Math 257B: Day 1. 3/28/2016.

Logistics: ~~[illegible]~~ changing the course to ~~[illegible]~~ 2x/week, 1:30–2:50pm, starting ~~next week~~ ASAP. (this week: 3x).

- no class Monday ~~3/28~~ 4/5/2016 (instead, class W+F that week)
  ideally Wed. + Fri. (4/7 + 4/9) classes
- after that, Monday + Wednesday 1:30–2:50 pm
- Seeking a note taker for each class.

Aspects of Fukaya categories: Overview.

§1. Defns & Motivations:

(X^{2n}, ω) symplectic manifold. $L^n \subseteq X$ is Lagrangian if $\omega|_L = 0$.

$H : X \to \mathbb{R}$ fcn Hamiltonian $\rightsquigarrow X_H$ Hamiltonian vec. field ($i_{X_H}\omega = dH$).

ϕ_H^t time t Ham. flow.

(a) Impressionistically, Lagrangian Floer homology associates (in nice cases) to a pair L_0, L_1 a group $HF^*(L_0, L_1)$ which (in nicer cases) is a Ham.-isotopy invariant, meaning

$$HF^*(\phi_H^1 L_0, L_1) \cong HF^*(L_0, \phi_{H'}^1 L_1)$$
$$\cong HF^*(L_0, L_1) \cong$$

[Q: • over what field/ring? $\mathbb{Z}_2$ for now, in general, ~~[illegible]~~ can grp loc. L / extra choices
• what grading? also $\mathbb{Z}_2$ for now, smooth topology often to refine to $\mathbb{Z}$.

& categorifies intersection number, i.e., $\chi(HF^*(L_0, L_1)) = L_0 \cdot L_1$.

~~one [illegible] is~~ If L_0, L_1 transverse, can arrange $HF^*(~~[illegible]~~(L_0, L_1))$, ~~[illegible]~~ to be homology of a chain complex gen by intersection points $L_0 \pitchfork L_1$.

$\Rightarrow \# L_0 \cap L_1 \geq \operatorname{rk} HF^*(L_0, L_1) \geq L_0 \cdot L_1$

improved estimate depends on symplectic structure.

(i) Ex: $L := S^1 \subseteq \mathbb{C}$ (unit). Note that $H(x,y) = 2y$ induces a Ham. vector field $-2\partial_x$ with $\phi_H^1(L) \cap L = \emptyset$. ($L$ displaceable) dy $dx \wedge dy$

Thus, if $HF^*(L,L)$ exists, it had better be zero.

(ii) Ex: ~~[illegible] Say L is exact if~~ ~~(if ω is exact, meaning ω = dλ)~~

Floer proved that

If $\pi_2(M, L) = \phi$ then $HF^*(L,L)$ exists & one has $HF^*(L,L) \cong H^*(L)$.

It follows that L is not displaceable, & $\operatorname{rk} \phi_H ~~L~~ \cap L \geq \operatorname{rk} H^*(L)$. if transverse (Note: assumption fails for S^1).

(iii) Ex: ("Everything is a Lagrangian" - Weinstein):

A famous conjecture of Arnold states that if ~~[illegible]~~ $H : S^1 = \mathbb{R}/\mathbb{Z} \times X \to \mathbb{R}$ is a (possibly time-dependent) generic Hamiltonian, then $\# \operatorname{Fix}(\phi_{H_t}^1 : M \to M) \geq \operatorname{rk} H^*(M)$.

Note that given a symplectomorphism $\phi: X \to X$, the graph $\Gamma_\phi = \{(x, \phi x) \subseteq X^- \times X\}$ is a Lagrangian (in particular, so is ~~$\mathcal{P}$~~ $\Delta := \Gamma_{id}$), and $\Delta \cap \Gamma_\phi \overset{1:1}{=} \{fix\ \phi\}$. $X^- = (X, -\omega)$

In particular, Arnold's conjecture would follow ~~from~~ if one knew

$$\# \Delta \cap \Gamma_{\phi^1_{H_t}} \geq rk\, H^*(\Delta) = rk\, H^*(M)$$

which would follow if we knew $HF^*(\Delta, \Gamma_{\phi^1_{H_t}}) \cong H^*(\Delta)$.

