To my respected parents and teachers

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TABLE OF CONTENTS

ACKNOWLEDGMENTS……………………..……………………….……….......…..3

TABLE OF CONTENTS……………………………………….……….....……4

LIST OF FIGURES…………………………………………………..…....……6

LIST OF TABLES…………………………………………………….…..……..7

LIST OF ABBREVIATIONS……………………………………..……...8

ABSTRACT……………………………………………………..………………9

1. CHAPTER ONE: INTRODUCTION 10

1.1 Background 10

1.2 Outlines of the remaining chapters 12

2. CHAPTER TWO: LITERATURE REVIEW 14

2.1 Literature review on inverse distribution 14

2.2 Literature review on *Maxwell* distribution 15

2.3 Literature review on inverse *Maxwell* distribution 16

2.4 Research questions 16

3. CHAPTER THREE: INVERSE MAXWELL DISTRIBUTION 17

3.1 Preliminaries of inverse distribution 17

3.1.1 Moments of Inverse distribution 17

3.1.2 Moment generating function of inverse distribution 18

3.1.3 Characteristics function of inverse distribution 19

3.1.4 Survival function of inverse distribution 20

3.1.5 Hazard function of inverse distribution 20

3.1.6 Mode of inverse distribution 20

3.1.7 Entropy of the inverse distribution 21

3.2 Properties of inverse *Maxwell* distribution 21

3.2.1 Moments of inverse *Maxwell* distribution 23

3.2.2 MGF of inverse *Maxwell* distribution 25

3.2.3 CF of inverse *Maxwell* distribution 25

3.2.4 Survival function of inverse *Maxwell* distribution 25

3.2.5 Hazard function of inverse *Maxwell* distribution 26

3.2.6 Mode of inverse *Maxwell* distribution 27

3.2.7 Entropy of the inverse *Maxwell* distribution 27

3.2.8 Some special characteristics of inverse *Maxwell* distribution 28

3.3 Summary of the inverse *Maxwell* distribution 31

4. CHAPTER FOUR: PARAMETER ESTIMATION OF INVERSE MAXWELL DISTRIBUTION 33

4.1 MLE of the parameter of inverse distribution 33

4.1.1 MLE of the parameter of inverse *Maxwell* distribution 34

4.2 Method of moment estimation of parameter of inverse *Maxwell* distribution 35

4.3 Fisher information of a distribution 35

4.3.1 Fisher information of the inverse *Maxwell* distribution 35

4.4 The bounds of the variance estimate of the inverse *Maxwell* parameter 37

4.5 Summary of the parameter estimation of inverse *Maxwell* distribution 38

5. CHAPTER FIVE: APPLICATION OF THE INVERSE MAXWELL DISTRIBUTION 39

5.1 Application of inverse *Maxwell* distribution in process monitoring 39

5.1.1 Derivation of the distribution of V (the estimate of scale parameter) 40

5.1.2 Process Monitoring 42

5.1.3 Simulation Study 48

5.1.4 Real Life Example 52

6. CHAPTER SIX: CONCLUSIONS AND RECOMMENDATIONS 55

6.1 Recommendations 56

REFERENCES 57

LIST OF FIGURES

Figure 1.1: The PDF of *Maxwell* distribution for different value of parameter [6]. 11

Figure 3.1: The PDF of inverse *Maxwell* distribution for different value of parameter. ..22

Figure 3.2: The CDF of inverse *Maxwell* distribution for different value of parameter. .22

Figure 3.3: The Survival curve of inverse *Maxwell* distribution for different value of parameter 26

Figure3.4: The Hazard curve of inverse *Maxwell* distribution for different value of parameter. 26

Figure 5.1: Power curves of control charts for different n at $\alpha = 0.027$. 44

Figure 5.2: RL curves for different shifts δ and sample size n ....……………………….47

Figure 5.3: V-chart for inverse *Maxwell* parameter using probability limits (In control situation). 50

Figure 5.4: V −chart for inverse *Maxwell* parameter using L sigma limits (In control situation). 50

Figure 5.5: V −chart for inverse *Maxwell* parameter using L sigma limits (Out of control situation). 51

Figure 5.6: V −chart for inverse *Maxwell* parameter using probability limits (Out of control situation)..51

Figure 5.7: Car's brake pads………………………………………..…………………52

Figure 5.8: V-Chart for inverse *Maxwell* parameter using probability limits. 53

Figure 5.9: V-Chart for inverse *Maxwell* parameter using L-sigma limits. 54

LIST OF TABLES

Table 5.1: Gamma quantiles for different n and α. 43

Table 5.2: ARL of RL of V −chart for different n at $\alpha = 0.0027$. 45

Table 5.3: SDRL and MDRL of RL of V-chart for different n at $\alpha = 0.0027$. 46

Table 5.4: Percentiles of RL of V-chart for different n at $\alpha = 0.0027$ …………….………47

Table 5.5: L coefficients...48

Table 5.6: Factor for constructing Control Chart for inverse Maxwell parameter. 48

Table 5.7: Lifetime of car's brake pad (in km) 53

LIST OF ABBREVIATIONS

MLE	Maximum Likelihood Estimation
PDF	Probability Density Function
CDF	Cumulative Distribution Function
CF	Characteristics Function
MGF	Moment Generating Function
UCL	Upper Control Limit
LCL	Lower Control Limit
CL	Central Line
UPL	Upper Probability Limit
LPL	Lower Probability Limit
ARL	Average Run Length
SDRL	Standard Deviation of Run Length
MDRL	Median Deviation of Run Length
RL	Run Length
SPC	Statistical Process Control
EWMA	Exponentially Weighted Moving Average
CUSMU	Cumulative Sum

ABSTRACT

Control charts have been popularly used as a user-friendly yet technically sophisticated tool to monitor whether a process is in statistical control or not. These charts are basically constructed under the normality assumption. But in many practical situations in real life this normality assumption may be violated. One such non-normal situation is to monitor the process variability from a skewed parent distribution where we propose the use of an inverse *Maxwell* control chart. We develop the proposed distribution from *Maxwell* distribution and discuss its various distributional properties. We also introduce a pivotal quantity for the scale parameter of the inverse *Maxwell* distribution which follows a *gamma* distribution. Probability limits and $L-$sigma limits are studied along with performance measure based on average run length and power curve. We provide simulated data to illustrate the inverse *Maxwell* control chart. Finally, a real-life example has been given to show the importance of such a control chart.

CHAPTER ONE

INTRODUCTION

1.1 Background

In this project we introduce a new type of *Maxwell* distribution, named *inverse Maxwell* distribution. We develop different properties the distribution such as probability density function (PDF), cumulative distribution function (CDF), skewness and kurtosis, Fisher information, moment generating function (MGF), characteristics function (CF), Entropy etc. We also estimate the parameter using the maximum likelihood method under simple random sampling and by the method of moments. Finally, the application of this distribution on real life is given.

The *Maxwell* distribution with scale parameter σ has the following PDF and CDF

$$f(x) = \sqrt{\frac{2}{\pi}}\,\sigma^{-3}x^2 e^{-x^2/2\sigma^2}\ ;\ x > 0, \text{ and} \tag{1.1}$$

$$F(x) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{x^2}{2\sigma^2}\right), \tag{1.2}$$

where $\int_0^u x^{v-1}e^{-\mu x}dx = \mu^{-v}\gamma(v, \mu u)$. (1.3)

In Figure 1.1 we show the PDF of *Maxwell* distribution which is skewed. Let X_i $(i = 1,2,\ldots\ldots n)$ be a random sample of size n then the likelihood and log-likelihood function of the *Maxwell* distribution are respectively given by

$$L(\sigma, x) = \left(\sqrt{\frac{2}{\pi}}\sigma^{-3}\right)^n \prod_{i=1}^n x_i^2 e^{-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}}, \text{ and} \tag{1.4}$$

$$log(L) = nlog\left(\sqrt{\frac{2}{\pi}}\right) - 3nlog(\sigma) + \sum_{i=1}^n log(x_i^2) - \frac{\sum_{i=1}^n x_i^2}{2\sigma^2}. \tag{1.5}$$

Taking derivation with respect to the parameter σ and equating to zero in equation (1.5) the MLE of σ is given by,

$$\hat{\sigma} = \sqrt{(3n)^{-1}\sum_{i=1}^n x_i^2}\ .$$

In 1860, James Clerk Maxwell introduced *Maxwell* distribution in the field of kinetic energy of gases which was further modified by Ludwig Boltzmann [1]. According to the name of these two pioneers the distribution is named as *Maxwell- Boltzmann* distribution or simply *Maxwell* distribution. Since the invention of this distribution it is used in lifetime modelling, chemistry, physics as well as statistical mechanics [2] [3] [4] [5].

