MATH PROBLEMS

Shef's Scholars

March 2024

1 Problems

Take as much time as you want with the problems. Make sure you give yourself enough time before the deadline to write up and upload your solutions in English (doesn't have to be perfect English, just something I can understand).
Make sure you write legibly and if you have diagrams make sure they are legible. Feel free to go through all the different levels and problems here and choose which one you want to apply to.

1.1 The Beginner

1.1.1 Algebra

1. Let

$$f(x) = \begin{cases} (a-6)x + a + 4 & \text{if } x \geq 4, \\ (a-3)x + b - 3 & \text{if } x < 4. \end{cases}$$

It is known that $f(6) = 3$ and that $f(f(6)) = 5$. Find $f(2) * f(4)$.

2. Nonzero reals a, b, c satisfy the equality $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 0$. Determine the value of the expression

$$w = \frac{3b + 2c}{6a} + \frac{2c + 6a}{3b} + \frac{6a + 3b}{2c}$$

3. Let a, b, c, d be reals such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 7$ and $\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} = 12$. Find the value of $w = \frac{a}{b} + \frac{c}{d}$

4. If a, b, c are real numbers such that $a+b+c = 0$, prove that $ab+bc+ca \leq 0$.

1.1.2 Number Theory

5. Let p and q be two consecutive primes. Prove that $p + q$ is either 5 or has at least 3 not necessarily distinct prime factors.(Eg: $3 + 5 = 8 = 2 * 2 * 2, 5 + 7 = 12 = 2 * 2 * 3$ both have 3 prime factors)

6. If n is a integer, prove that $6|n^3 + 5n$.

7. Find the smallest positive integer that is divisible by 7 and leaves the remainder 1 when divided by each of $2, 3, 4, 5, 6$. Prove that it's the smallest.

8. Find all positive integers n, such that $\sqrt{13 - \sqrt{n - 13}}$ is a positive integer.

1.1.3 Geometry

9.

10. In an acute triangle $ABC, C = 60°$. If AA' and BB' are two of the altitudes and C_1 is the midpoint of AB, prove that triangle $C_1A'B'$ is equilateral.

11. Let ABC be a triangle with $\angle ABC = 90$ and $\angle BAC = 15$. Let D be the foot of perpendicular from B to AC. Prove that $4 * BD = AC$

12. In an acute triangle ABC, let D, E be the midpoints of CA, CB, respectively. The angle bisector of angle $\angle BAC$ intersects DE at point S. Prove that $\angle ASC = 90$

1.1.4 Combinatorics

13. Every point in a Cartesian plane is colored either white or black. Prove that there exist two points A and B which are of the same color and at a distance of 1 from each other.

14. Every real number is colored either red or blue with the following rules: If n is colored red then so is $n + 15$, if n is colored blue then so is $n + 10$. Prove the following claims:

    (a) If n is colored blue then so is $n + 10$

    (b) If $n + 5$ is colored red then so is n

15. How many 3 element subsets of the set $S = \{1, 2, 3, ..., 19, 20\}$ exist such that the product of their elements is divisible by 4?

16. In the year 2024, certain individuals will find that their age is exactly equal to the sum of the digits in their year of birth. Determine the year or years of birth of these individuals. Assume the current year is 2024 for your calculation

1.2 The Apprentice

1.2.1 Algebra

1. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that
$$\frac{a}{b} + \frac{a}{c} + \frac{c}{b} + \frac{c}{a} + \frac{b}{c} + \frac{b}{a} + 6 \geq 2\sqrt{2}\left(\sqrt{\frac{1-a}{a}} + \sqrt{\frac{1-b}{b}} + \sqrt{\frac{1-c}{c}}\right).$$

2. If $x, y > 0$ prove that $\left(x + \frac{2}{y}\right)\left(\frac{y}{x} + 2\right) \geq 8$. When do we have equality?

3. Prove that the inequality
$$\frac{x+y}{x^2 - xy + y^2} \leq \frac{2\sqrt{2}}{\sqrt{x^2 + y^2}}$$
holds for all real numbers x and y, not both equal to 0.

4. Let x be a real number such that x^3 and $x^2 + x$ are rational numbers. Prove that x is rational.

1.2.2 Number Theory

5. For each nonnegative integer n we define $A_n = 2^{3n} + 3^{6n+2} + 5^{6n+2}$. Find the greatest common divisor of the numbers $A_0, A_1, \ldots, A_{1999}$.

6. Find all positive integers which have exactly 16 positive divisors $1 = d_1 < d_2 < \ldots < d_{16} = n$ such that the divisor d_k, where $k = d_5$, equals $(d_2 + d_4)d_6$.

7. Find all prime numbers of the form $\frac{1}{11} \cdot \underbrace{11\ldots1}_{2n \text{ ones}}$, where n is a natural number.

8. Find all primes p, q such that p divides $30q - 1$ and q divides $30p - 1$.

1.2.3 Geometry

9. Let ABC be a triangle with $\angle ABC = \angle CAB = 50^\circ$. Inside this triangle a point O is given with $\angle BAO = 10^\circ$ and $\angle ABO = 30^\circ$. Calculate the angle $\angle ACO$.

