

Symplectic cohomology from Hochschild (co)homology

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ABSTRACT. Consider the wrapped Fukaya category \mathcal{W} of a collection of exact Lagrangians in a Liouville manifold. Under a non-degeneracy condition implying the existence of enough Lagrangians, we show that natural geometric maps from the Hochschild homology of \mathcal{W} to symplectic cohomology and from symplectic cohomology to the Hochschild cohomology of \mathcal{W} are isomorphisms, in a manner compatible with ring and module structures. This is a consequence of a more general duality for the wrapped Fukaya category, which should be thought of as a (weak) non-compact Calabi-Yau structure. The new ingredients are: (1) new geometric operations, coming from discs with two negative punctures and arbitrary many positive punctures, (2) a generalization of the Cardy condition, (3) the use of homotopy units to relate non-degeneracy to a resolution of the diagonal Lagrangian, and (4) Fourier-Mukai theory for \mathcal{W} via a wrapped version of holomorphic quilts, developed in a prequel. As an application, we use our new operations geometric to give a formula for the symplectic cohomology ring structure on the Hochschild homology of \mathcal{W} .

1. Introduction

It is a conjecture of Kontsevich [K] (inspired by mirror symmetry) that the quantum cohomology ring of a compact symplectic manifold M should be isomorphic to the *Hochschild cohomology*

$$(1.1) \quad \mathrm{HH}^*(\mathcal{F}(M))$$

of the Fukaya category $\mathcal{F}(M)$. There are at least two strong motivations for understanding this conjecture. For one, such an isomorphism would allow one to algebraically recover quantum cohomology along with its ring structure from computations of the Fukaya category. In another direction, Hochschild cohomology measures deformations of a category, so the conjecture has implications for the deformation theory of Fukaya categories; see e.g. [S1].

We address a non-compact version of Kontsevich's conjecture, in the setting of exact (non-compact) symplectic manifolds. The relevant symplectic objects are **Liouville manifolds**, exact symplectic manifolds with a convexity condition at infinity. Examples include cotangent bundles, affine complex varieties, and more general Stein manifolds. In this setting, there is an enlargement of the Fukaya category known as the **wrapped Fukaya category**

$$(1.2) \quad \mathcal{W} := \mathcal{W}(M),$$

which includes as objects non-compact Lagrangians, and whose morphism spaces include intersection points as well as *Reeb chords* between Lagrangians at infinity. The wrapped Fukaya category is expected to be the correct mirror category to coherent sheaves on non-proper varieties, see e.g. [AS2] [AAE⁺]. Moreover, it is the *open-string*, or Lagrangian, counterpart to a relatively classical invariant of non-compact symplectic manifolds, **symplectic cohomology**

$$(1.3) \quad SH^*(M),$$

first defined by Cieliebak, Floer, and Hofer [FH] [CFH].

There are also existing geometric maps from the *Hochschild homology*

$$(1.4) \quad \mathrm{HH}_*(\mathcal{W}(M))$$

to symplectic cohomology [A1] and from symplectic cohomology to the Hochschild cohomology [S1]

$$(1.5) \quad \mathrm{HH}^*(\mathcal{W}(M)).$$

Thus, one can posit that a version of Kontsevich's conjecture holds in this setting; a version of such a conjecture first appeared in work of Seidel [S6].

In Theorem 1.1 below, we prove a version of Kontsevich's conjecture for a Liouville manifold M of dimension $2n$, assuming a non-degeneracy condition for M first introduced by Abouzaid [A1]. The reason for the non-degeneracy assumption is this: to have any hope that symplectic cohomology be recoverable from the wrapped Fukaya category, it is important that the target manifold contain "enough Lagrangians."

DEFINITION 1.1. *A finite collection of Lagrangians $\{L_i\}$ is said to be **essential** if the natural map from Hochschild homology of the wrapped Fukaya category generated by $\{L_i\}$ to symplectic cohomology hits the identity element. Call M **non-degenerate** if it admits any essential collection of Lagrangians.*

By work of Abouzaid [A1], any collection of essential Lagrangians split-generates the wrapped Fukaya category. Our main result is:

THEOREM 1.1. *If M is non-degenerate, then the natural geometric maps*

$$(1.6) \quad \mathrm{HH}_{*-n}(\mathcal{W}(M)) \xrightarrow{[\circ\mathcal{C}]} \mathrm{SH}^*(M) \xrightarrow{[\mathcal{C}\circ]} \mathrm{HH}^*(\mathcal{W}(M))$$

are all isomorphisms, compatible with Hochschild ring and module structures.

The non-degeneracy condition is explicitly known for cotangent bundles [A2], some punctured Riemann surfaces, and upcoming work will establish it for total spaces of Lefschetz fibrations [AS1]. It is generally expected that work of Bourgeois, Ekholm, and Eliashberg [BEE2], suitably translated into the setting of the wrapped Fukaya category, would imply that every Stein manifold is non-degenerate, with essential Lagrangians given by the ascending co-cores of a plurisubharmonic Morse function.

One consequence of Theorem 1.1 is an induced duality-type isomorphism between Hochschild homology and cohomology for the wrapped Fukaya category. A key step in proving Theorem 1.1 involves giving a direct geometric Poincaré duality isomorphism

$$(1.7) \quad \mathrm{HH}_{*-n}(\mathcal{W}(M)) \xrightarrow{\sim} \mathrm{HH}^*(\mathcal{W}(M)),$$

that does not pass through $\mathrm{SH}^*(M)$ and is an instance of a more general duality for Hochschild homology and cohomology with coefficients, stated in Theorem 1.3.

Such dualities have appeared before in the context of the algebraic geometry of smooth (not necessarily proper) varieties. Van den Bergh [vdB1] [vdB2] was the first to observe a duality between Hochschild homology and cohomology for the coordinate ring of a *smooth Calabi-Yau* affine variety; see also [Kr]. Informally, a smooth Calabi-Yau variety X admits by definition a *holomorphic volume form* $\mathrm{vol}_X \in H^0(X, \Omega_X^n)$; contracting against vol_X induces an identification between polyvector fields and differential forms

$$(1.8) \quad \iota_{(\cdot)} \mathrm{vol}_X : H^*(X, \Lambda^* T_X) \xrightarrow{\sim} H^*(X, \Omega_X^*).$$

Via a version of the Hochschild-Kostant-Rosenberg theorem, the above two groups are identified with Hochschild cohomology and homology of the category of coherent sheaves.

The relevant notion for us is a purely categorical version of *smooth and Calabi-Yau*, generalizing a smooth (not necessarily proper) Calabi-Yau variety. As in the algebro-geometric setting, *smoothness* is the prerequisite property that must be defined first.

DEFINITION 1.2 (Kontsevich-Soibelman [KS]). *An A_∞ category \mathcal{C} is **homologically smooth** if its diagonal bimodule is **perfect**, that is, built out of simple split Yoneda bimodules via taking a finite number of mapping cones and summands.*

A key ingredient in our proof is relating homological smoothness to the non-degeneracy condition.

THEOREM 1.2. *If M is non-degenerate, then \mathcal{W} is homologically smooth.*

In fact, we show that M is non-degenerate if and only if the diagonal Δ is split generated by products of essential Lagrangians in the (wrapped) Fukaya category of the product $M^- \times M$. Via a functor from Lagrangians in $M^- \times M$ to bimodules developed in the prequel paper [G2], this immediately implies the Theorem.

DEFINITION 1.3. An A_∞ category \mathcal{C} is a (weak) **non-compact Calabi-Yau category** if it is homologically smooth and there is a Poincaré duality-type natural transformation

$$(1.9) \quad \mathrm{HH}_{*-n}(\mathcal{C}, \mathcal{B}) \xrightarrow{\sim} \mathrm{HH}^*(\mathcal{C}, \mathcal{B})$$

of functors from bimodules to chain complexes, inducing isomorphisms on homology. Such a natural transformation should be induced by the existence of a perfect bimodule

$$(1.10) \quad \mathcal{C}^!$$

representing, via tensoring, Hochschild cohomology, and an equivalence

$$(1.11) \quad \mathcal{C} \xrightarrow{\sim} \mathcal{C}^![n].$$

The bimodule $\mathcal{C}^!$, defined in Section 2.4, is known as the **inverse dualizing bimodule**. The *non-compact Calabi-Yau* terminology was introduced by Kontsevich and Soibelman [KS] as a categorical abstraction of perfect complexes on a smooth, not necessarily proper Calabi-Yau variety.

THEOREM 1.3 (Duality for the wrapped Fukaya category). *There is a geometric morphism of bimodules*

$$(1.12) \quad \mathcal{CY} : \mathcal{W} \rightarrow \mathcal{W}^![n].$$

If M is non-degenerate, then \mathcal{W} is homologically smooth and the map (1.12) gives \mathcal{W} the structure of a **non-compact Calabi-Yau category** (in particular, it is a quasi-isomorphism).

The relationship between Theorem 1.3 and Theorem 1.1 comes from a study of degenerations of annuli with many inputs on both boundary components and one output on one boundary component, which is a version of the *Cardy condition* from topological field theory:

THEOREM 1.4 (Generalized Cardy Condition). *The following diagram commutes up to an overall sign of $(-1)^{n(n+1)/2}$:*

$$(1.13) \quad \begin{array}{ccc} \mathrm{HH}_{*-n}(\mathcal{W}, \mathcal{W}) & \xrightarrow{[\mathcal{CY}_\#]} & \mathrm{HH}_*(\mathcal{W}, \mathcal{W}^!) \\ \downarrow [\mathcal{OC}] & & \downarrow [\bar{\mu}] \\ SH^*(M) & \xrightarrow{[\mathcal{CO}]} & \mathrm{HH}^*(\mathcal{W}, \mathcal{W}) \end{array}$$

Degenerations of annuli have appeared earlier in holomorphic curve theory (with fewer boundary marked points and sometimes in different settings) in work of Biran-Cornea [BC], Abouzaid [A1], and Abouzaid-Fukaya-Oh-Ohta-Ono [AFO⁺]. The moduli spaces in the second work are more related to the ones appearing here, appearing (nearly) as a particular case of our construction. At any rate, assuming Theorems 1.2, 1.3, and 1.4 we can give a short proof of Theorem 1.1:

PROOF OF THEOREM 1.1. In Propositions 3.3 and 3.4, we prove that $[\mathcal{CO}]$ is a ring map and $[\mathcal{OC}]$ is an $SH^*(M)$ module map with respect to the $SH^*(M)$ module structure induced by the $\mathrm{HH}^*(\mathcal{W}, \mathcal{W})$ module structure and $[\mathcal{CO}]$. In particular, for any s , we have that

$$(1.14) \quad s = s \cdot 1 = [\mathcal{OC}]([\sigma] \cap [\mathcal{CO}](s)),$$

where $[\sigma]$ is the pre-image of 1 on homology. Thus, $[\mathcal{OC}]$ is surjective and $[\mathcal{CO}]$ is injective, and in particular Theorem 1 is equivalent to the composition $[\mathcal{CO}] \circ [\mathcal{OC}]$ being an isomorphism.

Turning to Theorem 1.4, we see that $[\mathcal{CO}] \circ [\mathcal{OC}] = [\bar{\mu}] \circ [\mathcal{CY}_\#]$. But $[\mathcal{CY}_\#]$ is an isomorphism since $[\mathcal{CY}]$ is (this is a general property about induced maps on Hochschild complexes). Finally, in Corollary 2.3, we show that $\bar{\mu}$, which is a sort of multiplication map, is an isomorphism whenever \mathcal{W} is smooth (which it is by Theorem 1.2). \square

In Section 2, we introduce various algebraic preliminaries, some of which are standard, (facts about A_∞ (bi)modules, split-generation and homological smoothness), and some of which are new (an A_∞ notion of bimodule duals $\mathcal{B}^!$ and their first properties). In Section 3, we recall the definitions of symplectic cohomology, the wrapped Fukaya category, and the open-closed and closed-open string maps from and to Hochschild chains

and co-chains. In section 4 we constructing the morphism \mathcal{CY} appearing in Theorem 1.3 and prove Theorem 1.4, reducing the remaining work to establishing Theorem 1.3.

Homological smoothness, which is an algebraic form of having a finite resolution of the diagonal in the sense of [Beuillon:1978bh], should follow from a geometric statement about resolving the diagonal Lagrangian in the product $M^- \times M$. In Section 5, we recall the results of our prequel paper, which builds a functor from Lagrangians in $M^- \times M$ and bimodules. In particular, whenever there is a resolution of the geometric diagonal Δ , this functor, and additional symmetry in Floer theory for $M^- \times M$, immediately implies Theorem 1.3.

Finally, in Section 6 we are left with the most technical part of the proof—to show that whenever M is non-degenerate, that Δ is split-generated (or *resolved*) by product Lagrangians. We give two proofs of this fact, both of which make use of an implementation of the *homotopy units* of [FOOO2] for the wrapped Fukaya category of $M^- \times M$ in order to construct geometric maps relating A_∞ structures in a product with A_∞ structures of each factor...

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2. Finiteness and duality for modules and bimodules

2.1. A series of warm-ups. To start, let V be a vector space over \mathbb{K} . There is a trivial such vector space, \mathbb{K} , characterized by the fact that $V \otimes \mathbb{K} \cong V$ for any V . Using \mathbb{K} , one defines the vector space dual $V^\vee = \text{hom}(V, \mathbb{K})$. If V is *finite-dimensional* (or built out of finitely many copies of \mathbb{K}), then the duality operation is well-behaved, meaning that V^\vee is also finite dimensional, $(V^\vee)^\vee \cong V$ and moreover the canonical map $V^\vee \otimes W \rightarrow \text{hom}(V, W)$ is an isomorphism. So a choice of isomorphism $V \rightarrow V^\vee$ induces isomorphisms $V \otimes W \rightarrow \text{hom}(V, W)$, for any W .

Next, consider left modules over an associative graded algebra A . If B is any A – A bimodule, then (derived) tensor product with B takes A modules to A modules:

$$(2.1) \quad M \otimes_A^L B \in A\text{-mod}.$$

Tensoring with the bimodule $B = A$ induces the identity operation, which suggests that one should define the dual of a module M as

$$(2.2) \quad M^\vee := R\text{Hom}_{A\text{-mod}}(M, A).$$

Since (2.2) only uses the left module structure on the target A , the right A module structure survives, giving M^\vee the structure of a right A module. If M is *perfect* (roughly, built out of finitely many copies of A as a module), then this duality operation is again well-behaved; namely, M^\vee is also perfect, $M \simeq (M^\vee)^\vee$, and the canonical map $M^\vee \otimes_A^L N \rightarrow R\text{Hom}_A(M, N)$ is an isomorphism.

Finally one can replicate the duality procedure for bimodules B over A . Recall that an A bimodule structure on B is equivalent to a left $A^e := A \otimes A^{op}$ module structure, meaning that one can emulate the above process replacing A by A^e . If P is any A^e -bimodule, then (derived) tensor product with P takes A^e modules to A^e modules as in (2.1), and moreover the identity operation is represented by the trivial A^e bimodule, $A^e := A \otimes A^{op}$ itself. Rephrasing everything in the language of bimodules over A , the *A^e -bimodule structure* on the graded vector space

$$(2.3) \quad A \otimes_{\mathbb{K}} A^{op}$$

is given by the data of two A bimodule structures, an *outer* structure and an *inner* structure, which is an *A -quadrmodule structure*. Defining the dual of a bimodule B in a fashion as above, we arrive at a definition that seems to have been first studied by Van Den Bergh [vdB1].

DEFINITION 2.1 (Van Den Bergh [vdB1], Kontsevich-Soibelman [KS], Ginzburg [G3]). Let B be an A - A bimodule. The **dual bimodule** of B is, as a graded vector space

$$(2.4) \quad B^! := \bigoplus_i \text{Ext}_{A^e}^i(B, A \otimes_{\mathbb{K}} A^{op})[-i].$$

where Ext is taken with respect to the outer bimodule structure on $A \otimes A$. The inner bimodule structure of $A \otimes A$ survives and gives the bimodule structure on $A^!$.

We refer to the bimodule dual of A , denoted $A^!$, as the **inverse dualizing bimodule**.

If B is *perfect*, (e.g., sufficiently finite) as a bimodule, then $(-)^!$ is a well-behaved operation; namely $(B^!)^! \simeq B$ and tensoring with $B^!$ represents taking hom from B :

$$(2.5) \quad B^! \otimes_{A^e} P \cong R\text{Hom}_{A^e}(B, P).$$

Our goal in this section is to generalize the above construction to A_∞ categories and their modules and bimodules, defining the relevant finiteness notion along the way. The main subtlety is that an A_∞ \mathcal{A} - \mathcal{A} bimodule \mathcal{B} can no longer be thought of as an $\mathcal{A} \otimes \mathcal{A}^{op}$ module (for one, there is not a clean notion of tensor product of A_∞ categories, and secondly, left and right multiplying no longer commutes on the chain level). This means that some care is required in the analogues of (2.2) and (2.4), and that A^e should genuinely be thought of as an A_∞ *quadrmodule* (a *4-module* in the sense of [M]). We will bypass the general such notions, referring the reader to [M], and directly define the relevant dual (2.4).

2.2. Modules, bimodules, and tensor products. Recall that an A_∞ category \mathcal{C} consists of the data of

- a collection of objects $\text{ob } \mathcal{C}$
- for each pair of objects X, X' , a graded vector space $\text{hom}_{\mathcal{C}}(X, X')$
- for any set of $d + 1$ objects X_0, \dots, X_d , higher composition maps

$$(2.6) \quad \mu^d : \text{hom}_{\mathcal{C}}(X_{d-1}, X_d) \otimes \dots \otimes \text{hom}_{\mathcal{C}}(X_0, X_1) \rightarrow \text{hom}_{\mathcal{C}}(X_0, X_d)$$

of degree $2 - d$, satisfying the quadratic relations

$$(2.7) \quad \sum_{i,l} (-1)^{\mathfrak{X}_i} \mu_{\mathcal{A}}^{k-l+1}(x_k, \dots, x_{i+l+1}, \mu_{\mathcal{A}}^l(x_{i+l}, \dots, x_{i+1}), x_i, \dots, x_1) = 0.$$

with sign

$$(2.8) \quad \mathfrak{X}_i := \|x_1\| + \dots + \|x_i\|.$$

where $\|x\| := |x| - 1$ denotes the reduced degree.

The first two equations imply that μ^1 is a differential and μ^2 descends to a cohomology-level composition map, denoted $[\mu^2]$. We will suppose that any such \mathcal{C} is *homologically unital*, meaning that the cohomology of the morphism spaces $H^*(\text{hom}_{\mathcal{C}}(X, X))$ contain identity elements for the composition $[\mu^2]$. For notational ease, we will frequently index our objects by $i \in I$, and conflate our categories \mathcal{C} with algebras over a semi-simple ring $R := \bigoplus_{i \in I} \mathbb{K}e_i$ by thinking of \mathcal{C} as an A_∞ algebra $\bigoplus_{i,j} \text{hom}(X_i, X_j)$ over R , with idempotents acting as $e_s \cdot \mathcal{C} \cdot e_t := \text{hom}(X_t, X_s)$. In particular, tensor products of this algebra, always thought of as being over R , allows us some notational simplification:

$$(2.9) \quad \mathcal{C}^{\otimes r} := \bigoplus_{r, X_0, \dots, X_r \in \text{ob } \mathcal{C}} \text{hom}_{\mathcal{C}}(X_{r-1}, X_r) \otimes \dots \otimes \text{hom}_{\mathcal{C}}(X_0, X_1).$$

Similarly, vector space homomorphisms $\mathcal{C}^{\otimes r} \rightarrow \mathcal{C}$ should be R -linear, so a map $\mathcal{C}^{\otimes r}$ is exactly the data of, for each r, X_0, \dots, X_r , maps $\text{hom}_{\mathcal{C}}(X_{r-1}, X_r) \otimes \dots \otimes \text{hom}_{\mathcal{C}}(X_0, X_1) \rightarrow \text{hom}_{\mathcal{C}}(X_0, X_r)$ (See e.g., [S6, p. 7] or [S3, §6]).

Now, let \mathcal{C} and \mathcal{D} be a pair of A_∞ categories.

DEFINITION 2.2. An A_∞ \mathcal{C} - \mathcal{D} **bimodule** \mathcal{B} consists of the following data:

- for $V \in \text{ob } \mathcal{C}$, $V' \in \text{ob } \mathcal{D}$, a graded vector space $\mathcal{B}(V, V')$

- for $r, s \geq 0$, and objects $V_0, \dots, V_r \in \text{ob } \mathcal{C}$, $W_0, \dots, W_s \in \text{ob } \mathcal{C}'$, bimodule structure maps

$$(2.10) \quad \mu_{\mathcal{B}}^{r|1|s} : \text{hom}_{\mathcal{C}}(V_{r-1}, V_r) \otimes \dots \otimes \text{hom}_{\mathcal{C}}(V_0, V_1) \otimes \mathcal{B}(V_0, W_0) \otimes \\ \otimes \text{hom}_{\mathcal{D}}(W_1, W_0) \otimes \dots \otimes \text{hom}_{\mathcal{D}}(W_s, W_{s-1}) \longrightarrow \mathcal{B}(V_r, W_s)$$

of degree $1 - r - s$,

such that the following equations are satisfied, for each $r \geq 0$, $s \geq 0$:

$$(2.11) \quad \sum (-1)^{\mathfrak{X}_{-s}^{-(j+1)}} \mu_{\mathcal{B}}^{r-i|1|s-j}(v_r, \dots, v_{i+1}, \mu_{\mathcal{B}}^{i|1|j}(v_i, \dots, v_1, \mathbf{b}, w_1, \dots, w_j), w_{j+1}, \dots, w_s) \\ + \sum (-1)^{\mathfrak{X}_{-s}^k} \mu_{\mathcal{B}}^{r-i+1|1|s}(v_r, \dots, v_{k+i+1}, \mu_{\mathcal{C}}^i(v_{k+i}, \dots, v_{k+1}), v_k, \dots, v_1, \mathbf{b}, w_1, \dots, w_s) \\ + \sum (-1)^{\mathfrak{X}_{-s}^{-(l+j+1)}} \mu_{\mathcal{B}}^{r|1|s-j+1}(v_r, \dots, v_1, \mathbf{b}, w_1, \dots, w_l, \mu_{\mathcal{D}}^j(w_{l+1}, \dots, w_{l+j}), w_{l+j+1}, \dots, w_s) \\ = 0.$$

The signs above are given by the sum of the degrees of elements to the right of the inner operation, with the convention that we use **reduced degree** for elements of \mathcal{C} or \mathcal{D} and **full degree** for elements of \mathcal{B} . Thus,

$$(2.12) \quad \mathfrak{X}_{-s}^{-(j+1)} := \sum_{i=j+1}^s \|w_i\|,$$

$$(2.13) \quad \mathfrak{X}_{-s}^k := \sum_{i=1}^s \|w_i\| + |\mathbf{b}| + \sum_{j=1}^k \|v_j\|.$$

The first few equations imply that $\mu^{0|1|0}$ is a differential, and the left and right multiplications $\mu^{1|1|0}$ and $\mu^{0|1|1}$ descend to homology. The bimodule \mathcal{B} is said to be **homologically-unital** if the homology level multiplications $[\mu^{1|1|0}]$ and $[\mu^{0|1|1}]$ are unital, i.e. homology units in \mathcal{C} and \mathcal{D} act as the identity. We implicitly work only with homologically unital bimodules. If R and R' are the semi-simple rings correspondingly to \mathcal{C} and \mathcal{D} in the sense of above, then we can think of \mathcal{B} as a bimodule over R and R' ; equipped with a collection of morphisms $\mathcal{C}^r \otimes \mathcal{B} \otimes \mathcal{D}^s \rightarrow \mathcal{B}$ satisfying (2.11); again, the tensor product and morphism are now interpreted R and R' linearly, so this exactly reproduces the direct product of (2.10) for all ordered tuples of objects $V_0, \dots, V_r, W_0, \dots, W_s$. We give a condensed version of the next definition, using the semi-simple ring perspective:

DEFINITION 2.3. A **pre-morphism of \mathcal{C} - \mathcal{D} bimodules of degree k**

$$(2.14) \quad \mathcal{F} : \mathcal{B} \longrightarrow \mathcal{B}'$$

is the data of maps

$$(2.15) \quad \mathcal{F}^{r|1|s} : \mathcal{C}^{\otimes r} \otimes \mathcal{B} \otimes \mathcal{D}^{\otimes s} \longrightarrow \mathcal{B}', \quad r, s \geq 0.$$

of degree $k - r - s$. These can be packaged together into a total pre-morphism map

$$(2.16) \quad \mathcal{F} := \oplus \mathcal{F}^{r|1|s} : T\mathcal{C} \otimes \mathcal{B} \otimes T\mathcal{D} \longrightarrow \mathcal{B}',$$

where $T\mathcal{C}$ denotes the tensor algebra

$$(2.17) \quad T\mathcal{C} := \bigoplus_{r \geq 0} \mathcal{C}^{\otimes r} := \bigoplus_{r \geq 0; X_0, \dots, X_r} \text{hom}_{\mathcal{C}}(X_{r-1}, X_r) \otimes \dots \otimes \text{hom}_{\mathcal{C}}(X_0, X_1),$$

and the tensor product (2.16) is R linear on the left and R' -linear on the right; the associated sequence of morphisms in (2.16) is always composable.

Pre-morphisms from \mathcal{B} to \mathcal{B}' form a chain complex, denoted $\text{hom}_{\mathcal{C}-\mathcal{D}}(\mathcal{B}, \mathcal{B}')$ with differential

$$(2.18) \quad \delta(\mathcal{F}) := \mu_{\mathcal{N}} \circ \hat{\mathcal{F}} - (-1)^{|\mathcal{F}|} \mathcal{F} \circ \hat{\mu}_{\mathcal{M}}.$$

where

$$\begin{aligned}
(2.19) \quad \hat{\mu}_{\mathcal{B}}(c_k, \dots, c_1, \mathbf{b}, d_1, \dots, d_l) := & \\
& \sum (-1)^{\mathfrak{X}_{-l}^{-(t+1)}} c_k \otimes \dots \otimes c_{s+1} \otimes \mu_{\mathcal{B}}^{s|1|t}(c_s, \dots, c_1, \mathbf{b}, d_1, \dots, d_t) \otimes d_{t+1} \otimes \dots \otimes d_l \\
& + \sum (-1)^{\mathfrak{X}_{-l}^s} c_k \otimes \dots \otimes c_{s+i+1} \otimes \mu_{\mathcal{C}}^i(c_{s+i}, \dots, c_{s+1}) \otimes c_s \otimes \dots \otimes c_1 \otimes \\
& \quad \mathbf{b} \otimes d_1 \otimes \dots \otimes d_l \\
& + \sum (-1)^{\mathfrak{X}_{-l}^{-(j+t+1)}} c_k \otimes \dots \otimes c_1 \otimes \mathbf{b} \otimes d_1 \otimes \dots \otimes d_j \otimes \\
& \quad \mu_{\mathcal{D}}^t(d_{j+1}, \dots, d_{j+t}) \otimes d_{j+t+1} \otimes \dots \otimes d_l
\end{aligned}$$

and

$$(2.20) \quad \hat{\mathcal{F}}(c_k, \dots, c_1, \mathbf{b}, d_1, \dots, d_l) := \sum (-1)^{|\mathcal{F}| \cdot \mathfrak{X}_{-l}^{-(t+1)}} c_k \otimes \dots \otimes c_{s+1} \otimes \mathcal{F}(c_s, \dots, c_1, \mathbf{b}, d_1, \dots, d_t) \otimes d_{t+1} \otimes \dots \otimes d_l.$$

One can also compose pre-morphisms:

$$(2.21) \quad \mathcal{F}_2 \text{ "o" } \mathcal{F}_1 := \mathcal{F}_2 \circ \hat{\mathcal{F}}_1.$$

The fact that $\delta^2 = 0$ and that more generally that composition and δ defines a differential graded category

$$(2.22) \quad \mathcal{C}\text{-mod-}\mathcal{D}$$

follows from the A_∞ bimodule equations (2.11).

DEFINITION 2.4. *The **diagonal bimodule** \mathcal{A}_Δ is specified by the following data:*

$$(2.23) \quad \mathcal{A}_\Delta(X, Y) := \text{hom}_{\mathcal{A}}(Y, X)$$

$$(2.24) \quad \mu_{\mathcal{A}_\Delta}^{r|1|s}(c_r, \dots, c_1, \mathbf{c}, c'_1, \dots, c'_s) := (-1)^{\mathfrak{X}_{-s}^{-1}+1} \mu_{\mathcal{A}}^{r+1+s}(c_r, \dots, c_1, \mathbf{c}, c'_1, \dots, c'_s).$$

with

$$(2.25) \quad \mathfrak{X}_{-s}^{-1} := \sum_{i=1}^s \|c'_i\|.$$

As a special case of this construction, one obtains *left* and *right modules* over a single A_∞ category. Let \mathbb{K} be the trivial category with one object $*$, with $\text{hom}(*, *) = \mathbb{K}$, generated by a strict unit. Then, define the category of **left \mathcal{C} modules**

$$(2.26) \quad \mathcal{C}\text{-mod}$$

to be the category of $\mathcal{C}\text{-}\mathbb{K}$ bimodules with structure maps $\mu^{r|1|s} = 0$ if $s > 0$, and pre-morphisms $\mathcal{F}^{k|1|l} = 0$ for $l > 0$. For such a \mathcal{C} module \mathcal{M} , we abbreviate $\mathcal{M}(X) := \mathcal{M}(X, *)$, $\mu^{r|1} := \mu^{r|1|0}$ and $\mathcal{F}^{k|1} := \mathcal{F}^{k|1|0}$. Similarly, define the category of **right \mathcal{C} modules**

$$(2.27) \quad \text{mod-}\mathcal{C}$$

to be the category of $\mathbb{K}\text{-}\mathcal{C}$ bimodules with structure maps $\mu^{r|1|s} = 0$ if $r > 0$, and pre-morphisms $\mathcal{F}^{k|1|l} = 0$ for $k > 0$, with abbreviations $\mathcal{N}(X) := \mathcal{N}(X, *)$, $\mu^{1|s} := \mu^{0|1|s}$ and $\mathcal{F}^{1|l} := \mathcal{F}^{0|1|l}$.

REMARK 2.1. *Right/left \mathbb{K} modules or equivalently $\mathbb{K}\text{-}\mathbb{K}$ bimodules are simply chain complexes, via the identification $\mathcal{B} \leftrightarrow \mathcal{B}(*, *)$.*

DEFINITION 2.5. *Given an object $X \in \text{ob } \mathcal{C}$, the **left Yoneda-module** \mathcal{Y}_X^l over \mathcal{C} is defined by the following data:*

$$(2.28) \quad \mathcal{Y}_X^l(Y) := \text{hom}_{\mathcal{C}}(X, Y) \text{ for any } Y \in \text{ob } \mathcal{C}$$

$$\begin{aligned}
(2.29) \quad \mu^{r|1} : \text{hom}_{\mathcal{C}}(Y_{r-1}, Y_r) \times \text{hom}_{\mathcal{C}}(Y_{r-2}, Y_{r-1}) \times \dots \times \text{hom}_{\mathcal{C}}(Y_0, Y_1) \times \mathcal{Y}_X^l(Y_0) \longrightarrow \mathcal{Y}_X^l(Y_r) \\
(y_r, \dots, y_1, \mathbf{x}) \longmapsto (-1)^{\mathfrak{X}_0^r} \mu^{r+1}(y_r, \dots, y_1, \mathbf{x}),
\end{aligned}$$

with sign

$$(2.30) \quad \mathfrak{X}_0^r = \|\mathbf{x}\| + \sum_{i=1}^r \|y_i\|.$$

Similarly, the **right Yoneda-module** \mathcal{Y}_X^r over \mathcal{C} is defined by the following data:

$$(2.31) \quad \mathcal{Y}_X^r(Y) := \text{hom}_{\mathcal{C}}(Y, X) \text{ for any } Y \in \text{ob } \mathcal{C}$$

$$(2.32) \quad \begin{aligned} \mu^{1|s} : \mathcal{Y}_X^r(Y_s) \times \text{hom}_{\mathcal{C}}(Y_{s-1}, Y_s) \times \text{hom}_{\mathcal{C}}(Y_{s-2}, Y_{s-1}) \times \cdots \times \text{hom}_{\mathcal{C}}(Y_0, Y_1) &\rightarrow \mathcal{Y}_X^r(Y_0) \\ (\mathbf{x}, y_s, \dots, y_1) &\mapsto \mu^{s+1}(\mathbf{x}, y_s, \dots, y_1). \end{aligned}$$

DEFINITION 2.6. Given a pair $K \in \text{ob } \mathcal{C}$, $L \in \text{ob } \mathcal{D}$, the **Yoneda bimodule** (or **split bimodule**) over the pair K, L , denoted by $\mathcal{Y}_K^l \otimes_{\mathbb{K}} \mathcal{Y}_L^r$ or simply Y_{KL} , is the $\mathcal{C}-\mathcal{D}$ bimodule defined by the following data

$$(2.33) \quad \mathcal{Y}_K^l \otimes \mathcal{Y}_L^r(X, Y) := \text{hom}_{\mathcal{C}}(X, K) \otimes_{\mathbb{K}} \text{hom}_{\mathcal{D}}(L, Y) \text{ for any } (X, Y) \in \text{ob } \mathcal{C} \times \text{ob } \mathcal{D}$$

with structure maps

$$(2.34) \quad \mu^{r|1|s}(x_r, \dots, x_1, \mathbf{a} \otimes \mathbf{b}, y_1, \dots, y_s) = \begin{cases} 0 & r > 0 \text{ and } s > 0 \\ (-1)^{|\mathbf{b}|+1} \mu_{\mathcal{C}}^1(\mathbf{a}) \otimes \mathbf{b} + (-1)\mathbf{a} \otimes \mu_{\mathcal{D}}^1(\mathbf{b}) & r = s = 0 \\ (-1)^{|\mathbf{b}|+1} \mu_{\mathcal{Y}_K^l}^{r|1}(x_r, \dots, x_1, \mathbf{a}) \otimes \mathbf{b} & s = 0 \\ (-1)\mathbf{a} \otimes \mu_{\mathcal{Y}_L^r}^{1|s}(\mathbf{b}, y_1, \dots, y_s) & r = 0 \end{cases}$$

There are two relevant notions of tensor product for bimodules. The first notion, that of tensoring two bimodules over one side, can be thought of as composition of 1-morphisms in the 2-category of A_{∞} categories, where the 1-morphisms are categories of bimodules.

DEFINITION 2.7. Given a $\mathcal{C}-\mathcal{D}$ bimodule \mathcal{M} and an $\mathcal{D}-\mathcal{E}$ bimodule \mathcal{N} , the **(convolution) tensor product over \mathcal{D}**

$$(2.35) \quad \mathcal{M} \otimes_{\mathcal{D}} \mathcal{N}$$

is the $\mathcal{C}-\mathcal{E}$ bimodule given by

- underlying graded vector space

$$(2.36) \quad \mathcal{M} \otimes T\mathcal{D} \otimes \mathcal{N},$$

interpreted as an R' -linear tensor product, in the sense described earlier. Explicitly for any collection $A \in \text{ob } \mathcal{C}$, $B \in \text{ob } \mathcal{E}$, this is the vector space

$$(2.37) \quad \mathcal{M} \otimes_{\mathcal{D}} \mathcal{N}(A, B) := \bigoplus_{r \geq 0; X_0, \dots, X_r \in \text{ob } \mathcal{D}} \mathcal{M}(A, X_r) \otimes \text{hom}_{\mathcal{D}}(X_{r-1}, X_r) \otimes \cdots \otimes \text{hom}_{\mathcal{D}}(X_0, X_1) \otimes \mathcal{N}(X_0, B).$$

- differential

$$(2.38) \quad \mu_{\mathcal{M} \otimes_{\mathcal{D}} \mathcal{N}}^{0|1|0} : \mathcal{M} \otimes T\mathcal{D} \otimes \mathcal{N} \rightarrow \mathcal{M} \otimes T\mathcal{D} \otimes \mathcal{N}$$

given by

$$(2.39) \quad \begin{aligned} \mu^{0|1|0}(\mathbf{m}, d_1, \dots, d_k, \mathbf{n}) &= \\ &\sum (-1)^{\mathfrak{X}_{-(k+1)}^{-(t+1)}} \mu_{\mathcal{M}}^{0|1|t}(\mathbf{m}, d_1, \dots, d_t) \otimes d_{t+1} \otimes \cdots \otimes d_k \otimes \mathbf{n} \\ &+ \sum \mathbf{m} \otimes d_1 \otimes \cdots \otimes d_{k-s} \otimes \mu_{\mathcal{N}}^{s|1|0}(d_{k-s+1}, \dots, d_k, \mathbf{n}) \\ &+ \sum (-1)^{\mathfrak{X}_{-(k+1)}^{-(j+i+1)}} \mathbf{m} \otimes d_1 \otimes \cdots \otimes d_j \otimes \mu_{\mathcal{A}}^i(d_{j+1}, \dots, d_{j+i}) \otimes \\ &\quad d_{j+i+1} \otimes \cdots \otimes d_k \otimes \mathbf{n}. \end{aligned}$$

- for r or $s > 0$, higher bimodule maps

$$(2.40) \quad \mu_{\mathcal{M} \otimes_{\mathcal{D}} \mathcal{N}}^{r|1|s} : \mathcal{C}^{\otimes r} \otimes \mathcal{M} \otimes T\mathcal{D} \otimes \mathcal{N} \otimes \mathcal{E}^{\otimes s} \longrightarrow \mathcal{M} \otimes T\mathcal{D} \otimes \mathcal{N}$$

given by:

$$(2.41) \quad \begin{aligned} \mu^{r|1|0}(c_1, \dots, c_r, \mathbf{m}, d_1, \dots, d_k, \mathbf{n}) = \\ \sum_t (-1)^{\mathfrak{X}_{-(k+1)}^{-(t+1)}} \mu_{\mathcal{M}}^{r|1|t}(c_1, \dots, c_r, \mathbf{m}, d_1, \dots, d_t) \otimes d_{t+1} \otimes \dots \otimes d_k \otimes \mathbf{n} \end{aligned}$$

$$(2.42) \quad \begin{aligned} \mu^{0|1|s}(\mathbf{m}, d_1, \dots, d_k, \mathbf{n}, e_1, \dots, e_s) = \\ \sum_j \mathbf{m} \otimes d_1 \otimes \dots \otimes d_{k-j} \otimes \mu_{\mathcal{N}}^{j|1|s}(d_{k-j+1}, \dots, d_k, \mathbf{n}, e_1, \dots, e_s) \end{aligned}$$

and

$$(2.43) \quad \mu^{r|1|s} = 0 \text{ if } r > 0 \text{ and } s > 0.$$

In all equations above, the sign is the sum of degrees of all elements to the right, using reduced degree for elements of \mathcal{A} and full degree for elements of \mathcal{N} :

$$(2.44) \quad \mathfrak{X}_{-(k+1)}^{-t} := |\mathbf{n}| + \sum_{i=t}^k ||d_i||.$$

One can check that these maps indeed give $\mathcal{M} \otimes_{\mathcal{D}} \mathcal{N}$ the structure of an $\mathcal{C}-\mathcal{E}$ bimodule. In line with the two-categorical perspective, convolution with \mathcal{N} gives a dg functor (whose effect on morphisms we omit)

$$(2.45) \quad \cdot \otimes_{\mathcal{D}} \mathcal{N} : \mathcal{C}\text{-mod-}\mathcal{D} \longrightarrow \mathcal{C}\text{-mod-}\mathcal{E}.$$

One of the standard complications in theory of A_{∞} bimodules is that tensor product with the diagonal bimodule is only quasi-isomorphic to the identity. However, these quasi-isomorphisms are explicit, at least in one direction.

PROPOSITION 2.1. *Let \mathcal{B} be a homologically unital A_{∞} bimodule over \mathcal{A} . Then, there are quasi-isomorphisms of bimodules*

$$(2.46) \quad \begin{aligned} \mathcal{F}_{\Delta, right} : \mathcal{B} \otimes_{\mathcal{A}} \mathcal{A}_{\Delta} &\longrightarrow \mathcal{B} \\ \mathcal{F}_{\Delta, right}^{r|1|s} : \mathcal{A}^{\otimes r} \otimes \mathcal{B} \otimes T\mathcal{A} \otimes \mathcal{A}_{\Delta} \otimes \mathcal{A}^{\otimes s} &\longrightarrow \mathcal{B} \\ (a_1, \dots, a_r, \mathbf{b}, a_l^1, \dots, a_l^1, \mathbf{a}, a_1^2, \dots, a_s^2) &\longmapsto \\ (-1)^{\circ_{-s}^l} \mu_{\mathcal{B}}^{r|1|l+s+1}(a_1, \dots, a_r, \mathbf{b}, a_l^1, \dots, a_l^1, \mathbf{a}, a_1^2, \dots, a_s^2) \end{aligned}$$

given by the following data:

$$(2.47) \quad \begin{aligned} \mathcal{F}_{\Delta, left} : \mathcal{A}_{\Delta} \otimes_{\mathcal{A}} \mathcal{B} &\longrightarrow \mathcal{B} \\ \mathcal{F}_{\Delta, left}^{r|1|s} : \mathcal{A}^{\otimes r} \otimes \mathcal{A}_{\Delta} \otimes T\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{A}^{\otimes s} &\longrightarrow \mathcal{B} \\ (a_1, \dots, a_r, \mathbf{a}, a_l^1, \dots, a_l^1, \mathbf{b}, a_1^2, \dots, a_s^2) &\longmapsto \\ (-1)^{\star_{-s}^l} \mu_{\mathcal{B}}^{r+l+1|1|s}(a_1, \dots, a_r, \mathbf{a}, a_l^1, \dots, a_l^1, \mathbf{b}, a_1^2, \dots, a_s^2) \end{aligned}$$

with signs

$$(2.48) \quad \circ_{-s}^l := \sum_{n=1}^s ||a_n^2|| + |\mathbf{a}| - 1 + \sum_{m=1}^l ||a_m^1||$$

$$(2.49) \quad \star_{-s}^l := \sum_{n=1}^s ||a_n^2|| + |\mathbf{b}| - 1 + \sum_{m=1}^l ||a_m^1||.$$

To prove this, one takes the cone complex and filters by *length*; acyclicity then follows from relevant page 1 statement, which is the classical case of unital associative algebras and bimodules (see e.g., [S5, (2.20)], though note slightly different sign conventions). Now, suppose we took the tensor product with respect to the diagonal bimodule on the right and left of a bimodule \mathcal{B} . Then one compose two of the above quasi-isomorphisms to obtain a direct quasi-isomorphism

$$(2.50) \quad \mathcal{F}_{\Delta, left, right} := \mathcal{F}_{\Delta, left} \circ \mathcal{F}_{\Delta, right} : \mathcal{A}_{\Delta} \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{A}_{\Delta} \xrightarrow{\sim} \mathcal{B},$$

which we record is given by:

$$(2.51) \quad \begin{aligned} & \mathcal{F}_{\Delta, left, right}^{r|1|s} : \\ & a_1 \otimes \cdots \otimes a_r \otimes \mathbf{a} \otimes a'_1 \otimes \cdots \otimes a'_l \otimes \mathbf{b} \otimes a''_1 \otimes \cdots \otimes a''_k \otimes \mathbf{a}' \otimes a'''_1 \otimes \cdots \otimes a'''_s \\ & \longmapsto \sum \mu_{\mathcal{B}}^{r+1+i|1|s-j} (a_1, \dots, a_r, \mathbf{a}, a'_1, \dots, a'_i, \mu_{\mathcal{B}}^{l-i|1|k+1+j} (a'_{i+1}, \dots, a'_l, \\ & \quad \mathbf{b}, a''_1, \dots, a''_k, \mathbf{a}', a'''_1, \dots, a'''_j), a'''_{j+1}, \dots, a'''_s). \end{aligned}$$

up to signs that have already been discussed. There is an analogous morphism $\mathcal{F}_{\Delta, right, left}$ given by collapsing on the left first before collapsing to the right.

Next, given an $\mathcal{A}-\mathcal{B}$ bimodule \mathcal{M} and a $\mathcal{B}-\mathcal{A}$ bimodule \mathcal{N} , we can simultaneously tensor over the \mathcal{A} and \mathcal{B} module structures to obtain a chain complex:

DEFINITION 2.8. *The bimodule tensor product of \mathcal{M} and \mathcal{N} as above, denoted*

$$(2.52) \quad \mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}$$

is a chain complex defined as follows: As a vector space,

$$(2.53) \quad \mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N} := (\mathcal{M} \otimes T\mathcal{B} \otimes \mathcal{N} \otimes T\mathcal{A})^{diag},$$

where the diag superscript means to restrict to cyclically composable elements. The degree of a tensor is the sum of the degrees of the bimodule elements plus the reduced degrees of the category elements:

$$(2.54) \quad \deg(\mathbf{m} \otimes b_k \otimes \cdots \otimes b_1 \otimes \mathbf{n} \otimes a_1 \otimes \cdots \otimes a_l) := |\mathbf{m}| + \sum_{m=1}^k ||b_m|| + |\mathbf{n}| + \sum_{n=1}^l ||a_n||.$$

The differential on $\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}$ is

$$(2.55) \quad \begin{aligned} d_{\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}} : \mathbf{m} \otimes b_k \otimes \cdots \otimes b_1 \otimes \mathbf{n} \otimes a_1 \otimes \cdots \otimes a_l \longmapsto \\ \sum_{r,s} (-1)^{\#_{r,s}} \mu_{\mathcal{M}}^{l-r|1|k-s} (a_{r+1}, \dots, a_l, \mathbf{m}, b_k, \dots, b_{s+1}) \otimes b_s \otimes \cdots \otimes b_1 \otimes \\ \mathbf{n} \otimes a_1 \otimes \cdots \otimes a_r \\ + \sum_{i,r} (-1)^{\mathfrak{X}^i_{-i}} \mathbf{m} \otimes b_k \otimes \cdots \otimes b_{i+r+1} \otimes \mu_{\mathcal{B}}^r (b_{i+r}, \dots, b_{i+1}) \otimes b_i \otimes \cdots \otimes b_1 \otimes \\ \mathbf{n} \otimes a_1 \otimes \cdots \otimes a_l \\ + \sum_{j,s} (-1)^{\mathfrak{X}^{-(j+s+1)}_{-l}} \mathbf{m} \otimes b_k \otimes \cdots \otimes b_1 \otimes \mathbf{n} \otimes a_1 \otimes \cdots \otimes a_j \otimes \\ \mu_{\mathcal{A}}^s (a_{j+1}, \dots, a_{j+s}) \otimes a_{j+s+1} \otimes \cdots \otimes a_l \\ + \sum_{r,s} (-1)^{\mathfrak{X}^{-(s+1)}_{-l}} \mathbf{m} \otimes b_k \otimes \cdots \otimes b_{r+1} \otimes \mu_{\mathcal{N}}^{r|1|s} (b_r, \dots, b_1, \mathbf{n}, a_1, \dots, a_s) \otimes \\ a_{s+1} \otimes \cdots \otimes a_l \end{aligned}$$

with signs given by:

$$(2.56) \quad \mathfrak{K}_{-l}^{-t} := \sum_{n=t}^l \|a_n\|$$

$$(2.57) \quad \mathfrak{K}_{-l}^i := \sum_{n=1}^l \|a_n\| + |\mathbf{n}| + \sum_{m=1}^i \|b_m\|$$

$$(2.58) \quad \#_{i,r} := \left(\sum_{n=r+1}^l \|a_n\| \right) \cdot \left(|\mathbf{m}| + \sum_{m=1}^k \|b_m\| + |\mathbf{n}| + \sum_{n=1}^r \|a_n\| \right) + \mathfrak{K}_{-r}^s.$$

The sign (2.58) should be thought of as the Koszul sign coming from moving a_{r+1}, \dots, a_l past all the other elements, applying $\mu_{\mathcal{M}}$ (which acts from the right), and then moving the result to the left.

The bimodule tensor product is functorial in the following sense. If

$$(2.59) \quad \mathcal{F} : \mathcal{N} \longrightarrow \mathcal{N}'$$

is any pre-morphism of $\mathcal{B}-\mathcal{A}$ bimodules, then there is an induced morphism

$$(2.60) \quad \mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N} \xrightarrow{\mathcal{F}_{\#}} \mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}'$$

given by summing with signs over all ways to collapse some of the terms around the element of \mathcal{N} by the various $\mathcal{F}^{r|1|^s}$, which can be concisely written as

$$(2.61) \quad \mathcal{F}_{\#}(\mathbf{m} \otimes b_1 \otimes \dots \otimes b_k \otimes \mathbf{n} \otimes a_1 \otimes \dots \otimes a_l) := \mathbf{m} \otimes \hat{\mathcal{F}}(b_1, \dots, b_k, \mathbf{n}, a_1, \dots, a_l),$$

where $\hat{\mathcal{F}}$ is defined in (2.20). A direct computation implies:

LEMMA 2.1. *The association (2.61) gives a chain map*

$$(2.62) \quad \begin{aligned} \text{hom}_{\mathcal{B}-\mathcal{A}}(\mathcal{N}, \mathcal{N}') \otimes (\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}) &\rightarrow \mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N} \\ (\mathcal{F}, \beta) &\mapsto \mathcal{F}_{\#}(\beta), \end{aligned}$$

which is natural with respect to composition of bimodules in the sense that the following diagram of chain maps strictly commutes:

$$(2.63) \quad \begin{array}{ccc} \text{hom}_{\mathcal{B}-\mathcal{A}}(\mathcal{N}_1, \mathcal{N}_2) \otimes \text{hom}_{\mathcal{B}-\mathcal{A}}(\mathcal{N}_0, \mathcal{N}_1) \otimes (\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}_0) & \longrightarrow & \text{hom}_{\mathcal{B}-\mathcal{A}}(\mathcal{N}_0, \mathcal{N}_2) \otimes (\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}_0) \\ \downarrow & & \downarrow \\ \text{hom}_{\mathcal{B}-\mathcal{A}}(\mathcal{N}_1, \mathcal{N}_2) \otimes (\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}_1) & \longrightarrow & (\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} \mathcal{N}_2) \end{array}$$

Put another way $(\mathcal{M} \otimes_{\mathcal{A}-\mathcal{B}} -)$ is a dg functor from $\mathcal{B}-\mathcal{A}$ bimodules to chain complexes.

COROLLARY 2.1. *If \mathcal{F} is a closed bimodule morphism, then (2.61) is a chain map. If \mathcal{F} is a bimodule quasi-isomorphism, then the induced map (2.61) is a quasi-isomorphism of chain complexes.*

We will also need to understand operations on modules and bimodules induced by A_{∞} functors. Recall that an A_{∞} **functor**

$$(2.64) \quad \mathbf{F} : \mathcal{C} \longrightarrow \mathcal{C}'$$

consists of the following data:

- For each object X in \mathcal{C} , an object $\mathbf{F}(X)$ in \mathcal{C}' ,
- for any set of $d+1$ objects X_0, \dots, X_d , higher maps

$$(2.65) \quad \mathbf{F}^d : \text{hom}_{\mathcal{C}}(X_{d-1}, X_d) \times \dots \times \text{hom}_{\mathcal{C}}(X_0, X_1) \longrightarrow \text{hom}_{\mathcal{C}'}(\mathbf{F}(X_0), \mathbf{F}(X_d))$$

of degree $1-d$, satisfying the relations:

$$(2.66) \quad \begin{aligned} \sum_{j; i_1 + \dots + i_j = k} \mu_{\mathcal{B}}^j(\mathbf{F}^{i_j}(x_k, \dots, x_{k-i_j+1}), \dots, \mathbf{F}^{i_1}(x_{i_1}, \dots, x_1)) = \\ \sum_{s \leq k, t} (-1)^{\mathfrak{K}_t} \mathbf{F}^{k-s+1}(x_k, \dots, x_{t+s+1}, \mu_{\mathcal{A}}^s(x_{t+s}, \dots, x_{t+1}), x_t, \dots, x_1). \end{aligned}$$

Here, $\star_t = \sum_{i=1}^t ||x_i||$ is the same (Koszul) sign as before.

The equations (2.66) imply that the first-order term of any morphism or functor descends to a cohomology level functor $[\mathbf{F}^1]$. We say that a morphism \mathbf{F} is a **quasi-isomorphism** if $[\mathbf{F}^1]$ is an isomorphism. Call a functor \mathbf{F} **(quasi-)full and faithful** if $[\mathbf{F}^1]$ is an isomorphism onto a full subcategory of the cohomology of the image, and call it a **quasi-equivalence** if it $[\mathbf{F}^1]$ is also essentially surjective.

EXAMPLE 2.1. In the context of modules, the first example of an A_∞ functor comes from the **Yoneda embeddings**: namely, there are A_∞ functors

$$(2.67) \quad \begin{aligned} \mathbf{Y}_L : \mathcal{C}^{op} &\longrightarrow \mathcal{C}\text{-mod} \\ \mathbf{Y}_R : \mathcal{C} &\longrightarrow \text{mod-}\mathcal{C} \end{aligned}$$

defined on objects as $\mathbf{Y}_L(X) := \mathcal{Y}_X^l$ and $\mathbf{Y}_R(X) := \mathcal{Y}_X^r$. We will not spell out the maps \mathbf{Y}_L^d , \mathbf{Y}_R^d on morphism spaces yet, but note that as long as \mathcal{C} is homologically unital, \mathbf{Y}_L and \mathbf{Y}_R are full and faithful (see [S1, Lemma 2.13] or [G2, Def. 2.21-2.22] for more details on both points).