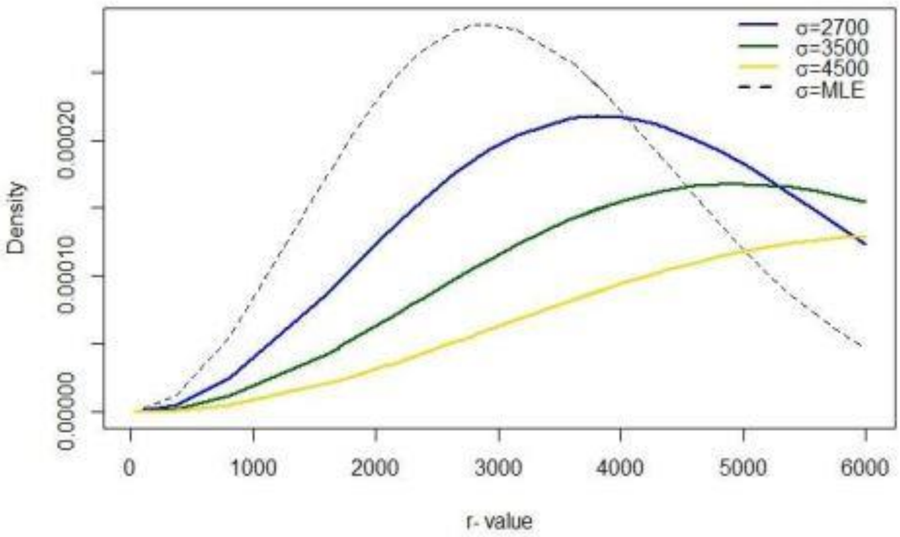

Figure 1.1: The PDF of *Maxwell* distribution for different value of parameter [6].

In statistics we have different types of distributions to model the life time data such as *Weibull* distribution, *Gamma* distribution, *Exponential* distribution, *Extreme* value distribution etc. But in real life we got different kinds of situations where these types of popular distributions are not suitable. For this we have to use other statistical distributions to fit the life time data. Tyagi & Bhattacharya [7] [8] used *Maxwell* distribution as a possible life time distribution.

In physics, the velocity distribution function of force-free granular gases was developed from *Maxwell* distribution [5]. Laser ablation is a process or removing materials from a solid surface by irradiating it with laser beam. In laser ablation, discarded molecules emitted from a solid surface have streams with certain velocities. These molecules are shown to follow the *Maxwell* distribution [9].

In chemistry, *Maxwell* distribution is also frequently used [10]. In the literature we see that airbags are used to help the passenger to reduce their speed in crash without getting injured. Airbags contain a mixture different gas. During an accident, these gases undergo a chemical reaction and produce a new and harmless gas which fills the airbag and saves passengers from getting injured. Airbags gases speed follow *Maxwell* distribution [9] [11].

Now, in statistics, an inverse distribution is a probability distribution of the reciprocal or inverse of a random variable. This type of distribution is used not only in mathematical statistics but also in various fields such as engineering, agricultural and industrial sector to describe different phenomena and to make quantitative analysis [12].

For example, in the study of a renewal process for human population, it is applied to describe the net maternity function in demography [13]. In the literature, we see that the maternity function for successive generations is expressed as the inverse distribution except for first generations. [14].

In aerial surveys of agricultural land, the crop field size distributions play a major role in the survey design. Field size are measured in length and width and in general these data are tending to be positively skewed. The inverse distributions are used for modelling crop field size [15].

In industry the length of employee service is also determined by the inverse distribution. These types of data are most of positively or negatively skewed. In the literature, we also see that inverse distribution are used as a candidate distribution. In risk analysis it is also applicable [16].

Now the question is, how can we develop the inverse distribution? Let X_i $(i = 1,2,\ldots\ldots n)$ be a random sample of size n having a PDF $f(x,\sigma)$ and CDF $F(x)$ where σ is the scale parameter. Now the PDF and CDF of the reciprocal of the random variable X_i is respectively defined by

$$f_R(r) = f_X(r) \times |J| \quad \text{and} \tag{1.6}$$

$$F_R(r) = \int_{-\infty}^{r} f_R(r)dr, \tag{1.7}$$

where $R_i = \frac{1}{X_i}$ $(i = 1,2,\ldots\ldots n)$ and $|J|$ is the modulus of the jacobian transformation in which the old variable is differentiated with respect to the new variable.

1.2 Outlines of the remaining chapters

In the following chapters we studied inverse *Maxwell* distribution that is derived from *Maxwell* distribution along with the different properties of this distribution. Also, a detail literature survey in Chapter 2. Chapter 3 consists of preliminaries such as moment generating function, characteristics function, survival function, hazard function and entropy of inverse distribution in general. These properties for inverse *Maxwell* distribution has also been discussed. Besides this, some special characteristics of the inverse *Maxwell* distribution has also been discussed. In Chapter 4, we discuss the estimation procedure of the inverse *Maxwell* parameter by MLE and the method of moments. The Fisher information and Cramer-Rao lower bound for this distribution is also studied in Chapter 4. The application of this distribution in the field of statistical process monitoring is studied in Chapter 5. In addition, with this the real-life example of this distribution is also given. Finally, the conclusion and recommendation is provided in Chapter 6.

CHAPTER TWO

LITERATURE REVIEW

In the recent years the inverse distributions become more popular among the researcher that was in before. Most of the researchers investigated different fundamental properties of the distributions, for example, derivation of PDF and CDF of the inverse distribution, different moments, moment generating function, characteristics function etc. Real life application of the inverse distribution has been provided by a few researchers. In the subsequent section a detail literature of inverse distributions is provided to get clear concept about different properties of the distributions.

2.1 Literature review on inverse distribution

In 1943 Stephan discussed the expected values and moments of the inverse *binomial* distribution and inverse *hypergeometric* distribution. They also expanded a series of inverse factorials and provided simple examples [17]. In [18] Edwin and Savage provide a table that includes several values of the *binomial* and *Poisson* variable for different values of the parameter. In [19] they obtained the mean and variance of the MLE of the scale parameter of a *Weibull* distribution and showed that these moments are the functions of the negative moments of the *binomial* distribution. In 1964 M. L. Tiku developed a Laguerre series and negative moment of a *Poisson* variate for truncation and comparison between this numerical value and exact value are given [20]. In [21] the cumulative round off error for the inverse moments of the decapitated negative variable is estimated and its approximation with the *Gamma* distribution is also discussed along with the application of these moments. The inverse moments of order r of the truncated *binomial* and *Poisson* distribution and their applications are provided by D. G. Kabe in [22]. In 1972 M.T. Chao and W.E. Strawderman developed a general procedure for finding out the negative moments and shown that by using these procedures how can we estimate the moments of the inverse *binomial* and *Poisson* distribution [23]. The moments generating function contains not only the information about the integer moments but also contains the information about the inverse or negative moments. The prove of this statement is given in [24]. In 1957 M.C.K. Tweedie developed different types statistical properties of the Inverse *Gaussian* Distribution [25], [26]. In [27], Folks and Chhikara showed the application of inverse *Gaussian* distribution in case of Brownian motion and provided the estimates of the parameter by MLE. As an alternative of the inverse *Gaussian* distribution, Christian Robert proposed the inverse *normal* distribution and he also discussed the basic properties of the generalized inverse *normal* distributions [28]. In 2011 A.G. Glen presented the inverse *Gamma* distribution as a Survival Distribution, described the shape of the hazard function and shown various statistical properties [29]. In case of maintenance data, the comparison for the reliability function of inverse *Gamma* distribution with *log-normal* and inverse *Gaussian* distribution is studied in [30]. Kleiber and Kotz list some basic properties of the inverse *Gamma* distribution also model incomes with the distributions [31]. Milevsky and Posner derived the inverse *Gamma* distribution and point out that estimation by the method of moments is tractable algebraically [32]. The characteristics functions of the inverse *Gamma* distribution is developed by Paul Witkovsky [33]. The inverted *exponential* distribution is introduced by Keller and Kamath [34] and applied in the field of survival analysis by Dey and Pradhan [35]. The parameters of the inverse *Weibull* distribution is estimated by using Bayesian procedures and MLE and a comparison between them is studied in case of censored sample for the simulation and real life data [36], [37], [38], [39], [40]. The conditional confidence intervals for the parameters and reliability of the inverse *Weibull* distribution is studied in [41]. The inverse of the *Rayleigh* distribution is also studied by the various researchers. The PDF and CDF of the inverse *Rayleigh* distribution along with its others statistical properties and the estimation of the parameters is found in [42], [43], [44]. The Bootstrap confidence interval for inverse *Rayleigh* and *log-logistic* distribution is studied in [45].

2.2 Literature review on *Maxwell* distribution

The *Maxwell* distribution was derived first in the 1870 [46] and different statistical properties of the *Maxwell* distribution are discussed in [47] , [48], [49]. Krishna and Malik proposed *Maxwell* distribution as a life time model and important distributional properties and reliability characteristics are derived under Type-II censored samples [50], [51]. The point and interval estimate of the *Maxwell* distribution in case of Type-I progressively hybrid censored data and the MLE and Bayesian estimates of the parameters are obtained in [52]. Under different loss function the Bayes estimators of the parameter of *Maxwell* distribution has been obtained in [53].

