Math 641 Homework 5: Characteristic classes

Due Wednesday, April 21, 2021 by 5 pm

Please remember to write down your name and ID number. We will refer to pages/sections from Milnor-Stasheff's *Characteristic classes* by [MilnorStasheff], and sections in Hatcher's *Algebraic Topology* by [HatcherAT], and Cohen's *The topology of fiber bundles* by [Cohen].

1. *Euler class and Euler characteristic.* For what follows, let M be a smooth compact oriented manifold of dimension n. The Euler class of M is by definition $e(M) := e(TM)$. There is an associated *Euler number* $e[M] := \langle e(M), [M] \rangle$ which is indepdnent of orientation (as $e(\bar{M}) = -e(M)$ and $[\bar{M}] = -[M]$. The goal of this exercise is to show that $e[M] = \chi(M)$ where $\chi(M) = \sum_{i=1}^{\dim(M)} (-1)^i \dim_{\mathbb{Q}} H^i(X; \mathbb{Q}) = \sum_{i=1}^{\dim(M)} (-1)^i \dim_{\mathbb{Q}} H_i(X; \mathbb{Q})$ is the Euler characteristic.
   (a) First, prove that the normal bundle to the diagonal embedding $\Delta : M \to M \times M$ is precisely the tangent bundle to M. Deduce using assertions made in class that the Poincaré dual of $\Delta_*[M]$ in $H^*(M \times M)$ is the image of the Thom class $u \in H^*(TM, (TM)^0) \cong H^*_c(TM)$ using a choice of tubular neighborhood of $\Delta(M)$, $\psi : TM \cong U \hookrightarrow M \times M$.

   (b) Deduce that the Euler number $e[M]$ is equal to $\Delta_*[M] \bullet \Delta_*[M]$, where for α, β integer homology classes in an oriented manifold Q with $\deg(\alpha) + \deg(\beta) = \dim(Q)$, recall that the *intersection* number $\alpha \bullet \beta := \langle PD(\alpha), \beta \rangle = \langle PD(\alpha) \cup PD(\beta), [Q] \rangle \in \mathbb{Z}$.

   (c) Under the Künneth equivalence (assume we're working over $\mathbb{Q}$) $H^*(M \times M) \cong H^*(M) \otimes H^*(M)$, show that $PD(\Delta_*[M])$ is equal to $\sum_i \alpha_i \otimes \alpha^i$, where $\{\alpha_i \in H^{j_i}(M)\}$ is any basis for rational cohomology and $\alpha^i \in H^{n-j_i}(M)$ denotes the dual (with respect to the Poincaré duality pairing) basis. *Hint: Prove and use the fact that for any α, β γ in $H^*(M)$ (in particular $\gamma = [M]$), $(\alpha \otimes \beta) \cap \Delta_*(\gamma) = \Delta_*((\alpha \cup \beta) \cap \gamma)$.*

   (d) Conclude that $\Delta_*[M] \bullet \Delta_*[M] = \sum_{i=0}^{\dim(M)} (-1)^i \dim H^i(M; \mathbb{Q}) = \chi(M)$. Hence, $\langle e(M), [M] \rangle = \chi(M)$.

   (e) Prove using Euler classes the *hairy ball theorem*: S^n has a nowhere vanishing vector field if and only if n is odd.

2. *Pontryagin classes, Euler classes, Chern classes.* For what follows, recall that any complex vector bundle E, when thought of as a real vector bundle $E_{\mathbb{R}}$, is canonically oriented (using the complex orientation), and therefore has an Euler class.
   (i) Let $L_{taut} \to \mathbb{CP}^k$ be the tautological complex line bundle. If $S(L_{taut})$ denotes the associated S^1-bundle, observe first that $S(L_{taut}) = S^{2k+1}$, and the bundle map $S^{2k+1} \to \mathbb{CP}^k$ is precisely the quotient by the multiplication of S^1 = unit complex numbers. Using this, show (using the Gysin sequence for $S(L_{taut})$) that $e(L_{taut}) \in H^2(\mathbb{CP}^k; \mathbb{Z})$ must be a generator, i.e., must be $\pm h$. Find a way to further check that in fact $e(L_{taut}) = -h = c_1(L_{taut})$ (one option is to appeal to problem #1 to pin down

$e(T\mathbb{CP}^1)$ and to deduce $e((L_{taut})_{\mathbb{R}})$ from there.