Given an A_∞ functor

$$(2.68) \quad \mathfrak{F} : \mathcal{A} \rightarrow \mathcal{B}$$

there is an associated pull-back functor on modules

$$(2.69) \quad \mathfrak{F}^* : \mathcal{B}\text{-mod} \longrightarrow \mathcal{A}\text{-mod},$$

defined as follows:

DEFINITION 2.9. *Given a right \mathcal{B} module \mathcal{M} with structure maps $\mu_{\mathcal{M}}^{1|r}$, and an A_∞ functor $\mathfrak{F} : \mathcal{A} \rightarrow \mathcal{B}$, define the **pullback of \mathcal{M} along \mathfrak{F}** to be the right \mathcal{A} module*

$$(2.70) \quad \mathfrak{F}^*\mathcal{M}(Y) := \mathcal{M}(\mathfrak{F}(Y)), \quad Y \in \text{ob } \mathcal{A}$$

with module structure maps

$$(2.71) \quad \mu_{\mathfrak{F}^*\mathcal{M}}^{1|r}(\mathbf{a}, a_1, \dots, a_r) := \sum_{k, i_1 + \dots + i_k = r} \mu_{\mathcal{M}}^{1|k}(\mathbf{a}, \mathfrak{F}^{i_1}(a_1, \dots), \dots, \mathfrak{F}^{i_k}(\dots, a_r)).$$

Here on the right side, \mathbf{a} is simply \mathbf{a} thought of as living in some $\mathcal{M}(\mathfrak{F}(Y))$ instead of $\mathfrak{F}^\mathcal{M}(Y)$.*

EXAMPLE 2.2. Given an A_∞ category \mathcal{C} and a collection of objects $\{X_i\}$ in \mathcal{C} , let \mathcal{X} be the full subcategory of \mathcal{C} with objects $\{X_i\}$. Then the naive inclusion functor

$$(2.72) \quad \iota : \mathcal{X} \hookrightarrow \mathcal{C},$$

induces a pullback on modules

$$(2.73) \quad \begin{aligned} \iota^* : \mathcal{C}\text{-mod} &\longrightarrow \mathcal{X}\text{-mod} \\ \iota^* : \text{mod-}\mathcal{C} &\longrightarrow \text{mod-}\mathcal{X} \end{aligned}$$

which is the ordinary restriction. Namely, a \mathcal{C} module such as

$$(2.74) \quad \mathcal{Y}_Z^r; \quad Z \in \text{ob } \mathcal{C}$$

induces a module

$$(2.75) \quad \iota^* \mathcal{Y}_Z^r$$

over \mathcal{X} , in which one pairs Z only with objects in \mathcal{X} . We will often refer to this module simply as \mathcal{Y}_Z^r when the category \mathcal{X} is explicit.

We can repeat these definitions for contravariant functors, which we will need. Namely, let

$$(2.76) \quad \tilde{\mathfrak{F}} : \mathcal{A}^{op} \rightarrow \mathcal{B}$$

be a contravariant A_∞ functor, which consists of maps

$$(2.77) \quad \tilde{\mathfrak{F}}^1 : \mathcal{A}(X, Y) \rightarrow \mathcal{B}(\tilde{\mathfrak{F}}(Y), \tilde{\mathfrak{F}}(X))$$

and higher order maps

$$(2.78) \quad \tilde{\mathfrak{F}}^d : \mathcal{A}(X_{d-1}, X_d) \otimes \mathcal{A}(X_{d-2}, X_{d-1}) \otimes \dots \otimes \mathcal{A}(X_0, X_1) \longrightarrow \mathcal{B}(\tilde{\mathfrak{F}}(X_d), \tilde{\mathfrak{F}}(X_0))$$

satisfying the following equations

$$(2.79) \quad \sum \mu_{\mathcal{B}} (\tilde{\mathfrak{F}}^{i_k}(\dots a_k) \tilde{\mathfrak{F}}^{i_{k-1}}(\dots) \dots \tilde{\mathfrak{F}}^{i_1}(a_1, \dots, a_{i_1})) = \sum \tilde{\mathfrak{F}}(a_1 \dots \mu_{\mathcal{A}}(\dots) \dots a_k).$$

(notice the order reversal in the a_i). In this case, pull-back changes the direction of the module action

$$(2.80) \quad \tilde{\mathfrak{F}}^* : \mathcal{B}\text{-mod} \longrightarrow \text{mod-}\mathcal{A}$$

DEFINITION 2.10. *Given a left \mathcal{B} module \mathcal{M} , and a contravariant functor $\tilde{\mathfrak{F}} : \mathcal{A}^{op} \rightarrow \mathcal{B}$ as above, the **pulled-back right module** $\tilde{\mathfrak{F}}^*\mathcal{M}$ is defined by*

$$(2.81) \quad \begin{aligned} \tilde{\mathfrak{F}}^*\mathcal{M}(X) &= \mathcal{M}(\tilde{\mathfrak{F}}(X)), \quad X \in \text{ob } \mathcal{A} \\ \mu_{\tilde{\mathfrak{F}}^*\mathcal{M}}^{1|r}(\mathbf{m}, a_1, \dots, a_r) &= \sum_{k, i_1 + \dots + i_k = r} \mu_{\mathcal{M}}^{k|1}(\tilde{\mathfrak{F}}^{i_k}(\dots, a_r) \dots \tilde{\mathfrak{F}}^{i_1}(a_1, \dots, a_{i_1}), \mathbf{m}) \end{aligned}$$

This entire process can be repeated for bimodules, with contravariant or covariant functors. Given A_∞ categories $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ and functors

$$(2.82) \quad \begin{aligned} \mathfrak{F} : \mathcal{A}_1 &\longrightarrow \mathcal{A}_2 \\ \mathfrak{G} : \mathcal{B}_1 &\longrightarrow \mathcal{B}_2 \end{aligned}$$

there is an associated pull-back functor

$$(2.83) \quad (\mathfrak{F} \otimes \mathfrak{G})^* : \mathcal{A}_2\text{-mod-}\mathcal{B}_2 \longrightarrow \mathcal{A}_1\text{-mod-}\mathcal{B}_1$$

defined as follows: If $\mathcal{M} \in \text{ob } \mathcal{A}_2\text{-mod-}\mathcal{B}_2$, then

$$(2.84) \quad (\mathfrak{F} \otimes \mathfrak{G})^*\mathcal{M}(A, B) := \mathcal{M}(\mathfrak{F}(A), \mathfrak{G}(B)), \quad A \in \text{ob } \mathcal{A}_1, \quad B \in \text{ob } \mathcal{B}_1$$

with structure maps

$$(2.85) \quad \begin{aligned} \mu_{(\mathfrak{F} \otimes \mathfrak{G})^*\mathcal{M}}^{r|1|s}(a_r, \dots, a_1, \mathbf{m}, b_1, \dots, b_s) &:= \\ \sum_{k, i_1 + \dots + i_k = r} \sum_{l, j_1 + \dots + j_l = s} \mu_{\mathcal{M}}^{k|1|l}(\tilde{\mathfrak{F}}^{i_k}(a_r, \dots), \dots, \tilde{\mathfrak{F}}^{i_1}(\dots a_1), \\ &\quad \mathbf{m}, \mathfrak{G}^{j_1}(b_1, \dots), \dots, \mathfrak{G}^{j_l}(\dots, b_s)). \end{aligned}$$

Once more, \mathbf{m} is simply the element \mathbf{m} thought of as living in $\mathcal{M}(\mathfrak{F}(A), \mathfrak{G}(B))$ for some A, B . Finally, abbreviate the pull-back $(\mathfrak{F} \otimes \mathfrak{G})^*$ by simply \mathfrak{F}^* .

The Yoneda embeddings \mathbf{Y}_L and \mathbf{Y}_R behave compatibly with pullback, in the sense that for any functor $\mathbf{F} : \mathcal{C} \longrightarrow \mathcal{D}$ there are natural transformation of functors

$$(2.86) \quad \begin{aligned} \mathfrak{T}_L^{\mathbf{F}} : (\mathbf{Y}_L)_c &\longrightarrow \mathbf{F}^* \circ (\mathbf{Y}_L)_d \circ \mathbf{F} \\ \mathfrak{T}_R^{\mathbf{F}} : (\mathbf{Y}_R)_c &\longrightarrow \mathbf{F}^* \circ (\mathbf{Y}_R)_d \circ \mathbf{F}. \end{aligned}$$

We will not need the full data of this natural transformation (the interested reader is referred to [S5, eq. (1.23)]), but to first order, we obtain morphisms of modules

$$(2.87) \quad \begin{aligned} (\mathfrak{T}_L^{\mathbf{F}})_X : \mathcal{Y}_X^l &\longrightarrow \mathbf{F}^* \mathcal{Y}_{\mathbf{F}(X)}^l \\ (\mathfrak{T}_R^{\mathbf{F}})_X : \mathcal{Y}_X^r &\longrightarrow \mathbf{F}^* \mathcal{Y}_{\mathbf{F}(X)}^r. \end{aligned}$$

given by

$$(2.88) \quad \begin{aligned} (\mathfrak{T}_L^{\mathbf{F}})_X^{r|1} &(a_1, \dots, a_r, \mathbf{x}) := \mathbf{F}^{r+1}(a_1, \dots, a_r, \mathbf{x}). \\ (\mathfrak{T}_R^{\mathbf{F}})_X^{1|s} &(\mathbf{x}, a_1, \dots, a_s) := \mathbf{F}^{s+1}(\mathbf{x}, a_1, \dots, a_s). \end{aligned}$$

Finally, tensoring $(\mathfrak{T}_L^{\mathbf{F}})_X$ and $(\mathfrak{T}_R^{\mathbf{F}})_Z$, we obtain associated morphisms of Yoneda bimodules

$$(2.89) \quad (\mathfrak{T}_{LR}^{\mathbf{F}})_{X,Z} := (\mathfrak{T}_L^{\mathbf{F}})_X \otimes (\mathfrak{T}_R^{\mathbf{F}})_Z : \mathcal{Y}_X^l \otimes \mathcal{Y}_Z^r \longrightarrow \mathbf{F}^* \mathcal{Y}_{\mathbf{F}(X)}^l \otimes \mathbf{F}^* \mathcal{Y}_{\mathbf{F}(Z)}^r.$$

These maps are quasi-isomorphisms if \mathbf{F} is. There are also analogous versions of these natural transformations for contravariant functors \mathbf{G} , in which naturally, Yoneda lefts and rights get reversed:

$$(2.90) \quad \begin{aligned} (\mathfrak{T}_L^{\mathbf{G}})_X : \mathcal{Y}_X^l &\longrightarrow \mathbf{G}^* \mathcal{Y}_{\mathbf{G}(X)}^r \\ (\mathfrak{T}_R^{\mathbf{G}})_X : \mathcal{Y}_X^r &\longrightarrow \mathbf{G}^* \mathcal{Y}_{\mathbf{G}(X)}^l. \end{aligned}$$

2.3. Perfection and smoothness. Let $X \subset C$ be a full subcategory of a triangulated category. We say that X **split-generates** C if every element of C is isomorphic to a summand of a finite iterated cone of elements in X . We say a triangulated category is **split-closed** if any idempotent endomorphism of an object Z leads to a splitting of that object as a direct sum $X \oplus Y$.

Now recall that, for an A_∞ category \mathcal{C} , the category of (right, but also left) modules $\text{mod-}\mathcal{C}$ is naturally **pre-triangulated**, meaning the cohomology level category $H^0(\text{mod-}\mathcal{C})$ is triangulated—we can take sums, shifts, mapping cones, and general complexes of modules are all still modules. Moreover, as long as \mathcal{C} is homologically unital, the natural **Yoneda embeddings** (2.67) of \mathcal{C} into $\text{mod-}\mathcal{C}$ or $\mathcal{C}\text{-mod}$ are full and faithful, as noted in Example 2.1. Thus, one can think of objects of \mathcal{C} as belonging in a functorial way to the pre-triangulated split-closed categories $\text{mod-}\mathcal{C}$ or $\mathcal{C}\text{-mod}$. In particular, $H^0(\mathcal{C})$ is naturally a sub-category of $H^0(\mathcal{C}\text{-mod})$ and $H^0(\text{mod-})$, so one can talk about its split-closure within these categories.

Next, there is a notion of an **idempotent up to homotopy** [S5, (4b)] which would allow us to properly extend discussion of split-generation to the chain level. However, it is also known that a cohomology level idempotent endomorphism can always be lifted, essentially uniquely, to an idempotent up to homotopy [S5, Lemma 4.2]. Thus, for our purposes, it is sufficient to use the following definition:

DEFINITION 2.11. *Let \mathcal{C} be an A_∞ category, and $\mathcal{X} \subset \mathcal{C}$ a full subcategory. We say that \mathcal{X} **split-generates** \mathcal{C} if any Yoneda module in $\text{mod-}\mathcal{C}$ admits a homologically left-invertible morphism into a (finite) complex of Yoneda modules of objects of \mathcal{X} .*

If i is the (homology-level) morphism and p is the (homology) left inverse, then the reverse composition $i \circ p$ is the idempotent that exhibits the target module as a homological summand of the larger complex.

DEFINITION 2.12. *Call a (right or left) module \mathcal{M} over \mathcal{X} **perfect** if it admits a homologically left-invertible morphism into a finite complex of Yoneda modules of objects of \mathcal{X} .*

Given a collection of objects $\{X_i\}$ in an A_∞ category \mathcal{C} , it is then natural to ask when they split-generate another object Z . There is a criterion for split-generation, known to category theorists and first introduced in the symplectic/ A_∞ setting by Abouzaid [A1], as follows. Denote by

$$(2.91) \quad \mathcal{X}$$

the full sub-category of \mathcal{C} with objects $\{X_i\}$. Then, one can form the chain complex

$$(2.92) \quad \mathcal{Y}_Z^r \otimes_{\mathcal{X}} \mathcal{Y}_Z^l$$

where the above notation indicates that the Yoneda modules $\mathcal{Y}_Z^r, \mathcal{Y}_Z^l$ are thought of as modules over \mathcal{X} via the inclusion $\mathcal{X} \subset \mathcal{C}$ as in Example 2.2. Concretely, this bar complex is given as

$$(2.93) \quad \bigoplus_{k \geq 1} \bigoplus_{X_{i_1}, \dots, X_{i_k} \in \text{ob } \mathcal{X}} \text{hom}_{\mathcal{C}}(X_{i_k}, Z) \otimes \text{hom}_{\mathcal{X}}(X_{i_{k-1}}, X_{i_k}) \otimes \dots \otimes \text{hom}_{\mathcal{X}}(X_{i_1}, X_{i_2}) \otimes \text{hom}_{\mathcal{C}}(Z, X_{i_1})$$

with differential given by summing over all ways to collapse some (but not all) of the terms with a μ :

$$(2.94) \quad \begin{aligned} d(\mathbf{a} \otimes x_k \otimes \dots \otimes x_1 \otimes \mathbf{b}) &= \sum_i (-1)^{\mathfrak{X}_0^i} \mu^i(\mathbf{a}, x_k, \dots, x_{i+1}) \otimes x_i \otimes \dots \otimes x_k \otimes \mathbf{b} \\ &+ \sum_{j,r} (-1)^{\mathfrak{X}_0^r} \mathbf{a} \otimes \dots \otimes \mu^j(x_{r+j}, \dots, x_{r+1}) \otimes x_r \otimes \dots \otimes x_1 \otimes \mathbf{b} \\ &+ \sum_s (-1)^{\mathfrak{X}_0^s} \mathbf{a} \otimes \dots \otimes x_{s+1} \otimes \mu^{s+1}(x_s, \dots, x_1, \mathbf{b}), \end{aligned}$$

where the sign is the usual

$$(2.95) \quad \mathfrak{X}_0^t := |\mathbf{b}| + \sum_{j=1}^t \|x_j\|.$$

There is a collapsing morphism

$$(2.96) \quad \begin{aligned} \mathcal{Y}_Z^r \otimes_{\mathcal{X}} \mathcal{Y}_Z^l &\xrightarrow{\mu} \text{hom}_{\mathcal{C}}(Z, Z) \\ \mathbf{a} \otimes x_k \otimes \dots \otimes x_1 \otimes \mathbf{b} &\longmapsto (-1)^{\mathfrak{X}_0^k} \mu^{k+2}(\mathbf{a}, x_k, \dots, x_1, \mathbf{b}). \end{aligned}$$

which is a chain map, inducing a homology level morphism

$$(2.97) \quad [\mu] : H^*(\mathcal{Y}_Z^r \otimes_{\mathcal{X}} \mathcal{Y}_Z^l) \longrightarrow H^*(\mathrm{hom}_{\mathcal{C}}(Z, Z)).$$

This map can be thought of as the first piece of information involving Z from the category \mathcal{C} that is not already contained in the \mathcal{X} modules \mathcal{Y}_Z^r and \mathcal{Y}_Z^l . The following proposition, relates a checkable criterion involving the map $[\mu]$ to the split-generation of Z .

PROPOSITION 2.2 ([A1, Lemma 1.4]). *The following two statements are equivalent:*

$$(2.98) \quad \text{The identity element } [e_Z] \in H^*(\mathrm{hom}_{\mathcal{C}}(Z, Z)) \text{ is in the image of } [\mu].$$

$$(2.99) \quad \text{The object } Z \text{ is split generated by the } \{X_i\}.$$

There is a direct proof of this fact given in [A1, §A], but morally, the reason is this: the cone of the map μ computes the Hom in the quotient \mathcal{C}/\mathcal{X} . The above condition is equivalent to saying the identity morphism $[e_Z]$ in this quotient is trivial, or that the object Z is trivial in \mathcal{C}/\mathcal{X} . But by general properties of quotients [D, Thm 3.4], only objects split generated by \mathcal{X} are trivial in the quotient.

From this perspective, it is perhaps unsurprising that the criterion (2.98) in turn implies that we completely understand the map $[\mu]$:

PROPOSITION 2.3. *If the identity element $[e_Z]$ is in the image of $[\mu]$, then the map $[\mu]$ is an isomorphism.*

We omit a proof from the paper, as it is not immediately relevant to our goals, but an explicit proof can be found in [G1, Prop. 2.7]. Of course, it also follows from the general perspective on quotients discussed above. Since the converse of the Proposition 2.3 is trivially true, we see that

COROLLARY 2.2. *The following three statements are equivalent:*

$$(2.100) \quad \text{The identity element } [e_Z] \in H^*(\mathrm{hom}_{\mathcal{C}}(Z, Z)) \text{ is in the image of } [\mu].$$

$$(2.101) \quad \text{The object } Z \text{ is split generated by the } \{X_i\}.$$

$$(2.102) \quad \text{The map } [\mu] \text{ is an isomorphism.}$$

PROPOSITION 2.4. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be an A_∞ functor, such that for a full subcategory $\mathcal{X} \subseteq \mathcal{C}$, \mathcal{F} is quasi-full. Then, if \mathcal{X} split generates \mathcal{C} , \mathcal{F} is quasi-full on \mathcal{C} .*

PROOF. This result does not have a clean proof in the literature. The case that \mathcal{X} (non-split)-generates \mathcal{C} can be found in [S5, Lemma 3.25]. Parts of the argument for the split-closed case are in [S2, Lemma 2.5]. \square

There are some final definitions we will need in the next subsection.

DEFINITION 2.13. *A \mathcal{C} – \mathcal{D} bimodule \mathcal{B} is said to be **perfect** if it is split-generated by a finite collection of Yoneda bimodules*

$$(2.103) \quad \mathcal{Y}_{X_i}^l \otimes_{\mathbb{K}} \mathcal{Y}_{Y_j}^r.$$

DEFINITION 2.14 (*homological smoothness*, compare [KS, §8]). *An A_∞ category \mathcal{C} is said to be **homologically smooth** if the diagonal bimodule \mathcal{C}_Δ is a perfect \mathcal{C} – \mathcal{C} bimodule.*

REMARK 2.2. *It is known that the dg category of perfect complexes on a variety X is homologically smooth if and only if X itself is smooth in the ordinary sense. This provides some justification for the usage of the term “smooth.” See for example [Kr, Thm. 3.8] for an exposition of this result in the case of affine varieties.*

2.4. Module and bimodule duality. We have seen that the convolution tensor product with an \mathcal{C} – \mathcal{D} bimodule \mathcal{B} induces dg functors of the form

$$(2.104) \quad \begin{aligned} \cdot \otimes_{\mathcal{C}} \mathcal{B} : \mathrm{mod}\text{--}\mathcal{C} &\longrightarrow \mathrm{mod}\text{--}\mathcal{D} \\ \mathcal{B} \otimes_{\mathcal{D}} \cdot : \mathcal{D}\text{--}\mathrm{mod} &\longrightarrow \mathcal{C}\text{--}\mathrm{mod} \end{aligned}$$

We can also define dual, or adjoint functors

$$(2.105) \quad \mathrm{hom}_{\mathcal{C}\text{--}\mathrm{mod}}(\cdot, \mathcal{B}) : \mathcal{C}\text{--}\mathrm{mod} \longrightarrow \mathrm{mod}\text{--}\mathcal{D}$$

$$(2.106) \quad \mathrm{hom}_{\mathrm{mod}\text{--}\mathcal{D}}(\cdot, \mathcal{B}) : \mathrm{mod}\text{--}\mathcal{D} \longrightarrow \mathcal{C}\text{--}\mathrm{mod}.$$

DEFINITION 2.15. Let \mathcal{M} be an A_∞ left module over a category \mathcal{C} , and \mathcal{B} a $\mathcal{C}-\mathcal{D}$ bimodule. The right \mathcal{D} module

$$(2.107) \quad \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{M}, \mathcal{B})$$

is specified by the following data:

- For each object Y of \mathcal{D} , a graded vector space

$$(2.108) \quad \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{M}, \mathcal{B})(Y) := \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{M}, \mathcal{B}(\cdot, Y))$$

whose elements \mathcal{F} are collections of maps

$$(2.109) \quad \mathcal{F} = \oplus_r \mathcal{F}^{r|1} : \oplus_{X_0, \dots, X_r} \text{hom}_{\mathcal{C}}(X_{r-1}, X_r) \times \cdots \times \text{hom}_{\mathcal{C}}(X_0, X_1) \times \mathcal{M}(X_0) \longrightarrow \mathcal{B}(X_r, Y).$$

- A differential

$$(2.110) \quad \mu_{\text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{M}, \mathcal{B})}^{1|0} : \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{M}, \mathcal{B})(Y) \longrightarrow \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{M}, \mathcal{B})(Y)$$

given by the differential in the dg category of left \mathcal{C} modules

$$(2.111) \quad d\mathcal{F} = \mathcal{F} \circ \hat{\mu}_{\mathcal{M}} - \mu_{\mathcal{B}} \circ \hat{\mathcal{F}}.$$

Above, $\mu_{\mathcal{B}}$ is the total left-sided bimodule structure map $\oplus \mu_{\mathcal{B}}^{r|1|0}$.

- Higher right multiplications:

$$(2.112) \quad \mu^{1|s} : \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{M}, \mathcal{B})(Y_0) \times \text{hom}_{\mathcal{D}}(Y_1, Y_0) \times \cdots \times \text{hom}_{\mathcal{D}}(Y_s, Y_{s-1}) \longrightarrow \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{M}, \mathcal{B})(Y_s)$$

given by

$$(2.113) \quad \mu^{1|s}(\mathcal{F}, y_1, \dots, y_s) := \mathcal{F}_{y_1, \dots, y_s} \in \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{M}, \mathcal{B}(\cdot, Y_s))$$

where $\mathcal{F}_{y_1, \dots, y_s}$ is the morphism specified by the following data

$$(2.114) \quad \mathcal{F}_{y_1, \dots, y_s}^{k|1}(x_1, \dots, x_k, \mathbf{m}) = \sum_i \mu_{\mathcal{B}}^{i|1|s}(x_1, \dots, x_i, \mathcal{F}^{k-i|1}(x_{i+1}, \dots, x_k, \mathbf{m}), y_1, \dots, y_s).$$

REMARK 2.3. Note that for any $\mathcal{C}-\mathcal{D}$ bimodule \mathcal{B} , $\mathcal{B}(\cdot, Y)$ is a left \mathcal{C} module with structure maps $\mu_{\mathcal{B}}^{r|1|0}$, for any object $Y \in \text{ob } \mathcal{D}$. This is implicit in our construction above. Similarly, $\mathcal{B}(X, \cdot)$ is a right \mathcal{D} module with structure maps $\mu_{\mathcal{B}}^{0|1|s}$.

DEFINITION 2.16. Let \mathcal{N} be an A_∞ right module over a category \mathcal{D} , and \mathcal{B} a $\mathcal{C}-\mathcal{D}$ bimodule. The left \mathcal{C} module

$$(2.115) \quad \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{N}, \mathcal{B})$$

is specified by the following data:

- For each object X of \mathcal{C} , a graded vector space

$$(2.116) \quad \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{N}, \mathcal{B})(X) := \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{N}, \mathcal{B}(X, \cdot))$$

whose elements \mathcal{G} are collections of maps

$$(2.117) \quad \mathcal{G} = \oplus_s \mathcal{G}^{1|s} : \oplus_{Y_0, \dots, Y_s} \mathcal{N}(Y_0) \times \text{hom}_{\mathcal{D}}(Y_1, Y_0) \times \cdots \times \text{hom}_{\mathcal{D}}(Y_s, Y_{s-1}) \longrightarrow \mathcal{B}(X, Y_s).$$

- A differential

$$(2.118) \quad \mu_{\text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{N}, \mathcal{B})}^{1|0} : \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{N}, \mathcal{B})(Y) \longrightarrow \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{N}, \mathcal{B})(Y)$$

given by the differential in the dg category of right \mathcal{D} modules

$$(2.119) \quad d\mathcal{G} = \mathcal{G} \circ \hat{\mu}_{\mathcal{N}} - \mu_{\mathcal{B}} \circ \hat{\mathcal{G}}.$$

Above, $\mu_{\mathcal{B}}$ is the total right-sided bimodule structure map $\oplus \mu_{\mathcal{B}}^{0|1|s}$.

- *Higher left multiplications:*

$$(2.120) \quad \mu^{r|1} : \text{hom}_{\mathcal{C}}(X_{r-1}, X_r) \times \cdots \times \text{hom}_{\mathcal{C}}(X_0, X_1) \times \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{N}, \mathcal{B})(X_0) \longrightarrow \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{N}, \mathcal{B})(X_r)$$

given by

$$(2.121) \quad \mu^{r|1}(x_r, \dots, x_1, \mathcal{G}) := \mathcal{G}_{x_r, \dots, x_1} \in \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{N}, \mathcal{B}(X_r, \cdot))$$

where $\mathcal{G}_{x_r, \dots, x_1}$ is the morphism specified by the following data

$$(2.122) \quad \mathcal{G}_{x_r, \dots, x_1}^{1|l}(\mathbf{n}, y_l, \dots, y_1) = \sum_j (-1)^{\star} \mu_{\mathcal{B}}^{r|1|l-j}(x_r, \dots, x_1, \mathcal{G}^{1|l-j}(\mathbf{n}, y_l, \dots, y_{j+1}), y_j, \dots, y_1)$$

with sign

$$(2.123) \quad \star = |\mathcal{G}| \cdot \left(\sum_{i=1}^j \|y_i\| \right).$$

DEFINITION 2.17. When the bimodule in question above is the diagonal bimodule \mathcal{C}_{Δ} , we call the resulting left or right module $\text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{M}, \mathcal{C}_{\Delta})$ or $\text{hom}_{\text{mod-}\mathcal{C}}(\mathcal{N}, \mathcal{C}_{\Delta})$ the **module dual** of \mathcal{M} or \mathcal{N} respectively.

REMARK 2.4. The terminology **module dual** is in contrast to linear dual, another operation that can frequently be performed on modules and bimodules that are finite rank over \mathbb{K} (see e.g. [S3]).

Now suppose our target bimodule splits as a tensor product of a left module with a right module

$$(2.124) \quad \mathcal{B} = \mathcal{M} \otimes_{\mathbb{K}} \mathcal{N}.$$

Then, given another left module \mathcal{P} , the definitions imply that there is a natural inclusion

$$(2.125) \quad \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{P}, \mathcal{M}) \otimes_{\mathbb{K}} \mathcal{N} \hookrightarrow \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{P}, \mathcal{M} \otimes_{\mathbb{K}} \mathcal{N})$$

LEMMA 2.2. When \mathcal{P} and \mathcal{N} are Yoneda modules (or perfect modules), the inclusion (2.125) is a quasi-equivalence.

PROOF. We suppose that \mathcal{P} and \mathcal{N} are Yoneda modules \mathcal{Y}_X^l , \mathcal{Y}_Z^r , and compute the underlying chain complexes, for an object B :

$$(2.126) \quad \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{Y}_X^l, \mathcal{M} \otimes_{\mathbb{K}} \mathcal{Y}_Z^r)(B) := \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{Y}_X^l, \mathcal{M} \otimes_{\mathbb{K}} \text{hom}_{\mathcal{C}}(B, Z))$$

$$(2.127) \quad \simeq \mathcal{M}(X) \otimes_{\mathbb{K}} \text{hom}_{\mathcal{C}}(B, Z)$$

where in (2.127) we've used a slight generalization of the Yoneda lemma [S5, Lem 2.12]. Using the same lemma,

$$(2.128) \quad \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{Y}_X^l, \mathcal{M}) \otimes_{\mathbb{K}} \mathcal{Y}_Z^r(B) \simeq \mathcal{M}(X) \otimes_{\mathbb{K}} \text{hom}(B, Z)$$

The inclusion (2.125) commutes with the quasi-isomorphisms used in Proposition 2.3.

We deduce the result for more general perfect modules by noting that as we vary \mathcal{P} and \mathcal{N} in (2.125) we obtain natural transformations that commute with finite colimits, hence they remain isomorphisms for perfect objects. \square

Similarly, given a right module \mathcal{Q} , there are natural inclusions

$$(2.129) \quad \mathcal{M} \otimes \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{Q}, \mathcal{N}) \hookrightarrow \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{Q}, \mathcal{M} \otimes_{\mathbb{K}} \mathcal{N}).$$

LEMMA 2.3. When \mathcal{Q} and \mathcal{M} are Yoneda modules or (perfect modules), the inclusion (2.129) is a quasi-equivalence.

REMARK 2.5. There are also analogously defined functors on modules given by *Hom* from a bimodule:

$$(2.130) \quad \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{B}, \cdot) : \mathcal{C}\text{-mod} \longrightarrow \mathcal{D}\text{-mod}$$

$$(2.131) \quad \text{hom}_{\text{mod-}\mathcal{D}}(\mathcal{B}, \cdot) : \text{mod-}\mathcal{D} \longrightarrow \text{mod-}\mathcal{C}.$$

We will not need them here.

The following proposition in some sense verifies that module duality is a sane operation for homologically unital A_∞ categories.

PROPOSITION 2.5. *Let X be an object of a homologically unital A_∞ category \mathcal{C} and $\mathcal{Y}_X^r, \mathcal{Y}_X^l$ the corresponding Yoneda modules. Then, there is a quasi-isomorphism between the module dual of \mathcal{Y}_X^r and \mathcal{Y}_X^l , and vice versa:*

$$(2.132) \quad \mathrm{hom}_{\mathcal{C}\text{-mod}}(\mathcal{Y}_X^l, \mathcal{C}_\Delta) \simeq \mathcal{Y}_X^r$$

$$(2.133) \quad \mathrm{hom}_{\mathrm{mod}\text{-}\mathcal{C}}(\mathcal{Y}_X^r, \mathcal{C}_\Delta) \simeq \mathcal{Y}_X^l.$$

PROOF. We verify first that the module dual

$$(2.134) \quad \mathrm{hom}_{\mathcal{C}\text{-mod}}(\mathcal{Y}_X^l, \mathcal{C}_\Delta)$$

is identical in definition to the pulled back right module

$$(2.135) \quad \mathcal{N} := \mathbf{Y}_L^* \mathcal{Y}_{\mathbf{Y}_L(X)}^l,$$

using the definition of pullback in Definition 2.10. By definition (2.135) is the right module is given by the following data:

- a graded vector space

$$(2.136) \quad \mathcal{N}(Y) := \mathrm{hom}_{\mathcal{C}\text{-mod}}(\mathbf{Y}_L(X), \mathbf{Y}_L(Y)) = \mathrm{hom}_{\mathcal{C}\text{-mod}}(\mathbf{Y}_L(X), \mathcal{C}_\Delta(\cdot, Y))$$

- differential

$$(2.137) \quad \mu_{\mathcal{N}}^{1|0}$$

given by the standard differential in the dg category of modules in (2.136).

- bimodule structure maps given by

$$(2.138) \quad \begin{aligned} \mu_{\mathcal{N}}^{1|s} : \mathcal{N}(Y_0) \times \mathrm{hom}_{\mathcal{C}}(Y_1, Y_0) \times \cdots \times \mathrm{hom}_{\mathcal{C}}(Y_s, Y_{s-1}) &\longrightarrow \mathcal{N}(Y_s) \\ \mu_{\mathcal{N}}^{1|s}(\mathbf{N}, y_1, \dots, y_s) &:= \sum_k \mu_{\mathcal{Y}_{\mathbf{Y}_L(X)}^l}^{k|1}(\mathbf{Y}_L^{i_k}(y_{s-i_k+1}, \dots, y_s), \dots, \mathbf{Y}_L^{i_1}(y_1, \dots, y_{i_1}), \underline{\mathbf{N}}) \\ &= \mu_{\mathcal{C}\text{-mod}}^2(\mathbf{Y}_L^s(y_1, \dots, y_s), \underline{\mathbf{N}}). \end{aligned}$$

The last equality in (2.138) used the fact that since $\mathcal{C}\text{-mod}$ is a dg category, $\mu_{\mathcal{Y}_{\mathbf{Y}_L(X)}^l}^{1|k}$ is μ^1 in the category of modules when $k = 0$, μ^2 when $k = 1$, and 0 otherwise. Now, recall from [S5] or [G2, §2.6] that the higher order term of the Yoneda functor

$$(2.139) \quad \mathbf{Y}_L^s(y_1, \dots, y_s) := \phi_{y_1, \dots, y_s} \in \mathrm{hom}_{\mathcal{C}\text{-mod}}(\mathbf{Y}_L(Y_0), \mathbf{Y}_L(Y_s))$$

is the morphism given by the data

$$(2.140) \quad \phi_{y_1, \dots, y_s}^{r|1}(x_1, \dots, x_r, \mathbf{a}) = \mu^{r+1+s}(x_1, \dots, x_r, \mathbf{a}, y_1, \dots, y_s)$$

so by definition the composition

$$(2.141) \quad \mu_{\mathcal{C}\text{-mod}}^2(\phi_{y_1, \dots, y_s}, \underline{\mathbf{N}}) \in \mathcal{N}(Y_s) = \mathrm{hom}_{\mathcal{C}\text{-mod}}(\mathcal{Y}_X^l, \mathcal{Y}_{Y_s}^l)$$

is $\phi_{y_1, \dots, y_s} \circ \hat{\mathbf{N}}$, i.e.

$$(2.142) \quad \begin{aligned} \mu_{\mathcal{C}\text{-mod}}^2(\phi_{y_1, \dots, y_s}, \underline{\mathbf{N}})^{r|1}(x_1, \dots, x_r, \mathbf{n}) &= \\ \sum_k \mu_{\mathcal{C}}^{r-k+s+1}(x_1, \dots, x_{r-k}, \underline{\mathbf{N}}^{k|1}(x_{r-k+1}, \dots, x_r, \mathbf{n}), y_1, \dots, y_s). \end{aligned}$$

This is evidently the same as the definition of $\mathrm{hom}_{\mathcal{C}\text{-mod}}(\mathcal{Y}_X^l, \mathcal{C}_\Delta)$.

Thus, the first order term of the contravariant natural transformation

$$(2.143) \quad (\mathfrak{T}_R^{\mathbf{Y}_L})_X : \mathcal{Y}_X^r \longrightarrow \mathbf{Y}_L^* \mathcal{Y}_{\mathbf{Y}_L(X)}^l.$$

defined by an order reversal of (2.87) provides the desired quasi-isomorphism when \mathcal{C} is homologically unital.

An analogous check verifies that $\text{hom}_{\text{mod-}\mathcal{C}}(\mathcal{Y}_X^r, \mathcal{C}_\Delta)$ is exactly $\mathbf{Y}_R^* \mathcal{Y}_{\mathbf{Y}_R(X)}^l$. In a similar fashion, the first order term of the natural transformation defined in (2.87)

$$(2.144) \quad (\mathfrak{T}_L^{\mathbf{Y}_R})_X : \mathcal{Y}_X^l \longrightarrow \mathbf{Y}_R^* \mathcal{Y}_{\mathbf{Y}_R(X)}^l$$

gives the desired quasi-isomorphism. \square

PROPOSITION 2.6 (Hom-tensor adjunction). *Let \mathcal{M} and \mathcal{N} be left and right \mathcal{C} modules, and \mathcal{B} a \mathcal{C} bimodule. Then there are natural adjunction isomorphisms, as chain complexes*

$$(2.145) \quad \text{hom}_{\mathcal{C-}\mathcal{C}}(\mathcal{M} \otimes_{\mathbb{K}} \mathcal{N}, \mathcal{B}) = \text{hom}_{\mathcal{C-}\text{mod}}(\mathcal{M}, \text{hom}_{\text{mod-}\mathcal{C}}(\mathcal{N}, \mathcal{B}))$$

$$(2.146) \quad \text{hom}_{\mathcal{C-}\mathcal{C}}(\mathcal{M} \otimes_{\mathbb{K}} \mathcal{N}, \mathcal{B}) = \text{hom}_{\text{mod-}\mathcal{C}}(\mathcal{N}, \text{hom}_{\mathcal{C-}\text{mod}}(\mathcal{M}, \mathcal{B})).$$

PROOF. Up to a sign check, we will show that the two expressions in (2.145) contain manifestly the same amount of data as chain complexes; the case (2.146) is the same. A premorphism

$$(2.147) \quad \mathcal{F} : \mathcal{M} \longrightarrow \text{hom}_{\text{mod-}\mathcal{C}}(\mathcal{N}, \mathcal{B})$$

is the data of morphisms

$$(2.148) \quad \mathcal{F}^{r|1} : \text{hom}_{\mathcal{C}}(X_{r-1}, X_r) \times \cdots \times \mathcal{M}(X_0) \longrightarrow \text{hom}_{\text{mod-}\mathcal{C}}(\mathcal{N}, \mathcal{B})$$

sending

$$(2.149) \quad c_1 \otimes \cdots \otimes c_k \otimes \mathbf{x} \longmapsto \mathcal{F}^{r|1}(c_1, \dots, c_k, \mathbf{x}) \in \text{hom}_{\text{mod-}\mathcal{C}}(\mathcal{N}, \mathcal{B}(X_r, \cdot)).$$

Associate to this the morphism

$$(2.150) \quad \underline{\mathcal{F}} \in \text{hom}_{\mathcal{C-}\mathcal{C}}(\mathcal{M} \otimes_{\mathbb{K}} \mathcal{N}, \mathcal{B})$$

specified by

$$(2.151) \quad \underline{\mathcal{F}}^{r|1|s}(a_1, \dots, a_r, (\mathbf{x} \otimes \mathbf{y}), b_1, \dots, b_s) := \left(\mathcal{F}^{r|1}(a_1, \dots, a_r, \mathbf{x}) \right)^{1|s} (\mathbf{y}, b_1, \dots, b_s).$$

This identification is clearly reversible, so it will suffice to quickly check that the differential agrees. We compute that

$$(2.152) \quad \delta \mathcal{F} = \mathcal{F} \circ \hat{\mu}_{\mathcal{M}} - \mu_{\text{hom}(\mathcal{N}, \mathcal{B})} \circ \hat{\mathcal{F}}.$$

By the correspondence given above, $\mathcal{F} \circ \hat{\mu}_{\mathcal{M}}$ is the morphism whose $1|s$ terms correspond to

$$(2.153) \quad \sum \underline{\mathcal{F}}^{r-r'|1|s}(a_1, \dots, a_{r'}, \mu_{\mathcal{M}}^{r-r'|1}(a_{r'+1}, \dots, a_r, \mathbf{x}) \otimes \mathbf{y}, b_1, \dots, b_s) \\ + \sum \underline{\mathcal{F}}^{r-k'+1|1|s}(a_1, \dots, a_i, \mu_{\mathcal{C}}^k(a_{i+1}, \dots, a_{i+k}), a_{i+k+1}, \dots, a_r, \mathbf{x} \otimes \mathbf{y}, b_1, \dots, b_s).$$

In the second term of (2.152), there are two cases. First, $\mu_{\text{hom}(\mathcal{N}, \mathcal{B})}^{0|1}(\mathcal{F}(a_1, \dots, a_r, \mathbf{x}))$ is the differential

$$(2.154) \quad \mathcal{F}(a_1, \dots, a_r, \mathbf{x}) \circ \hat{\mu}_{\mathcal{N}} - \mu_{\mathcal{B}} \circ \hat{\mathcal{F}}(a_1, \dots, a_r, \mathbf{x}),$$

a morphism whose $1|s$ terms correspond to

$$(2.155) \quad \sum \underline{\mathcal{F}}^{r|1|s-s'}(a_1, \dots, a_r, \mathbf{x} \otimes \mu_{\mathcal{N}}^{1|s'}(\mathbf{y}, b_1, \dots, b_{s'}), b_{s'+1}, \dots, b_s) \\ + \sum \underline{\mathcal{F}}^{r|1|l'+1|s}(a_1, \dots, a_r, \mathbf{x} \otimes \mathbf{y}, b_1, \dots, b_j, \mu_{\mathcal{C}}^l(b_{j+1}, \dots, b_{j+l}), b_{j+l+1}, \dots, b_s)$$

and

$$(2.156) \quad \sum \mu_{\mathcal{B}}^{0|1|s-s'}(\underline{\mathcal{F}}^{r|1|s'}(a_1, \dots, a_r, \mathbf{x} \otimes \mathbf{y}, b_1, \dots, b_{s'}), b_{s'+1}, \dots, b_s)$$

respectively. Finally, there are higher terms

$$(2.157) \quad \sum \mu_{\text{hom}(\mathcal{N}, \mathcal{B})}^{r'|1|s}(a_1, \dots, a_{r'}, \mathcal{F}^{r-r'|1|s}(a_{r'+1}, \dots, a_r, \mathbf{x}))$$

whose $1|s$ terms correspond exactly to

$$(2.158) \quad \sum \mu_{\mathcal{B}}^{r'|1|s'}(a_1, \dots, a_{r'}, \underline{\mathcal{F}}^{r-r'|1|s'}(a_{r'+1}, \dots, a_r, \mathbf{x} \otimes \mathbf{y}, b_1, \dots, b_{s'}), b_{s'+1}, \dots, b_s).$$

Thus, the differentials agree. \square

Using adjunction, we can rapidly prove a few facts about bimodules.

PROPOSITION 2.7. *There is a quasi-isomorphism of chain complexes*

$$(2.159) \quad \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{Y}_X^l \otimes \mathcal{Y}_Z^r, \mathcal{C}_\Delta) \simeq \mathrm{hom}_{\mathcal{C}}(Z, X).$$

PROOF. By adjunction and module duality, we have that

$$(2.160) \quad \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{Y}_X^l \otimes \mathcal{Y}_Z^r, \mathcal{C}_\Delta) \simeq \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_X^l, \mathrm{hom}_{\mathrm{mod}-\mathcal{C}}(\mathcal{Y}_Z^r, \mathcal{C}_\Delta))$$

$$(2.161) \quad \simeq \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_X^l, \mathcal{Y}_Z^l)$$

$$(2.162) \quad \simeq \mathrm{hom}_{\mathcal{C}}(Z, X).$$

□

Strictly speaking, the next fact is not about bimodules, but it will be useful for what follows.

PROPOSITION 2.8. *Let A and B be objects of a homologically unital category \mathcal{C} . Then, the collapse map*

$$(2.163) \quad \mu : \mathcal{Y}_A^r \otimes_{\mathcal{C}} \mathcal{Y}_B^l \simeq \mathrm{hom}(B, A)$$

defined by

$$(2.164) \quad \mathbf{a} \otimes c_1 \otimes \cdots \otimes c_k \otimes \mathbf{b} \longmapsto \mu_{\mathcal{C}}^{k+2}(\mathbf{a}, c_1, \dots, c_k, \mathbf{b}).$$

is a quasi-isomorphism.

PROOF. One can see this result as a consequence of Corollary 2.2, as \mathcal{C} split-generates itself. Or one could examine the cone of μ and note that it is exactly the usual A_∞ bar complex for \mathcal{C} . Alternatively, here is a conceptual computation using module duality that the chain complexes compute the same homology:

$$(2.165) \quad \mathcal{Y}_A^r \otimes_{\mathcal{C}} \mathcal{Y}_B^l \simeq \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_A^l, \mathcal{C}_\Delta) \otimes_{\mathcal{C}} \mathcal{Y}_B^l$$

$$(2.166) \quad \simeq \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_A^l, \mathcal{C}_\Delta \otimes_{\mathcal{C}} \mathcal{Y}_B^l)$$

$$(2.167) \quad \simeq \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_A^l, \mathcal{Y}_B^l)$$

$$(2.168) \quad \simeq \mathrm{hom}_{\mathcal{C}}(B, A).$$

Here, the justification for our ability to bring in \mathcal{Y}_B^l in (2.166) is analogous to Lemma 2.2. □

PROPOSITION 2.9 (Künneth Formula for Bimodules). *There are quasi-isomorphisms*

$$(2.169) \quad \begin{aligned} \mathrm{hom}_{\mathcal{C}}(X', X) \otimes \mathrm{hom}_{\mathcal{D}}(Z, Z') &\simeq \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_X^l, \mathcal{Y}_{X'}^l) \otimes_{\mathbb{K}} \mathrm{hom}_{\mathrm{mod}-\mathcal{D}}(\mathcal{Y}_Z^r, \mathcal{Y}_{Z'}^r) \\ &\simeq \mathrm{hom}_{\mathcal{C}-\mathcal{D}}(\mathcal{Y}_X^l \otimes \mathcal{Y}_Z^r, \mathcal{Y}_{X'}^l \otimes \mathcal{Y}_{Z'}^r). \end{aligned}$$

PROOF. Using adjunction and the Yoneda lemma, we compute

$$(2.170) \quad \mathrm{hom}_{\mathcal{C}-\mathcal{D}}(\mathcal{Y}_X^l \otimes \mathcal{Y}_Z^r, \mathcal{Y}_{X'}^l \otimes \mathcal{Y}_{Z'}^r) = \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_X^l, \mathrm{hom}_{\mathrm{mod}-\mathcal{D}}(\mathcal{Y}_Z^r, \mathcal{Y}_{X'}^l \otimes \mathcal{Y}_{Z'}^r))$$

$$(2.171) \quad \simeq \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_X^l, \mathcal{Y}_{X'}^l \otimes \mathrm{hom}_{\mathrm{mod}-\mathcal{D}}(\mathcal{Y}_Z^r, \mathcal{Y}_{Z'}^r)) \quad (\text{Lemma 2.3})$$

$$(2.172) \quad \simeq \mathrm{hom}_{\mathcal{C}}(X', X) \otimes \mathrm{hom}_{\mathrm{mod}-\mathcal{D}}(\mathcal{Y}_Z^r, \mathcal{Y}_{Z'}^r)$$

$$(2.173) \quad \simeq \mathrm{hom}_{\mathcal{C}}(X', X) \otimes \mathrm{hom}_{\mathcal{D}}(Z, Z').$$

One can check that the maps in this computation are compatible with the natural inclusions

$$(2.174) \quad \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_X^l, \mathcal{Y}_{X'}^l) \otimes_{\mathbb{K}} \mathrm{hom}_{\mathrm{mod}-\mathcal{D}}(\mathcal{Y}_Z^r, \mathcal{Y}_{Z'}^r) \hookrightarrow \mathrm{hom}_{\mathcal{C}-\mathcal{D}}(\mathcal{Y}_X^l \otimes \mathcal{Y}_Z^r, \mathcal{Y}_{X'}^l \otimes \mathcal{Y}_{Z'}^r).$$

given by sending a pair of morphisms \mathcal{F}, \mathcal{G} to the morphism

$$(2.175) \quad (\mathcal{F} \otimes \mathcal{G})^{r|1|s} := \mathcal{F}^{r|1} \otimes \mathcal{G}^{1|s}.$$

□

We now define, for a bimodule \mathcal{B} over an A_∞ category \mathcal{C} , a **bimodule dual**

$$(2.176) \quad \mathcal{B}^{!} = {}^{\circ} \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{B}, \mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta),$$

generalizing (2.4).

REMARK 2.6. The reason we have put the above equality in quotes is that “ $\mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta$ ” is not a bimodule or even a space with commuting outer and inner bimodule structures, unlike the associative case. It can, however, be thought of as a **4-module**, a special case of a theory of A_∞ **n-modules** recently introduced by Ma'u [M]. A 4-module associates to any four-tuple of objects (X, Y, Z, W) a chain complex, and to any four-tuples of composable sequences of objects

$$(2.177) \quad \left((X_0, \dots, X_k), (Y_0, \dots, Y_l), (Z_0, \dots, Z_s), (W_0, \dots, W_t) \right)$$

operations $\mu^{k|s|t|l}$ satisfying a generalization of the A_∞ bimodule equations. In the same way that one can tensor/hom modules with bimodules to obtain new modules, one can tensor/hom bimodules with 4-modules to obtain new bimodules. The process we are about to describe is a special case of a general such theory.

DEFINITION 2.18. Let \mathcal{B} be an A_∞ bimodule over an A_∞ category \mathcal{C} . The **bimodule dual** of \mathcal{B} is the bimodule

$$(2.178) \quad \mathcal{B}^! := \text{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{B}, \mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta)$$

over \mathcal{C} defined by the following data:

- For pairs of objects, (X, Y) , $\mathcal{B}^!(X, Y)$ is the chain complex

$$(2.179) \quad \mathcal{B}^!(X, Y) := \text{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{B}, \mathcal{Y}_X^l \otimes_{\mathbb{K}} \mathcal{Y}_Y^r)$$

which we recall is the data of, for $k, l \geq 0$ and objects $A_0, \dots, A_k, B_0, \dots, B_l$, maps

$$(2.180) \quad \begin{aligned} \mathcal{F}^{k|1|s} : \text{hom}_{\mathcal{C}}(A_{k-1}, A_k) \otimes \dots \otimes \text{hom}_{\mathcal{C}}(A_0, A_1) \otimes \mathcal{B}(A_0, B_0) \\ \otimes \text{hom}_{\mathcal{C}}(B_1, B_0) \otimes \dots \otimes \text{hom}_{\mathcal{C}}(B_l, B_{l-1}) \\ \longrightarrow \mathcal{Y}_X^l(A_k) \otimes_{\mathbb{K}} \mathcal{Y}_Y^r(B_l). \end{aligned}$$

As usual, package this into a single map

$$(2.181) \quad \mathcal{F} : T\mathcal{C} \otimes \mathcal{B} \otimes T\mathcal{C} \longrightarrow \mathcal{Y}_X^l \otimes_{\mathbb{K}} \mathcal{Y}_Y^r.$$

Then, the differential is given by the usual bimodule hom differential

$$(2.182) \quad \mu_{\mathcal{B}^!}^{0|1|0}(\mathcal{F}) = \mathcal{F} \circ \hat{\mu}_{\mathcal{B}} \pm \mu_{\mathcal{Y}_X^l \otimes_{\mathbb{K}} \mathcal{Y}_Y^r} \circ \hat{\mathcal{F}}.$$

- for collections of objects $(X_0, \dots, X_r, Y_0, \dots, Y_s)$, maps

$$(2.183) \quad \begin{aligned} \mu_{\mathcal{B}^!}^{r|1|s} : \text{hom}_{\mathcal{B}}(X_{r-1}, X_r) \otimes \dots \otimes \text{hom}_{\mathcal{C}}(X_0, X_1) \otimes \mathcal{B}^!(X_0, Y_0) \\ \otimes \text{hom}_{\mathcal{C}}(Y_1, Y_0) \otimes \dots \otimes \text{hom}_{\mathcal{C}}(Y_s, Y_{s-1}) \\ \longrightarrow \mathcal{B}^!(X_r, Y_s). \end{aligned}$$

defined as follows:

$$(2.184) \quad \mu_{\mathcal{B}^!}^{r|1|s} = 0 \text{ if both } r, s > 0.$$

$$(2.185) \quad \mu_{\mathcal{B}^!}^{r|1|0}(x_r, \dots, x_1, \phi) = \Phi_{(x_r, \dots, x_1, \phi)} \in \mathcal{B}^!(X_r, Y_0)$$

where $\Phi_{(x_r, \dots, x_1, \phi)}$ is the bimodule map whose $k|1|l$ term is:

$$(2.186) \quad \begin{aligned} \Phi_{(x_r, \dots, x_1, \phi)}^{k|1|l}(a_k, \dots, a_1, \mathbf{b}, b_l, \dots, b_1) := \\ \sum_{l' \leq l} (-1)^{\mathfrak{K}_1^{l'}} (\mu_{\mathcal{C}_\Delta}^{r|1|l'}(x_r, \dots, x_1, \cdot, b_{l'}, \dots, b_1) \otimes id) \\ \circ \phi^{k|1|l-l'}(a_k, \dots, a_1, \mathbf{b}, b_l, \dots, b_{l'+1}). \end{aligned}$$

with sign

$$(2.187) \quad \mathfrak{K}_1^{l'} := \sum_{i=1}^{l'} ||b_i||.$$

Also,

$$(2.188) \quad \mu_{\mathcal{B}^!}^{0|1|^s}(\phi, y_1, \dots, y_s) = \Phi_{(\phi, y_1, \dots, y_s)} \in \mathcal{B}^!(X_0, Y_s)$$

where $\Phi_{(\phi, y_1, \dots, y_s)}$ is the bimodule map whose $k|1|l$ term is:

$$(2.189) \quad \begin{aligned} & \Phi_{(\phi, y_1, \dots, y_s)}^{k|1|l}(a_k, \dots, a_1, \mathbf{b}, b_l, \dots, b_1) := \\ & \sum_{k' \leq k} (id \otimes \mu_{\mathcal{C}_\Delta}^{k'|1|^s}(a_k, \dots, a_{k'+1}, \cdot, y_1, \dots, y_s)) \\ & \quad \circ \phi^{k-k'|1|l}(a_{k'}, \dots, a_1, \mathbf{b}, b_1, \dots, b_l). \end{aligned}$$

REMARK 2.7. There is a more intrinsic definition of $\mathcal{B}^!$, analogous to the relationship described in (2.134)-(2.135) in terms of Yoneda pullbacks. Let $\mathcal{B} \in \text{ob } \mathcal{C}\text{-mod-}\mathcal{C}$ be a specified bimodule. Take the right Yoneda module over this bimodule

$$(2.190) \quad \mathcal{Y}_{\mathcal{B}}^r \in \text{ob mod-}(\mathcal{C}\text{-mod-}\mathcal{C}).$$

By restricting via the natural embedding

$$(2.191) \quad \mathcal{C}\text{-mod} \otimes \text{mod-}\mathcal{C} \hookrightarrow \mathcal{C}\text{-mod-}\mathcal{C}$$

we think of $\mathcal{Y}_{\mathcal{B}}^r$ as a right dg module over the category of split bimodules, e.g. the tensor product

$$(2.192) \quad \mathcal{C}\text{-mod} \otimes \text{mod-}\mathcal{C}.$$

Since a right dg module \mathcal{M} over a tensor product of dg categories $\mathcal{C} \otimes \mathcal{D}$ is tautologically a $\mathcal{C}^{op} - \mathcal{D}$ bimodule,

$$(2.193) \quad \mathcal{Y}_{\mathcal{B}}^r \in \text{ob } (\mathcal{C}\text{-mod})^{op}\text{-mod-}(\text{mod-}\mathcal{C}).$$

Now, recall in §2.2 that for functors $\mathfrak{F} : \mathcal{A} \rightarrow \mathcal{C}^{op}$, $\mathfrak{G} : \mathcal{B} \rightarrow \mathcal{D}$ we defined a pullback functor

$$(2.194) \quad (\mathfrak{F} \otimes \mathfrak{G})^* : \mathcal{C}^{op}\text{-mod-}\mathcal{D} \rightarrow \mathcal{A}\text{-mod-}\mathcal{B}.$$

We can now take the pullback of (2.193) using the left and right Yoneda embeddings

$$(2.195) \quad \begin{aligned} \mathbf{Y}_L : \mathcal{C} &\rightarrow (\mathcal{C}\text{-mod})^{op} \\ \mathbf{Y}_R : \mathcal{C} &\rightarrow (\text{mod-}\mathcal{C}) \end{aligned}$$

Then, the **bimodule dual** of \mathcal{B} is equivalent to the $\mathcal{C} - \mathcal{C}$ bimodule

$$(2.196) \quad \mathcal{B}^! := (\mathbf{Y}_L \otimes \mathbf{Y}_R)^*(\mathcal{Y}_{\mathcal{B}}^r).$$

The proof follows by comparing definitions as in Proposition 2.5.