An extension of the *Maxwell* distribution is also discussed by few researchers like as Kumar, Hassan and Suthar proposed a new distribution named extended *Maxwell* distribution which contains *Maxwell-X* family of distributions and they developed different distributional properties of the proposed distribution [54], Saghir, Khadim and Lin proposed a length-biased *Maxwell* distribution which is usually considered as a weighted distribution and its several properties such as shape, moments, reliability function and hazard function are derived in [55]. Haung and Chen investigated the asymptotic property of the generalized *Maxwell* distribution which is known as the tail behavior [56]. The higher order asymptotic expansion of the moments of the generalized *Maxwell* distribution are obtained in [57]. The MLE, Bayes and empirical Bayes estimator of *Maxwell* distribution are discussed in [58].

2.3 Literature review on inverse *Maxwell* distribution

Recent theoretical development of the inverse *Maxwell* distribution appears in [59], [60], [61]. In [59] they derived the PDF of inverse *Maxwell* distribution and estimate its mean, variance, harmonic mean and mode. They check the suitability of inverse *Maxwell* distribution as a survival model by obtaining the hazard and survival functions. In [61] they have discussed Bayesian estimation of the parameter of an inverse *Maxwell* distribution via Size- Biased sampling. They obtained the Bayes estimators of the scale parameter of this distribution under squared error, precautionary, entropy and another two loss functions. The risk functions of these estimators relative to loss function have also been estimated. In 2014 Lata and Srivastava discussed the estimation problem of size-biased inverse *Maxwell* distribution in [60]. In that paper they estimated the parameter by MLE and method equation estimation. Several properties of this distribution are obtained along with its survival and hazard curve.

2.4 Research questions

Based on the literature given in the previous section we may consider following problems.

Problem 1: **In depth study of the characteristics of inverse *Maxwell* distribution.**

Few properties of inverse *Maxwell* distribution is known from the literature so we need to find out the other distributional properties of inverse *Maxwell* distribution by using the standard form of the *Maxwell* distribution.

Problem 2: **Solving the problems in plotting of inverse *Maxwell* distribution.**

In the literature we know that the actual graph of the PDF, CDF, survival and hazard functions are not correctly drawn. So, we have to draw the actual figure of the graphs.

Problem 3: **Application of the distribution in industrial process monitoring.**

There is no real-life applications of inverse *Maxwell* distribution is available in the literature. So, we need to find out the applications.

CHAPTER THREE

INVERSE MAXWELL DISTRIBUTION

In this chapter we will introduce the inverse *Maxwell* distribution. We will also demonstrate different properties of this inverse distribution such as moment generating function, characteristics function, survival function, hazard function and mode. Some preliminaries of the distribution are given in the following sections.

3.1 Preliminaries of inverse distribution

In the chapter one, inverse distribution has been defined. Here we introduce different properties of inverse distribution in general.

3.1.1 Moments of Inverse distribution

Moments are important characteristics of an inverse distribution. Let, R be a random variable having PDF $f(r)$. The sth moment of R about origin or sth raw moment of the inverse distribution, denoted by μ_s', is defined as

$$\mu_s' = E[R^s] = \int_{-\infty}^{\infty} r^s f(r) dr, \tag{3.1}$$

provided the integral in equation (3.1) exist. If we put, $s = 1$ then $\mu_1' = E[R] = \mu$ is the mean of the distribution. Moments about the mean of an inverse distribution are known as the central moments or corrected moments of the inverse distribution. The central or corrected moment about mean μ, usually denoted by μ_s is defined as

$$\mu_s = E[R-\mu]^S = \int_{-\infty}^{\infty} (r-\mu)^s f(r) dr, \tag{3.2}$$

provided the integral in equation (3.2) exist.

If $s = 2, \mu_2 = E[R-\mu]^2$ is called the variance of the distribution which measures the spread and dispersion of an inverse distribution. If $s = 3$ and 4 in equation (3.1) and equation (3.2), we get the third and fourth raw and central moments of an inverse distribution. That is,

$$\mu_3' = E[R^3],\ \mu_4' = E[R^4],\ \mu_3 = E[R-\mu]^3\ and\ \mu_4 = E[R-\mu]^4.$$

The third central moment μ_3 and the fourth central moment μ_4 are sometimes called a measure of skewness and kurtosis respectively. The coefficient of skewness and kurtosis based on moments is respectively given by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2}. \tag{3.3}$$

If β_1 is positive then the inverse distribution is positively skewed, if β_1 is negative then the inverse distribution is negatively skewed and if β_1 is equal to zero then the distribution is called symmetric. If β_2 is equal to 3 then the inverse distribution is meso kurtic, if greater than 3 then the inverse distribution is lepto kurtic and if less than 3 then the inverse distribution is platy kurtic.

There is a relationship between raw moments and central moments and by using this relationship we can find the central moments from the raw moments. By definition,

$$\mu_s = E[R - \mu_1']^s, \text{ (where } \mu_{1'} \text{ is the mean of the distribution)}$$

$$= E\left[R^s - \binom{s}{1} R^{s-1}\mu_1' + \binom{s}{2} R^{s-2}(\mu_1')^2 - \binom{s}{3} R^{s-3}(\mu_1')^3 + \cdots + (-1)^s(\mu_1')^s\right],$$

$$= \mu_s' - \binom{s}{1} \mu_{s-1}'\mu_1' + \binom{s}{2} \mu_{s-2}'(\mu_1')^2 - \binom{s}{3} \mu_{s-3}'(\mu_1')^3 + \cdots + (-1)^s(\mu_1')^s.$$

In particular, if we put $s = 1,2,3$ *and* 4, we get $\mu_1 = 0$, since $\mu_0' = 1$.

$$\mu_2 = \mu_2' - (\mu_1')^2, \tag{3.4}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3, \tag{3.5}$$

$$\mu_4 = \mu_4' - (4\mu_3'\mu_1') + (6\mu_2'(\mu_1')^2) - 3(\mu_1')^4. \tag{3.6}$$

3.1.2 Moment generating function of inverse distribution

Theorem 3.1 The moment generating function of inverse distribution can be presented as

$$M_R(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!}\mu_s'. \tag{3.7}$$

Proof. Let R be a random variable having PDF $f(r)$. The moment generating function of R is defined by

$$M_R(t) = E[e^{tR}] = \int_{-\infty}^{\infty} e^{tr} f(r)dr, \tag{3.8}$$

provided $E[e^{tr}]$ exists for all values in the limit $-h < t < h, h > 0$. If $f(r)$ is an inverse distribution, as defined in Chapter one equation (1.7), the equation (3.8) then can be written as

$$M_R(t) = \int_{-\infty}^{\infty} \left[1 + tR + \frac{(tR)^2}{2!} + \frac{(tR)^3}{3!} + \cdots + \frac{(tR)^s}{s!} + \cdots\right] f(r)dr$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^s}{s!}\mu_s' + \cdots$$

$$= \sum_{s=0}^{\infty} \frac{t^s}{s!}\mu_s'.$$

Hence, the theorem proved. □

3.1.3 Characteristics function of inverse distribution

When we consider the moment generating function it is observed that moments of a distribution can be obtained from its MGF. But in certain cases, for a particular distribution all the moments may not be found out from MGF. In addition, for some distributions MGF does not exit. To avoid such issues characteristics function may be used to find the moments and the probability function of the distribution. The characteristics function of the distribution of R is defined as

$$\phi_R(t) = E\left[e^{itR}\right] = \int_{-\infty}^{\infty} e^{itr} f(r)dr; \tag{3.9}$$

provided $E[e^{tr}]$ exists for all values in the limit $-h < t < h, h > 0$ and $i = \sqrt{-1}$, the imaginary unit. Using characteristics function, the probability density function of a random variable R can be obtained as

$$f(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itR}\phi_R(t)dt. \tag{3.10}$$

This is called Levy's theorem or inverse Mellin transformation formula [62].

