Week 7, Wednesday.

Def: An A_∞ cat. $\mathcal{C}$ is pre-tr. if

- every closed c, $X \xrightarrow{c} Y$ extends to an exact triangle.

∃ shifts

- given any $K \in \operatorname{ob}\mathcal{C}$, ∃ an object $K[1]$ with

$$\hom^\bullet_{\mathcal{C}}(X, K[1]) = \hom^\bullet(X, K)\,[1]$$

$$= \cancel{\hom^{\bullet+1}(X[1], K)}$$

$$\& \ \hom^\bullet_{\mathcal{C}}(X[1], K)[1] = \hom^\bullet_{\mathcal{C}}(X, K).$$

Prop: Any (A_∞) category $\mathcal{C}$ (cohomologically unital), has an (~~essentially~~ ~~quasi~~ unique up to q.equiv.) pre-tr. hull.

Two methods of doing this:

(1) A_∞ modules over $\mathcal{C}$. The construction produces a category, $\mathrm{Mod}(\mathcal{C})$, pre-triangulated & moreover split-closed, ~~equip~~ with a cohom. full & faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Mod}(\mathcal{C})$.

Recall given $\mathbb{K}$ field, have a dg cat. of Chain cplxes $\mathrm{Chain}_{\mathbb{K}}$ ($\mathrm{Ch}_{\mathbb{K}}$).

$$\operatorname{ob} \mathrm{Ch}_{\mathbb{K}} = \underset{\text{gr. vect}}{C^\bullet} \circlearrowleft \underset{\text{deg. 1}}{d} \ \text{chain cplx. over } \mathbb{K}$$

$$\mathrm{Mor}(C_1, C_2) = \hom_{\text{vect}}(C_1, C_2) \circlearrowleft d \quad \text{all homs}, \quad \text{Composition as usual.}$$

$$d(F) = F \circ d_{C_1} \mp d_{C_2} \circ F.$$ So closed morphisms are chain maps.

Def: A (right) A_∞ module over $\mathcal{C}$ ($\mathcal{C}$ A_∞ cat./$\mathbb{K}$) is an A_∞ functor $F : \mathcal{C}^{op} \longrightarrow \mathrm{Ch}_{\mathbb{K}}$.

Rmk: The cat. of A_∞ functors $\mathcal{C} \to \mathcal{D}$ ~~is~~ are objects of another A_∞ cat., $\text{nu-fun}(\mathcal{C}, \mathcal{D})$; this cat. is dg if $\mathcal{D}$ is.

In particular, $\mathrm{Mod}(\mathcal{C}) := \text{nu-fun}(\mathcal{C}^{op}, \mathrm{Ch}_{\mathbb{K}})$ is a dg category.

Explicitly spell this out:

Def (redux): A right A_∞ module is the data of —

Rmk: If $\mathcal{C}$ has one object X $\longleftrightarrow$ $A := \hom_{\mathcal{C}}(X,X)$ A_∞ alg.

Then, an A_∞ module over $\mathcal{C}$ $\longleftrightarrow$ A_∞ module over A

gr. vector space $M := \mathcal{M}(X) \circlearrowleft \mu_M^{1|0}$ w/

maps $\mu^{1|d} : M \otimes A^{\otimes d} \to M\,[1-d]$

satisfying - -

⊗ what are the morphisms in $\mathrm{Mod}(\mathcal{C})$? (they're "$A_\infty$ premodule homomorphisms").

Def: A pre-morphism of A_∞ modules $M_0 \to M_1$.

⋮

Yoneda embedding: There is a natural A_∞ functor.

$$Y_R : \mathcal{C} \longrightarrow \mathrm{Mod}(\mathcal{C})$$

which on objects sends ~~Sending~~ $K \longmapsto Y_K^{(r)} := \hom_{\mathcal{C}}(-,K) : \mathcal{C}^{op} \to Ch_{\mathbb{K}}$.

