Two interesting points in Geometry

Shef Scholars Winter Math Camp

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Introduction

In this handout, we explore two points within a triangle that exhibit elegant geometric properties and often simplify seemingly challenging problems. While these points are vertex-dependent and lack widely recognized names in mathematical literature, we will adopt our own terminology: the Zlatko's and Salko's points. These points are isogonal conjugates of each other, and understanding their properties can provide powerful insights into triangle geometry. We will delve into their definitions and uncover their intriguing behaviors, but first a story on multiple discovery.

I was preparing for my final IMO in 2017 and I kept seeing the same point (which we'll call Salko's Point) come up again in certain problems. I realized the fact that I could notice it, gave me an edge in geometry so I tried to come up with a list of properties it satisfied. I shared this with my IMO team and we called it "Salko's point" (initially it was called "the mysterious point Q" because it did some up often as point Q in the problems) because there was no point in the literature that we knew of at the time. At one of our prep-tests for the IMO, I solved a 3rd problem that was a G using some large combination of trigonometry and projective geometry meanwhile my teammate, Adisa Bolic, just asked me how I didn't notice my own point which lead to a much simpler solution. Around this time I also saw that the other point kept popping up but never thought of writing anything about it. Imagine my sadness when someone wrote an article about it and named the points "Humpty" and "Dumpty", thereby exemplifying multiple discovery.

Now

The Salko Point

Definition 1: In $\triangle ABC$, the **A-Salko point**, denoted as Q_A, is defined as the point inside the triangle such that:

$$\angle Q_A BC = \angle Q_A AB \quad \text{and} \quad \angle Q_A CB = \angle Q_A AC.$$

Facts about Q_A:

1) The Salko point lies on the **median** of $\triangle ABC$ from vertex A to side BC.
2) It lies on the **A-Apollonius circle**, meaning the ratio of distances satisfies:

$$\frac{AB}{AC} = \frac{Q_A B}{Q_A C}.$$

3) The points B, Q_A, H, and C are **concyclic**, where H is the orthocenter of $\triangle ABC$.

4) The line segment HQ_A is **perpendicular** to AQ_A.

Discovery Questions:

- Can you use the properties of medians to locate Q_A geometrically?
- How does the Salko point's relationship with the Apollonius circle connect to the triangle's side ratios?
- What does the cyclicity of B, Q_A, H, C reveal about the orthocenter and the Salko point?

The Zlatko Point

Definition 2: In $\triangle ABC$, the **A-Zlatko point**, denoted as S_A, is defined as the point inside the triangle such that:

$$\angle S_A BA = \angle S_A AC \quad \text{and} \quad \angle S_A AB = \angle S_A CA.$$

Facts about S_A:

1) The Zlatko point lies on the **symmedian** of $\triangle ABC$ from vertex A.
2) It is the **center of spiral similarity** sending $\triangle AS_AC$ to $\triangle CS_AB$, or equivalently, mapping side AC to AB.
3) The points B, S_A, O, and C are **concyclic**, where O is the circumcenter of $\triangle ABC$.
4) The line segment OS_A is **perpendicular** to AS_A.

Discovery Questions:

- How does the Zlatko point's position on the symmedian differ from the Salko point's position on the median?
- What does the property of spiral similarity imply about Q_A and its relationship with the triangle's sides?
- Can you prove the concyclicity of B, S_A, O, C by examining the angles subtended by BC at O and S_A?

Connection Between the Zlatko and Salko Points

The Zlatko and Salko points are **isogonal conjugates** of each other. This means the reflections of the lines AP_A and AQ_A over the angle bisectors of $\angle BAC$ will coincide. Exploring this relationship further will uncover deeper symmetries in the triangle.

Challenge: Prove that the Zlatko and Salko points are indeed isogonal conjugates.

Problems

1. Let M be the midpoint of side BC in the acute angled triangle ABC. Let Q be the point on the line segment AM such that $\angle BAC + \angle BQC = 180°$. Prove that $MQ \cdot MA = MC \cdot MB$.

2. Let M be the midpoint of side BC in the acute angled triangle ABC and let T be on the chord BC of the circumcircle of ABC such that $\angle BAT = \angle CAM$. Let S be the midpoint of AT. Prove that $2 \cdot \angle CAB = \angle CSB$.

3. Let ABC be an acute angled triangle and P a variable point inside the triangle such that $\angle BPC + \angle BAC = 180°$. Let BP and CP intersect AC and AB at points X and Y respectively. Prove that the circumcircle around AXY passes through a fixed point as P varies.

4. Let ABC be an acute angled triangle and P a variable point on side BC. The line through P parallel to AB intersects AC at point D and the line through P parallel to AC intersects AB at point E. Prove that the circumcircle of triangle AED passes through a fixed point as P varies.

5. Let ABC be an acute angled triangle, let BE and CF heights of the triangle, and let M be the midpoint of BC. Line through A parallel to BC intersects line EF at point T. Let P and Q be the midpoints of ME and MF, respectively. The line PQ intersects BC at point S. Prove that ST is tangent to the circumcircle of triangle AEF.

6. Let ABC be an acute triangle such that $AB \neq AC$,with circumcircle Γ and circumcenter O. Let M be the midpoint of BC and D be a point on Γ such that $AD \perp BC$. let T be a point such that $BDCT$ is a parallelogram and Q a point on the same side of BC as A such that $\angle BQM = \angle BCA$ and $\angle CQM = \angle CBA$. Let the line AO intersect Γ at E ($E \neq A$) and let the circumcircle of $\triangle ETQ$ intersect Γ at point $X \neq E$. Prove that the point A, M and X are collinear.

7. Let ABC be an acute-angled triangle S circumcenter O. Let Γ be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that $BP \cdot CQ = AP \cdot AQ$. Prove that Γ is tangent to the circumcircle of triangle BOC.

8. Let P be a point inside the triangle ABC such that $\angle PBC = \angle PAB$ and $\angle PCB = \angle PAC$. Let Q be on line BC such that $AQ = PQ$. If O is the circumcenter of triangle ABC prove that $2 \cdot \angle OQB = \angle PQA$.

9. Let ABC be an acute angled triangle with circumcenter O and $AB < AC$. Let Q be the intersection of the external angle bisector of $\angle BAC$ and BC. Finally let P be a point inside triangle ABC such that $\angle PBA = \angle PAC$ and $\angle PCA = \angle PAB$. Prove that $\angle QPA + \angle OQB = 90°$.