Theorem 3.2 The characteristics function of inverse distribution can be presented as

$$\phi_R(t) = \sum_{s=0}^{\infty} \frac{(it)^s}{s!}\mu_s'. \tag{3.11}$$

Proof. Let R be a random variable having PDF $f(r)$. If $f(r)$ is an inverse distribution, as defined in Chapter one equation (1.7), the equation (3.9) then can be written as

$$\phi_R(t) = \int_{-\infty}^{\infty} \left[1 + itR + \frac{(itR)^2}{2!} + \frac{(itR)^3}{3!} + \cdots + \frac{(itR)^s}{s!} + \cdots\right] f(r)dr$$

$$= 1 + it\mu_1' + \frac{(it)^2}{2!}\mu_2' + \frac{(it)^3}{3!}\mu_3' + \cdots + \frac{(it)^s}{s!}\mu_s' + \cdots$$

$$= \sum_{s=0}^{\infty} \frac{(it)^s}{s!}\mu_s'.$$

Hence, the theorem proved. □

Theorem 3.3 The Mellin transformation formula for inverse distribution can be presented as

$$f(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itR}\sum_{s=0}^{\infty}\frac{(it)^s}{s!}\mu_s'\,dt. \tag{3.12}$$

Proof. Due to equation (3.11), Mellin transformation formula, given in equation (3.10), for inverse distribution can be presented as the form given in equation (3.12). □

3.1.4 Survival function of inverse distribution

Survival function is defined as the probability that an individual life longer than time t. It is symbolized by $S(t)$. Hence, $S(t) = P[An\ indivudal\ stay\ more\ than\ time\ t] = P[T > t] = \int_t^{\infty} f(r)dr$,where, T is survival time. From the definition of CDF $F(t)$ of T, we have $S(t) = 1 - P[An\ individual\ dies\ before\ time\ t] = 1 - F(t)$. Here $S(t)$ is a non-increasing function of time t such that $S(t) = 1$ for $t = 0$ and $S(t) = 0$ for $t = \infty$. That is, the probability than an individual will survive at time zero is at least 1 and will survive up to infinite time is zero. The function $S(t)$ is also known as the cumulative survival rate. The graph of $S(t)$ is called the survival curve. So, $S(t)$ is a monotonic decreasing continuous function. In terms of inverse distribution this function can be defined as

$$S(t) = 1 - F(t). \tag{3.13}$$

3.1.5 Hazard function of inverse distribution

The hazard function of survival time T provides the conditional failure rate. It is denoted by $h(t)$ and defined in terms of inverse distribution as

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{S(t)}, \tag{3.14}$$

where, $f(t)$is the PDF of the inverse distribution and $S(t)$ is the survival function. Here, $h(t) = 0$ for $t = 0$ and $h(t) = 1$ for $t = \infty$. The hazard function is also known as the instantaneous failure rate, force of mortality, conditional mortality rate and age-specific failure rate.

3.1.6 Mode of inverse distribution

The mode of an inverse distribution is the point at which the probability density function attains its maximum value. That means the value or r for which $f(r)$ is maximum is called the mode of that inverse distribution. The mode will be obtained from the following relations

$$f'(r) = 0 \text{ and } f''(r) < 0. \tag{3.15}$$

3.1.7 Entropy of the inverse distribution

Entropy is a measure of unpredictability of the state, or equivalently, of its average information content. For an inverse distribution entropy is calculated by the expectation of the negative logarithm of the PDF. Mathematically it is defined by

$$H(R) = E(-log f(r)). \tag{3.16}$$

3.2 Properties of inverse *Maxwell* distribution

Consider a *Maxwell* distribution with PDF and CDF are given in the equation (1.1) and (1.2) respectively in Chapter one. Hence the inverse *Maxwell* distribution, according to the definition given in equation (1.6) in chapter one, is presented by the PDF

$$f(r) = \sqrt{\frac{2}{\pi}}\, \sigma^{-3} r^{-4} e^{-\frac{1}{2r^2\sigma^2}}, \tag{3.17}$$

for $r > 0$ and $\sigma > 0$. The following Figure 3.1 shows the PDF of the distribution graphically. It shows that the inverse *Maxwell* distribution is right skewed like as the *Maxwell* distribution. The figure is plotted using R 3.4.2 We draw the figure for $\sigma = 0.01, 0.02, 0.03, 0.04$ and 0.05. And, the CDF of the above mentioned distribution is obtained using the equation (1.7) given in Chapter one as

$$F(r) = \frac{2}{\sqrt{\pi}}\left[1 - \gamma\left(\frac{3}{2}, \frac{1}{2r^2\sigma^2}\right)\right]. \tag{3.18}$$

Again, the following Figure 3.2 demonstrated the CDF of the distribution. The figure is plotted for different values of the scale parameter where $\sigma = 0.01, 0.02, 0.03, 0.04$ and 0.05.

Theorem 3.4 The integration of the equation (3.17) over the entire range is unity. Hence it is a probability density function.

Proof. Taking integration in equation (3.17) we get,

$$\int_0^\infty f(r)dr = \int_0^\infty \sqrt{\frac{2}{\pi}}\, \sigma^{-3} r^{-4} e^{-\frac{1}{2r^2\sigma^2}} dr. \tag{3.19}$$

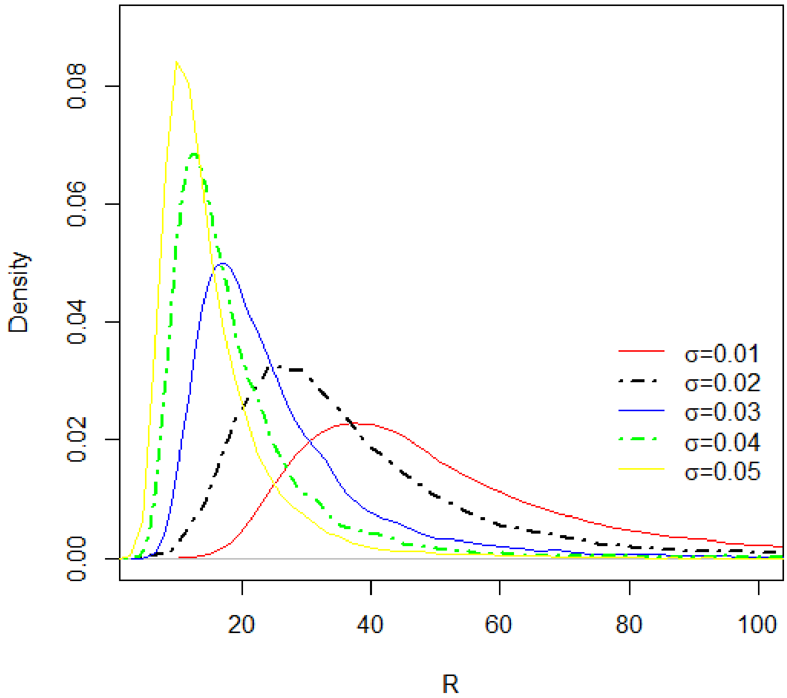

Figure 3.1: The PDF of inverse *Maxwell* distribution for different value of parameter.

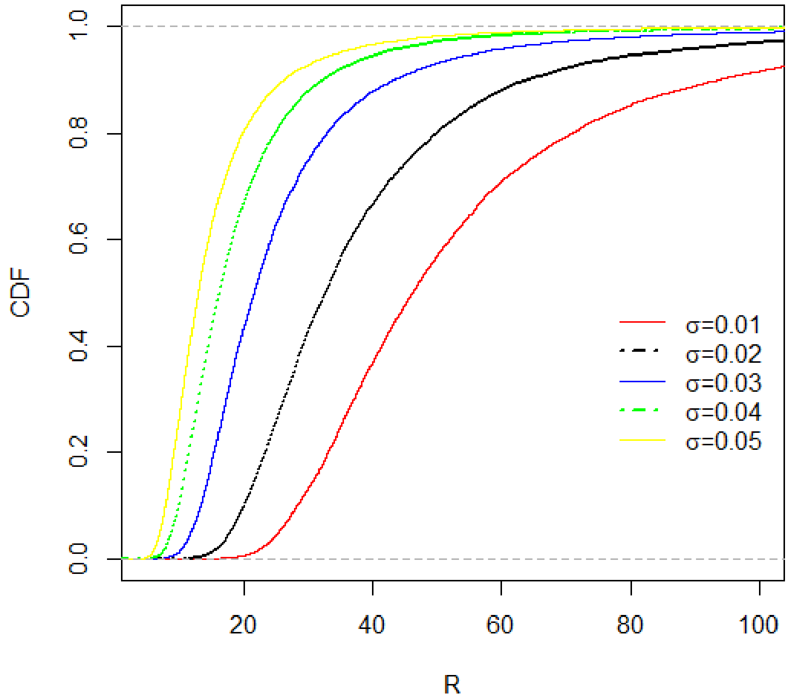

Figure 3.2: The CDF of inverse *Maxwell* distribution for different value of parameter.

Now let $\frac{1}{2r^2\sigma^2} = p$ which implies that $r = \frac{1}{\sigma\sqrt{2p}}$. Taking differentiation on both sides with respect to r we get, $\frac{dp}{dr} = -\frac{2}{2r^3\sigma^2}$. This can be written as $dr = -r^3\sigma^2 dp$. Limit becomes when $r = 0$ then $p = \infty$ and when $r = \infty$ then $p = 0$.

