \cdots \le \overbrace{r_n + \cdots + r_1}^{k_n}.$$

<u>Pf of Thm:</u> Let $f_k : \underbrace{\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty}_{k} \to BU(k)$ be the splitting map from above. (so $f_k^* E_{taut} \cong \bigoplus_{i=1}^{k} \pi_i^* L_{taut}$ / f_k^* injective.

By above:

$$f_k^* : H^*(BU(k);\mathbb{Z}) \xhookrightarrow{\text{injective}} H^*((\mathbb{CP}^\infty)^k;\mathbb{Z}) \underset{\text{künneth}}{\cong} \mathbb{Z}[h_1, \dots, h_k]$$

$|h_i| = 2$ for each i.

So just need to calc. im(f_k^*).

Now consider action of symmetric group $\Sigma_k \circlearrowright (\mathbb{CP}^\infty)^k$ permuting factors..

$\Rightarrow$ action on $H^*((\mathbb{CP}^\infty)^k)$ permutes $(h_1, \dots, h_k)$.

Note $E = \bigoplus_{i=1}^{k} \pi_i^* L_{taut}$ is invariant under such an action,

that is $\sigma^* E \cong E$ for any $\sigma \in \Sigma_k$.

$\Rightarrow f_k \circ \sigma$ still classifies E $\quad ((f_k \circ \sigma)^* E_{taut} \cong E)$

$\underset{\text{classifying map uniqueness}}{\Rightarrow}$ $f_k \circ \sigma \simeq f_k$ i.e., $\sigma^* f_k^* = f_k^*$

i.e., im(f_k^*) lands in <u>symmetric polynomials</u> in <u>$h_1, \dots, h_k$</u>.

Let's calculate $f_k^*(c(E_{taut})) = c(f_k^* E_{taut} = E)$

$$= c\left(\bigoplus_{i=1}^{k} \pi_i^* L_{taut}\right)$$

$$\underset{\text{Whitney sum}}{=} \prod_{i=1}^{k} c(\pi_i^* L_{taut}) = \prod \pi_i^* c(L_{taut})$$

$$= \prod \pi_i^* (1+h)$$

$$= \prod (1+h_i)$$

$$= \underbrace{(1+h_1)\cdots(1+h_k)}$$

so $f_k^* c_i = \deg 2i \text{ part of }$ ↑

$$= \left(\sum_{\substack{J \subseteq \{1,\dots,k\} \\ |J| = i}} \prod_{j \in J} h_j \right) = \sigma_i =$$

i^{th} elementary sym. poly. in $h_1, \dots, h_k$

<u>Fact</u>: There are no algebraic relations between elementary sym. polys, & every symmetric polynomial can be uniquely written as a poly. in $\sigma_1, \dots, \sigma_k$

$\Rightarrow$

$$\{\text{all symmetric polys.}\} \supseteq \operatorname{im}(f_k^*) \supseteq \{\text{subring of } \mathbb{Z}[h_1, \ldots, h_k] \text{ gen. by } \sigma_1, \ldots, \sigma_k\}$$

$$= \qquad\qquad = \qquad\qquad \cong \mathbb{Z}[\sigma_1, \ldots, \sigma_k]$$

$\deg \sigma_i = 2i.$

w/ $c_i \xmapsto{f_k^*} \sigma_i$. As f_k^* injective,

$$\Rightarrow H^\circ(BU(k); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_k]$$