

Math 535a Homework 1

Due Wednesday, February 2, 2022 by 5 pm

Please remember to write down your name on your assignment, and submit it by uploading a (scanned or typed) to *Gradescope*.

1. Show that the topological spaces $S^1 \subset \mathbb{R}^2$ (with topology induced by the inclusion into \mathbb{R}^2 and $[0, 1]/\{0, 1\}$ (with the quotient topology from the topology on $[0, 1] \subset \mathbb{R}$) are homeomorphic.
2. Show that the derivative of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, if it exists at a point $a \in \mathbb{R}^n$, is unique.
3. Produce, with proofs, examples of the following topological spaces which are not topological manifolds:
 - a) A space X which is locally Euclidean¹ and second countable, but not Hausdorff.
 - b) A space X which is Hausdorff and second countable, but not locally Euclidean.
4. (*double weight problem*) Let V be a vector space over a field k and let $W \subset V$ be a subspace (meaning a subset satisfying: for any $\mathbf{a}, \mathbf{b} \in W, c \cdot \mathbf{a} + d \cdot \mathbf{b} \in W$ for any scalars $c, d \in k$. Such a subset is automatically a vector space over k ; with all operations coming from those in V).

In class, we defined the quotient V/W to be the set-theoretic quotient of V by the equivalence relation $x \sim y$ iff $x - y \in W$. Equivalently, the quotient is the partition of V consisting of sets of the form $[\mathbf{a}] = \mathbf{a} + W = \{\mathbf{a} + w | w \in W\}$.

- (a) Show that V/W is a vector space, with operations induced by those of V in the following sense: for α and β in V/W , choose elements \mathbf{a} and \mathbf{b} with $[\mathbf{a}] = \alpha$, $[\mathbf{b}] = \beta$ and define $\alpha + \beta = [\mathbf{a} + \mathbf{b}]$ and $c \cdot \alpha = [c \cdot \mathbf{a}]$ (one needs to first show that these definitions, which require choosing representatives \mathbf{a} and \mathbf{b} of equivalence classes $\alpha = [\mathbf{a}]$, $\beta = [\mathbf{b}]$, is independent of choice, and then show that the axioms of being a vector space are satisfied).

- (b) The quotient comes equipped with a natural linear map

$$\begin{aligned}\pi : V &\longrightarrow V/W \\ \mathbf{v} &\longmapsto [\mathbf{v}] = \mathbf{v} + W,\end{aligned}$$

called the *projection*, which has $\ker \pi = W$ (check that π is linear and has kernel as desired). Suppose V is finite-dimensional, and let U be a subspace complementary to W , that is a subspace such that $U \cap W = \{\mathbf{0}\}$ and $V = W + U = W \oplus U$. Show that the restriction of projection to U

$$\pi_U : U \longrightarrow V/W$$

¹in the sense that each point $p \in X$ has a neighborhood homeomorphic to an open set of \mathbb{R}^n .

is an isomorphism.

- (c) Let $C^\infty(\mathbb{R})$ denote the vector space of *smooth* functions $f : \mathbb{R} \rightarrow \mathbb{R}$ (namely, $f \in C^\infty(\mathbb{R})$ if f' exists and is continuous, f'' exists and is continuous, and so on². Examples include many of the standard functions you know—sin, cos, polynomials, exponentials, etc.)

Let U denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at 3 and 5

$$U = \{f \in C^\infty(\mathbb{R}) \mid f(3) = f(5) = 0\};$$

you may take for granted that U is a subspace.³ Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite-dimensional. What is its dimension? In contrast, note that $C^\infty(\mathbb{R})$ is infinite dimensional.⁴

- (d) Let V be a vector space and $W \subset V$ be a vector subspace. We denote by $i : W \rightarrow V$ the inclusion map. Recall we defined the dual of a vector space as $V^* = \text{Hom}_k(V, k)$. There is a natural induced map $i^* : V^* \rightarrow W^*$ dual to the inclusion sending a linear map $\phi \mapsto \phi|_W$. The kernel of i^* is called the *annihilator* of W and denoted

$$\text{Ann}(W) = \{\phi \in V^* \mid \phi|_W = \mathbf{0} \in W^*\}$$

it is the set of linear maps from V to k that return 0 on any element in W .

Prove that there is a canonical isomorphism

$$\text{Ann}(W) \cong (V/W)^*$$

Hint: there is a natural map $\pi^* : (V/W)^* \rightarrow V^*$ dual to the projection $\pi : V \rightarrow V/W$. Describe this map and prove that π^* is an isomorphism onto its image, which is $\text{Ann}(W)$, or equivalently that π^* is injective and surjective onto $\text{Ann}(W)$.

5. Let $S^n = \{(x_1, \dots, x_{n+1}) \mid x_1^2 + \dots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$. Prove that S^n has the structure of a smooth manifold, using charts associated to the cover $U_N = \{x_1 \neq -1\}$, $U_S = \{x_1 \neq 1\}$. (Hint: as in the case of S^1 in class, use *stereographic projection* to map U_N , respectively U_S to \mathbb{R}^n). Then prove that the antipodal map $S^n \rightarrow S^n$, $\mathbf{x} \mapsto -\mathbf{x}$ is a diffeomorphism of manifolds.
6. Prove (as stated in class) that the product of two smooth manifolds $(M^m, \mathcal{A}_M = \{(U_\alpha, \phi_\alpha : U_\alpha \rightarrow \mathbb{R}^m)\}_{\alpha \in I})$, $(N, \mathcal{A}_N = \{(V_\beta, \psi_\beta : V_\beta \rightarrow \mathbb{R}^n)\}_{\beta \in J})$ naturally has the structure of a smooth manifold, with atlas given by $\mathcal{A}_{M \times N} = \{(U_\alpha \times V_\beta, (\phi_\alpha, \psi_\beta) : U_\alpha \times V_\beta \rightarrow \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n})\}_{(\alpha, \beta) \in I \times J}$.

²You may take for granted this is a vector space, with operations $(f+g)(x) = f(x) + g(x)$ and $(c \cdot f)(x) = c \cdot (f(x))$; the key point is that these operations preserve the condition of being smooth

³Proof: If $f, g \in U$, then note that $c \cdot f + d \cdot g \in U$ for any scalars c, d , because e.g., $(c \cdot f + d \cdot g)(3) = c \cdot f(3) + d \cdot g(3) = c \cdot 0 + d \cdot 0 = 0$ (and similarly when one evaluates at 5)

⁴Proof: Let $f_i = x^i \in C^\infty(\mathbb{R})$ for each $i \in \mathbb{Z}_{\geq 0}$. Suppose a finite collection f_{i_1}, \dots, f_{i_k} spanned $C^\infty(\mathbb{R})$; therefore a collection of the form f_0, f_1, \dots, f_N spans $C^\infty(\mathbb{R})$ for some $N = \max(i_1, \dots, i_k)$. Now, note that the smooth function x^{N+1} is linearly independent from x^0, x^1, \dots, x^N , a contradiction. (why is this the case?)

7. Finish the proof from class that $\mathbb{R}P^n$ is a smooth manifold (of dimension n).
8. Finish the proof from class that $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a smooth 2-manifold.
9. *Different smooth structures on a topological manifold, Lee problem 1-6.* Let M be a non-empty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: First show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from the n -ball to itself, which is a diffeomorphism if and only if $s = 1$.]