Hence,

$$\int_0^\infty f(r)dr = \sqrt{\frac{2}{r}}\,\sigma^{-3}\int_\infty^0 \left(\frac{1}{\sigma\sqrt{2p}}\right)^{-4} e^{-p}(-r^3\sigma^2 dp)$$

$$=\sqrt{\frac{2}{r}}\,\sigma^{-1}\int_0^\infty \left(\sigma\sqrt{2p}\right)^4 \left(\frac{1}{\sigma\sqrt{2p}}\right)^3 e^{-p}dp$$

$$=\sqrt{\frac{2}{r}}\,\sigma^{-1}\int_0^\infty \left(\sigma\sqrt{2p}\right)^4 \left(\sigma\sqrt{2p}\right)^{-3} e^{-p}dp$$

$$=\sqrt{\frac{2}{r}}\,\sigma^{-1}\int_0^\infty \sigma\sqrt{2p}\, e^{-p}dp$$

$$=\frac{2}{\sqrt{\pi}}\int_0^\infty p^{\frac{1}{2}}e^{-p}dp$$

$$=\frac{2}{\sqrt{\pi}}\int_0^\infty p^{\frac{3}{2}-1}e^{-p}dp$$

$$=\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)$$

$$=\frac{2}{\sqrt{\pi}}\times\frac{\sqrt{\pi}}{2}=1.$$

Therefore, equation (3.4) can be written as $\int_0^\infty f(r)dr = 1$. Thus, the function in equation (3.17) is a probability density function. □

3.2.1 Moments of inverse *Maxwell* distribution

Theorem 3.5 The $s-th$ moments of inverse *Maxwell* distribution is given as

$$\mu_s' = \frac{2^{1-\frac{s}{2}}}{\sqrt{\pi}}\sigma^{-s}\Gamma\left(\frac{3-s}{2}\right). \tag{3.20}$$

Proof. Recall equation (3.1) and equation (3.17) to obtain the $s-th$ raw moment of inverse *Maxwell* distribution as

$$\mu_s' = E(R^s) = \int_0^\infty \sqrt{\frac{2}{\pi}}\,\sigma^{-3}r^{s-4}e^{-1/2r^2\sigma^2}\,dr.$$

Now, let $\frac{1}{2r^2\sigma^2} = u$ which implies that $r = \sqrt{\frac{1}{2\sigma^2 u}}$. Taking differentiation on both sides with respect to u we get, $\frac{dr}{du} = -\frac{1}{2\sqrt{2}\sigma u^{\frac{3}{2}}}$. This can be written as $dr = -\frac{du}{2\sqrt{2}\sigma u^{\frac{3}{2}}}$. Limit becomes when $r=0$ then $u=\infty$ and when $r=\infty$ then $u=0$.

Hence,

$$\mu_s' = \sqrt{\frac{2}{\pi}}\sigma^{-3}\int_{\infty}^{0}\left(\sqrt{\frac{1}{2\sigma^2 u}}\right)^{s-4} e^{-u}\frac{-1}{2\sqrt{2}\sigma u^{\frac{3}{2}}}\,du$$

$$= -\frac{1}{2\sqrt{\pi}}\sigma^{-3}\int_{\infty}^{0}\left(\frac{1}{2\sigma^2 u}\right)^{\frac{s}{2}-2} e^{-u}\sigma^{-1}u^{-\frac{3}{2}}\,du$$

$$= \frac{1}{2\sqrt{\pi}}\sigma^{-4}\int_{0}^{\infty}(2\sigma^2 u)^{-\frac{s}{2}+2}e^{-u}u^{-\frac{3}{2}}\,du$$

$$= \frac{1}{2\sqrt{\pi}}\sigma^{-4}\int_{0}^{\infty}2^{2-\frac{s}{2}}\sigma^{4-s}e^{-u}u^{-\frac{3}{2}-\frac{s}{2}+2}\,du$$

$$= \frac{1}{\sqrt{\pi}}\int_{0}^{\infty}2^{1-\frac{s}{2}}\sigma^{-s}e^{-u}u^{\frac{1}{2}-\frac{s}{2}}\,du$$

$$= \frac{2^{1-\frac{s}{2}}}{\sqrt{\pi}}\sigma^{-s}\int_{0}^{\infty}e^{-u}\,u^{\left(\frac{3}{2}-\frac{s}{2}\right)-1}\,du$$

$$= \frac{2^{1-\frac{s}{2}}}{\sqrt{\pi}}\sigma^{-s}\Gamma\left(\frac{3-s}{2}\right).$$

The above equation is sufficient for prove the theorem. □

Corollary 3.1 The first four raw moments of inverse *Maxwell* distribution are

$$\mu_1' = \sqrt{2/\pi}\sigma^{-1},$$

$$\mu_2' = \sigma^{-2},$$

$$\mu_3' = \left(\sqrt{2\pi}\right)^{-1}\sigma^{-3}, \text{ and}$$

$$\mu_4' = \sigma^{-4}.$$

Proof. By substituting $s = 1, 2, 3$ and 4 in equation (3.7) the first four raw moments of inverse *Maxwell* distribution can be obtaining. □

Corollary 3.2 The 2nd, 3rd and 4th central moments of inverse *Maxwell* distribution are respectively,

$$\mu_2 = \left(\frac{\pi-2}{\pi}\right)\sigma^{-2},$$

$$\mu_3 = \left(\frac{4-5\pi}{\sqrt{2}\pi\sqrt{\pi}}\right)\sigma^{-3}, \text{ and}$$

$$\mu_4 = \left(\frac{8\pi-\pi^2-12}{\pi^2}\right)\sigma^{-4}.$$

Proof. By putting the values of the raw moments in equation (3.4), (3.5) and (3.6) the desired central moments of inverse *Maxwell* distribution can be obtained. □

Corollary 3.3 The coefficient of skewness and kurtosis of inverse *Maxwell* distribution are

$$\beta_1 = \left(\frac{4-5\pi}{\sqrt{2\pi}\sqrt{\pi}}\right)^2 / \left(\frac{\pi-2}{\pi}\right)^3 \text{ and } \beta_2 = \left(\frac{8\pi-\pi^2-12}{\pi^2}\right) / \left(\frac{\pi-2}{\pi}\right)^2.$$

Proof. By putting the values of the raw moments in equation (3.3) the skewness and kurtosis of inverse *Maxwell* distribution can be obtaining. □

3.2.2 MGF of inverse *Maxwell* distribution

Theorem 3.6 The moment generating function for inverse *Maxwell* distribution is

$$M_R(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{2^{1-\frac{s}{2}}}{\sqrt{\pi}} \sigma^{-s} \Gamma\left(\frac{3-s}{2}\right). \tag{3.21}$$

Proof. Recalling equation (3.7) and in that equation put the value of μ_s' from equation (3.10), then we will get our desired moment generating function of the inverse *Maxwell* distribution.

Hence, prove. □

3.2.3 CF of inverse *Maxwell* distribution

Theorem 3.7 The characteristics function for inverse Maxwell distribution is

$$\phi_R(t) = \sum_{s=0}^{\infty} \frac{(it)^s}{s!} \frac{2^{1-\frac{s}{2}}}{\sqrt{\pi}} \sigma^{-s} \Gamma\left(\frac{3-s}{2}\right). \tag{3.22}$$

Proof. Recalling equation (3.11) and in that equation put the value of μ_s' from equation (3.10), then we will get our desired CF of the inverse *Maxwell* distribution.

Hence, prove. □

3.2.4 Survival function of inverse *Maxwell* distribution

Theorem 3.8 Survival function of inverse *Maxwell* distribution is given as

$$S(t) = 1 - \frac{2}{\sqrt{\pi}}\left[1 - \gamma\left(\frac{3}{2}, \frac{1}{2r^2\sigma^2}\right)\right]. \tag{3.23}$$

Proof. Using definition given in equation (3.13) and the CDF given in equation (3.8) we get the result given in equation (3.23). □

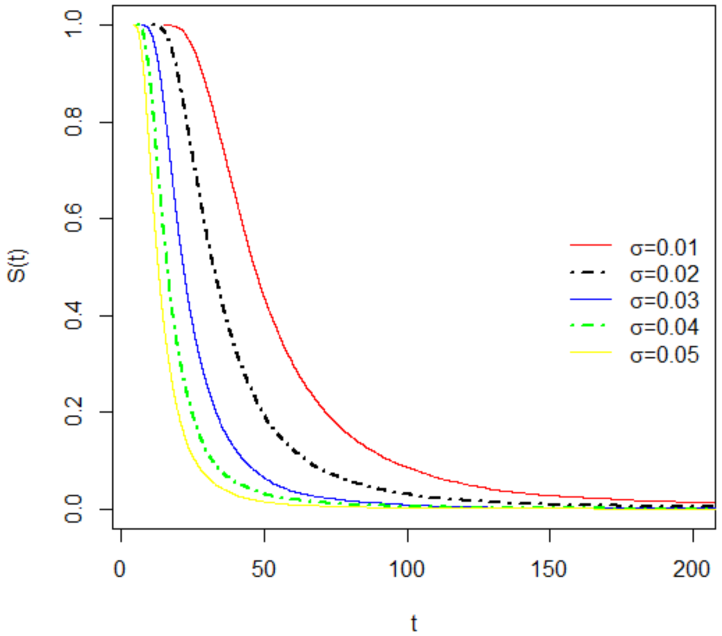

Figure 3.3: The Survival curve of inverse *Maxwell* distribution for different value of parameter.

