1 What's a group?

At PROMYS, we have spent the bulk of our time investigating the ring $\mathbb{Z}_m$[1], with many interesting properties. But in some sense, rings are too complicated — after all, they have two operations, addition and multiplication. If we look at just the addition properties, we can define a new mathematical structure which is important and interesting in its own right.

Definition 1.1. A **group** is a pair $(G, *)$, with $*$ a binary operation on G, such that the following properties hold:

- G is **associative** under $*$: for all $a, b, c \in G$,

$$(a * b) * c = a * (b * c).$$

- There exists an **identity element** $e \in G$ such that $e * a = a * e = a$ for all $a \in G$.
- For each element $a \in G$, there exists an **inverse element** $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

Remark 1.2. Note we do not require $*$ to be commutative (i.e. $a * b = b * a$ for all $a, b \in G$). If it is, we say that G is **commutative** or **abelian**, after the Norwegian mathematician Niels Henrik Abel.

Remark 1.3. This is a pedantic note, but some texts add an additional property that G be **closed** under $*$. This is already implied from the definition of a binary operation, but for completeness' sake we mention it here.

2 References

The standard reference for anything abstract algebra is Dummit and Foote, but it's an admittedly huge, dense book which isn't the best for beginners. I would instead recommend Gallian, which has all around friendly exposition, as well as Artin, a very idiosyncratic book which I learned from and still use as a reference. It is probably a bit too wordy for PROMYS students, but for absolute beginners to proof-based math I would recommend Pinter. If you're feeling up to the task, go for Dummit and Foote.

For a more unique approach, I would recommend Matt Macauley's notes here (`http://www.math.clemson.edu/~macaule/classes/s22_math4120/index.html`), which he is currently turning into a full-fledged textbook with projected completion by the end of 2022. His approach is far more visual, which he believes is the correct and more intuitive way to teach the subject, and I am excited to see his final product. This is definitely worth looking at if you are interested in another perspective than the one presented here and in other textbooks.

I am also a fan of Evan Chen's Napkin: `https://web.evanchen.cc/napkin.html`. It is a more casual, succinct explanation of a good deal of modern mathematics, and I would recommend it to get a broad understanding and intuition for a topic, which can prove helpful to know for further research. As Evan Chen comes from an olympiad background, his style is biased towards that direction, but I still find the exposition quite helpful.

[1]This is known by most non-PROMYS mathematicians as $\mathbb{Z}/m\mathbb{Z}$, notation which will make sense after this talk.