For instance $Y_R^1 : \hom(K,L) \longrightarrow \hom(Y_K^r, Y_L^r)$

$\phi \longmapsto$ associates the ~~map~~ chain $\{\mu^{d}(\phi, -, -, -)$:

$\hom(X,K) \to \hom(X,L)$

$\mu^3(\phi, -, -)$:

$\hom(X_0,K) \otimes \hom(X_1,X_0) \to \hom(X_1,L)$

etc.

Prop: If $\mathcal{C}$ is hom. unital, Y is fully faithful.

(exercise: prove this when $\mu^1, \mu^{d \geq 3} = 0$, & $\mathcal{C}$ has strict identity morphisms, so is a usual category enriched in vector spaces).

Note that $\mathrm{Mod}(\mathcal{C})$ inherits the following operations from $Ch_{\mathbb{K}}$.

- can take direct sums of objects

$$M_0, M_1 \rightsquigarrow M_0 \oplus M_1 \text{ is a } A_\infty \text{ module.}$$

- can tensor w/ a fixed chain complex/or vect space:

$$M \text{ } A_\infty \text{ module} \rightsquigarrow V \otimes M \text{ } A_\infty \text{ module.}$$

- can shift objects:

$$M[1](X) := \underbrace{M(X)}^{\text{ch. cplx.}}[1].$$

- Mapping cones: Given a closed morphism $f : K^\bullet \to L^\bullet$ of ch. cplxes, have a ch. cplx. $\mathrm{Cone}(f) := K^\bullet[1] \oplus L^\bullet$ with $d_{\mathrm{Cone}(f)} = \begin{pmatrix} d_K & 0 \\ f & d_L \end{pmatrix}$.

fitting into a triangle:

$$\begin{tikzcd} K^\bullet \arrow[rr, "f"] & & L^\bullet \arrow[dl, "i"', "\text{inclusion}"] \\ & \mathrm{Cone}(f) \arrow[ul, "pr", "\text{projection}"'] & \end{tikzcd}$$

(check: in this exact triangle, there's a non-trivial Massey product ~~between~~ between pr, i & f.
Namely, $pr \circ i = 0$ on the nose, but $i \circ f = \delta_{\mathrm{Cone}(f)} \circ pr_{K^\bullet}$

& $\mathrm{Massey}(pr, i, f) := \mathrm{id}_{K^\bullet}$.

This is why, essentially, the abstract triangle has a μ^3.)

Similarly given M_0, M_1 & $F: M_0 \to M_1$: closed,

have $\mathrm{Cone}(F) := M_0[1] \oplus M_1$, with

$$\mu^{1|d}_{\mathrm{Cone}(F)}(m_0 \oplus m_1, \_\_\_) = \begin{pmatrix} \mu^{1|d}_{M_0}(m_0, \_\_\_) \\ F^{1|d}(m_0, \_\_\_) + \mu^{1|d}_{M_1}(m_1, \_\_\_) \end{pmatrix} \oplus$$

- <u>idempotents</u>: Given $p \in \hom(C^\bullet, C^\bullet)$ <u>closed</u> w/ $p^2 = p$, can split off

$$C^\bullet := \text{~~[illegible]~~}\ \mathrm{im}(p) \circlearrowleft_{p\partial} \oplus \ker(p) \circlearrowleft_{\partial - p\partial}.$$

Similarly, given M, equipped with an "idempotent up to homotopy" $\overset{\text{equiv.}}{\underset{\text{[Seidel]}}{\Longleftrightarrow}}$ a morphism $[p] \in \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, M)$ w/ $[p]^2 = [p]$,

can split off from M its <u>abstract image</u> module $\bar{\imath} \frown$

$\Rightarrow \mathrm{Mod}(\mathcal{C})$ is triangulated & split-closed.

