Divisibility

Shef Scholars Winter Math Camp

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Divisibility Rules

- A number is divisible by **2** if its last digit is even.
- A number is divisible by **3** if the sum of its digits is divisible by 3.
- A number is divisible by **4** if its last two digits form a number divisible by 4.
- A number is divisible by **5** if its last digit is 0 or 5.
- A number is divisible by **8** if its last three digits form a number divisible by 8.
- A number is divisible by **9** if the sum of its digits is divisible by 9.
- A number is divisible by **11** if the difference between the sum of the digits in even positions and the sum of the digits in odd positions is divisible by 11.

Prime Factorizations and Divisibility

Every number can be broken down into smaller "building blocks" called prime factors. For example:

- The prime factorization of 5 is just 5, because 5 is already a prime number.
- The prime factorization of 60 is $2 \cdot 2 \cdot 3 \cdot 5$, or $2^2 \cdot 3 \cdot 5$.

If a number passes the divisibility test for all the prime factors of another number, raised to the exact powers found in its factorization, then it is divisible by that number. Here's how it works:

- To check if a number is divisible by 60:
  - First, check if it is divisible by 4 (because $4 = 2 \cdot 2$),
  - Then check if it is divisible by 3,
  - Finally, check if it is divisible by 5.
- If all these tests pass, the number is divisible by 60.

This method breaks a big divisibility test into smaller, easier steps, so you don't have to test everything at once. Just focus on the smaller prime pieces!

Problems

1. Find all digits a, b, with $a \neq 0$, such that the number $\overline{2021a2021b}$ is divisible by 36, and then determine these digits so that it is divisible by 55.

2. Find all natural numbers n such that the given expression is an integer:

   (a) $\frac{n+12}{n}$

   (b) $\frac{3n+9}{n+2}$

   (c) $\frac{8n+12}{2n+3}$

   (d) $\frac{5n+11}{2n-3}$

3. Are there any natural numbers x, y, z for which $5x^3 + 16y^2 + 40z^3 = 1213141516$?

4. Prove that a three-digit number whose last two digits are the same is divisible by 7 if the sum of its digits is divisible by 7.

5. Find all pairs of digits (a, b) such that one of the seven-digit numbers $\overline{ab2abba}$ and $\overline{2ab0abb}$ is divisible by 9 and the other by 5 (it is not necessary that the first one is divisible by 9 and the second by 5, it could be the opposite).

6. For integers a, b, c, it holds that $4a + 5b = 6c$. Prove that $(3c - 2a)(a + 2b)(b + 2a)$ is an integer multiple of 30.

7. Determine the digits a and b, with $a \neq 0$, such that the number $\overline{a1995} + \overline{1996b}$ is divisible by 44.

8. The sum of the digits of a four-digit number is 27. Prove that the sum of this number and the number written with the same digits but in reverse order is divisible by 27.

9. A number is called beautiful if it consists only of the digits 3 and 4, with the digit 3 appearing at least once, and the digit 4 exactly once (e.g., 43, 343, 3334 are some of the beautiful numbers).

   (a) Prove that a beautiful number cannot be divisible by more than two numbers from the set $A = \{2, 3, 4, 5, 6, 7, 8, 9\}$.

   (b) Determine the smallest beautiful number that is divisible by exactly two numbers from the set A. Justify your answer!