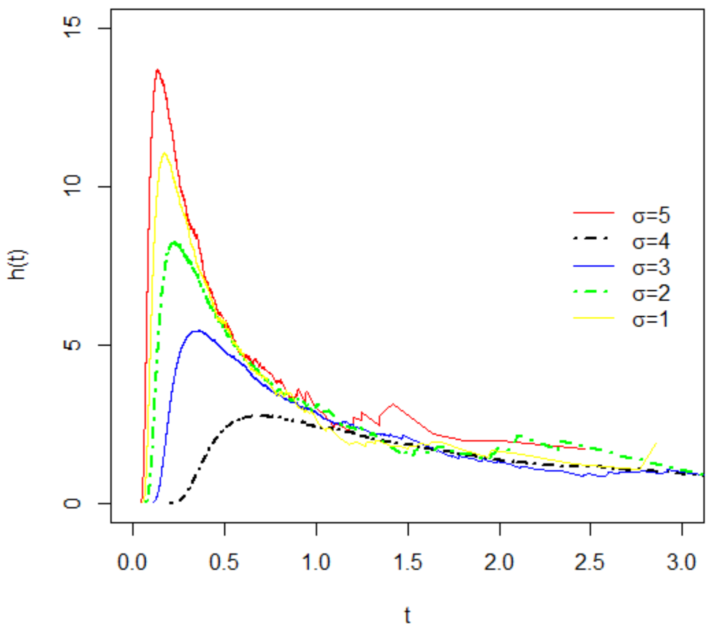

Figure 3.4: The Hazard curve of inverse *Maxwell* distribution for different value of parameter.

3.2.5 Hazard function of inverse Maxwell distribution

Theorem 3.9 Hazard function of inverse Maxwell distribution is given as

$$h(t) = \sqrt{\frac{2}{\pi}}\,\sigma^{-3}r^{-4}e^{-1/2r^2\sigma^2}/1 - \frac{2}{\sqrt{\pi}}\left[1 - \gamma\left(\frac{3}{2},\frac{1}{2r^2\sigma^2}\right)\right]. \tag{3.24}$$

Proof. Using definition given in equation (3.14) and from equation (3.7) and (3.23) we put the value of PDF and survival function then we will get the result in equation (3.24).□

3.2.6 Mode of inverse *Maxwell* distribution

Theorem 3.10 If the PDF of the inverse *Maxwell* distribution is according to equation (3.17) then the mode of the distribution is $\frac{1}{2\sigma}$.

Proof. Recalling the relationships given in equation (3.15) and differentiate equation (3.17) with respect to r and putting is equal to zero. We get,

$$f(r)' = \sqrt{\frac{2}{\pi}}\,\sigma^{-3}\left(\frac{d}{dr}(r^{-4})e^{-\frac{1}{2r^2\sigma^2}} + \frac{d}{dr}\left(e^{-\frac{1}{2r^2\sigma^2}}\right)r^{-4}\right)$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma^{-3}\left(\left(-\frac{4}{r^5}\right)e^{-\frac{1}{2r^2\sigma^2}} + \left(\frac{e^{-\frac{1}{2r^2\sigma^2}}}{\sigma^2 r^3}\right)r^{-4}\right)$$

$$= \frac{\sqrt{2}\left(-4e^{-\frac{1}{2r^2\sigma^2}}r^2\sigma^2 + e^{-\frac{1}{2r^2\sigma^2}}\right)}{\sqrt{\pi}\sigma^5 r^7}$$

$$= \frac{(1-4r^2\sigma^2)}{r^3\sigma^2}\,f(r)$$

Now, we can write,

$$f(r)' = 0$$

$$\Rightarrow \frac{(1-4r^2\sigma^2)}{r^3\sigma^2}\,f(r) = 0$$

$$\Rightarrow \frac{(1-4r^2\sigma^2)}{r^3\sigma^2} = 0 \quad [\text{Since } f(r) \neq 0\ ]$$

$$\Rightarrow (1-4r^2\sigma^2) = 0$$

$$\Rightarrow r = \frac{1}{2\sigma}.$$

Therefore, $r = \frac{1}{2\sigma}$ is the mode of the distribution. □

3.2.7 Entropy of the inverse *Maxwell* distribution

Theorem 3.11 If the PDF of the inverse *Maxwell* distribution is according to equation (3.17) then the entropy of the inverse *Maxwell* distribution is $1.3 + \log(\sigma)$.

Proof. Using the formula described in equation (3.16) and put the value of $f(r)$ from equation (3.17) we get,

$$H(R) = \int_0^\infty f(r)\big(-log f(r)\big)dr$$

$$= \int_0^\infty f(r)\left[-log\left(\sqrt{\frac{2}{\pi}}\ \sigma^{-3}r^{-4}e^{-\frac{1}{2r^2\sigma^2}}\right)\right]dr$$

$$= \int_0^\infty f(r)\left[-\frac{1}{2}\log(2) + \frac{1}{2}\log(\pi) + 3\log(\sigma) + 4\log(r) + \frac{1}{2r^2\sigma^2}\right]dr$$

$$= -\frac{1}{2}\log(2)\int_0^\infty f(r)dr \quad + \quad \frac{1}{2}\log(\pi)\int_0^\infty f(r)dr \quad + \quad 3\log(\sigma)\int_0^\infty f(r)dr \quad +$$

$$4\int_0^\infty \log(r)\, f(r)dr \quad + \quad \frac{1}{2\sigma^2}\int_0^\infty r^{-2}f(r)dr$$

$$= -\frac{1}{2}\log(2) + \frac{1}{2}\log(\pi) + 3\log(\sigma) + 4\int_0^\infty \log(r)\, f(r)dr + \frac{1}{2\sigma^2}\int_0^\infty r^{-2}f(r)dr$$

$$= -\frac{1}{2}\log(2) + \frac{1}{2}\log(\pi) + 3\log(\sigma) + 4\int_0^\infty \log(r)\, f(r)dr + \frac{1}{2\sigma^2}\times\frac{4}{\sqrt{\pi}}\sigma^2\,\frac{3\sqrt{\pi}}{4}$$

$$= -\frac{1}{2}\log(2) + \frac{1}{2}\log(\pi) + 3\log(\sigma) + 4\int_0^\infty \log(r)\, f(r)dr + \frac{3}{2}$$

$$= -\frac{1}{2}\log(2) + \frac{1}{2}\log(\pi) + 3\log(\sigma) + \frac{3}{2} + 4\int_0^\infty \log(r)\, f(r)dr$$

$$= -\frac{1}{2}\log(2) + \frac{1}{2}\log(\pi) + 3\log(\sigma) + \frac{3}{2} + 4\left(0.5\log\left(\frac{1}{\sigma}\right) - 0.084\right)$$

$$= -0.15 + 0.29 + 3\log(\sigma) + 1.5 + 2\log\left(\frac{1}{\sigma}\right) - 0.34$$

$$= 1.3 + 3\log(\sigma) + 2\log\left(\frac{1}{\sigma}\right)$$

$$= 1.3 + (3-2)\log(\sigma)$$

$$= 1.3 + \log(\sigma)$$

Hence, the theorem proved. □

3.2.8 Some special characteristics of inverse *Maxwell* distribution

Theorem 3.12 If r_1 and r_2 are two independent random variables having density functions

$$f(r_1) = \sqrt{\frac{2}{\pi}}\ \sigma^{-3}r_1^{-4}e^{-1/2r_1^2\sigma^2} \quad \text{and} \quad f(r_2) = \sqrt{\frac{2}{\pi}}\ \sigma^{-3}r_2^{-4}e^{-1/2r_2^2\sigma^2}. \qquad (3.25)$$

Then the PDF of $u = r_1 + r_2$ is given by

$$f(u) = \frac{2}{\pi}\sigma^{-6}u^{-4}e^{-\frac{1}{2u^2\sigma^2}}I(uv), \tag{3.26}$$

where, $I(uv) = \int_0^\infty \frac{1}{6u^2v^6 - 4u^3v^5 - 4uv^7 + v^8}e^{-\frac{1}{v^2\sigma^2}+\frac{1}{4uv\sigma^2}}dv$ and $r_1 > 0$ and $r_2 > 0$.

Proof. Here r_1 and r_2 are two independent random variables so their joint density function is calculated by

$f(r_1, r_2) = f(r_1) \times f(r_2)$

$$= \sqrt{\frac{2}{\pi}}\ \sigma^{-3}r_1^{-4}e^{-1/2r_1^2\sigma^2} \times \sqrt{\frac{2}{\pi}}\ \sigma^{-3}r_2^{-4}e^{-1/2r_2^2\sigma^2}$$

$$= \frac{2}{\pi}\sigma^{-6}r_1^{-4}r_2^{-4}e^{-\left(\frac{1}{2r_1^2\sigma^2}+\frac{1}{2r_2^2\sigma^2}\right)}, r_1 > 0 \text{ and } r_2 > 0. \tag{3.27}$$

Let, $u = r_1 + r_2$ and $v = r_1$ or $r_2 = u - v$. The Jacobian transformation is, $|J| = 1$.

