TEST 1

Shef's Scholars

April 2022

1 Problems

Make sure to block out enough time for the test.

You may use your phone or mechanical clock as a timer.

Once you begin the exam, do not use the internet until you've finished the exam. Do not consult textbooks or outside sources. Any cheating may result in your application automatically being rejected and you being unable to apply to future iterations of Shef's Scholars or even other events organized by Shefs of Problem Solving.

Write up your solutions in English and put your email address (the same one you're putting in your application) on top of the paper on each submitted page.

The 3 of the problems are of roughly equal difficulty with 1 problem being a bit harder than other problems in all 3 exams. Note that problem difficulty is ultimately subjective and what might be the easiest problem in my mind might be the hardest for you and vice versa. The first problem in all 3 exams is considered to be roughly the easiest.

You have:

1. 210 minutes (3 and a half hours) of exam time for the Apprentice Level exam which is on page 3

2. 240 minutes (4 hours) of exam time for the Machine level exam which is on page 5

3. 270 minutes (4 and a half hours) of exam time for the Shef level exam which is on page 7

Feel free to use the bathroom any amount of time you like during the exam and eat and drink as much as you like. My advice would be to treat this like a competition exam, so that you can practice your problem solving skills under some artificial exam pressure.

Label the problems and problem pages on top of every paper you submit.

Note that the exam isn't the determining factor in your application. We will try to look at your application as a whole, which may include an interview.

Just focus on doing your best !

Finally, enjoy the problems, they are fun :)

The Apprentice

The Apprentice Problems

1. Find all quadratic polynomials P such that $P(P(-1)) = P(P(0)) = P(P(1))$.
2. We are given a 2022-gon $A_1A_2...A_{2022}$. Its vertices have numbers written on them such that the sum of the numbers on any 9 consecutive vertices is 300. If the numbers written on A_{22} and A_{29} are 22 and 29 respectively, what is the number written on vertex A_{2022}?
3. Let ABC be an acute angled triangle. Assume the height from A to BC, the angle bisector of $\angle ABC$, and the median from C to side BC form a triangle with positive area. Prove that this triangle isn't equilateral.
4. Find all pairs of positive integers (x, y) such that $2x+1$ and $2y-1$ are relatively prime and $x+y|4xy+1$.

The Machine

The Machine Problems

1. Let $x_1 = 4, x_2 = 6$ and let x_n be the smallest composite integer greater than $2 * x_{n-1} - x_{n-2}$, Determine x_{2022}.

2. Let a, b, c be positive reals such that $a + b + c = 1$. Prove that the inequality

$$a\sqrt[4]{1+b-c} + b\sqrt[4]{1+c-a} + c\sqrt[4]{1+a-b} \leq 1$$

   holds.

3. Every point (x, y) in the Cartesian plane, where x and y are both integers, is colored in one of 5 colors (these are also known as lattice points). Prove there exist 20 mutually congruent triangles whose vertices are all of the same color.
   (Elaboration: When we look at the at most 60 points from those 20 triangles the points are all colored with the same color)

4. Consider triangle ABC such that $AB \leq AC$. Point X on the arc BC of thecircumcirle of ABC not containing point A and point Y on side BC are such that $\angle BAX = \angle CAY < \frac{1}{2}\angle BAC$. Let Z be the midpoint of segment AX. If $\angle AXY = \angle ABC - \angle ACB$ prove that $\angle BZC = 2\angle BAC$.

The Shef

The Shef Problems

1. Let n be a positive integer. Find the largest nonnegative real number $g(n)$ (depending on n) with the following property: whenever $a_1, a_2, ..., a_n$ are real numbers such that $a_1+a_2+\cdots+a_n$ is an even integer, there exists some i such that $\left|a_i - \frac{1}{3}\right| \geq g(n)$.

2. Let $a_1 = 1, a_{2n} = a_n, a_{2n+1} = a_n + a_{n+1}$ for all $n \geq 1$. Prove that for each $k \in \mathbb{N}$, the number of odd m such that $a_m = k$ is equal to the number of positive integers less than or equal to k and relatively prime to k.

3. Let ω be a circumcircle of triangle ABC ($AC < BC$). Also, let CD be an angle bisector of angle ACB ($D \in AB$), M be a midpoint of arc AB of circle ω containing the point C, and let I be an incenter of a triangle ABC. Circle ω cuts line MI at point E and circle with diameter CI at H. If the circumcircle of triangle CDE intersects AB again at T, prove that T, H and C are collinear.

4. Every point (x, y) in the Cartesian plane, where x and y are both integers, is colored in one of 20 colors (these are also known as lattice points). Prove there exist 10 mutually congruent triangles whose vertices are all of the same color and which have at least one side length that is divisible by 2011.

   (Elaboration: When we look at the at most 30 points from those 10 triangles the points are all colored with the same color)