10. Let $ABCDE$ be a convex pentagon such that $AB = AE = CD = 1$, $\angle ABC = \angle DEA = 90^\circ$ and $BC + DE = 1$. Compute the area of the pentagon.

11. Determine the triangle with sides a, b, c and circumradius R for which $R(b + c) = a\sqrt{bc}$.

12. Let ABC be a triangle with $\angle ABC = \angle ACB = 40^\circ$ Line bisector of $\angle ABC$ intersects AC at point D. Prove that $BD + DA = BC$.

1.2.4 Combinatorics

13. One pupil has 7 cards of paper. He takes a few of them and tears each in 7 pieces. Then, he choses a few of the pieces of paper that he has and tears it again in 7 pieces. He continues the same procedure many times with the pieces he has every time. Will he be able to have 2022 pieces of paper after a certain number of operations?

14. Every real number is colored either red or blue with the following rules: If n is colored red then so is $n+15$, if n is colored blue then so is $n+10$. Prove or disprove the following claims:

    (a) If n is colored blue then so is $n+10$

    (b) If $n+5$ is colored red then so is n

    (c) If n is colored blue then so is $n+2$

15. Let S be a finite set of numbers, such that among any three of its elements there are two whose sum belongs to S. Find the greatest possible number of elements of S.

16. Two players play the Shef's game: On a 1×100 board there is a coin on each end. Player 1 can move the coin on the right end either 1, 2, or 3 places to the left at every turn. Player 2 can move the coin on the left end either 1, 2, or 3 places to the right at every turn. The player who is forced to place their coin on or over the other player's coin loses the game. Determine which player has the winning strategy

1.3 The Machine

1.3.1 Algebra

1. Let x, y, and z be positive real numbers. Prove that
   $\sqrt{\frac{xy}{x^2+y^2+2z^2}} + \sqrt{\frac{yz}{y^2+z^2+2x^2}} + \sqrt{\frac{zx}{z^2+x^2+2y^2}} \leq \frac{3}{2}$.
   When does equality hold?

2. Show that
   $$\left(a+2b+\frac{2}{a+1}\right)\left(b+2a+\frac{2}{b+1}\right) \geq 16$$
   for all positive real numbers a and b such that $ab \geq 1$.

3. Determine all real constants t such that whenever a, b and c are the lengths of sides of a triangle, then so are a^2+bct, b^2+cat, c^2+abt.

4. Let x_0, y_0 be positive integers, and define $x_{i+1} = x_i + \lfloor\sqrt{y_i}\rfloor$ and $y_{i+1} = y_i + \lfloor\sqrt{x_i}\rfloor$ for all $i \geq 0$. Show that there exists a positive integer n such that $x_n = y_n$.

1.3.2 Number Theory

5. Determine all prime numbers of the form $1 + 2^p + 3^p + ... + p^p$ where p is a prime number.

6. a) Find all prime numbers p, q, r satisfying $3 \nmid p + q + r$ and $p + q + r$ and $pq + qr + rp + 3$ are both perfect squares.

   b) Do there exist prime numbers p, q, r such that $3 \mid p+q+r$ and $p+q+r$ and $pq + qr + rp + 3$ are both perfect squares?

7. Determine all four-digit numbers $\overline{abcd}$ such that $(a + b)(a + c)(a + d)(b + c)(b + d)(c + d) = \overline{abcd}$:

8. For positive integer n we define $f(n)$ as sum of all of its positive integer divisors (including 1 and n). Find all positive integers c such that there exists strictly increasing infinite sequence of positive integers $n_1, n_2, n_3, ...$ such that for all $i \in \mathbb{N}$ holds $f(n_i) - n_i = c$

1.3.3 Geometry

9. Let ABC be an acute-angled triangle with $AB < AC$ and let O be the centre of its circumcircle ω. Let D be a point on the line segment BC such that $\angle BAD = \angle CAO$. Let E be the second point of intersection of ω and the line AD. If M, N and P are the midpoints of the line segments BE, OD and AC, respectively, show that the points M, N and P are collinear.

10. It is given triangle ABC and points P and Q on sides AB and AC, respectively, such that $PQ \mid\mid BC$. Let X and Y be intersection points of lines BQ and CP with circumcircle k of triangle APQ, and D and E intersection points of lines AX and AY with side BC. If $2 \cdot DE = BC$, prove that circle k contains intersection point of angle bisector of $\angle BAC$ with BC

11. Let D and E be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$. Let the lines CD and BE meet at F. Prove that the incentre I of triangle ABC, the orthocentre H of triangle DEF and the midpoint M of the arc BAC of the circumcircle of triangle ABC are collinear.

12. Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

1.3.4 Combinatorics

13. Let n be a positive integer. Two players, Alice and Bob, are playing the following game: - Alice chooses n real numbers; not necessarily distinct. - Alice writes all pairwise sums on a sheet of paper and gives it to Bob. (There are $\frac{n(n-1)}{2}$ such sums; not necessarily distinct.) - Bob wins if he finds correctly the initial n numbers chosen by Alice with only one guess. Can Bob be sure to win for the following cases?

