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Pearson Edexcel Qualifications

MATHEMATICS

eBook included

PEARSON EDEXCEL INTERNATIONAL A LEVEL

PURE MATHEMATICS 2

STUDENT BOOK

PEARSON EDEXCEL INTERNATIONAL A LEVEL

PURE MATHEMATICS 2

Student Book

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ABOUT THIS BOOK

The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

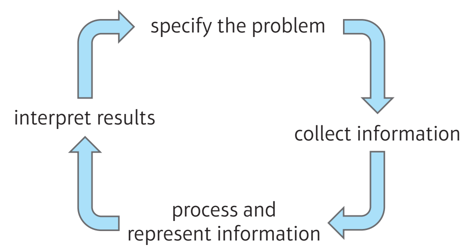
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Challenge questions provide extra stretch

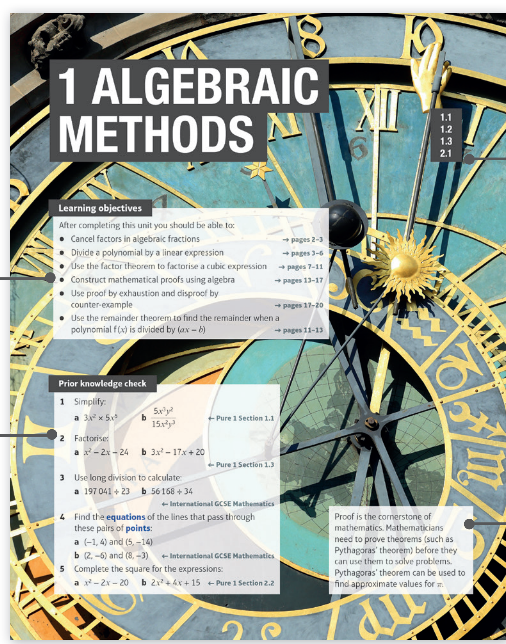
The Mathematical Problem-Solving Cycle



3. Transferable skills

- Transferable skills are embedded throughout this book, in the exercises and in some examples
- These skills are signposted to show students which skills they are using and developing

Finding your way around the book



Each chapter starts with a list of Learning objectives

The *Prior knowledge check* helps make sure you are ready to start the chapter

Glossary terms will be identified by bold blue text on their first appearance.

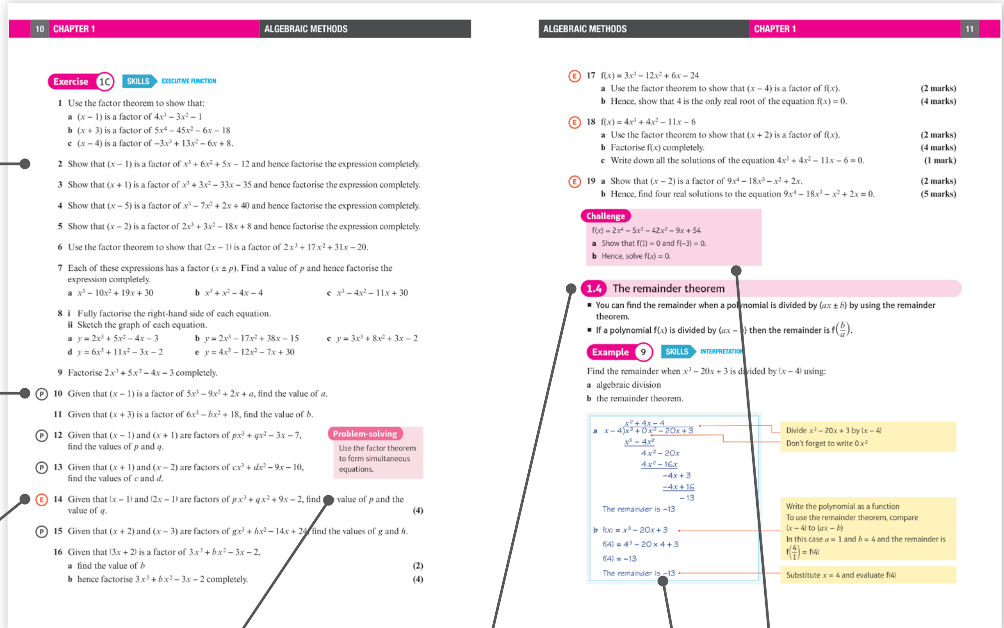
Each chapter is mapped to the specification content for easy reference

The real world applications of the maths you are about to learn are highlighted at the start of the chapter.

Transferable skills are signposted where they naturally occur in the exercises and examples

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

Exercises are packed with exam-style questions to ensure you are ready for the exams



Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

Each section begins with explanation and key learning points

Challenge boxes give you a chance to tackle some more difficult questions

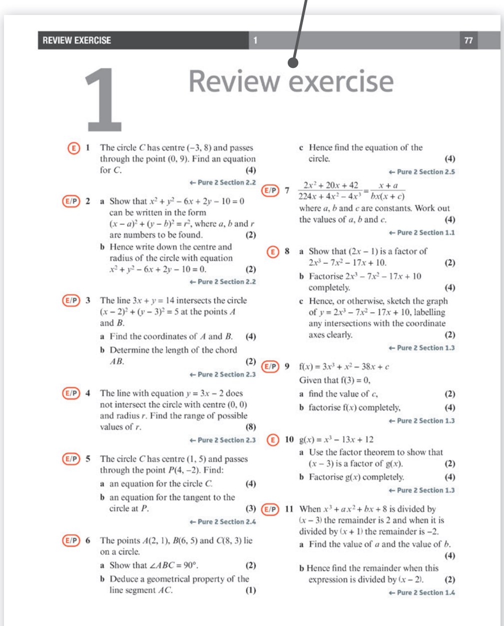
Step-by-step worked examples focus on the key types of questions you'll need to tackle

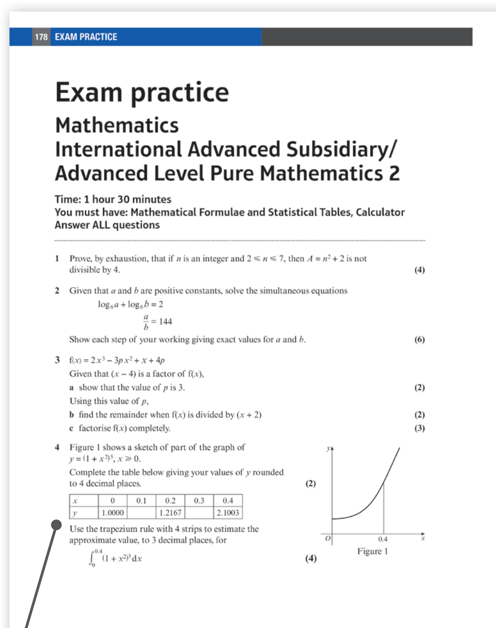
Exam-style questions are flagged with Ⓔ

Problem-solving questions are flagged with Ⓟ

Each chapter ends with a *Chapter review* and a *Summary of key points*

After every few chapters, a *Review exercise* helps you consolidate your learning with lots of exam-style questions





A full practice paper at the back of the book helps you prepare for the real thing

QUALIFICATION AND ASSESSMENT OVERVIEW

Qualification and content overview

Pure Mathematics 2 (P2) is a **compulsory** unit in the following qualifications:

International Advanced Subsidiary in Mathematics

International Advanced Subsidiary in Pure Mathematics

International Advanced Level in Mathematics

International Advanced Level in Pure Mathematics

Assessment overview

The following table gives an overview of the assessment for this unit.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

Unit	Percentage	Mark	Time	Availability
P2: Pure Mathematics 2 Paper code WMA12/01	$33\frac{1}{3}$ % of IAS $16\frac{2}{3}$ % of IAL	75	1 hour 30 mins	January, June and October First assessment June 2019

IAS: International Advanced Subsidiary, IAL: International Advanced A Level.

Assessment objectives and weightings

		Minimum weighting in IAS and IAL
AO1	Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.	30%
AO2	Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.	30%
AO3	Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.	10%
AO4	Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.	5%
AO5	Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.	5%

Relationship of assessment objectives to units

P2	Assessment objective				
	AO1	AO2	AO3	AO4	AO5
Marks out of 75	25–30	25–30	5–10	5–10	5–10
%	$33\frac{1}{3}$–40	$33\frac{1}{3}$–40	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$

Calculators

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements given in the table below.

Students are expected to have available a calculator with at least the following keys: +, –, ×, ÷, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, ln x, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

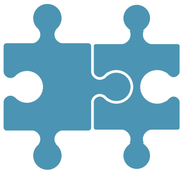
Prohibitions

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet

Extra online content

Whenever you see an *Online* box, it means that there is extra online content available to support you.



SolutionBank

SolutionBank provides a full worked solution for every question in the book. Download all the solutions as a PDF or quickly find the solution you need online.

