Reference Problem Set

Shef Scholars

If you're unsure which level to pick, there are reference practice problems that you can find here.

If you can solve 50% of practice problems for a level, you should consider taking the test and applying for that level. If you cannot solve more than 50% of the practice problems at a particular level, I would apply to the highest level for which you can solve at least 50%.

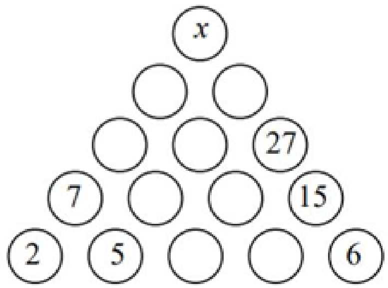
If you're unsure which level you would be best suited for, take the test for the level you feel most comfortable with, attach both the test and practice problems solutions, and note in your application.

You're allowed as much time as you want with these problems in evaluating whether or not you can solve them.

There is a final level we call Super-Olympic-Shef that we don't have classes for this time. If you can solve over 50% of those problems, you might not get much out of our content at the Shef level. If you're able to get all those right, do send me us an email at shefscholars@gmail.com so you can teach us :P

The Beginner-Problem Set

1. In the figure, each circle contains a number equal to the sum of the two numbers in the circles directly below it. What is the value of x?



2. If the operation $a * b$ is defined by

$$a * b = \frac{a \times b}{a + b}$$

for positive integers a, b, then what is $5 * 10$?

Possible answers:

(a) $\frac{3}{10}$
(b) 1
(c) 2
(d) $\frac{10}{3}$
(e) 50

3. The movie ticket printing machine always prints 4 digits for the ticket number. If the ticket number is a one-digit, two-digit, or three-digit number, a certain number of zeros is added in front of the number. For example, ticket number 5 is printed as 0005, and ticket number 343 is printed as 0343. How many zeros does the printer print in total for ticket numbers from 1 to 212?

4. John is 26 years older than Anna, and in 10 years, he will be three times as old. How old are John and Anna?

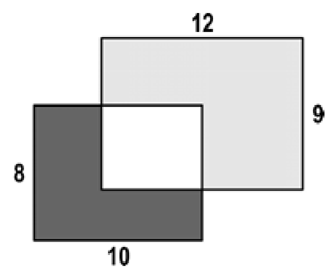
5. Today is Sunday. Francis starts to read a book with 290 pages today. On Sundays, he reads 25 pages, and on all other days, he reads 4 pages, with no exception. How many days does it take him to read the entire book?

   Possible answers:

   (a) 5
   (b) 46
   (c) 40
   (d) 35
   (e) 41

6. Nina wrote the integers from 1 to 9 in some order into the cells of a 3×3 square. She calculated the sum of the numbers in each row and each column. Five of these sums are 12, 13, 15, 16, and 17. What is the sixth sum?

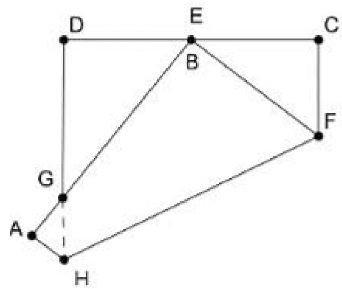
7. Two rectangles with measurements 8×10 and 9×12 overlap to some extent. The dark grey area is 37. What is the area of the light grey part?



   Possible answers:

   (a) 60
   (b) 62
   (c) 62.5
   (d) 64
   (e) 65

8. A square $ABCD$ with side length 9 cm is given. The square is folded, as shown in the figure, so that vertex B falls onto point E, which is the midpoint of side CD. This creates three triangles: CEF, EDG, and GAH. What is the sum of the perimeters of these three triangles?

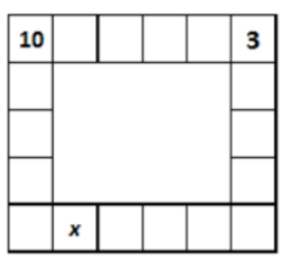


9. Goku, Piccolo, Vegeta, and Roshi are people who either always lie or always tell the truth. Goku claims that Piccolo is lying. Roshi says that Goku is lying. Vegeta says that both Goku and Piccolo are lying. Vegeta also says that Roshi is a liar. Which of them always lie, and which always tell the truth?