    (a) $n = 5$

    (b) $n = 6$

    (c) $n = 8$

    Justify your answer(s).

    [For example, when $n = 4$, Alice may choose the numbers 1, 5, 7, 9, which have the same pairwise sums as the numbers 2, 4, 6, 10, and hence Bob cannot be sure to win.]

14. Let $a_1, a_2, \cdots, a_{2n}$ be $2n$ numbers such that is is a member of the set $A = \{1, 2, 3, \cdots, 2n-1\}$ $(n > 3)$. It is known that $a_1 + a_2 + \cdots + a_{2n} = 4n$. Prove that we can choose some of the numbers a_i such that their sum is $2n$.

15. It is given sequence wih length of 2017 which consists of first 2017 positive integers in arbitrary order (every number occus exactly once). Let us consider a first term from sequence, let it be k. From given sequence we form a new sequence of length 2017, such that first k elements of new sequence are same as first k elements of original sequence, but in reverse order while other elements stay unchanged. Prove that if we continue transforming a sequence, eventually we will have sequence with first element 1.

16. Let a, b, c, d and e be distinct positive real numbers such that $a^2 + b^2 + c^2 + d^2 + e^2 = ab + ac + ad + ae + bc + bd + be + cd + ce + de$

    (a) Prove that among these 5 numbers there exists triplet such that they cannot be sides of a triangle

    (b) Prove that, for a), there exists at least 6 different triplets

1.4 The Shef

1.4.1 Algebra

1. Let a, b, c be positive real numbers. Prove the inequality

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}.$$

When does equality occur?

2. Let n be a given positive integer. Solve the system

$$x_1 + x_2^2 + x_3^3 + \cdots + x_n^n = n,$$

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = \frac{n(n+1)}{2}$$

in the set of nonnegative real numbers.

3. Find all real constants c for which there exist strictly increasing infinite sequence $a_1, a_2, ...$ of positive integers such that $(a_{2n-1} + a_{2n})/a_n = c$ for all positive intgers n.

4. For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

1.4.2 Number Theory

5. For $m \in \mathbb{N}$ define $m?$ be the product of first m primes. Determine if there exists positive integers m, n with the following property :

$$m? = n(n+1)(n+2)(n+3)$$

6. Prove that for all odd prime numbers p there exist a natural number $m < p$ and integers x_1, x_2, x_3 such that:

$$mp = x_1^2 + x_2^2 + x_3^2.$$

7. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function for which the expression $af(a) + bf(b) + 2ab$ for all $a, b \in \mathbb{N}$ is always a perfect square. Prove that $f(a) = a$ for all $a \in \mathbb{N}$.

8. Prove that there exist infinitely many composite positive integers n such that n divides $3^{n-1} - 2^{n-1}$.

1.4.3 Geometry

9. Let $ABCD$ be a convex quadrilateral, and write $\alpha = \angle DAB$; $\beta = \angle ADB$; $\gamma = \angle ACB$; $\delta = \angle DBC$; and $\epsilon = \angle DBA$. Assuming that $\alpha < \pi/2$, $\beta + \gamma = \pi/2$, and $\delta + 2\epsilon = \pi$, prove that

$$(DB + BC)^2 = AD^2 + AC^2$$

10. Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects

sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

11. Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

12. Given is a triangle $\triangle ABC$ and points M and K on lines AB and CB such that $AM = AC = CK$. Prove that the length of the radius of the circumcircle of triangle $\triangle BKM$ is equal to the lenght OI, where O and I are centers of the circumcircle and the incircle of $\triangle ABC$, respectively. Also prove that $OI \perp MK$.

1.4.4 Combinatorics

13. Let $n > 1$ be a fixed integer. Alberto and Barbara play the following game:

    (i) Alberto chooses a positive integer,

    (ii) Barbara chooses an integer greater than 1 which is a multiple or submultiple of the number Alberto chose (including itself),

    (iii) Alberto increases or decreases the Barbara's number by 1.

    Steps (ii) and (iii) are alternatively repeated. Barbara wins if she succeeds to reach the number n in at most 50 moves. For which values of n can she win, no matter how Alberto plays?

14. It is given regular n-sided polygon, $n \geq 6$. How many triangles they are inside the polygon such that all of their sides are formed by diagonals of polygon and their vertices are vertices of polygon?

15. A unit square is divided into polygons, so that all sides of a polygon are parallel to sides of the given square. If the total length of the segments inside the square (without the square) is $2n$ (where n is a positive real number), prove that there exists a polygon whose area is greater than $\frac{1}{(n+1)^2}$.

16. 30 people are sitting at round table. $30 - N$ of them always tell the truth ("truth speakers") while the other N of them tell the truth sometimes and lie other times ("flip speakers"). The question: "Who is your right neighbour - "truth speaker" or "flip speaker" ?" is asked to all 30 people and 30 answers are collected. What is maximal number N for which (with knowledge of these answers) we can always deduce at least one person who is "truth speaker".