Math 520 Take-home Midterm

Due Thursday, April 9, 2020 at 5 pm (by e-mail to Instructor and TA)

Please remember to write down your name on your take-home midterm. You are welcome to use any results we have used/stated/proved in class or the homework. You may consult the textbook, course notes, and previous homework assignments. (You can consult other references for background, but you must solve these problems completely on your own) *You may not discuss your solutions with other people except for the course instructor and TA.*

1. Let $f : B_1(0) \to \mathbb{C}$ be a non-constant complex differentiable map with $\mathrm{Re}(f(z)) \geq 0$. Prove $f(z)$ is never purely imaginary.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic (i.e., complex differentiable) function. Recall that a *period* of f is any complex number c such that $f(z+c) = f(z)$ for all c. Let $P \subset \mathbb{C}$ denote the set of all periods of f; note that P is necessarily a *lattice* in $\mathbb{C}$: if $c, d \in P$ then $mc + nd \in P$ too for all integers m, n.
   a. Prove that if the function f is non-constant, then the intersection of P with any bounded subset of $\mathbb{C}$ is necessarily finite.
   b. Suppose that one can find $\omega_1, \omega_2 \in P$ non-zero complex numbers which are linearly independent over $\mathbb{R}$ (i.e., neither element is a real scalar multiple of the other). Prove that f is constant.

3. Let f be a meromorphic function on $\mathbb{C}$ such that, for $|z| > R$, f has no poles and satisfies $|f(z)| \leq M$ for some constant M. Prove that f must be a rational function, i.e., there must exist polynomials p and q with $f(z) = \frac{p}{q}$.

4. Determine the type of every isolated singularity of the following function (including at infinity, and order of the pole if there is a pole), with proof:
$$f(z) = \frac{1}{e^z - 1} + \frac{z^3}{(z-2)^2(z+i)}.$$

5. Say $f(z) = \sum_n a_n z^n$ has radius of convergence $R > 0$. Prove that $g(z) = \sum \frac{a_n}{n!} z^n$ has radius of convergence infinity and show that for all $0 < r < R$, there exists an M such that $|g(z)| \leq Me^{\frac{|z|}{r}}$.