As a first step, we can take the bimodule dual of a Yoneda bimodule.

PROPOSITION 2.10. If \mathcal{B} is the Yoneda bimodule

$$(2.197) \quad \mathcal{Y}_X^l \otimes_{\mathbb{K}} \mathcal{Y}_Z^r$$

then $\mathcal{B}^!$ is quasi-isomorphic to the Yoneda bimodule

$$(2.198) \quad \mathcal{Y}_Z^l \otimes_{\mathbb{K}} \mathcal{Y}_X^r.$$

PROOF. By definitions, there is a natural inclusion

$$(2.199) \quad \text{hom}_{\text{mod-}\mathcal{C}}(\mathcal{Y}_Z^r, \mathcal{C}_\Delta) \otimes_{\mathbb{K}} \text{hom}_{\mathcal{C}\text{-mod}}(\mathcal{Y}_X^l, \mathcal{C}_\Delta) \hookrightarrow \text{hom}_{\mathcal{C}\text{-mod-}\mathcal{C}}(\mathcal{Y}_X^l \otimes \mathcal{Y}_Z^r, \mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta).$$

inducing a quasi-isomorphism by Proposition 2.9. It follows immediately from inspection of Definition 2.18 that this inclusion can be extended to a morphism of A_∞ bimodules. Thus, by Proposition 2.5, we conclude. \square

The precise version of (2.5) is

PROPOSITION 2.11. If \mathcal{Q} is a perfect $\mathcal{C} - \mathcal{C}$ bimodule, then $\mathcal{Q}^!$ is also perfect and for perfect \mathcal{B} there is a natural quasi-isomorphism

$$(2.200) \quad \mathcal{Q}^! \otimes_{\mathcal{C}\text{-}\mathcal{C}} \mathcal{B} \simeq \text{hom}_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{Q}, \mathcal{B}).$$

PROOF. The perfectness of $\mathcal{Q}^!$ follows from the fact (Proposition 2.10) that duals of Yoneda bimodules are Yoneda bimodules; hence we see that if \mathcal{Q} is a summand of a complex of Yoneda bimodules, $\mathcal{Q}^!$ is too. (Implicitly, we are using the fact that the duality functor commutes with finite cones and summands.)

Now, there is a natural transformation of functors

$$(2.201) \quad \mathfrak{C} : \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\cdot, \mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta) \otimes_{\mathcal{C}-\mathcal{C}} \mathcal{B} \longrightarrow \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\cdot, \mathcal{C}_\Delta \otimes_{\mathcal{C}} \mathcal{B} \otimes_{\mathcal{C}} \mathcal{C}_\Delta)$$

given by, for a bimodule \mathcal{Q} , the natural inclusion of chain complexes,

$$(2.202) \quad \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{Q}, \mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta) \otimes_{\mathcal{C}-\mathcal{C}} \mathcal{B} \hookrightarrow \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{Q}, \mathcal{C}_\Delta \otimes_{\mathcal{C}} \mathcal{B} \otimes_{\mathcal{C}} \mathcal{C}_\Delta).$$

Concretely, this is the map

$$(2.203) \quad \mathfrak{C}_{\mathcal{Q}} : \underline{\phi} \otimes x_1 \otimes \cdots \otimes x_k \otimes \mathbf{b} \otimes y_1 \otimes \cdots \otimes y_l \longmapsto \hat{\phi}_{x_1, \dots, x_k, \mathbf{b}, y_1, \dots, y_l}$$

where $\hat{\phi}_{x_1, \dots, x_k, \mathbf{b}, y_1, \dots, y_l} \in \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{Q}, \mathcal{C}_\Delta \otimes_{\mathcal{C}} \mathcal{B} \otimes_{\mathcal{C}} \mathcal{C}_\Delta)$ is specified by the following data:

$$(2.204) \quad \begin{aligned} \hat{\phi}_{x_1, \dots, x_k, \mathbf{b}, y_1, \dots, y_l}(a_1, \dots, a_r, \mathbf{q}, b_1, \dots, b_s) := \\ \phi(a_1, \dots, a_r, \mathbf{q}, b_1, \dots, b_s) \bar{\otimes} (x_1 \otimes \cdots \otimes x_k \otimes \mathbf{b} \otimes y_1 \otimes \cdots \otimes y_l). \end{aligned}$$

Here the operation $\bar{\otimes}$ is the reversed tensor product

$$(2.205) \quad (a \otimes b) \bar{\otimes} (c_1 \otimes \cdots \otimes c_k) := b \otimes c_1 \otimes \cdots \otimes c_k \otimes a,$$

extended linearly. For \mathcal{Q} and \mathcal{B} both Yoneda bimodules of the form $\mathcal{Y}_{XZ} := \mathcal{Y}_X^l \otimes \mathcal{Y}_Z^r$ and $\mathcal{Y}_{X'}^l \otimes \mathcal{Y}_{Z'}^r$, we claim the natural transformation \mathfrak{C} is a quasi-isomorphism. This follows from the computations

$$(2.206) \quad \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{Y}_X^l \otimes \mathcal{Y}_Z^r, \mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta) \otimes_{\mathcal{C}-\mathcal{C}} (\mathcal{Y}_{X'}^l \otimes \mathcal{Y}_{Z'}^r)$$

$$(2.207) \quad \simeq (\mathcal{Y}_X^r \otimes_{\mathbb{K}} \mathcal{Y}_Z^l) \otimes_{\mathcal{C}-\mathcal{C}} (\mathcal{Y}_{X'}^l \otimes_{\mathbb{K}} \mathcal{Y}_{Z'}^r)$$

$$(2.208) \quad \simeq (\mathcal{Y}_X^r \otimes_{\mathcal{C}} \mathcal{Y}_{X'}^l) \otimes_{\mathbb{K}} (\mathcal{Y}_{Z'}^r \otimes_{\mathcal{C}} \mathcal{Y}_Z^l)$$

$$(2.209) \quad \simeq \mathrm{hom}(X', X) \otimes_{\mathbb{K}} \mathrm{hom}(Z, Z') \quad (\text{Proposition 2.8}).$$

and

$$(2.210) \quad \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{Y}_X^l \otimes \mathcal{Y}_Z^r, \mathcal{C}_\Delta \otimes_{\mathcal{C}} (\mathcal{Y}_{X'}^l \otimes \mathcal{Y}_{Z'}^r) \otimes_{\mathcal{C}} \mathcal{C}_\Delta)$$

$$(2.211) \quad \simeq \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_X^l, \mathcal{C}_\Delta \otimes_{\mathcal{C}} \mathcal{Y}_{X'}^l) \otimes_{\mathbb{K}} \mathrm{hom}_{\mathrm{mod}-\mathcal{C}}(\mathcal{Y}_Z^r, \mathcal{Y}_{Z'}^r \otimes_{\mathcal{C}} \mathcal{C}_\Delta)$$

$$(2.212) \quad \simeq \mathrm{hom}_{\mathcal{C}-\mathrm{mod}}(\mathcal{Y}_X^l, \mathcal{Y}_{X'}^l) \otimes_{\mathbb{K}} \mathrm{hom}_{\mathrm{mod}-\mathcal{C}}(\mathcal{Y}_Z^r, \mathcal{Y}_{Z'}^r)$$

$$(2.213) \quad \simeq \mathrm{hom}(X', X) \otimes_{\mathbb{K}} \mathrm{hom}(Z, Z'),$$

which are compatible with the morphism \mathfrak{C} .

Since the natural transformation \mathfrak{C} commutes with finite cones and summands, we see that for perfect \mathcal{Q} and \mathcal{B} , there must be a quasi-isomorphism

$$(2.214) \quad \mathfrak{C}_{\mathcal{Q}} : \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{Q}, \mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta) \otimes_{\mathcal{C}-\mathcal{C}} \mathcal{B} \xrightarrow{\sim} \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{Q}, \mathcal{C}_\Delta \otimes_{\mathcal{C}} \mathcal{B} \otimes_{\mathcal{C}} \mathcal{C}_\Delta).$$

Now, postcomposing with, e.g. the quasi-isomorphism

$$(2.215) \quad \mathcal{F}_{\Delta, \text{left}, \text{right}} : \mathcal{C}_\Delta \otimes_{\mathcal{C}} \mathcal{B} \otimes_{\mathcal{C}} \mathcal{C}_\Delta \longrightarrow \mathcal{B}$$

defined in (2.50) gives the desired quasi-isomorphism. □

We now specialize to the case $\mathcal{B} = \mathcal{C}_\Delta$.

DEFINITION 2.19. *The inverse dualizing bimodule*

$$(2.216) \quad \mathcal{C}^!$$

is by definition the bimodule dual of the diagonal bimodule \mathcal{C}_Δ .

As an immediate corollary of Proposition 2.11,

COROLLARY 2.3 ($\mathcal{C}^!$ represents Hochschild cohomology). *If \mathcal{C} is homologically smooth, then the complex*

$$(2.217) \quad \mathcal{C}^! \otimes_{\mathcal{C}-\mathcal{C}} \mathcal{B}$$

computes the Hochschild cohomology $\mathrm{HH}^(\mathcal{C}, \mathcal{B})$. More precisely, there is an explicit quasi-isomorphism induced by a “multiplication map:”*

$$(2.218) \quad \bar{\mu} : \mathcal{C}^! \otimes_{\mathcal{C}-\mathcal{C}} \mathcal{B} \longrightarrow {}_2\mathrm{CC}^*(\mathcal{C}, \mathcal{B})$$

Explicitly, this map is given by

$$(2.219) \quad \bar{\mu} : \phi \otimes x_1 \otimes \cdots \otimes x_k \otimes \mathbf{b} \otimes y_1 \otimes \cdots \otimes y_l \longmapsto \Psi_{\phi, x_1, \dots, x_k, \mathbf{b} y_1, \dots, y_l} \in \mathrm{hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{C}_\Delta, \mathcal{B})$$

where $\Psi := \Psi_{\phi, x_1, \dots, x_k, \mathbf{b} y_1, \dots, y_l}$ is the morphism given by

$$(2.220) \quad \begin{aligned} \Psi(a_1, \dots, a_r, \mathbf{c}, b_1, \dots, b_s) &:= \sum_{r', s'} \mathcal{F}_{\Delta, \text{left}, \text{right}}^{r'|1|s'}(a_1, \dots, a_{r'}, \\ &(\phi^{r-r'|1|s-s'}(a_{r'+1}, \dots, a_r, \mathbf{c}, b_1, \dots, b_{s'}) \bar{\otimes} (x_1 \otimes \cdots \otimes x_k \otimes \mathbf{b} \otimes y_1 \otimes \cdots \otimes y_l)), \\ &b_{s'+1}, \dots, b_s). \end{aligned}$$

Here $\bar{\otimes}$ is the reverse tensor product defined in (2.205), and $\mathcal{F}_{\Delta, \text{left}, \text{right}}$ is the bimodule morphism given in (2.50).

3. Open-closed and closed-open maps

3.1. Symplectic cohomology and wrapped Floer cohomology. Let $M := (M, \theta)$ be a *Liouville manifold*, that is a $2n$ dimensional manifold M equipped with a one form θ such that $\omega = d\theta$ is symplectic, satisfying a *convexity* condition: away from a compact set, M should be modeled on the cone (semi-infinite symplectization) of a contact manifold

$$(3.1) \quad M = \bar{M} \cup_{\partial \bar{M}} \partial \bar{M} \times [1, +\infty)_r,$$

such that the flow of the *Liouville vector field* (the symplectic dual of θ , defined by $\iota_Z \omega = \theta$) is transverse to $\bar{M} \times \{1\}$ and in fact equal to

$$(3.2) \quad Z = r \partial_r$$

on the conical, or *cylindrical* region. We explicitly fix a representation (3.1).

Fix a class of *Hamiltonians*

$$(3.3) \quad \mathcal{H}(M) \subset C^\infty(M, \mathbb{R}),$$

functions H that, away from some compact subset of M are *strictly quadratic*:

$$(3.4) \quad H(r, y) = r^2.$$

Also, consider a class of almost-complex structures $\mathcal{J}_1(M)$ that are **rescaled contact type** on the conical end, meaning that

$$(3.5) \quad \frac{1}{r} \theta \circ J = dr.$$

We assume that θ has been chosen generically so that

$$(3.6) \quad \text{all Reeb orbits of } \bar{\theta} \text{ are non-degenerate.}$$

REMARK 3.1. *Our class of complex structures differs from those used by Abouzaid [A1] and Abouzaid-Seidel [AS2], who consider almost complex structures satisfying $\theta \circ J = dr$. The difference is backwards compatible with the operations constructed in [A1], as can be seen by using interpolating complex structures. However, the class we used has slightly better compactness properties for operations involving multiple orbit inputs [G2, §B].*

Observe that non-trivial closed orbits of H (for instance, those on the cylindrical end) occur in S^1 -families of Reeb orbits and are therefore degenerate. We break the S^1 symmetry by choosing $F : S^1 \times M \rightarrow \mathbb{R}$ a smooth non-negative function, satisfying

- F and $\theta(X_F)$ uniformly bounded in absolute value, and

- all time-1 periodic orbits of X_{S^1} , the (time-dependent) Hamiltonian vector field corresponding to $H_{S^1}(t, m) = H(m) + F(t, m)$, are non-degenerate.

Fixing such a choice, define

$$\mathcal{O}$$

to be the set of (time-1) periodic orbits of H_{S^1} . To fix an integer grading on \mathcal{O} , we

$$(3.7) \quad \text{Fix a trivialization of } (\Lambda_{\mathbb{C}}^n T^* M)^{\otimes 2}.$$

(in particular, we assume for simplicity that $2c_1(M) = 0$, though the results of our paper hold over $\mathbb{Z}/2\mathbb{Z}$ as well).

Given an element $y \in \mathcal{O}$, define the *degree* of y to be

$$(3.8) \quad \deg(y) := n - CZ(y)$$

where CZ is the Conley-Zehnder index of y . Now, define the **symplectic co-chain complex** over \mathbb{K} to be

$$(3.9) \quad CH^i(M; H, F, J_t) = \bigoplus_{y \in \mathcal{O}, \deg(y)=i} |o_y|_{\mathbb{K}},$$

where the **orientation line** o_y is defined using the determinant line of a linearization of Floer's equation in [G2, Appendix A] or [A1, Def. C.3]. Also, for any one-dimensional real vector space V (such as o_y) and any field \mathbb{K} ,

$$(3.10) \quad |V|_{\mathbb{K}}$$

is \mathbb{K} -vector space generated by the two orientations on V , modulo the relation that the sum of the orientations is zero [S5, §12e] (if $\mathbb{K} = \mathbb{Z}_2$, note that $|o_y|_{\mathbb{K}} \cong \mathbb{Z}_2$ canonically, so (3.9) is literally generated by a copy of \mathbb{K} for each orbit).

Given an S^1 dependent family $J_t \in \mathcal{J}_1(M)$, consider maps

$$(3.11) \quad u : (-\infty, \infty) \times S^1 \rightarrow M$$

converging exponentially at each end to a time-1 periodic orbit of H_{S^1} and satisfying Floer's equation

$$(3.12) \quad (du - X_{S^1} \otimes dt)^{0,1} = 0.$$

Here, as above the cylinder $A = (-\infty, \infty) \times [0, 1]$ is equipped with coordinates s, t and a complex structure j with $j(\partial s) = \partial t$. In coordinates, the above equation is the usual

$$(3.13) \quad \partial_s u = -J_t(\partial_t u - X).$$

Given time 1 orbits $y_0, y_1 \in \mathcal{O}$, denote by $\tilde{\mathcal{M}}(y_0; y_1)$ the set of maps u converging to y_0 when $s \rightarrow -\infty$ and y_1 when $s \rightarrow +\infty$. This set is equipped with a topology and a natural \mathbb{R} action coming from translation in the s direction. Following standard transversality arguments, one sees that for generic J_t , the moduli space is smooth of dimension $\deg(y_0) - \deg(y_1)$ with free \mathbb{R} action unless it is of dimension 0.

Define

$$(3.14) \quad \mathcal{M}(y_0; y_1)$$

to be the quotient of $\tilde{\mathcal{M}}(y_0; y_1)$ by the \mathbb{R} action whenever it is free, and the empty set when the \mathbb{R} action is not free.

Construct the bordification $\overline{\mathcal{M}}(y_0; y_1)$ by adding **broken cylinders**

$$(3.15) \quad \overline{\mathcal{M}}(y_0; y_1) = \coprod \mathcal{M}(y_0; x_1) \times \mathcal{M}(x_1; x_2) \times \cdots \times \mathcal{M}(x_k; y_1)$$

A maximum principle for solutions to this (3.13) for our particular choices of H and J is proven in [A1] [G2, §B]. Along with standard arguments, one concludes:

LEMMA 3.1. *For generic J_t , the moduli space $\overline{\mathcal{M}}(y_0, y_1)$ is a compact manifold with boundary of dimension $\deg(y_0) - \deg(y_1) - 1$. The boundary is covered by the closure of the images of natural inclusions*

$$(3.16) \quad \mathcal{M}(y_0; y) \times \mathcal{M}(y; y_1) \rightarrow \overline{\mathcal{M}}(y_0; y_1).$$

Moreover, for each y_1 , $\overline{\mathcal{M}}(y_0; y_1)$ is empty for all but finitely many choices of y_0 .

For a regular $u \in \mathcal{M}(y_0; y_1)$ with $\deg(y_0) = \deg(y_1) + 1$, standard arguments (which are reviewed in [G2, Appendix A]) give us an isomorphism of orientation lines

$$(3.17) \quad \mu_u : o_{y_1} \longrightarrow o_{y_0}$$

and hence a morphism also denoted μ_u between associated \mathbb{K} vector spaces. So we can define a differential

$$(3.18) \quad d : CH^*(M; H, F_t, J_t) \longrightarrow CH^{*+1}(M; H, F_t, J_t)$$

whose matrix coefficients are sums of such associated morphisms:

$$(3.19) \quad d([y_1]) = \sum_{y_0; \deg(y_0) = \deg(y_1) + 1} \sum_{u \in \mathcal{M}(y_0; y_1)} (-1)^{\deg(y_1)} \mu_u([y_1]).$$

LEMMA 3.2.

$$d^2 = 0.$$

Call the cohomology of d $SH^*(M)$.

REMARK 3.2. *Our grading conventions for symplectic cohomology follow Seidel [S4], Abouzaid [A1], and Ritter [R]. These conventions are essentially determined by the fact that the identity element lives in degree zero, and the product map is also a degree zero operation, making $SH^*(M)$ a graded ring.*

DEFINITION 3.1. *An **admissible Lagrangian brane** consists of an exact properly embedded Lagrangian submanifold $L \subset M$, satisfying*

$$(3.20) \quad L \text{ is oriented and Spin;}$$

$$(3.21) \quad 2c_1(M, L) = 0, \text{ where } c_1(M, L) \in H^2(M, L) \text{ is the relative first Chern class; and}$$

$$(3.22) \quad \lambda|_L \text{ vanishes away from a compact set,}$$

and equipped with the following extra data:

$$(3.23) \quad \text{a primitive } f_L : L \rightarrow \mathbb{R} \text{ for } \lambda|_L;$$

$$(3.24) \quad \text{an orientation and spin structure; and}$$

$$(3.25) \quad \text{a grading.}$$

REMARK 3.3. *The condition (3.22) implies that with respect to the conical structure on M , L is modeled on the cone of a Legendrian near ∞ , and that the chosen primitive f_L is locally constant away from a conical set.*

Consider a finite collection $\text{ob } \mathcal{W}$ of admissible Lagrangian branes.

We assume a further genericity condition for θ :

$$(3.26) \quad \text{all Reeb chords between Lagrangians in } \text{ob } \mathcal{W} \text{ are non-degenerate.}$$

Recalling our fixed choice of $H \in \mathcal{H}(M)$ define

$$\chi(L_0, L_1)$$

to be the set of time 1 Hamiltonian flows of H between L_0 and L_1 . Given the data specified in Definition 3.1, the **Maslov index** defines an absolute grading on $\chi(L_0, L_1)$, which we will denote by

$$(3.27) \quad \deg : \chi(L_0, L_1) \rightarrow \mathbb{Z}.$$

Then, given a family $J_t \in \mathcal{J}_1(M)$ parametrized by $t \in [0, 1]$, define the **wrapped Floer co-chain complex** over \mathbb{K} to be, as a graded vector space,

$$(3.28) \quad CW^i(L_0, L_1, H, J_t) = \bigoplus_{x \in \chi(L_0, L_1), \deg(x)=i} |o_x|_{\mathbb{K}}.$$

Here $|o_x|_{\mathbb{K}}$, henceforth abbreviated $|o_x|$, is the one-dimensional \mathbb{K} -vector space associated via (3.10) to the one-dimensional real **orientation line** o_x of x , which in turn is the determinant line of a linearization of Floer's equation associated to x (see e.g., see [S5, §11h] for more details on both constructions, or the prequel article [G2, Appendix A]).

Now, consider maps

$$(3.29) \quad u : (-\infty, \infty) \times [0, 1] \rightarrow M$$

converging exponentially at each end to time-1 chords of H , satisfying boundary conditions

$$\begin{aligned} u(s, 0) &\in L_0 \\ u(s, 1) &\in L_1 \end{aligned}$$

and satisfying Floer's equation

$$(3.30) \quad (du - X \otimes dt)^{0,1} = 0.$$

Above, X is the Hamiltonian vector field of H and we think of the strip

$$(3.31) \quad Z = (-\infty, \infty) \times [0, 1]$$

as equipped with coordinates s, t and the canonical complex structure j ($j(\partial_s) = \partial_t$). With this prescription one can rewrite the above equation in coordinates in the more familiar form

$$(3.32) \quad \partial_s u = -J_t(\partial_t u - X).$$

Given time 1 chords $x_0, x_1 \in \chi(L_0, L_1)$, denote by

$$(3.33) \quad \tilde{\mathcal{R}}^1(x_0; x_1)$$

the set of maps u converging to x_0 when $s \rightarrow -\infty$ and x_1 when $s \rightarrow +\infty$. As a component of the zero-locus of an elliptic operator on the space of smooth functions from Z into M , this set carries a natural topology. Moreover, the natural \mathbb{R} action on $\tilde{\mathcal{R}}^1(x_0; x_1)$, coming from translation in the s direction, is continuous with respect to this topology. Following standard arguments, we conclude:

LEMMA 3.3. *For generic J_t , the moduli space $\tilde{\mathcal{R}}^1(x_0; x_1)$ is a compact manifold of dimension $\deg(x_0) - \deg(x_1)$. The action of \mathbb{R} is smooth and free unless $\deg(x_0) = \deg(x_1)$.*

PROOF. See [A1, Lemma 2.3]. □

DEFINITION 3.2. *Define*

$$(3.34) \quad \mathcal{R}(x_0; x_1)$$

to be the quotient of $\tilde{\mathcal{R}}^1(x_0; x_1)$ by the \mathbb{R} action whenever it is free, and the empty set when the \mathbb{R} action is not free.

Also following now-standard arguments, one may construct a bordification $\overline{\mathcal{R}}(x_0; x_1)$ by adding **broken strips**

$$(3.35) \quad \overline{\mathcal{R}}(x_0; x_1) = \coprod \mathcal{R}(x_0; y_1) \times \mathcal{R}(y_1; y_2) \times \cdots \times \mathcal{R}(y_k; x_1)$$

LEMMA 3.4. *For generic J_t , the moduli space $\overline{\mathcal{R}}(x_0; x_1)$ is a compact manifold with boundary of dimension $\deg(x_0) - \deg(x_1) - 1$. The boundary is covered by the closure of the images of natural inclusions*

$$(3.36) \quad \mathcal{R}(x_0; y) \times \mathcal{R}(y; x_1) \rightarrow \overline{\mathcal{R}}(x_0; x_1).$$

LEMMA 3.5. *Moreover, for each x_1 , the $\overline{\mathcal{R}}(x_0; x_1)$ is empty for all but finitely many x_0 .*

PROOF. A proof of this is given in [A1, Lemma 2.5] but it is not quite applicable as it involves a general compactness result proven for complex structures J satisfying $\theta \circ J = dr$, see [A1, Lemma B.1-2]. In fact, the arguments from this general compactness result directly carry over for our J_t but we can alternately apply the arguments in [G2, Appendix B]. □

Now, for regular $u \in \mathcal{R}(x_0; x_1)$, if $\deg(x_0) = \deg(x_1) + 1$, standard arguments reviewed in [G2, Appendix A] give an isomorphism

$$(3.37) \quad \mu_u : o_{x_1} \longrightarrow o_{x_0}.$$

Thus we can define a differential

$$(3.38) \quad d : CW^*(L_0, L_1; H, J_t) \longrightarrow CW^*(L_0, L_1; H, J_t)$$

$$d([x_1]) = \sum_{x_0; \deg(x_0) = \deg(x_1) + 1} \sum_{u \in \mathcal{R}(x_0; x_1)} (-1)^{\deg(x_1)} \mu_u([x_1]).$$

LEMMA 3.6.

$$d^2 = 0.$$

Call the resulting group $HW^*(L_0, L_1)$.

3.2. Floer-theoretic operations from bordered surfaces. We recall briefly the construction of Floer-theoretic operations for families of genus 0 bordered Riemann surfaces given in [G2, §4]. A **genus-0 open-closed string** of type h with $n, \vec{m} = (m^1, \dots, m^h)$ marked points Σ is a sphere with h disjoint discs removed, with n interior marked points and m^i boundary marked points on the i th boundary component $\partial^i \Sigma$. Fix some subset $\mathbf{I} \subset \{1, \dots, n\}$ and a vector of subsets $\vec{\mathbf{K}} = (K^1, \dots, K^h)$ with $K^i \subset \{1, \dots, m^i\}$. Σ has **sign-type** $(\mathbf{I}, \vec{\mathbf{K}})$ if

- interior marked points p_i , with $i \in I$ are negative,
- boundary marked points $z_{j,k} \in \partial^j \Sigma$, $k \in K^j$ are negative, and
- all other marked points are positive.

Also, a genus-0 open-closed string comes equipped with the data of

- a choice of **normal vector** or **asymptotic marker** at each interior marked point.

For our applications, we consider at most one negative interior marked point or at most two negative boundary marked points, i.e. the cases

$$(3.39) \quad \begin{cases} |I| = 1 \text{ and } \sum |K^i| = 0 \\ |I| = 0 \text{ and } \sum |K^i| = 1 \text{ or } 2. \end{cases}$$

The **(non-compactified) moduli space of genus-0 open-closed strings** of type h with n, \vec{m} marked points and sign-type $(\mathbf{I}, \vec{\mathbf{K}})$ is denoted

$$(3.40) \quad \mathcal{N}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}}.$$

There is a corresponding moduli space $\tilde{\mathcal{N}}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}}$ without asymptotic markers, and the map which forgets the asymptotic markers

$$(3.41) \quad \pi : \mathcal{N}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}} \rightarrow \tilde{\mathcal{N}}_{h,n,\vec{m}}^{\mathbf{I},\vec{\mathbf{K}}}$$

is a fibration in the stable range, with fiber non-canonically isomorphic to $(S^1)^n$.

DEFINITION 3.3. Define $\mathcal{J}_c(M)$ to be the space of almost-complex structures J that are **c-rescaled contact type**, i.e.

$$(3.42) \quad \frac{c}{r} \theta \circ J = dr.$$

Also, define $\mathcal{J}(M)$ to be the space of almost-complex structures J that are *c-rescaled contact type* for any $c > 0$.

DEFINITION 3.4. A **collection of strip and cylinder data** for a surface S with some boundary and interior marked pointed removed is a choice of

- **strip-like ends** $\epsilon_{\pm}^k : Z_{\pm} \rightarrow S$,
- **finite strips** $\epsilon^l : [a^l, b^l] \times [0, 1] \rightarrow S$,
- **cylindrical ends** $\delta_{\pm}^j : A_{\pm} \times S^1 \rightarrow S$, and
- **finite cylinders** $\delta^r : [a_r, b_r] \times S^1 \rightarrow S$

all with disjoint image in S . Such a collection is said to be **weighted** if each cylinder and strip above comes equipped with a choice of positive real number, called a *weight*. Label these weights as follows:

- $w_{S,k}^{\pm}$ is the weight associated to the strip-like end ϵ_{\pm}^k ,

- $w_{S,l}$ is associated to the finite strip ϵ^l ,
- $v_{S,j}^\pm$ is associated to the cylindrical end δ_\pm^j , and
- $v_{S,r}$ is associated to the finite cylinder δ^r .

Finally, such a collection is said to be **δ -bounded** if

- the length of each finite cylinder $(b_r - a_r)$ is larger than 3δ .

DEFINITION 3.5. A one-form α_S on S is said to be **compatible with the weighted strip and cylinder data \mathfrak{S}** if, for each finite or semi-infinite cylinder or strip κ of S , with associated weight ν_κ ,

$$(3.43) \quad \kappa^* \alpha_S = \nu_\kappa dt.$$

Above, t is the coordinate of the second component of the associated strip or cylinder.

DEFINITION 3.6. Fix a Hamiltonian $H \in \mathcal{H}(M)$. An S -dependent Hamiltonian $H_S : S \rightarrow \mathcal{H}(M)$ is said to be **H-compatible with the weighted strip and cylinder data \mathfrak{S}** if, for each cylinder or strip κ with associated weight ν_κ ,

$$(3.44) \quad \kappa^* H_S = \frac{H \circ \psi^{\nu_\kappa}}{\nu_\kappa^2}.$$

DEFINITION 3.7. An **\mathfrak{S} -adapted rescaling function** is a map $a_S : S \rightarrow [1, \infty)$ that is constant on each cylinder and strip of \mathfrak{S} , equal to the associated weight of that cylinder or strip.

DEFINITION 3.8. Fix a time-dependent almost-complex structure $J_t : S^1 \rightarrow \mathcal{J}_1(M)$, and an adapted rescaling function a_S . An **(\mathfrak{S}, a_S, J_t) -adapted complex structure** is a map $J_S : S \rightarrow \mathcal{J}(M)$ such that

- at each point $p \in S$, $J_p \in \mathcal{J}_{a_S(p)}(M)$,
- at each cylinder or strip κ with associated weight ν_κ ,

$$(3.45) \quad \kappa^* J_S = (\psi^{\nu_\kappa})^* J_t.$$

Here, if κ is a strip, we mean the $[0, 1]$ dependent complex structure given by pulling back J_t by the projection map $[0, 1] \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$.

DEFINITION 3.9. A **Floer datum \mathbf{F}_S** on a stable genus zero open-closed string S consists of the following choices on each component:

- (1) A collection of **weighted strip and cylinder data \mathfrak{S}** that is δ -bounded;
- (2) **sub-closed 1-form**: a one-form α_S with

$$d\alpha_S \leq 0,$$

compatible with the weighted strip and cylinder data;

- (3) A **primary Hamiltonian $H_S : S \rightarrow \mathcal{H}(M)$** that is H -compatible with the weighted strip and cylinder data \mathfrak{S} for some fixed H ;
- (4) An **\mathfrak{S} -adapted rescaling function a_S** ;
- (5) An **almost-complex structure J_S** that is (\mathfrak{S}, a_S, J_t) -adapted for some J_t .
- (6) An **S^1 -perturbation F_S adapted to (F_T, ϕ_ϵ)** for some F_T, ϕ_ϵ as above.

There is a notion of equivalence of Floer data, weaker than strict equality, which will imply by the rescaling correspondence (3.47) that the resulting operations are identical.

DEFINITION 3.10. Say that Floer data D_S^1 and D_S^2 are **conformally equivalent** if there exist constants C, K, K' such that

$$(3.46) \quad \begin{aligned} a_S^2 &= C \cdot a_S^1, \\ \alpha_S^2 &= C \cdot \alpha_S^1, \\ J_S^2 &= (\psi^C)^* J_S^1, \\ H_S^2 &= \frac{H_S^1 \circ \psi^C}{C^2} + K, \text{ and} \\ F_S^2 &= \frac{F_S^1 \circ \psi^C}{C^2} + K', \end{aligned}$$

where ψ^ρ denotes the Liouville flow for time $\log(\rho)$. In other words, the Floer D_S^2 is a rescaling by Liouville flow of the Floer data D_S^1 , up to a constant ambiguity in the Hamiltonian terms.

To see that the above definition gives a suitable notion of equivalence, note that the pullback of Floer's equation (3.30) by the Liouville flow for time $\log(\rho)$ defines a canonical isomorphism

$$(3.47) \quad CW^*(L_0, L_1; H, J_t) \simeq CW^*\left(\psi^\rho L_0, \psi^\rho L_1; \frac{H}{\rho} \circ \psi^\rho, (\psi^\rho)^* J_t\right)$$

LEMMA 3.7 ([A1, Lemma 3.1]). *The function $\frac{H}{\rho^2} \circ \psi^\rho$ lies in $\mathcal{H}(M)$.*

PROOF. Away from a compact set, the Liouville flow is given by

$$(3.48) \quad \psi^\rho(r, y) = (\rho \cdot r, y),$$

and $r^2 \circ \psi^\rho = \rho^2 r^2$. □

Now, fix a submanifold with corners of dimension d ,

$$(3.49) \quad i : \mathcal{Q}^d \hookrightarrow \mathcal{N}_{h,n,\vec{m}}^{\mathbf{I},\vec{K}}$$

We assume that \mathcal{Q} has no asymptotic marker freedom, meaning that the position of any asymptotic marker is uniquely determined by the positions of the marked points. Put another way, we require $\pi \circ i$ to also be an embedding, so $i(\mathcal{Q}^d)$ is a section of the fibration (3.40) restricted to $\pi \circ i(\mathcal{Q})$.

REMARK 3.4. *In fact, in the examples we consider, the asymptotic direction is always determined uniquely by requiring it to point towards one particular distinguished boundary point.*

The main point here is that in the absence asymptotic marker freedom, one can compactify \mathcal{Q} using the usual Deligne-Mumford compactification instead of the real blow-up compactification [KS^V] naturally associated to moduli spaces of surfaces with marked points equipped with asymptotic markers.

Denote by $\bar{\mathcal{Q}}$ the natural Gromov-type compactification, generally constructed by allowing certain stable degenerations (which will be constructed in the various examples to follow).

Define

$$(3.50) \quad \bar{\mathcal{Q}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$$

to be the space of maps

$$(3.51) \quad \{u : S \longrightarrow M : S \in \bar{\mathcal{Q}}^d\}$$

satisfying the inhomogenous Cauchy-Riemann equation with respect to the complex structure J_S :

$$(3.52) \quad (du - X_S \otimes \alpha_S)^{0,1} = 0$$

and asymptotic and boundary conditions:

$$(3.53) \quad \begin{cases} \lim_{s \rightarrow \pm\infty} u \circ \epsilon_\pm^{i,j}(s, \cdot) = x_i^j, \\ \lim_{s \rightarrow \pm\infty} u \circ \delta_\pm^l(s, \cdot) = y_l, \\ u(z) \in \psi^{a_S(z)} L_i^j, \end{cases} \quad z \in \partial_i^j S.$$

Above, X_S is the (surface-dependent) Hamiltonian vector field corresponding to $H_S + F_S$.

LEMMA 3.8. *The moduli spaces $\bar{\mathcal{Q}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ are compact and there are only finitely many collections $\vec{x}_{out}, \vec{y}_{out}$ for which they are non-empty given input $\vec{x}_{in}, \vec{y}_{in}$. For a generic universal and conformally consistent Floer data they form manifolds of dimension*

$$(3.54) \quad \begin{aligned} \dim \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in}) &:= \sum_{x_- \in \vec{x}_{out}} \deg(x_-) + \sum_{y_- \in \vec{y}_{out}} \deg(y_-) + \\ &\quad (2 - h - |\vec{x}_{out}| - 2|\vec{y}_{out}|)n + d - \sum_{x_+ \in \vec{x}_{in}} \deg(x_+) - \sum_{y_+ \in \vec{y}_{in}} \deg(y_+). \end{aligned}$$

PROOF. This is a slight adaptation of the usual results in the literature, the proof of this exact statement appears in [G2, Lem. 4.3]. □

When $\mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ has dimension zero, we conclude that its elements are rigid. For any such element $u \in \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$, we obtain an isomorphism of orientation lines by arguments reviewed in [G2, Appendix A]:

$$(3.55) \quad \mathcal{Q}_u : \bigotimes_{x \in \vec{x}_{in}} o_x \otimes \bigotimes_{y \in \vec{y}_{in}} o_y \longrightarrow \bigotimes_{x \in \vec{x}_{out}} o_x \otimes \bigotimes_{y \in \vec{y}_{out}} o_y.$$

Thus, we can define a map

$$(3.56) \quad \mathbf{F}_{\overline{\mathcal{Q}}^d} : \bigotimes_{(i,j); 1 \leq i \leq m_j; i \notin K^j} CW^*(L_i^j, L_{i+1}^j) \otimes \bigotimes_{1 \leq k \leq n; k \notin \mathbf{I}} CH^*(M) \longrightarrow \bigotimes_{(i,j); 1 \leq i \leq m_j; i \in K^j} CW^*(L_{i+1}^j, L_i^j) \otimes \bigotimes_{1 \leq k \leq n; k \in \mathbf{I}} CH^*(M)$$

given by:

$$(3.57) \quad \mathbf{F}_{\overline{\mathcal{Q}}^d}([y_t], \dots, [y_1], [x_s], \dots, [x_1]) := \sum_{\dim \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \{x_1, \dots, x_s\}, \{y_1, \dots, y_t\})=0} \sum_{u \in \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \{x_1, \dots, x_s\}, \{y_1, \dots, y_t\})} \mathcal{Q}_u([x_s], \dots, [x_1], [y_t], \dots, [y_1]).$$

This construction naturally associates, to any submanifold $\mathcal{Q}^d \in \mathcal{N}_{h,n,\vec{m}}^{\mathbf{I},\vec{K}}$, a chain-level map $\mathbf{F}_{\overline{\mathcal{Q}}^d}$, depending on a sufficiently generic choice of Floer data for open-closed strings. We need to modify this construction by signs depending on the relative positions and degrees of the inputs.

DEFINITION 3.11. *Given such a submanifold \mathcal{Q} , a **sign twisting datum** \vec{t} for \mathcal{Q} is a vector of integers, one for each input boundary or interior marked point on an element of \mathcal{Q} .*

To a pair (\mathcal{Q}, \vec{t}) one can associate a twisted operation

$$(3.58) \quad (-1)^{\vec{t}} \mathbf{F}_{\overline{\mathcal{Q}}^d},$$

defined as follows. If $\{\vec{x}, \vec{y}\} = \{x_1, \dots, x_s, y_1, \dots, y_t\}$ is a set of asymptotic inputs, the **vector of degrees** is denoted

$$(3.59) \quad \vec{\deg}(\vec{x}, \vec{y}) := \{\{\deg(x_1), \dots, \deg(x_s)\}, \{\deg(y_1), \dots, \deg(y_t)\}\}.$$

The corresponding sign twisting datum \vec{t} is of the form

$$(3.60) \quad \vec{t} := \{\{v_1, \dots, v_s\}; \{w_1, \dots, w_t\}\}.$$

Then, the operation (3.58) is defined to be

$$(3.61) \quad (-1)^{\vec{t}} \mathbf{F}_{\overline{\mathcal{Q}}^d}([y_t], \dots, [y_1], [x_s], \dots, [x_1]) := \sum_{\dim \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}, \vec{y})=0} \sum_{u \in \mathcal{Q}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}, \vec{y})} (-1)^{\vec{t} \cdot \vec{\deg}(\vec{x}, \vec{y})} \mathcal{Q}_u([x_1], \dots, [x_s], [y_1], \dots, [y_t]).$$

The zero vector $\vec{t} = (0, \dots, 0)$ recovers the original operation $\mathbf{F}_{\overline{\mathcal{Q}}^d}$.

3.3. The A_∞ structure on the wrapped Fukaya category. For $d \geq 2$, denote by

$$(3.62) \quad \mathcal{R}^d$$

the moduli space of discs with $d+1$ marked points modulo reparametrization, with one point z_0^- marked as negative and the remainder z_1^+, \dots, z_d^+ (labeled counterclockwise from z_0^-) marked as positive. (3.62), a quotient, can be identified with the space of unit discs with z_0^- , z_1^+ , and z_2^+ in fixed position; the positions of the remaining ordered points identify \mathcal{R}^d with an open subset of \mathbb{R}^{d-2} . Following [S5, 12g], we orient \mathcal{R}^d by pulling back the \mathbb{R}^{d-2} orientation $dz_2 \wedge \dots \wedge dz_{d-2}$.

The natural (Deligne-Mumford) compactification

$$(3.63) \quad \overline{\mathcal{R}}^d,$$

consisting of trees of stable discs with a total of d exterior positive marked points and 1 exterior negative marked point, modulo compatible reparametrization of each disc in the tree. The compactification (3.63) inherits the structure of a manifold with corners, coming from local gluing charts

$$(3.64) \quad (0, +\infty]^k \times \sigma \rightarrow \overline{\mathcal{R}}^d.$$

near (nodal) strata of codimension k , which can be explicitly described in terms of connect-sums with respect to *strip-like ends* [S1, 9f].

DEFINITION 3.12. *A **consistent choice of Floer data for the A_∞ structure** is a (n inductive) choice of Floer data, for each $d \geq 2$ and for each representative S of $\overline{\mathcal{R}}^d$, smoothly varying in S , whose restriction to each boundary stratum is conformally equivalent to the product of Floer data coming from lower-dimensional spaces. With respect to the boundary gluing charts, the Floer data should agree to infinite order at boundary strata with Floer data obtained via gluing.*

Inductively, since there is a contractible space of choices consistent with lower levels,

PROPOSITION 3.1. *Universal and consistent choices of Floer data for the A_∞ structure exist.*

Fixing a universal and consistent Floer datum \mathbf{D}_μ , for L_0, \dots, L_d be objects of \mathcal{W} , and a sequence of chords $\vec{x} = \{x_k \in \chi(L_{k-1}, L_k)\}$ as well as another chord $x_0 \in \chi(L_0, L_d)$, we obtain moduli spaces of solutions to Floer's equation with source an arbitrary element of $\overline{\mathcal{R}}^d$, using the specified Floer datum:

$$(3.65) \quad \mathcal{R}^d(x_0; \vec{x})$$

The consistency of our Floer data with respect to the codimension one boundary of the abstract moduli spaces $\overline{\mathcal{R}}^d$ implies that the (Gromov-type) compactification $\overline{\mathcal{R}}^d(x_0; \vec{x})$ is obtained by adding the images of the natural inclusions

$$(3.66) \quad \overline{\mathcal{R}}^{d_1}(x_0; \vec{x}_1) \times \overline{\mathcal{R}}^{d_2}(y; \vec{x}_2) \rightarrow \overline{\mathcal{R}}^d(x_0; \vec{x})$$

where y agrees with one of the elements of \vec{x}_1 and \vec{x} is obtained by removing y from \vec{x}_1 and replacing it with the sequence \vec{x}_2 . Here, we let d_1 range from 1 to d , with $d_2 = d - d_1 + 1$, with the stipulation that $d_1 =$ or $d_2 = 1$ is the semistable case:

$$(3.67) \quad \overline{\mathcal{R}}^1(x_0; x_1) := \overline{\mathcal{R}}(x_0; x_1)$$

The following is a special case of Lemma 3.8:

LEMMA 3.9. *For a generically chosen Floer data \mathbf{D}_μ , the moduli space $\overline{\mathcal{R}}^d(x_0; \vec{x})$ is a smooth compact manifold of dimension*

$$\deg(x_0) + d - 2 - \sum_{1 \leq k \leq d} \deg(x_k),$$

and for fixed \vec{x} is empty for all but finitely many x_0 .

We obtain, in the manner of §3.2, a map

$$(3.68) \quad \mathbf{F}_{\overline{\mathcal{R}}^d} : CW^*(L_{d-1}, L_d) \otimes \dots \otimes CW^*(L_0, L_1) \longrightarrow CW^*(L_0, L_d),$$

of degree $2 - d$, by for each \vec{x} , x_0 summing over rigid elements of $\overline{\mathcal{R}}^d(x_0; \vec{x})$ the induced maps on orientation lines. We define the d th A_∞ operation to be (3.68) up to a sign twist: define the **incremental sign twisting datum of length d** , denoted \vec{t}_d , to be the vector $(1, 2, \dots, d-1, d)$, and set

$$(3.69) \quad \mu^d := (-1)^{\vec{t}_d} \mathbf{F}_{\overline{\mathcal{R}}^d},$$

More concretely once, we have that

$$(3.70) \quad \mu^d([x_d], \dots, [x_1]) := \sum_{\deg(x_0)=2-d+\sum \deg(x_k)} \sum_{u \in \overline{\mathcal{R}}^d(x_0; \vec{x})} (-1)^{\vec{t}_d \cdot \deg(\vec{x})} \mathcal{R}_u^d([x_d], \dots, [x_1])$$

where the sign is

$$(3.71) \quad \vec{t}_d \cdot (\deg(x_1), \dots, \deg(x_d)) = \star_d = \sum_{i=1}^d i \cdot \deg(x_i).$$

Analyzing boundaries of 1-dimensional moduli spaces (3.66), along with the signs of operations, establishes:

LEMMA 3.10 ([S5, Prop. 12.3]). *The maps μ^d satisfy the A_∞ relations.*

3.4. Hochschild invariants from bimodules. In this section, we recall chain level models of Hochschild invariants coming from operations on A_∞ bimodules. We remark that these differ from the more popular *cyclic bar* chain level models for Hochschild homology and cohomology, but in a harmless fashion (by explicit quasi-isomorphisms, see e.g., [G2, §2.7]). Because they can be thought of as a version of the cyclic bar complex in which there are two special elements of a tensor which cannot simultaneously be collapsed by a differential, we call them *two-pointed complexes*.

In the two definitions that follow, let \mathcal{A} denote an A_∞ category and \mathcal{B} an $\mathcal{A}-\mathcal{A}$ bimodule.

DEFINITION 3.13. *The **two-pointed Hochschild chain complex***

$$(3.72) \quad {}_2\mathrm{CC}_*(\mathcal{A}, \mathcal{B})$$

is the chain complex computing the bimodule tensor product with the diagonal bimodule:

$$(3.73) \quad {}_2\mathrm{CC}_*(\mathcal{A}, \mathcal{B}) := \mathcal{A}_\Delta \otimes_{\mathcal{A}-\mathcal{A}} \mathcal{B}.$$

REMARK 3.5. *Even though this is a chain complex, the grading inherited from the description of bimodule tensor product given in Definition 2.8 is cohomological, so the differential δ has degree $+1$.*

DEFINITION 3.14. *The **two-pointed Hochschild co-chain complex***

$$(3.74) \quad {}_2\mathrm{CC}^*(\mathcal{A}, \mathcal{B})$$

is the chain complex computing the bimodule morphism space:

$$(3.75) \quad {}_2\mathrm{CC}^*(\mathcal{A}, \mathcal{B}) := \mathrm{hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{A}_\Delta, \mathcal{B}).$$

REMARK 3.6. *For the reader familiar with the usual definition of Hochschild chains and co-chains of $(\mathcal{A}, \mathcal{B})$ as in e.g., [A1] or [G2, §2.7], we observe that the complex ${}_2\mathrm{CC}_*(\mathcal{A}, \mathcal{B})$ is equivalent to the ordinary (cyclic bar) Hochschild chain complex of a bimodule equivalent to \mathcal{B} : $\mathrm{CC}_*(\mathcal{A}, \mathcal{A}_\Delta \otimes_{\mathcal{A}} \mathcal{B})$. Thus, the quasi-isomorphism of bimodules $\mathcal{A}_\Delta \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\sim} \mathcal{B}$ functorially induces a quasi-isomorphism of complexes $\Phi : {}_2\mathrm{CC}_*(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{CC}_*(\mathcal{A}, \mathcal{B})$.*

Similarly, as one natural interpretation of Hochschild cohomology is as endomorphisms of the identity functor or the (derived) self-ext of the diagonal bimodule, one expects a quasi-isomorphism of complexes between (3.75). $\Psi : \mathrm{CC}^(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{A}_\Delta, \mathcal{B})$. Both quasi-isomorphisms Φ and Ψ admit explicit straightforward chain-level descriptions (see for instance [G2, §2.7]).*

In light of the above remark we refer to the cohomology of the two-pointed Hochschild chains and co-chains complexes simply as *Hochschild homology and cohomology*, denoted

$$(3.76) \quad \mathrm{HH}_*(\mathcal{A}, \mathcal{B}), \mathrm{HH}^*(\mathcal{A}, \mathcal{B}).$$

In the case, $\mathcal{B} = \mathcal{A}_\Delta$, abbreviate ${}_2\mathrm{CC}_*(\mathcal{A}, \mathcal{A}_\Delta)$ and ${}_2\mathrm{CC}^*(\mathcal{A}, \mathcal{A}_\Delta)$ by ${}_2\mathrm{CC}_*(\mathcal{A}, \mathcal{A})$ and ${}_2\mathrm{CC}^*(\mathcal{A}, \mathcal{A})$ respectively. We will sometimes omit the *two-pointed* prefix, and simply refer to these complexes as Hochschild chains and co-chains.

The Hochschild co-chains of \mathcal{A} is actually a dg-algebra, with Hochschild chains of \mathcal{A} a dg-module over it. A convenient aspect of using the theory of bimodules to give the above definitions is that the formulae for these operations is a special case of the discussion in §2.2:

LEMMA 3.11. *The composition of bimodule pre-morphisms*

$$(3.77) \quad \cup := " \circ " : \mathrm{hom}_{\mathcal{A}}(\mathcal{A}_\Delta, \mathcal{A}_\Delta) \otimes \mathrm{hom}_{\mathcal{A}}(\mathcal{A}_\Delta, \mathcal{A}_\Delta) \rightarrow \mathrm{hom}_{\mathcal{A}}(\mathcal{A}_\Delta, \mathcal{A}_\Delta).$$

gives Hochschild co-chains the structure of a dg algebra. The functoriality of two-sided tensor products with respect to bimodule pre-morphisms (2.62):

$$(3.78) \quad \begin{aligned} \cap : \text{hom}_{\mathcal{A}}(\mathcal{A}_{\Delta}, \mathcal{A}_{\Delta}) \otimes (\mathcal{A}_{\Delta} \otimes_{\mathcal{A}-\mathcal{A}} \mathcal{A}_{\Delta}) &\rightarrow (\mathcal{A}_{\Delta} \otimes_{\mathcal{A}-\mathcal{A}} \mathcal{A}_{\Delta}) \\ (\mathcal{F}, \alpha) &\mapsto \mathcal{F}_{\#}(\alpha). \end{aligned}$$

gives Hochschild chains the structure of a dg module over Hochschild co-chains.

PROOF. This is immediate from §2.2. \square

REMARK 3.7. *It is not obvious from these formulae, but the Hochschild co-chain algebra structure is actually commutative on cohomology. In fact, by a form of Deligne's conjecture, Hochschild co-chains are an E_2 algebra, and Hochschild chains an E_2 module over them, a fact we will not need.*

3.5. Two-pointed open-closed and closed-open maps. We recall from [G2] the construction of the variant of the open-closed and closed-open maps using two-pointed Hochschild chains and co-chains:

$$(3.79) \quad {}_2\mathcal{OC} : {}_2\mathcal{CC}_*(\mathcal{W}, \mathcal{W}) \longrightarrow SH^*(M)$$

$$(3.80) \quad {}_2\mathcal{CO} : SH^*(M) \longrightarrow {}_2\mathcal{CC}^*(\mathcal{W}, \mathcal{W})$$

The *two-pointed open-closed moduli space* with (k, l) marked points

$$(3.81) \quad \mathcal{R}_{k,l}^1$$

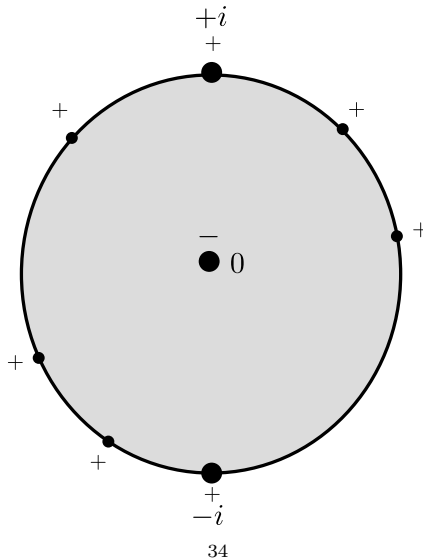
is the space of discs with one interior negative puncture labeled y_{out} , and $k + l + 2$ boundary punctures, labeled in counterclockwise order $z_0, z_1, \dots, z_k, z'_0, z'_1, \dots, z'_l$, such that:

$$(3.82) \quad \text{up to automorphism, } z_0, z'_0, \text{ and } y_{out} \text{ are constrained to lie at } -i, i \text{ and } 0 \text{ respectively.}$$

Call z_0 and z'_0 the **special inputs** of any such disc.

