# Incoming Applicant Test

Shef Scholars

Due July 22nd 2025

## 1 Problems

Make sure to block out enough time for the test.

You may use your phone or mechanical clock as a timer.

Once you begin the exam, do not use the internet until you've finished the exam. Do not consult textbooks, chatbots, or outside sources. Any cheating may result in your application automatically being rejected and you being unable to apply to future iterations of Shef Scholars, Shef Scholars Competitive Math Academy, or even other events organized by Shefs of Problem Solving.

Write up your solutions in English and put your email address (the same one you're putting in your application) on top of the paper on each submitted page.

The 3 of the problems are of roughly equal difficulty with 1 problem being a bit harder than other problems in all 4 exams. Note that problem difficulty is ultimately subjective and what might be the easiest problem in my mind might be the hardest for you and vice versa. The first problem in all 4 exams is considered to be roughly the easiest.

You have:

1. 150 minutes (2 hours and a half hours) of exam time for the Beginner Level exam which in on page 4
2. 180 minutes (3 hours) of exam time for the Apprentice Level exam which is on page 7
3. 210 minutes (3 hours and a half hours) of exam time for the Machine level exam which is on page 9
4. 270 minutes (4 and a half hours) of exam time for the Shef level exam which is on page 11

Unless you are submitting your exam in word or LaTeXthe write up is part of the time you have for the problem. If you are submitting your exam in word or LaTeX, write the solutions on paper first and then rewrite them in LaTeXor word. Feel free not to submit papers you don't think are important. Note how long you took the exam on one of the papers.

Feel free to use the bathroom for any amount of time during the exam and eat and drink as much as you like. My advice would be to treat this like a competition exam, so that you can practice your problem solving skills under some artificial exam pressure.

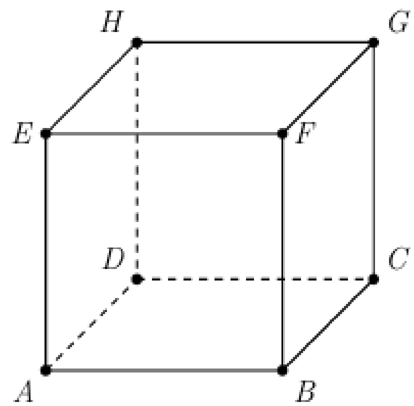
Label the problems and problem pages on top of each paper you submit.

Focus on doing your best !

Finally, enjoy the problems, they are fun :)

# The Beginner

1. In how many different ways can an ant move from point $A$ to point $G$ if it moves only along the edges of a cube and passes through each edge and vertex at most once? List all possible paths.
(For example, $AEHG$ is one such path!)




2. Calculate:

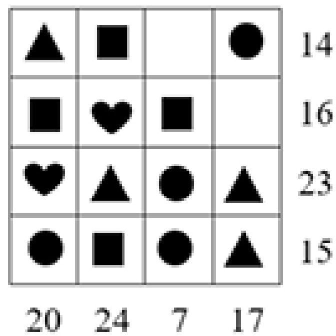
$$-\frac{20}{\sqrt{2}}+\frac{21}{\sqrt{4}}-\frac{22}{\sqrt{8}}+\frac{23}{\sqrt{16}}-\frac{24}{\sqrt{32}}+\frac{25}{\sqrt{64}}-\frac{26}{\sqrt{128}}+\frac{27}{\sqrt{256}}$$

3. Let $ABCD$ be a convex quadrilateral such that:

$$|AB|=|BC|=|BD|, \quad \angle BAC=\angle CBD, \quad \angle ADB=\angle DCA.$$

Determine the measures of the angles of quadrilateral $ABCD$.

4. In the image, a table is shown. Some cells in the table contain symbols. Each symbol always represents the same value wherever it appears in the table, and the sum of the values of all symbols in each row and column is given. If a square is empty, the "value" of that square is 0. Determine the value of each symbol in the table.




5. On each square of a $5 \times 5$ board, there is either a knight or a liar. A knight always tells the truth, and a liar always lies.

   At one moment, they all shouted: "Among my neighbors, there is an equal number of knights and liars."

   What is the maximum number of knights that can be on the board?