10. There are three pens labeled A, B, and C. One is red, one is white, and one is blue. Determine the color of each pen if only one of the following statements is true:

    - "A is red"
    - "B is not red"
    - "C is not blue"

11. Maya wants to write numbers in each cell of the board shown in the figure. Any integer could be written in a cell and an integer could be written multiple times in the board. Each number she writes is equal to the sum of two numbers in the neighboring cells that share a side with it. Two numbers are already written, as shown in the figure. What number will she write in the cell marked x?



12. There are three identical boxes on the table, differing only in color: black, white, and gray. One of them contains two black balls, another contains one black and one white ball, and the third contains two white balls.

    The black box is labeled “two black”, the white box “two white”, and the gray box “one black and one white”. However, it is known that none of these labels are correct.

    Is it possible, without looking into any box, to draw just one ball from one box and, based on its color, determine the exact contents of the black, white, and gray boxes?

The Apprentice-Problem Set

1. Every day at school, Jo climbs a flight of 6 stairs. Jo can take the stairs 1, 2, or 3 at a time. For example, Jo could climb 3, then 1, then 2. In how many different ways can Jo climb the stairs?

2. Out of 70 fifth-grade students, 27 are members of the drama club, 32 sing in the choir, and 22 are athletes. There are 16 students who are both in the drama club and the choir, 6 who are both in the choir and athletes, and 8 who are both in the drama club and athletes. Three athletes are also in both the drama club and the choir.

   (a) How many students are not in any of the three groups?

   (b) How many students are only athletes?

3. Determine all $n \in \mathbb{N}$ such that the following expression is a natural number:
$$\frac{5n+3}{2n+1}$$

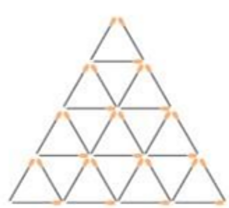
4. At the Olympiad, there are participants from four countries: Spain, Belgium, Germany, and France. There are three times as many participants from Spain as from Belgium, and three times as many from Germany as from France. Five contestants counted all participants at the Olympiad (including themselves) and obtained the following totals: 366, 367, 368, 369, and 370. Sheldon immediately realized that only one of them counted correctly. How did Sheldon figure this out, and what is the correct number of participants?

5. (a) Can the set $\{1, 2, 3, \ldots, 2024\}$ be divided into 1012 pairs such that the sum of the numbers in each pair is divisible by 3?

   (b) Can the set $\{1, 2, 3, \ldots, 2026\}$ be divided into 1013 pairs such that the sum of the numbers in each pair is divisible by 3?

   (c) Can the set $\{1, 2, 3, \ldots, 2028\}$ be divided into 1014 pairs such that the sum of the numbers in each pair is divisible by 3?

6. We arrange matchsticks to form a configuration like the one shown in the figure below. How many matchsticks are needed if the border consists of 100 matchsticks?



7. A 16×16 board is given. Each row and each column is labeled with numbers from 1 to 16. On each square, we place coins such that the number of coins on a square is equal to the product of its row and column numbers.

   For example, square $(4, 2)$ will have 8 coins, since $4 \cdot 2 = 8$. The notation $(4, 2)$ refers to the square in the 4^{th} row and 2^{nd} column. How many coins are placed on the entire board?

8. The sum of two three-digit numbers is divisible by 37. Prove that the six-digit number formed by writing one number immediately after the other (in any order) is also divisible by 37.

9. If $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$, what is the value of $a^4 + b^4 + c^4$?

10. A magic square is a square of side length n, divided into $n \times n$ unit squares, where each unit square contains a number such that the sums of the numbers in all rows, all columns, and both main diagonals are equal. All numbers in a standard magic square are distinct. The sum of the elements in any row, column, or diagonal is called the *magic sum* (or characteristic sum).

    a) In every 3×3 magic square, the magic sum is three times the number written in the center cell of the square. Prove this!

    a) Determine the magic sum for the basic magic square of size 12×12 (which has 144 cells).

11. James thought of 5 numbers. He then wrote down 10 numbers, which are all the possible sums of each pair of those numbers. For example, if James thinks of the numbers 1, 2, 3, 4, 5, he would write the numbers: 3, 4, 5, 5, 6, 6, 7, 7, 8, 9.

    James wrote down the following numbers: 129, 159, 172, 181, 194, 224, 240, 253, 283, 305. Which 5 numbers did he originally think of?