Now the joint density function of u and v is given by

$f(u, v) = f(r_1, r_2) \times |J|$

$$= \frac{2}{\pi}\sigma^{-6}r_1^{-4}r_2^{-4}e^{-\left(\frac{1}{2r_1^2\sigma^2}+\frac{1}{2r_2^2\sigma^2}\right)} \times 1$$

$$= \frac{2}{\pi}\sigma^{-6}v^{-4}(u-v)^{-4}e^{-\left(\frac{1}{2v^2\sigma^2}+\frac{1}{2(u-v)^2\sigma^2}\right)}; u > 0 \text{ and } v > 0.$$

So, the density function of u is given by

$$f(u) = \int_0^\infty \frac{2}{\pi}\sigma^{-6}v^{-4}(u-v)^{-4}e^{-\left(\frac{1}{2v^2\sigma^2}+\frac{1}{2(u-v)^2\sigma^2}\right)}dv$$

$$= \int_0^\infty \frac{2}{\pi}\sigma^{-6}v^{-4}\frac{1}{u^4 - 4u^3v + 6u^2v^2 - 4uv^3 + v^4}e^{-\left(\frac{1}{2v^2\sigma^2}+\frac{1}{2(u-v)^2\sigma^2}\right)}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-4}\int_0^\infty v^{-4}\frac{1}{-4u^3v + 6u^2v^2 - 4uv^3 + v^4}e^{-\left(\frac{1}{2v^2\sigma^2}+\frac{1}{2(u^2-2uv+v^2)\sigma^2}\right)}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-4}\int_0^\infty v^{-4}\frac{1}{-4u^3v + 6u^2v^2 - 4uv^3 + v^4}e^{-\frac{1}{2v^2\sigma^2}-\frac{1}{2(u^2-2uv+v^2)\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-4}\int_0^\infty v^{-4}\frac{1}{-4u^3v + 6u^2v^2 - 4uv^3 + v^4}e^{-\frac{1}{2v^2\sigma^2}-\frac{1}{(2u^2-4uv+2v^2)\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-4}e^{-\frac{1}{2u^2\sigma^2}}\int_0^\infty v^{-4}\frac{1}{-4u^3v+6u^2v^2-4uv^3+v^4}e^{-\frac{1}{2v^2\sigma^2}-\frac{1}{(-4uv+2v^2)\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-4}e^{-\frac{1}{2u^2\sigma^2}}\int_0^\infty v^{-4}\frac{1}{-4u^3v+6u^2v^2-4uv^3+v^4}e^{-\frac{1}{2v^2\sigma^2}+\frac{1}{4uv\sigma^2}-\frac{1}{2v^2\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-4}e^{-\frac{1}{2u^2\sigma^2}}\int_0^\infty v^{-4}\frac{1}{-4u^3v+6u^2v^2-4uv^3+v^4}e^{-\frac{2}{2v^2\sigma^2}+\frac{1}{4uv\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-4}e^{-\frac{1}{2u^2\sigma^2}}\int_0^\infty \frac{1}{v^4}\frac{1}{-4u^3v+6u^2v^2-4uv^3+v^4}e^{-\frac{1}{v^2\sigma^2}+\frac{1}{4uv\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-4}e^{-\frac{1}{2u^2\sigma^2}}\int_0^\infty \frac{1}{6u^2v^6-4u^3v^5-4uv^7+v^8}e^{-\frac{1}{v^2\sigma^2}+\frac{1}{4uv\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-4}e^{-\frac{1}{2u^2\sigma^2}}I(uv)$$

Where, $I(uv) = \int_0^\infty \frac{1}{6u^2v^6-4u^3v^5-4uv^7+v^8}e^{-\frac{1}{v^2\sigma^2}+\frac{1}{4uv\sigma^2}}dv$

Hence, the theorem proved. □

Theorem 3.13 If r_1 and r_2 are two independent random variables having density functions like equation (3.25). Then the PDF of $u = r_1 - r_2$ is given by

$$f(u) = \frac{2}{\pi}\sigma^{-6}u^{-4}e^{-\frac{1}{2u^2\sigma^2}}I(uv), \tag{3.28}$$

where, $I(uv) = \int_0^\infty \frac{1}{6u^2v^6-4u^3v^5-4uv^7+v^8}e^{-\frac{1}{v^2\sigma^2}+\frac{1}{4uv\sigma^2}}dv$,$r_1 > 0$ and $r_2 > 0$.

Proof. This theorem can also be proved like as Theorem 3.12**.** □

Corollary 3.4 If r_1 and r_2 are two independent random variables having densities like equation (3.25) then the joint PDF of the summation and subtraction of these two random variables are same.

Proof. By comparing the equation (3.26) and equation (3.28) we can get out desired result. □

Theorem 3.14 If r_1 and r_2 are two independent random variables having density functions like equation (3.25). Then the PDF of $u = r_1/r_2$ is given by

$f(u) = \frac{2}{\pi}\sigma^{-6}u^{-6}e^{-\frac{1}{2u^2\sigma^2}}I(uv)$; where, $I(uv) = \int_0^\infty \frac{1}{6u^2v^5-4u^3v^4-4uv^6+v^7}e^{-\frac{1}{2v^2\sigma^2}-\frac{v^2}{\sigma^2}}dv$

$r_1 > 0$ and $r_2 > 0$.

Proof. From equation (3.27) we get the joint density of r_1 and r_2. Let, $u = \frac{r_1}{r_2}$ and $u = r_1$ or $r_2 = \frac{v}{u}$. The Jacobean transformation is $|J| = \frac{v}{u^2}$. Now, the joint density of u and v *is*

$$g(u,v) = \frac{2}{\pi}\sigma^{-6}u^{-2}v^{-3}(u-v)^{-4}e^{-\left(\frac{1}{2v^2\sigma^2}+\frac{1}{2\left(\frac{v}{u}\right)^2\sigma^2}\right)}; u > 0 \text{ and } v > 0.$$

So, the density function of u is given by

$$g(u) = \int_0^\infty \frac{2}{\pi}\sigma^{-6}u^{-2}v^{-3}(u-v)^{-4}e^{-\left(\frac{1}{2v^2\sigma^2}+\frac{1}{2\left(\frac{v}{u}\right)^2\sigma^2}\right)}dv$$

$$= \int_0^\infty \frac{2}{\pi}\sigma^{-6}u^{-2}v^{-3}\frac{1}{u^4-4u^3v+6u^2v^2-4uv^3+v^4}e^{-\left(\frac{1}{2v^2\sigma^2}+\frac{1}{2\left(\frac{v}{u}\right)^2\sigma^2}\right)}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-6}\int_0^\infty v^{-3}\frac{1}{-4u^3v+6u^2v^2-4uv^3+v^4}e^{-\left(\frac{1}{2v^2\sigma^2}+\frac{1}{2\frac{u^2}{v^2}\sigma^2}\right)}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-6}\int_0^\infty v^{-3}\frac{1}{-4u^3v+6u^2v^2-4uv^3+v^4}e^{-\frac{1}{2v^2\sigma^2}-\frac{1}{2\frac{u^2}{v^2}\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-6}e^{-\frac{1}{2u^2\sigma^2}}\int_0^\infty v^{-3}\frac{1}{-4u^3v+6u^2v^2-4uv^3+v^4}e^{-\frac{1}{2v^2\sigma^2}-\frac{1}{\frac{1}{v^2}\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-6}e^{-\frac{1}{2u^2\sigma^2}}\int_0^\infty v^{-3}\frac{1}{-4u^3v+6u^2v^2-4uv^3+v^4}e^{-\frac{1}{2v^2\sigma^2}-\frac{v^2}{\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-6}e^{-\frac{1}{2u^2\sigma^2}}\int_0^\infty \frac{1}{6u^2v^5-4u^3v^4-4uv^6+v^7}e^{-\frac{1}{2v^2\sigma^2}-\frac{v^2}{\sigma^2}}dv$$

$$= \frac{2}{\pi}\sigma^{-6}u^{-6}e^{-\frac{1}{2u^2\sigma^2}}I(uv)$$

Where, $I(uv) = \int_0^\infty \frac{1}{6u^2v^5-4u^3v^4-4uv^6+v^7}e^{-\frac{1}{2v^2\sigma^2}-\frac{v^2}{\sigma^2}}dv.$

Hence, the theorem is proved. □

3.3 Summary of the inverse *Maxwell* distribution

In this chapter, we developed inverse distribution in general. We studied different properties of the distribution, we implemented these properties for a particular distribution named inverse *Maxwell* distribution. The mentionable properties of the distribution are MGF, CF, survival function, hazard function, mode and entropy. We derive the functional form of the MGF and CF. As moments are very important to study the shape characteristics of a distribution, we derived first four raw and central moments for the inverse *Maxwell* distribution. From this moments we calculate the skewness and kurtosis of the desired distribution. We considered some special cases of the proposed distribution which may be implemented in some real-life scenario. In the next chapter, we will discuss about this distribution in light of estimating its distributional parameters.