<u>Def</u>: • The pre-triangulated envelope ~~of~~ of $\mathcal{C}$ is the closure of $\mathcal{Y}_R(\mathcal{C})$ in $\mathrm{Mod}(\mathcal{C})$ under finitely many mapping cones & shifts, <u>$tw(\mathcal{C})$</u>

- The <u>split-closed pre-triangulated</u> envelope, denoted $\mathrm{perf}(\mathcal{C})$ or $tw^\pi(\mathcal{C})$ is the closure of $\mathcal{Y}(\mathcal{C})$ in $\mathrm{mod}(\mathcal{C})$ under all of the above & idempotent $\overset{\text{splittings}}{\underline{\text{decompositions}}}$.

Another <u>explicit</u> construction: (~~[illegible]~~ best behaved when $\mathcal{C}$ is strictly unital): twisted complexes, $tw(\mathcal{C})$ & its idempotent closure $tw^\pi(\mathcal{C})$.

Step 1: Additive enlargement:

Define $\mathrm{ob}(\Sigma \mathcal{C}) = \left(\bigoplus V_i \otimes L_i \right)$. , morphisms the linear extension of morphisms in $\mathcal{C}$.

$\uparrow$ gr. vect or Chain cplexes ; $\downarrow$ objects of $\mathcal{C}$

Ex: $\hom_{\Sigma\mathcal{C}}(V_0 \otimes L_0, V_1 \otimes L_1)$

$:= \hom_{\mathrm{vect}}(V_0, V_1)$

$\otimes \hom_{\mathcal{C}}(L_0, L_1)$,

etc.

Notes $\Sigma\mathcal{C}$ is closed by shifts already

$$\left(L[1] := L \otimes \mathbb{k}[1] \right).$$

$\uparrow$ concentrated in degree 1.

(Kontsevich, Bondal-Kapranov) by rest.

Def: A twisted complex in $\mathcal{C}$ is ~~collection~~ a pair ~~of ob~~ (X, δ_X), X an object of $\Sigma\mathcal{C}$, & $\delta_X \in \hom^1_{\Sigma\mathcal{C}}(X, X)$ an upper-triangular element (w.r.t. some ordering) satisfying:

$$\underbrace{\sum_{k \geq 0} \mu^k(\delta_X, \dots, \delta_X)}_{\text{finite sum, by upper-triangularity}} = 0.$$

Ex:

$$\begin{tikzcd}
X_0 \arrow[r, "\delta_X^{01}"] \arrow[rr, bend right, "\delta_X^{02}"'] & X_1 \arrow[r, "\delta_X^{12}"] & X_2
\end{tikzcd}$$

the equation is

$$\mu^2(\delta_X^{12}, \delta_X^{01}) = \mu^1(\delta_X^{02})$$

means, say if X_0, X_1, X_2 chain cplxes, that

$$\delta_X^{12} \circ \delta_X^{01} = d_{X_2} F + \overset{\overset{\delta_X^{02}}{\shortparallel}}{F} d_{X_0}$$

is ch. htopic to 0, with fixed chain htopy.

There is a natural A_∞ str. ~~Twisted~~ complexes induced on twisted cplexes ~~for~~ in $\mathcal{C}$, ~~so~~ call result $\mathrm{Tw}(\mathcal{C})$

$$\mu^k(x_k, \dots, x_1)$$

$$\rotatebox{90}{$\in$}$$

$$\hom((X_{k-1}, \delta_{k-1}), (X_k, \delta_k)) \otimes \dots \otimes \hom((X_0, \delta_0), (X_1, \delta_1))$$

exercise: A_∞ cat.

$$:= \sum \mu(\delta_{X_k}, \dots, \delta_{X_k}, x_k, \delta_{X_{k-1}}, \dots, \delta_{X_{k-1}}, x_{k-1}, \dots, x_1, \delta_{X_0}, \dots, \delta_{X_0})$$

arbitrary

Ex: $f : X \to Y$ closed morphism.