12. When dividing the numbers 287 and 431 by a natural number n, the remainders are 1 and 2 respectively. Also, when dividing the number 231 by $n + 1$, the remainder is 3. Determine all such values of n.

13. A 5×5 table is filled with numbers from the set $\{-1, 0, 1\}$. Then, the sums are calculated for each row, each column, and both diagonals. Prove that no matter how the table is filled, there will always be at least two of these sums that are equal.

14. Let AD, BE, and CF be the altitudes of an acute triangle ABC, and let M and N be the reflections of point D across lines AC and AB, respectively. Prove that the points E, F, M, and N are collinear.

    Note: The reflection of a point P across a line ℓ is the point P' such that line ℓ is the perpendicular bisector of the segment PP'. In other words, ℓ

is the line of symmetry, and the distances from P and P' to the line ℓ are equal, with PP' perpendicular to ℓ.

The Machine-Problem Set

1. There are 17 scientists, each communicating with every other scientist. They discuss exactly three topics in total, but every pair of scientists communicates about exactly one topic. Prove that there exist at least three scientists who mutually communicate about the same topic.

2. Given the table shown in the figure below, prove that the sum of the numbers above the empty diagonal is twice the sum of the numbers below it.

	2	3	4	...	n
1		3	4	...	n
1	2		4	...	n
1	2	3		...	n
...	...	...	...		...
1	2	3	4	...	

3. Find the difference between the smallest six-digit number divisible by 45 and the smallest six-digit number divisible by 18.

4. At a party, some people were dancing. Each dance involved one male and one female. Every male danced with 3 girls, and every female danced with 4 boys. If there were 9 girls at the party, how many boys were there?

5. How many 6-tuples of positive integers $(x_1, x_2, x_3, x_4, x_5, x_6)$ exist such that all the following hold:

   (a) $x_1 \geq 5$

   (b) $x_2 \leq 10$

   (c) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20$

6. Prove that if p is a prime number, then $7p + 3^p - 4$ is not a perfect square.

7. Find all triplets (p, a, b) of positive integers such that

$$p = b\sqrt{\frac{a - 8b}{a + 8b}}$$

   is prime

8. In a tennis tournament, each player played against every other player exactly once (in tennis, there is no tie). After the tournament, each player created a list that included:

- All the players they defeated.
- All the players who were defeated by someone they defeated (i.e., if A defeated B and B defeated C, then C is also on A's list).

Prove that there is at least one player whose list contains all the other players.

9. Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

10. There is a queue of 99 people. Some of them are knights (who always tell the truth), and some are liars (who always lie). Each person shouted one of two statements:

- "To my left, there are twice as many knights as liars."
- "To my left, there are as many knights as liars."

It is known that there are more knights than liars in total, and that more than 50 people said the first statement. How many knights said the first statement?

11. item The set of natural numbers is divided into subsets in the following way:

$$\{1, 2\},\ \{3, 4, 5\},\ \{6, 7, 8, 9\},\ \{10, 11, 12, 13, 14\}, \ldots$$

Each new subset has one more element than the previous one.

- What is the smallest element in the 100^{th} subset?
- Is the number 2015 the last element of some subset?

12. Let $a_0, a_1, \ldots, a_{2024}$ be real numbers such that $|a_{i+1} - a_i| \leq 1$ for $i = 0, 1, \ldots, 2023$.

a) Find the minimum possible value of

$$a_0a_1 + a_1a_2 + \cdots + a_{2023}a_{2024}$$

b) Does there exist a real number C such that

$$a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + \cdots + a_{2022}a_{2023} - a_{2023}a_{2024} \geq C$$

for all real numbers $a_0, a_1, \ldots, a_{2024}$ such that $|a_{i+1} - a_i| \leq 1$ for $i = 0, 1, \ldots, 2023$.

13. Let $ABCD$ be a parallelogram with an acute angle at vertex A. Let E be the foot of the perpendicular from C to line AB, and let F be the foot of the perpendicular from C to line AD. Prove that the following holds:

$$AB \cdot AE + AD \cdot AF = AC^2.$$

14. Let ABC be a triangle with $\angle ABC = \angle CAB = 50^\circ$. Inside this triangle a point O is given with $\angle BAO = 10^\circ$ and $\angle ABO = 30^\circ$. Calculate the angle $\angle ACO$.