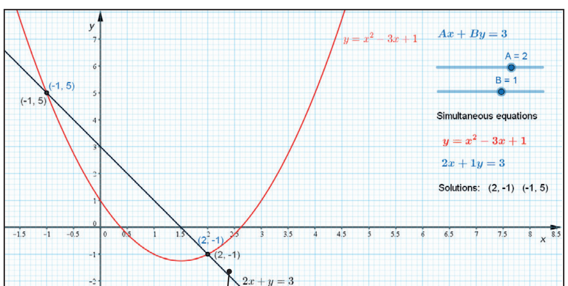
Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

Online Find the point of intersection graphically using technology.

GeoGebra

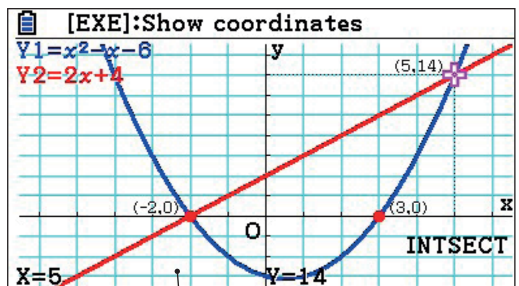
GeoGebra-powered interactives



Interact with the maths you are learning using GeoGebra's easy-to-use tools

CASIO

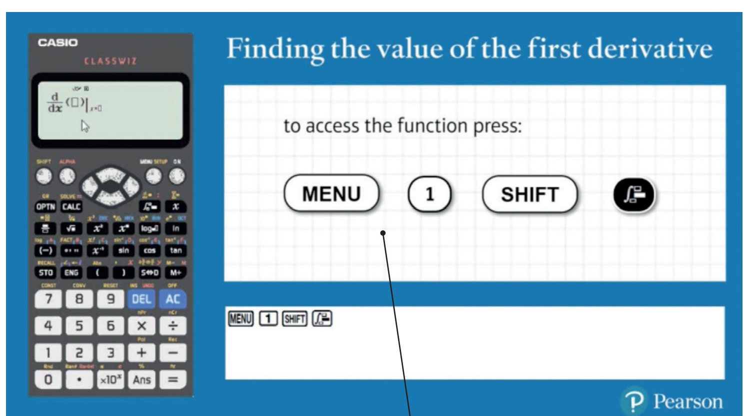
Graphic calculator interactives



Explore the maths you are learning and gain confidence in using a graphic calculator

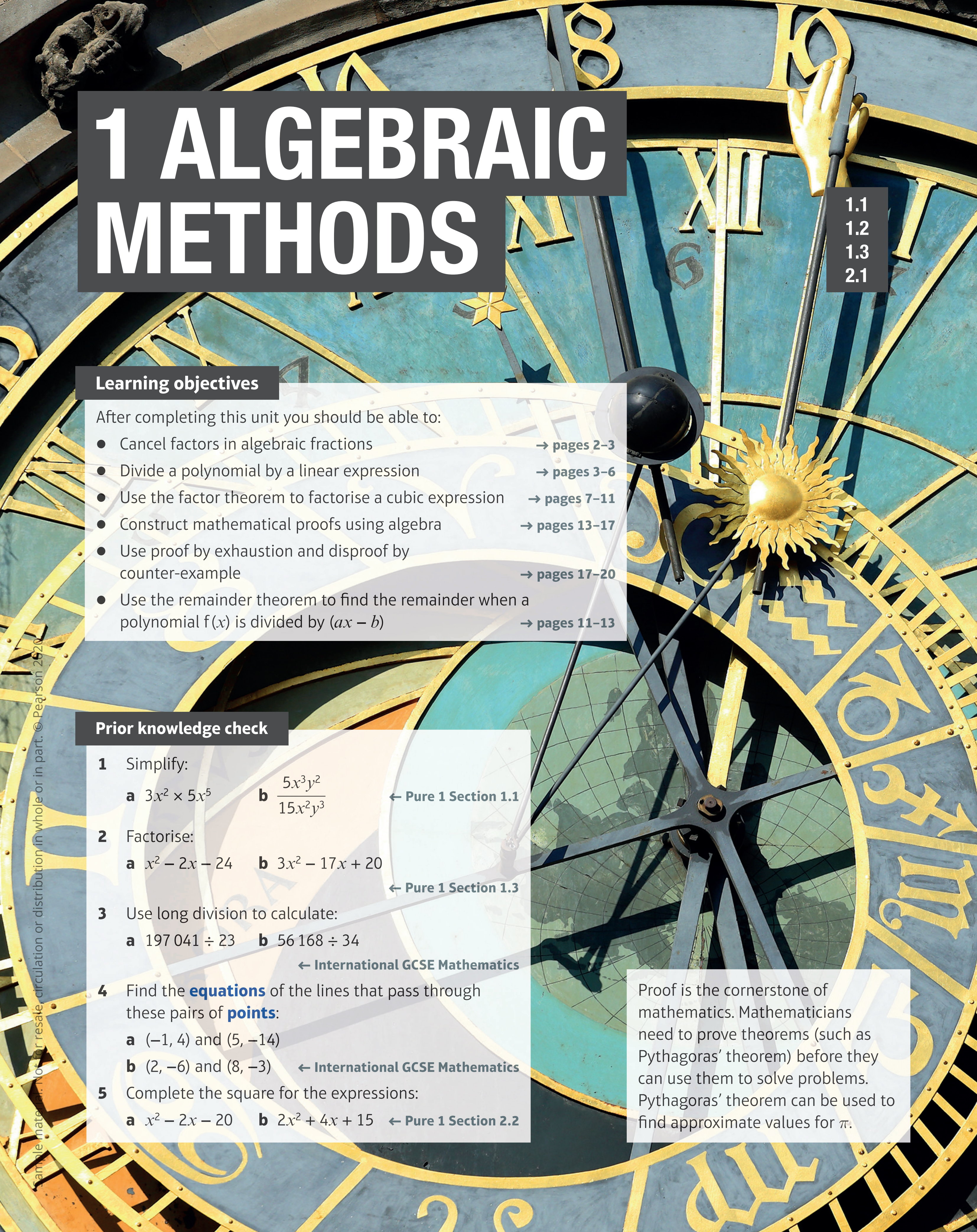
Calculator tutorials

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.



Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen

Online Work out each coefficient quickly using the ${}^nC_r$ and power functions on your calculator.



1 ALGEBRAIC METHODS

1.1
1.2
1.3
2.1

Learning objectives

After completing this unit you should be able to:

- Cancel factors in algebraic fractions → pages 2–3
- Divide a polynomial by a linear expression → pages 3–6
- Use the factor theorem to factorise a cubic expression → pages 7–11
- Construct mathematical proofs using algebra → pages 13–17
- Use proof by exhaustion and disproof by counter-example → pages 17–20
- Use the remainder theorem to find the remainder when a polynomial $f(x)$ is divided by $(ax - b)$ → pages 11–13

Prior knowledge check

1 Simplify:

a $3x^2 \times 5x^5$ **b** $\dfrac{5x^3y^2}{15x^2y^3}$ ← Pure 1 Section 1.1

2 Factorise:

a $x^2 - 2x - 24$ **b** $3x^2 - 17x + 20$

← Pure 1 Section 1.3

3 Use long division to calculate:

a 197 041 ÷ 23 **b** 56 168 ÷ 34

← International GCSE Mathematics

4 Find the **equations** of the lines that pass through these pairs of **points**:

a $(-1, 4)$ and $(5, -14)$

b $(2, -6)$ and $(8, -3)$ ← International GCSE Mathematics

5 Complete the square for the expressions:

a $x^2 - 2x - 20$ **b** $2x^2 + 4x + 15$ ← Pure 1 Section 2.2

Proof is the cornerstone of mathematics. Mathematicians need to prove theorems (such as Pythagoras' theorem) before they can use them to solve problems. Pythagoras' theorem can be used to find approximate values for π.

1.1 Algebraic fractions

You can simplify algebraic fractions using division.

- When simplifying an algebraic fraction, where possible factorise the numerator and denominator and then cancel common factors.

$$\frac{5x^2-245}{2x^2-15x+7} \xrightarrow{\text{Factorise}} \frac{5(x+7)(x-7)}{(2x-1)(x-7)} \xrightarrow{\text{Cancel common factor}} \frac{5(x+7)}{2x-1}$$

Example 1 SKILLS PROBLEM-SOLVING

Simplify these fractions:

a $\frac{7x^4-2x^3+6x}{x}$ **b** $\frac{(x+7)(2x-1)}{(2x-1)}$ **c** $\frac{x^2+7x+12}{(x+3)}$ **d** $\frac{x^2+6x+5}{x^2+3x-10}$ **e** $\frac{2x^2+11x+12}{(x+3)(x+4)}$

a $\frac{7x^4-2x^3+6x}{x}$

$= \frac{7x^4}{x} - \frac{2x^3}{x} + \frac{6x}{x}$ — Divide each part of the numerator by x.