Then $\mathrm{Cone}(f) := X[-1] \oplus Y$, is a twisted cplx, w/

$$\delta_{\mathrm{Cone}(f)} = \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}.$$

why? $\mu^1(\delta) = 0$ exactly b/c f is closed.

$\mathrm{Cone}(f) \xrightarrow{g} Z$ a morphism

what is this? $\hom(\mathrm{Cone}(f), Z) = \hom^{\bullet+1}(X, Z) \oplus \hom^{\bullet}(Y, Z)$,

$g := (g_1, g_2)$

g closed if $\mu^1(g_1) = 0$

$\mu^1(g_2) + \mu^2(f, g_1) = 0$.

$$\circlearrowright \quad \mu^1 := \begin{pmatrix} \mu^1(\cdot) & - \\ \mu^2(f, -) & \mu^1(\cdot) \end{pmatrix}$$

Given such a g, get a new twisted complex

$$\mathrm{Cone}(g) := X[-2] \oplus Y[-1] \oplus Z : \quad \delta_X = \begin{pmatrix} 0 & 0 & 0 \\ f[-1] & 0 & 0 \\ g_1 & g_2 & 0 \end{pmatrix}.$$

$$\begin{tikzcd} X \arrow[rr, hook] & & Y \arrow[dl] \\ & Z \arrow[ul] & \end{tikzcd}$$

Lem: A triangle in $\mathrm{Tw}\mathcal{A}$ is an exact triangle iff there is a ~~quasi~~ (quasi-)isomorphism of objects in $\mathrm{Tw}\mathcal{A}$ $Z \simeq \mathrm{Cone}(c)$.

Lem: $\mathrm{Tw}\mathcal{A}$ is pre-triangulated, & $H^0(\mathrm{Tw}\mathcal{A})$ is a triangulated category. ($\mathrm{Tw}\, H^0(\mathcal{A})$!!)

$\mathrm{Tw}^{\pi}\mathcal{A}$ idempotent closure.

~~Def~~: Notes: Any embedding $\mathcal{A} \subseteq \mathcal{C}$ induces an embedding $\mathrm{tw}^{\pi}(\mathcal{A}) \subseteq \mathrm{tw}^{\pi}(\mathcal{C})$ ~~$\mathrm{perf}(\mathcal{A}) \subseteq \mathrm{perf}(\mathcal{C})$~~

& $\mathrm{tw}(\mathcal{A}) \subseteq \mathrm{tw}(\mathcal{C})$.

Def: $\mathcal{A}$ split-generates (generates, resp.) $\mathcal{C}$ is this embedding is a q. equivalence.

In a given A_∞ category $\mathcal{C}$, it's not true that every closed morphism $X \xrightarrow{f} Y$ extends to an exact-triangle.

$$\begin{tikzcd} 0 \arrow[rr] & & L \arrow[dl, "?"] \\ & L \arrow[ul, "{[1]}"] & \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r] & 0 \arrow[dl, dashed] \\ L[1] \arrow[u] & \end{tikzcd}$$

Def: An A_∞ category is <u>pre-triangulated</u> if

- every closed morphism extends to an exact-triangle
- ~~and~~ there exist <u>shifts</u>: given any $K \in \text{ob}\,\mathcal{C}$, $\exists$ an object "the shift" $K[1)$ with $\forall X, K$ $\hom^\bullet_{\mathcal{C}}$ ~~$(K[1], \mathcal{C})$~~ $=$

$$\hom^\bullet_{\mathcal{C}}(X, K[1]) = \hom^{\bullet-1}_{\mathcal{C}}(X, K) = \hom^{\bullet+1}(X[1], K)$$

$$\hom^\bullet_{\mathcal{C}}(X, K)[1]$$

(~~is a~~ ~~manner~~ & compatibility of higher products.)