15. Let ABC be an acute-angled triangle with $AB < AC$ and let O be the centre of its circumcircle ω. Let D be a point on the line segment BC such that $\angle BAD = \angle CAO$. Let E be the second point of intersection of ω and the line AD. If M, N and P are the midpoints of the line segments BE, OD and AC, respectively, show that the points M, N and P are collinear.

The Shef-Problem Set

1. Let ABC be a triangle with incentre I and circumcircle Ω such that $|AC| \neq |BC|$. The internal angle bisector of $\angle CAB$ intersects side BC at D and the external angle bisectors of $\angle ABC$ and $\angle BCA$ intersect Ω at E and F respectively. Let G be the intersection of lines AE and FI and let Γ be the circumcircle of triangle BDI. Show that E lies on Γ if and only if G lies on Γ.

2. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E , respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

3. Find all positive integers a and b for which there are three consecutive integers at which the polynomial

$$P(n) = \frac{n^5 + a}{b}$$

takes integer values.

4. Let a_1, a_2, $\ldots$, a_n be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1$, $3a_2$, $\ldots$, $3a_n$.

5. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$

6. Let ABC be a triangle and M be an interior point. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + AC + BC.$$

7. Let n be a positive integer. Two players, Alice and Bob, are playing the following game: - Alice chooses n real numbers; not necessarily distinct. - Alice writes all pairwise sums on a sheet of paper and gives it to Bob. (There are $\frac{n(n-1)}{2}$ such sums; not necessarily distinct.) - Bob wins if he finds correctly the initial n numbers chosen by Alice with only one guess. Can Bob be sure to win for the following cases?

   (a) $n = 5$
   (b) $n = 6$
   (c) $n = 8$

8. Let a, b, c, d and e be distinct positive real numbers such that $a^2 + b^2 + c^2 + d^2 + e^2 = ab + ac + ad + ae + bc + bd + be + cd + ce + de$

(a) Prove that among these 5 numbers there exists triplet such that they cannot be sides of a triangle

(b) Prove that, for a), there exists at least 6 different triplets

9. Show that

$$\left(a+2b+\frac{2}{a+1}\right)\left(b+2a+\frac{2}{b+1}\right)\geq 16$$

for all positive real numbers a and b such that $ab \geq 1$.

The Super-Olympic-Shef-Problem Set

1. Prove for all $k > 1$ equation $(x+1)(x+2)...(x+k) = y^2$ has finite solutions.

2. Prove that there are infinitely many positive integers a such that

$$a! + (a+2)! \mid (2a+2)!$$

3. Let x, y, z be real numbers from the interval $[0, 1]$. Determine the maximal value of

$$W = y \cdot \sqrt{1-x} + z \cdot \sqrt{1-y} + x \cdot \sqrt{1-z}$$

4. Simple graph G has 19998 vertices. For any subgraph $\bar{G}$ of G with 9999 vertices, $\bar{G}$ has at least 9999 edges. Find the minimum number of edges in G.

5. Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

6. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function for which the expression $af(a) + bf(b) + 2ab$ for all $a, b \in \mathbb{N}$ is always a perfect square. Prove that $f(a) = a$ for all $a \in \mathbb{N}$.

7. A unit square is divided into polygons, so that all sides of a polygon are parallel to sides of the given square. If the total length of the segments inside the square (without the square) is $2n$ (where n is a positive real number), prove that there exists a polygon whose area is greater than $\frac{1}{(n+1)^2}$.

8. 30 people are sitting at round table. $30 - N$ of them always tell the truth ("truth speakers") while the other N of them tell the truth sometimes and lie other times ("flip speakers"). The question: "Who is your right neighbour - "truth speaker" or "flip speaker" ?" is asked to all 30 people and 30 answers are collected. What is maximal number N for which (with knowledge of these answers) we can always deduce at least one person who is "truth speaker".

9. Let $A_1C_2B_1A_2C_1B_2$ be an equilateral hexagon. Let O_1 and H_1 denote the circumcenter and orthocenter of $\triangle A_1B_1C_1$, and let O_2 and H_2 denote the circumcenter and orthocenter of $\triangle A_2B_2C_2$. Suppose that $O_1 \neq O_2$ and $H_1 \neq H_2$. Prove that the lines O_1O_2 and H_1H_2 are either parallel or coincide.