(of course there are more direct approaches!) specialcase of Arnold-Givental conjecture

~~(b) Fukaya category~~

(a) There will be cases in which one can define $CF^*(L_0, L_1) \circlearrowleft \delta$ but $\delta^2 \neq 0$. (ex: L_1, L_0)

Following the monumental work of Fukaya-Oh-Ohta-Ono, we say (L_0, L_1) (or L if $L_0 = L_1 = L$) is obstructed. In many cases, can quantify these obstructions.

(b) Towards the Fukaya category: collects ~~the~~ the relationship between L_0, L_1 as L_0, L_1 vary.

Donaldson noticed: there is a ~~way~~ ~~of multiplying~~ (when defined), multiplication operation "composition" arising from geometry.

$$[\mu^2]: HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \to HF^*(L_0, L_2).$$ Moreover, composition is associative

therefore we can think of the following as a category, the Donaldson-Fukaya category

ob $H^0\mathcal{F}$: L Lagrangians (w/ extra data), ~~for which~~ which are unobstructed.

Mor $H^0\mathcal{F}$: $Hom^0(L_0, L_1) := HF^0(L_0, L_1)$.

composition given by $[\mu^2]$.

To check: $\exists$ identity morphisms

Unfortunately, this is insufficient for many purposes, i.e., to build LES's in Floer homology, or to iterate these sequences, to study further relations, we will need chain level structure.

Problem: μ^2: - - - not associative

~~Solution~~ Obstruct [Fukaya]: there is a hierarchy ..

$$\mu^k: CF^*(L_{k-1}, L_k) \otimes \cdots \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_k), \quad k \geq 1$$

satisfying $$\sum_{i,l} (-1)^{*} \mu^{k-l+1}(x_k, \ldots, x_{i+l+1}, \mu^l(x_{i+l}, \ldots, x_{i+1}), x_i, \ldots, x_1) = 0$$

+ some sign $(= \sum_{j=1}^{i} \deg(x_j) - i)$.

First equation: $(\mu^1)(\mu^1)(x) = 0$. $\Rightarrow$ μ^1 is a differential

• $\mu^2(\mu^1(x), y) \pm \mu^2(x, \mu^1(y)) = \mu^1(\mu^2(x,y))$

μ^2 is a chain map

• μ^2 is associative up to chain htpy.

Notes: • each μ^k, $k > 2$ is not invariant; but the collection $\{\mu^k\}$ is invariant up to quasi-isomorphism

• some μ^k's are invariants (cf. Massey products in topology).

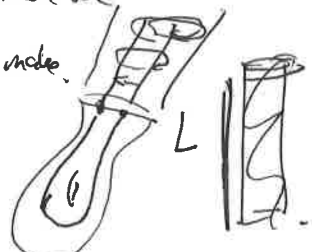
we will see that A_∞ categories are more straightforward than they initially seem to manipulate.

One of the main goals will be constructing these categories, verifying some sort of invariance, & computing in certain cases. Call the result $\mathcal{F}(M)$

§2. ~~Flavors~~ Variations on the Fukaya category.

~~Especially One can imagine that we only allow compact Lagrangians in ob $\mathcal{F}(M)$~~

~~However,~~ when M is non-compact (Ex: T^*Q, $\mathbb{C}$-affine variety), there are interesting non-trivial enlargements involving non-compact Lag'ns. choices to make.



(c) Relatives of $\mathcal{F}(M)$: Pass to split-closed derived category

$D^\pi\mathcal{F}(M)$: a formal enlargement involving ~~[illegible]~~ complexes

the right place to talk about long-exact sequences between Floer homology groups.

Ex: Saying $L_0 \simeq \{L_1 \to L_2\}$ complex is saying sth. about a LES between $HF^*(L_0,K)$, $HF^*(L_1,K)$ & $HF^*(L_2,K)$ $\forall K$.