REMARK 3.8. *The moduli space $\mathcal{R}_{k,l}^1$ is a codimension one submanifold of \mathcal{R}_{k+l+2}^1 , the moduli space of discs with marked points as above, moduli automorphism, with no cross ratio constraint (3.82). Hence $\dim \mathcal{R}_{k,l}^1 = k + l$.*

FIGURE 1. A representative of an element of the moduli space $\mathcal{R}_{3,2}^1$ with special points at 0 (output), $-i$, and i .



The boundary strata of Deligne-Mumford compactification $\overline{\mathcal{R}}_{k,l}^1$ is covered by the images of the natural inclusions of the following products:

$$(3.83) \quad \overline{\mathcal{R}}^{k'} \times_{n+1} \overline{\mathcal{R}}_{k-k'+1,l}^1, \quad 0 \leq n < k - k' + 1$$

$$(3.84) \quad \overline{\mathcal{R}}^{l'} \times_{(n+1)'} \overline{\mathcal{R}}_{k,l-l'+1}^1, \quad 0 \leq n' < l - l' + 1$$

$$(3.85) \quad \overline{\mathcal{R}}^{k'+l'+1} \times_0 \overline{\mathcal{R}}_{k-k',l-l'}^1$$

$$(3.86) \quad \overline{\mathcal{R}}^{l'+k'+1} \times_{0'} \overline{\mathcal{R}}_{k-k',l-l'}^1.$$

Here the notation \times_j indicates that one glues the distinguished output of the first factor to the input z_j , and the notation $\times_{j'}$ indicates that one glues the distinguished output of the first factor to the input z'_j . Moreover, in (3.85), after gluing the output of the first disc to to the first special point z_0 , the $k' + 1$ st input becomes the new special point z_0 . Similarly in (3.86), after gluing the output of the first stable disc to the second special point z'_0 , the $l' + 1$ st input becomes the new special point z'_0 . Thinking of $\mathcal{R}_{k,l}^1$ as a submanifold of open-closed strings, we obtain, given a compatible Lagrangian labeling $\{L_0, \dots, L_k, L'_0, \dots, L'_l\}$ asymptotic input chords $\{x_0, x_1, \dots, x_k, x'_0, x'_1, \dots, x'_l\}$ and output orbit y , Floer theoretic moduli spaces

$$(3.87) \quad \overline{\mathcal{R}}_{k,l}^1(y; x_0, x_1, \dots, x_k, x'_0, x'_1, \dots, x'_l)$$

of dimension

$$(3.88) \quad k + l - n + \deg(y) - \deg(x_0) - \deg(x'_0) - \sum_{i=1}^k \deg(x_i) - \sum_{j=1}^l \deg(x'_j).$$

Here, L_k, L'_0 are adjacent to the second special point z'_0 and L'_l, L_0 are adjacent to z_0 (with corresponding inputs x'_0, x_0). Using the sign twisting datum

$$(3.89) \quad \vec{t}_{2\mathcal{OC}_{k,l}} = (1, 2, \dots, k+1, k+3, k+4, \dots, k+2+l)$$

with respect to the ordering of inputs $(z_0, \dots, z_k, z'_0, \dots, z'_l)$, define associated Floer-theoretic operations

$$(3.90) \quad {}_2\mathcal{OC}_{k,l} := (-1)^{\vec{t}_{2\mathcal{OC}}} \mathbf{F}_{\overline{\mathcal{R}}_{k,l}^1} : (\mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes l} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes k})^{diag} \longrightarrow CH^*(M).$$

The two-pointed open-closed map is defined to be the sum of these operations:

$$(3.91) \quad {}_2\mathcal{OC} = \sum_{k,l} {}_2\mathcal{OC}_{k,l} : (\mathcal{W}_\Delta \otimes T\mathcal{W} \otimes \mathcal{W}_\Delta \otimes T\mathcal{W})^{diag} \longrightarrow CH^*(M).$$

With respect to the grading on the 2-pointed Hochschild complex (2.54), ${}_2\mathcal{OC}$ is chain map a map of degree n . By analyzing the boundary of the one-dimensional components of $\overline{\mathcal{R}}_{k,l}^1$, seeing that the relevant boundary behavior is governed by the codimension-1 boundary of the abstract moduli space $\overline{\mathcal{R}}_{k,l}^1$ described from (3.83)-(3.86) and strip-breaking, and performing a sign verification in [G2, §A], we conclude that

COROLLARY 3.1. *The map ${}_2\mathcal{OC} : {}_2\mathcal{CC}_*(\mathcal{W}, \mathcal{W}) \longrightarrow CH^{*+n}(M)$ is a chain map.*

Next, the *two-pointed closed-open moduli space* with (r, s) marked points

$$(3.92) \quad \mathcal{R}_{r,s}^{1,1}$$

is the space of discs with one interior positive puncture labeled y_{in} , one negative boundary puncture z_{out} , and $r+s+1$ positive boundary punctures, labeled in clockwise order from z_{out} as $z_1, \dots, z_r, z_{fixed}, z'_1, \dots, z'_s$, subject to the following constraint:

$$(3.93) \quad \text{up to automorphism, } z_{out}, z_{fixed}, \text{ and } y_{in} \text{ lie at } -i, i \text{ and } 0 \text{ respectively.}$$

The boundary strata of the Deligne-Mumford compactification $\overline{\mathcal{R}}_{r,s}^{1,1}$ is covered by the natural inclusions of the following products:

$$(3.94) \quad \overline{\mathcal{R}}^{r'} \times_{n+1} \overline{\mathcal{R}}_{r-r'+1,s}^{1,1}, \quad 0 \leq n < r - r' + 1$$

$$(3.95) \quad \overline{\mathcal{R}}^{s'} \times_{(m+1)'} \overline{\mathcal{R}}_{r,s-s'+1}^{1,1}, \quad 0 \leq m < s - s' + 1$$

$$(3.96) \quad \overline{\mathcal{R}}^{r'+s'+1} \times_0 \overline{\mathcal{R}}_{r-r',s-s'}^{1,1}$$

$$(3.97) \quad \overline{\mathcal{R}}_{r-a',s-b'}^{1,1} \times_{a'+1} \overline{\mathcal{R}}^{a'+b'+1}.$$

Here in (3.96), the output of the stable disc is glued to the special input z_{fixed} with the $r' + 1$ st point becoming the new distinguished z_{fixed} . Similarly, in (3.97), the output of the two-pointed closed-open disc z_{out} is glued to the $a + 1$ st input of the stable disc.

Thinking of $\mathcal{R}_{r,s}^{1,1}$ as a submanifold of open-closed strings, we obtain, given a compatible Lagrangian labeling $\{L_0, \dots, L_r, L'_0, \dots, L'_s\}$ and input chords $\{x_1, \dots, x_r, x_{fixed}, x'_1, \dots, x'_s\}$ input orbit y , and output chord x_{out} , a moduli space

$$(3.98) \quad \overline{\mathcal{R}}_{r,s}^{1,1}(x_{out}; y, x_1, \dots, x_r, x_{fixed}, x'_1, \dots, x'_s)$$

of dimension

$$(3.99) \quad r + s + \deg(x_{out}) - \deg(y) - \deg(x_{fixed}) - \sum_{i=1}^r \deg(x_i) - \sum_{j=1}^s \deg(x'_j).$$

Here, L_r, L'_0 are adjacent to the second special output z_{out} and L'_s, L_0 are adjacent to z_{fixed} (with corresponding asymptotic conditions x_{out}, x_{fixed}). We also obtain associated Floer-theoretic operations

$$(3.100) \quad \mathbf{F}_{\overline{\mathcal{R}}_{r,s}^{1,1}} : CH^*(M) \otimes (\mathcal{W}^{\otimes s} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes r}) \longrightarrow \mathcal{W}_\Delta.$$

Now, define

$$(3.101) \quad {}_2\mathcal{CO}_{r,s} : CH^*(M) \longrightarrow \text{hom}_{Vect}(\mathcal{W}^{\otimes s} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes r}, \mathcal{W}_\Delta)$$

as

$$(3.102) \quad {}_2\mathcal{CO}_{r,s}(y)(y_s, \dots, y_1, \mathbf{b}, x_s, \dots, x_1) := (-1)^{\vec{t}_{{}_2\mathcal{CO}_{r,s}}} \mathbf{F}_{\overline{\mathcal{R}}_{r,s}^{1,1}}(y, y_s, \dots, x_1, \mathbf{b}, x_r, \dots, x_1)$$

where $\vec{t}_{{}_2\mathcal{CO}_{r,s}}$ is the sign twisting datum

$$(3.103) \quad \vec{t}_{{}_2\mathcal{CO}_{r,s}} := (-1, 0, \dots, r-1, r+1, r+2, \dots, r+s+1)$$

with respect to the input ordering $(y_{in}, z_1, \dots, z_r, z_{fixed}, z'_1, \dots, z'_s)$. Define the two-pointed closed-open map to be the sum of these operations

$$(3.104) \quad {}_2\mathcal{CO} = \sum_{r,s} {}_2\mathcal{CO}_{r,s} : CH^*(M) \longrightarrow \text{hom}_{\mathcal{W}-\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{W}_\Delta)$$

With respect to the grading on the 2-pointed Hochschild co-chain complex, ${}_2\mathcal{CO}$ a degree zero map. An analysis of the boundary of the one-dimensional components of $\overline{\mathcal{R}}_{k,l}^{1,1}$ coming from strip-breaking and the codimension-1 boundary of the abstract moduli space $\overline{\mathcal{R}}_{k,l}^{1,1}$ described in (3.94)-(3.97), along with a sign verification discussed in [G2, §A], we conclude that

COROLLARY 3.2. *The map ${}_2\mathcal{CO} : CH^*(M) \longrightarrow {}_2\mathcal{CC}^*(\mathcal{W}, \mathcal{W})$ is a chain map.*

These *two-pointed maps* are slightly different chain-level implementations of the open-closed and closed-open maps, but in the prequel article we showed:

PROPOSITION 3.2 ([G2, Prop. 4.3]). *On the cohomology level, we have equalities $[{}_2\mathcal{OC}] = [\mathcal{OC}]$, $[{}_2\mathcal{CO}] = [\mathcal{CO}]$, where \mathcal{OC} and \mathcal{CO} are the earlier chain level implementations using the cyclic bar complexes for Hochschild chains and co-chains, found in [A1] [S1] [G2].*

Hence, we refer to the cohomology level maps simply as $[\mathcal{CO}]$ and $[\mathcal{OC}]$.

REMARK 3.9. In fact, [G2, Prop. 4.3] exhibited chain homotopies $\mathcal{OC} \circ \Phi \simeq {}_2\mathcal{OC}$ and $\Psi \circ \mathcal{CO} \simeq {}_2\mathcal{CO}$, where Φ and Ψ are the explicit quasi-isomorphisms from Remark 3.6.

3.6. Ring and module structure compatibility. We make two assertions about the maps $[\mathcal{CO}]$ and $[\mathcal{OC}]$, both of which follow from an analysis of similar-looking moduli spaces.

PROPOSITION 3.3. $[\mathcal{CO}]$ is an algebra homomorphism.

Pulling back the module structure over $\mathrm{HH}^*(\mathcal{W}, \mathcal{W})$ via the algebra homomorphism $[\mathcal{CO}]$, Hochschild homology $\mathrm{HH}_*(\mathcal{W}, \mathcal{W})$ obtains the structure of a module over $SH^*(M)$.

PROPOSITION 3.4. $[\mathcal{OC}]$ is a map of $SH^*(M)$ -modules.

The $SH^*(M)$ module structure property of $[\mathcal{OC}]$ was observed independently by the author [G1] and (in the monotone setting and with a somewhat different technical setup) by Ritter and Smith [RS]. The particular proof given here differs slightly from the original proof given in [G1] in that it is adapted to two-pointed open-closed maps. The chain-level statement is that the following diagram homotopy commutes:

$$(3.105) \quad \begin{array}{ccc} {}_2\mathrm{CC}_*(\mathcal{W}) \times CH^*(M) & \xrightarrow{({}_2\mathcal{OC}, \mathrm{id})} & CH^*(M) \times CH^*(M) \\ \downarrow (\mathrm{id}, {}_2\mathcal{CO}) & & \downarrow * \\ {}_2\mathrm{CC}_*(\mathcal{W}) \times {}_2\mathrm{CC}^*(\mathcal{W}) & \xrightarrow{{}_2\mathcal{OC} \circ \cap} & CH^*(M) \end{array}$$

To prove Proposition 3.4, for $r \in (0, 1)$, define the auxiliary moduli space

$$(3.106) \quad \mathcal{P}_{k,l}^2(r)$$

to consist of the unit disc in \mathbb{C} with the following data:

- $k + l + 2$ boundary punctures, labeled in counterclockwise order $z_0, z_1, \dots, z_k, z'_0, z'_1, \dots, z'_l$, with z_0 and z'_0 marked as distinguished, and
- two interior marked points (κ_+, κ_-) , one positive and one negative

such that,

$$(3.107) \quad \text{after automorphism, the points } z_0, z'_0, \kappa_+, \kappa_- \text{ lie at } -i, i, 0, \text{ and } ir \text{ respectively.}$$

These spaces vary smoothly with r and their union

$$(3.108) \quad \mathcal{P}_{k,l}^2 := \bigcup_{r \in (0,1)} \mathcal{P}_{k,l}^2(r)$$

is naturally a codimension 2 submanifold of genus 0 open-closed strings consisting of a single disc with $k + l + 2$ positive boundary punctures and two interior punctures, one positive and one negative. Compactifying, we obtain a family that submerses over $r \in [0, 1]$ and see that with codimension-1 boundary of $\overline{\mathcal{P}}_{k,l}^2$ is covered by the images of the natural inclusions of the following products (some living over the endpoints $r \in \{0, 1\}$ and some living over the entire interval):

$$(3.109) \quad \overline{\mathcal{R}}_{d_1, d_2}^{1,1} \times_{0'} \overline{\mathcal{R}}_{k-d_1, l-d_2}^1, \quad d_1 \leq k, \quad d_2 \leq l \quad (r = 1)$$

$$(3.110) \quad \overline{\mathcal{R}}_{k,l}^1 \times_2 \overline{\mathcal{S}}_2 \quad (r = 0)$$

$$(3.111) \quad \overline{\mathcal{R}}^{k'} \times_{n+1} \overline{\mathcal{P}}_{k-k'+1, l}^2, \quad 0 \leq n < k - k' + 1$$

$$(3.112) \quad \overline{\mathcal{R}}^{l'} \times_{(n+1)'} \overline{\mathcal{P}}_{k, l-l'+1}^2, \quad 0 \leq n' < l - l' + 1$$

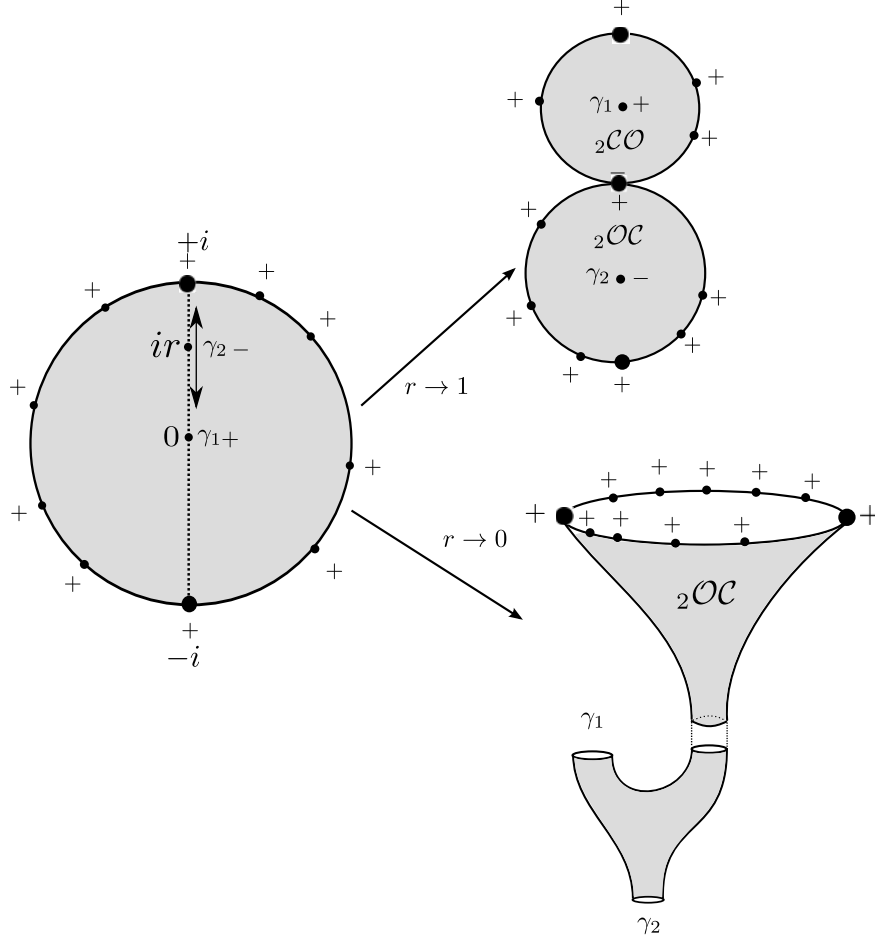
$$(3.113) \quad \overline{\mathcal{R}}^{k'+l'+1} \times_0 \overline{\mathcal{P}}_{k-k', l-l'}^2$$

$$(3.114) \quad \overline{\mathcal{R}}^{l'+k'+1} \times_{0'} \overline{\mathcal{P}}_{k-k', l-l'}^2.$$

Fix a universal and consistent Floer data for all $\overline{\mathcal{P}}_{k,l}^2$, for $k, l \geq 0$. Given a set of Lagrangian labels

$$(3.115) \quad L_0, \dots, L_k, L'_0, \dots, L'_l$$

FIGURE 2. The space $\mathcal{P}_{4,4}^2$ and examples of its possible $r \rightarrow \{0, 1\}$ degenerations.



and compatible asymptotic conditions $\vec{x} = \{x_0, \dots, x_k, x'_0, \dots, x'_l\}$ and $\gamma_-, \gamma_+ \in \mathcal{O}$, we obtain a moduli space of maps

$$(3.116) \quad \mathcal{P}_{k,l}^2(\gamma_-; \gamma_+, \vec{x}).$$

which are smooth compact manifolds of dimension

$$(3.117) \quad \deg(\gamma_-) - n + k + l + 1 - \deg(\gamma_+) - \sum_{i=0}^k \deg(x_i) - \sum_{j=0}^l \deg(x'_j).$$

Consistency of our Floer data implies that the Gromov bordification $\overline{\mathcal{P}}_{k,l}^2(\gamma_-; \gamma_+, \vec{x})$ is obtained by adding the images of the natural inclusions

$$(3.118) \quad \mathcal{M}(\gamma_0; \gamma_+) \times \overline{\mathcal{P}}_{k,l}^2(\gamma_-; \gamma_0, \vec{x}) \rightarrow \partial \mathcal{P}_{k,l}^2(\gamma_-; \gamma_+, \vec{x})$$

$$(3.119) \quad \overline{\mathcal{P}}_{k,l}^2(\gamma_0; \gamma_+, \vec{x}) \times \mathcal{M}(\gamma_-; \gamma_0) \rightarrow \partial \mathcal{P}_{k,l}^2(\gamma_-; \gamma_+, \vec{x})$$

$$(3.120) \quad \overline{\mathcal{R}}^{d_1}(x_a; \vec{x}^2) \times \overline{\mathcal{P}}_{k-d_1+1,l}^2(\gamma_-; \gamma_+, \vec{x}^1) \rightarrow \partial \mathcal{P}_{k,l}^2(\gamma_-; \gamma_+, \vec{x})$$

$$(3.121) \quad \overline{\mathcal{R}}^{d_1}(x_a; \vec{x}^2) \times \overline{\mathcal{P}}_{k,l-d_1+1}^2(\gamma_-; \gamma_+, \vec{x}^1) \rightarrow \partial \mathcal{P}_{k,l}^2(\gamma_-; \gamma_+, \vec{x})$$

$$(3.122) \quad \overline{\mathcal{R}}^{d_1+d_2+1}(x_a; \vec{x}^2) \times \overline{\mathcal{P}}_{k-d_1+1,l-d_2+1}^2(\gamma_-; \gamma_+, \vec{x}^1) \rightarrow \partial \mathcal{P}_{k,l}^2(\gamma_-; \gamma_+, \vec{x})$$

$$(3.123) \quad \overline{\mathcal{R}}^{d_1+d_2+1}(x_a; \vec{x}^2) \times \overline{\mathcal{P}}_{k-d_1+1,l-d_2+1}^2(\gamma_-; \gamma_+, \vec{x}^1) \rightarrow \partial \mathcal{P}_{k,l}^2(\gamma_-; \gamma_+, \vec{x})$$

$$(3.124) \quad \overline{\mathcal{R}}_{k,l}^1(\gamma_1; \vec{x}) \times \overline{\mathcal{S}}_2(\gamma_-; \gamma_+, \gamma_1) \rightarrow \partial \mathcal{P}_{k,l}^2(\gamma_-; \gamma_+, \vec{x})$$

$$(3.125) \quad \overline{\mathcal{R}}_{d_1,d_2}^{1,1}(x_b; \gamma_+, \vec{x}^2) \times \overline{\mathcal{R}}_{k-d_1,l-d_2}^1(\gamma_-; \vec{x}^1) \rightarrow \partial \mathcal{P}_{k,l}^2(\gamma_-; \gamma_+, \vec{x})$$

where

- in (3.120) and (3.121), \vec{x}^2 is a subvector of $\{x_1, \dots, x_k\}$ and $\{x'_1, \dots, x'_l\}$ respectively, and \vec{x}^1 is obtained from \vec{x} by replacing \vec{x}^2 by x_a
- in (3.122), \vec{x}^2 is a subvector of \vec{x} containing x'_0 but not x_0 , and \vec{x}^1 is obtained from \vec{x} by replacing \vec{x}^2 by x_a , thought of as the new distinguished x'_0 .
- in (3.123), \vec{x}^2 is a subvector containing x_0 but not x'_0 of the cyclically permuted $\vec{x} = \{x'_0, \dots, x'_l, x_0, \dots, x_k\}$, and \vec{x}^1 is obtained from \vec{x} by replacing all the elements of \vec{x}^2 by the single x_a , thought of as the new distinguished x_0 .
- in (3.125), \vec{x}^2 is a subvector of \vec{x} containing x'_0 but not x_0 . $\widehat{\vec{x}^1}$ is obtained from \vec{x} by replacing \vec{x}^2 by x_b , thought of as the new distinguished x'_0 .

Now, define the map

$$(3.126) \quad \mathcal{H}_{k,l} : CH^*(M) \otimes (\mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes k} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes l})^{diag} \rightarrow CH^*(M)$$

as

$$(3.127) \quad \mathcal{H}_{k,l} := (-1)^{\vec{t}_{\mathcal{P}_{k,l}^2}} \mathbf{F}_{\overline{\mathcal{P}}_{k,l}^2},$$

where we use sign twisting datum

$$(3.128) \quad \vec{t}_{\mathcal{P}_d^2} := (-1, 1, 2, \dots, k+1, k+3, \dots, k+l+2)$$

corresponding to the ordering of inputs $(\kappa_+, z_0, \dots, z_k, z'_0, \dots, z'_l)$. The composite map $\mathcal{H} = \sum_{k,l} \mathcal{H}_{k,l}$ gives a map

$$(3.129) \quad \mathcal{H} : CH^*(M) \times {}_2\text{CC}_*(\mathcal{W}, \mathcal{W}) \longrightarrow CH^*(M).$$

By the above result about Gromov bordifications and a sign verification discussed in [G2, Appendix A] we conclude that

LEMMA 3.12. *For any $(\alpha, s) \in {}_2\text{CC}_*(\mathcal{W}) \times CH^*(M)$,*

$$(3.130) \quad d_{SH} \circ \mathcal{H}(\alpha, s) \pm \mathcal{H}(\delta(\alpha), s) \pm \mathcal{H}(\alpha, d_{SH}(s)) = {}_2\mathcal{O}\mathcal{C}(\alpha) * s - {}_2\mathcal{O}\mathcal{C}({}_2\mathcal{O}\mathcal{C}(s) \cap \alpha).$$

Thus, \mathcal{H} is the desired chain homotopy inducing (3.105), concluding the proof of Proposition 3.4. We briefly indicate how to change this argument to prove Proposition 3.3. One considers operation associated to the same abstract moduli space as $\mathcal{P}_{k,l}^2$, where both interior punctures are marked as positive marked points and the distinguished boundary input z'_0 is now marked as an output. The associated Floer theoretic operation with a similar sign twist gives a chain-homotopy between the algebra structure applied to ${}_2\mathcal{O}\mathcal{C}(s_2)$ “ \circ'' ” ${}_2\mathcal{O}\mathcal{C}(s_1)$ (the limit $r \rightarrow 1$) and ${}_2\mathcal{O}\mathcal{C}(s_1 * s_2)$ (the limit $r \rightarrow 0$).

4. The Cardy condition

By the results in §2.4, there is a bimodule $\mathcal{W}^!$ known as the **inverse dualizing bimodule**

$$(4.1) \quad \mathcal{W}^! := \text{hom}_{\mathcal{W}-\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{W}_\Delta \otimes_{\mathbb{K}} \mathcal{W}_\Delta),$$

which comes equipped with a natural map

$$(4.2) \quad \mathcal{W}^! \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{B} \xrightarrow{\bar{\mu}} {}_2\text{CC}^*(\mathcal{W}, \mathcal{B})$$

defined in (2.219). In this section, we first construct a geometric (closed) morphism of bimodules.

$$(4.3) \quad \mathcal{C}\mathcal{Y} : \mathcal{W}_\Delta \longrightarrow \mathcal{W}^![n].$$

Recall from Lemma 2.1 that any morphism such as $\mathcal{C}\mathcal{Y}$ induces functorial (degree n) chain maps between (two pointed) Hochschild complexes:

$$(4.4) \quad \mathcal{C}\mathcal{Y}_\# : {}_2\text{CC}_*(\mathcal{W}, \mathcal{W}) := \mathcal{W}_\Delta \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{W}_\Delta \longrightarrow \mathcal{W}_\Delta \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{W}^!.$$

When composed with (4.2) we obtain a map from (two-pointed) Hochschild chains to Hochschild cochains. The main result of this Section prove that the resulting map is homotopic to the map from Hochschild chains to co-chains which passes through $SH^*(M)$, using the closed-open and open-closed maps:

THEOREM 4.1 (Generalized Cardy Condition). *There is a (homotopy)-commutative diagram up to an overall sign of $(-1)^{n(n+1)/2}$:*

$$(4.5) \quad \begin{array}{ccc} {}_2\text{CC}_*(\mathcal{W}, \mathcal{W}) & \xrightarrow{\mathcal{C}\mathcal{Y}_\#} & \mathcal{W}_\Delta \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{W}^! \\ \downarrow {}_2\mathcal{O}\mathcal{C} & & \downarrow \bar{\mu} \\ SH^*(M) & \xrightarrow{{}_2\mathcal{C}\mathcal{O}} & {}_2\text{CC}^*(\mathcal{W}, \mathcal{W}) \end{array}$$

4.1. The non-compact Calabi-Yau structure. We consider operations arising from discs with two negative punctures and arbitrary numbers of positive punctures. We require that there be a *distinguished* positive puncture on each component of the boundary of the disc minus negative punctures; namely, we require there to be at least two inputs. Then, we interpret one of the distinguished positive punctures as belonging to \mathcal{W} and the remaining distinguished input and two outputs as belonging to $\mathcal{W}^!$.

DEFINITION 4.1. *The moduli space of discs with two negative punctures, two positive punctures, and $(k, l; s, t)$ positive marked points*

$$(4.6) \quad \mathcal{R}_2^{k, l; s, t}$$

is the abstract moduli space of discs with

- *two distinguished negative marked points z_-^1, z_-^2 ,*
- *two distinguished positive marked points z_+^1, z_+^2 , one removed from each boundary component cut out by z_-^1 and z_-^2 ,*
- *k positive marked points a_1, \dots, a_k between z_-^1 and z_+^1*
- *l positive marked points b_1, \dots, b_l between z_+^1 and z_-^2*
- *s positive marked points c_1, \dots, c_s between z_-^2 and z_+^2 ; and*
- *t positive marked points d_1, \dots, d_t between z_+^2 and z_-^1 .*

Moreover, the distinguished points z_-^1, z_-^2, z_+^1 , and z_+^2 are constrained to lie (after automorphism) at 1, -1 , i and $-i$ respectively. Namely, we fix the cross-ratios of these 4 points.

Fixing a slice of the automorphism action for which $z_-^1, z_-^2, z_+^1, z_+^2$ are fixed at $i, -i, 1$ and -1 and using the positions of the remaining coordinates

$$(4.7) \quad (z_1, \dots, z_k, z_1^1, \dots, z_l^1, z_1^2, \dots, z_s^2, z_1^3, \dots, z_t^3)$$

as a chart, orient (4.6) with the volume form

$$(4.8) \quad -dz_1 \wedge \dots \wedge dz_k \wedge dz_1^1 \wedge \dots \wedge dz_l^1 \wedge dz_1^2 \wedge \dots \wedge dz_s^2 \wedge dz_1^3 \wedge \dots \wedge dz_t^3.$$

The boundary strata of the Deligne-Mumford compactification

$$(4.9) \quad \overline{\mathcal{R}}_2^{k,l;s,t}$$

is covered by the images of natural inclusions of the following products:

$$(4.10) \quad \overline{\mathcal{R}}^{k'+1+l'} \times_{1+} \overline{\mathcal{R}}_2^{k-k',l-l';s,t}$$

$$(4.11) \quad \overline{\mathcal{R}}^{k'} \times_{n+1}^a \overline{\mathcal{R}}_2^{k-k'+1,l;s,t}, \quad 0 \leq n < k - k' + 1$$

$$(4.12) \quad \overline{\mathcal{R}}^{l'} \times_{n+1}^b \overline{\mathcal{R}}_2^{k,l-l'+1;s,t}, \quad 0 \leq n < l - l' + 1$$

$$(4.13) \quad \overline{\mathcal{R}}^{s'+1+t'} \times_{2+} \overline{\mathcal{R}}_2^{k,l,s-s',t-t'}$$

$$(4.14) \quad \overline{\mathcal{R}}^{s'} \times_{n+1}^c \overline{\mathcal{R}}_2^{k,l;s-s'+1,t}, \quad 0 \leq n < s - s' + 1$$

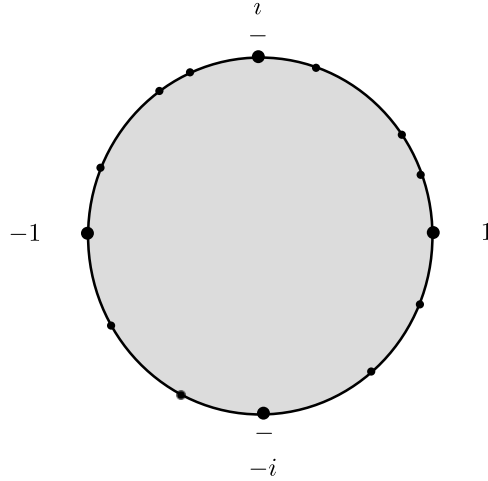
$$(4.15) \quad \overline{\mathcal{R}}^{t'} \times_{n+1}^d \overline{\mathcal{R}}_2^{k,l;s,t-t'+1}, \quad 0 \leq n < t - t' + 1$$

$$(4.16) \quad \overline{\mathcal{R}}_2^{k-k',l;s',t-t'} \times_{(1,k'+1)} \overline{\mathcal{R}}^{k'+1+t'}$$

$$(4.17) \quad \overline{\mathcal{R}}_2^{k,l-l';s-s',t} \times_{(2,s'+1)} \overline{\mathcal{R}}^{l'+1+s'}.$$

Above, the notation \times_{j+} in (4.10) and (4.13) indicates that the output of the first component is glued to the special input z_j^+ of the second component, \times_j^a , \times_j^b , \times_j^c and \times_j^d indicate gluing to the input a_j , b_j , c_j , and d_j respectively, and $\times_{(i,j)}$ in (4.16) and (4.17) indicate gluing the i th output of the first component to the j th input of the second. Also, in (4.10) and (4.13), the $k' + 1$ st and $s' + 1$ st input points of the first component become the special points z_+^1 and z_+^2 after gluing respectively.

FIGURE 3. A schematic of the moduli space $\mathcal{R}_2^{2,3;3,2}$. All non-signed marked points are inputs.



DEFINITION 4.2. A **Floer datum** for a disc S with two positive, two negative, and $(k, l; s, t)$ positive boundary marked points is a Floer datum of S thought of as an open-closed string.

Fix a sequence of Lagrangians

$$(4.18) \quad A_0, \dots, A_k, B_0, \dots, B_l, C_0, \dots, C_s, D_0, \dots, D_t,$$

corresponding to a labeling of the boundary of an element of $\mathcal{R}_2^{k,l;s,t}$ by specifying that a_i be the intersection point between A_{i-1} and A_i , and so on for b_i , c_i , and d_i . In the manner described in (3.58), the space $\overline{\mathcal{R}}_2^{k,l;s,t}$, along with the sign twisting datum

$$(4.19) \quad \vec{t}_{\mathcal{C}y_{k,l,s,t}} := (1, 2, \dots, k, k, k+1, \dots, k+l, 1, 2, \dots, s, s, s+1, \dots, s+t)$$

corresponding to inputs $(a_1, \dots, a_k, z_1^+, b_1, \dots, b_l, c_1, \dots, c_s, z_2^+, d_1, \dots, d_t)$, determines an operation

$$\begin{aligned}
(4.20) \quad \mathbf{CY}_{l,k,t,s} : & \left(\text{hom}(B_{l-1}, B_l) \otimes \dots \otimes \text{hom}(B_0, B_1) \right. \\
& \otimes \mathbf{hom}(A_k, B_0) \otimes \text{hom}(A_{k-1}, A_k) \otimes \dots \otimes \text{hom}(A_0, A_1) \\
& \otimes \left(\text{hom}(D_{t-1}, D_t) \otimes \dots \otimes \text{hom}(D_0, D_1) \right. \\
& \otimes \mathbf{hom}(C_s, D_0) \otimes \text{hom}(C_{s-1}, C_s) \otimes \dots \otimes \text{hom}(C_0, C_1) \\
& \left. \left. \longrightarrow \mathbf{hom}(A_0, D_t) \otimes \mathbf{hom}(C_0, B_l) \right) \right)
\end{aligned}$$

DEFINITION 4.3. *The Calabi-Yau morphism*

$$(4.21) \quad \mathcal{CY} : \mathcal{W}_\Delta \longrightarrow \mathcal{W}^l[n]$$

is given by the following data:

- For objects (X, Y) , a map

$$\begin{aligned}
(4.22) \quad \mathcal{CY}^{0|1|0} : & \mathcal{W}_\Delta(X, Y) \longrightarrow \text{hom}_{\mathcal{W}\text{-mod-}\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{Y}_X^l \otimes \mathcal{Y}_Y^r) \\
& \mathbf{a} \longmapsto \phi_{\mathbf{a}}
\end{aligned}$$

where $\phi_{\mathbf{a}}$ is the morphism whose $t|1|s$ term is

$$\begin{aligned}
(4.23) \quad \phi_{\mathbf{a}}^{t|1|s}(d_t, \dots, d_1, \mathbf{b}, c_s, \dots, c_1) := \\
\mathcal{CY}_{0,0,t,s}(\mathbf{a}, d_t, \dots, d_1, \mathbf{b}, c_s, \dots, c_1)
\end{aligned}$$

- Higher morphisms

$$\begin{aligned}
(4.24) \quad \mathcal{CY}^{l|1|k} : & \text{hom}_{\mathcal{W}}(Y_{k-1}, Y_k) \otimes \dots \otimes \text{hom}_{\mathcal{W}}(Y_0, Y_1) \otimes \mathcal{W}_\Delta(X_0, Y_0) \\
& \otimes \text{hom}_{\mathcal{W}}(X_1, X_0) \otimes \dots \otimes \text{hom}_{\mathcal{W}}(X_l, X_{l-1}) \longrightarrow \text{hom}_{\mathcal{W}\text{-mod-}\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{Y}_{X_k}^l \otimes \mathcal{Y}_{Y_l}^r) \\
& (b_l, \dots, b_1, \mathbf{a}, a_k, \dots, a_1) \longmapsto \psi_{b_l, \dots, b_1, \mathbf{a}, a_k, \dots, a_1}
\end{aligned}$$

where $\psi = \psi_{b_l, \dots, b_1, \mathbf{a}, a_k, \dots, a_1}$ is the morphism whose $t|1|s$ term is

$$\begin{aligned}
(4.25) \quad \phi^{t|1|s}(d_t, \dots, d_1, \mathbf{b}, c_s, \dots, c_1) := \\
\mathcal{CY}_{l,k,t,s}(b_l, \dots, b_1, \mathbf{a}, a_k, \dots, a_1; d_t, \dots, d_1, \mathbf{b}, c_s, \dots, c_1).
\end{aligned}$$

Put another way, we can in a single breath say that

$$\begin{aligned}
(4.26) \quad (\mathcal{CY}^{l|1|k}(b_l, \dots, b_1, \mathbf{a}, a_k, \dots, a_1))^{t|1|s}(d_t, \dots, d_1, \mathbf{b}, c_s, \dots, c_1) \\
:= \mathcal{CY}_{l,k,t,s}(b_l, \dots, b_1, \mathbf{a}, a_k, \dots, a_1; d_t, \dots, d_1, \mathbf{b}, c_s, \dots, c_1).
\end{aligned}$$

The Gromov bordification $\overline{\mathcal{R}}_2^{k,l;s,t}(\vec{x}_{in}, \vec{x}_{out})$ has boundary covered by the images of the Gromov bordifications of spaces of maps from the nodal domains (4.10) - (4.17), along with standard strip breaking, which put together implies that:

PROPOSITION 4.1. *\mathcal{CY} is a closed morphism of A_∞ bimodules of degree n .*

PROOF. The degree of $+n$ comes from the dimension formula in Lemma 3.8 (using the fact that there are two outputs). We will briefly indicate how to convert the strata (4.10)-(4.17) to the equation

$$(4.27) \quad \delta \mathcal{CY} = \mathcal{CY} \circ \hat{\mu}_{\mathcal{W}_\Delta} - \mu_{\mathcal{W}^!} \circ \hat{\mathcal{C}}\mathcal{Y} = 0.$$

The strata (4.10) - (4.12) correspond to \mathcal{CY} composed with various A_∞ bimodule differentials for \mathcal{W}_Δ . The strata (4.13) - (4.15) all correspond to the internal differential $\mu_{\mathcal{W}^!}^{0|1|0}$, which itself involves various pieces of the \mathcal{W}_Δ A_∞ bimodule differentials for the second string of inputs. Finally, the strata (4.16)-(4.17) for fixed k', l' , and varying over all s', t' correspond to the terms of the form $\mu_{\mathcal{W}^!}^{k'|1|0} \circ \mathcal{CY}$ and $\mu^{0|1|l'} \circ \mathcal{CY}$. The ingredients to verify signs are discussed in [G2, Appendix A]. \square

4.2. A geometric bimodule quasi-isomorphism. In (2.50) of §2.2, we gave a construction of a quasi-isomorphism of bimodules

$$(4.28) \quad \mathcal{F}_{\Delta, left, right} : \mathcal{C}_{\Delta} \otimes_{\mathcal{C}} \mathcal{B} \otimes_{\mathcal{C}} \mathcal{C}_{\Delta} \xrightarrow{\sim} \mathcal{B},$$

where \mathcal{C} was an arbitrary A_{∞} category, \mathcal{C}_{Δ} the diagonal bimodule, and \mathcal{B} a $\mathcal{C}-\mathcal{C}$ bimodule. The morphism involved collapsing on the right followed by collapsing on the left by the bimodule structure maps $\mu_{\mathcal{B}}$. The order of collapsing is of course immaterial; we have just picked one.

Let us now suppose that $\mathcal{C} = \mathcal{W}$ and $\mathcal{B} = \mathcal{W}_{\Delta}$. We would like to give a direct geometric quasi-isomorphism

$$(4.29) \quad \mu_{LR} : \mathcal{W}_{\Delta} \otimes_{\mathcal{W}} \mathcal{W}_{\Delta} \otimes_{\mathcal{W}} \mathcal{W}_{\Delta} \longrightarrow \mathcal{W}_{\Delta},$$

homotopic to $\mathcal{F}_{\Delta, left, right}$, but not involving counts of degenerate surfaces.

DEFINITION 4.4. *The moduli space of discs with four special points of type (r, k, l, s)*

$$(4.30) \quad \mathcal{R}^{r, k, l, s}$$

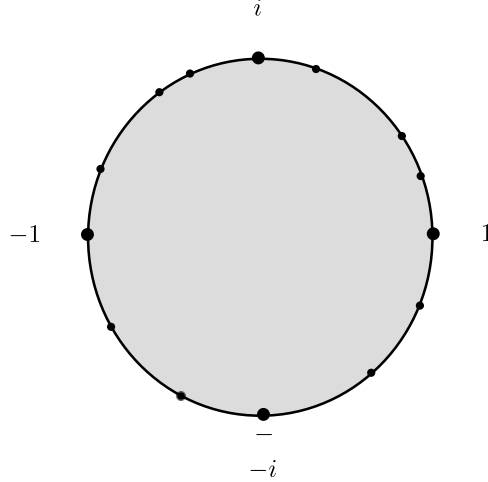
is the abstract moduli space of discs with $r + k + l + s + 3$ positive boundary marked points and one negative boundary marked point labeled in counterclockwise order from the negative point as

$$(z_{out}^-, z_1, \dots, z_r, \bar{z}^1, z_1^1, \dots, z_k^1, \bar{z}^2, z_1^2, \dots, z_l^2, \bar{z}^3, z_1^3, \dots, z_s^3),$$

such that

$$(4.31) \quad \begin{aligned} &\text{after automorphism, the points } z_{out}^-, \bar{z}^1, \bar{z}^2, \bar{z}^3 \text{ lie at} \\ &-i, -1, i, \text{ and } 1 \text{ respectively.} \end{aligned}$$

FIGURE 4. A schematic of the moduli space of discs with four special points of type $(2, 3, 3, 2)$. All non-signed marked points are inputs.



The associated Floer theoretic operation to the space $\overline{\mathcal{R}}^{r, k, l, s}$ with sign twisting datum

$$(4.32) \quad \begin{aligned} \vec{t}_{LR, r, k, l, s} = & (1, 2, \dots, r, r, r+1, \dots, r+k, r+k, \\ & r+k+1, \dots, r+k+l, r+k+l, r+k+l+1, \dots, \\ & r+k+l+s). \end{aligned}$$

is

$$(4.33) \quad (\mu_{LR})_{r, k, l, s} := (-1)^{\vec{t}_{LR, r, k, l, s}} \mathbf{F}_{\mathcal{R}^{r, k, l, s}} : \mathcal{W}^{\otimes r} \otimes \mathcal{W}_{\Delta} \otimes \mathcal{W}^{\otimes k} \otimes \mathcal{W}_{\Delta} \otimes \mathcal{W}^{\otimes l} \otimes \mathcal{W}_{\Delta} \otimes \mathcal{W}^{\otimes s} \longrightarrow \mathcal{W}_{\Delta}$$

Then, define the morphism

$$(4.34) \quad \mu_{LR}^{r|1|^s} := \bigoplus_{k \geq 0, l \geq 0} (\mu_{LR})_{r,k,l,s} : \mathcal{W}^{\otimes r} \otimes (\mathcal{W}_\Delta \otimes T\mathcal{W} \otimes \mathcal{W}_\Delta \otimes T\mathcal{W} \otimes \mathcal{W}_\Delta) \otimes \mathcal{W}^{\otimes s} \longrightarrow \mathcal{W}_\Delta.$$

One can calculate that the morphism is degree zero, as desired.

PROPOSITION 4.2. *The pre-morphism of bimodules*

$$(4.35) \quad \mu_{LR} \in \text{hom}_{\mathcal{W}-\mathcal{W}}(\mathcal{W}_\Delta \otimes_{\mathcal{W}} \mathcal{W}_\Delta \otimes_{\mathcal{W}} \mathcal{W}_\Delta, \mathcal{W}_\Delta)$$

is **closed**, *i.e.*

$$(4.36) \quad \delta \mu_{LR} = 0.$$

PROOF. We leave this mostly as an exercise, but this follows from analyzing the equations arising from the boundary of the one-dimensional space of maps with domain $\overline{\mathcal{R}}^{r,k,l,s}$. The relevant codimension 1 boundary components involve strip-breaking and the codimension 1 boundary strata of the abstract moduli space $\overline{\mathcal{R}}^{r,k,l,s}$, which is covered by

$$(4.37) \quad \overline{\mathcal{R}}^{r'} \times_{n+1} \overline{\mathcal{R}}^{r-r'+1,k,l,s}, \quad 0 \leq n < r - r' + 1$$

$$(4.38) \quad \overline{\mathcal{R}}^{k'} \times_{n+1}^1 \overline{\mathcal{R}}^{r,k-k'+1,l,s}, \quad 0 \leq n < k - k' + 1$$

$$(4.39) \quad \overline{\mathcal{R}}^{l'} \times_{n+1}^2 \overline{\mathcal{R}}^{r,k,l-l'+1,s}, \quad 0 \leq n < l - l' + 1$$

$$(4.40) \quad \overline{\mathcal{R}}^{s'} \times_{n+1}^3 \overline{\mathcal{R}}^{r,k,l,s-s'+1}, \quad 0 \leq n < s - s' + 1$$

$$(4.41) \quad \overline{\mathcal{R}}^{r'+1+k'} \times_{1+} \overline{\mathcal{R}}^{r-r',k-k',l,s}$$

$$(4.42) \quad \overline{\mathcal{R}}^{k'+1+l'} \times_{2+} \overline{\mathcal{R}}^{r,k-k',l-l',s}$$

$$(4.43) \quad \overline{\mathcal{R}}^{l'+1+s'} \times_{3+} \overline{\mathcal{R}}^{r,k,l-l',s-s'}$$

$$(4.44) \quad \overline{\mathcal{R}}^{r-r',k,l,s-s'} \times_{r'+1} \overline{\mathcal{R}}^{r'+1+s'}.$$

Above, the notation \times_k^j means the output of the first component is glued to the input point z_k^j of the second component, and the notation \times_{i+} means the output of the first component is glued to the special point \bar{z}^i . Also, in (4.41), (4.42), and (4.43), the $r' + 1$ st, $k' + 1$ st, and $l' + 1$ st inputs on the first component become the distinguished point \bar{z}^1 , \bar{z}^2 , and \bar{z}^3 respectively after gluing. \square

Now, we show that μ_{LR} was in fact homotopic to $\mathcal{F}_{\Delta, \text{left}, \text{right}}$. We construct a geometric homotopy using

DEFINITION 4.5. *The moduli space*

$$(4.45) \quad \mathcal{S}^{r,k,l,s}$$

is the abstract moduli space of discs with $r + k + l + s + 3$ positive boundary marked points and one negative boundary marked point labeled in counterclockwise order from the negative point as

$$(z_{\text{out}}^-, z_1, \dots, z_r, \bar{z}^1, z_1^1, \dots, z_k^1, \bar{z}^2, z_1^2, \dots, z_l^2, \bar{z}^3, z_1^3, \dots, z_s^3),$$

such that, for any $t \in (0, 1)$,

$$(4.46) \quad \begin{aligned} &\text{after automorphism, the points } z_{\text{out}}^-, \bar{z}^1, \bar{z}^2, \bar{z}^3 \text{ lie at} \\ &-i, -1, \exp(i\frac{\pi}{2}(1-t)), \text{ and } 1 \text{ respectively.} \end{aligned}$$

$\mathcal{S}^{r,k,l,s}$ fibers over the open interval $(0, 1)$ given by the value of t , and thus has dimension one greater than $\mathcal{R}^{r,k,l,s}$. Compactifying, we see that $\overline{\mathcal{S}}^{r,k,l,s}$ submerses over $[0, 1]$ and has codimension one boundary

covered by the natural inclusions of the following strata, the first two of which correspond to fibers at the endpoints 0 and 1, and the remainder of which lie over the entire interval:

$$(4.47) \quad \overline{\mathcal{R}}^{r'+k+l'+2} \times_{r-r'+1} \mathcal{R}^{(r-r')+s+(l-l')+1} \quad (t = 1 \text{ fiber})$$

$$(4.48) \quad \overline{\mathcal{R}}^{r,k,l,s} \quad (t = 0 \text{ fiber})$$

$$(4.49) \quad \overline{\mathcal{R}}^{r'} \times_{n+1} \overline{\mathcal{S}}^{r-r'+1,k,l,s}, \quad 0 \leq n < r - r' + 1$$

$$(4.50) \quad \overline{\mathcal{R}}^{k'} \times_{n+1}^1 \overline{\mathcal{S}}^{r,k-k'+1,l,s}, \quad 0 \leq n < k - k' + 1$$

$$(4.51) \quad \overline{\mathcal{R}}^{l'} \times_{n+1}^2 \overline{\mathcal{S}}^{r,k,l-l'+1,s}, \quad 0 \leq n < l - l' + 1$$

$$(4.52) \quad \overline{\mathcal{R}}^{s'} \times_{n+1}^3 \overline{\mathcal{S}}^{r,k,l,s-s'+1}, \quad 0 \leq n < s - s' + 1$$

$$(4.53) \quad \overline{\mathcal{R}}^{r'+1+k'} \times_{1+} \overline{\mathcal{S}}^{r-r',k-k',l,s}$$

$$(4.54) \quad \overline{\mathcal{R}}^{k'+1+l'} \times_{2+} \overline{\mathcal{S}}^{r,k-k',l-l',s}$$

$$(4.55) \quad \overline{\mathcal{R}}^{l'+1+s'} \times_{3+} \overline{\mathcal{S}}^{r,k,l-l',s-s'}$$

$$(4.56) \quad \overline{\mathcal{S}}^{r-r',k,l,s-s'} \times_{r'+1} \mathcal{R}^{r'+1+s'}.$$

Above, in (4.47), the $r' + 1$ st and $r' + k + 2$ nd inputs of the first component and the $(r - r') + (l - l') + 2$ nd input of the second component become the three special points \bar{z}^1 , \bar{z}^2 , and \bar{z}^3 respectively after gluing. Also, the notation for strata (4.49)-(4.56) exactly mirrors the notation in (4.37)-(4.44). There is an associated Floer operation

$$(4.57) \quad \mathcal{H}_{r,k,l,s} = \mathbf{F}_{\overline{\mathcal{S}}^{r,k,l,s}},$$

and we can thus define a morphism of bimodules, of degree -1

$$(4.58) \quad \mathcal{H} \in \text{hom}_{\mathcal{W}-\mathcal{W}}(\mathcal{W}_{\Delta} \otimes_{\mathcal{W}} \mathcal{W}_{\Delta} \otimes_{\mathcal{W}} \mathcal{W}_{\Delta}, \mathcal{W}_{\Delta}),$$

defined by

$$(4.59) \quad \mathcal{H}^{r|1|s} = \bigoplus_{k,l} \mathcal{H}_{r,k,l,s}.$$

An analysis of the boundaries of the one-dimensional moduli space of maps given by $\overline{\mathcal{S}}_{r,k,l,s}$ reveals

PROPOSITION 4.3. *\mathcal{H} is a chain homotopy between $\mathcal{F}_{\Delta, \text{left}, \text{right}}$ and μ_{LR} .*

PROOF. The $t = 1$ strata (4.47) correspond to $\mathcal{F}_{\Delta, \text{left}, \text{right}}$, the $t = 0$ strata (4.48) correspond to μ_{LR} , and the other strata (4.49)-(4.56) correspond to the chain homotopy terms $\mathcal{H} \circ d - d \circ \mathcal{H}$. \square

4.3. A family of annuli. By Proposition 4.3, Theorem 4.1 now follows from

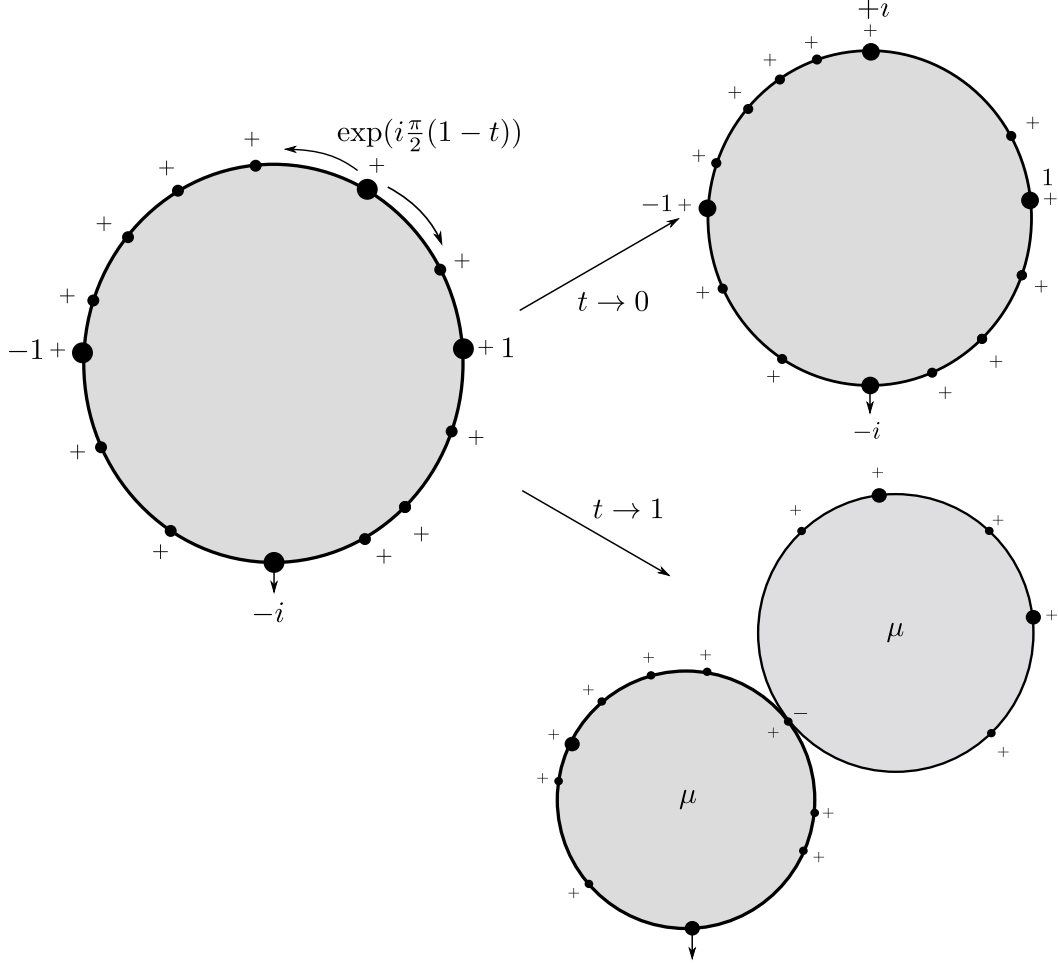
PROPOSITION 4.4. *This diagram homotopy-commutes up to sign $(-1)^{n(n+1)/2}$:*

$$(4.60) \quad \begin{array}{ccc} \mathcal{W}_{\Delta} \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{W}_{\Delta} & \xrightarrow{\mathcal{C}\mathcal{Y}_{\#}} & \mathcal{W}_{\Delta} \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{W}^l, \\ \downarrow 2\mathcal{O}\mathcal{C} & & \downarrow \bar{\mu}_{LR} \\ CH^*(M) & \xrightarrow{2\mathcal{C}\mathcal{O}} & {}_2CC^*(\mathcal{W}, \mathcal{W}) \end{array}$$

where $\bar{\mu}_{LR}$ is as in (2.219) with $\mathcal{F}_{\Delta, \text{left}, \text{right}}^{r'|1|s'}$ replaced by $\mu_{LR}^{r'|1|s'}$.

