Guillermou III -

- pf of Thm. 1
- pf. thm. Viterbo $\Rightarrow H^\bullet(\overset{\text{Lagrangian}}{\Lambda}) = H^\bullet(\overset{\text{base}}{M})$

Thm: [-KS] Given $\phi : \dot{T}^*M \times I \longrightarrow \dot{T}^*M$ homog. ham. isotopy.

associate graph

$$\Lambda \subset \dot{T}^*(M \times M \times I)$$

$$\Lambda = \{(\phi_t(x,\xi), x, -\xi, -f(\phi_t(x,\xi), t))\}$$

$\exists !\ K \in D(k_{M \times M \times I})$

s.t. $SS(K) \subset \Lambda \cup (M \times M \times I)$ & $K_0 = k_{\Delta_M}$.

How?

- Proof of existence of K on $M \times M \times (-\varepsilon, \varepsilon)$.

Λ_t = graph of $\phi_t \subset \dot{T}^*(M \times M)$

= deformation of $\Lambda_0 = T^*_\Delta(M \times M)$ conormal.

① We deform $T_{\partial U}(M \times M)$ = exterior conormal bundle of ∂U

$U \subset M \times M$ ∂U smooth.

$$N_0 = T^{*,\overset{\text{exterior}}{e}}_{\partial U}(M \times M)$$

$$\Lambda' = \Lambda \circ \Lambda'_0$$

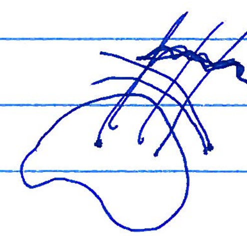
$$\Lambda'_t := \Lambda' \circ T^*_{\{t\}} I.$$

$$= \Lambda_t \circ \Lambda' = \{(\phi_t(x,\xi), y, \eta) ;\ (x,\xi,y,\eta) \in \Lambda'_0\}.$$

We assume $\phi_t = \mathrm{id}$ outside T^*W where $W \subset M$ open, $\overline{W}$ compact.

Then, $\exists\ \varepsilon$ s.t. $\forall t \in (-\varepsilon, \varepsilon)$

$$\Lambda'_t = T^{*,e}_{\partial U_t}(M \times M), \quad \partial U_t = \pi_{M \times M}(\Lambda'_t)$$



$$V = \coprod U_t \times \{t\} \subset M \times M \times (-\varepsilon, \varepsilon)$$

$$SS(k_V) \cap \dot{T}^*(M \times M \times (-\varepsilon, \varepsilon)) = \Lambda'$$
$$= \Lambda \circ SS(k_U)$$

If $\Lambda = SS(K) = SS(K \circ k_U)$

maybe $k_1 = K \circ k_U$

We look for an L such that $k_U \circ L = k_{\Delta_M}$.

In fact, if $T^{*,e}_{\partial U}(M \times M) \subset \dot{T}^*(M \times M)$

is a graph of a diffeo., then

$v(k_{\bar{U}})$ is $M \times M \to M \times M$
$(x, y) \longmapsto (y, x)$

is an inverse to k_U.

In general: for any kernel $K \in D^b(k_{M \times M})$,

$$K^{-1} := v(R\mathcal{H}om(K, \underset{\substack{\shortparallel \\ \mathcal{O}_M[d_M]}}{\omega_1} \boxtimes k_M))$$

we have a morphism

$$K \circ K^{-1} \longrightarrow k_{\Delta_M}.$$

We have $\boxed{k_{\bar{U}}[n] \circ k_U} \xrightarrow{a} k_{\Delta_M}$.

Choose a smooth metric on M

$$U = \{(x, y) ;\ d(x, y) < \eta\}$$

For η small enough, a is an isomorphism.

We set $K = k_V \circ k_{\bar{U}} [d_N]$, where V is obtained in step 1.

Then,

- $K_0 = k_U \circ k_{\bar{V}} (d_N) = k_{\Delta_M}$
- $SS(K) \subset SS(k_V) \circ SS(k_{\bar{U}})$
  $= \Lambda$.

<u>step 2</u>: unicity of K.

if have two

$$K, L \in D^b(k_{M \times M \times I}) \text{ with}$$

$$SS(K), SS(L) \subset \Lambda \cup (M \times M \times I)$$

$$K_0 = L_0 = k_{\Delta_M},$$

then $K \simeq L$.