The Fukaya category is not a priori pre-triangulated. ~~But,~~ It is often very advantageous to take its <u>triangulated hull</u>:

Two ways of doing this:

(1) A_∞ modules:

Def: A (right) (~~left~~) <u>A_∞ module</u> over $\mathcal{C}$ is an A_∞ functor $\mathcal{M}$ ~~$\mathcal{C}$~~ $: \mathcal{C}^{op} \to \text{Chain}$ (chain complexes (is a dg, hence $\exists$ A_∞ category).

↑ <u>contravariant</u>.

concretely, this is the data of

- For every object $X \in \text{ob}\,\mathcal{C}$, a chain complex

$$\mathcal{M}^\bullet(X) \circlearrowleft \mu^{0|1}_{\mathcal{M}} \leftarrow \text{squares to zero}$$

- For every $X_0, X_1 \in \text{ob}\,\mathcal{C}$, a "multiplication map"

$$(\mathcal{M}^1 : \hom_{\mathcal{C}}(X, Y) \to \hom_{ch.}(\mathcal{M}(Y), \mathcal{M}(X)))$$

$$\longleftrightarrow \mu^{1|1}_{M} : \cancel{\hom}\,\mathcal{M}(Y) \otimes \hom(X, Y) \to \mathcal{M}(X)$$

- higher multiplications (M^d) : $\forall X_0, \ldots, X_d$

$$\longleftrightarrow \mu^{1|d}_{M} : \mathcal{M}(X_d) \otimes \hom(X_{d-1}, X_d) \otimes \cdots \hom(X_0, X_1) \to M(X_0)$$

satisfying A_∞ module equations.

$$\sum \pm \mu_M^{1|d-j+1}(\underline{m}, x_d, \ldots, \mu^{i*}(x_{i+s}, \ldots, x_{s+1}), x_s, \ldots, x_1)$$

$$+ \sum \pm \mu_M^{1|d-i}(\mu_M^{1|i}(\underline{m}, x_d, \ldots, x_{\cancel{i+s+1}\ d-i+1}), x_{d-i}, \ldots, x_1) \qquad (*)$$

$$= 0.$$

A∞ modules themselves form an A∞ (in fact, dg) category.

(General fact: ~~the category of fu~~ A∞ functors between $\mathcal{C}$ and $\mathcal{D}$ form an A∞ category $\mathcal{X}$, denote nu-fun$(\mathcal{C}, \mathcal{D})$. ← when $\mathcal{D}$ is DG, this category is too.)

(If $\mathcal{C}$ has one object X $\longleftrightarrow$ $A := \hom_{\mathcal{C}}(X, X)$ A∞ algebra.

Then, an A∞ module is a graded ~~vector space~~ chain complex

$M \quad (:= M(X))$

equipped w/ maps $\mu_M^{1|d} : M \otimes A^d \to M$ deg $1-d$

satisfying $(*)$.)

A pre-morphism of modules $M_0 \to M_1$ of degree k is the data of maps, ∀ tuples of objects $X_0, \ldots, X_d$.

$$M_0(X_d) \otimes \hom_{\mathcal{C}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom(X_0, X_1) \longrightarrow M_1(X_1)$$

deg $k-d$.

shorthand ~~F~~ an elt. of $\prod_d \hom_{\mathrm{vect}}(M_0 \otimes \mathcal{C}^{\otimes d}, M_1)$

Can compose pre-morphisms: $F_0 : \bigoplus_d M_0 \otimes \mathcal{C}^d \longrightarrow M_2$ induces

$\wedge \quad M_0 \otimes T\mathcal{C}_d \longrightarrow M_1$

~~$F_0 : \bigoplus_d M_0 \otimes \mathcal{C}^d \to M_1$~~

$$\hat{F}_0 : M_0 \otimes T\mathcal{C} \to M_1 \otimes T\mathcal{C}$$

by $\hat{F}_0(\underline{m}, x_d, \ldots, x_1) = \sum F_0(\underline{m}, x_d, \ldots, x_{d-i+1}) \otimes x_i \otimes \cdots \otimes x_1$.