In order to establish this we introduce some auxiliary moduli spaces of annuli. The operations induced by this (compactified) family, which fibers over $[0, 1]$, induces a chain homotopy between operations associated to its 1 and 0 fibers, which we would like to say are $2\mathcal{C}\mathcal{O} \circ 2\mathcal{O}\mathcal{C}$ and $\bar{\mu}_{LR} \circ \mathcal{C}\mathcal{Y}_{\#}$ respectively. In fact, the operation induced by the 0 fiber generally does not coincide with $\bar{\mu}_{LR} \circ \mathcal{C}\mathcal{Y}_{\#}$ and needs to be further chain-homotoped, for a reason already observed in the degeneration of annuli in [A1, §6.2] (this is the reason for

FIGURE 5. The moduli space $\mathcal{S}^{r,k,l,s}$ and its $t \rightarrow \{0, 1\}$ degenerations.



the “first homotopy” in the language of [A1]). To explain, given an input $\alpha \in \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes k} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes l}$ and $\beta \in \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes s} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes t} \otimes \mathcal{W}_\Delta$, the element

$$(4.61) \quad (\bar{\mu}_{LR} \circ \mathcal{C}\mathcal{Y}_\#)(\alpha)(\beta) \in \mathcal{W}_\Delta$$

is the operation associated to the (compactified) spaces of broken curves:

$$(4.62) \quad \bigcup_{s', t', l', k'} \mathcal{R}_2^{s', t', l', k'} \times_{(1, 1+), (2, 3+)} \mathcal{R}^{s-s', k-k', l-l', t-t'}$$

(with sign twists already described) where the notation indicates that the outputs z_-^1 and z_-^2 of $\mathcal{R}_2^{s', t', l', k'}$ are (degenerately) attached to the special inputs \bar{z}^1 and \bar{z}^3 of $\mathcal{R}^{s-s', k-k', l-l', t-t'}$ respectively. These broken curves cannot in general be glued with the existing Floer data along both break points, unless there is a simultaneous agreement of the weights chosen near z_-^1 and z_-^2 with the weights chosen near \bar{z}^1 and \bar{z}^3 respectively (in the sense of Definition 3.4), at least up to conformal equivalence (so the ratio of weights should agree at least). The requirement that e.g., the Floer data on $\mathcal{R}_2^{s', t', l', k'}$ and $\mathcal{R}^{s-s', k-k', l-l', t-t'}$ be chosen compatibly with breakings in order to guarantee $\mathcal{C}\mathcal{Y}$ and μ_{LR} are individually closed morphism of bimodules precludes such a matching condition. Thus, we should first homotope the Floer data on the family (4.62) to one where the ratio of weights agrees (which will give an intermediate bimodule morphism ${}_2CC_*(\mathcal{W}, \mathcal{W}) \rightarrow {}_2CC^*(\mathcal{W}, \mathcal{W})$ which does not factor through $\mathcal{W}_\Delta \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{W}^l$), and then glue this degenerate surface to obtain a family of annuli annuli. The other boundary degeneration of this family of annuli will then

be the operation ${}_2\mathcal{O}\mathcal{C} \circ {}_2\mathcal{C}\mathcal{O}$. To slightly optimize, we perform both homotopies in single step by considering operations for a family over $(-1, 1)$ which is the disjoint union of a trivial product family $([-1, 0] \times (4.62))$ over $[-1, 0]$ and the desired family of annuli over $[0, 1]$.

DEFINITION 4.6. *The moduli space*

$$(4.63) \quad \mathcal{A}_1$$

consists of annuli with two positive punctures on the inner boundary, one positive puncture on the outer boundary, and one negative puncture on the outer boundary. The codimension 3 subspace

$$(4.64) \quad \mathcal{A}_1^-$$

consists of those annuli that are conformally equivalent to

$$(4.65) \quad \{z | 1 \leq |z| \leq R\} \subset \mathbb{C},$$

for any (varying) R , with inner positive marked points at $\pm i$, outer positive marked point at Ri , and outer negative marked point at $-Ri$.

DEFINITION 4.7. *Define*

$$(4.66) \quad \mathcal{A}_{k,l;s,t}$$

to be the moduli space of annuli with

- $k + l + 2$ positive marked points on the inner boundary, labeled $a_0, a_1, \dots, a_k, a'_0, a'_1, \dots, a'_l$ in counterclockwise order,
- one negative marked point on the outer boundary, labeled z_{out} , and
- $s + t + 1$ positive marked points on the outer boundary, labeled counterclockwise from z_{out} as $b_1, \dots, b_s, b'_0, b'_1, \dots, b'_t$.

Fixing a representative of each element of (4.66) in which a_0, a'_0, b'_0 , and z_{out} are at $\pm i, Ri$ and $-Ri$ respectively, the remaining coordinates include the positions of the remaining boundary points $a_1, \dots, a_k, a'_1, \dots, a'_l, b_1, \dots, b_s, b'_1, \dots, b'_t$, and the radial parameter $r = \frac{R}{R+1}$. With respect to these coordinates, orient (4.66) with the volume form

$$(4.67) \quad -dr \wedge da_1 \wedge \dots \wedge da_k \wedge da'_1 \wedge \dots \wedge da'_l \wedge db_1 \wedge \dots \wedge db_s \wedge db'_1 \wedge \dots \wedge db'_t.$$

There is a map

$$(4.68) \quad \pi : \mathcal{A}_{k,l;s,t} \longrightarrow \mathcal{A}_1$$

given by forgetting all of the marked points except for a_0, a'_0, b'_0 , and z_{out} .

DEFINITION 4.8. *Define*

$$(4.69) \quad \mathcal{A}_{k,l;s,t}^-$$

to be the pre-image of \mathcal{A}_1^- under π .

Via the map

$$(4.70) \quad \mathcal{A}_1^- \longrightarrow (0, 1)$$

which associates to any annulus the scaling parameter $\frac{R}{1+R}$, the space $\mathcal{A}_{k,l;s,t}^-$ also fibers over $(0, 1)$. Compactifying, we see that

$$(4.71) \quad \overline{\mathcal{A}}_{k,l;s,t}^-$$

submerses over $[0, 1]$, with fiber over 0 covered by the natural image of the inclusion of and has boundary stratum covered by the natural images of the inclusions of $\bigcup_{s',t',l',k'} \overline{\mathcal{R}}_2^{s',t',l',k'} \times_{(1,1+),(2,3+)} \overline{\mathcal{R}}^{s-s',k-k',l-l',t-t'}$. In particular, the fiber over 0, denoted $\mathcal{F}_{k,l;s,t}^0$, is by definition the quotient of the compactifications of (4.62)

by the equivalence relation which identifies points which have the same image in $\partial \overline{\mathcal{A}}_{k,l;s,t}^-$ (such points come from coincident higher strata of different factors of (4.62)). Denote this space

$$(4.72) \quad \overline{\mathcal{F}}_{k,l;s,t}^0 := \bigcup_{s',t',l',k'} \overline{\mathcal{R}}_2^{s',t',l',k'} \times_{(1,1+),(2,3+)} \overline{\mathcal{R}}^{s-s',k-k',l-l',t-t'} / \sim$$

We then define a trivial family of pairs $\{(t, S) | t \in [-1, 0], S \in (4.72)\}$:

$$(4.73) \quad \overline{\mathcal{F}}_{k,l;s,t} := [-1, 0] \times \overline{\mathcal{F}}_{k,l;s,t}^0.$$

Elements of the top strata of (4.72) inherit a *boundary orientation* η from (4.67). We orient (4.73) using the volume form

$$(4.74) \quad (-1)^{n(n+1)/2} dt \wedge \eta$$

which has the property that

$$(4.75) \quad \text{the boundary orientation on the } t = 0 \text{ boundary of (4.73) is } -(-1)^{n(n+1)/2} \eta.$$

Now define the *modified abstract moduli space of annuli* to be the disjoint union

$$(4.76) \quad \widetilde{\overline{\mathcal{A}}}_{k,l;s,t}^- := \overline{\mathcal{F}}_{k,l;s,t} \cup \overline{\mathcal{A}}_{k,l;s,t}^-$$

The projection to the first factor on (4.73) along with the submersion to $[0, 1]$ of (4.69) equip $\widetilde{\overline{\mathcal{A}}}_{k,l;s,t}^-$ with a submersion to $[-1, 1]$. The boundary of (4.76) is covered by the natural images of the following strata:

$$(4.77) \quad \overline{\mathcal{R}}_{k,l}^1 \times \overline{\mathcal{R}}_{s,t}^{1,1} \quad (\text{fiber of (4.71) over } 1)$$

$$(4.78) \quad \overline{\mathcal{R}}_2^{s',t',l',k'} \times_{(1,1+),(2,3+)} \overline{\mathcal{R}}^{s-s',k-k',l-l',t-t'} \quad (\text{fiber of (4.73) over } -1)$$

$$(4.79) \quad \overline{\mathcal{F}}_{k,l;s,t}^0 \quad (\text{fiber of (4.71) over } 0)$$

$$(4.80) \quad \overline{\mathcal{F}}_{k,l;s,t}^0 \quad (\text{fiber of (4.73) over } 0)$$

$$(4.81) \quad \overline{\mathcal{R}}^{k'} \times_{n+1}^a \widetilde{\overline{\mathcal{A}}}_{k-k'+1,l;s,t}^-, \quad 0 \leq n < k - k' + 1$$

$$(4.82) \quad \overline{\mathcal{R}}^{l'} \times_{(n+1)'}^a \widetilde{\overline{\mathcal{A}}}_{k,l-l'+1;s,t}^-, \quad 0 \leq n < l - l' + 1$$

$$(4.83) \quad \overline{\mathcal{R}}^{k'+l'+1} \times_0^a \widetilde{\overline{\mathcal{A}}}_{k-k',l-l';s,t}^-$$

$$(4.84) \quad \overline{\mathcal{R}}^{l'+k'+1} \times_{0'}^a \widetilde{\overline{\mathcal{A}}}_{k-k',l-l';s,t}^-$$

$$(4.85) \quad \overline{\mathcal{R}}^{s'} \times_{n+1}^b \widetilde{\overline{\mathcal{A}}}_{k,l;s-s'+1,t}^-, \quad 0 \leq n < s - s' + 1$$

$$(4.86) \quad \overline{\mathcal{R}}^{t'} \times_{(n+1)'}^b \widetilde{\overline{\mathcal{A}}}_{k,l;s,t-t'+1}^-, \quad 0 \leq n < t - t' + 1$$

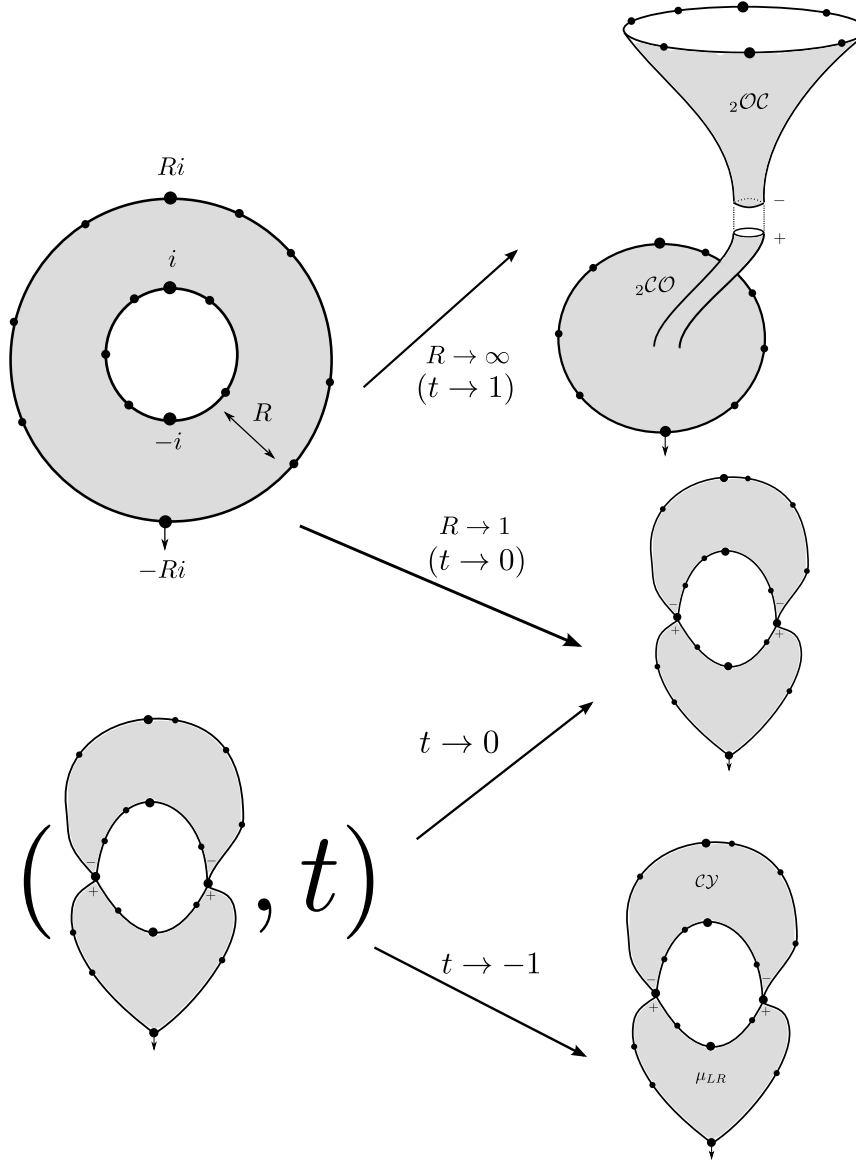
$$(4.87) \quad \overline{\mathcal{R}}^{s'+t'+1} \times_{0'}^b \widetilde{\overline{\mathcal{A}}}_{k,l;s-s',t-t'}^-$$

$$(4.88) \quad \widetilde{\overline{\mathcal{A}}}_{k,l;s-s',t-t'}^- \times_{s'+1} \overline{\mathcal{R}}^{t'+s'+1}.$$

Here, the notation \times_j^a means the output of the first stratum is glued to the input point a_j , $\times_{j'}^a$ means glue to $a'_{j'}$, and \times_j^b , $\times_{j'}^b$ mean the same for b_j , $b'_{j'}$. Also, in (4.78), the two special inputs of the first factor and the second special input of the second factor become the special input points on the annulus after gluing. Also, in (4.83) and (4.84), the $k' + 1$ st and $l' + 1$ st marked points of the first component become the special point (a_0 or a'_0 respectively) after gluing. See Figure 6 for an image of the this abstract moduli space and some of the strata (4.77)-(4.80)

Given $S \in \widetilde{\overline{\mathcal{A}}}_{k,l;s,t}^-$, the *underlying surface associated to* S , denoted \bar{S} , is just S if $S \in (4.71)$ and is S' if $S = (t, S') \in (4.73)$. A *Floer datum* for an element $S \in \widetilde{\overline{\mathcal{A}}}_{k,l;s,t}^-$ is a choice of Floer datum in the sense of Definition 3.9 for the underlying surface \bar{S} . A *universal and conformally consistent choice of Floer data*

FIGURE 6. The modified abstract space of annuli $\widetilde{\mathcal{A}}_{k,l;s,t}^-$ and its degenerations associated to the endpoints $\{-1, 0, 1\}$.



for the annulus argument is as usual an inductive choice of Floer data for the every element of the spaces $\widetilde{\mathcal{A}}_{k,l;s,t}^-$, which agrees with previously made choices on strata and converges to infinite order to (conformal representatives) of these choices with respect to gluing charts near strata. In particular, with respect to a

few of the boundary strata described above, consistency means that

$$(4.89) \quad \text{The Floer data chosen on stratum (4.77) agrees with the data of the operation } {}_2\mathcal{CO} \circ {}_2\mathcal{OC}$$

$$(4.90) \quad \text{The Floer data chosen on stratum (4.78) agrees with the data of the operation } \bar{m}u_{LR} \circ \mathcal{CY}_\#$$

$$(4.91) \quad \text{The Floer data chosen on the strata (4.79) and (4.80) agree;}$$

in particular the last condition implies that the Floer data on $(1, T) \in (4.73)$ must have the simultaneous coincidence of weights discussed above, up to conformal equivalence, as it coincides with the Floer data $T \in (4.71)$ which is the degenerate limit of Floer data on the annulus. By studying parametrized families of solutions to Floer's equation from \bar{S} to M , for $S \in \bar{\mathcal{A}}_{k,l;s,t}^-$ with its accompanying Floer data, we obtain a moduli space of maps

$$(4.92) \quad \bar{\mathcal{A}}_{k,l;s,t}^-(y_{out}; x_0, x_1, \dots, x_k, x'_0, x'_1, \dots, x'_l, y_1, \dots, y_s, y'_0, y'_1, \dots, y'_t)$$

given a compatible Lagrangian labeling and input chords $\vec{x} = (x_0, x_1, \dots, x_k, x'_0, x'_1, \dots, x'_l, y_1, \dots, y_s, y'_0, y'_1, \dots, y'_t)$ and output chord \vec{y}_{out} as above.

Given sign twisting datum

$$(4.93) \quad \vec{t}_{\mathbf{A},k,l,s,t} = \{(1, \dots, k, k+1, \dots, k+l, k+l+1, \dots, s, s+1, \dots, s+t, s+t)\}$$

with respect to the ordering of boundary inputs

$$(4.94) \quad a_1, \dots, a_k, a'_0, a'_1, \dots, a'_l, a_0, b_1, \dots, b_s, z_{in}, b'_1, \dots, b'_t$$

we therefore obtain associated Floer operations

$$(4.95) \quad \mathbf{A}_{k,l;s,t} := (-1)^{\vec{t}_{\mathbf{A},k,l,s,t}} \mathbf{F}_{\bar{\mathcal{A}}_{k,l;s,t}^-} : (\mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes l} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes k})^{diag} \otimes \mathcal{W}^{\otimes t} \otimes \mathcal{W}_\Delta \otimes \mathcal{W}^{\otimes s} \longrightarrow \mathcal{W}_\Delta$$

where we have indicated the inputs corresponding to the special points a_0 , a'_0 , and b'_0 by the first, second, and third \mathcal{W}_Δ input factor, and the output z_{out} by the output \mathcal{W}_Δ factor. As usual, the *diag* superscript indicates that the first set of $k+l+2$ inputs must be cyclically composable. Using these operations, define a map

$$(4.96) \quad \mathbf{A} : {}_2\mathcal{CC}_*(\mathcal{W}, \mathcal{W}) \longrightarrow {}_2\mathcal{CC}^*(\mathcal{W}, \mathcal{W})$$

by

$$(4.97) \quad \mathbf{A} : (\mathbf{x} \otimes x_r \otimes \dots \otimes x_1 \otimes \mathbf{y} \otimes y_q \otimes \dots \otimes y_1) \longmapsto \Phi$$

where Φ is the 2-Hochschild co-chain given by

$$(4.98) \quad \Phi(c_m, \dots, c_1, \mathbf{c}, d_n, \dots, d_1) = \mathbf{A}_{q,r;n,m}(\mathbf{x} \otimes x_r \otimes \dots \otimes x_1 \otimes \mathbf{y} \otimes y_q \otimes \dots \otimes y_q; c_m, \dots, c_1, \mathbf{c}, d_n, \dots, d_1).$$

A dimension computation shows that the operation \mathbf{A} has degree $n-1$ as a map from Hochschild homology to Hochschild cohomology. An analysis of the boundary of the one-dimensional moduli spaces of maps with source domain the various $\bar{\mathcal{A}}_{k,l;s,t}^-$, along with the consistency condition imposed on Floer data, reveals:

PROPOSITION 4.5. \mathbf{A} gives a chain homotopy between ${}_2\mathcal{CO} \circ {}_2\mathcal{OC}$ and $(-1)^{n(n+1)/2} \mu_{LR} \circ \mathcal{CY}_\#$.

PROOF. The strata over the endpoints of the interval $\{-1, 1\}$ correspond exactly to the operations ${}_2\mathcal{CO} \circ {}_2\mathcal{OC}$ and $\mu_{LR} \circ \mathcal{CY}_\#$. The two strata over 0 cancel, and the various intermediate strata give terms corresponding to $d_{{}_2\mathcal{CC}^*} \circ \mathbf{A} \pm \mathbf{A} \circ d_{{}_2\mathcal{CC}_*}$. In the prequel article [G2, §A] we discussed the ingredients necessary to check the signs of this equation. In particular, the orientation over the 0 stratum of the moduli space of maps with domain $\bar{\mathcal{A}}_{k,l;s,t}^-$ differs from the product orientation by a sign of $(-1)^{n(n+1)/2}$ as computed in [A1, Lemma 6.8] (this difference in sign was first noticed in a different setting by Fukaya, Oh, Ohta, and Ono [FOOO3, Prop. 3.9.1]), which also accounts for our placement of $(-1)^{n(n+1)/2}$ in the orientation (4.74) (so that the (signed) operations associated to the two strata over 0 to cancel). \square

By postcomposing with the chain homotopy in Proposition 4.3 between $\bar{\mu}$ and μ_{LR} , Theorem 4.1 follows.

REMARK 4.1. If bimodules $\mathcal{B}_0, \mathcal{B}_1$ come from any Lagrangian in the product $M^- \times M$ for which we are able to define the quilt functor described in the prequel [G2] and §5,

$$(4.99) \quad \begin{aligned} \mathcal{B}_0 &= \mathbf{M}(L_0) \\ \mathcal{B}_1 &= \mathbf{M}(L_1) \end{aligned}$$

then there is an analogue of the Cardy condition, which looks like (4.5):

$$(4.100) \quad \begin{array}{ccc} \mathcal{B}_0 \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{B}_1 & \xrightarrow{\mathcal{C}\mathcal{Y}_{\#}^{\mathcal{B}_0}} & \mathcal{B}_0^! \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{B}_1 \\ \downarrow \circ \mathcal{C} & & \downarrow \bar{\mu} \\ \mathrm{hom}_{\mathcal{W}^2}(L_0, L_1) & \xrightarrow{\mathcal{C}\mathcal{O}} & \mathrm{hom}_{\mathcal{W}-\mathcal{W}}(\mathcal{B}_0, \mathcal{B}_1) \end{array}$$

Here, $\mathcal{B}_0^!$ is the bimodule dual of \mathcal{B}_0 , as defined in Section 2.4, and $\mathcal{C}\mathcal{Y}^{\mathcal{B}_0}$ is a generalization of our $\mathcal{C}\mathcal{Y}$ morphism. The moduli space controlling the relevant commutative diagram is a quilted generalization of the annulus. The only obstacle to the existence of this diagram for arbitrary pairs of Lagrangians in M^2 is our current inability to define the functor \mathbf{M} in complete generality, due to issues of admissibility and compactness of moduli spaces in $M^- \times M$.

5. The wrapped Fukaya category of the product

In this section, we recall the main constructions of the prequel article [G2]:

- Using split data and operations controlled by families of open-closed strings, one can define a model of the wrapped Fukaya category of the product $M^- \times M$ with objects $\{A \times B \mid A, B \in \mathrm{ob} \mathcal{W}\} \cup \{\Delta\}$, which we abbreviate

$$(5.1) \quad \mathcal{W}^2.$$

In the compact setting under the usual technical hypotheses this is genuinely the sub-category of the Fukaya category of the product with these objects, constructed with split data. As noted in [G2, §6, see e.g., Rmk. 6.1], in the non-compact setting our construction bypasses technical issues arising from admissibility of product Lagrangians with respect to the Liouville structure induced by sums of Liouville coordinates.

- We summarize the main result of [G2] giving a functor \mathbf{M} from \mathcal{W}^2 to $\mathcal{W}-\mathcal{W}$ bimodules. We also recall the relationship of \mathbf{M} to closed open maps and the first order term of the Calabi-Yau morphism, proved at the end of [G2].

In fact, we will need to go into some detail in defining \mathcal{W}^2 , as a later construction will equip \mathcal{W}^2 with additional data, analogous to that of a homotopy unit. Therefore, we cover the second point (which can be treated as more of a formal input) first.

5.1. Quilts and the Calabi-Yau morphism. In this section, we assume we have defined a *split-data wrapped category of the product* $M^- \times M$.

$$(5.2) \quad \mathcal{W}^2,$$

which has objects products of objects in \mathcal{W} and the diagonal Δ , and review the main result of [G2], which we treat as a black box input into our paper.

THEOREM 5.1 ([G2]). *There is an A_∞ functor*

$$(5.3) \quad \mathbf{M} : \mathcal{W}^2 \rightarrow \mathcal{W}\text{-mod-}\mathcal{W}$$

which has the following properties:

- On objects, \mathbf{M} sends

$$(5.4) \quad \begin{aligned} L_i \times L_j &\mapsto \mathcal{Y}_{L_i}^l \otimes_{\mathbb{K}} \mathcal{Y}_{L_j}^r \\ \Delta &\mapsto \mathcal{W}_\Delta. \end{aligned}$$

- \mathbf{M} is cohomologically full and faithful on the subcategory of product Lagrangians.

- Under the correspondence $\mathrm{hom}_{\mathcal{W}^2}^*(\Delta, A \times B) = \mathrm{hom}_{\mathcal{W}}^{*-n}(A, B)$,

$$\mathbf{M}_{\Delta, A \times B}^1 : \mathrm{hom}_{\mathcal{W}^2}(\Delta, A \times B) \rightarrow \mathrm{hom}_{\mathcal{W}-\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{Y}_A^l \otimes_{\mathbb{K}} \mathcal{Y}_B^r)$$

agrees on homology with the first order term of the Calabi-Yau morphism $\mathcal{CY}^{0|1|0} : \mathcal{W}_\Delta(A, B) \rightarrow \mathrm{hom}_{\mathcal{W}-\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{Y}_A^l \otimes_{\mathbb{K}} \mathcal{Y}_B^r)$.

- Under the correspondence $\mathrm{hom}_{\mathcal{W}^2}(\Delta, \Delta) = CH^*(M)$,

$$\mathbf{M}_{\Delta, \Delta}^1 : \mathrm{hom}_{\mathcal{W}^2}(\Delta, \Delta) \rightarrow \mathrm{hom}_{\mathcal{W}-\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{W}_\Delta)$$

agrees on homology with the closed open map $[\mathcal{CO}] : SH^*(M) \rightarrow HH^*(\mathcal{W}, \mathcal{W}) \cong H^*(\mathrm{hom}_{\mathcal{W}-\mathcal{W}}(\mathcal{W}_\Delta, \mathcal{W}_\Delta))$.

It follows immediately that

COROLLARY 5.1. *If Δ is split-generated by product Lagrangians in \mathcal{W}^2 , then \mathcal{W} is homologically smooth and \mathcal{CY} is a quasi-isomorphism of bimodules. Also, $[\mathcal{CO}]$ is an isomorphism (though that follows anyway from the rest of our argument).*

PROOF. We note that the condition of being split-generated is preserved under applying (cohomologically unital) A_∞ functors (and it is easy to see that \mathbf{M} is cohomologically unital), hence applying \mathbf{M} to the hypotheses shows that \mathcal{W}_Δ is split-generated by Yoneda bimodules, the definition of (homological) smoothness.

Next, it was observed in [G2, Cor 9.1] that if an A_∞ functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is cohomologically full and faithful on a full subcategory $\mathcal{X} \subset \mathcal{C}$, and \mathcal{X} split-generates \mathcal{C} , then \mathbf{F} is in fact (cohomologically) full and faithful on all of \mathcal{C} . In particular, $[\mathbf{M}^1]$ must be an isomorphism on all of \mathcal{W}^2 , and specifically, $[\mathbf{M}_{\Delta, A \times B}^1] = [\mathcal{CY}^{0|1|0}]$ is an isomorphism. Similarly, $[\mathbf{M}_{\Delta, \Delta}^1] = [\mathcal{CO}]$ is an isomorphism. \square

REMARK 5.1. *This similarly implies $[\mathbf{M}_{A \times B, \Delta}^1]$ is isomorphism too, from $H^*(\mathrm{hom}_{\mathcal{W}^2}^*(A \times B, \Delta)) = H^*(\mathrm{hom}_{\mathcal{W}}(B, A))$ to $H^*(\mathrm{hom}_{\mathcal{W}-\mathcal{W}}(\mathcal{Y}_A^l \otimes_{\mathbb{K}} \mathcal{Y}_B^r, \mathcal{W}_\Delta))$. However, this particular map, which can be given a purely algebraic description in terms of the A_∞ structure on \mathcal{W} , is always an isomorphism (even if Δ is not split-generated by product Lagrangians), so one hasn't learned anything new.*

5.2. An A_∞ category from pairs of discs. The construction of \mathcal{W}^2 passes through the construction of moduli spaces of *pairs of glued discs*, which we briefly recall. Let (k, l) be a pair of non-negative integers with one of k or l to be ≥ 2 (this is the **stable range**, in which the below moduli space is defined):

DEFINITION 5.1. *The moduli space of pairs of discs with (k, l) marked points, denoted*

$$(5.5) \quad \mathcal{R}_{k,l}$$

is the moduli space of pairs of discs with k and l positive marked points and one negative marked point each in the same position, modulo simultaneous automorphisms.

REMARK 5.2. *This definition is **not** identical to the product of associahedra $\mathcal{R}^k \times \mathcal{R}^l$. The latter space is a further quotient of the former space by automorphisms of the right or left disc, at least when both k and l are in the stable range.*

The Stasheff associahedron embeds in $\mathcal{R}_{k,l}$ via the **diagonal embedding**

$$(5.6) \quad \mathcal{R}^d \xrightarrow{\Delta_d} \mathcal{R}_{d,d},$$

which (suitably labeled) will control operations in \mathcal{W}^2 . But it will help to first recall more general facts about (5.5). For instance, the open moduli space $\mathcal{R}_{k,l}$ admits a stratification by *coincident points* between factors:

DEFINITION 5.2. *A (k, l) -point identification \mathfrak{P} is a sequence of tuples*

$$(5.7) \quad \{(i_1, j_1), \dots, (i_s, j_s)\} \subset \{1, \dots, k\} \times \{1, \dots, l\}$$

which are strictly increasing, i.e.

$$(5.8) \quad \begin{aligned} i_r &< i_{r+1} \\ j_r &< j_{r+1} \end{aligned}$$

The number of coincidences of \mathfrak{P} is the size $|\mathfrak{P}|$.

DEFINITION 5.3. Take a representative (S_1, S_2) of a point in $\mathcal{R}_{k,l}$. A boundary input marked point p_1 on S_1 is said to **coincide** with a boundary marked input marked point p_2 on S_2 if they are at the same position when S_1 is superimposed upon S_2 . This notion is independent of the representative (S_1, S_2) , as we act only by simultaneous automorphism.

The space of \mathfrak{P} -coincident pairs of discs with (k, l) marked points

$$(5.9) \quad \mathfrak{P}\mathcal{R}_{k,l}$$

is the subspace of $\mathcal{R}_{k,l}$ where pairs of input marked points on each factor specified by \mathfrak{P} are required to coincide, and no other input marked points are allowed to coincide. Here the indices in \mathfrak{P} coincide with the **counter-clockwise ordering** of input marked points on each factor.

EXAMPLE 5.1. When $k = l$ and $|\mathfrak{P}| = k$ is maximal, the associated space $\mathfrak{P}\mathcal{R}_{k,l}$ is just the diagonal associahedron $\Delta_k(\mathcal{R}^k)$. This is the main example we will consider here.

The next identification specifies the data required to glue a pair of discs.

DEFINITION 5.4. A (k, l) **boundary identification** is a (possibly empty) subset \mathfrak{S} of the set of pairs $\{0, \dots, k\} \times \{0, \dots, l\}$ satisfying the following conditions:

- $(0, 0)$ and (k, l) are the only admissible pairs in \mathfrak{S} containing extrema.
- (**monotonicity**) \mathfrak{S} can be written as $\{(i_1, j_1), \dots, (i_s, j_s)\}$ with $i_r < i_{r+1}$ and $j_r < j_{r+1}$.

DEFINITION 5.5. Let S and T be unit discs in \mathbb{C} with k and l incoming boundary marked points respectively, and one outgoing boundary point each. Assume further that the outgoing boundary points of S and T are in the same position. Label the boundary components of S

$$(5.10) \quad \{\partial^0 S, \dots, \partial^k S\}$$

in **counterclockwise** order from the outgoing point, and label the components of T

$$(5.11) \quad \{\partial^0 T, \dots, \partial^l T\}$$

in **counterclockwise** order from the outgoing point. Let \mathfrak{S} be a (k, l) boundary identification. S and T are said to be **\mathfrak{S} -compatible** if

- the outgoing points of S and T are at the same position.
- The identity map induces a one-to-one identification of $\partial^x S$ with $\partial^y T$ for each $(x, y) \in \mathfrak{S}$.

DEFINITION 5.6. Let \mathfrak{S} be a (k, l) boundary identification. The **associated (k, l) point identification**

$$(5.12) \quad p(\mathfrak{S})$$

is defined as follows:

$$(5.13) \quad p(\mathfrak{S}) := \{(i, j) | (i, j) \in \mathfrak{S} \text{ or } (i-1, j-1) \in \mathfrak{S}\}.$$

Given a pair $\mathfrak{S}, \mathfrak{T}$ of a (k, l) boundary identification and a (k, l) point identification, we say \mathfrak{S} is **compatible with \mathfrak{T}** if

$$(5.14) \quad p(\mathfrak{S}) \subseteq \mathfrak{T}.$$

Thus, in the same manner that we have already spoken about boundary-labeled moduli spaces, we can define the **moduli space of \mathfrak{S} identified pairs of discs with \mathfrak{T} point identifications and (k, l) marked points**

$$(5.15) \quad \mathfrak{T}, \mathfrak{S} \mathcal{R}_{k,l}$$

to be exactly $\mathfrak{T}, \mathfrak{S} \mathcal{R}_{k,l}$ with the additional boundary labellings that we described above.

Our reason for defining boundary identification is so that we can speak more easily about gluings.

DEFINITION 5.7. Let S and T be compatible with a (k, l) boundary identification datum \mathfrak{S} . The **\mathfrak{S} -gluing**

$$(5.16) \quad \pi_{\mathfrak{S}} := S \coprod_{\mathfrak{S}} T$$

is the genus 0 open-closed string defined as follows: view $-S$, i.e. S with the **opposite complex structure** as being the south half of a sphere bounding the equator via the complex doubling procedure, one of the

methods of constructing the moduli of bordered surfaces [L1, §3.1]. Similarly, view T as the north half of the sphere. Then

$$(5.17) \quad (S \coprod_{\mathfrak{S}} T) := (-S) \coprod T / \sim$$

where \sim identifies $\partial^x(-S)$ to $\partial^y T$ ($\partial^x(-S)$ is the same boundary component of S as before, now with the reverse orientation) under the identification coming from inclusion into the sphere if and only if $(x, y) \in \mathfrak{S}$. Boundary marked points are identified as follows: Let z_{-S}^i be the boundary marked point between $\partial^{i-1}(-S)$ and $\partial^i(-S)$, z_{-S}^0 the outgoing marked point, and z_T^j similar. Then:

- if $(x-1, y-1), (x, y) \in \mathfrak{S}$, then $z_{-S}^x \sim z_T^y$ becomes a single interior marked point.
- if $(x-1, y-1) \in \mathfrak{S}$ but (x, y) is not, then $z_{-S}^x \sim z_T^y$ becomes a single boundary marked point, between $\partial^y T$ and $\partial^x(-S)$
- if $(x, y) \in \mathfrak{S}$ but $(x-1, y-1)$ is not, then $z_{-S}^x \sim z_T^y$ becomes a single boundary marked point, between $\partial^x(-S)$ and $\partial^y T$
- otherwise, z_{-S}^x and z_T^y are kept distinct, becoming two boundary marked points.

REMARK 5.3. We equip S with the opposite complex structure to account for the symplectic form (and hence almost complex structure) reversal in the first factor of $M^- \times M$.

By \mathfrak{S} -compatibility, S and T can be viewed as the south and north halves of a sphere in a manner preserving the alignment of outgoing marked points and boundary components specified by \mathfrak{S} , so the above definition is sensible. One can read off the characteristics of the resulting bordered surface from k, l , and \mathfrak{S} , which we leave as an exercise. Denote the resulting number of boundary components of the open-closed string

$$(5.18) \quad h(k, l, \mathfrak{S}).$$

DEFINITION 5.8. A **Lagrangian labeling from \mathbf{L}** for a glued pair of discs (P, \mathfrak{S}) is a Lagrangian labeling from \mathbf{L} for the gluing $\pi_{\mathfrak{S}}(P) = S \coprod_{\mathfrak{S}} T$, thought of as a (possibly disconnected) open-closed string. Given a fixed labeling \vec{L} , denote by

$$(5.19) \quad (\mathfrak{T}, \mathfrak{S} \overline{\mathcal{R}}_{k,l})_{\vec{L}}$$

the space of labeled \mathfrak{S} -identified pairs of discs with \mathfrak{T} point coincidences.

Now, fix a compact oriented submanifold with corners of dimension d ,

$$(5.20) \quad \overline{\mathcal{L}}^d \hookrightarrow \mathfrak{T}, \mathfrak{S} \overline{\mathcal{R}}_{k,l}$$

Fix a Lagrangian labeling

$$(5.21) \quad \vec{L} = \{\{L_0^1, \dots, L_{m_1}^1\}, \{L_0^2, \dots, L_{m_2}^2\}, \dots, \{L_0^h, \dots, L_{m_h}^h\}\}.$$

Also, fix chords

$$(5.22) \quad \vec{x} = \{\{x_1^1, \dots, x_{m_1}^1\}, \dots, \{x_1^h, \dots, x_{m_h}^h\}\}$$

and orbits $\vec{y} = \{y_1, \dots, y_n\}$ with

$$(5.23) \quad x_i^j \in \begin{cases} \chi(L_{i+1}^j, L_i^j) & i \in K^j \\ \chi(L_i^j, L_{i+1}^j) & \text{otherwise.} \end{cases}$$

Above, the index i in L_i^j is counted mod m_j . Collectively, the \vec{x}, \vec{y} are called a set of **asymptotic conditions** for the labeled moduli space $\overline{\mathcal{L}}_{\vec{L}}^d$. The **outputs** $\vec{x}_{out}, \vec{y}_{out}$ are by definition those x_i^j and y_s for which $i \in K^j$ and $s \in \mathbf{I}$, corresponding to negative marked points. The **inputs** $\vec{x}_{in}, \vec{y}_{in}$ are the remaining chords and orbits from \vec{x}, \vec{y} . Fixing a chosen universal and consistent Floer datum, denote $\epsilon_{\pm}^{i,j}$ and δ_{\pm}^l the strip-like and cylindrical ends corresponding to x_i^j and y_l respectively.

Finally, define

$$(5.24) \quad \overline{\mathcal{L}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$$

to be the space of maps

$$(5.25) \quad \{u : \pi_{\mathfrak{S}}(P) \longrightarrow M : P \in \overline{\mathcal{L}}^d\}$$

satisfying, at each element P , Floer's equation for $(\mathbf{D}_{\mathfrak{S}})_P$ with boundary and asymptotic conditions

$$(5.26) \quad \begin{cases} \lim_{s \rightarrow \pm\infty} u \circ \epsilon_{\pm}^{i,j}(s, \cdot) = x_i^j, \\ \lim_{s \rightarrow \pm\infty} u \circ \delta_{\pm}^l(s, \cdot) = y_l, \\ u(z) \in \psi^{as(z)} L_i^j, \end{cases} \quad z \in \partial_i^j S.$$

We have the usual transversality and compactness results:

LEMMA 5.1. *The moduli spaces $\overline{\mathcal{L}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ are compact and there are only finitely many collections $\vec{x}_{out}, \vec{y}_{out}$ for which they are non-empty given input $\vec{x}_{in}, \vec{y}_{in}$. For a generic universal and conformally consistent Floer data they form manifolds of dimension*

$$(5.27) \quad \begin{aligned} \dim \mathcal{L}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in}) &:= \sum_{x_- \in \vec{x}_{out}} \deg(x_-) + \sum_{y_- \in \vec{y}_{out}} \deg(y_-) \\ &+ (2 - h(k, l, \mathfrak{S}) - |\vec{x}_{out}| - 2|\vec{y}_{out}|)n + d - \sum_{x_+ \in \vec{x}_{in}} \deg(x_+) - \sum_{y_+ \in \vec{y}_{in}} \deg(y_+). \end{aligned}$$

Using this we define a map

$$(5.28) \quad \mathbf{G}_{\overline{\mathcal{L}}^d} : \bigotimes_{(i,j); 1 \leq i \leq m_j; i \notin K^j} CW^*(L_i^j, L_{i+1}^j) \otimes \bigotimes_{1 \leq k \leq n; k \notin \mathbf{I}} CH^*(M) \longrightarrow \bigotimes_{(i,j); 1 \leq i \leq m_j; i \in K^j} CW^*(L_{i+1}^j, L_i^j) \otimes \bigotimes_{1 \leq k \leq n; k \in \mathbf{I}} CH^*(M)$$

given by, as usual (abbreviating $\vec{x}_{in} = \{x_1, \dots, x_s\}$, $\vec{y}_{in} = \{y_1, \dots, y_t\}$)

$$(5.29) \quad \mathbf{G}_{\overline{\mathcal{L}}^d}([y_t], \dots, [y_1], [x_s], \dots, [x_1]) := \sum_{\dim \mathcal{L}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})=0} \sum_{u \in \mathcal{L}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})} \mathcal{L}_u([x_1], \dots, [x_s], [y_1], \dots, [y_t]).$$

This construction naturally associates, to any submanifold $\mathcal{L}^d \in \mathfrak{S}, \mathfrak{T} \mathcal{R}_{k,l}$, a map $\mathbf{G}_{\mathcal{L}^d}$, depending on a sufficiently generic choice of Floer data for glued pairs of discs. In a similar fashion, this can be done for a submanifold of the labeled space

$$(5.30) \quad \mathcal{L}_{\vec{L}}^d \subset (\mathfrak{S}, \mathfrak{T} \mathcal{R}_{k,l})_{\vec{L}},$$

in which case the result is an operation defined only for a specific labeling,

$$(5.31) \quad \mathbf{G}_{\mathcal{L}_{\vec{L}}^d},$$

This operation can also be constructed with a sign twisting datum to create an operation

$$(5.32) \quad (-1)^{\vec{f}} \mathbf{G}_{\mathcal{L}^d}$$

in an identical fashion to (3.61).

Define the **objects** of \mathcal{W}^2 as

$$(5.33) \quad \text{ob } \mathcal{W}^2 := \{L_i \times L_j \mid L_i, L_j \in \text{ob } \mathcal{W}\} \cup \{\Delta\}.$$

For objects $X_k, X_l \in \text{ob } \mathcal{W}^2$, define the **generators of the hom complexes**

$$(5.34) \quad \chi_{M^2}(X_k, X_l) := \begin{cases} \chi(L_j, L_i, H) \times \chi(L'_i, L'_j, H) & X_k = L_i \times L'_i, X_l = L_j \times L'_j \\ \chi(L_j, L_i) & X_k = L_i \times L_j, X_l = \Delta \\ \chi(L_i, L_j) & X_k = \Delta, X_l = L_i \times L_j \\ \mathcal{O} & X_k = X_l = \Delta \end{cases}$$

The *morphism spaces* are the (graded) chain complexes generated by (5.34) with differential μ^1 specified below:

$$(5.35) \quad \text{hom}_{\mathcal{W}^2}^*(\Delta, \Delta) := (CH^*(M, H_t, J), \mu^1 := d)$$

$$(5.36) \quad \text{hom}_{\mathcal{W}^2}^*(L_j \times L_i, \Delta) := (CW^*(L_i, L_j, H, J_t), \mu^1 := \mu_{\mathcal{W}}^1)$$

$$(5.37) \quad \text{hom}_{\mathcal{W}^2}^*(\Delta, L_i \times L_j) := (CW^{*+n}(L_j, L_i, H, J_t), \mu^1 := \mu_{\mathcal{W}}^1)$$

$$(5.38) \quad \text{hom}_{\mathcal{W}^2}^*(L_i \times L_i, L'_i \times L'_j) := (CW^*(L'_i, L_i, H, J_t) \otimes CW^*(L_j, L'_j, H, J_t), \mu^1 := \mu_{\mathcal{W}}^1 \otimes id + id \otimes \mu_{\mathcal{W}}^1).$$

We will frequently distinguish between generators of morphism spaces in \mathcal{W}^2 versus those in \mathcal{W} and/or elements of \mathcal{O} via the following correspondences:

$$(5.39) \quad \{x \in \chi_{M^2}(L_0 \times L_1, L'_0 \times L'_1)\} \longleftrightarrow \{\hat{x} = (x_1, x_2) \in \chi(L'_0, L_0) \times \chi(L_1, L'_1)\}$$

$$(5.40) \quad \{z \in \chi_{M^2}(L_0 \times L_1, \Delta)\} \longleftrightarrow \{\hat{z} \in \chi(L_1, L_0)\}$$

$$(5.41) \quad \{w \in \chi_{M^2}(\Delta, L_0 \times L_1)\} \longleftrightarrow \{\hat{w} \in \chi(L_0, L_1)\}$$

$$(5.42) \quad \{y \in \chi_{M^2}(\Delta, \Delta)\} \longleftrightarrow \{\hat{y} \in \mathcal{O}\}$$

$$(5.43)$$

assign gradings as follows:

$$(5.44) \quad \deg x = \deg \hat{x} = \deg x_1 + \deg x_2$$

$$(5.45) \quad \deg z = \deg \hat{z}$$

$$(5.46) \quad \deg w = \deg \hat{w} + n$$

$$(5.47) \quad \deg y = \deg \hat{y}.$$

To complete the construction of \mathcal{W}^2 , we construct higher A_∞ operations $\mu_{\mathcal{W}^2}^d$, $d \geq 2$. First, suppose we have *fixed a universal and conformally consistent Floer datum for pairs of glued discs and genus-0 open closed strings*. Now, consider the space of labeled associahedra

$$(5.48) \quad \mathcal{R}_{\mathbf{L}^2}^d$$

with label set the relevant Lagrangians in M^2 :

$$(5.49) \quad \mathbf{L}^2 = \{\Delta\} \cup \{L_i \times L_j | L_i, L_j \in \text{ob } \mathcal{W}\}.$$

Let S be a disc in \mathcal{R}^d with labels \vec{L}^2 from \mathbf{L}^2 . Let

$$(5.50) \quad D(\vec{L}^2)$$

be the set of indices of boundary components of S labeled Δ (this could possibly be empty). Then, let

$$(5.51) \quad \mathfrak{T}_{max} = \{(1, 1), (2, 2), \dots, (d, d)\}$$

be the maximal boundary identification data and let

$$(5.52) \quad \mathfrak{S}(\mathbf{L}^2) = \{(i, i) | i \in D(\vec{L}^2)\}$$

be the set of boundary components determined by the positions of Δ . Finally, define

$$(5.53) \quad \Phi_{\vec{L}^2}(\mathcal{R}^d) :=_{\mathfrak{S}(\mathbf{L}^2), \mathfrak{T}_{max}} \mathcal{R}_{d,d}$$

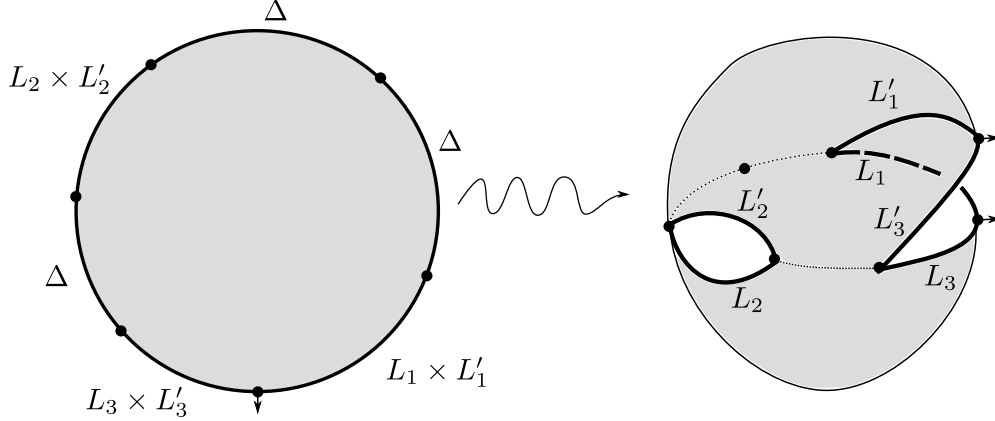
Label the boundary components of the resulting pair of discs as follows: if $\partial_k S$ was labeled $L_i \times L_j$, then in $\Phi_{\vec{L}^2}(S)$, the left image of $\partial_k S$ will be labeled L_i and the right of $\partial_k S$ will be labeled L_j . If $\partial_k S$ was labeled Δ , then it will become part of a boundary identification and disappear under gluing so there is nothing to label.

DEFINITION 5.9. *Define the operation*

$$(5.54) \quad \mu_{\mathcal{W}^2}^d,$$

for sequences of Lagrangians \vec{L}^2 in \mathbf{L}^2 , to be the operation controlled by the image of $\Phi_{\vec{L}^2}$ as in Equation (5.31).

FIGURE 7. An example of the labeled gluing $\Phi_{\vec{L}^2}$.



Because the unfolding maps $\Phi_{\vec{L}^2}$ are embeddings of labeled associahedra,

PROPOSITION 5.1. *The operations $\mu_{\mathcal{W}^2}^d$ as constructed satisfy the A_∞ equations.*

5.3. Other operations from glued discs. When $l = 1$ and $k \geq 2$ (respectively $k = 1$ and $l \geq 2$), there are one-sided embeddings

$$(5.55) \quad \mathcal{R}^k \xrightarrow{\mathcal{J}_k} \mathcal{R}_{k,1},$$

$$(5.56) \quad \mathcal{R}^k \xrightarrow{\mathcal{J}_k} \mathcal{R}_{1,k}.$$

where $\mathcal{J}_k = (id, For_{k-1})$ is the pair of maps corresponding to inclusion and forgetting the first $k-1$ boundary marked points respectively, and $\mathcal{J}_k = (For_{k-1}, id)$. We call (5.55) and (5.56) the **left** and **right semi-stable embeddings** respectively.

We often group these spaces $\mathfrak{P}\mathcal{R}_{k,l}$ by the number of coincident points. The **space of pairs of discs with (k, l) marked points and i coincident points** is defined to be

$$(5.57) \quad {}_i\mathcal{R}_{k,l} := \coprod_{|\mathfrak{P}|=i} \mathfrak{P}\mathcal{R}_{k,l}.$$

The closure of a stratum ${}_i\mathcal{R}_{k,l}$ in $\mathcal{R}_{k,l}$ is $\coprod_{j \geq i} {}_j\mathcal{R}_{k,l}$. Moreover, each stratum ${}_i\mathcal{R}_{k,l}$ can be explicitly described as a set theoretically a union of associahedra. By overlaying a pair of discs with (k, l) marked points and i coincidences, one obtains a disc with $k + l - i$ input marked points (and an output marked point), whose inputs inherit one of three labels, depending on whether the marked point came from the left factor (L), the right factor (R), or both (LR). In this way, one associates to a pair of disc with coincidences a *tricolored disc*, discussed in great detail in the prequel [G2]. By beginning with the natural Deligne-Mumford compactifications of tricolored discs (as labeled associahedra), and recreating the topology with which points labeled L and R are allowed to “coincide and/or go past each other,” in [G2] a model for the Deligne-Mumford compactification

$$(5.58) \quad \overline{\mathcal{R}}_{k,l},$$

(and hence the subspaces $\mathfrak{P}\overline{\mathcal{R}}_{k,l}$) was constructed.

DEFINITION 5.10. *A point identification \mathfrak{P} is said to be **sequential** if it is of the form*

$$(5.59) \quad \mathfrak{S} = \{(i_1, j_1), (i_1 + 1, j_1 + 1), \dots, (i_1 + s, j_1 + s)\}.$$

*It is further said to be **initial** if $(i_1, j_1) = (1, 1)$.*

DEFINITION 5.11. *A **cyclic sequential point identification** of type (r, s) is one of the form*

$$(5.60) \quad \mathfrak{S} = \{(1, 1), (2, 2), \dots, (r, r), (k - s, l - s), (k - s + 1, l - s + 1), \dots, (k, l)\}.$$

In other words, it is a sequential point identification where we need to take indices mod (k, l) .

One of the core technical constructions in [G2, §5] is the construction of a model for the Deligne-Mumford compactification of the spaces

$$(5.61) \quad \mathfrak{P} \overline{\mathcal{R}}_{k,l}.$$

We do not need to really recall the construction here, aside...

PROPOSITION 5.2. *Let \mathfrak{S} be a boundary identification, with compatible point identification \mathfrak{T} . Then, the gluing operation $S \coprod_{\mathfrak{S}} T$ extends to an operation on the Deligne-Mumford compactifications $\mathfrak{T} \overline{\mathcal{R}}_{k,l}$.*

Beyond the category \mathcal{W}^2 , we note a few other examples of operations from glued pairs of discs that arise later. As a first example, consider the case $\mathfrak{S} = \emptyset$ and \mathcal{L} equal to the full $\overline{\mathcal{R}}_{k,l}$.