$= 7x^3 - 2x^2 + 6$

b $\frac{(x+7)(2x-1)}{(2x-1)} = x+7$ — Simplify by cancelling the common factor of $(2x-1)$.

c $\frac{x^2+7x+12}{(x+3)} = \frac{(x+3)(x+4)}{(x+3)}$ — Factorise: $x^2+7x+12=(x+3)(x+4)$.

$= x+4$ — Cancel the common factor of $(x+3)$.

d $\frac{x^2+6x+5}{x^2+3x-10} = \frac{(x+5)(x+1)}{(x+5)(x-2)}$ — Factorise: $x^2+6x+5=(x+5)(x+1)$ and $x^2+3x-10=(x+5)(x-2)$.

$= \frac{x+1}{x-2}$ — Cancel the common factor of $(x+5)$.

e $2x^2+11x+12 = 2x^2+3x+8x+12$

$= x(2x+3)+4(2x+3)$

$= (2x+3)(x+4)$ — Factorise $2x^2+11x+12$

So $\frac{2x^2+11x+12}{(x+3)(x+4)}$

$= \frac{(2x+3)(x+4)}{(x+3)(x+4)}$

$= \frac{2x+3}{x+3}$ — Cancel the common factor of $(x+4)$.

Exercise 1A SKILLS PROBLEM-SOLVING

1 Simplify these fractions:

a $\frac{4x^4+5x^2-7x}{x}$ **b** $\frac{7x^5-5x^5+9x^3+x^2}{x}$ **c** $\frac{-x^4+4x^2+6}{x}$

d $\dfrac{7x^5 - x^3 - 4}{x}$ **e** $\dfrac{8x^4 - 4x^3 + 6x}{2x}$ **f** $\dfrac{9x^2 - 12x^3 - 3x}{3x}$

g $\dfrac{7x^3 - x^4 - 2}{5x}$ **h** $\dfrac{-4x^2 + 6x^4 - 2x}{-2x}$ **i** $\dfrac{-x^8 + 9x^4 - 4x^3 + 6}{-2x}$

j $\dfrac{-9x^9 - 6x^6 + 4x^4 - 2}{-3x}$

2 Simplify these fractions as far as possible:

a $\dfrac{(x+3)(x-2)}{(x-2)}$ **b** $\dfrac{(x+4)(3x-1)}{(3x-1)}$ **c** $\dfrac{(x+3)^2}{(x+3)}$

d $\dfrac{x^2 + 10x + 21}{(x+3)}$ **e** $\dfrac{x^2 + 9x + 20}{(x+4)}$ **f** $\dfrac{x^2 + x - 12}{(x-3)}$

g $\dfrac{x^2 + x - 20}{x^2 + 2x - 15}$ **h** $\dfrac{x^2 + 3x + 2}{x^2 + 5x + 4}$ **i** $\dfrac{x^2 + x - 12}{x^2 - 9x + 18}$

j $\dfrac{2x^2 + 7x + 6}{(x-5)(x+2)}$ **k** $\dfrac{2x^2 + 9x - 18}{(x+6)(x+1)}$ **l** $\dfrac{3x^2 - 7x + 2}{(3x-1)(x+2)}$

m $\dfrac{2x^2 + 3x + 1}{x^2 - x - 2}$ **n** $\dfrac{x^2 + 6x + 8}{3x^2 + 7x + 2}$ **o** $\dfrac{2x^2 - 5x - 3}{2x^2 - 9x + 9}$

E/P **3** $\dfrac{6x^3 + 3x^2 - 84x}{6x^2 - 33x + 42} = \dfrac{ax(x+b)}{x+c}$, where a, b and c are **constants**.

Work out the values of a, b and c. **(4 marks)**

1.2 Dividing polynomials

A **polynomial** is a **finite** expression with positive whole number **indices**.

- You can use long division to divide a polynomial by $(x \pm p)$, where p is a constant.
- You can use long division to divide a polynomial by $(ax \pm b)$, where a and b are constants.

Polynomials	Not polynomials
$2x + 4$	$\sqrt{x}$
$4xy^2 + 3x - 9$	$6x^{-2}$
8	$\dfrac{4}{x}$

Example 2

Divide $x^3 + 2x^2 - 17x + 6$ by $(x - 3)$.

1

$$\begin{array}{r l}
 & x^2 \\
x - 3 \,\big) & \overline{x^3 + 2x^2 - 17x + 6} \\
 & \underline{x^3 - 3x^2} \\
 & \quad\; 5x^2 - 17x
\end{array}$$

Start by dividing the first term of the polynomial by x, so that $x^3 \div x = x^2$.

Next multiply $(x - 3)$ by x^2, so that $x^2 \times (x - 3) = x^3 - 3x^2$.

Now subtract, so that $(x^3 + 2x^2) - (x^3 - 3x^2) = 5x^2$.

Copy $-17x$.

②

$$\begin{array}{r} x^2 + 5x \phantom{{}+6} \\ x-3\overline{)\,x^3 + 2x^2 - 17x + 6} \\ \underline{x^3 - 3x^2} \phantom{{}-17x+6} \\ 5x^2 - 17x \phantom{{}+6} \\ \underline{5x^2 - 15x} \phantom{{}+6} \\ -2x + 6 \end{array}$$

Repeat the method. Divide $5x^2$ by x, so that $5x^2 \div x = 5x$.

Multiply $(x - 3)$ by $5x$, so that $5x \times (x - 3) = 5x^2 - 15x$.

Subtract, so that $(5x^2 - 17x) - (5x^2 - 15x) = -2x$.

Copy $+6$.

③

$$\begin{array}{r} x^2 + 5x - 2 \\ x-3\overline{)\,x^3 + 2x^2 - 17x + 6} \\ \underline{x^3 - 3x^2} \phantom{{}-17x+6} \\ 5x^2 - 17x \phantom{{}+6} \\ \underline{5x^2 - 15x} \phantom{{}+6} \\ -2x + 6 \\ \underline{-2x + 6} \\ 0 \end{array}$$

Repeat the method. Divide $-2x$ by x, so that $-2x \div x = -2$.

Multiply $(x - 3)$ by -2, so that $-2 \times (x - 3) = -2x + 6$.

Subtract, so that $(-2x + 6) - (-2x + 6) = 0$.

No numbers left to copy, so you have finished.

So $\dfrac{x^3 + 2x^2 - 17x + 6}{x - 3} = x^2 + 5x - 2$

This is called the **quotient**.

Example 3 SKILLS INTERPRETATION

$f(x) = 4x^4 - 17x^2 + 4$

Divide $f(x)$ by $(2x + 1)$, giving your answer in the form $f(x) = (2x + 1)(ax^3 + bx^2 + cx + d)$.

Find $(4x^4 - 17x^2 + 4) \div (2x + 1)$

$$\begin{array}{r} 2x^3 - x^2 - 8x + 4 \\ 2x+1\overline{)\,4x^4 + 0x^3 - 17x^2 + 0x + 4} \\ \underline{4x^4 + 2x^3} \phantom{{}-17x^2+0x+4} \\ -2x^3 - 17x^2 \phantom{{}+0x+4} \\ \underline{-2x^3 - x^2} \phantom{{}+0x+4} \\ -16x^2 + 0x \phantom{{}+4} \\ \underline{-16x^2 - 8x} \phantom{{}+4} \\ 8x + 4 \\ \underline{8x + 4} \\ 0 \end{array}$$

So

$4x^4 - 17x^2 + 4 = (2x + 1)(2x^3 - x^2 - 8x + 4)$.

Use long division. Include the terms $0x^3$ and $0x$ when you write out $f(x)$.

You need to multiply $(2x + 1)$ by $2x^3$ to get the $4x^4$ term, so write $2x^3$ in the answer, and write $2x^3(2x + 1) = 4x^4 + 2x^3$ below. Subtract and copy the next term.

You need to multiply $(2x + 1)$ by $-x^2$ to get the $-2x^3$ term, so write $-x^2$ in the answer, and write $-x^2(2x + 1) = -2x^3 - x^2$ below. Subtract and copy the next term.

Repeat the method.

$(4x^4 - 17x^2 + 4) \div (2x + 1) = 2x^3 - x^2 - 8x + 4$.

Example 4

Find the **remainder** when $2x^3 - 5x^2 - 16x + 10$ is divided by $(x - 4)$.

$$
\begin{array}{r}
2x^2 + 3x - 4 \\
x - 4\,\overline{)\,2x^3 - 5x^2 - 16x + 10} \\
\underline{2x^3 - 8x^2} \\
3x^2 - 16x \\
\underline{3x^2 - 12x} \\
-4x + 10 \\
\underline{-4x + 16} \\
-6
\end{array}
$$

So the remainder is -6.

$(x - 4)$ is not a factor of $2x^3 - 5x^2 - 16x + 10$ as the remainder $\neq 0$.

This means you cannot write the expression in the form $(x - 4)(ax^2 + bx + c)$.