& define F_1 "∘" F_0 by $F_1 \circ \hat{F}_0$.

~~Moreover~~

define $\hom_{mod(\mathcal{C})}(M_0, M_1) :=$ space of pre-morphisms $M_0 \to M_1$

$$= \prod_{X_0, \dots, X_d} \hom_{vect}\left(M_0(X_d) \otimes \hom_{\mathcal{C}}(X_{d-1}, X_d) \otimes \dots \otimes \hom_{\mathcal{C}}(X_0, X_1), M_1(X_0) \right) .$$

composition defined.

Differential: given a pre-morphism F, define $\mu^1_{mod(\mathcal{C})}(F) := F \circ \widehat{(\mu^{1|\cdot}_M)} \mp \mu^{1|\cdot} \circ \widehat{F}$.

~~closed~~ = namely $\mu^1_{mod(\mathcal{C})}(F)$.

Yoneda embedding: There is a natural A_∞ functor

$$\mathcal{C} \longrightarrow Mod(\mathcal{C})$$

(right) Yoneda module over K.

sending $K \longmapsto \mathcal{Y}^{(r)}_K$

ex: $\hom(K, L) \mapsto \hom(\mathcal{Y}^r_K, \mathcal{Y}^r_L)$

$\phi \longmapsto \oplus \hom(\hom(X, K), \hom(X, L))$ "mult. by ϕ".

A_∞ Yoneda lem.

Lem: this extends to the data of an A_∞ functor $Y_R : \mathcal{C} \longrightarrow Mod(\mathcal{C})$ where

Lem: Moreover, if $\mathcal{C}$ is hom unital, Y is fully faithful.

ob $Chain_{\mathbb{K}}$ = $C^\bullet$ & d chain cplx over $\mathbb{K}$ $d^2 = 0$

$Mor(C_1, C_2) := \hom_{vect}(C_1, C_2)$ all hom

$d(F) = F \circ d_{C_1} \mp d_{C_2} \circ F$.

so closed morphisms are chain maps.

Note that $Mod(\mathcal{C})$ inherits the following operations from $Chain_{\mathbb{K}}$:

- can take direct sums of ~~objects~~ objects $\qquad \oplus : chain_{\mathbb{K}} \otimes chain_{\mathbb{K}} \to chain_{\mathbb{K}}$.
- can tensor an object with a fixed chain cplx. $\qquad \otimes V : Chain_{\mathbb{K}} \longrightarrow Chain_{\mathbb{K}}$ dg functor
- can shift objects $(S : Chain_{\mathbb{K}} \to Chain_{\mathbb{K}})$

$$(SC)^\bullet := C^\bullet[1] = C^{\bullet - 1} .$$

- Mapping cones: given ~~f~~ $f : K^\bullet \to L^\bullet$, define

$$Cone(f) := K^\bullet[1] \oplus L^\bullet, \quad \text{with } d_{Cone(f)} = \begin{pmatrix} d_K & 0 \\ f & d_L \end{pmatrix}$$

& $K^\bullet \to L^\bullet$ fit into an exact triangle.

coeffs

- idempotents: Given $p \in \hom(C^\bullet, C^\bullet)$ closed w/ $p^2 = p$, can ~~take~~ split.

$$C^\bullet := \ker p \oplus im(p)$$

$\circlearrowleft$ ~~p~~ $\ker p \qquad \circlearrowleft$ on $im\, p$

Similarly, given an M, ~~ack~~ ~~equipped~~ w/ "an idempotent up to homotopy", ~~eqv~~ $\xleftrightarrow[\text{[Seidel]}]{\text{equiv. data}}$ an $\cdot 0$ morphism $[p] \in H^0_{mod}(M, M)$ w/ $[p]^2 = [p]$,