PROPOSITION 5.3. *The operation associated to $\mathcal{L} = \overline{\mathcal{R}}_{k,l}$ with arbitrary Lagrangian labeling is zero if both k and l are ≥ 1 and one of (k, l) is ≥ 2 .*

PROOF. Let u be a rigid element in the associated moduli space $\overline{\mathcal{R}}_{k,l}(\vec{x}_{in}; \vec{x}_{out})$; since we are in the transverse situation, we can assume the domain of u is a point in the interior $p \in \mathcal{R}_{k,l}$. On the interior, the projection map

$$(5.62) \quad \pi_{\emptyset} : \mathcal{R}_{k,l} \longrightarrow \mathcal{R}^k \times \mathcal{R}^l$$

has fibers of dimension at least 1, parametrized by automorphisms of one factor relative to the other. (when $k = 1$, we implicitly replace \mathcal{R}^k by a point, and same for l —stabilization in this case completely collapses the left or right component). Since our Floer data was chosen to only depend on $\pi_{\emptyset}(p)$, we conclude that any map from an element of the fiber $\pi_{\emptyset}^{-1}(\pi_{\emptyset}(p))$ also satisfies Floer's equation; hence u cannot be rigid. \square

Now, consider the case of a single gluing adjacent to the outgoing marked points, i.e. $\mathfrak{S} = \{(1, 1)\}$ or $\mathfrak{S} = \{(k, l)\}$ with the induced point identification.

PROPOSITION 5.4. *The resulting operation in either case is μ^{k+l+1} .*

PROOF. We will without loss of generality do $\mathfrak{S} = \{(1, 1)\}$; the associated point identification is also $p(\mathfrak{S}) = \{(1, 1)\}$. The gluing morphism is of the form

$$(5.63) \quad \pi_{\mathfrak{S}} : \mathfrak{S}, p(\mathfrak{S}) \mathcal{R}_{k,l} \longrightarrow \mathcal{R}^{k+l+1}.$$

if k or l is ≥ 1 , the unreduced gluing is automatically stable, implying that (5.63) is an isomorphism. We obtain a corresponding identification of moduli spaces. \square

Our next example is the case $\mathfrak{S} = \{(1, 1), (k, l)\}$ with the induced point identification.

PROPOSITION 5.5. *The resulting operation is exactly ${}_2\mathcal{OC}^{k-2, l-2}$.*

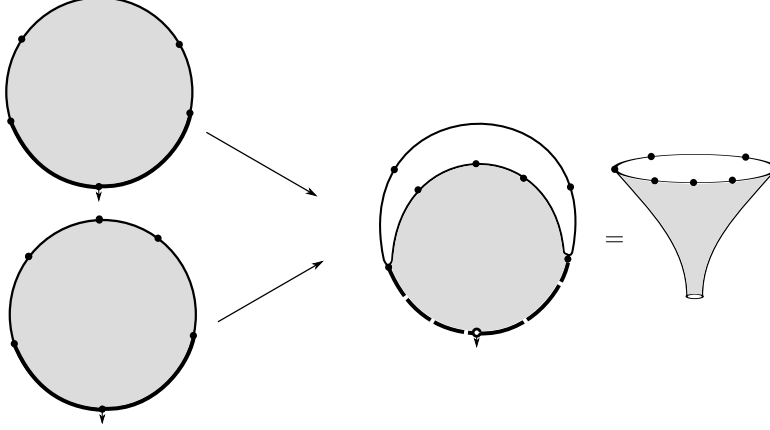
PROOF. The surface obtained by gluing the $(1, 1)$ and (k, l) boundary components together in $\mathcal{R}_{k,l}$ is stable, and has one interior output marked point. There are also $k + l$ boundary marked points, two of which are special. In cyclic order on the boundary, there is the identified point p_1 coming from the $(1, 1)$ boundary points, the $k - 2$ non-identified points from the left disc, the identified point p_2 coming from the (k, l) boundary points, and the $l - 2$ non-identified points from the right disc. Moreover, the identified points p_1, p_2 , and the interior boundary point are required to, up to equivalence, lie at the points $-i, 0$, and i respectively. We conclude that the projection is an isomorphism onto

$$(5.64) \quad \pi_{\mathfrak{S}} : \mathfrak{S}, p(\mathfrak{S}) \mathcal{R}_{k,l} \longrightarrow \mathcal{R}_{k-2, l-2}^1,$$

the moduli space controlling ${}_2\mathcal{OC}^{k-2, l-2}$.

See also Figure 8 for an image of this situation. \square

FIGURE 8. An example of the gluing $\pi_{\mathfrak{E}}$ associated to a $\{(1,1), (k,l)\}$ boundary identification.



6. Forgotten points and homotopy units

6.1. Geometric motivation. In this section, we introduce an important technical tool used in our result: a version of homotopy units for glued pairs of discs. We can motivate the need and/or application of such a tool as follows:

Suppose for a moment that we are in an idealized setting of Lagrangian Floer theory for a single Lagrangian $L \subset M$, in which we may ignore all issues of perturbations, transversality of moduli spaces, and obstructedness of Floer groups. Let us also for a moment reason using the conceptually intuitive singular chain variant of Floer theory as developed by [FOOO1]. In this framework, generators of the Floer chain complex $CF(L, L)$ are given by *equivalence classes* of geometric (singular) cycles in L . Given cycles $\mathbf{b}_1, \dots, \mathbf{b}_k$, we define the A_∞ structure map μ^k to be:

$$(6.1) \quad \mu^k(\mathbf{b}_k, \dots, \mathbf{b}_1) := (ev_0)_*[\mathcal{M}^k(\mathbf{b}_k, \dots, \mathbf{b}_1)]$$

Here $[\mathcal{M}^k(\mathbf{b}_k, \dots, \mathbf{b}_1)]$ is a “virtual fundamental chain” for the moduli space of holomorphic maps

$$u : (D, \partial D, z_1^+, \dots, z_k^+, z_0^-) \rightarrow (M, L, \mathbf{b}_1, \dots, \mathbf{b}_k, \cdot)$$

with positive boundary marked points z_i^+ constrained to lie on the cycles \mathbf{b}_i , and negative boundary marked point z_0^- unconstrained. The notation from (6.1) simply means that we take as result the cycle “swept out” by the marked point z_0^- in this moduli space.

In this (unfortunately imaginary) setting, there is a canonical choice of *strict unit* for the A_∞ algebra $CF(L, L)$: the fundamental class $[L]$. This cycle satisfies the fundamental property that for $u : (D, \partial D) \rightarrow (M, L)$, the condition that $z_i \in \partial D$ lies on the cycle $[L]$ is an empty constraint.

Let us very informally show that this property gives $[L]$ the structure of a strict unit. First, work in the stable range $d \geq 3$. There is a projection map

$$(6.2) \quad \pi_j : \mathcal{R}^d \rightarrow \mathcal{R}^{d-1},$$

forgetting the j th marked point. In the above setting, π_j extends to a map between moduli spaces of stable maps:

$$(6.3) \quad (\pi_j)_* : \mathcal{M}^d(\mathbf{b}_1, \dots, \mathbf{b}_k) \rightarrow \mathcal{M}^{d-1}(\mathbf{b}_1, \dots, \mathbf{b}_{j-1}, \mathbf{b}_{j+1}, \dots, \mathbf{b}_k)$$

Suppose $\mathbf{b}_j = [L]$, an empty constraint on the marked point z_j . This implies that $(\pi_j)_*$ is a submersion with one-dimensional fibers, corresponding to the location of the j th marked point. In particular,

$$\dim \mathcal{M}^d(\mathbf{b}_1, \dots, \mathbf{b}_k) = \dim \mathcal{M}^{d-1}(\mathbf{b}_1, \dots, \mathbf{b}_{j-1}, \dots, \mathbf{b}_{j+1}, \dots, \mathbf{b}_k) + 1,$$

which implies that $(ev_0)_*\mathcal{M}^d(\mathbf{b}_1, \dots, \mathbf{b}_k)$ is a degenerate chain (in the sense that it factors through a lower-dimensional submanifold) and thus should be zero on homology. Hence hopefully

$$(6.4) \quad \mu^d(\dots, [L], \dots) = 0.$$

When $d = 2$, we leave it as a pictorial exercise to the interested reader to “prove” that

$$\mu^2([L], x) = \pm \mu^2(x, [L]) = \pm x$$

(or see Proposition 6.2). Even in this setting, there are a number of issues:

- in order to obtain transversality, one needs to coherently perturb the holomorphic curve equations in a domain-dependent manner and there is no known way to make the forgetful map compatible with these perturbations. These perturbations occur in the setting of *Kuranishi structures*, making them even less likely to be compatible with the forgetful map.
- strictly speaking, this moral argument only proved the equality (6.4) modulo some subcomplex of degenerate chains (it is not obvious for instance that the equality holds e.g., in normalized singular chains). To move to an A_∞ structure on $H^*(L)$, a host of additional arguments are required, including homological perturbation theory. The pay-off is that after some additional work one obtains a strictly unital structure on $H^*(L)$.

The reader is referred to [FOOO2, Ch. 7, §31] for details on Fukaya-Oh-Ohta-Ono’s approach to these problems.

In our setting, generators are time-1 Hamiltonian chords, so we have an additional issue:

- even if transversality were not an issue, there is no time-1 chord(s) x with the property that imposing an asymptotic condition to x is in any sense the same as adding the marked point back with an empty constraint.

The remedy that seems to have been used in the literature most is this: construct a homology level unit geometrically, and then apply algebraic results of Seidel to obtain a quasi-isomorphic A_∞ algebra that is strictly unital.

However, we are in a setting where we do not just care about algebraic properties of strictly unital A_∞ categories. We would like to carefully analyze certain operations on \mathcal{W}^2 controlled by forgetful maps applied to submanifolds of moduli spaces of open-closed surfaces and pairs of discs. To be able to use such operations in \mathcal{W}^2 , we will need them to be homotopic to existing operations.

The eventual punchline of this section is this: *Given some operations controlled by a submanifold \mathcal{Q} of open-closed strings, the construction of homotopy units gives us a quasi-isomorphic category with additional elements $e_j^+ \in \chi(L_j, L_j)$ such that the operation $\mathcal{Q}(\cdots e_j^+ \cdots)$ is controlled by the submanifold $\pi_j(\mathcal{Q})$.*

6.2. The algebra of homotopy units. There are three a posteriori equivalent definitions of units in the A_∞ setting. The weakest version, following [S5], is the notion of a *homology-level unit*, already defined at the beginning of §2.2. A category equipped with homologically-level units is said to be *homologically unital* (or *c-unital* in [S5]). One could further impose requirements of the chain level representative of this unit, leading to a stricter notion:

DEFINITION 6.1. *An A_∞ category \mathcal{C} is **strictly unital** if for each $X \in \text{ob } \mathcal{C}$ there is an element $e_X^+ \in \text{hom}_{\mathcal{C}}(X, X)$, called a **strict unit for X** such that*

$$(6.5) \quad \begin{aligned} \mu^1(e_X^+) &= 0, \\ (-1)^{|x|} \mu^2(x, e_X^+) &= x, \text{ for any } x \in \text{hom}_{\mathcal{C}}(X, Y), \\ \mu^2(e_X^+, y) &= y \text{ for any } y \in \text{hom}_{\mathcal{C}}(Y, X), \text{ and} \\ \mu^k(\dots, e_X^+, \dots) &= 0, \quad k \geq 3. \end{aligned}$$

Fukaya-Oh-Ohta-Ono [FOOO1] observed (for the A_∞ algebra on a single Lagrangian $CF^*(L, L)$) that there is a richer structure which can be constructed geometrically which interpolates between these two notions. We will give the straightforward categorical variant of their definition:

DEFINITION 6.2. *Let \mathcal{C} be an A_∞ category with chosen chain-level representatives of homological units $e_X \in \text{hom}_{\mathcal{C}}(X, X)$ for each X . A **homotopy unit** for $(\mathcal{C}, \{e_X\})$ is an A_∞ category \mathcal{C}' with $\text{ob } \mathcal{C}' = \text{ob } \mathcal{C}$ and morphisms*

$$(6.6) \quad \text{hom}_{\mathcal{C}'}(X, Y) := \begin{cases} \text{hom}_{\mathcal{C}}(X, X) \oplus \mathbb{K}f_X[1] \oplus \mathbb{K}e_X^+ & X = Y \\ \text{hom}_{\mathcal{C}}(X, Y) & X \neq Y \end{cases}$$

such that the $\mu_{\mathcal{C}'}^d$ restrict to the original A_∞ operations μ^d on \mathcal{C} whenever the input elements all lie in \mathcal{C} , and additionally:

$$(6.7) \quad \text{Each } e_X^+ \text{ is a strict unit for } X;$$

$$(6.8) \quad \mu^1(f_X) = e_X^+ - e_X,$$

We sometimes omit the chain level choices $\{e_X\}$ made and call \mathcal{C}' the **homotopy units associated to** \mathcal{C} , or once \mathcal{C}' is specified, say \mathcal{C} is **equipped with homotopy units**. As a sanity check, we note that in the definition above, the functor which is the identity on objects and inclusion of morphism spaces

$$(6.9) \quad \mathcal{C} \hookrightarrow \mathcal{C}'$$

is a quasi-isomorphism. Moreover, the condition that e_X^+ be a strict unit determines all A_∞ structure maps $\mu_{\mathcal{C}'}$ involving an occurrence of e_X^+ . Thus, the additional data involved in constructing the required A_∞ structure on \mathcal{C} is exactly contained in operations with occurrences of f_X morphisms. Thus, (assuming f_X and e_X^+ are never the targets of such an A_∞ structure map) the data of a homotopy unit translates into the data of maps

$$(6.10) \quad \mathfrak{h}_k : (T\mathcal{C})^{\otimes k} \rightarrow \mathcal{C}$$

such that the operations

$$(6.11) \quad \begin{aligned} \mu_{\mathcal{C}'}^{i_1 + \dots + i_k + k - 1}(x_1^1, \dots, x_{i_1}^1, f_{X^1}, x_1^2, \dots, x_{i_2}^2, \dots, f_{X^{k-1}}, x_1^k, \dots, x_{i_k}^k) := \\ \mathfrak{h}_k(x_1^1 \otimes \dots \otimes x_{i_1}^1; \dots; x_1^k \otimes \dots \otimes x_{i_k}^k) \end{aligned}$$

satisfy the A_∞ relations. The A_∞ relations for these particular operations can be straightforwardly translated into equations for the \mathfrak{h}_k , which we omit; see [FOOO1, §3.3] for greater detail, and c.f. [S5, (2a), e.g., (2.3)], where the first few equations of (6.10) are indexed slightly differently.

By definition any A_∞ category \mathcal{C} equipped with homotopy units is quasi-isomorphic to a strictly unital one (namely \mathcal{C}' itself). Conversely, it is shown in [S5, Lemma 2.1] that any homologically unital A_∞ category is quasi-equivalent to a strictly unital or homotopy unital A_∞ category.

6.3. Forgotten marked points. We begin with a notion of what it means to have forgotten a boundary marked point in Floer-theoretic operations. Since the construction is identical for discs and pairs of glued discs, we initiate them in parallel. Strictly speaking, we do not need the single-disc construction in our paper, but it is no additional work and may be foundationally useful. Also, we only consider forgotten points on pairs of *identical discs modulo simultaneous automorphism*, the only case that arises for us.

DEFINITION 6.3. *The moduli space of discs with d marked points and $F \subseteq \{1, \dots, d\}$ forgotten marked points, denoted*

$$(6.12) \quad \mathcal{R}^{d,F}$$

is exactly the moduli space of discs \mathcal{R}^d with marked points labeled as belonging to F .

DEFINITION 6.4. *The moduli space of \mathfrak{S} -glued pairs of discs with (k, l) marked points, \mathfrak{T} point identifications, and*

$$(6.13) \quad (F_1, F_2) \subseteq (\{1, \dots, k\}, \{1, \dots, l\}),$$

forgotten points, denoted

$$(6.14) \quad \mathfrak{S}, \mathfrak{T} \mathcal{R}_{k,l}^{(F_1, F_2)}$$

*is the image of the diagonal associahedron in the moduli space of glued pairs of discs $\mathcal{R}_{k,k,\mathfrak{S}}$ with positive marked points on each disc corresponding to F_1 and F_2 labeled as **forgotten points**. Crucially, F_1 and F_2 must satisfy the following conditions:*

- F_1 and F_2 are subsets of the left and right identified points respectively. Namely,

$$(6.15) \quad F_i \subset \pi_i(\mathfrak{T}),$$

where π_i is projection onto the i th component.

- F_1 and F_2 are not associated to a boundary identification, i.e.

$$(6.16) \quad F_i \cap \pi_i(p(\mathfrak{S})) = \emptyset$$

- F_1 and F_2 do not contain both the left and right points of any identification, i.e.

$$(6.17) \quad (F_1 \times F_2) \cap \mathfrak{T} = \emptyset$$

REMARK 6.1. Put another way, the conditions F_1 and F_2 must satisfy correspond to the following from the viewpoint of moduli spaces of tricolored discs developed in [G2, §5]: F_1 and F_2 correspond to disjoint subsets of the points colored LR, such that neither F_1 or F_2 is adjacent to a boundary component labeled as identified.

For the purpose of solving Floer's equations, we will be putting the marked points labeled by F , F_i back in. Such points should be thought of as *markers* rather than punctures.

DEFINITION 6.5. Let $I \subseteq F$. The **I-forgetful map**

$$(6.18) \quad \mathcal{F}_I : \mathcal{R}^{d,F} \longrightarrow \mathcal{R}^{d-|I|,F'}$$

associates to any S the surface obtained by putting the points of I back in and forgetting them. F' in the equation above is the set of forgetful points $F - I$, re-indexed appropriately.

There is a similar forgetful map for pairs of glued discs,

$$(6.19) \quad \mathcal{F}_{I_1, I_2} : \mathfrak{S}, \mathfrak{T} \mathcal{R}_{k,l}^{F_1, F_2} \longrightarrow \mathfrak{S}', \mathfrak{T}' \mathcal{R}_{k-|I_1|, l-|I_2|}^{F'_1, F'_2}$$

We need a notion that corresponds to stability of the underlying disc once we have forgotten points.

DEFINITION 6.6. A disc with d marked points and F forgotten points is **f-stable** or **f-semistable** if $d - |F| \geq 2$ or $d - |F| = 1$ respectively. A pair of discs with (k, l) marked points and F_1, F_2 forgotten points is **f-stable** if $k - |F_1|$, $l - |F_2|$ are both greater than or equal to 1 and one is greater than or equal to 2. It is **f-semistable** if both of these quantities are equal to 1.

In the f-stable range, there are maximally forgetful maps, collectively denoted \mathcal{F}_{max} :

$$(6.20) \quad \mathcal{F}_{max} = \mathcal{F}_F : \mathcal{R}^{d,F} \longrightarrow \mathcal{R}^{d-|F|}$$

$$(6.21) \quad \mathcal{F}_{max} = \mathcal{F}_{F_1, F_2} : \mathfrak{S}, \mathfrak{T} \mathcal{R}_{k,l}^{F_1, F_2} \longrightarrow \mathfrak{S}', \mathfrak{T}' \mathcal{R}_{k-|F_1|, l-|F_2|}$$

The (Deligne-Mumford) compactifications

$$(6.22) \quad \overline{\mathcal{R}}^{d,F}$$

$$(6.23) \quad \mathfrak{S}, \mathfrak{T} \overline{\mathcal{R}}_{k,l}^{F_1, F_2}$$

are exactly the usual Deligne-Mumford compactifications, along with the data of *forgotten* labels for the relevant boundary marked points. Interior positive nodes inherit the label of *forgotten* in the following fashion:

DEFINITION 6.7. An interior positive node of a stable representative S of a disc or pair of glued discs is said to be a **forgotten node** if and only if every boundary marked point in every component above p is a forgotten marked point and there are no interior marked points in any component above p .

In the f-stable range, stable discs with forgotten marked points have underlying stable representatives with forgotten points removed.

DEFINITION 6.8. A component of a stable representative S of a disc or a pair of glued discs is said to be **forgettable** if all of its positive boundary marked points (including nodal ones) are forgotten points and it has no interior marked points.

Using the above definitions, one can extend the maximally forgetful map to compactifications.

DEFINITION 6.9. Let S be a nodal bordered f-stable surface with forgotten marked points. The **associated reduced surface** \hat{S} is the nodal surface obtained by

- eliminating all forgettable components

- putting back in all forgotten boundary points and forgetting them
- if in the f -stable range, eliminating any non-main component with only one non-forgotten marked point p , and labeling the positive marked point below this component by p .

Define the **induced marked points** of \hat{S} to be the boundary marked points that survive this procedure.

In other words, the nodal surface \hat{S} is obtained from the nodal surface S by forgetting the points with an F label and then stabilizing the resulting bubble tree.

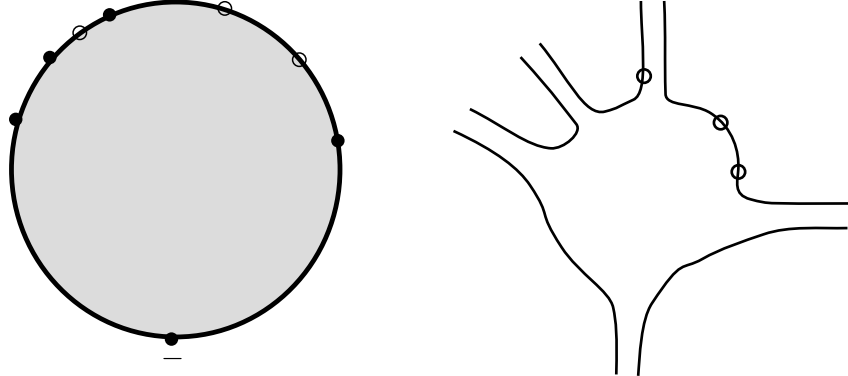
DEFINITION 6.10. The **maximally forgetful map** \mathcal{F}_{\max} , defined for any nodal f -semistable disc or pair of glued discs is defined to be the map that associate to a nodal surface with forgotten marked points S the associated reduced surface \hat{S} .

DEFINITION 6.11. A **Floer datum** for a stable, f -semistable disc or pair of glued discs with forgotten marked points consists of a Floer datum for the associated reduced surface $\hat{S} = \mathcal{F}_{\max}(S)$, in the sense of [G2, Def. 4.11] or [G2, Def. 5.16] satisfying the following conditions:

- in the **f-stable range**, it is identical to our previously chosen Floer datum for \hat{S} thought of as an open-closed string.
- in the **f-semistable range**, it is given by the unique translation-invariant Floer datum on the strip \hat{S} .

This implies in particular that the Floer datum only depends on the point $\mathfrak{F}_{\max}(S)$.

FIGURE 9. Two drawings of a disc with forgotten points (denoted by hollow points). The drawing on the right emphasizes the choice of strip-like ends.



REMARK 6.2. By the above definition, a Floer datum for a pair of discs P with \mathfrak{S} boundary identifications, \mathfrak{T} point identifications, and F_1, F_2 forgotten points is a Floer datum for the open-closed string obtained by forgetting the marked points corresponding to F_1 and F_2 , stabilizing, and gluing the resulting pairs of discs via $\pi_{\mathfrak{S}}$.

Because we have chosen our Floer data to be the one we have already chosen for the underlying reduced open-closed string, we immediately obtain:

PROPOSITION 6.1. There exists a universal and consistent choice of Floer data for discs or pairs of glued discs with forgotten marked points.

DEFINITION 6.12. An **admissible Lagrangian labeling** for a surface S with forgotten marked points is a choice of Lagrangian labeling that descends to a well-defined labeling on the associated reduced surface $\mathcal{F}_{\max}(S)$. Namely, if p is any forgotten boundary marked point of S , then the labels before and after p must coincide. The **reduced labeling** is the corresponding labeling on the underlying reduced surface.

Now, suppose we have fixed a universal and consistent choice of Floer data for discs. Consider a compact submanifold with corners of dimension d

$$(6.24) \quad \bar{\mathcal{Z}}^d \hookrightarrow \mathfrak{S}, \mathfrak{T} \mathcal{R}_{k,l}^{F_1, F_2}.$$

with an admissible Lagrangian labeling \vec{L} . In the usual fashion, fix input and output chords $\vec{x}_{in}, \vec{x}_{out}$ and orbits $\vec{y}_{in}, \vec{y}_{out}$ for the induced marked points of the gluing $\pi_{\mathfrak{S}}(\mathcal{F}_{max}(\bar{\mathcal{Z}}^d))$, which forgets all points labeled as forgotten and glues along the boundary components \mathfrak{S} . Define

$$(6.25) \quad \bar{\mathcal{Z}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$$

to be the space of maps

$$(6.26) \quad \{u : \pi_{\mathfrak{S}}(\mathcal{F}_{max}(S)) \longrightarrow M : S \in \bar{\mathcal{Z}}^d\}$$

satisfying Floer's equation with respect to the Floer datum and asymptotic and boundary conditions specified by the Lagrangian labeling \vec{L} and asymptotic conditions $(\vec{x}_{out}, \vec{y}_{out}, \vec{x}_{in}, \vec{y}_{in})$.

As before, $h(\mathfrak{S}, k, l)$ denote the number of boundary components of any resulting surface obtained from the gluing.

LEMMA 6.1. *The moduli spaces $\bar{\mathcal{Z}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ are compact, and empty for all but finitely many $(\vec{x}_{out}, \vec{y}_{out})$ given fixed inputs $(\vec{x}_{in}, \vec{y}_{in})$. For generically chosen Floer data, they form smooth manifolds of dimension*

$$(6.27) \quad \dim \bar{\mathcal{Z}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in}) := \sum_{x_- \in \vec{x}_{out}} \deg(x_-) + \sum_{y_- \in \vec{y}_{out}} \deg(y_-) \\ + (2 - h(\mathfrak{S}, k, l) - |\vec{x}_{out}| - 2|\vec{y}_{out}|)n + d - \sum_{x_+ \in \vec{x}_{int}} \deg(x_+) - \sum_{y_+ \in \vec{y}_{in}} \deg(y_+).$$

In the usual fashion, when the dimension of the spaces $\bar{\mathcal{Z}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ are zero, we can use natural isomorphisms of orientation lines to count (with signs) the number of points in such spaces, and associate operations

$$(6.28) \quad (-1)^{\vec{r}} \mathbf{H}_{\bar{\mathcal{Z}}^d}$$

from the tensor product of wrapped Floer complexes and symplectic cochain complexes where $\vec{x}_{in}, \vec{y}_{in}$ reside to the tensor product of the complexes where $\vec{x}_{out}, \vec{y}_{out}$ reside.

We can specify certain submanifolds of the space of forgotten marked points by applying forgotten labels to various boundary points on spaces of open-closed strings.

DEFINITION 6.13. *The forget map*

$$(6.29) \quad \mathfrak{f}_F : \mathcal{R}^d \longrightarrow \mathcal{R}^{d, F}$$

$$(6.30) \quad \mathfrak{f}_{F_1, F_2} : \mathfrak{S}, \mathfrak{T} \mathcal{R}_{k,l} \longrightarrow \mathfrak{S}, \mathfrak{T} \mathcal{R}_{k,l}^{F_1, F_2}$$

simply marks boundary points with indices in F (or (F_1, F_2)) as forgotten.

6.4. Operations with forgotten points. Our main application is of course to think of forgotten points as formal units, either for a disc or pair of discs. It is thus illustrative to see how operations with forgotten marked points either vanish or reduce to other known operations.

PROPOSITION 6.2. *Let $F \subset \{1, \dots, d\}$ be a non-empty subset of size $0 < |F| < d$. Then the operation associated to $\bar{\mathcal{R}}^{d, F}$ is zero if $d > 2$ and the identity operation $I(\cdot)$ (up to a sign) when $d = 2$.*

PROOF. Suppose first that $d > 2$, and let u be any solution to Floer's equation over the space $\mathcal{R}^{d, F}$ with domain S . Let $p \in F$ be the last element of F . Since the Floer data on S only depends on $\mathcal{F}_p(S)$, we see that maps from S' with $S' \in \mathcal{F}_p^{-1}(\mathcal{F}_p(S))$ also give solutions to Floer's equation with the same asymptotics. Moreover, the fibers of the map \mathcal{F}_p are one-dimensional, implying that u cannot be rigid, and thus the associated operation is zero.

Now suppose that $d = 2$, and without loss of generality $F = \{1\}$. Then the forgetful map associates to the single point $[S] \in \mathcal{R}^{2,F}$ the unstable strip with its translation invariant Floer datum. We conclude, based on Section B.1, that the resulting operation is the identity. \square

REMARK 6.3. *Actually, one would like this operation to be zero when $|F| = d$ as well. However, we have not defined an operation with $|F| = d$, due to the unstability of the underlying reduced surface. Our solution will be to declare this operation to be zero, and check that our declaration is compatible with the behavior of boundaries of one dimensional moduli spaces.*

PROPOSITION 6.3. *Let $\Delta_d \subset \mathcal{R}_{d,d}$ be the diagonal associahedron. Let $[d]$ denote the set $\{1, \dots, d\}$, and (k, l) such that pairs of discs with k, l marked points are stable. Then, the operation given by the disjoint union*

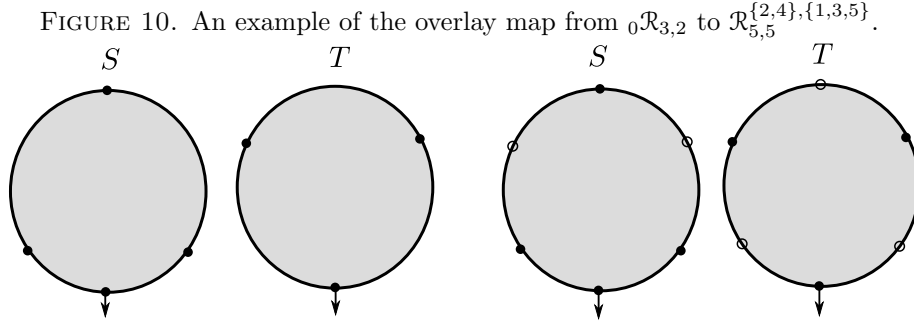
$$(6.31) \quad \coprod_{I \subset [k+l] \mid |I|=k} (\mathfrak{f}_{I, [k+l]-I})(\Delta_{k+l})$$

with appropriate orientations is identical to the operation given by $\mathcal{R}_{k,l}$. In other words, by Proposition 5.3 it is equal to zero when $k, l \geq 1$ and one is ≥ 2 .

PROOF. On the open locus ${}_0\mathcal{R}_{k,l}$ where none of the k points on the first disc and the l points on the second disc are in identical positions, we can consider the **overlay** map:

$$(6.32) \quad {}_0\mathcal{R}_{k,l} \longrightarrow \coprod_{I \subset [k+l] \mid |I|=k} (\mathfrak{f}_{I, [k+l]-I})(\Delta_{k+l}(\mathcal{R}^{k+1}))$$

given by marking the l positive marked points on S_2 as extra forgotten points on S_1 and vice versa. On the level of tri-colored discs, the overlay makes L points LR points with the R component marked as forgotten, and makes R points LR points with the L component marked as forgotten. By construction, this map is compatible with Floer data, and covers the entire interior of the target. Since after a perturbation zero-dimensional solutions to Floer's equation come from a representative on the interior of any source abstract moduli space, we conclude that the two operations in the Proposition are identical, modulo sign. See Figure 10 for an example of this overlay map. \square



PROPOSITION 6.4. *Take boundary identification $\mathfrak{S} = \{(1, 1)\}$ and maximal point identification $\mathfrak{T}_{max} = \{(1, 1), \dots, (k+l, k+l)\}$. Then, letting $S = \{2, \dots, k+l-2\}$, the operation corresponding to*

$$(6.33) \quad \coprod_{I \subset S \mid |I|=k-1} \mathfrak{f}_{I, S-I}(\mathfrak{S}, \mathfrak{T}_{max} \mathcal{R}_{k,l})$$

is μ^{k+l-1} (with suitable sign twisting datum).

PROOF. On the open locus of $_{(1,1),(1,1)}\mathcal{R}_{k,l}$ where the only coincident points are $(1, 1)$ and no other points coincide, denoted

$$(6.34) \quad {}_{(1,1),(1,1)}\mathcal{R}_{k,l}^0$$

there is once more an *overlay map*

$$(6.35) \quad {}_{(1,1),(1,1)}\mathcal{R}_{k,l}^0 \longrightarrow \coprod_{I \subset S \mid |I|=k-1} \mathfrak{f}_{I,S-I}(\mathfrak{S}, \mathfrak{T}_{max} \mathcal{R}_{k,l})$$

given by superimposing left and right discs, and marking points from the right as forgotten points on the left and vice versa. This gives an isomorphism of spaces with Floer data on the open locus, so we conclude by applying Proposition 5.4 to calculate the operation associated to ${}_{(1,1),(1,1)}\mathcal{R}_{k,l}$. \square

PROPOSITION 6.5. *Take boundary identification $\mathfrak{S} = \{(k+l, k+l)\}$ and maximal point identification $\mathfrak{T}_{max} = \{(1,1), \dots, (k+l, k+l)\}$. Then, letting $S = \{1, \dots, k+l-1\}$, the operation corresponding to*

$$(6.36) \quad \coprod_{I \subset S \mid |I|=k-1} \mathfrak{f}_{I,S-I}(\mathfrak{S}, \mathfrak{T}_{max} \mathcal{R}_{k,l})$$

is μ^{k+l-1} (with suitable sign twisting datum).

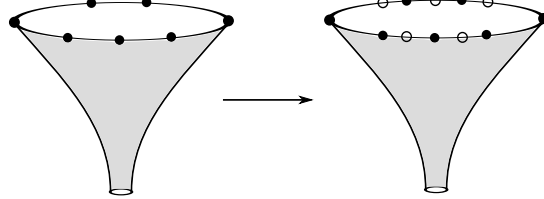
PROOF. The proof is identical to the above case, using an overlay map and reducing to Proposition 5.4. \square

PROPOSITION 6.6. *Take boundary identification $\mathfrak{S} = \{(1,1), (k+l, k+l)\}$ and maximal point identification $\mathfrak{T}_{max} = \{(1,1), \dots, (k+l, k+l)\}$. Then, letting $S = \{2, \dots, k+l-1\}$, the operation corresponding to*

$$(6.37) \quad \coprod_{I \subset S \mid |I|=k-2} \mathfrak{f}_{I,S-I}(\mathfrak{S}, \mathfrak{T}_{max} \mathcal{R}_{k,l})$$

is ${}_2\mathcal{OC}^{k-2,l-2}$ (with suitable sign twisting datum).

FIGURE 11. The overlay map from ${}_{\mathfrak{S},p(\mathfrak{S})}\mathcal{R}_{5,4}$ to ${}_{\mathfrak{S}',p(\mathfrak{S}')}\mathcal{R}_{7,7}^{\{3,5\},\{2,4,6\}}$, where $\mathfrak{S} = \{(1,1), (5,4)\}$ and $\mathfrak{S}' = \{(1,1), (7,7)\}$. Forgotten points are marked with rings.



PROOF. The same arguments using overlay maps as before apply, only now we compare to $\{_{(1,1),(k,l)}, \{_{(1,1),(k,l)}\}\mathcal{R}_{k,l}$. The associated operation is, by Proposition 5.5, ${}_2\mathcal{OC}^{k-2,l-2}$. See Figure 11 for an example of this particular overlay map. \square

6.5. A local model. Our definition of forgetful operations and homotopy units is based upon the following local model. Let \mathbb{H} denote the upper half plane and \mathbb{H}^o the upper half plane with the origin removed. Viewing \mathbb{H} as a disc with a point removed, and \mathbb{H}^o as a disc with two points removed, there is the natural “forgetful” map

$$(6.38) \quad F : \mathbb{H}^o \hookrightarrow \mathbb{H}$$

which forgets the special point 0. Consider the following negative strip-like end around ∞ :

$$(6.39) \quad \epsilon_{\mathbb{H}} : (-\infty, 0] \times [0, 1] \longrightarrow \mathbb{H}$$

$$(6.40) \quad (s, t) \longmapsto \exp(-\pi(s + it))$$

which has image $\{(r, \theta) \mid r \geq 1\} \subset \mathbb{H}$. For \mathbb{H}^o , define the following basic positive strip-like end around 0:

$$(6.41) \quad \epsilon_{\mathbb{H}^o} : [0, \infty) \times [0, 1] \longrightarrow \mathbb{H}^o$$

$$(6.42) \quad (s, t) \longmapsto 2 \cdot \exp(-\pi(s + it)).$$

This end has image $\{(r, \theta) | 0 < r \leq 2\} \subset \mathbb{H}^o$. With these special choices of strip-like ends, we observe that 0-connect sum is exactly the forgetful map F . The precise statement is this:

PROPOSITION 6.7. *Let \mathbb{T} be the associated thick part in \mathbb{H}^0 of the 0 connect sum*

$$C := \mathbb{H}^o \#_{(\epsilon_{\mathbb{H}^o}, 0), (\epsilon_{\mathbb{H}}, \infty)}^0 \mathbb{H}$$

and let C^0 denote the complement of $0 \in \mathbb{H}$ in the connect sum C . Then, there is a commutative diagram

$$(6.43) \quad \begin{array}{ccccc} \mathbb{T} & \xrightarrow{\quad} & C^0 & \xrightarrow{\quad} & C \\ & \searrow & \downarrow & & \\ & & \mathbb{H}^0 & \xrightarrow{\quad} & \mathbb{H} \end{array}$$

PROOF. This is obvious via viewing each of these regions and the connect sum itself as subsets of \mathbb{H} . \square

6.6. Revisiting the unit. We revisit our choice of Floer datum for the explicit geometric unit map, defined in Section B.2. Let $\Sigma_0 = \mathbb{H}$ once more denote the upper half plane, and fix an outgoing striplike end at ∞ given by $\epsilon_{\mathbb{H}}$, defined in (6.39).

Let $\psi : [0, \infty) \rightarrow [0, 1]$ be a smooth function equaling 0 in a neighborhood of 0 and 1 in a neighborhood of $[1, \infty)$.

Given a weight $w = 1$, and a Hamiltonian H , fix the following Floer datum on \mathbb{H} :

- one form $\alpha_{\mathbb{H}}$ given by $-\frac{1}{\pi} \cdot \psi(r) d\theta$
- rescaling function $a_{\mathbb{H}}$ equal to 1.
- any primary Hamiltonian, $H_{\mathbb{H}}$ that is compatible with the strip-like end $\epsilon_{\mathbb{H}}$
- any almost complex structure that is compatible with $\epsilon_{\mathbb{H}}$.
- some constant perturbation term $F_{\mathbb{H}}$.

Define the Floer datum for arbitrary weight w to be the w conformal rescaling of the above Floer datum. It has the following properties:

- one form $\alpha_{\mathbb{H}}^w$ given by $-\frac{1}{\pi} \cdot w\psi(r) d\theta$
- rescaling function $a_{\mathbb{H}}^w$ equal to w .
- primary Hamiltonian $H_{\mathbb{H}}^w$ given by $\frac{H_{\mathbb{H}} \circ \psi^w}{w^2}$
- almost complex structure $J_{\mathbb{H}}^w$ given by $(\psi^w)^* J_{\mathbb{H}}$
- perturbation term $F_{\mathbb{H}}^w$ given by $w \cdot F_{\mathbb{H}}$, another constant.

Call the above datum a **standard unit datum of type w** . By design, $\epsilon_{\mathbb{H}}^*(\alpha_{\mathbb{H}}^w) = w dt$.

6.7. Damped connect sums. We describe a local model, depending on a time parameter

$$(6.44) \quad \tau \in [0, 1],$$

that gives a homotopy relating a “formal unit,” or forgotten marked point, to the geometric unit described in Section B and again above. We would like such a homotopy, which we call a **τ -damped connect sum**, to have the following properties in a neighborhood of a given forgotten point p on a surface S .

- at time $\tau = 0$, the Floer datum is essentially unconstrained in a neighborhood of p , agreeing with whatever Floer datum we obtained by forgetting p and compactifying.
- at intermediate time τ , the Floer datum is modeled on a growing connect sum of a neighborhood of p with a disc with one output, thought of as \mathbb{H} with output at ∞ .
- the $\tau = 1$ limit is the nodal connect sum $\mathbb{H} \#_p^1 S$. The Floer datum on the \mathbb{H} component should agree with the Floer datum on the geometric unit, and the Floer data on the p side should agree with a standard, previously chosen Floer datum.

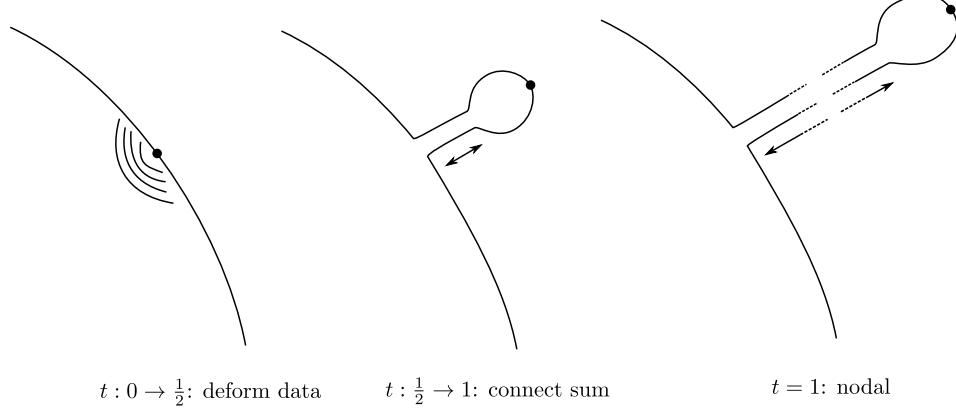
Readers who wish to skip this section should treat the **τ -damped connect sum** along a boundary point p as a formal operation on surfaces with Floer data, satisfying the property that at $\tau = 0$, one has the forgetful map, and at $\tau = 1$, one has nodally glued on an \mathbb{H} .

In reality, we will need to construct such an operation in two steps:

- for $\tau \in (0, \frac{1}{2})$, the Floer datum on S goes from arbitrary with respect to p to (partially) compatible with respect to a strip-like end around p .

- for $\tau \in (\frac{1}{2}, 1)$, the datum is modeled as a growing connect sum as before.

FIGURE 12. A schematic of a damped connect sum (though in reality, the conformal structure of S will stay the same).



The basic setup is as follows: Let S be a Riemann surface with boundary, with some boundary marked points removed. Fix one such positive boundary marked point z , with strip-like end around z

$$(6.45) \quad \epsilon_z : [0, \infty) \times [0, 1] \longrightarrow S.$$

DEFINITION 6.14. Let \hat{S}_z be S with the point z filled back in. Call the strip-like end ϵ_z **rational** if it extends to a holomorphic map $\bar{\epsilon}_z : ([0, 1] \times [0, \infty)) \cup \{\infty\} \longrightarrow \hat{S}_z$.

REMARK 6.4. Working with rational strip-like ends does not impose any additional trouble in choosing Floer data. We can implicitly choose all of our strip-like ends $\epsilon : Z_+ \rightarrow S$ to be rational. See e.g. [AS2, Addendum 2.3].

Now, let ϵ_z be any rational striplike end. Let D_2^0 denote the punctured upper half radius two disc $\{0 < |z| \leq 2\} \subset \mathbb{C}$ and D_2 the domain arising from D_2^0 by filling in the origin, i.e. $D_2 = \{0 \leq |z| \leq 2\}$. By precomposing with the standard map

$$(6.46) \quad \begin{aligned} \epsilon_{\mathbb{H}_0}^{-1} : D_2^0 &\longrightarrow [0, \infty) \times [0, 1] \\ z &\longmapsto \left(\ln\left(\frac{2}{|z|}\right), 1 - \frac{\arg(z)}{\pi} \right) \end{aligned}$$

we may equivalently suppose ϵ_z is a map $\tilde{\epsilon}_z$ from D_1^0 to S that extends to a map from D_2 to \hat{S}_z . Call $\tilde{\epsilon}_z$ the associated **disc-like end** of ϵ_z , and let $\bar{\tilde{\epsilon}}_z$ be the associated map from D_2 to \hat{S}_z .

Now, fix a time-parameter $\tau \in [0, 1]$. In a manner depending on τ , we weaken the notion of compatibility with respect to the strip-like end ϵ_z .

DEFINITION 6.15. A Floer datum $(\alpha_S, a_S, J_S, H_S, F_S)$ is said to be τ -**partially compatible** with a strip-like end (z, ϵ) , for $\tau \in [0, 1]$, if the datum extends to one on the compactification \hat{S}_z for $\tau \in [0, \frac{1}{2}]$ and the conditions

$$(6.47) \quad \begin{aligned} \epsilon^* a_S &= w_1(\tau) \\ \epsilon^* \alpha_S &= w_2(\tau) dt \\ \epsilon^* H_S &= \frac{H \circ \psi^{w_2(\tau)}}{w_2(\tau)^2} \end{aligned}$$

only hold for $\tau \geq \frac{1}{2}$. Furthermore, we require that at $\tau = 1$, τ -compatibility is genuine compatibility; in other words,

$$(6.48) \quad w_1(1) = w_2(1).$$

REMARK 6.5. In the limit $\tau = 0$, τ -partial compatibility is an empty condition for the Floer data on S . Note that in contrast to normal Floer data, we are using two potentially different functions $w_1(\tau)$ and $w_2(\tau)$. In other words, we are not requiring the value of the one-form α_S or the amount flowed by the Hamiltonian to always be the same as the amount of rescaling or time-shifting performed by the almost complex structure or Lagrangian boundary. This is sensible—on a boundary point that is not a priori a striplike end, the one form α_S is asymptotically 0, but α_S is always non-zero.

DEFINITION 6.16. Let S have rational strip-like end ϵ_z around z with associated disc-like end $\tilde{\epsilon}_z$, and suppose we have chosen a Floer datum \mathbf{D}^τ that is τ -compatible with $\tilde{\epsilon}_z$. An associated τ -structure on \mathbb{H} , denoted $\mathbf{D}_{\mathbb{H}}^\tau(z)$ consists of the following Floer datum on \mathbb{H} , depending on τ :

- for $\tau \in [0, \frac{1}{2}]$, the Floer datum \mathbf{D} extends to the compactification \hat{S}_z . The pullback $(\tilde{\epsilon}_z)^*\mathbf{D}$ gives some Floer datum on D_2 . Define the Floer datum on \mathbb{H} to be any datum extending this one to all of \mathbb{H} .
- For $\tau \in [\frac{1}{2}, 1]$, the Floer datum is defined as follows:
 - one-form $\alpha_{\mathbb{H}}^\tau(\mathbf{D})$ given by $-\frac{1}{\pi}w_2(\tau) \cdot \psi(r)d\theta$.
 - any primary Hamiltonian $H^\tau(\mathbf{D})$ equal to $\frac{H \circ \psi^{w_2(\tau)}}{w_2(\tau)^2}$ on the striplike end $\epsilon_{\mathbb{H}}$.
 - any rescaling function $a_S^\tau(\mathbf{D})$ equal to $w_1(\tau)$ when restricted to $\epsilon_{\mathbb{H}}$.
 - any complex structure equal to $(\psi^{a_S^\tau(\mathbf{D})})^*J_t$ on $\epsilon_{\mathbb{H}}$.

Furthermore, we mandate that when $\tau = 1$, the Floer datum on \mathbb{H} must be the **standard unit datum** of type $w_1(1) = w_2(1)$.

Pick a smooth non-decreasing function $\kappa : [0, 1] \rightarrow [0, 1]$ that is 0 in a neighborhood of $[0, \frac{1}{2}]$ and 1 exactly at 1.

DEFINITION 6.17. Let S have rational strip-like end ϵ_z around z with associated disc-like end $\tilde{\epsilon}_z$, and suppose we have chosen a Floer datum \mathbf{D}^τ that is τ -compatible with $\tilde{\epsilon}_z$, and an associated τ -structure on \mathbb{H} , $\mathbf{D}_{\mathbb{H}}^\tau(z)$. Define the τ -damped connect sum

$$(6.49) \quad S \#_z^\tau \mathbb{H},$$

to be the surface

$$(6.50) \quad S \#_{z, \infty}^{\kappa(\tau)} \mathbb{H}$$

equipped with the Floer datum \mathbf{D}^τ on $S - \epsilon_z([0, \infty) \times [0, 1])$ and the Floer datum $\mathbf{D}_{\mathbb{H}}^\tau(z)$ on \mathbb{H} elsewhere.

By construction, this is a smooth Floer datum on $S \#_z^\tau \mathbb{H}$, and satisfies the following properties:

- For $\tau \in [0, \frac{1}{2}]$ it agrees with the compactified Floer datum $\hat{\mathbf{D}}^\tau$ on \hat{S}_z .
- For $\tau = 1$, it is the nodal connect sum

$$(6.51) \quad S \#^1 \mathbb{H}$$

where \mathbb{H} is equipped with the **standard unit datum** of type $w_1(\tau) = w_2(\tau)$, and S has some Floer datum \mathbf{D}^1 that is genuinely compatible with S, z, ϵ_z in the usual sense.

REMARK 6.6. We should note that any intermediate damped connect sum with a copy of \mathbb{H} for our choices of standard strip-like ends (6.39) is conformally equivalent to the forgetful map. All that changes as the damped connect sum parameter approaches 1 is that the standard unit Floer datum is rescaled and shrunk into a smaller and smaller neighborhood of the marked point, until eventually at time 1 it is forced to break off. Despite this, it is useful sometimes to visualize the process as a topological connect sum.

6.8. Abstract moduli spaces and operations.

DEFINITION 6.18. The **moduli space of discs** with d marked points, $F \subset [d]$ forgotten points, and $H \subset [d] - F$ **homotopy units**

$$(6.52) \quad \mathfrak{H}^{d, F, H}$$

is exactly the moduli space of discs \mathcal{R}^d , with points in F or H labeled as belonging to F or H times a copy of $[0, 1]$ for each element of H :

$$(6.53) \quad \mathfrak{H}^{d, F, H} \simeq \mathcal{R}^d \times [0, 1]^{|H|}.$$

When $H = \emptyset$, we define $\mathfrak{H}^{d,F,\emptyset} = \mathcal{R}^{d,F}$.

We think of a point in this moduli space as a pair $(S, \vec{v} = (v_1, \dots, v_{|H|}))$. We associate the i th copy of the interval to the i th ordered point in H in the following sense: Suppose H is ordered $\{p_{n_1}, \dots, p_{n_{|H|}}\}$. Then, for each element of H , there are **endpoint maps**

$$(6.54) \quad \pi_{p_{n_i}}^1 : \mathfrak{H}^{d,F,H}|_{v_i=1} \longrightarrow \mathfrak{H}^{d,F,H-\{p_{n_i}\}}$$

$$(6.55) \quad \pi_{p_{n_i}}^0 : \mathfrak{H}^{d,F,H}|_{v_i=0} \longrightarrow \mathfrak{H}^{d,F+\{p_{n_i}\},H-\{p_{n_i}\}}$$

defined as follows: given an element (S, \vec{v}) $\pi_{p_{n_i}}^1$ removes the label H from the point p_{n_i} in S , and projects \vec{v} away from the i th component (which is 1). $\pi_{p_{n_i}}^0$ removes the label of H but assigns the label of F to p_{n_i} , and projects \vec{v} away from the i th component (which is 0).

DEFINITION 6.19. *The moduli space of \mathfrak{S} -glued pairs of discs with (k, l) marked points, \mathfrak{T} point identifications, $F_1, F_2 \subset ([k], [l])$ forgotten points, and $H_1, H_2 \subset ([k], [l])$ homotopy units, denoted*

$$(6.56) \quad \mathfrak{S}, \mathfrak{T} \mathfrak{H}_{k,l}^{F_1, F_2, H_1, H_2}$$

is exactly the moduli space $\mathfrak{S}, \mathfrak{T} \mathcal{R}_{k,l}$ with points in F_1, F_2, H_1, H_2 labeled accordingly, times a copy of $[0, 1]$ for each element of H_1 or H_2 :

$$(6.57) \quad \mathfrak{S}, \mathfrak{T} \mathfrak{H}_{k,l}^{F_1, F_2, H_1, H_2} \simeq_{\mathfrak{S}, \mathfrak{T}} \mathcal{R}_{k,l} \times [0, 1]^{|H_1|} \times [0, 1]^{|H_2|}.$$

As with forgotten marked points, we have the following constraints:

- F_1, H_1 and F_2, H_2 are disjoint subsets of the left and right identified points respectively. Namely,

$$(6.58) \quad F_i, H_i \subset \pi_i(\mathfrak{T}), F_i \cap H_i = \emptyset$$

where π_i is projection onto the i th component.

- F_1, H_1 and F_2, H_2 are not associated to a boundary identification, i.e.

$$(6.59) \quad (F_i \cup H_i) \cap \pi_i(p(\mathfrak{S})) = \emptyset$$

- F_1, H_1 and F_2, H_2 do not contain both the left and right points of any identification, i.e.

$$(6.60) \quad ((F_1 \cup H_1) \times (F_2 \cup H_2)) \cap \mathfrak{T} = \emptyset$$

We think of a point of $\mathfrak{S}, \mathfrak{T} \mathfrak{H}_{k,l}^{F_1, F_2, H_1, H_2}$ as a tuple

$$(6.61) \quad (P, \vec{v}, \vec{w}).$$

Suppose $H_1, H_2 = \{p_{n_1}, \dots, p_{n_{|H_1|}}\}, \{p_{m_1}, \dots, p_{m_{|H_2|}}\}$. For any point $p_{n_i} \in H_1$ or $p_{m_i} \in H_2$, there are analogously defined **endpoint maps**

$$(6.62) \quad \pi_{L, p_{n_i}}^1 : \mathfrak{S}, \mathfrak{T} \mathfrak{H}_{k,l}^{F_1, F_2, H_1, H_2} \longrightarrow_{\mathfrak{S}, \mathfrak{T}} \mathfrak{H}_{k,l}^{F_1, F_2, H_1 - \{p_{n_i}\}, H_2}$$

$$(6.63) \quad \pi_{L, p_{n_i}}^0 : \mathfrak{S}, \mathfrak{T} \mathfrak{H}_{k,l}^{F_1, F_2, H_1, H_2} \longrightarrow_{\mathfrak{S}, \mathfrak{T}} \mathfrak{H}_{k,l}^{F_1 + \{p_{n_i}\}, F_2, H_1 - \{p_{n_i}\}, H_2}$$

$$(6.64) \quad \pi_{R, p_{m_i}}^1 : \mathfrak{S}, \mathfrak{T} \mathfrak{H}_{k,l}^{F_1, F_2, H_1, H_2} \longrightarrow_{\mathfrak{S}, \mathfrak{T}} \mathfrak{H}_{k,l}^{F_1, F_2 + \{p_{m_i}\}, H_1, H_2 - \{p_{m_i}\}}$$

$$(6.65) \quad \pi_{R, p_{m_i}}^0 : \mathfrak{S}, \mathfrak{T} \mathfrak{H}_{k,l}^{F_1, F_2, H_1, H_2} \longrightarrow_{\mathfrak{S}, \mathfrak{T}} \mathfrak{H}_{k,l}^{F_1, F_2, H_1, H_2 - \{p_{m_i}\}},$$

which change the labelings of P , and apply a projection map to (\vec{v}, \vec{w}) in the following way: for $\pi_{L, p_{n_i}}^b$: given a point (P, \vec{v}, \vec{w}) , remove the point p_{n_i} from the set H_1 , and add it to F_1 if $b = 0$. Also, project \vec{v} away from the i th factor and do nothing to \vec{w} . For $\pi_{R, p_{m_j}}^b$: given a point (P, \vec{v}, \vec{w}) , remove the point p_{m_j} from the set H_2 , and add it to F_2 if $b = 0$. Also, project \vec{w} away from the j th factor and do nothing to \vec{v} .