Exercise 1B SKILLS INTERPRETATION

1 Write each polynomial in the form $(x \pm p)(ax^2 + bx + c)$ by dividing:

a $x^3 + 6x^2 + 8x + 3$ by $(x + 1)$

b $x^3 + 10x^2 + 25x + 4$ by $(x + 4)$

c $x^3 - x^2 + x + 14$ by $(x + 2)$

d $x^3 + x^2 - 7x - 15$ by $(x - 3)$

e $x^3 - 8x^2 + 13x + 10$ by $(x - 5)$

f $x^3 - 5x^2 - 6x - 56$ by $(x - 7)$

2 Write each polynomial in the form $(x \pm p)(ax^2 + bx + c)$ by dividing:

a $6x^3 + 27x^2 + 14x + 8$ by $(x + 4)$

b $4x^3 + 9x^2 - 3x - 10$ by $(x + 2)$

c $2x^3 + 4x^2 - 9x - 9$ by $(x + 3)$

d $2x^3 - 15x^2 + 14x + 24$ by $(x - 6)$

e $-5x^3 - 27x^2 + 23x + 30$ by $(x + 6)$

f $-4x^3 + 9x^2 - 3x + 2$ by $(x - 2)$

3 Divide:

a $x^4 + 5x^3 + 2x^2 - 7x + 2$ by $(x + 2)$

b $4x^4 + 14x^3 + 3x^2 - 14x - 15$ by $(x + 3)$

c $-3x^4 + 9x^3 - 10x^2 + x + 14$ by $(x - 2)$

d $-5x^5 + 7x^4 + 2x^3 - 7x^2 + 10x - 7$ by $(x - 1)$

4 Divide:

a $3x^4 + 8x^3 - 11x^2 + 2x + 8$ by $(3x + 2)$

b $4x^4 - 3x^3 + 11x^2 - x - 1$ by $(4x + 1)$

c $4x^4 - 6x^3 + 10x^2 - 11x - 6$ by $(2x - 3)$

d $6x^5 + 13x^4 - 4x^3 - 9x^2 + 21x + 18$ by $(2x + 3)$

e $6x^5 - 8x^4 + 11x^3 + 9x^2 - 25x + 7$ by $(3x - 1)$

f $8x^5 - 26x^4 + 11x^3 + 22x^2 - 40x + 25$ by $(2x - 5)$

g $25x^4 + 75x^3 + 6x^2 - 28x - 6$ by $(5x + 3)$

h $21x^5 + 29x^4 - 10x^3 + 42x - 12$ by $(7x - 2)$

5 Divide:

a $x^3 + x + 10$ by $(x + 2)$

b $2x^3 - 17x + 3$ by $(x + 3)$

c $-3x^3 + 50x - 8$ by $(x - 4)$

Hint Include $0x^2$ when you write out $f(x)$.

6 Divide:

a $x^3 + x^2 - 36$ by $(x - 3)$

b $2x^3 + 9x^2 + 25$ by $(x + 5)$

c $-3x^3 + 11x^2 - 20$ by $(x - 2)$

7 Show that $x^3 + 2x^2 - 5x - 10 = (x + 2)(x^2 - 5)$

8 Find the remainder when:

a $x^3 + 4x^2 - 3x + 2$ is divided by $(x + 5)$

b $3x^3 - 20x^2 + 10x + 5$ is divided by $(x - 6)$

c $-2x^3 + 3x^2 + 12x + 20$ is divided by $(x - 4)$

9 Show that when $3x^3 - 2x^2 + 4$ is divided by $(x - 1)$ the remainder is 5.

10 Show that when $3x^4 - 8x^3 + 10x^2 - 3x - 25$ is divided by $(x + 1)$ the remainder is -1.

11 Show that $(x + 4)$ is a factor of $5x^3 - 73x + 28$.

12 Simplify $\dfrac{3x^3 - 8x - 8}{x - 2}$

Hint Divide $3x^3 - 8x - 8$ by $(x - 2)$.

13 Divide $x^3 - 1$ by $(x - 1)$.

Hint Write $x^3 - 1$ as $x^3 + 0x^2 + 0x - 1$.

14 Divide $x^4 - 16$ by $(x + 2)$.

E 15 $f(x) = 10x^3 + 43x^2 - 2x - 10$

Find the remainder when $f(x)$ is divided by $(5x + 4)$. **(2 marks)**

E/P 16 $f(x) = 3x^3 - 14x^2 - 47x - 14$

SKILLS EXECUTIVE FUNCTION

a Find the remainder when $f(x)$ is divided by $(x - 3)$. **(2 marks)**

b Given that $(x + 2)$ is a factor of $f(x)$, factorise $f(x)$ completely. **(4 marks)**

Problem-solving

Write $f(x)$ in the form $(x + 2)(ax^2 + bx + c)$ then factorise the quadratic factor.

E/P 17 a Find the remainder when $x^3 + 6x^2 + 5x - 12$ is divided by

SKILLS EXECUTIVE FUNCTION

i $x - 2$,

ii $x + 3$. **(3 marks)**

b Hence, or otherwise, find all the solutions to the equation $x^3 + 6x^2 + 5x - 12 = 0$. **(4 marks)**

E/P 18 $f(x) = 2x^3 + 3x^2 - 8x + 3$

SKILLS EXECUTIVE FUNCTION

a Show that $f(x) = (2x - 1)(ax^2 + bx + c)$ where a, b and c are constants to be found. **(2 marks)**

b Hence factorise $f(x)$ completely. **(4 marks)**

c Write down all the real roots of the equation $f(x) = 0$. **(2 marks)**

E/P 19 $f(x) = 12x^3 + 5x^2 + 2x - 1$

a Show that $(4x - 1)$ is a factor of $f(x)$ and write $f(x)$ in the form $(4x - 1)(ax^2 + bx + c)$. **(6 marks)**

b Hence, show that the equation $12x^3 + 5x^2 + 2x - 1 = 0$ has exactly one real solution. **(2 marks)**

1.3 The factor theorem

The factor theorem is a quick way of finding simple linear factors of a polynomial.

- The factor theorem states that if $f(x)$ is a polynomial then:
  - If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$.
  - If $(x - p)$ is a factor of $f(x)$, then $f(p) = 0$.
  - If $f\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $f(x)$.
  - If $(ax - b)$ is a factor of $f(x)$ then $f\left(\frac{b}{a}\right) = 0$.

Watch out

The first two statements are not the same. Here are two similar statements, only one of which is true:

If $x = -2$ then $x^2 = 4$ ✓

If $x^2 = 4$ then $x = -2$ ✗

You can use the factor theorem to quickly factorise a cubic function, $g(x)$:

1 Substitute values into the function until you find a value p such that $g(p) = 0$.

2 Divide the function by $(x - p)$. — The remainder will be 0 because $(x - p)$ is a factor of $g(x)$.

3 Write $g(x) = (x - p)(ax^2 + bx + c)$ — The other factor will be quadratic.

4 Factorise the quadratic factor, if possible, to write $g(x)$ as a product of three linear factors.

Example 5 SKILLS ANALYSIS

Show that $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$ by:

a algebraic division **b** the factor theorem

a

$$\begin{array}{r} x^2 + 3x + 2 \\ x - 2\overline{)x^3 + x^2 - 4x - 4} \\ \underline{x^3 - 2x^2} \qquad\qquad \\ 3x^2 - 4x \qquad \\ \underline{3x^2 - 6x} \qquad \\ 2x - 4 \\ \underline{2x - 4} \\ 0 \end{array}$$

Divide $x^3 + x^2 - 4x - 4$ by $(x - 2)$.

The remainder is 0, so $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

So $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

b $f(x) = x^3 + x^2 - 4x - 4$ — Write the polynomial as a function.

$f(2) = (2)^3 + (2)^2 - 4(2) - 4$ — Substitute $x = 2$ into the polynomial.

$= 8 + 4 - 8 - 4$

$= 0$

So $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Use the factor theorem:
If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$.
Here $p = 2$, so $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Example 6 SKILLS EXECUTIVE FUNCTION

a Fully factorise $2x^3 + x^2 - 18x - 9$ **b** Hence sketch the graph of $y = 2x^3 + x^2 - 18x - 9$

a $f(x) = 2x^3 + x^2 - 18x - 9$

Write the polynomial as a function.

$f(-1) = 2(-1)^3 + (-1)^2 - 18(-1) - 9 = 8$
$f(1) = 2(1)^3 + (1)^2 - 18(1) - 9 = -24$
$f(2) = 2(2)^3 + (2)^2 - 18(2) - 9 = -25$
$f(3) = 2(3)^3 + (3)^2 - 18(3) - 9 = 0$

Try values of x, e.g. −1, 1, 2, 3, … until you find $f(p) = 0$.

$f(p) = 0$.

So $(x - 3)$ is a factor of $2x^3 + x^2 - 18x - 9$.

Use statement 1 from the factor theorem: If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$. Here $p = 3$.

$$
\begin{array}{r}
2x^2 + 7x + 3 \\
x - 3 \overline{)\, 2x^3 + x^2 - 18x - 9} \\
\underline{2x^3 - 6x^2 \quad\quad\quad\quad} \\
7x^2 - 18x \quad\quad \\
\underline{7x^2 - 21x \quad\quad} \\
3x - 9 \\
\underline{3x - 9} \\
0
\end{array}
$$

Use long division to find the quotient when dividing by $(x - 3)$.