(ii) Using the above fact, prove that for any complex vector bundle E of complex rank k over any space X (so $E_{\mathbb{R}}$ is a real oriented rank $2k$ bundle), $c_k(E) = e(E_{\mathbb{R}})$. That is, the Euler class of $E_{\mathbb{R}}$ equals the top Chern class of E in $H^{2k}(X;\mathbb{Z})$. *Hint: first check this for rank 1, then use the splitting principle.*

(iii) Let F be now any oriented $2k$-dimensional real vector bundle over a space X. Prove that $p_k(F) = e(F) \cup e(F) \in H^{4k}(X;\mathbb{Z})$. *Hint: Prove and use the fact that the isomorphism of real vector bundles* $F_{\mathbb{C}} := F \otimes_{\mathbb{R}} \mathbb{C} \cong (F \oplus F)$ *fiberwise taking* $v \otimes (a + bi) \mapsto (av, bv)$ *takes the complex orientation of* $F_{\mathbb{C}}$ *to* $(-1)^{n(n-1)/2}$ *times the orientation on* $F \oplus F$ *induced by direct sum from the orientation on* F.

3. *Euler classes and Stiefel-Whitney classes.* Note for any not-necessarily orientable rank k real bundle $E \to X$, one can define the $\mathbb{Z}_2$-Euler class $e_{\mathbb{Z}_2}(E) \in H^k(X;\mathbb{Z}_2)$, via the $\mathbb{Z}_2$ Thom isomorphism: i.e., one looks at the image of the $\mathbb{Z}_2$ Thom class $u_{\mathbb{Z}_2} \in H^k((E, E^0);\mathbb{Z}_2)$ under restriction to X along the zero section $(X, \emptyset) \to (E, E^0)$.
   (i) Show that if E is oriented then $e_{\mathbb{Z}_2}(E)$ is the $\mathbb{Z}_2$ reduction of $e(E)$.
   (ii) Let $L_{taut} \to \mathbb{RP}^k$ be the tautological real line bundle. If $S(L_{taut})$ denotes the associated S^0 bundle, show (using the $\mathbb{Z}_2$-Euler class version of the Gysin sequence and the topological fact that $S(L_{taut}) = S^k$ and the projection map is just the quotient by the antipodal map) that $e_{\mathbb{Z}_2}(L_{taut}) = w_1(L_{taut})$.
   (iii) Using the above fact, prove that for any real rank k vector bundle E over any X, that $e_{\mathbb{Z}_2}(E) = w_k(E)$. Conclude that if E is oriented, that $w_k(E)$ is the mod 2 reduction of $e(E)$. *Hint: first check this for rank 1, then use the splitting principle.*

4. *[Exercise from Cohen Chapter 3.3] New characteristic classes via the splitting principle.* As mentioned in class, any cohomology element of $H^*(BU(n);\mathbb{Z})$ determines a characteristic class for rank n vector complex vector bundles. In class we showed there is an identification of $H^*(BU(n);\mathbb{Z})$ with *symmetric polynomials* in $\mathbb{Z}[h_1, \dots, h_n] = H^*((\mathbb{CP}^\infty)^n;\mathbb{Z})$, where $|h_i| = 2$. In particular, given any power series $f = \sum_I \alpha_I h^I$ (using the multi-index notation; if $I = (i_1, \dots, i_n), h^I := h_1^{i_1} \cdots h_n^{i_n}$) which is *symmetric*, the degree $2i$ part of this series $f_i \in H^{2i}(BU(n);\mathbb{Z})$ gives a characteristic class for complex rank n vector bundles taking values in $2i$th integral cohomology. One particularly natural source of symmetric power series are series of the form $\prod_{i=1}^n p(h_i) = p(h_1)p(h_2)\cdots p(h_n)$, where $p(h) = \sum_{i=0}^\infty a_i h^i$ is any single-variable power series in h.

   We can in particular, for any smooth function $g : \mathbb{R} \to \mathbb{R}$, associate a collection of characteristic classes as follows. Taylor expand g at zero to get a power series $p_g(x) = \sum_k \frac{f^{(k)}(0)}{k!} x^k$, and now look at the symmetric power series $f := \prod_{i=1}^n p_f(h_i) := p_f(h_1) \cdots p_f(h_n)$; by above the homogeneous degree $2i$ part determines a characteristic class, which we call g_i.

   Consider the examples $g(x) = e^x$ and $g(x) = tanh(x)$. Write the low dimensional characteristic classes $g_i \in H^*(BU(n);\mathbb{Z})$ for $i = 1, 2, 3$ as explicit polynomials in the Chern classes.

5. *The Poincaré dual of the Euler class.* Let E be an oriented smooth rank k vector bundle over a smooth compact oriented manifold M of dimension m. Let $s \in \Gamma(E)$ be any section which is transverse to the zero section, and let $Z := s^{-1}(0)$ be the zero set of S. Prove that Z an oriented submanifold of dimension $m - k$, and also prove that $[Z]$ is Poincaré dual (in M) to $e(M)$. *Hint: using a tubular neighborhood of Z in M, first appeal to the fact that in M, the Poincaré dual to $[Z]$ is the push forward of the Thom class of the normal bundle to Z in M.*