How?: $F : K^{-1} \underset{I}{\circ} L$ Easy to bound & see that $SS(F) \subset T^*_{\Delta_{M \times I}}(M \times M \times I)$
$\cup (M \times M \times I)$

$$p : M \times M \times I \longrightarrow M \times M.$$

$$F \simeq p^{-1}(F').$$

$$F_0 = k_{\Delta_M}, \text{ so } \overset{K^{-1} \circ L}{\overset{\shortparallel}{F}} = k_{\Delta_M \times I}$$

<u>Step 3</u>: glue:

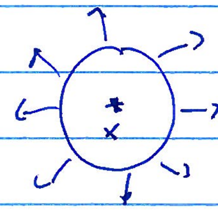
$$K^{-1} \circ K = k_{\Delta_M} = K \circ K^{-1}.$$

<u>example</u>: $M = \mathbb{R}^n$,

$$\phi_t : T^*M \to T^*M.$$

$$(x, \xi) \longmapsto (x + t\frac{\xi}{|\xi|}, \xi)$$

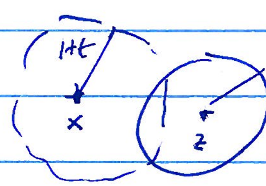
$\cdot\ U = \{(x,y);\ \|x-y\| < 1\} \subset M \times M$



$V = \{(x,y,t);\ \|x-y\| < 1+t\}$

$K = K_U \circ k_{\bar{U}}[n]$

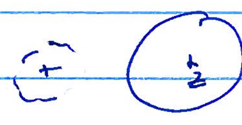
$K_{(x,\cancel{z},t)} = R\Gamma_c(k_{\{;\ \|x-y\| < 1+t\}} \otimes k_{\{y;\ \|\cancel{z}-y\| < 1\}})$



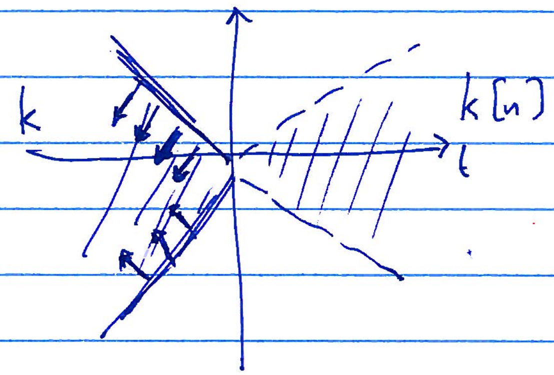
For $t \geq 0$

$$K_{(x,z,t)} = \left|\begin{array}{ll} 0 & |x-z| \geq t \\ k[n] & - < - \end{array}\right.$$

for $t \leq 0$:



$$= \begin{cases} 0 & |x-\cancel{z}| \geq -t. \\ k & \cdot \leq \cdot \end{cases}$$

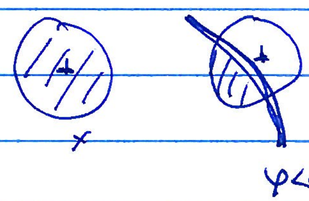


$$k_U[n] \to K \to k_Z \xrightarrow{+1} k_U[n+1]$$

<u>simple sheaves</u>

$\Lambda \subset \dot{T}^*M$ smooth conic Lagrangian submanifold

assume $F \in D^b(k_M)$ with $SS(F) \cap \dot{T}^*M \subset \Lambda$.



For $(x,\xi) \in \Lambda$, $\varphi : M \to \mathbb{R}$ with $d\varphi_x = \xi$.

$$(R\Gamma_{(\varphi \geq 0)}(F))_x \to F_x \to R\Gamma_{(\varphi \leq 0)}(F)_x.$$

<u>Thm</u> [K-S]: $p = (x,\xi) \in \Lambda$.

If φ, φ' satisfies

$\Lambda_\varphi \pitchfork \Lambda$ at p

$\{(x, d\varphi_x)\}$,

the same for φ'

Maslov index shift.

$$R\Gamma_{(\varphi \leq 0)}(F)_x \simeq R\Gamma_{(\varphi' \leq 0)}(F)_x \left[\frac{\tau_\varphi - \tau_{\varphi'}}{2}\right]$$

$T_p(T^*M) \supset T_p\Lambda_p,\ T_p\Lambda,\ T_p(\pi^{-1}(x))$,

and

define $\tau_p = \tau(T_p\Lambda_\varphi,\ T_p\Lambda,\ T_p\pi^{-1}(x))$.