You can check your division here: $(x - 3)$ is a factor of $2x^3 + x^2 - 18x - 9$, so the remainder must be 0.

$$2x^3 + x^2 - 18x - 9 = (x - 3)(2x^2 + 7x + 3)$$
$$= (x - 3)(2x + 1)(x + 3)$$

$2x^2 + 7x + 3$ can also be factorised.

b $0 = (x - 3)(2x + 1)(x + 3)$

Set $y = 0$ to find the points where the curve crosses the x-axis.

So the curve crosses the x-axis at (3, 0), $(-\frac{1}{2}, 0)$ and (−3, 0).

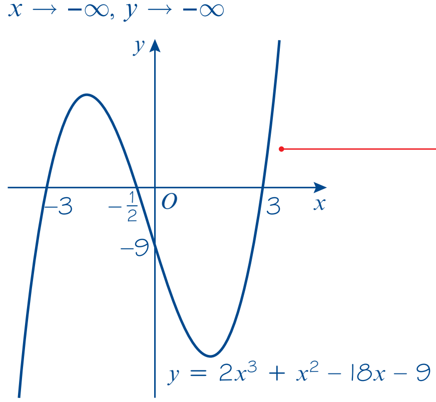
When $x = 0$, $y = (-3)(1)(3) = -9$

Set $x = 0$ to find the y-intercept.

The curve crosses the y-axis at (0, −9).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



This is a cubic graph with a positive **coefficient** of x^3 and three distinct roots. You should be familiar with its general shape. ← Pure 1 Section 4.1

Example 7

Given that $(x + 1)$ is a factor of $4x^4 - 3x^2 + a$, find the value of a.

$f(x) = 4x^4 - 3x^2 + a$

Write the polynomial as a function.

$f(-1) = 0$

Use statement 2 from the factor theorem.
$(x - p)$ is a factor of $f(x)$, so $f(p) = 0$
Here $p = -1$.

$4(-1)^4 - 3(-1)^2 + a = 0$

$4 - 3 + a = 0$

$a = -1$

Substitute $x = -1$ and **solve the equation** for a.
Remember $(-1)^4 = 1$.

Example 8

$f(x) = px^3 + x^2 - 19x + p$

Given that $(2x - 3)$ is a factor of $f(x)$

a find the value of p,

b hence factorise $f(x)$ completely.

a $f(x) = px^3 + x^2 - 19x + p$

$f\left(\frac{3}{2}\right) = 0$

Use the factor theorem
$(2x - 3)$ is a factor of $f(x)$, so $f\left(\frac{3}{2}\right) = 0$

$p\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 - 19 \times \left(\frac{3}{2}\right) + p = 0$

Substitute $x = \frac{3}{2}$ into $f(x) = 0$

$p\left(\frac{27}{8}\right) + \frac{9}{4} - \frac{57}{2} + p = 0$

$p\left(\frac{27}{8} + 1\right) = \frac{57}{2} - \frac{9}{4}$

$p\left(\frac{35}{8}\right) = \frac{105}{4}$

$p = 6$

Solve the equation for p

b
$$\begin{array}{r} 3x^2 + 5x - 2 \\ 2x - 3 \overline{)\,6x^3 + x^2 - 19x + 6} \\ \underline{6x^3 - 9x^2} \\ 10x^2 - 19x \\ \underline{10x^2 - 15x} \\ -4x + 6 \\ \underline{-4x + 6} \\ 0 \end{array}$$

Now divide $6x^3 + x^2 - 19x + 6$ by $2x - 3$

$3x^2 + 5x - 2 = (x + 2)(3x - 1)$

Factorise the quadratic quotient

$f(x) = (2x - 3)(x + 2)(3x - 1)$

Remember to write $f(x)$ as a product of its factors

Exercise 1C

SKILLS EXECUTIVE FUNCTION

1 Use the factor theorem to show that:

a $(x - 1)$ is a factor of $4x^3 - 3x^2 - 1$

b $(x + 3)$ is a factor of $5x^4 - 45x^2 - 6x - 18$

c $(x - 4)$ is a factor of $-3x^3 + 13x^2 - 6x + 8$.

2 Show that $(x - 1)$ is a factor of $x^3 + 6x^2 + 5x - 12$ and hence factorise the expression completely.

3 Show that $(x + 1)$ is a factor of $x^3 + 3x^2 - 33x - 35$ and hence factorise the expression completely.

4 Show that $(x - 5)$ is a factor of $x^3 - 7x^2 + 2x + 40$ and hence factorise the expression completely.

5 Show that $(x - 2)$ is a factor of $2x^3 + 3x^2 - 18x + 8$ and hence factorise the expression completely.

6 Use the factor theorem to show that $(2x - 1)$ is a factor of $2x^3 + 17x^2 + 31x - 20$.

7 Each of these expressions has a factor $(x \pm p)$. Find a value of p and hence factorise the expression completely.

a $x^3 - 10x^2 + 19x + 30$ **b** $x^3 + x^2 - 4x - 4$ **c** $x^3 - 4x^2 - 11x + 30$

8 **i** Fully factorise the right-hand side of each equation.

ii Sketch the graph of each equation.

a $y = 2x^3 + 5x^2 - 4x - 3$ **b** $y = 2x^3 - 17x^2 + 38x - 15$ **c** $y = 3x^3 + 8x^2 + 3x - 2$

d $y = 6x^3 + 11x^2 - 3x - 2$ **e** $y = 4x^3 - 12x^2 - 7x + 30$

9 Factorise $2x^3 + 5x^2 - 4x - 3$ completely.

P **10** Given that $(x - 1)$ is a factor of $5x^3 - 9x^2 + 2x + a$, find the value of a.

11 Given that $(x + 3)$ is a factor of $6x^3 - bx^2 + 18$, find the value of b.

P **12** Given that $(x - 1)$ and $(x + 1)$ are factors of $px^3 + qx^2 - 3x - 7$, find the values of p and q.

P **13** Given that $(x + 1)$ and $(x - 2)$ are factors of $cx^3 + dx^2 - 9x - 10$, find the values of c and d.

Problem-solving

Use the factor theorem to form simultaneous equations.

E **14** Given that $(x - 1)$ and $(2x - 1)$ are factors of $px^3 + qx^2 + 9x - 2$, find the value of p and the value of q. **(4)**

P **15** Given that $(x + 2)$ and $(x - 3)$ are factors of $gx^3 + hx^2 - 14x + 24$, find the values of g and h.

16 Given that $(3x + 2)$ is a factor of $3x^3 + bx^2 - 3x - 2$,

a find the value of b **(2)**

b hence factorise $3x^3 + bx^2 - 3x - 2$ completely. **(4)**

E **17** $f(x) = 3x^3 - 12x^2 + 6x - 24$

a Use the factor theorem to show that $(x - 4)$ is a factor of $f(x)$. **(2 marks)**

b Hence, show that 4 is the only real root of the equation $f(x) = 0$. **(4 marks)**

E **18** $f(x) = 4x^3 + 4x^2 - 11x - 6$

a Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$. **(2 marks)**

b Factorise $f(x)$ completely. **(4 marks)**

c Write down all the solutions of the equation $4x^3 + 4x^2 - 11x - 6 = 0$. **(1 mark)**

E **19** **a** Show that $(x - 2)$ is a factor of $9x^4 - 18x^3 - x^2 + 2x$. **(2 marks)**

b Hence, find four real solutions to the equation $9x^4 - 18x^3 - x^2 + 2x = 0$. **(5 marks)**

Challenge

$f(x) = 2x^4 - 5x^3 - 42x^2 - 9x + 54$

a Show that $f(1) = 0$ and $f(-3) = 0$.

b Hence, solve $f(x) = 0$.

1.4 The remainder theorem

- **You can find the remainder when a polynomial is divided by $(ax \pm b)$ by using the remainder theorem.**
- **If a polynomial $f(x)$ is divided by $(ax - b)$ then the remainder is $f\left(\frac{b}{a}\right)$.**

Example 9 SKILLS INTERPRETATION

Find the remainder when $x^3 - 20x + 3$ is divided by $(x - 4)$ using:

a algebraic division

b the remainder theorem.

a

$$\begin{array}{r} x^2 + 4x - 4 \\ x - 4 \overline{\smash{)}\, x^3 + 0x^2 - 20x + 3} \\ \underline{x^3 - 4x^2} \qquad\qquad \\ 4x^2 - 20x \qquad \\ \underline{4x^2 - 16x} \qquad \\ -4x + 3 \\ \underline{-4x + 16} \\ -13 \end{array}$$

The remainder is -13

Divide $x^3 - 20x + 3$ by $(x - 4)$

Don't forget to write $0x^2$

b $f(x) = x^3 - 20x + 3$

$f(4) = 4^3 - 20 \times 4 + 3$

$f(4) = -13$

The remainder is -13

Write the polynomial as a function

To use the remainder theorem, compare $(x - 4)$ to $(ax - b)$

In this case $a = 1$ and $b = 4$ and the remainder is $f\left(\frac{4}{1}\right) = f(4)$

Substitute $x = 4$ and evaluate $f(4)$

Example 10

When $8x^4 - 4x^3 + ax^2 - 1$ is divided by $(2x + 1)$ the remainder is 3. Find the value of a.