As before, there are **forgetful maps**

$$(6.66) \quad \mathfrak{F}_I : \mathfrak{H}^{d,F,H} \longrightarrow \mathfrak{H}^{d-|I|,F',H'}$$

$$(6.67) \quad \mathfrak{F}_{I_1, I_2} : \mathfrak{S}, \mathfrak{T} \mathfrak{H}_{k,l}^{F_1, F_2, H_1, H_2} \longrightarrow_{\mathfrak{S}', \mathfrak{T}'} \mathfrak{H}_{k-|I_1|, l-|I_2|}^{F'_1, F'_2, H'_1, H'_2}$$

for $I \subset F$ or $I_1, I_2 \subset F_1, F_2$. F'_1 and F'_2 are F_1 and F_2 sans I_1 and I_2 , reindexed appropriately, and H'_1 and H'_2 are just H_1 and H_2 reindexed. On the $[0, 1]^{|H_i|}$ components, the forgetful maps are the identity.

DEFINITION 6.20. Fix some very small number $\epsilon \ll 1$. let (S, \vec{v}) denote an element of the moduli space $\mathfrak{H}^{d,F,H}$. This element is said to be **$\mathbf{h}(\epsilon)$ -semistable** if

$$(6.68) \quad d - |F| - |H| + \#\{j | v_j > \epsilon\} = 1.$$

It is said to be **$\mathbf{h}(\epsilon)$ -stable** if the equality above is replaced by the strict inequality $>$.

Similarly, let (P, \vec{v}, \vec{w}) denote an element of the moduli space $\mathfrak{S}, \mathfrak{T} \mathfrak{H}_{k,l}^{F_1, F_2, H_1, H_2}$. This element is said to be **$\mathbf{h}(\epsilon)$ -semistable** if

$$(6.69) \quad \begin{aligned} k - |F_1| - |H_1| + \#\{j | v_j > \epsilon\} &= 1 \\ l - |F_2| - |H_2| + \#\{k | w_k > \epsilon\} &= 1, \end{aligned}$$

and **$\mathbf{h}(\epsilon)$ -stable** if the equalities above are replaced by inequalities \geq , with one of the inequalities being strict.

The Deligne-Mumford compactifications

$$(6.70) \quad \overline{\mathfrak{H}}^{d,F,H}$$

$$(6.71) \quad \mathfrak{S}, \mathfrak{T} \overline{\mathfrak{H}}_{k,l}^{F_1, F_2, H_1, H_2}$$

exist, equal as abstract spaces to the product of the compactifications

$$(6.72) \quad \overline{\mathcal{R}}^{d,F} \times [0, 1]^{|H|}$$

$$(6.73) \quad \mathfrak{S}, \mathfrak{T} \overline{\mathcal{R}}_{k,l}^{F_1, F_2} \times [0, 1]^{|H_1| + |H_2|}$$

respectively. The codimension 1 boundaries of these spaces are given by the codimension 1 boundary of the various underlying spaces of discs, along with restrictions to various endpoints.

$$(6.74) \quad \partial^1 \overline{\mathfrak{H}}^{d,F,H} = (\partial^1 \overline{\mathcal{R}}^{d,F}) \times [0, 1]^H \cup \coprod \overline{\mathcal{R}}^{d,F} \times [0, 1]^i \times \{0, 1\} \times [0, 1]^{|H| - i - 1}$$

$$(6.75) \quad \begin{aligned} \partial^1 \mathfrak{S}, \mathfrak{T} \overline{\mathfrak{H}}_{k,l}^{F_1, F_2, H_1, H_2} &= \partial^1 (\mathfrak{S}, \mathfrak{T} \overline{\mathcal{R}}_{k,l}^{F_1, F_2}) \times [0, 1]^{|H_1| + |H_2|} \\ &\cup \coprod (\mathfrak{S}, \mathfrak{T} \overline{\mathcal{R}}_{k,l}^{F_1, F_2}) \times [0, 1]^i \times \{0, 1\} \times [0, 1]^{|H_1| + |H_2| - i - 1} \end{aligned}$$

In a manner identical to the previous section, in the **\mathbf{f} -stable** range (which is independent of H or H_1, H_2), the maximal forgetful map extends to a map on compactifications:

$$(6.76) \quad \mathfrak{F}_{max} : \overline{\mathfrak{H}}^{d,F,H} \longrightarrow \overline{\mathfrak{H}}^{d-|F|, \emptyset, H'}$$

$$(6.77) \quad \mathcal{F}_{max} : \mathfrak{S}, \mathfrak{T} \overline{\mathfrak{H}}_{k,l}^{F_1, F_2, H_1, H_2} \longrightarrow \mathfrak{S}', \mathfrak{T}' \overline{\mathfrak{H}}_{k-|F_1|, l-|F_2|}^{\emptyset, \emptyset, H'_1, H'_2}$$

In what follows, we will only construct Floer data for glued pairs of discs—though the case for a single disc is identical (and in fact simpler).

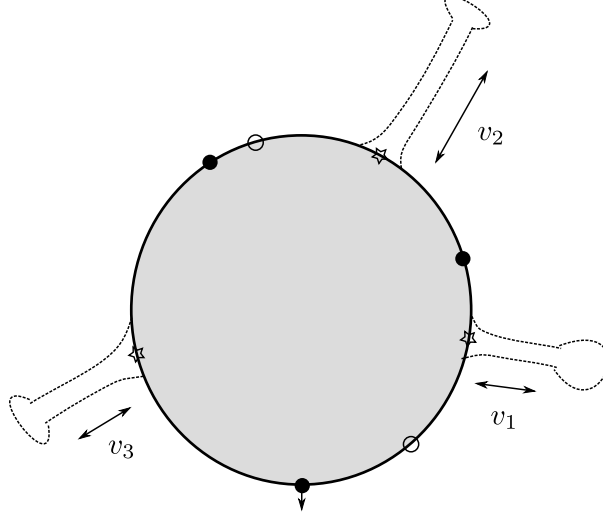
DEFINITION 6.21. A **Floer datum** for a pair of glued discs with homotopy units and forgotten points (P, \vec{v}, \vec{w}) is a Floer datum for the reduced gluing $\pi_{\mathfrak{S}}(\mathcal{F}_{max}(P, \vec{v}, \vec{w}))$ in the usual sense, with the following exceptions:

- For boundary point $p_{n_i} \in H_1$, thought of as a point in $\pi_{\mathfrak{S}}(P)$, the Floer datum only needs to be **v_i -partially compatible** with the associated strip-like end $\epsilon_{p_{n_i}}$, in the sense of Definition 6.15.
- Similarly, for boundary point $p_{m_j} \in H_2$, thought of as a point in the gluing $\pi_{\mathfrak{S}}(P)$, the Floer datum only needs to be **w_j -partially compatible** with the striplike end $\epsilon_{p_{m_j}}$.

We additionally fix, for each element of H_1 and H_2 , a copy $(\mathbb{H}, \epsilon_{\mathbb{H}})$. Call $\mathbb{H}_{p_{n_i}}$ and $\mathbb{H}_{p_{m_j}}$ the copies of \mathbb{H} corresponding to points $p_{n_i} \in H_1$ and $p_{m_j} \in H_2$ respectively. Then, a Floer datum also consists of a choice of **associated v_i and w_j structures** on $\mathbb{H}_{p_{n_i}}$ and $\mathbb{H}_{p_{m_j}}$ for p_{n_i} and p_{m_j} respectively, in the sense of Definition 6.16.

DEFINITION 6.22. A **universal and conformally consistent choice of Floer data for glued pairs of discs with homotopy units** is a choice $\mathbf{D}_{(P, \vec{v}, \vec{w})}$, for every boundary identification \mathfrak{S} and compatible sequential point identification \mathfrak{T} , and every representative (P, \vec{v}, \vec{w}) of $\mathfrak{S}, \mathfrak{T} \overline{\mathfrak{H}}_{k,l}^{F_1, F_2, H_1, H_2}$, varying smoothly over

FIGURE 13. A single disc with forgotten points (marked with hollow circles) and homotopy units (marked with stars and dotted connect sums). The connect sums should be thought of simply as a schematic picture; really the conformal structure on the disc stays the same.



this space, whose restriction to a boundary stratum is conformally equivalent to a Floer datum coming from lower dimensional moduli spaces. Moreover, Floer data agree to infinite order at the boundary stratum with the Floer datum obtained by gluing. Finally, we require that

- **(forgotten points are forgettable)** In the $\mathbf{h}(\epsilon)$ -stable range, the choice of Floer datum only depends on the reduced surface $\mathcal{F}_{\max}(P, \vec{v}, \vec{w})$. In the $\mathbf{h}(\epsilon)$ -semistable range, the Floer datum agrees with the translation-invariant Floer datum on the strip.
- **(0 endpoint is forgetting)** In the $\mathbf{h}(\epsilon)$ -stable range, if $v_i = 0$ or $w_j = 0$, then after forgetting the copy of \mathbb{H} corresponding to p_{n_i} or p_{m_j} respectively, the Floer datum should be isomorphic to the Floer datum on $\pi_{L, p_{n_i}}^0(S, \vec{v}, \vec{w})$ or $\pi_{R, p_{m_j}}^0(S, \vec{v}, \vec{w})$ respectively. In the $\mathbf{h}(\epsilon)$ -semistable case, the Floer datum should be isomorphic to the translation invariant Floer datum on the respective surface.
- **(1 endpoint is gluing in a unit)** if $v_i = 1$ or $w_j = 1$, then $\mathbb{H}_{p_{n_i}}$ or $\mathbb{H}_{p_{m_j}}$ should have the standard unit datum Floer data, and the Floer datum on the main component should be isomorphic to a Floer datum on $\pi_{L, p_{n_i}}^1(P, \vec{v}, \vec{w})$ or $\pi_{R, p_{m_j}}^1(P, \vec{v}, \vec{w})$ respectively.

PROPOSITION 6.8. *There exists a universal and conformally consistent choice of Floer data for glued pairs of discs with homotopy units.*

PROOF. One proceeds inductively on the number of homotopy units. Suppose that we have universally and conformally consistently chosen Floer data for $|H_1| + |H_2| \leq k$ and Floer data for glued pairs of discs with at least $r + s$ marked points, with homotopy units such that $|H_1| + |H_2| = k + 1$. Using the endpoint constraints described above, we have already described constraints on our Floer data on the endpoints, and codimension-1 boundary strata, so we pick some Floer datum extending these cases. Recall that this is possible because all of the spaces of choices are contractible. \square

REMARK 6.7. *Notice that the notion of $\mathbf{h}(\epsilon)$ -stability depends in some cases on the chosen point in the moduli space. For example, an element (S, \vec{v}) of $\mathfrak{H}^{d, \{1, \dots, d\}, \emptyset}$ is only $\mathbf{h}(\epsilon)$ -stable if at least two of the components of \vec{v} are greater than ϵ . It will not be possible to consistently inherit Floer data at zero-endpoints from the forgetful map when all (or all but one) \vec{v} equal to zero. Thus we are forced to turn off the notion of stability in a neighborhood of this case.*

DEFINITION 6.23. *Let (P, \vec{v}, \vec{w}) be a pair of glued discs with H_1, H_2 homotopy units, and suppose we have fixed a Floer datum \mathbf{D} for $T = (P, \vec{v}, \vec{w})$. Then the **associated homotopy-unit surface**, denoted*

$$(6.78) \quad \mathbf{h}(T),$$

is the **iterated damped connect sum**

$$(6.79) \quad \mathfrak{h}(T) := \pi_{\mathfrak{S}}(\mathfrak{F}_{max}(P)) \sharp_{p_{n_1}}^{v_1} \mathbb{H}_{p_{n_1}} \cdots \sharp_{p_{n|H_1|}}^{v|H_1|} \mathbb{H}_{p_{n|H_1|}} \sharp_{p_{m_1}}^{w_1} \mathbb{H}_{p_{m_1}} \cdots \sharp_{p_{m|H_2|}}^{w|H_2|} \mathbb{H}_{p_{m|H_2|}}.$$

This is a (potentially nodal) surface with associated Floer data.

DEFINITION 6.24. An **admissible Lagrangian labeling** for pairs of glued discs with homotopy units and forgotten point is a labeling in the usual sense, satisfying the conditions that labelings before and after H points and F points must coincide.

The admissibility condition implies that there is an **induced labeling** on the **associated homotopy-unit surface**.

Now, suppose we have fixed a universal and consistent choice of Floer data for homotopy units. Consider a compact submanifold with corners of dimension d

$$(6.80) \quad \overline{\mathcal{E}}^d \hookrightarrow \mathfrak{S}, \mathfrak{T} \mathcal{H}_{k,l}^{F_1, F_2, H_1, H_2}.$$

with an admissible Lagrangian labeling \vec{L} . In the usual fashion, fix input and output chords $\vec{x}_{in}, \vec{x}_{out}$ and orbits $\vec{y}_{in}, \vec{y}_{out}$ for the induced marked points of the associated homotopy-unit surface $\mathfrak{h}(P, \vec{v}, \vec{w})$. Define

$$(6.81) \quad \overline{\mathcal{E}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$$

to be the space of maps

$$(6.82) \quad \{(S, u) | S \in \overline{\mathcal{E}}^d, u : \mathfrak{h}(S) \rightarrow M\}$$

satisfying Floer's equation with respect to the Floer datum and asymptotic and boundary conditions specified by the Lagrangian labeling \vec{L} and asymptotic conditions $(\vec{x}_{out}, \vec{y}_{out}, \vec{x}_{in}, \vec{y}_{in})$.

As before, $h(\mathfrak{S}, k, l)$ denote the number of boundary components of any resulting surface $\mathfrak{h}(P)$.

LEMMA 6.2. The moduli spaces $\overline{\mathcal{E}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ are compact, and empty for all but finitely many $(\vec{x}_{out}, \vec{y}_{out})$ given fixed inputs $(\vec{x}_{in}, \vec{y}_{in})$. For generically chosen Floer data, they form smooth manifolds of dimension

$$(6.83) \quad \begin{aligned} \dim \overline{\mathcal{E}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in}) := & \sum_{x_- \in \vec{x}_{out}} \deg(x_-) + \sum_{y_- \in \vec{y}_{out}} \deg(y_-) \\ & + (2 - h(\mathfrak{S}, k, l) - |\vec{x}_{out}| - 2|\vec{y}_{out}|)n + d - \sum_{x_+ \in \vec{x}_{in}} \deg(x_+) - \sum_{y_+ \in \vec{y}_{in}} \deg(y_+). \end{aligned}$$

PROOF. The usual transversality arguments, dimension calculation, and compactness results apply. \square

In the usual fashion, when the dimension of the spaces $\overline{\mathcal{E}}^d(\vec{x}_{out}, \vec{y}_{out}; \vec{x}_{in}, \vec{y}_{in})$ are zero, we use orientation lines to count (with signs) the number of points in such spaces, and associate operations

$$(6.84) \quad (-1)^{\vec{t}} \mathbf{I}_{\overline{\mathcal{E}}^d}$$

from the tensor product of wrapped Floer complexes and symplectic cochain complexes where $\vec{x}_{in}, \vec{y}_{in}$ reside to the tensor product of the complexes where $\vec{x}_{out}, \vec{y}_{out}$ reside, where \vec{t} is a chosen sign twisting datum.

An interesting source of submanifolds for operations comes from the entire moduli spaces

$$(6.85) \quad \mathfrak{S}, \mathfrak{T} \mathcal{H}_{k,l}^{F_1, F_2, H_1, H_2}$$

for an initial sequential point identification \mathfrak{T} .

6.9. New operations. Up until now, we have been somewhat imprecise when specifying correspondences between inputs and asymptotic boundary conditions on moduli spaces associated with operations. Let us fix some notation for a specific class of moduli spaces.

Let \mathfrak{S} be a boundary identification, and let \mathfrak{T} be an initial sequential boundary identification that is compatible with \mathfrak{S} ; say it is

$$(6.86) \quad \mathfrak{T} = \{(1, 1), (2, 2), \dots, (r, r)\}$$

We previously defined an operation $\mathbf{G}_{\mathfrak{S}, \mathfrak{T}}$ corresponding to the entire moduli space

$$(6.87) \quad \mathfrak{S}, \mathfrak{T} \overline{\mathcal{R}}_{k, l}.$$

Let us be precise about inputs. Given *boundary marked points* $z_1, \dots, z_k, z'_1, \dots, z'_l$ on each factor of our pair of discs, if $i \leq r$, define

$$(6.88) \quad g_{\mathfrak{S}}(z_i, z'_i)$$

to be the image of the pair of identified points under the gluing $\pi_{\mathfrak{S}}$. The possibilities are

- a **pair of boundary input points** $(\tilde{z}_i, \tilde{z}'_i)$ if z_i, z'_i were not adjacent to a boundary identification;
- a **single boundary input point** $\tilde{z}_{i, i'}$ if z_i, z'_i were adjacent to a single boundary identification; or
- a **single interior input point** $\tilde{y}_{i, i'}$ if z_i, z'_i were adjacent to two boundary identifications.

Denote by

$$(6.89) \quad g_{\mathfrak{S}}(z_j), g_{\mathfrak{S}}(z'_j)$$

the images of non-identified points under the gluing $\pi_{\mathfrak{S}}$. Then, the associated operation takes the form

$$(6.90) \quad \mathbf{G}_{\mathfrak{S}, \mathfrak{T}}((\bar{x}_1, \dots, \bar{x}_r), (x_{r+1}, \dots, x_k), (x'_{r+1}, \dots, x'_l)),$$

where \bar{x}_i is an asymptotic condition of the same basic type as $g_{\mathfrak{S}}(z_i, z'_i)$, x_j is a boundary asymptotic condition corresponding to $g_{\mathfrak{S}}(z_j)$, and x'_j is a boundary asymptotic condition associated to $g_{\mathfrak{S}}(z'_j)$. This operation returns a sum of boundary asymptotic condition of the same type as $g_{\mathfrak{S}}(z_{out}, z'_{out})$, the gluing of the outputs.

To be even a bit more precise, let us move now to the operations of the above form arising in \mathcal{W}^2 . Given a tuple of Lagrangians $\vec{X} = X_1, \dots, X_d$ in $\text{ob } \mathcal{W}^2$, and morphisms

$$(6.91) \quad x_i \in \text{hom}(X_i, X_{i+1}),$$

identified via the correspondences

$$(6.92) \quad x_i \leftrightarrow \hat{x}_i$$

with a boundary asymptotic condition, pair of boundary asymptotic conditions, or interior asymptotic condition respectively, described in (5.39) - (5.42). $\mu^d(x_d, \dots, x_1)$ is by definition the labeled operation

$$(6.93) \quad \mathbf{G}_{\mathfrak{S}(\vec{X}), \mathfrak{T}}(\hat{x}_1, \dots, \hat{x}_d)$$

in the sense of above, where we are implicitly composing with the reverse identification

$$(6.94) \quad \hat{x} \leftrightarrow x$$

to obtain the correct output, and using the usual sequential sign twisting datum $\vec{t}_d = (1, \dots, d)$. This is sensible because the boundary asymptotic type of the input \hat{x}_i is compatible with the type of the glued marked point $g_{\mathfrak{S}(\vec{X})}(z_i, z'_i)$ by construction.

With this notation in place, let us now incorporate homotopy units and forgotten points. Define the A_{∞} category

$$(6.95) \quad \tilde{\mathcal{W}}^2$$

to have the same objects as \mathcal{W}^2 . Its morphisms will be identical to \mathcal{W}^2 as graded vector spaces, except for each L , it also contains the following formal generators:

$$(6.96) \quad f_L \otimes x, e_L^+ \otimes x \in \text{hom}_{\tilde{\mathcal{W}}^2}(L \times L_j, L \times L_k) \text{ for all } x \in CW^*(L_k, L_j)$$

$$(6.97) \quad x \otimes f_L, x \otimes e_L^+ \in \text{hom}_{\tilde{\mathcal{W}}^2}(L_j \times L, L_k \times L) \text{ for all } x \in CW^*(L_j, L_k).$$

The degrees of these generators are

$$(6.98) \quad \deg(f_L \otimes x) = \deg(x \otimes f_L) = \deg(x) - 1.$$

$$(6.99) \quad \deg(e_L^+ \otimes x) = \deg(x \otimes e_L^+) = \deg(x),$$

i.e. f_L and e_L^+ should be thought of as having degrees -1 and 0 respectively. Denote the generators of the morphism space between X and X' in \tilde{W}^2 by $\tilde{\chi}(X, X')$. The operations on \tilde{W}^2 are as follows: Fix a label-set $\vec{X} = X_0, \dots, X_d$. As in §5.2, there is an associated boundary identification

$$(6.100) \quad \mathfrak{S}(\vec{X}) = \{(i, i) | X_i = \Delta.\}$$

Now, let x_1, \dots, x_d be a sequence of asymptotic boundary conditions, i.e. $x_i \in \tilde{\chi}(X_{i-1}, X_i)$. Let

$$(6.101) \quad F_1, F_2, H_1, H_2 \subset \{1, \dots, d\}$$

denote the subset of these of the form $e_L^+ \otimes x$, $x \otimes e_L^+$, $f_L \otimes x$, and $x \otimes f_L$ respectively. By construction we have that

$$(6.102) \quad \begin{aligned} (F_1 \cup H_1) \cap (F_2 \cup H_2) &= \emptyset \\ F_i \cup H_i &= \emptyset. \end{aligned}$$

Then, define

$$(6.103) \quad \mu^d(x_d, \dots, x_1)$$

to be the operation controlled by the moduli space

$$(6.104) \quad \mathfrak{S}(\vec{X}), \mathfrak{T}_{d,d}^{\overline{F_1, F_2, H_1, H_2}},$$

with labeling induced by the labeling of \vec{X} as in Section 5.2 as follows: If the k th lagrangian X_k was labeled $L_i \times L_j$, then in the gluing $\pi_{\mathfrak{S}}(P)$, the left image of $\partial^k S$ will be labeled L_i and the right image $\bar{\partial}^k S$ will be labeled L_j . If $\partial_k S$ was labeled Δ , then it disappears under gluing so there is nothing to label. This induces a labeling for the associated homotopy-unit surface $\mathfrak{h}(P, \vec{v}, \vec{w})$: since our labeling was by choice *admissible*, any boundary point which we forget or take damped connect sum is adjacent to boundary components with the same label.

The asymptotic conditions in the gluing

$$(6.105) \quad \mathfrak{h}(P)$$

are as follows: in the glued surface $\pi_{\mathfrak{S}}(P)$, let $g_{\mathfrak{S}}(z_i, z'_i)$ be the resulting inputs (or pair of inputs) obtained by the gluing. Then if x_i is not a formal element, one requires these inputs to be asymptotic to the associated \hat{x}_i as before. If x_i is a formal element of any form, then $g_{\mathfrak{S}}(z_i, z'_i)$ is a pair of boundary marked points $(\tilde{z}_i, \tilde{z}'_i)$.

- if x is of the form $f_L \otimes x$, then \tilde{z}_i is marked as one of the H_1 points, and disappears under the damped connect sum operation. We require the other point \tilde{z}'_i to be asymptotic to x .
- if x is of the form $x \otimes f_L$, then \tilde{z}'_i is marked as one of the H_2 points, and disappears under the damped connect sum operation. We require the other point \tilde{z}_i to be asymptotic to x .
- if x is of the form $e_L^+ \otimes x$, then \tilde{z}_i is marked as one of the F_1 points, and disappears under the forgetful map. We require the other point \tilde{z}'_i to be asymptotic to x .
- if x is of the form $x \otimes e_L^+$, then \tilde{z}'_i is marked as one of the F_2 points, and disappears under the forgetful map. We require the other point \tilde{z}_i to be asymptotic to x .

This gives rise to a well-defined operation

$$(6.106) \quad \mu^d(x_d, \dots, x_1)$$

where x_1, \dots, x_d are allowed to be formal elements—implicitly again, we are taking the output of this operation, and composing under the reverse association

$$(6.107) \quad \hat{x} \leftrightarrow x.$$

In this case, we use sign twisting datum $\vec{t}_d = (1, \dots, d)$, including the degrees and presence of formal elements.

One can check that degree of the associated operation is $2 - d$, under the choice of gradings of the formal elements (6.98) and (6.99), for the following reason: there are no Maslov type contributions of the form $\deg(f_L)$, but this is compensated for by any additional factor of the interval $[0, 1]$ in the source abstract moduli space.

As we have constructed it, this operation is only well defined for $(d, d, F_1, F_2, H_1, H_2)$ in the **f-semistable range**. Hand-declare the following operations, corresponding to the **f-unstable range**:

$$(6.108) \quad \mu_{\mathcal{W}^2}^k(x_1 \otimes e_L^+, \dots, x_k \otimes e_L^+) := \mu_{\mathcal{W}^{op}}^k(x_1, \dots, x_k) \otimes e_L^+ \\ = (-1)^* \mu_{\mathcal{W}}^k(x_k, \dots, x_1) \otimes e_L^+$$

$$(6.109) \quad \mu_{\mathcal{W}^2}^k(e_L^+ \otimes x_k, \dots, e_L^+ \otimes x_1) := e_L^+ \otimes \mu_{\mathcal{W}}^k(x_k, \dots, x_1)$$

$$(6.110) \quad \mu_{\mathcal{W}^2}^1(x \otimes e_L^+) := \mu_{\mathcal{W}}^1(x) \otimes e_L^+$$

$$(6.111) \quad \mu^1(e_L^+ \otimes x) := e_L^+ \otimes \mu_{\mathcal{W}}^1(x)$$

$$(6.112) \quad \mu_{\mathcal{W}^2}^1(f_L \otimes x) := (e_L^+ - e_L) \otimes x \pm f_L \otimes \mu_{\mathcal{W}}^1(x)$$

$$(6.113) \quad \mu_{\mathcal{W}^2}^1(x \otimes f_L) := x \otimes (e_L^+ - e_L) \pm \mu_{\mathcal{W}}^1(x) \otimes f_L.$$

PROPOSITION 6.9. *The resulting category $\tilde{\mathcal{W}}^2$ is an A_∞ category.*

PROOF. We need to verify the A_∞ equations hold on sequences of morphisms that include the formal elements $x \otimes e^+$, $e^+ \otimes x$, $f \otimes x$, $x \otimes f$. This is mostly a consequence of the codimension-1 boundary of moduli spaces of homotopy-unit maps, although some cases (corresponding to bubbling of f -unstable components) will need to be checked by hand. Without loss of generality, we can assume that our original category contained just one Lagrangian L , so $\text{ob } \mathcal{W}^2 = \{L \times L, \Delta\}$; the multi-Lagrangians case is identical but slightly more notationally complex. The codimension 1 boundary of the abstract moduli space

$$(6.114) \quad \mathfrak{S}, \mathfrak{T}_{max} \overline{\mathfrak{H}}_{k,l}^{F_1, F_2, H_1, H_2}$$

is covered by the following strata:

- **0 and 1 endpoints**

$$(6.115) \quad \mathfrak{S}, \mathfrak{T}_{max} \overline{\mathfrak{H}}_{d,d}^{F_1, F_2, H_1, H_2} \big|_{v_i \in \{0,1\}}, \mathfrak{S}, \mathfrak{T}_{max} \overline{\mathfrak{H}}_{d,d}^{F_1, F_2, H_1, H_2} \big|_{w_j \in \{0,1\}},$$

- **nodal degenerations:**

$$(6.116) \quad \mathfrak{S}', \mathfrak{T}_{max} \overline{\mathfrak{H}}_{d',d'}^{F'_1, F'_2, H'_1, H'_2} \times \mathfrak{S}'', \mathfrak{T}_{max} \overline{\mathfrak{H}}_{d-d'+1, d-d'+1}^{F''_1, F''_2, H''_1, H''_2}$$

Here, the boundary marked points in $\mathfrak{S}', \mathfrak{T}_{max} \overline{\mathfrak{H}}_{d',d'}^{F'_1, F'_2, H'_1, H'_2}$ consist of some subsequence of length d' of $(z_1, z'_1), \dots, (z_d, z'_d)$ along with inherited F/H labels, and the boundary marked points of $\mathfrak{S}'', \mathfrak{T}_{max} \overline{\mathfrak{H}}_{d-d'+1, d-d'+1}^{F''_1, F''_2, H''_1, H''_2}$ consist of the sequence $(z_1, z'_1), \dots, (z_d, z'_d)$ where the chosen subsequence is replaced by a single new point (z_{new}, z'_{new}) (again with inherited F/H labels).

This implies that the boundary of the one-dimensional space of maps will consist of compositions of operations coming from these strata as well as various strip-breaking operations, corresponding to pre and post-composing with μ^1 in all possible ways.

By the choices we have made in our Floer datum, the 0 endpoint for a point $p_{n_i} \in H_1$ correspond to the operation of forgetting the point p_{n_i} , which changes the formal asymptotic condition from f_L to e_L^+ . The 1 end point corresponds to gluing in a geometric unit to an existing A_∞ operation, i.e. the formal condition f_L is replaced by an actual asymptotic condition e_L . In conjunction we see that the endpoint strata account for the occurrences of μ^1 for the f_L as formally defined above.

The nodal degenerations strata ensure that the associated operation is a genuine composition of the form $\mu^{d-d'+1}(\dots \mu^{d'}(\dots) \dots)$ when both components of the strata are **f-semistable**. Let us without loss of generality suppose an f -unstable component bubbles off, consisting of a subsequence of the form $(z_{i+1}, z'_{i+1}), \dots, (z_{i+d'}, z'_{i+d'})$, with all of the right factored pointed labeled as forgotten, with adjacent boundary components labeled by L . By construction such a sequence corresponds to inputs $x_{i+1} \otimes e_L^+, \dots, x_{i+d'} \otimes e_L^+$. In the induced forgetful/gluing map, the right disc consists entirely of points labeled forgotten and is thus deleted by f -stabilization. Moreover, the right input of the lower disc z'_{new} is marked as forgotten. The left disc survives, contributing a $(\mu_{\mathcal{W}}^{d'})^{op}$. We conclude that the operation associated to the top stratum is $(-1)^* \mu^{d'}(x_{i+1}, \dots, x_{i+d'}) \otimes e_L^+$, which equals $\mu^{d'}(x_{i+d'} \otimes e_L^+, \dots, x_{i+1} \otimes e_L^+)$ as desired. \square

We call the data that we have just constructed the structure of **one-sided homotopy units** for the category \mathcal{W}^2 .

PROPOSITION 6.10. *The modified category $\tilde{\mathcal{W}}^2$ is quasi-equivalent to \mathcal{W}^2 .*

PROOF. By construction, the inclusion

$$(6.117) \quad \mathcal{J} : \mathcal{W}^2 \hookrightarrow \tilde{\mathcal{W}}^2$$

is the desired quasi-isomorphism. If $\mu_{\mathcal{W}}^1(x) = 0$, the elements $e_L^+ \otimes x, x \otimes e_L^+$, which are the only potentially new elements of cohomology, are homologous to $e_L \otimes x, x \otimes e_L$. \square

In order to simplify notation, define the **total homotopy unit**

$$(6.118) \quad e^+ := \sum_{L \in \text{ob } \mathcal{W}} e_L^+,$$

thought of as an element in the semi-simple ring version of \mathcal{W} . The corresponding elements in \mathcal{W}^2 are the **total one-sided units**

$$(6.119) \quad \begin{aligned} e^+ \otimes x &:= \sum_{L \in \text{ob } \mathcal{W}} e_L^+ \otimes x \\ x \otimes e^+ &:= \sum_{L \in \text{ob } \mathcal{W}} x \otimes e_L^+ \end{aligned}$$

6.10. Shuffle identities. The technology we have introduced, and the analyses of the previous section give some morphisms involving the e^+ desirable properties. To state them, we first recall the combinatorial notion of a **shuffle**:

DEFINITION 6.25. *Let V be a graded vector space. The (k, l) **shuffle** of an ordered collections of elements $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_l\}$ is defined to be following element in the tensor algebra TV :*

$$(6.120) \quad \mathcal{S}_{k,l}(\{a_i\}, \{b_j\}) := \sum_{\sigma \in \text{shuff}(\{a_i\}, \{b_j\})} (-1)^{\text{sgn}(\sigma)} \sigma(a_1 \otimes \dots \otimes a_k \otimes b_1 \otimes \dots \otimes b_l).$$

Above, $\text{shuff}(\{a_i\}, \{b_j\})$ is the collection of permutations of the set $\{a_1, \dots, a_k, b_1, \dots, b_l\}$ that preserve the relative orderings of the a_i and b_j , σ is the corresponding permutation on the tensor algebra, and the sign $\text{sgn}(\sigma)$ is the sign of the graded permutation, i.e. the ordinary sign of the permutation plus a sign of parity the sums of degrees of elements that have been permuted past one another.

The following Proposition, essential for our forthcoming argument, is the main consequence of the technology of one-sided homotopy units.

PROPOSITION 6.11. *We have the following identities in $\tilde{\mathcal{W}}^2$:*

$$(6.121) \quad \mu_{\tilde{\mathcal{W}}^2}^{k+l}(\mathcal{S}_{k,l}(\{x_i \otimes e^+\}_{i=k}^1; \{e^+ \otimes y_j\}_{j=1}^l)) = 0, \text{ for } k, l > 0$$

$$(6.122) \quad \mu_{\tilde{\mathcal{W}}^2}^{k+l+1}(\hat{\mathbf{a}}, \mathcal{S}_{k,l}(\{x_i \otimes e^+\}_{i=k}^1; \{e^+ \otimes y_j\}_{j=1}^l)) = \mu_{\mathcal{W}}^{k+l+1}(x_1, \dots, x_k, \mathbf{a}, y_1, \dots, y_l)$$

$$(6.123) \quad \mu_{\tilde{\mathcal{W}}^2}^{k+l+1}(\mathcal{S}_{k,l}(\{x_i \otimes e^+\}_{i=k}^1; \{e^+ \otimes y_j\}_{j=1}^l), \hat{\mathbf{b}}) = \mu_{\mathcal{W}}^{k+l+1}(y_1, \dots, y_l, \mathbf{b}, x_1, \dots, x_k)$$

$$(6.124) \quad \mu_{\tilde{\mathcal{W}}^2}^{k+l+2}(\hat{\mathbf{a}}, \mathcal{S}_{k,l}(\{x_i \otimes e^+\}_{i=k}^1; \{e^+ \otimes y_j\}_{j=1}^l), \hat{\mathbf{b}}) = {}_2\mathcal{OC}(\mathbf{a}, y_1, \dots, y_l, \mathbf{b}, x_1, \dots, x_k)$$

where $\mathbf{b} \in \text{hom}(\Delta, L_i \times L_j)$ and $\mathbf{a} \in \text{hom}(L_i \times L_j, \Delta)$ respectively.

PROOF. This is the content of Propositions 6.3, 6.4, 6.5, and 6.6 except for the case of (6.121) when $k = l = 1$. In that case, we have that

$$(6.125) \quad \begin{aligned} \mu_{\tilde{\mathcal{W}}^2}^2(\mathcal{S}_{1,1}(\{x \otimes e^+\}; \{e^+ \otimes y\})) &= \mu_{\tilde{\mathcal{W}}^2}^2(x \otimes e^+, e^+ \otimes y) - \mu_{\tilde{\mathcal{W}}^2}^2(e^+ \otimes y, x \otimes e^+) \\ &= x \otimes y - x \otimes y \\ &= 0. \end{aligned}$$

\square

7. Split-resolving the diagonal

In this section, we prove the following theorem.

THEOREM 7.1. *If M is non-degenerate, the product Lagrangians $\{L_i \times L_j\}$ split-generate Δ in the category \mathcal{W}^2 .*

The proof uses a criterion for split generation discussed in Section 2.3, which we now recall. Let \mathcal{W}_{split}^2 be the full sub-category of \mathcal{W}^2 with objects given by the product Lagrangians $\{L_i \times L_j\}$. There is a natural bar complex

$$(7.1) \quad \mathcal{Y}_\Delta^r \otimes_{\mathcal{W}_{split}^2} \mathcal{Y}_\Delta^l$$

and collapse map

$$(7.2) \quad \mu : \mathcal{Y}_\Delta^r \otimes_{\mathcal{W}_{split}^2} \mathcal{Y}_\Delta^l \longrightarrow \text{hom}_{\mathcal{W}^2}(\Delta, \Delta).$$

If $[\mu]$ hits the unit element $[e] \in \text{hom}_{\mathcal{W}^2}(\Delta, \Delta) = SH^*(M)$, then we can conclude that the product Lagrangians split-generate Δ .

Because split-generation is invariant under quasi-isomorphisms, it will suffice to establish the above claim in the category

$$(7.3) \quad \tilde{\mathcal{W}}^2$$

which is quasi-isomorphic to \mathcal{W}^2 .

Define a map

$$(7.4) \quad \Gamma : {}_2\text{CC}_*(\mathcal{W}, \mathcal{W}) \longrightarrow \mathcal{Y}_\Delta^r \otimes_{\tilde{\mathcal{W}}_{split}^2} \mathcal{Y}_\Delta^l$$

as follows:

$$(7.5) \quad \Gamma : \mathbf{a} \otimes b_1 \otimes \dots \otimes b_l \otimes \mathbf{b} \otimes a_1 \otimes \dots \otimes a_k \longmapsto (-1)^{\blacksquare} \hat{\mathbf{a}} \otimes \mathcal{S}_{k,l}((a_k \otimes e^+, \dots, a_1 \otimes e^+); (e^+ \otimes b_1, \dots, e^+ \otimes b_l)) \otimes \hat{\mathbf{b}}.$$

where $\mathcal{S}_{k,l}$ is the (k, l) **shuffle product** defined in the previous section, and $\hat{\mathbf{a}}$ refers to \mathbf{a} thought of as an element of $\text{hom}(L_i \times L_j, \Delta)$ instead of $\text{hom}(L_j, L_i)$, and similarly for $\hat{\mathbf{b}}$ under the usual correspondences (5.39)-(5.42). The Koszul sign

$$(7.6) \quad \blacksquare := \sum_{j=k}^1 \left(\|a_j\| \cdot \left(\sum_{i=j-1}^1 \|a_i\| + |\mathbf{b}| \right) \right)$$

can be thought of as arising from rearranging the substrings of the Hochschild chain $\mathbf{a}, b_1, \dots, b_l$ and $\mathbf{b}, a_1, \dots, a_k$ so that they are superimposed, with the latter sequence in reverse order.

PROPOSITION 7.1. *Γ is a chain map.*

PROOF. We verify this proposition up to sign. Using Proposition 6.11, we must show that Γ intertwines the two-pointed Hochschild differential with the bar complex differential on $\tilde{\mathcal{W}}^2$. Abbreviate the shuffle product

$$(7.7) \quad \mathcal{S}_{i,j}(\{a_s \otimes e^+\}_{s=r+i}^{r+1}; \{e^+ \otimes b_t\}_{t=n+1}^{n+j})$$

by

$$(7.8) \quad \mathcal{S}(a_{r+i \rightarrow r+1}; b_{n+1 \rightarrow n+j})$$

The bar differential applied to

$$(7.9) \quad \Gamma(\mathbf{a} \otimes b_1 \otimes \dots \otimes b_l \otimes \mathbf{b} \otimes a_1 \otimes \dots \otimes a_k)$$

is the sum of the following terms (with Koszul signs described in (2.94) that are omitted):

$$(7.10) \quad \sum_{i \geq 0, j \geq 0} \mu_{\tilde{W}^2}(\hat{\mathbf{a}}, \mathcal{S}(a_{k \rightarrow k-i+1}; b_{1 \rightarrow j})) \otimes \mathcal{S}(a_{k-i \rightarrow 1}; b_{j \rightarrow l}) \otimes \hat{\mathbf{b}} \quad (\text{collapse on left})$$

$$(7.11) \quad \sum_{i \geq 0, j \geq 0} \hat{\mathbf{a}} \otimes \mathcal{S}(a_{k \rightarrow k-i+1}; b_{1 \rightarrow j}) \otimes \mu_{\tilde{W}^2}(\mathcal{S}(a_{k-i \rightarrow 1}; b_{j+1 \rightarrow l}), \hat{\mathbf{b}}) \quad (\text{collapse on right})$$

$$(7.12) \quad \sum_{i_0, i_1, j_0, j_1} \hat{\mathbf{a}} \otimes \mathcal{S}(a_{k \rightarrow k-i_0+1}; b_{1 \rightarrow j_0}) \otimes \mu_{\tilde{W}^2}(\mathcal{S}(a_{k-i_0 \rightarrow k-i_0-i_1+1}; b_{j_0+1 \rightarrow j_0+j_1})) \\ \otimes \mathcal{S}(a_{k-i_0-i_1 \rightarrow 1}; b_{j_0+j_1+1 \rightarrow l}) \otimes \hat{\mathbf{b}} \quad (\text{collapse in middle}).$$

By Proposition 6.11,

$$(7.13) \quad \mu_{\tilde{W}^2}(\hat{\mathbf{a}}, \mathcal{S}(a_{k \rightarrow k-i+1}; b_{1 \rightarrow j})) = \mu_{\tilde{W}}^{i+j+1}(a_{k-i+1}, \dots, a_k, \mathbf{a}, b_1, \dots, b_j)$$

$$(7.14) \quad \mu_{\tilde{W}^2}(\mathcal{S}(a_{k-i \rightarrow 1}; b_{j+1 \rightarrow l}), \hat{\mathbf{b}}) = \mu_{\tilde{W}}^{(k-i)+(l-j)+1}(b_{j+1}, \dots, b_l, \mathbf{b}, a_1, \dots, a_{k-i})$$

and

$$(7.15) \quad \mu_{\tilde{W}^2}(\mathcal{S}(a_{k-i_0 \rightarrow k-i_0-i_1+1}; b_{j_0+1 \rightarrow j_0+j_1})) = \begin{cases} 0 & i_1 \geq 1 \text{ and } j_1 \geq 1 \\ \mu_{\tilde{W}}^{i_1}(a_{k-i_0-i_1}, \dots, a_{k-i_0}) \otimes e^+ & j_1 = 0 \\ e^+ \otimes \mu_{\tilde{W}}^{j_1}(b_{j_0+1}, \dots, b_{j_0+j_1}) & i_1 = 0 \end{cases}$$

Putting this all together, we see that the non-zero terms above comprise exactly the terms in

$$(7.16) \quad \Gamma \circ d_{2\text{CC}}(\mathbf{a} \otimes a_1 \otimes \dots \otimes a_k \otimes \mathbf{b} \otimes b_1 \otimes \dots \otimes b_l).$$

□

The following Proposition completes the proof of Theorem 7.1.

PROPOSITION 7.2. *There is a commutative diagram of chain complexes*

$$(7.17) \quad \begin{array}{ccc} {}_2\text{CC}_*(\mathcal{W}, \mathcal{W}) & \xrightarrow{\Gamma} & \mathcal{Y}_{\Delta}^r \otimes_{\tilde{W}_{split}^2} \mathcal{Y}_{\Delta}^l \\ \downarrow {}_2\mathcal{O}\mathcal{C} & & \downarrow H^*(\mu) \\ CH^*(M) & \xrightarrow{D} & \text{hom}_{\tilde{W}^2}(\Delta, \Delta) \end{array}$$

where D is the identity map (by our definition of $\text{hom}(\Delta, \Delta)$).

PROOF. This is also a corollary of Proposition 6.11. Namely, we showed there that

$$(7.18) \quad \mu(\hat{\mathbf{a}}, \mathcal{S}_{k,l}(a_k \otimes e^+, \dots, a_1 \otimes e^+); (e^+ \otimes b_1, \dots, e^+ \otimes b_l), \hat{\mathbf{b}}) = {}_2\mathcal{O}\mathcal{C}(\hat{\mathbf{a}}, b_1, \dots, b_l, \hat{\mathbf{b}}, a_1, \dots, a_k),$$

a restatement of (7.17). □

PROOF OF THEOREM 7.1. If the map $[\mathcal{C}\mathcal{O}] : \text{HH}_*(\mathcal{W}, \mathcal{W}) \rightarrow SH^*(M)$ hits $[e]$, we conclude Thus by the existence of the diagram (7.17), $[\mu]$ hits $[e] \in HW^*(\Delta, \Delta)$. Applying Proposition 2.2, we conclude that Δ is split-generated in \tilde{W}^2 by products of objects in $\text{ob } \mathcal{W}$; hence it is in \mathcal{W}^2 as well. □

8. Some consequences

8.1. A converse result. In Section 7, we proved that if M is non-degenerate, then the product Lagrangians $L_i \times L_j$ split-generate Δ in \mathcal{W}^2 . The proof went via analyzing a homotopy commutative diagram

$$(8.1) \quad \begin{array}{ccc} {}_2\text{CC}_*(\mathcal{W}, \mathcal{W}) & \xrightarrow{\Gamma} & \mathcal{Y}_{\Delta}^r \otimes_{\tilde{W}_{split}^2} \mathcal{Y}_{\Delta}^l \\ \downarrow {}_2\mathcal{O}\mathcal{C} & & \downarrow \mu \\ CH^*(M) & \xrightarrow{D} & \text{hom}_{\tilde{W}^2}(\Delta, \Delta) \end{array}$$

where $\tilde{\mathcal{W}}_{split}^2$ was a category quasi-isomorphic to \mathcal{W}_{split}^2 , the full subcategory of product Lagrangians in \mathcal{W}^2 .

COROLLARY 8.1. *Under the same hypotheses, Γ is a quasi-isomorphism.*

PROOF. The map $H^*(\mu)$ hits the unit, so by Proposition 2.3 $H^*(\mu)$ is an isomorphism. D is an isomorphism by definition, and we then note, thanks to Theorem 1.4, that ${}_2\mathcal{OC}$ is also a quasi-isomorphism. Hence, Γ is a quasi-isomorphism. \square

REMARK 8.1. *It seems believable that the map Γ is always an isomorphism, via the existence of an explicit quasi-inverse instead of such circuitous arguments. At the time of writing we have not come up with a simple proof.*

With the technology we have established, we can also see that non-degeneracy is in fact *equivalent* to split-generation of the diagonal:

PROPOSITION 8.1. *If Δ is split-generated by product Lagrangians in \mathcal{W}^2 , then M is non-degenerate.*

PROOF. If so, then, applying Theorem 5.1, and Corollary 5.1, we conclude that \mathcal{W} is homologically smooth and the maps $\mathcal{CY}_\#$, \mathcal{CO} , and $\bar{\mu}$ are isomorphisms in the Cardy Condition diagram (1.13) (by applying Corollaries 5.1 and 2.3 respectively). Hence \mathcal{OC} is an isomorphism as well; in particular, it hits the unit. \square

In particular, Γ is once more an isomorphism.

8.2. A ring structure on Hochschild homology. We can pull back the ring structure from Hochschild cohomology to Hochschild homology. Thanks to Theorem 1.4, this can be done without passing through symplectic cohomology.

COROLLARY 8.2. *Let σ be the pre-image of the unit, and let α, β be two classes in $\mathrm{HH}_*(\mathcal{W}, \mathcal{W})$ that map to elements a and b of symplectic cohomology via \mathcal{OC} . Then, the following Hochschild homology classes are equal in homology and map to $a \cdot b$:*

$$(8.2) \quad \alpha \star^1 \beta := (\bar{\mu} \circ \mathcal{CY}_\#(\alpha)) \cap \beta$$

$$(8.3) \quad \alpha \star^2 \beta := \alpha \cap (\bar{\mu} \circ \mathcal{CY}_\#(\beta))$$

$$(8.4) \quad \alpha \star^3 \beta := ((\bar{\mu} \circ \mathcal{CY}_\#)(\alpha) * (\bar{\mu} \circ \mathcal{CY}_\#)(\beta)) \cap \sigma.$$

It is illustrative to write down an explicit expression for (8.2) in terms of operations. First, we note that for two-pointed complexes, cap-product has a very simple form

$$(8.5) \quad {}_2\mathrm{CC}_*(\mathcal{W}, \mathcal{W}) \times {}_2\mathrm{CC}^*(\mathcal{W}, \mathcal{B}) \longrightarrow {}_2\mathrm{CC}_*(\mathcal{W}, \mathcal{B})$$

$$(\alpha, \mathcal{F}) \longmapsto \mathcal{F}_\#(\alpha)$$

where $\mathcal{F}_\#$ is the pushforward operation on the tensor product $\mathcal{W}_\Delta \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{W}_\Delta$ that acts by collapsing terms around and including the first factor of \mathcal{W}_Δ (with usual Koszul reordering signs).

Now, if β is the Hochschild class represented by

$$(8.6) \quad \beta = \mathbf{a} \otimes b_1 \otimes \cdots \otimes b_t \otimes \mathbf{b} \otimes a_1 \otimes \cdots \otimes a_s$$

and α is represented by

$$(8.7) \quad \alpha = \mathbf{c} \otimes c_1 \otimes \cdots \otimes c_v \otimes \mathbf{d} \otimes d_1 \otimes \cdots \otimes d_w$$

then the formula (8.2) is, up to sign:

$$(8.8) \quad \alpha \star^1 \beta := \sum \left(\left(\mu_{\mathcal{W}}(a_{s-k''+1}, \dots, a_{s-k'}, \cdot, c_{r'+1}, \dots, c_{r''-1}, \right. \right.$$

$$\left. \mu_{\mathcal{W}}(c_{r''}, \dots, c_v \mathbf{d}, d_1, \dots, d_{w-q'}, \cdot, b_{l'+1}, \dots, b_{l'+l''}), \right.$$

$$\left. b_{l'+l''+1}, \dots, b_{l'+l''+l'''} \right)$$

$$\circ \mathcal{CY}_{k', l', q', r'}(d_{w-q'+1}, \dots, d_w, \mathbf{c}, c_1, \dots, c'_r;$$

$$a_{s-k'+1}, \dots, a_s, \mathbf{a}, b_1, \dots, b'_l)$$

$$\otimes b_{l'+l''+l''' +1} \otimes \cdots \otimes b_t \otimes \mathbf{b} \otimes a_1 \otimes \cdots \otimes a_{s-k'}.$$

REMARK 8.2. While this paper (as part of the author's thesis) was being prepared, a different model for the ring structure on symplectic cohomology, from the perspective of Legendrian surgery, was announced for Weinstein manifolds by Bourgeois, Ekholm, and Eliashberg in [BEE1]. Interestingly, their formula, in terms of a Hochschild type invariant of the Legendrian contact homology algebra of attaching cores of a handle presentation, also involves extra duality operations in Legendrian rational SFT that go beyond the Legendrian contact homology differential. It seems likely that such a surgery product formula could also be given in terms of the cocores of a plurisubharmonic function via an upside-down surgery perspective. In the special case of (8.8) that the Lagrangians in $\text{ob } \mathcal{W}$ are the cocores of a plurisubharmonic Morse function, we expect that a suitable equivalence between wrapped Fukaya and Legendrian SFT structures extends to involve the duality operations, and that the resulting formula of [BEE1] is then related to ours.

8.3. A non-commutative volume form. Assume M is non-degenerate, and let $\sigma \in \text{CC}_*(\mathcal{W}, \mathcal{W})$ be any pre-image of $[e] \in SH^*(M)$ under the map \mathcal{OC} . Because \mathcal{OC} is of degree n , σ is a degree $-n$ element. Following terminology from the introduction, call σ a **non-commutative volume form** for the wrapped Fukaya category. Our reason for this terminology is that, under the perspective in which Hochschild co-chains and chains are noncommutative analogues of (poly)vector-fields and differential forms, σ behaves like a holomorphic volume form:

COROLLARY 8.3. *Cap product with σ induces an isomorphism*

$$(8.9) \quad [\cdot \cap \sigma] : \text{HH}^*(\mathcal{W}, \mathcal{W}) \xrightarrow{\sim} \text{HH}_{*-n}(\mathcal{W}, \mathcal{W})$$

that is quasi-inverse to the geometric morphisms $\mathcal{CO} \circ \mathcal{OC}$. Thus, by Theorem 1.4, it is also quasi-inverse to $\mu_{LR} \circ \mathcal{CY}_\#$.