Def: $\mathcal{F}$, $SS(\mathcal{F}) \cap \dot{T}^*\Gamma = \Lambda$,

$\mathcal{F}$ is simple at $p \in \Lambda$ if

$R\Gamma_{(\varphi \le 0)}(\mathcal{F})_x$ is concentrated in some dimension and $\simeq k$.

Prop: If Λ connected,

$SS(\mathcal{F}) = \Lambda$, $\mathcal{F}$ simple at one p

$\Longleftrightarrow$ ——— any.

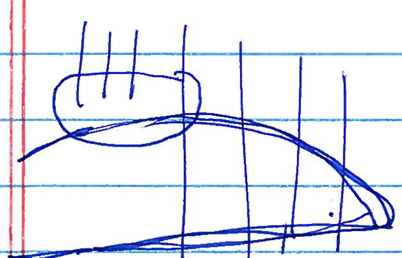
Thm [Viterbo]: $\Lambda \subset T^*M$ compact, exact, Maslov class $= 0$

$\rightsquigarrow \tilde{\Lambda} \subset T^*(M \times \mathbb{R})$

$\Rightarrow \exists F \in D^b(k_{M\times\mathbb{R}})$

s.t. • $SS(\mathcal{F}) \cap \dot{T}^*(M\times\mathbb{R}) = \tilde{\Lambda}$, $\mathcal{F}$ simple along $\tilde{\Lambda}$.

$$\mathcal{F}_t := \mathcal{F}|_{M\times\{t\}} = \begin{cases} 0 & \text{for } t \ll 0 \\ k_M^d & \text{for } t \gg 0 \end{cases}$$



Prop: $\Lambda \subset \dot{T}^*M$ (smooth), $\mathcal{F}$ simple/Λ. Then have exact sequence

ω_M?

dual of $\mathcal{F}$ $\to R\mathcal{H}om(\mathcal{F}, \mathcal{O}_M) \otimes \mathcal{F} \to R\mathcal{H}om(\mathcal{F}, \mathcal{F})$

$\to R\pi_* \mathbb{C}_\Lambda \xrightarrow{+1}$

Pf:

In general:

for $\mathcal{F}, \mathcal{G} \in D^b(k_M)$,

Saito's microlocalization

- $\mu hom(\mathcal{F}, \mathcal{G}) \in D^b(k_{T^*M})$

invariant by sympl. transformations

$R\pi_* \mu hom(\mathcal{F}, \mathcal{G}) = R\mathcal{H}om(\mathcal{F}, \mathcal{G})$

$R\Gamma_M \mu hom(\mathcal{F}, \mathcal{G}) = R\mathcal{H}om(\mathcal{F}, C_\Lambda) \otimes \mathcal{G}$

Finally:

$T_c : M \times \mathbb{R} \to M \times \mathbb{R}$

$(x, t) \longmapsto (x, t + c)$

$SS(\mathcal{F}) \subset \Lambda \subset \{\tau \geq 0\}$

$\exists\ F \xrightarrow{\tau_+} T_{c*} F.$

Prop: For $\mathcal{F}, \mathcal{G}$ s.t. $SS(\mathcal{F}), SS(\mathcal{G}) \subset \Lambda$,

$R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \to R\mathcal{H}om(\mathcal{F}, T_{c*}(\mathcal{G}))$ is an isom. for all $c \geq 0$.

Using $F_t = 0 \quad t \ll 0$,

$F_t = k_M^d \quad t \gg 0$ & iso. above, find:

$R\mathcal{H}om(\mathcal{F}, \mathcal{F}) = hom(k_M^d, k_M^d) = R\Gamma(M; k_M^{d^2})$

– we can prove in the same way that $R\Gamma(M \times \mathbb{R}, R\mathcal{H}om(\mathcal{F}, C_\Lambda) \otimes \mathcal{F})$

$= 0.$

In dimension 0, we get that $d^2 = 1$, implying

theorem: $H^i(M) = H^i(\Lambda)$