$f(x) = 8x^4 - 4x^3 + ax^2 - 1$

$$f\left(-\frac{1}{2}\right) = 3$$

$$8\left(-\frac{1}{2}\right)^4 - 4\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 - 1 = 3$$

$$8\left(\frac{1}{16}\right) - 4\left(-\frac{1}{8}\right) + a\left(\frac{1}{4}\right) - 1 = 3$$

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{4}a - 1 = 3$$

$$\frac{1}{4}a = 3$$

$$a = 12$$

Problem-solving

Use the remainder theorem: If $f(x)$ is divided by $(ax - b)$, then the reminder is $f\left(\frac{b}{a}\right)$.

Compare $(2x + 1)$ to $(ax - b)$, so $a = 2$, $b = -1$ and the remainder is $f\left(-\frac{1}{2}\right)$.

Using the fact that the remainder is 3, substitute $x = -\frac{1}{2}$ and solve the equation for a.

Exercise 1D

SKILLS INTERPRETATION

1 Find the remainder when:

a $4x^3 - 5x^2 + 7x + 1$ is divided by $(x - 2)$

b $2x^5 - 32x^3 + x - 10$ is divided by $(x - 4)$

c $-2x^3 + 6x^2 + 5x - 3$ is divided by $(x + 1)$

d $7x^3 + 6x^2 - 45x + 1$ is divided by $(x + 3)$

e $4x^4 - 4x^2 + 8x - 1$ is divided by $(2x - 1)$

f $243x^4 - 27x^3 - 3x + 7$ is divided by $(3x - 1)$

g $64x^3 + 32x^2 - 16x + 9$ is divided by $(4x + 3)$

h $81x^3 - 81x^2 + 9x + 6$ is divided by $(3x - 2)$

i $243x^6 - 780x^2 + 6$ is divided by $(3x + 4)$

j $125x^4 + 5x^3 - 9x$ is divided by $(5x + 3)$.

P **2** When $2x^3 - 3x^2 - 2x + a$ is divided by $(x - 1)$ the remainder is -4. Find the value of a.

P **3** When $-3x^3 + 4x^2 + bx + 6$ is divided by $(x + 2)$ the remainder is 10. Find the value of b.

P **4** When $216x^3 - 32x^2 + cx - 8$ is divided by $(2x - 1)$ the remainder is 1. Find the value of c.

5 Show that $(x - 3)$ is a factor of $x^6 - 36x^3 + 243$.

6 Show that $(2x - 1)$ is a factor of $2x^3 + 17x^2 + 31x - 20$.

E/P **7** $f(x) = x^2 + 3x + q$

Given $f(2) = 3$, find $f(-2)$. **(5 marks)**

Hint First find q.

E/P **8** $g(x) = x^3 + ax^2 + 3x + 6$

Given $g(-1) = 2$, find the remainder when $g(x)$ is divided by $(3x - 2)$. **(5 marks)**

E/P **9** The expression $2x^3 - x^2 + ax + b$ gives a remainder of 14 when divided by $(x - 2)$ and a remainder of -86 when divided by $(x + 3)$.
Find the value of a and the value of b. **(5 marks)**

E/P **10** The expression $3x^3 + 2x^2 - px + q$ is divisible by $(x - 1)$ but leaves a remainder of 10 when divided by $(x + 1)$. Find the value of a and the value of b. **(5 marks)**

Problem-solving Solve simultaneous equations.

1.5 Mathematical proof

A proof is a logical and **structured** argument to show that a mathematical statement (or **conjecture**) is always true.
A mathematical proof usually starts with previously **established** mathematical facts (or **theorems**) and then works through a series of logical steps. The final step in a proof is a **statement** of what has been proven.

Notation A statement that has been proven is called a **theorem**.
A statement that has yet to be proven is called a **conjecture**.

Known facts or theorems → Clearly shown logical steps → Statement of proof

A mathematical proof needs to show that something is true in every case.

- You can prove a mathematical statement is true by **deduction**. This means starting from known facts or definitions, then using logical steps to reach the **desired** conclusion.

Here is an example of proof by deduction:

Statement: The product of two **odd numbers** is odd.

Demonstration: $5 \times 7 = 35$, which is odd

This is demonstration but it is not a proof. You have only shown one case.

Proof: p and q are **integers**, so $2p + 1$ and $2q + 1$ are odd numbers.

You can use $2p + 1$ and $2q + 1$ to represent any odd numbers. If you can show that $(2p + 1) \times (2q + 1)$ is always an odd number then you have proved the statement for all cases.

$$\begin{aligned}(2p + 1) \times (2q + 1) &= 4pq + 2p + 2q + 1\\ &= 2(2pq + p + q) + 1\end{aligned}$$

Since p and q are integers, $2pq + p + q$ is also an integer.

So $2(2pq + p + q) + 1$ is one more than an **even number**.

So the product of two odd numbers is an odd number.

This is the statement of proof.

- In a mathematical proof you must
  - **State any information or assumptions you are using**
  - **Show every step of your proof clearly**
  - **Make sure that every step follows logically from the previous step**
  - **Make sure you have covered all possible cases**
  - **Write a statement of proof at the end of your working**

You need to be able to prove results involving **identities**, such as $(a + b)(a - b) \equiv a^2 - b^2$

- To prove an identity you should
  - **Start with the expression on one side of the identity**
  - **Manipulate that expression algebraically until it matches the other side**
  - **Show every step of your algebraic working**

Notation The symbol $\equiv$ means 'is always equal to'. It shows that two expressions are mathematically **identical**.

Watch out Don't try to 'solve' an identity like an equation. Start from one side and manipulate the expression to match the other side.

Example 11

SKILLS REASONING/ARGUMENTATION

Prove that $(3x + 2)(x - 5)(x + 7) \equiv 3x^3 + 8x^2 - 101x - 70$

$(3x + 2)(x - 5)(x + 7)$
$= (3x + 2)(x^2 + 2x - 35)$
$= 3x^3 + 6x^2 - 105x + 2x^2 + 4x - 70$
$= 3x^3 + 8x^2 - 101x - 70$
So
$(3x + 2)(x - 5)(x + 7) \equiv 3x^3 + 8x^2 - 101x - 70$

Start with the left-hand side and expand the brackets.

In proof questions you need to show all your working.

Left-hand side = right-hand side.

Example 12

Prove that if $(x - p)$ is a factor of $f(x)$ then $f(p) = 0$.

If $(x - p)$ is a factor of $f(x)$ then
$f(x) = (x - p) \times g(x)$
So $f(p) = (p - p) \times g(p)$
i.e. $f(p) = 0 \times g(p)$
So $f(p) = 0$ as required.

$g(x)$ is a polynomial expression.

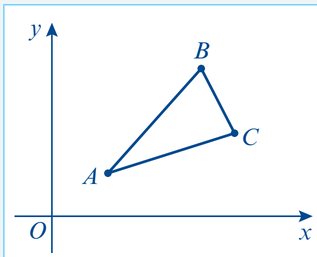
Substitute $x = p$.

$p - p = 0$

Remember $0 \times$ *anything* $= 0$

Example 13

Prove that $A(1, 1)$, $B(3, 3)$ and $C(4, 2)$ are the **vertices** of a right-angled triangle.



The gradient of line $AB = \frac{3 - 1}{3 - 1} = \frac{2}{2} = 1$

Problem-solving If you need to prove a geometrical result, it can sometimes help to sketch a diagram as part of your working.

The gradient of a line $= \frac{y_2 - y_1}{x_2 - x_1}$

The gradient of line $BC = \frac{2-3}{4-3} = \frac{-1}{1} = -1$

The gradient of line $AC = \frac{2-1}{4-1} = \frac{1}{3}$

The gradients are different so the three points are not collinear.

Hence ABC is a triangle.

Gradient of $AB \times$ gradient of $BC = 1 \times (-1)$
$= -1$

If the product of two gradients is −1 then the two lines are **perpendicular**.
Gradient of line $AB \times$ gradient of line $BC = -1$

So AB is perpendicular to BC,
and the triangle is a right-angled triangle.

Remember to state what you have proved.

Example 14 SKILLS REASONING/ARGUMENTATION

The equation $kx^2 + 3kx + 2 = 0$, where k is a constant, has no real roots.

Prove that k satisfies the inequality $0 \leqslant k < \frac{8}{9}$

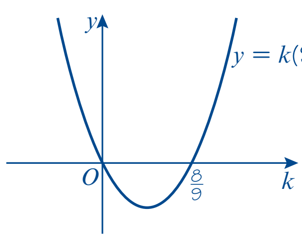
$kx^2 + 3kx + 2 = 0$ has no real roots,
so $b^2 - 4ac < 0$

State which assumption or information you are using at each stage of your proof.

