Inversion

Shef Scholars Winter Math Camp

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Introduction to Inversion

Inversion is a transformational tool in geometry that reimagines the plane through a fixed circle. This powerful technique allows us to turn circles into lines and vice versa, offering profound simplifications for intricate geometric problems. With inversion, we introduce the idea of **clines**—generalized circles that include both ordinary circles and lines.

Why Learn Inversion?

Inversion equips us with tools to:

- Transform complicated geometric setups into simpler configurations.
- Prove concyclicity, tangency, and collinearity with ease.
- Explore elegant symmetries and invariants in triangle geometry.
- Uncover hidden properties in problems involving angles and distances.

In this lecture, we will explore the fundamental properties of inversion, visualize how clines transform, and apply these ideas to solve challenging problems.

Definition and Basic Properties

Let ω be a circle with center O and radius r. An **inversion** with respect to ω is a transformation of the plane such that:

- The center O maps to the **point at infinity** (P_∞).
- Any point A maps to A^* such that $OA \cdot OA^* = r^2$, with A^* lying on the ray OA.
- The point P_∞ maps to O.

Key Observations

1) The points on the inversion circle ω remain **fixed points**.
2) Inversion is **involutory**, meaning $(A^*)^* = A$.
3) Inversion preserves **angles** but **reverses orientation**.

Geometric Interpretation: Inversion connects each point A inside ω with a corresponding point A^* outside ω, such that the product of distances to the center O is constant.

How Inversion Transforms Geometric Objects

Lines and Circles:

- A line passing through O inverts to itself.
- A line not passing through O inverts to a circle passing through O.
- A circle passing through O inverts to a line not passing through O.
- A circle not passing through O inverts to another circle not passing through O.

Tangent Clines: Tangency is preserved under inversion. For example, two tangent circles will invert to two parallel lines if we invert around the point of tangency.

Key Theorems

Inversion and Concyclic Points: If four points A, B, C, D are concyclic, then their inverses A^*, B^*, C^*, D^* are also concyclic. This follows from the fact that inversion preserves angles.

Inversion and Angles: If A^* and B^* are the inverses of A and B with respect to ω, then:

$$\angle OAB = \angle OB^*A^*.$$

Applications of Inversion

Simplifying Problems

By choosing the center of inversion strategically, complex configurations can be reduced:

- Circles can become lines, simplifying problems involving tangency.
- Geometric loci can transform into simpler shapes.

Example Problem

Problem: Prove that the tangents drawn from a point A^*, the inverse of A inside ω, to the circle ω are collinear with A.

Hint: Use the definition of inversion and properties of similar triangles.

Initial Exercises

1. Show that under inversion, the inverse of a circle with center C and radius r is a circle whose center lies on the line OC.
2. Prove that a line not passing through the center of inversion O inverts to a circle passing through O.
3. Prove that a line passing through the center of inversion inverts to itself.
4. Given two circles tangent at a point O, prove that inverting about the point O will invert the two circles to two parallel lines.

Problems

1. Prove that for any points A, B, C, D, the inequality

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD$$

holds, and that equality occurs if and only if points A, B, C, and D are on a circle or a line in this order.

2. Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

3. Let P' be the inverse of P with respect to a circle Ω. Show that for any point A on Ω, the ratio $\frac{AP'}{AP}$ holds constant. What is the locus of points K in $\triangle ABC$ such that $\frac{AB}{AC} = \frac{KB}{KC}$?

4. Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE meet at a point.

5. In acute triangle ABC, $AB > AC$. D and E are the midpoints of AB, AC respectively. The circumcircle of ADE intersects the circumcircle of BCE again at P. The circumcircle of ADE intersects the circumcircle BCD again at Q. Prove that $AP = AQ$.

6. Let be given a triangle ABC and its internal angle bisector BD ($D \in AC$). The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.

7. Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q.

   Prove that $\angle ACP = \angle QCB$.

8. Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

9. Let ω be the circumcircle of ABC, l be the tangent line to the circle ω at point A. The circles ω_1 and ω_2 touch lines l, BC and the circle ω externally. Denote by D, E the points where ω_1, ω_2 touch BC. Prove that the circumcircles of triangles ABC and ADE are tangent.