   (Note: Neighbors are those in squares that share a side. For example, a person in a corner has 2 neighbors, while a person in the middle has 4.)

6. The sum of two natural numbers is 4923. If we append the digit 7 to the right side of one of those numbers, and remove the units digit from the other, the resulting numbers will be equal. Determine the original two numbers.

7. Given a convex quadrilateral $ABCD$. Points $M$ and $N$ are the midpoints of the diagonals $AC$ and $BD$, respectively. Point $P$ is the intersection of the lines passing through the midpoints of the opposite sides of the quadrilateral. Prove that the points $M$, $N$, and $P$ are collinear.

8. Given a regular 2013-gon $A_1A_2 \ldots A_{2013}$. Its vertices are labeled with numbers such that the sum of the numbers assigned to any 9 consecutive vertices is constant and equals 300. (The vertices $A_1$ and $A_{2013}$ are also considered consecutive.)

   If $A_{13} = 13$ and $A_{20} = 20$, determine the value of $A_{2013}$.

# The Apprentice

1. From the set $\{1, 2, \ldots, 100\}$, 51 numbers are selected. Prove that:
   (a) Among the selected numbers, there exist two consecutive numbers.
   (b) Among the selected numbers, there exist two whose sum is 101.

2. Solve the equation in integers:
$$x^2 + 4y^2 = 4(x + y) + 5.$$

3. The set $\{1, 2, \ldots, 9\}$ is partitioned into two disjoint subsets $A$ and $B$.
Prove that in at least one of these subsets there exist three distinct numbers $x, y, z$ such that
$$x + y = z.$$
Show that this statement does not necessarily hold for the set $\{1, 2, \ldots, 8\}$.

4. Let point $D$ lie on the circumcircle of triangle $ABC$. Let $M, N, P$ be the feet of the perpendiculars from point $D$ to lines $BC, CA$, and $AB$, respectively. Prove that the points $M, N, P$ are collinear.

5. The internal and external angle bisectors at vertex $C$ of triangle $ABC$ intersect the line $AB$ at points $L$ and $M$, respectively. If $CL = CM$, prove that:
$$AC^2 + BC^2 = 4R^2,$$
where $R$ is the circumradius of triangle $ABC$.

6. Let $x$ and $y$ be non-negative real numbers such that
$$x + y = 1.$$
Determine the minimum and maximum value of the expression
$$x\sqrt{1 + y} + y\sqrt{1 + x}.$$

# The Machine

1. A circle has center on the side $AB$ of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

2. Prove that for every $n$ that's a natural number there exist natural numbers $a$ and $b$ such that $3 \nmid ab$ and $3^n = a^2 + 2b^2$

3. Let $S$ be the set of all positive integers from 1 to 100 included. Two players play a game. The first player removes any $k$ numbers he wants, from $S$. The second player's goal is to pick $k$ different numbers, such that their sum is 100. Which player has the winning strategy if:

   (a) $k = 9$?

   (b) $k = 8$?

4. Can the equation $f(g(h(x))) = 0$, where $f$, $g$, $h$ are quadratic polynomials, have the solutions $1, 2, 3, 4, 5, 6, 7, 8$?

# The Shef

1. A positive real number $a$ satisfies:

$$\left\{\frac{1}{a}\right\} = \{a^2\}$$

$$2 < a^2 < 3$$

Find the value of

$$a^{12} - \frac{144}{a}$$

Note: $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.

2. In triangle $ABC$, $AL$ is the angle bisector, and $P$ and $Q$ are the points where the incircle touches $AB$ and $BC$, respectively. If $X = AQ \cap LP$, prove that $BX$ is perpendicular to $AL$.

3. Let $f(m)$ denote the largest integer $k$ such that $2^k \mid m!$. Prove that for every natural number $n$, there exist infinitely many natural numbers $m$ such that:

$$m - f(m) = n.$$

4. Let $S = \{a_1, a_2, \ldots, a_{2n+1}\}$ be a set of natural numbers such that for every subset $S_i = S \backslash \{a_i\}$, it can be partitioned into two subsets of $n$ elements each, with equal sums of their elements. Prove that all elements of $S$ are equal.