- wrapped Fukaya category: $CF^*(L_i, L_j)$ gen. by $L_i \cap L_j$ and Reeb chords $\partial L_i \to \partial L_j$.
- infinitesimal Fukaya category. $CF^*(L_i, L_j)$ gen by $\phi_\epsilon L_i \cap L_j$, where ϕ_ϵ is a Ham. pushoff of symplectic which is "positive" near ∞. (asymmetric).
- Fukaya categories of $E \xrightarrow{W} \mathbb{C}$ ⇝ assoc. a category $\mathcal{F}(E,W)$ containing $\mathcal{F}(E)$ & some non-compact Lag'ns. Invariant of W.

(d) Mirror symmetry: strange dualities discovered in ~~physics~~ string theory between symplectic geometry of (X,ω) & complex geometry of $(\check{X}, J)$

~~Kontsevich~~: Ex: [CdOGP '91].

X = "quintic 3-fold"

$\{n_d^A := \#$ rational curves$\}$ can package as a GW invariant $\longleftrightarrow$ can be recovered by

$\check{X}$ mirror quintic (totally different space) some "mirror"

N_d^β "period integrals"

$\int_{\check{X}} \Omega \wedge \nabla_\alpha \nabla_\beta \nabla_\gamma \Omega$ (Ω: holomorphic vol. form)

Kontsevich: HMS, ~~[illegible]~~: For such $(X, \check{X})$, $D^\pi\mathcal{F}(X) \cong D^b Coh(\check{X})$ (+ this implies ⇑)!

Computational methods involving:

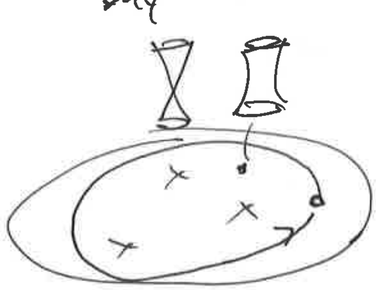
(3): Fukaya category of singularities

$$\begin{array}{c} E \\ \downarrow W \\ \mathbb{C} \end{array}$$

Say W is a symplectic fibration away from some critical points in base.

Ex: a Lefschetz fibration ("hol. Morse case").

$\mathbb{C}$



Then is an associated monodromy symplectomorphism $\phi_W : M \to M$ (general fibre) (ex. Dehn twist)

Thm [~~Seidel~~, Abouzaid–G.] : Can describe ϕ as a functor on $\mathcal{F}(M)$ via the category $\mathcal{F}(E,W)$.

Precisely, have $\mathcal{F}(E,W) \underset{\cup}{\overset{\cap}{\rightleftarrows}} \mathcal{F}(M)$

$$\& \quad \mathrm{id} \to \phi_W \to \cap\cup \to$$

Thm 2: [Abouzaid–G., Abouzaid–Seidel, ~~Seidel~~] Can understand the wrapped Fukaya category of E ~~wrapping in E~~ by a suitable localization process from $\mathcal{F}(E,W)$, at least for certain generating objects [Seidel ()].

In many cases, $\mathcal{F}(E,W)$ is more computable than $\mathcal{F}(E)$.

Thm 3: [Abouzaid–G.] In many cases, can construct generators for $\mathcal{F}(E,W)$. "directed Fukaya categories"

→ Thm 4: [Seidel]: In many cases, can determine $\mathcal{F}(M)$ from $\mathcal{F}(E,W)$ + small extra data, & $\mathcal{F}(E)$ from $\mathcal{F}(M)$.

Examples: • W hol. fcn, $\mathbb{C}^n \to \mathbb{C}$. $\mathcal{F}(W)$ invariant of the type of singularity of $\mathbb{C}^n$.

• Mirror symmetry says X non-Calabi–Yau $\Rightarrow$ mirror is pair $(\check{X}, w)$

$$``D^b Coh(X) \cong D^\pi \mathcal{F}(\check{X}, w)".$$

Next time: begin Floer homology: back to basics.