$(3k)^2 - 4k(2) < 0$
$9k^2 - 8k < 0$
$k(9k - 8) < 0$

Use the discriminant. ← Pure 1 Section 2.5

Solve this quadratic inequality by sketching the graph of $y = k(9k - 8)$ ← Pure 1 Section 3.5



The graph shows that when $k(9k - 8) < 0$, $0 < k < \frac{8}{9}$

$0 < k < \frac{8}{9}$

When $k = 0$:

Be really careful to consider all the possible situations. You can't use the discriminant if $k = 0$ so look at this case separately.

$(0)x^2 + 3(0)x + 2 = 0$
$2 = 0$

Which is impossible, so no real roots

Write out all of your conclusions clearly.

So combining these:

$0 \leqslant k < \frac{8}{9}$ as required

$0 < k < \frac{8}{9}$ together with $k = 0$, gives $0 \leqslant k < \frac{8}{9}$

Exercise 1E

SKILLS REASONING/ARGUMENTATION

Hint: The proofs in this exercise are all proofs by deduction.

P **1** Prove that $n^2 - n$ is an even number for all values of n.

P **2** Prove that $\frac{x}{1+\sqrt{2}} \equiv x\sqrt{2} - x$.

P **3** Prove that $(x + \sqrt{y})(x - \sqrt{y}) \equiv x^2 - y$.

P **4** Prove that $(2x - 1)(x + 6)(x - 5) \equiv 2x^3 + x^2 - 61x + 30$.

P **5** Prove that $x^2 + bx \equiv \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

P **6** Prove that the solutions of $x^2 + 2bx + c = 0$ are $x = -b \pm \sqrt{b^2 - c}$.

P **7** Prove that $\left(x - \frac{2}{x}\right)^3 \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$

P **8** Prove that $\left(x^3 - \frac{1}{x}\right)\left(x^{\frac{3}{2}} + x^{-\frac{5}{2}}\right) \equiv x^{\frac{1}{2}}\left(x^4 - \frac{1}{x^4}\right)$

P **9** Use completing the square to prove that $3n^2 - 4n + 10$ is positive for all values of n.

P **10** Use completing the square to prove that $-n^2 - 2n - 3$ is negative for all values of n.

Problem-solving: Any expression that is squared must be $\geqslant 0$.

E/P **11** Prove that $x^2 + 8x + 20 \geqslant 4$ for all values of x. **(3 marks)**

E/P **12** The equation $kx^2 + 5kx + 3 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \leqslant k < \frac{12}{25}$ **(4 marks)**

E/P **13** The equation $px^2 - 5x - 6 = 0$, where p is a constant, has two distinct real roots. Prove that p satisfies the inequality $p > -\frac{25}{24}$ **(4 marks)**

P **14** Prove that $A(3, 1)$, $B(1, 2)$ and $C(2, 4)$ are the vertices of a right-angled triangle.

P **15** Prove that **quadrilateral** $A(1, 1)$, $B(2, 4)$, $C(6, 5)$ and $D(5, 2)$ is a parallelogram.

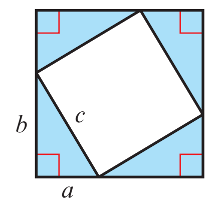
P **16** Prove that quadrilateral $A(2, 1)$, $B(5, 2)$, $C(4, -1)$ and $D(1, -2)$ is a rhombus.

P **17** Prove that $A(-5, 2)$, $B(-3, -4)$ and $C(3, -2)$ are the vertices of an isosceles right-angled triangle.

E/P **18** A **circle** has equation $(x - 1)^2 + y^2 = k$, where $k > 0$.
The straight line L with equation $y = ax$ cuts the circle at two distinct points.
Prove that $k > \dfrac{a^2}{1 + a^2}$ **(6 marks)**

E/P **19** Prove that the line $4y - 3x + 26 = 0$ is a **tangent** to the circle $(x + 4)^2 + (y - 3)^2 = 100$. **(5 marks)**

P **20** The diagram shows a square and four congruent right-angled triangles.
Use the diagram to prove that $a^2 + b^2 = c^2$.



Problem-solving
Find an expression for the area of the large square in terms of a and b.

Challenge

SKILLS CREATIVITY

1 Prove that $A(7, 8)$, $B(-1, 8)$, $C(6, 1)$ and $D(0, 9)$ are points on the same circle.

2 Prove that any odd number can be written as the **difference** of two squares.

1.6 Methods of proof

A mathematical statement can be proved by **exhaustion**. For example, you can prove that the sum of two **consecutive** square numbers between 100 and 200 is an odd number. The square numbers between 100 and 200 are 121, 144, 169, 196.

121 + 144 = 265 which is odd   144 + 169 = 313 which is odd   169 + 196 = 365 which is odd

So the sum of two consecutive square numbers between 100 and 200 is an odd number.

- You can prove a mathematical statement is true by exhaustion. This means breaking the statement into smaller cases and proving each case separately.

This method is better suited to a small number of results. You cannot use one example to prove a statement is true, as one example is only one case.

Example 15

Prove that all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

For odd numbers:

$(2n + 1)^2 = 4n^2 + 4n + 1 = 4n(n + 1) + 1$

$4n(n + 1)$ is a multiple of 4, so $4n(n + 1) + 1$ is 1 more than a multiple of 4.

Problem-solving
Consider the two cases, odd and even numbers, separately.

You can write any odd number in the form $2n + 1$ where n is a positive integer.

For even numbers:

$(2n)^2 = 4n^2$

You can write any even number in the form $2n$ where n is a positive integer.

$4n^2$ is a multiple of 4.
All integers are either odd or even, so all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

A mathematical statement can be disproved using a **counter-example**. For example, to prove that the statement '$3n + 3$ is a multiple of 6 for all values of n' is not true you can use the counter-example when $n = 2$, as $3 \times 2 + 3 = 9$ and 9 is not a multiple of 6.

- **You can prove a mathematical statement is not true by a counter-example. A counter-example is an example that does not work for the statement. You do not need to give more than one, as one is sufficient to disprove a statement.**

Example 16

Prove that the following statement is **not** true:

'The sum of two consecutive prime numbers is always even.'

2 and 3 are both prime
$2 + 3 = 5$
5 is odd

You only need one counter-example to show that the statement is false.

So the statement is not true.

Example 17

SKILLS REASONING/ARGUMENTATION

a Prove that for all positive values of x and y:

$$\frac{x}{y} + \frac{y}{x} \geqslant 2$$

b Use a counter-example to show that this is not true when x and y are not both positive.

Watch out You must always start a proof from **known facts**. Never start your proof with the statement you are trying to prove.

a Jottings:

$\frac{x}{y} + \frac{y}{x} \geqslant 2$

$\frac{x^2 + y^2}{xy} \geqslant 2$

$x^2 + y^2 - 2xy \geqslant 0$

$(x - y)^2 \geqslant 0$

Proof:

Consider $(x - y)^2$

$(x - y)^2 \geqslant 0$

$x^2 + y^2 - 2xy \geqslant 0$

$\frac{x^2 + y^2 - 2xy}{xy} \geqslant 0$

This step is valid because x and y are both positive so $xy > 0$.

$\frac{x}{y} + \frac{y}{x} - 2 \geqslant 0$

$\frac{x}{y} + \frac{y}{x} \geqslant 2$

b Try $x = -3$ and $y = 6$

$\frac{-3}{6} + \frac{6}{-3} = -\frac{1}{2} - 2 = -\frac{5}{2}$

This is not $\geqslant 2$ so the statement is not true.

Problem-solving

Use jottings to get some ideas for a good starting point. These don't form part of your proof, but can give you a clue as to what expression you can consider to begin your proof.

Now you are ready to start your proof. You know that any expression squared is $\geqslant 0$. This is a **known fact** so this is a valid way to begin your proof.

State how you have used the fact that x and y are positive in your proof. If $xy = 0$ you couldn't divide the LHS by xy, and if $xy < 0$, then the direction of the inequality would be reversed.

This was what you wanted to prove so you have finished.

Your working for part **a** tells you that the proof fails when $xy < 0$, so try one positive and one negative value.

Exercise 1F

SKILLS REASONING/ARGUMENTATION

P **1** Prove that when n is an integer and $1 \leqslant n \leqslant 6$, then $m = n + 2$ is not divisible by 10.

Hint You can try each integer for $1 \leqslant n \leqslant 6$.

P **2** Prove that every odd integer between 2 and 26 is either prime or the product of two primes.

P **3** Prove that the sum of two consecutive square numbers from 1^2 to 8^2 is an odd number.

E/P **4** Prove that all cube numbers are either a multiple of 9 or 1 more or 1 less than a multiple of 9. **(4 marks)**

P **5** Find a counter-example to disprove each of the following statements:

a If n is a positive integer then $n^4 - n$ is divisible by 4.

b Integers always have an even number of factors.

c $2n^2 - 6n + 1$ is positive for all values of n.

d $2n^2 - 2n - 4$ is a multiple of 3 for all integer values of n.