PROOF. We note that by the module structure compatibility of \mathcal{OC} the following holds on the level of homology:

$$(8.10) \quad \begin{aligned} \mathcal{OC}((\mathcal{CO} \circ \mathcal{OC}(x)) \cap \sigma) &= \mathcal{OC}(x) \cdot \mathcal{OC}(\sigma) \\ &= \mathcal{OC}(x) \cdot [e] \\ &= \mathcal{OC}(x). \end{aligned}$$

Since \mathcal{OC} is a homology-level isomorphism, we conclude that

$$(8.11) \quad (\mathcal{CO} \circ \mathcal{OC}(x)) \cap \sigma = x$$

as desired. \square

It is more generally true that for any (perfect) bimodule \mathcal{B} , cap product with σ induces an isomorphism

$$(8.12) \quad \cdot \cap \sigma : \text{HH}^*(\mathcal{W}, \mathcal{B}) \xrightarrow{\sim} \text{HH}_*(\mathcal{W}, \mathcal{B}).$$

This can be efficiently shown by noting that cap product with σ (using the formula from (8.5)) also induces a map, for each $K, L \in \text{ob } \mathcal{W}$:

$$(8.13) \quad \begin{aligned} \mathbf{F}_\sigma : \mathcal{W}^l(K, L) &= {}_2\text{CC}^*(\mathcal{W}, \mathcal{Y}_K^l \otimes \mathcal{Y}_L^l) \rightarrow {}_2\text{CC}_*(\mathcal{W}, \mathcal{Y}_K^l \otimes \mathcal{Y}_L^l) = (\mathcal{W}_\Delta \otimes_{\mathcal{W}} \mathcal{W}_\Delta \otimes_{\mathcal{W}} \mathcal{W}_\Delta)(K, L) \\ \phi &\mapsto \phi_\#(\sigma), \end{aligned}$$

which moreover is the first order term of a bimodule morphism

$$(8.14) \quad \mathbf{F}_\sigma : \mathcal{W}^l \rightarrow \mathcal{W}_\Delta \otimes_{\mathcal{W}} \mathcal{W}_\Delta \otimes_{\mathcal{W}} \mathcal{W}_\Delta$$

with vanishing higher order terms. Hence, we can tensor the morphism (8.14) with any bimodule \mathcal{B} , and note that there is a (strictly) commutative diagram

$$(8.15) \quad \begin{array}{ccc} \mathcal{W}^l \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{B} & \xrightarrow{\bar{\mu}} & {}_2\text{CC}^*(\mathcal{W}, \mathcal{B}) \\ \downarrow (\mathbf{F}_\sigma)_\# & & \downarrow \cdot \cap \sigma \\ (\mathcal{W}_\Delta \otimes_{\mathcal{W}} \mathcal{W}_\Delta \otimes_{\mathcal{W}} \mathcal{W}_\Delta) \otimes_{\mathcal{W}-\mathcal{W}} \mathcal{B} & \xrightarrow{(\mathcal{F}_{\Delta, \text{left}, \text{right}})_\#} & {}_2\text{CC}_*(\mathcal{W}, \mathcal{B}). \end{array}$$

Here, the right vertical arrow is cap product with σ again using the formula (8.5), and $\mathcal{F}_{\Delta, \text{left}, \text{right}}$ and $\bar{\mu}$ are as in (2.50) and (2.219); commutativity follows from unpacking the definitions. Moreover, as shown in

Proposition 2.1 and Corollary 2.3, the bottom horizontal map is always a quasi-isomorphism and the top horizontal map is a quasi-isomorphism for \mathcal{B} perfect. Hence, the right vertical map is a quasi-isomorphism for all \mathcal{B} by reduction to the following Lemma:

LEMMA 8.1. *Capping with σ followed by collapsing by $\mathcal{F}_{\Delta, \text{left}, \text{right}}$ induces a quasi-isomorphism of bi-modules, quasi-inverse to $\mathcal{C}\mathcal{Y}$*

$$(8.16) \quad \mathcal{F}_{\Delta, \text{left}, \text{right}} \circ \mathbf{F}_\sigma : \mathcal{W}^! \longrightarrow \mathcal{W}_\Delta$$

(Since $\mathcal{F}_{\Delta, \text{left}, \text{right}}$ is a quasi-isomorphism by Proposition 2.1, we conclude that \mathbf{F}_σ is too).

PROOF. We consider a special case of the Floer-theoretic operations constructed in (4.95) (setting $s = t = 0$):

$$(8.17) \quad \tilde{\mathbf{A}} := \oplus_{k,l} \mathbf{A}_{k,l;0,0} : {}_2\text{CC}_*(\mathcal{W}, \mathcal{W}) \otimes_{\mathbb{K}} \mathcal{W}_\Delta \rightarrow \mathcal{W}_\Delta.$$

The study of the codimension 1 boundary of the associated moduli space of maps performed (briefly) in Proposition 4.5 (and applied there to the adjoint map \mathbf{A} defined in (4.97)) imply that $\tilde{\mathbf{A}}$ is the chain homotopy making the following diagram homotopy commutative (up to an overall sign of $(-1)^{n(n+1)/2}$, which we disregard for this Lemma):

$$(8.18) \quad \begin{array}{ccc} {}_2\text{CC}_*(\mathcal{W}, \mathcal{W}) \otimes_{\mathbb{K}} \mathcal{W}_\Delta & \xrightarrow{\cap^{\circ(id, \mathcal{C}\mathcal{Y}^{0|1|0})}} & \mathcal{W}_\Delta \otimes_{\mathcal{W}} \mathcal{W}_\Delta \otimes_{\mathcal{W}} \mathcal{W}_\Delta \\ \downarrow {}_2\mathcal{O}\mathcal{C} \otimes id & & \downarrow \mu_{LR}^{0|1|0} \\ SC^*(M) \otimes \mathcal{W}_\Delta & \xrightarrow{{}_2\mathcal{C}\mathcal{O}_{0,0}(-)(-)} & \mathcal{W}_\Delta \end{array}$$

Here μ_{LR} is the morphism constructed in (4.34) and shown in Proposition 4.3 to be homotopic to $\mathcal{F}_{\Delta, \text{left}, \text{right}}$. In particular, using the fact that $[\mu_{LR}^{0|1|0}] = [\mathcal{F}_{\Delta, \text{left}, \text{right}}^{0|1|0}]$ and taking $[\sigma]$ to be the element hitting 1 in $SH^*(M)$, we have that, for any y ,

$$(8.19) \quad \begin{aligned} [\mathcal{F}_{\Delta, \text{left}, \text{right}}^{0|1|0}]([\sigma] \cap [\mathcal{C}\mathcal{Y}^{0|1|0}(y)]) &= {}_2\mathcal{C}\mathcal{O}_{0,0}([{}_2\mathcal{O}\mathcal{C}(\sigma)])(y) \\ &= {}_2\mathcal{C}\mathcal{O}_{0,0}(1)(y) \\ &= y. \end{aligned}$$

where the first equality used the hypothesis that $[{}_2\mathcal{O}\mathcal{C}(\sigma)] = 1$, and the second the fact that ${}_2\mathcal{C}\mathcal{O}$ is unital, sending $1 \in SH^*(M)$ to the identity morphism $\mathcal{W}_\Delta \rightarrow \mathcal{W}_\Delta$. This establishes that $[\mathcal{F}_{\Delta, \text{left}, \text{right}}^{0|1|0}] \circ [\mathbf{F}_\sigma^{0|1|0}]$ is a one-sided inverse to $[\mathcal{C}\mathcal{Y}^{0|1|0}]$, which establishes the Lemma, as $[\mathcal{C}\mathcal{Y}^{0|1|0}]$ was shown to be an isomorphism in Corollary 5.1. \square

This last Lemma is relevant to the idea that σ is a *non-degenerate co-trace* for the wrapped Fukaya category in the sense of [L3, Rmk. 4.2.17] or [KV]. This will be explored further elsewhere.

Appendix A. Abouzaid's generation criterion is preserved under products

The goal of this Appendix is to use the technology of **one-factor homotopy units**, developed in §6 to give a detailed sketch of the following Theorem, which shows that Abouzaid's generation criterion [A1], called *non-degeneracy* in the language of this paper, is preserved under products:

THEOREM A.1. *Let M and N be Liouville manifolds, and denote by $\mathcal{W}(M)$ and $\mathcal{W}(N)$ finite full subcategories of their wrapped Fukaya category. Denote by $\mathcal{W}_{\text{split}}(M \times N)$ the **split data** wrapped Fukaya category of the product whose objects are products of objects in $\mathcal{W}(M)$ and $\mathcal{W}(N)$. Then if the collections $\mathcal{W}(M)$ and $\mathcal{W}(N)$ are non-degenerate, so is $\mathcal{W}_{\text{split}}(M \times N)$.*

REMARK A.1. In their work on HMS for T^4 , Abouzaid-Smith [AbSm] proved a statement formally similar to Theorem A.1 for compact Calabi-Yau manifolds, under two additional hypotheses on the split-generating collections $\mathcal{A}(X)$, for $X = M, N$: first $\mathcal{A}(X)$ must be homologically smooth, and the closed-open maps $QH^*(X) \rightarrow HH^*(\mathcal{A}(X), \mathcal{A}(X))$ should be isomorphisms. Strictly speaking, [AbSm] consider $M = N$, but the difference is minor.

Since [AbSm] was written, work on generation criteria [A1] [AFO⁺] has shown that $\mathcal{A}(X)$ satisfies the Abouzaid’s non-degeneracy criterion if and only if $QH^{2n}(X) \rightarrow HH^{2n}(\mathcal{A}(X), \mathcal{A}(X))$ is non-zero [AFO⁺], and in particular if (but not only if) $QH^*(X) \rightarrow HH^*(\mathcal{A}(X), \mathcal{A}(X))$ is an isomorphism.

Our Theorem A.1, applied in the compact setting, shows that the assumption on homological smoothness for $\mathcal{A}(X)$ (and the closed-open map being an isomorphism) is extraneous, and can be a posteriori deduced for $\mathcal{A}(X)$ from the hypothesis of satisfying non-degeneracy by Theorem 5.1.

We also remark that the proof given in [AbSm] is actually not applicable under the stated hypotheses to the non-compact setting, as it uses in an essential way Poincaré duality for compact Lagrangians (see [AbSm, Lem. 7.11], which appeals [AbSm, Lem. 7.2]) in order to show a version of the quilt functor is full on morphisms of the form $HF^*(\Delta, A \times B)$. Theorem A.1 applies in either setting, at least under the usual technical hypotheses imposed to avoid virtual methods.

To clarify Theorem A.1, there is a *split data open-closed map*

$$(A.1) \quad \mathcal{OC}_{split}^2 : CC_*(\mathcal{W}_{split}(M \times N), \mathcal{W}_{split}(M \times N)) \rightarrow CH_{split}^*(M \times N)$$

where the group $CH_{split}^*(M \times N)$ is the Hamiltonian Floer complex of $H_M + H_N$ with respect to an almost complex structure $J_M \oplus J_N$, and the map \mathcal{OC}_{split}^2 is the open-closed map for $M \times N$ where all of the Floer data used is split. In the compact setting, split Floer data is an admissible choice of perturbation data, hence this is just the usual open-closed map. We say $\mathcal{W}_{split}(M \times N)$ is *non-degenerate* if (A.1) hits 1.

REMARK A.2. The main current application of Theorem A.1 to Liouville manifolds products of essential Lagrangians generate any object of the product that can be made admissible with split Floer data. This includes the diagonal but also graphs of exact compactly supported symplectomorphisms.

The relevance of Theorem A.1 is slightly limited in the wrapped setting since products of objects in $\mathcal{W}(M)$ and $\mathcal{W}(N)$ are not necessarily “admissible” in $\mathcal{W}(M \times N)$ using the Liouville coordinate $r_M + r_N$, and in particular one doesn’t in general have an the embedding $\mathcal{W}_{split}(M \times N) \hookrightarrow \mathcal{W}(M \times N)$. If all of the objects have strictly vanishing primitive, we expect such an embedding can be constructed in a manner generalizing [O], compatibly with open-closed string maps. This can be arranged for any finite collection of Lagrangians at the expense of passing to a stabilization [AS2, §5.2].

On the other hand, there is no such issue in the compact setting. Therefore, the adaptation of this Theorem to compact settings (under the usual hypotheses: monotone, etc.) should immediately imply that products of Lagrangians satisfying Abouzaid’s generation criterion again satisfy Abouzaid’s generation criterion.

From the point of view of Hochschild theory, such a result is not really surprising: it is a classical fact that for a pair of unital associative algebras A and A' , there is an Eilenberg-Zilber type isomorphism (see e.g., Loday [L2, Thm. 4.2.5])

$$(A.2) \quad sh : HH_*(A) \otimes HH_*(A') \xrightarrow{\sim} HH_*(A \otimes A').$$

We will describe an A_∞ version of the above morphism for Hochschild complexes of Fukaya categories, which requires a partial geometric strictification of units to carry through. (Note from [L2, Thm. 4.2.5] that sh involves many appearances of units). For what follows, abbreviate

$$(A.3) \quad \begin{aligned} \mathcal{W}_M &:= \mathcal{W}(M) \\ \mathcal{W}_N &:= \mathcal{W}(N) \\ \mathcal{W}_{split}^2 &:= \mathcal{W}_{split}(M \times N) \end{aligned}$$

It will be simpler to work with the usual cyclic bar (or *one-pointed*) Hochschild chain complex, denoted $CC_*(\mathcal{C}, \mathcal{B})$ for an A_∞ category \mathcal{C} with coefficients in a bimodule \mathcal{B} , and $CC_*(\mathcal{C}, \mathcal{C})$ for $CC_*(\mathcal{C}, \mathcal{C}_\Delta)$.

Explicitly this complex is the direct sum of, for any k and any $k+1$ -tuple of objects $X_0, \dots, X_k \in \text{ob } \mathcal{A}$, the vector spaces

$$(A.4) \quad \mathcal{B}(X_k, X_0) \otimes \text{hom}_{\mathcal{A}}(X_{k-1}, X_k) \otimes \cdots \otimes \text{hom}_{\mathcal{A}}(X_0, X_1).$$

The differential d_{CC_*} acts on Hochschild chains as follows:

$$(A.5) \quad \begin{aligned} d_{CC_*}(\mathbf{b} \otimes x_k \otimes \cdots \otimes x_1) = \\ \sum (-1)^{\#_j^i} \mu_B^{j|1|k-i}(x_j, \dots, x_1, \mathbf{b}, x_k, \dots, x_{i+1}) \otimes x_i \otimes \cdots \otimes x_{j+1} \\ + \sum (-1)^{\mathfrak{A}_1^s} \mathbf{b} \otimes x_k \otimes \cdots \otimes x_{s+j+1} \otimes \mu_A^j(x_{s+j} \otimes \cdots \otimes x_{s+1}) \otimes x_s \otimes \cdots \otimes x_1 \end{aligned}$$

with signs

$$(A.6) \quad \mathfrak{A}_s^t := \sum_{i=s}^t \|x_i\|$$

$$(A.7) \quad \#_j^i := \left(\sum_{s=1}^j \|x_s\| \right) \cdot \left(|\mathbf{b}| + \sum_{t=j+1}^k \|x_t\| \right) + \mathfrak{A}_{j+1}^i.$$

Hochschild chains are graded as follows:

$$(A.8) \quad \deg(\mathbf{b} \otimes x_k \otimes \cdots \otimes x_1) := \deg(\mathbf{b}) + \sum_i \deg(x_i) - k + 1.$$

Let $\tilde{\mathcal{W}}_{split}^2$ denote the category \mathcal{W}_{split}^2 enhanced with **one-factor homotopy units**, that is, extra formal morphism of the form $e^+ \otimes x$, $f \otimes x$, $x \otimes e^+$, $x \otimes f$, which satisfy the following identities:

$$(A.9) \quad \mu^k(x_k \otimes e^+, \dots, x_1 \otimes e^+) = \mu_M^k(x_k, \dots, x_1) \otimes e^+$$

$$(A.10) \quad \mu^l(e^+ \otimes x'_l, \dots, e^+ \otimes x'_1) = e^+ \otimes \mu_N^l(x'_l, \dots, x'_1)$$

$$(A.11) \quad \mu^2(e^+ \otimes x, y \otimes e^+) = y \otimes x$$

$$(A.12) \quad \mu^2(y \otimes e^+, e^+ \otimes x) = y \otimes x$$

$$(A.13) \quad \mu^s(\dots, e^+ \otimes x_1, \dots, x_2 \otimes e^+, \dots) = 0, \quad s > 2$$

$$(A.14) \quad \mu^s(\dots, x_1 \otimes e^+, \dots, e^+ \otimes x_2, \dots) = 0, \quad s > 2$$

$$(A.15) \quad \begin{aligned} \mu^{s+t+1}(x_{s+t+1} \otimes e^+, \dots, x_{s+1} \otimes e^+, x \otimes y, x_s \otimes e^+, \dots, x_1 \otimes e^+) = \\ \mu^{s+t+1}(x_{s+t+1}, \dots, x_{s+1}, x, x_s, \dots, x_1) \otimes y \end{aligned}$$

$$(A.16) \quad \begin{aligned} \mu^{k+l+1}(e^+ \otimes y_{k+l+1}, \dots, e^+ \otimes y_{k+1}, x \otimes y, e^+ \otimes y_k, \dots, e^+ \otimes y_1) = \\ x \otimes \mu^{k+l+1}(y_{k+l+1}, \dots, y_{k+1}, y, y_s, \dots, y_1) \end{aligned}$$

$$(A.17) \quad \text{INSERT MORE IDENTITIES WITH SIGN.}$$

The functor $\mathcal{W}_{split}^2 \hookrightarrow \tilde{\mathcal{W}}_{split}^2$ given by inclusion of morphism spaces is a quasi-equivalence, and moreover, we observe that

LEMMA A.1. *The restriction of the diagonal bimodule $(\tilde{\mathcal{W}}_{split}^2)_\Delta$ to the subcomplex $(\mathcal{W}_{split}^2)_\Delta$ is a subbimodule over $\tilde{\mathcal{W}}_{split}^2$.*

PROOF. By definition, any operation of the form $\mu^k(\dots, y, \dots)$ where y is not a *formal element* (e.g., equal to $f \otimes x$, $e^+ \otimes x$, $x \otimes f$, or $x \otimes e^+$) is defined using a geometric count, and hence by definition produces an element of \mathcal{W}_{split}^2 . \square

Thus, we can consider the Hochschild complex $CC_*(\tilde{\mathcal{W}}_{split}^2, \mathcal{W}_{split}^2)$ of $\tilde{\mathcal{W}}_{split}^2$ with coefficients in the bimodule \mathcal{W}_{split}^2 (in this complex, the first element is not allowed to be a formal element).

LEMMA A.2. *The natural inclusion map from the Hochschild complex of \mathcal{W}_{split}^2 gives a quasi-isomorphism*

$$(A.18) \quad CC_*(\mathcal{W}_{split}^2, \mathcal{W}_{split}^2) \xrightarrow{\sim} CC_*(\tilde{\mathcal{W}}_{split}^2, \mathcal{W}_{split}^2).$$

PROOF. The map $CC_*(\tilde{\mathcal{W}}_{split}^2, \mathcal{W}_{split}^2) \hookrightarrow CC_*(\tilde{\mathcal{W}}_{split}^2, \tilde{\mathcal{W}}_{split}^2)$ is a quasi-isomorphism, and the quasi-isomorphism $CC_*(\mathcal{W}_{split}^2, \mathcal{W}_{split}^2) \hookrightarrow CC_*(\tilde{\mathcal{W}}_{split}^2, \tilde{\mathcal{W}}_{split}^2)$ factors through (A.18). \square

We define the map

$$(A.19) \quad \Omega : CC_*(\mathcal{W}_M, \mathcal{W}_M) \otimes CC_*(\mathcal{W}_N, \mathcal{W}_N) \longrightarrow CC_*(\tilde{\mathcal{W}}_{split}^2, \mathcal{W}_{split}^2)$$

by sending a pair of Hochschild chains to the tensor of the first distinguished elements with the shuffle of tensors of the remaining morphisms with one-factor strict units:

$$(A.20) \quad (\mathbf{a}_k \otimes a_{k-1} \otimes \cdots \otimes a_1) \otimes (\mathbf{b}_l \otimes b_{l-1} \otimes \cdots \otimes b_1) \longmapsto (\mathbf{a}_k \otimes \mathbf{b}_l) \otimes \mathcal{S}_{k-1, l-1}(a_{k-1} \otimes e^+, \dots, a_1 \otimes e^+; e^+ \otimes b_{l-1}, \dots, e^+ \otimes b_1).$$

(c.f., [L2, Thm. 4.2.5]) The following is an immediate application of the shuffle identities:

PROPOSITION A.1. Ω is a chain map.

PROOF. We verify this Proposition up to sign. Similarly to earlier, abbreviate the shuffle product

$$(A.21) \quad \mathcal{S}_{i,j}(\{a_s \otimes e^+\}_{s=r+i}^{r+1}; \{e^+ \otimes b_t\}_{t=n+j}^{n+1})$$

by

$$(A.22) \quad \mathcal{S}(a_{r+i \rightarrow r+1}; b_{n+j \rightarrow n+1}).$$

This formally includes the case that there are no elements in one of the collections; e.g., we can use the shorthand

$$(A.23) \quad \mathcal{S}(a_{r+i \rightarrow r+1}; b_{n \rightarrow n+1}) := (a_{r+i} \otimes e^+) \otimes \cdots \otimes (a_{r+1} \otimes e^+),$$

Abbreviating $\alpha := \mathbf{a}_k \otimes a_{k-1} \otimes \cdots \otimes a_0$ and $\beta := \mathbf{b}_l \otimes b_{l-1} \otimes \cdots \otimes b_0$, the bar differential applied to $\Omega(\alpha \otimes \beta)$ is the sum of the following terms (with the usual Koszul signs, which are omitted):

$$(A.24) \quad \sum (\mathbf{a}_k \otimes \mathbf{b}_l) \otimes \mathcal{S}(a_{k-1 \rightarrow i+s+1}; b_{l-1 \rightarrow j+t+1}) \otimes \mu(\mathcal{S}(a_{i+s \rightarrow i+1}; b_{j+t \rightarrow j+1})) \otimes \mathcal{S}(a_{i \rightarrow 1}; b_{j \rightarrow 1})$$

$$(A.25) \quad \sum \mu(\mathcal{S}(a_{i \rightarrow 1}; b_{j \rightarrow 1}), \mathbf{a}_k \otimes \mathbf{b}_l, \mathcal{S}(a_{k-1 \rightarrow k-s}; b_{l-1 \rightarrow l-t})) \otimes \mathcal{S}(a_{k-s-1 \rightarrow i+1}; b_{l-t-1 \rightarrow j+1})$$

Applying the one-factor unit identities we see that

$$(A.26) \quad \mu(\mathcal{S}(a_{i+s \rightarrow i+1}; b_{j+t \rightarrow j+1})) = \begin{cases} 0 & s \geq 0, t \geq 0, s+t > 0 \\ a_{i+1} \otimes b_{j+1} - a_{i+1} \otimes b_{j+1} = 0 & s = t = 0 \\ \mu^s(a_{i+s}, \dots, a_{i+1}) \otimes e^+ & t = -1 \\ e^+ \otimes \mu^t(b_{j+t}, \dots, b_{j+1}) & s = -1 \end{cases}$$

where the first case used (A.13)-(A.14), the second case used (A.11)-(A.12), and the last two cases used (A.9) and (A.10) respectively. Similarly, we see that

$$(A.27) \quad \mu(\mathcal{S}(a_{i \rightarrow 1}; b_{j \rightarrow 1}), \mathbf{a}_k \otimes \mathbf{b}_l, \mathcal{S}(a_{k-1 \rightarrow k-s}; b_{l-1 \rightarrow l-t})) = \begin{cases} 0 & i+s > 0, j+t > 0 \\ (-1)^{|\mathbf{b}_l|} \mu^{i+s+1}(a_i, \dots, a_1, \mathbf{a}_k, a_{k-1}, \dots, a_{k-s}) \otimes \mathbf{b}_l & i+s > 0, j+t = 0 \\ \mathbf{a}_k \otimes \mu^{j+t+1}(b_j, \dots, b_1, \mathbf{b}_l, b_{l-1}, \dots, b_{l-t}) & i+s = 0, j+t > 0 \\ (-1)^{|\mathbf{b}_l|} \mu^1(\mathbf{a}_k) \otimes \mathbf{b}_l + \mathbf{a}_k \otimes \mu^1(\mathbf{b}_l) & i = s = j = t = 0, \end{cases}$$

by using the identities (A.13) and (A.14), (A.15), (A.16), and the definition of the differential on $\mathbf{a}_k \otimes \mathbf{b}_l$.

Hence, the non-zero terms are precisely in bijection with the elements of $\Omega((-1)^{|\beta|} d_{CC_*} \alpha \otimes \beta + \alpha \otimes d_{CC_*} \beta)$. \square

REMARK A.3. We conjecture Ω is always a quasi-isomorphism. We have two reasons for such a conjecture: First, in the associative unital setting, there is a well-defined so-called Alexander-Whitney inverse map to Ω . We conjecture that such quasi-inverses exist in the A_∞ setting as well. Constructing them may involve deforming the diagonal associahedron $\Delta_d \subset \mathbb{R}^d \times \mathbb{R}^d$ onto various copies of products of strata in order to obtain formulas for the tensor product of A_∞ algebras—see e.g., [SU].

A more direct proof could go as follows: strictifying $\mathcal{W}(M)$ and $\mathcal{W}(N)$ via the Yoneda embedding gives dg categories with quasi-isomorphic Hochschild complexes. One could strictify \mathcal{W}_{split}^2 via the quilt-type functor \mathbf{M} reviewed in §5 to also obtain a dg category, and check that Ω is compatible with strictifications, in that it is compatible with the usual shuffle product on the Hochschild complexes of the strictified categories. This

would involve enhancing the functor \mathbf{M} to one from $\tilde{\mathcal{W}}_{split}^2$, not just \mathcal{W}_{split}^2 , which can be done along the lines of our enhancement of \mathcal{OC}^2 below.

The main construction of this appendix extends the open-closed map \mathcal{OC}^2 to a map from the Hochschild complex of $\tilde{\mathcal{W}}_{split}^2$:

PROPOSITION A.2 (The open-closed map with homotopy units). *There is a map*

$$(A.28) \quad \tilde{\mathcal{OC}}^2 : CC_*(\tilde{\mathcal{W}}_{split}^2, \mathcal{W}_{split}^2) \rightarrow CH^*(M \times N)$$

extending \mathcal{OC}^2 , meaning that that $\tilde{\mathcal{OC}}^2$ restricted to cyclic chains of morphisms in \mathcal{W}_{split}^2

$$(A.29) \quad CC_*(\mathcal{W}_{split}^2, \mathcal{W}_{split}^2) \xrightarrow{\sim} CC_*(\tilde{\mathcal{W}}_{split}^2, \mathcal{W}_{split}^2) \xrightarrow{\tilde{\mathcal{OC}}^2} CH^*(M \times N)$$

is just \mathcal{OC}^2 . Moreover, this map fits into a strictly commutative diagram

$$(A.30) \quad \begin{array}{ccc} CC_*(\mathcal{W}_M, \mathcal{W}_M) \otimes_{\mathbb{K}} CC_*(\mathcal{W}_N, \mathcal{W}_N) & \xrightarrow{\Omega} & CC_*(\tilde{\mathcal{W}}_{split}^2, \mathcal{W}_{split}^2) , \\ \downarrow \mathcal{OC}_M \otimes \mathcal{OC}_N & & \downarrow \tilde{\mathcal{OC}}^2 \\ CH^*(M) \otimes_{\mathbb{K}} CH^*(N) & \xrightarrow{=} & CH^*(M \times N) \end{array}$$

Using this, we can immediately prove the main Theorem:

PROOF OF THM. A.1. The diagram (A.30) implies that if \mathcal{OC}_M and \mathcal{OC}_N hit 1, then $\tilde{\mathcal{OC}}^2$ hits 1 too. (as the map $CH^*(M) \otimes CH^*(N) \rightarrow CH_{split}^*(M \times N)$ sends $1_M \otimes 1_N$ to 1). The fact (A.29) that $\tilde{\mathcal{OC}}^2$ extends \mathcal{OC}^2 , implies (using (A.18)) that $[\mathcal{OC}^2] = [\tilde{\mathcal{OC}}^2]$; hence, $[\mathcal{OC}^2]$ hits [1] too. \square

In order to establish Proposition A.2, we need to introduce a series of auxiliary moduli spaces and their operations. First, we have the moduli space controlling $\mathcal{OC}_M \otimes \mathcal{OC}_N$, thought of as a single operation instead of a tensor product of operations. The *moduli space of pairs of discs with an interior marked points and (k, l) boundary marked points*

$$(A.31) \quad \mathcal{R}_{k,l}^1$$

is the space of pairs of standard discs (S^L, S^R) with one interior puncture each, labeled y_{out}^L and y_{out}^R , $k+1$ boundary punctures on S^L labeled z_0, \dots, z_k , $l+1$ boundary punctures on S^R labeled (z'_0, \dots, z'_l) with

$$(A.32) \quad \begin{aligned} & z_k \text{ and } z'_l \text{ in the same position, and} \\ & y_{out}^L \text{ and } y_{out}^R \text{ in the same position,} \end{aligned}$$

modulo *simultaneous automorphism* (noting that simultaneous automorphism preserves (A.32)).

By choosing a representative where z_k and z'_l are both at 1 and y_{out}^L and y_{out}^R are both at zero, one seems that this moduli space is in fact equal to the product of moduli spaces, with identification given by projecting onto either factor:

$$(A.33) \quad (\pi_L, \pi_R) : \mathcal{R}_{k,l}^1 \xrightarrow{\sim} \mathcal{R}_k^1 \times \mathcal{R}_l^1$$

This is in contrast to the case of pairs of discs with no interior marked points (discussed in detail in [G2, §5] and recalled in §5.2), where the projection was a further quotient by automorphisms on the right or left. (we remark that the notation \mathcal{R}_k^1 is not used in this paper, which uses *two-pointed* open-closed maps, but is in the prequel—see REF).

Hence, we could take as a compactification the product of compactifications

$$(A.34) \quad \overline{\mathcal{R}}_{k,l}^1 := \overline{\mathcal{R}}_k^1 \times \overline{\mathcal{R}}_l^1.$$

We remark here that there is a slightly different compactification with the same essential boundary strata that one can construct, following [G2, §5.2], recalled in REF. The essential point here is that either way, the maps (π_L, π_R) extend to a map between compactified spaces.

A *split Floer datum* for pairs of discs with an interior marked point is a choice, for each $T \in \overline{\mathcal{R}}_{k,l}^1$, of a Floer datum for $\pi_L(T)$ and one for $\pi_R(T)$. A universal and consistent choice is as in the usual definition an

inductive choice of such Floer data for all k, l , smoothly varying with respect to gluing charts and conformally consistent with previous choices made on boundary strata

Given a pair of orbits $y_0 \in \mathcal{O}_M$ and $y_1 \in \mathcal{O}_N$, as well as chords $x_i \in \chi_M(L_{i-1}, L_{i \bmod k})$, $x'_j \in \chi_N(L'_{i-1}, L_i)$, for $(i, j) \in \{1, \dots, k\} \times \{1, \dots, l\}$, we obtain a moduli space of maps into $M \times N$ satisfying Floer's equation with Floer datum. . .

$$(A.35) \quad \mathbf{F}_{\mathcal{R}_{k,l}^1}$$

LEMMA A.3. *The operation associated to $\mathbf{F}_{\mathcal{R}_{k,l}^1}$, up to sign twist INSERT, is $\mathcal{OC}_k^M \otimes \mathcal{OC}_l^N$.*

Next, if we restrict to operations controlled by certain nice submanifolds of $\mathcal{R}_{k,l}^1$ where points in either factor are *coincident*, we obtain new families of operations, such as \mathcal{OC}^2 . More specifically, a $(k-1, l-1)$ *point identification* is a subset of pairs of tuples $\mathfrak{P} \subset \{1, \dots, k-1\} \times \{1, \dots, l-1\}$ as in Def. 5.2. Associated to \mathfrak{P} is a submanifold of (A.31)

$$(A.36) \quad \mathfrak{P}\mathcal{R}_{k,l}^1$$

as in Def. 5.2 of pairs of discs such that in addition to the coincidence between z_k and z'_l and y_{out}^L and y_{out}^R , the marked points indexed by \mathfrak{P} are coincident.

For instance, when $k = l = d$, taking the *maximal point identification*, the resulting submanifold is the diagonal embedding of \mathcal{R}_d^1 :

$$(A.37) \quad (\mathcal{R}_{d,d}^1)_{diag} := \mathfrak{P}_{max} \mathcal{R}_{d,d}^1$$

LEMMA A.4. *The operation associated to $(\mathcal{R}_{d,d}^1)_{diag}$ is \mathcal{OC}_d^2 .*

Finally, to define $\tilde{\mathcal{OC}}^2$, we need to understand how to deal with asymptotic conditions of the form $\{e^+, f\} \otimes x$ or $x \otimes \{e^+, f\}$. The answer is to treat the corresponding marked points in our abstract moduli spaces as being equipped with the label of being a *forgotten point* or a *homotopy unit* respectively, exactly as in §6. The relation to $\mathcal{OC}_M \otimes \mathcal{OC}_N$ will follow immediately from the relationship between $(\mathcal{R}_{d,d}^1)_{diag}$ and various $\mathcal{R}_{k,l}^1$'s via forgetting points.

Let $F_1 \subset \{1, \dots, k-1\}$ and $F_2 \subset \{1, \dots, l-1\}$, $H_1 \subset \{1, \dots, k-1\}$, $H_2 \subset \{1, \dots, l-1\}$. The *moduli space of \mathfrak{P} -coincident pairs of open-closed discs with (k, l) boundary marked points (F_1, F_2) forgotten marked points, and (H_1, H_2) homotopy units* is exactly the moduli space $\mathfrak{P}\mathcal{R}_{k,l}^1$ with points in F_1, F_2, H_1, H_2 labeled accordingly, times a copy of $[0, 1]$ for each element of H_1 or H_2 .

$$(A.38) \quad \mathfrak{P}\mathfrak{H}_{k,l}^{1,F_1,F_2,H_1,H_2} := \mathfrak{P}\mathcal{R}_{k,l}^1 \times [0, 1]^{|H_1|} \times [0, 1]^{|H_2|},$$

subject to the following constraints:

- F_1, H_1 and F_2, H_2 are disjoint subsets of the left and right identified points respectively. Namely,

$$(A.39) \quad F_i, H_i \subset \pi_i(\mathfrak{P}), F_i \cap H_i = \emptyset$$

where π_i is projection onto the i th component (i.e., $\pi_1 = \pi_L$, $\pi_2 = \pi_R$).

- F_1, H_1 and F_2, H_2 do not contain both the left and right points of any identification, i.e.

$$(A.40) \quad ((F_1 \cup H_1) \times (F_2 \cup H_2)) \cap \mathfrak{T} = \emptyset$$

We think of a point of $\mathfrak{P}\mathfrak{H}_{k,l}^{1,F_1,F_2,H_1,H_2}$ as a tuple

$$(A.41) \quad (P, \vec{v}, \vec{w}).$$

Suppose $H_1, H_2 = \{p_{n_1}, \dots, p_{n_{|H_1|}}\}, \{p_{m_1}, \dots, p_{m_{|H_2|}}\}$. For any point $p_{n_i} \in H_1$ or $p_{m_i} \in H_2$, there are **endpoint maps** defined as in (6.54)-(6.55)

$$(A.42) \quad \pi_{L,p_{n_i}}^1 : \mathfrak{P}\mathfrak{H}_{k,l}^{1,F_1,F_2,H_1,H_2} \longrightarrow \mathfrak{P}\mathfrak{H}_{k,l}^{1,F_1,F_2,H_1-\{p_{n_i}\},H_2}$$

$$(A.43) \quad \pi_{L,p_{n_i}}^0 : \mathfrak{P}\mathfrak{H}_{k,l}^{1,F_1,F_2,H_1,H_2} \longrightarrow \mathfrak{P}\mathfrak{H}_{k,l}^{1,F_1+\{p_{n_i}\},F_2,H_1-\{p_{n_i}\},H_2}$$

$$(A.44) \quad \pi_{R,p_{m_i}}^1 : \mathfrak{P}\mathfrak{H}_{k,l}^{1,F_1,F_2,H_1,H_2} \longrightarrow \mathfrak{P}\mathfrak{H}_{k,l}^{1,F_1,F_2+\{p_{m_i}\},H_1,H_2-\{p_{m_i}\}}$$

$$(A.45) \quad \pi_{R,p_{m_i}}^0 : \mathfrak{P}\mathfrak{H}_{k,l}^{1,F_1,F_2,H_1,H_2} \longrightarrow \mathfrak{P}\mathfrak{H}_{k,l}^{1,F_1,F_2,H_1,H_2-\{p_{m_i}\}},$$

which change the labelings of P , and apply a projection map to (\vec{v}, \vec{w}) in the following way: for $\pi_{L, p_{n_i}}^b$: given a point (P, \vec{v}, \vec{w}) , remove the point p_{n_i} from the set H_1 , and add it to F_1 if $b = 0$. Also, project \vec{v} away from the i th factor and do nothing to \vec{w} . Analogously, for $\pi_{R, p_{m_j}}^b$: given a point (P, \vec{v}, \vec{w}) , remove the point p_{m_j} from the set H_2 , and add it to F_2 if $b = 0$. Also, project \vec{w} away from the j th factor and do nothing to \vec{v} .

There are also the **forgetful maps**

$$(A.46) \quad \mathcal{F}_{I_1, I_2} : \mathfrak{P} \mathfrak{H}_{k, l}^{1, F_1, F_2, H_1, H_2} \longrightarrow \mathfrak{P}' \mathfrak{H}_{k - |I_1|, l - |I_2|}^{1, F'_1, F'_2, H'_1, H'_2}$$

for $I_1, I_2 \subset F_1, F_2$. F'_1 and F'_2 are F_1 and F_2 sans I_1 and I_2 , reindexed appropriately, and H'_1 and H'_2 are just H_1 and H_2 reindexed. On the $[0, 1]^{|H_i|}$ components, the forgetful maps are the identity. The Deligne-Mumford compactification is the product of the Deligne-Mumford compactifications:

$$(A.47) \quad \mathfrak{P} \overline{\mathfrak{H}}_{k, l}^{1, F_1, F_2, H_1, H_2} := \mathfrak{P} \overline{\mathcal{R}}_{k, l}^1 \times [0, 1]^{|H_1|} \times [0, 1]^{|H_2|},$$

and the maximally forgetful map extends to a map on compactifications

$$(A.48) \quad \mathcal{F}_{max} : \mathfrak{P} \mathfrak{H}_{k, l}^{1, F_1, F_2, H_1, H_2} \longrightarrow \mathfrak{P}' \mathfrak{H}_{k - |F_1|, l - |F_2|}^{1, \emptyset, \emptyset, H'_1, H'_2}$$

(there is no stability issue in this case, at least on the main components—since the outputs y_{out}^L, y_{out}^R and distinguished inputs z_k, z'_l are not allowed to be forgotten or homotopy units).

We remark that the case $H_1, H_2 = \emptyset$ recovers the *moduli space of \mathfrak{P} -coincident pairs of open-closed discs with (k, l) boundary marked points and (F_1, F_2) forgotten marked points*

$$(A.49) \quad \mathfrak{P} \mathcal{R}_{k, l}^{1, (F_1, F_2)},$$

which is exactly $\mathfrak{P} \mathcal{R}_{k, l}^1$, with suitable *forgotten labels* associated to elements indexed by F_1 or F_2 . As a special case of the discussion with homotopy units, this moduli space comes equipped with forgetful maps

$$(A.50) \quad \mathcal{F}_{I_1, I_2} : \mathfrak{P} \mathcal{R}_{k, l}^{1, F_1, F_2} \longrightarrow \mathfrak{P}' \mathcal{R}_{k - |I_1|, l - |I_2|}^{1, F'_1, F'_2}$$

(A.49).

DEFINITION A.1. A **Floer datum** for a pair of discs with homotopy units and forgotten points (P, \vec{v}, \vec{w}) is a Floer datum for the pair of elements $(\pi_L, \pi_R)(\mathcal{F}_{max}(P))$ satisfying the following properties in addition to being a Floer datum:

- For boundary point $p_{n_i} \in H_1$, thought of as a point in $\pi_{\mathfrak{S}}(P)$, the Floer datum only needs to be **v_i -partially compatible** with the associated strip-like end $\epsilon_{p_{n_i}}$, in the sense of Definition 6.15.
- Similarly, for boundary point $p_{m_j} \in H_2$, thought of as a point in the gluing $\pi_{\mathfrak{S}}(P)$, the Floer datum only needs to be **w_j -partially compatible** with the striplike end $\epsilon_{p_{m_j}}$.

We additionally fix, for each element of H_1 and H_2 , a copy $(\mathbb{H}, \epsilon_{\mathbb{H}})$. Call $\mathbb{H}_{p_{n_i}}$ and $\mathbb{H}_{p_{m_j}}$ the copies of \mathbb{H} corresponding to points $p_{n_i} \in H_1$ and $p_{m_j} \in H_2$ respectively. Then, a Floer datum also consists of a choice of **associated v_i and w_j structures** on $\mathbb{H}_{p_{n_i}}$ and $\mathbb{H}_{p_{m_j}}$ for p_{n_i} and p_{m_j} respectively, in the sense of Definition 6.16.

A *universal and conformally consistent choice of Floer data for pairs of discs with homotopy units* is a choice $\mathbf{D}_{(P, \vec{v}, \vec{w})}$, for every boundary identification \mathfrak{S} and compatible sequential point identification \mathfrak{P} , and every representative (P, \vec{v}, \vec{w}) of $\mathfrak{P} \overline{\mathfrak{H}}_{k, l}^{1, F_1, F_2, H_1, H_2}$, varying smoothly over this space, whose restriction to a boundary stratum is conformally equivalent to a Floer datum coming from lower dimensional moduli spaces. Moreover, Floer data agree to infinite order at the boundary stratum with the Floer datum obtained by gluing. Most crucially, these choices satisfy the following properties:

- **(forgotten points are forgettable)** In the **$h(\epsilon)$ -stable** range, the choice of Floer datum only depends on the reduced surface $\mathcal{F}_{max}(P, \vec{v}, \vec{w})$. In the **$h(\epsilon)$ -semistable** range, the Floer datum agrees with the translation-invariant Floer datum on the strip.
- **(0 endpoint is forgetting)** In the **$h(\epsilon)$ -stable range**, if $v_i = 0$ or $w_j = 0$, then after forgetting the copy of \mathbb{H} corresponding to p_{n_i} or p_{m_j} respectively, the Floer datum should be isomorphic to the Floer datum on $\pi_{L, p_{n_i}}^0(S, \vec{v}, \vec{w})$ or $\pi_{R, p_{m_j}}^0(S, \vec{v}, \vec{w})$ respectively. In the $h(\epsilon)$ -semistable case,

the Floer datum should be isomorphic to the translation invariant Floer datum on the respective surface.

- **(1 endpoint is gluing in a unit)** if $v_i = 1$ or $w_j = 1$, then $\mathbb{H}_{p_{n_i}}$ or $\mathbb{H}_{p_{m_j}}$ should have the standard unit datum Floer data, and the Floer datum on the main component should be isomorphic to a Floer datum on $\pi_{L,p_{n_i}}^1(P, \vec{v}, \vec{w})$ or $\pi_{R,p_{m_j}}^1(P, \vec{v}, \vec{w})$ respectively.

Let (P, \vec{v}, \vec{w}) be a pair of discs with H_1, H_2 homotopy units, and suppose we have fixed a Floer datum \mathbf{D} for $T := (P, \vec{v}, \vec{w})$. Then the **associated homotopy-unit surface**, denoted

$$(A.51) \quad \mathfrak{h}(T),$$

is the **iterated damped connect sum**

$$(A.52) \quad \mathfrak{h}(T) := (\pi_L, \pi_R)(\mathfrak{F}_{max}(P)) \sharp_{p_{n_1}}^{v_1} \mathbb{H}_{p_{n_1}} \cdots \sharp_{p_{n_{|H_1|}}}^{v_{|H_1|}} \mathbb{H}_{p_{n_{|H_1|}}} \sharp_{p_{m_1}}^{w_1} \mathbb{H}_{p_{m_1}} \cdots \sharp_{p_{m_{|H_2|}}}^{w_{|H_2|}} \mathbb{H}_{p_{m_{|H_2|}}}.$$

This is a (potentially nodal) pair of discs with associated Floer data. An *admissible Lagrangian labeling* for pairs of discs with homotopy units and forgotten point is a labeling in the usual sense, satisfying the conditions that labelings before and after H points and F points must coincide. The admissibility condition implies that there is an **induced labeling** on the **associated homotopy-unit surface**.

For each H_1, H_2, F_1, F_2 and compatible labeling and asymptotics

$$(A.53) \quad \begin{aligned} \vec{x}^L &:= \{x_i \in \chi(L_i, L_{i+1}) | i \notin H_1 \cup F_1\} \\ \vec{x}^R &:= \{x_i \in \chi(L'_i, L'_{i+1}) | i \notin H_2 \cup F_2\} \end{aligned}$$

we obtain a (compactified) moduli space of maps

$$(A.54) \quad \mathfrak{P}_{max} \bar{\mathfrak{H}}_{d,d}^{1,F_1,F_2,H_1,H_2}(y_{out}^L, y_{out}^R; \vec{x}^L, \vec{x}^R).$$

which (for generic choices) are compact manifolds of dimension

$$(A.55) \quad \text{INSERT}$$

Finally, we can define

$$(A.56) \quad \tilde{\mathcal{O}}\mathcal{C}^2([\tilde{x}_d] \otimes \cdots \otimes [\tilde{x}_1])$$

for $\tilde{x}_i \in \text{hom}_{\mathcal{W}^2}(L_i \times L_{i'} \times L_{i+1 \bmod d} \times L_{i'+1 \bmod d})$ by linearly extending the map defined on generators of the form $\tilde{x}_i := x_i^L \otimes x_i^R$ as follows: let F_1 and H_1 denote the subsets of $\{1, \dots, d\}$ for which x_i^L is a formal element e^+ or f respectively, and similarly construct F_2 and H_2 . If \vec{x}^L denote the remaining x_i^L elements, and \vec{x}^R the remaining x_i^R elements, we can define

$$(A.57) \quad \tilde{\mathcal{O}}\mathcal{C}^2([\tilde{x}_d] \otimes \cdots \otimes [\tilde{x}_1]) := \sum_{y_{out}^L, y_{out}^R; (A.55)=0, u \in \mathfrak{P}_{max} \bar{\mathfrak{H}}_{d,d}^{1,F_1,F_2,H_1,H_2}(y_{out}^L, y_{out}^R; \vec{x}^L, \vec{x}^R)} (-1)^*(\mathfrak{P}_{max} \bar{\mathfrak{H}}_{d,d}^{1,F_1,F_2,H_1,H_2})_u([\vec{x}^L; \vec{x}^R])$$

where the sign twist is...

LEMMA A.5. $\tilde{\mathcal{O}}\mathcal{C}^2$ is a chain map.

The following completes the proof of Proposition A.2:

LEMMA A.6 (Shuffle identities for the open-closed map). *There is an equality of chain level operations:*

$$(A.58) \quad \tilde{\mathcal{O}}\mathcal{C}^2(\Omega(x_k \otimes \cdots \otimes x_0; x'_l \otimes \cdots \otimes x'_0)) = \mathcal{O}\mathcal{C}_M(x_k \otimes \cdots \otimes x_0) \otimes \mathcal{O}\mathcal{C}_N(x'_l \otimes \cdots \otimes x'_0)$$

PROOF. On the open stratum $\emptyset \mathcal{R}_{k,l}^1$ where none of the (non-distinguished) boundary marked points are coincident, as before there is an *overlay map*

$$(A.59) \quad \emptyset \mathcal{R}_{k,l}^1 \longrightarrow \coprod_{F_1 \cup F_2 = \{1, \dots, d\}} \mathfrak{P}_{max} \mathcal{R}_{d,d}^{1,F_1,F_2},$$

given by marking the $k-1$ boundary punctures on S_L as extra forgotten points on S_R and vice versa. By construction, this map is compatible with Floer data and covers the entire interior of the target. Since after perturbation zero-dimensional solutions to Floer's equation come from a representative on the interior, we conclude that the two operations are identical modulo sign...

We are done because the operation associated to the left moduli space is. . . □

Appendix B. Unstable operations

Some of the operations we would like to consider are parametrized not by underlying moduli spaces but instead a single surface.

B.1. Strips. Let Σ_1 denote a disc with two boundary punctures removed, thought of as a strip $(-\infty, \infty) \times [0, 1]$. We have already defined a Floer-theoretic operation using Σ_1 , namely the differential μ^1 . Let us recast this operation in terms of Floer data.

DEFINITION B.1. *A **Floer datum** for Σ_1 can be thought of a Floer datum in the sense of [G2, Def. 4.11] with the following additional constraints:*

- *The strip-like ends ϵ_+ and ϵ_- are given by inclusion of the positive and negative semi-infinite strips respectively.*
- *The incoming and outgoing weights are both equal to a single number w .*
- *the one-form α is $w \cdot dt$ everywhere, as is the rescaling map a_S .*
- *the Hamiltonian H_{Σ_1} is equal everywhere to $\frac{H \circ \psi^w}{w^2}$*
- *the almost complex structure J_{Σ_1} is equal everywhere to $(\psi^w)^* J_t$.*

REMARK B.1. *Upon fixing H and J_t , the Floer datum above only depends on w . Moreover, the data defined by any two different weights w and w' are conformally equivalent.*

Fix the Floer datum for Σ_1 with $w = 1$. This induces, for Lagrangians $L_0, L_1 \in \text{ob } \mathcal{W}$, and chords $x_0, x_1 \in \chi(L_0, L_1)$, a space of maps

$$\Sigma_1(x_0; x_1)$$

satisfying the usual asymptotic and boundary conditions, and solving the relevant version of Floer's equation for the Floer datum. Instead of dividing by \mathbb{R} -translation, we can also consider the operation induced by the space $\Sigma_1(x_0; x_1)$ itself, which has dimension

$$\deg(x_0) - \deg(x_1)$$

We get a map

$$(B.1) \quad I : CW^*(L_0, L_1) \longrightarrow CW^*(L_0, L_1)$$

defined by

$$(B.2) \quad I([x_0]) := \sum_{x_1 : \deg(x_1) = \deg(x_0)} \sum_{u \in \Sigma_1(x_0; x_1)} (-1)^{\deg(x_0)} (\Sigma_1)_u([x_0])$$

where $(\Sigma_1)_u : o_{x_0} \rightarrow o_{x_1}$ is the induced map on orientation lines (using the arguments reviewed in [G2, Appendix A]).

PROPOSITION B.1. *I is the identity map.*

PROOF. If u is any non-constant strip mapping into M , composing with the \mathbb{R} action on Σ_1 gives other maps into M solving the same equation by \mathbb{R} -invariance of our Floer data; hence u is not rigid. Therefore, dimension 0 strips must all be constant, concluding the proof. □

B.2. The unit. Let Σ_0 denote a once-punctured disc thought of as the upper half plane $\mathbb{H} \subset \mathbb{C}$ with puncture at ∞ , thought of as a negative puncture.

DEFINITION B.2. *A **Floer datum** for Σ_0 is a Floer datum in the sense of Definition [G2, Def. 4.11] Concretely, this consists of*

- *A **strip-like end** $\epsilon : (-\infty, 0] \times [0, 1] \rightarrow \Sigma_0$ around the puncture*
- *A **choice of weight** $w \in [1, \infty)$*
- *A **rescaling map** $a_{\Sigma_0} : \Sigma_0 \rightarrow [1, +\infty)$ equal to w on the strip-like end*
- ***Hamiltonian perturbation:** A map $H_{\Sigma_0} : \Sigma_0 \rightarrow \mathcal{H}(M)$ such that $\epsilon^* H_{\Sigma_0} = \frac{H \circ \psi^w}{w^2}$.*
- ***basic 1-form:** a sub-closed 1-form α_{Σ_0} , whose restriction to $\partial \Sigma_0$ vanishes, such that $\epsilon^* \alpha_{\Sigma_0} = w \cdot dt$.*
- ***Almost complex structure:** A map $J_{\Sigma_0} : \Sigma_0 \rightarrow \mathcal{J}(M)$ such that $J_{\Sigma_0} \in \mathcal{J}_{a_{\Sigma_0}}$ and $\epsilon^* J_{\Sigma_0} = (\psi^w)^* J_t$.*

REMARK B.2. *Note by Stokes' theorem that in the definition above, the one form α_{Σ_0} cannot be closed everywhere, i.e. there are points with $d\alpha_{\Sigma_0} < 0$.*

REMARK B.3. *Up to conformal equivalence, it suffices to take a Floer datum for Σ_0 with weight $w = 1$.*

Let L be an object of \mathcal{W} , and consider a chord $x_0 \in \chi(L, L)$. Fixing a Floer datum for Σ_0 , write

$$(B.3) \quad \Sigma_0(x_0;)$$

for the space of maps $u : \Sigma_0 \rightarrow E$ satisfying boundary and asymptotic conditions

$$(B.4) \quad \begin{cases} u(z) \in \psi^{as(z)} L & z \in \partial \Sigma_0 \\ \lim_{s \rightarrow -\infty} u \circ \epsilon(s, \cdot) = x \end{cases}$$

and differential equation

$$(B.5) \quad (du - X_{\Sigma_0} \otimes \alpha_{\Sigma_0})^{0,1} = 0$$

with respect to J_{Σ_0} .

LEMMA B.1. *The space of maps $\Sigma_0(x_0;)$ is compact and forms a manifold of dimension $\deg(x_0)$.*

Thus, we can define the element $e_L \in CW^*(L, L)$ to be the sum

$$(B.6) \quad e_L := \sum_{\deg(x_0)=0} \sum_{u \in \Sigma_0(x_0;)} (\Sigma_0)_u(1)$$

where $(\Sigma_0)_u : \mathbb{R} \rightarrow o_{x_0}$ is the induced map on orientation lines (using arguments reviewed in [G2, Appendix A])

PROPOSITION B.2. *The resulting elements $e_{L_i} \in CW^*(L_i, L_i)$ give the identity element on homology.*

PROOF. This is a classical result, but we briefly sketch a proof for completeness; see e.g. [R] for more details. One first checks via analyzing the boundary of the one dimensional moduli space of $\Sigma_0(x_0;)$ that $d(e_L) = 0$, so e_L descends to homology. Then, one needs to check that, up to sign

$$(B.7) \quad \mu^2([x], [e_{L_i}]) = [x]$$

$$(B.8) \quad \mu^2([e_{L_i}], [x]) = [x],$$

where the brackets denote homology classes. Since the arguments to establish (B.7) and (B.8) are identical, it suffices to construct a geometric chain homotopy between the maps

$$(B.9) \quad \mu^2(\cdot, e_{L_i})$$

and

$$(B.10) \quad I(\cdot)$$

where I is as in (B.1), which can be described as follows. Let Σ_2 be a disc with two incoming boundary marked points x_1, x_2 , and one outgoing point x_{out} , with x_1 marked as “forgotten”, or a ghost point. Fix a strip-like end around x_1 and consider a one parameter family of Floer data on Σ_2 with x_2 and x_{out} removed, over the interval $[0, 1)_t$, such that

- at $t = 0$, the Floer data agrees with the translation-invariant one on the Σ_1 arising by forgetting x_1 ,
- for general t , the Floer data is modeled on the connect sum of the Floer data for μ_2 with the Floer data for e_{L_i} over the strip-like ends at the output of Σ_0 with the one around x_1 , with connect sum length approaching ∞ as $t \rightarrow 1$.

Compactifying and looking at the associated Floer operation, one obtains a chain homotopy between the degenerate curve corresponding to $\mu(\cdot, e_{L_i})$ and the operation $I(\cdot)$. Finally, one performs a sign verification analogous to those in [G2, Appendix A]. \square

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