E/P **6** A student is trying to prove that $x^3 + y^3 < (x + y)^3$.

The student writes:

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
which is less than $x^3 + y^3$ since
$3x^2y + 3xy^2 > 0$

Problem-solving

For part **b** you need to write down suitable values of x and y and show that they do not satisfy the inequality.

a Identify the error made in the proof. **(1 mark)**

b Provide a counter-example to show that the statement is not true. **(2 marks)**

E/P **7** Prove that for all real values of x

$$(x + 6)^2 \geqslant 2x + 11$$ **(3 marks)**

E/P **8** Given that a is a positive real number, prove that:

$$a + \frac{1}{a} \geqslant 2$$

Watch out Remember to state how you use the condition that a is positive.

(2 marks)

E/P **9** **a** Prove that for any positive numbers p and q:

$$p + q \geqslant \sqrt{4pq}$$ **(3 marks)**

b Show, by means of a counter-example, that this inequality does not hold when p and q are both negative. **(2 marks)**

Problem-solving

Use jottings and work backwards to work out what expression to consider.

E/P **10** It is claimed that the following inequality is true for all negative numbers x and y:

$$x + y \geqslant \sqrt{x^2 + y^2}$$

The following proof is offered by a student:

$x + y \geqslant \sqrt{x^2 + y^2}$
$(x + y)^2 \geqslant x^2 + y^2$
$x^2 + y^2 + 2xy \geqslant x^2 + y^2$
$2xy > 0$ which is true because x and y are both negative, so xy is positive.

a Explain the error made by the student. **(2 marks)**

b By use of a counter-example, verify that the inequality is not satisfied if both x and y are negative. **(1 mark)**

c Prove that this inequality is true if x and y are both positive. **(2 marks)**

Chapter review 1

1 Simplify these fractions as far as possible:

a $\dfrac{3x^4 - 21x}{3x}$

b $\dfrac{x^2 - 2x - 24}{x^2 - 7x + 6}$

c $\dfrac{2x^2 + 7x - 4}{2x^2 + 9x + 4}$

2 Divide $3x^3 + 12x^2 + 5x + 20$ by $(x + 4)$.

3 Simplify $\dfrac{2x^3 + 3x + 5}{x + 1}$

E 4 a Show that $(x - 3)$ is a factor of $2x^3 - 2x^2 - 17x + 15$. **(2 marks)**

b Hence express $2x^3 - 2x^2 - 17x + 15$ in the form $(x - 3)(Ax^2 + Bx + C)$, where the values A, B and C are to be found. **(3 marks)**

E 5 Find the remainder when $16x^5 - 20x^4 + 8$ is divided by $(2x - 1)$. **(2 marks)**

E 6 a Show that $(x - 2)$ is a factor of $x^3 + 4x^2 - 3x - 18$. **(2 marks)**

b Hence express $x^3 + 4x^2 - 3x - 18$ in the form $(x - 2)(px + q)^2$, where the values p and q are to be found. **(4 marks)**

E 7 Factorise completely $2x^3 + 3x^2 - 18x + 8$. **(6 marks)**

E/P 8 Find the value of k if $(x - 2)$ is a factor of $x^3 - 3x^2 + kx - 10$. **(4 marks)**

E/P 9 $f(x) = 2x^2 + px + q$. Given that $f(-3) = 0$, and $f(4) = 21$:

a find the value of p and q **(6 marks)**

b factorise $f(x)$. **(3 marks)**

E/P 10 $h(x) = x^3 + 4x^2 + rx + s$. Given $h(-1) = 0$, and $h(2) = 30$:

a find the values of r and s **(6 marks)**

b factorise $h(x)$. **(3 marks)**

E 11 $g(x) = 2x^3 + 9x^2 - 6x - 5$.

a Factorise $g(x)$. **(6 marks)**

b Solve $g(x) = 0$. **(2 marks)**

E **12** **a** Show that $(x - 2)$ is a factor of $f(x) = x^3 + x^2 - 5x - 2$. **(2 marks)**

b Hence, or otherwise, find the exact solutions of the equation $f(x) = 0$. **(4 marks)**

E **13** Given that -1 is a root of the equation $2x^3 - 5x^2 - 4x + 3$, find the two positive roots. **(4 marks)**

E/P **14** The remainder obtained when $x^3 - 5x^2 + px + 6$ is divided by $(x + 2)$ is equal to the remainder obtained when the same expression is divided by $(x - 3)$.

Find the value of p. **(4 marks)**

E **15** $f(x) = x^3 - 2x^2 - 19x + 20$

a Show that $(x + 4)$ is a factor of $f(x)$. **(3 marks)**

b Hence, or otherwise, find all the solutions to the equation $x^3 - 2x^2 - 19x + 20 = 0$. **(4 marks)**

E **16** $f(x) = 6x^3 + 17x^2 - 5x - 6$

a Show that $f(x) = (3x - 2)(ax^2 + bx + c)$, where a, b and c are constants to be found. **(2 marks)**

b Hence factorise $f(x)$ completely. **(4 marks)**

c Write down all the real roots of the equation $f(x) = 0$. **(2 marks)**

17 Prove that $\dfrac{x - y}{\sqrt{x} - \sqrt{y}} \equiv \sqrt{x} + \sqrt{y}$.

P **18** Use completing the square to prove that $n^2 - 8n + 20$ is positive for all values of n.

P **19** Prove that the quadrilateral $A(1, 1)$, $B(3, 2)$, $C(4, 0)$ and $D(2, -1)$ is a square.

P **20** Prove that the sum of two consecutive positive odd numbers less than ten gives an even number.

P **21** Prove that the statement '$n^2 - n + 3$ is a prime number for all values of n' is untrue.

P **22** Prove that $\left(x - \dfrac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right) \equiv x^{\frac{1}{3}}\left(x^2 - \dfrac{1}{x^2}\right)$.

P **23** Prove that $2x^3 + x^2 - 43x - 60 \equiv (x + 4)(x - 5)(2x + 3)$.

E **24** The equation $x^2 - kx + k = 0$, where k is a positive constant, has two equal roots.
Prove that $k = 4$. **(3 marks)**

P **25** Prove that the distance between opposite edges of a regular hexagon of side length $\sqrt{3}$ is a rational value.

(P) **26** **a** Prove that the difference of the squares of two consecutive even numbers is always divisible by 4.

b Is this statement true for odd numbers? Give a reason for your answer.

(E) **27** A student is trying to prove that $1 + x^2 < (1 + x)^2$.
The student writes:

$(1 + x)^2 = 1 + 2x + x^2$.
So $1 + x^2 < 1 + 2x + x^2$.

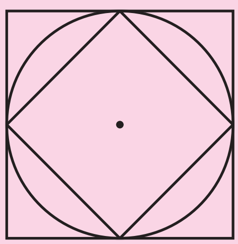
a Identify the error made in the proof. **(1 mark)**

b Provide a counter-example to show that the statement is not true. **(2 marks)**

Challenge

SKILLS INNOVATION

1 The diagram shows two squares and a circle.



a Given that π is defined as the circumference of a circle of **diameter** 1 unit, prove that $2\sqrt{2} < \pi < 4$.

b By similarly constructing regular hexagons inside and outside a circle, prove that $3 < \pi < 2\sqrt{3}$.

2 Prove that if $f(x) = ax^3 + bx^2 + cx + d$ and $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$.

Summary of key points

1 When simplifying an algebraic fraction, factorise the numerator and denominator where possible and then cancel common factors.

2 You can use long division to divide a polynomial by $(x \pm p)$, where p is a constant.

3 The **factor theorem** states that if f(x) is a polynomial then:

- If f(p) = 0, then $(x - p)$ is a factor of f(x)
- If $(x - p)$ is a factor of f(x), then f(p) = 0
- If $\mathrm{f}\left(\frac{b}{a}\right) = 0$ then $(ax - b)$ is a factor

4 The **remainder theorem** states that if a polynomial f(x) is divided by $(ax - b)$ then the remainder is $\mathrm{f}\left(\frac{b}{a}\right)$

5 You can prove a mathematical statement is true by **deduction**. This means starting from known facts or definitions, then using logical steps to reach the desired conclusion.

6 In a mathematical proof you must

- State any information or assumptions you are using
- Show every step of your proof clearly
- Make sure that every step follows logically from the previous step
- Make sure you have covered all possible cases
- Write a statement of proof at the end of your working

7 To prove an identity you should

- Start with the expression on one side of the identity
- Manipulate that expression algebraically until it matches the other side
- Show every step of your algebraic working

8 You can prove a mathematical statement is true by **exhaustion**. This means breaking the statement into smaller cases and proving each case separately.

9 You can prove a mathematical statement is not true by a **counter-example**. A counter-example is one example that does not work for the statement. You do not need to give more than one example, as one is sufficient to disprove a statement.