CHAPTER FOUR

PARAMETER ESTIMATION OF INVERSE MAXWELL DISTRIBUTION

Parameters are called the characteristics of population. Since it is not feasible to measure an entire population and in practice the value of the parameters are unknown, hence we need to estimate these parameters. For this purpose, we need to take a random sample from the desired population. One of the objectives of statistical analysis is to obtain estimates of these population parameters. The estimates of the population parameters are called sample statistics as they are calculated for a random sample. Now the question arise how much information can a sample of data provide about the unknown parameter?

In this chapter we present different estimation technique of population parameters for inverse *Maxwell* distribution. More specifically, maximum likelihood estimation and method of moments are discussed. As the answer of the question we will find out the Fisher information and the Cramer-Rao Inequality.

4.1 MLE of the parameter of inverse distribution

From the theoretical point of view, the most general method of estimation known is the method of Maximum Likelihood Estimation (MLE) which was initially formulated by C.F. Gauss but as a general method of estimation was first introduced by Prof. R.A. Fisher.

Let $x_1, x_2, \ldots . x_n$ be a random sample of size n from a population with density function $f(x, \sigma)$. Then the likelihood function of the sample values $x_1, x_2, \ldots . x_n$ usually denoted by $L = L(\sigma)$ is their joint density function, given by

$$L = f(x_1, \sigma) f(x_2, \sigma) \ldots . . f(x_n, \sigma) = \sum_{i=1}^{n} f(x_i, \sigma). \tag{4.1}$$

L becomes a function of the variable σ, the parameter. The principal of maximum likelihood consists in finding an estimator for the unknown parameter σ which maximizes the likelihood function $L(\sigma)$ for various in parameter, i.e. we wish to find $\hat{\sigma}$ so that

$$L(\hat{\sigma}) > L(\sigma) \;\; \forall \, \sigma \in \emptyset, i.e. L(\hat{\sigma}) = Sup \; L(\sigma) \;\; \forall \, \sigma \in \emptyset.$$

Thus, if there exists a function $\hat{\sigma} = \hat{\sigma}(x_1, x_2, \ldots . x_n)$ of the sample values which maximizes L for variations in σ, then $\hat{\sigma}$ is to be taken as an estimator of σ. $\hat{\sigma}$ is usually called Maximum Likelihood Estimator. Thus $\hat{\sigma}$ is the solution, if any, of

$$\frac{\partial L}{\partial \sigma} = 0 \text{ and } \frac{\partial^2 L}{\partial \sigma^2} < 0. \tag{4.2}$$

Since $L > 0$ and $\log L$ is a non-decreasing function of L, L and $\log L$ attain their extreme values at the same value of $\hat{\sigma}$. So, we can rewrite the above two equations as

$$\frac{1}{L}.\frac{\partial L}{\partial \sigma} = 0 \quad \Rightarrow \frac{\partial logL}{\partial \sigma} = 0,$$

a form which is more convenient from practical point of view.

4.1.1 MLE of the parameter of inverse *Maxwell* distribution

For a sample of size n, consider a set of observations to be $r_1, r_2 \dots . r_n$ with pdf $f(r_1, \sigma), f(r_2, \sigma), \dots \dots f(r_n, \sigma)$. Then according to the section 4.1, the likelihood function is given by,

$$L(\sigma, r) = \left(\sqrt{\frac{2}{\pi}}\sigma^{-3}\right)^n \prod_{i=1}^{n} r_i^{-4} e^{-\sum_{i=1}^{n}\frac{1}{2r_i^2\sigma^2}}$$

Now the log-likelihood function becomes,

$$log(L) = nlog\sqrt{\frac{2}{\pi}} - 3nlog(\sigma) + \sum_{i=1}^{n} log(r_i^{-4}) - \sum_{i=1}^{n}\frac{1}{2r_i^2\sigma^2} \tag{4.3}$$

Differentiating equation (4.3) with respect to parameter σ and then equating to zero. We get,

$$\frac{\partial}{\partial \sigma} log(L) = 0$$

$$\Rightarrow -\frac{3n}{\sigma} + \sum_{i=1}^{n}\frac{2}{2r_i^2\sigma^3} = 0$$

$$\Rightarrow \frac{3n}{\sigma} = \sum_{i=1}^{n}\frac{1}{r_i^2\sigma^3}$$

$$\Rightarrow \sigma^2 = (3n)^{-1}\sum_{i=1}^{n}\frac{1}{r_i^2}.$$

Finally, the MLE of σ is given by,

$$\hat{\sigma} = \sqrt{(3n)^{-1}\sum_{i=1}^{n}\frac{1}{r_i^2}} \ . \tag{4.4}$$

4.2 Method of moment estimation of parameter of inverse *Maxwell* distribution

Let, $R_1, R_2, \ldots\ldots R_n$ be a random sample of size n follows the inverse *Maxwell* distribution. The first sample raw moment is defined as

$$m_1' = \frac{\sum_{i=1}^{n} r_i}{n} = \bar{r}. \tag{4.5}$$

We already derived in Chapter three that the first raw moment of the inverse *Maxwell* distribution is

$$\mu_1' = \sqrt{\frac{2}{\pi}}\sigma^{-1}. \tag{4.6}$$

Now equating the equations (4.5) and (4.6) we get the estimate of the parameter σ as

$$\hat{\sigma} = \sqrt{\frac{2}{\pi}\frac{1}{\bar{r}}}. \tag{4.7}$$

4.3 Fisher information of a distribution

Fisher information tells us how much information about an unknown parameter we can get from a sample. In other words, it tells us how we can measure a parameter, given a certain amount of data. More formally, it measures he expected amount of information given by a random variable for a parameter of interest. If $R_1, R_2, \ldots\ldots R_n$ be a random sample of size n having pdf $f(r_1, \sigma), f(r_2, \sigma), \ldots\ldots f(r_n, \sigma)$. Then the fisher information for σ contained in the random variable R is denoted by $I(\sigma)$ and defined by

$$I(\sigma) = E\left(\frac{\partial \log \mathrm{f}(r, \sigma)}{\partial \sigma}\right)^2. \tag{4.8}$$

4.3.1 Fisher information of the inverse *Maxwell* distribution

Let, $R_1, R_2, \ldots\ldots R_n$ be a random sample of size n follows the inverse *Maxwell* distribution. According to equation (4.8) and (3.17) the fisher information of the inverse *Maxwell* distribution is given by

$$logf(r, \sigma) = \frac{1}{2}\log\left(\frac{2}{\pi}\right) - 3log\sigma - 4logr - \frac{1}{2r^2\sigma^2}. \tag{4.9}$$

The first derivative of equation (4.9) is given by,

$$\frac{\partial logf(r, \sigma)}{\partial \sigma} = \left(\frac{1}{r^2\sigma^3} - \frac{3}{\sigma}\right).$$

Putting these value into equation (4.8), we can write

$$I(\sigma) = E\left(\frac{1}{r^2\sigma^3} - \frac{3}{\sigma}\right)^2$$

$$=E\left(\frac{1}{r^4\sigma^6} - \frac{6}{r^2\sigma^4} + \frac{9}{\sigma^2}\right)$$

$$=\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{1}{r^4\sigma^6} - \frac{6}{r^2\sigma^4} + \frac{9}{\sigma^2}\right)\sigma^{-3}r^{-4}e^{-1/2r^2\sigma^2}dr$$

$$=\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{1}{r^8\sigma^9} - \frac{6}{r^6\sigma^7} + \frac{9}{r^4\sigma^5}\right)e^{-1/2r^2\sigma^2}dr$$

$$=\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{r^8\sigma^9}e^{-\frac{1}{2r^2\sigma^2}}dr - \int_0^\infty \sqrt{\frac{2}{\pi}}\frac{6}{r^6\sigma^7}e^{-\frac{1}{2r^2\sigma^2}}dr + \int_0^\infty \sqrt{\frac{2}{\pi}}\frac{9}{r^4\sigma^5}e^{-1/2r^2\sigma^2}dr$$

Let, $\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{r^8\sigma^9}e^{-\frac{1}{2r^2\sigma^2}}dr = I_1$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{6}{r^6\sigma^7}e^{-\frac{1}{2r^2\sigma^2}}dr = I_2$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{9}{r^4\sigma^5}e^{-\frac{1}{2r^2\sigma^2}}dr = I_3$$

Finally we can write, $I(\sigma) = I_1 - I_2 + I_3$ (4.10)

Where,

$$I_1 = \int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{r^8\sigma^9}e^{-\frac{1}{2r^2\sigma^2}}dr$$

$$=\frac{\sqrt{2}}{\sqrt{\pi}\sigma^9}\int_0^\infty \frac{e^{-\frac{1}{2r^2\sigma^2}}}{r^8}dr$$

Let, $u = \frac{1}{2r^2\sigma^2}$ $\Rightarrow 2r^2\sigma^2 u = 1$ $\Rightarrow r = \sqrt{\frac{1}{2\sigma^2 u}}$ $\Rightarrow \frac{dr}{du} = -\frac{1}{2\sqrt{2}\sigma u^{\frac{3}{2}}}$

Now, we can write,

$$I_1 = \frac{\sqrt{2}}{\sqrt{\pi}\sigma^9}\int_\infty^0 \left(\frac{1}{2\sqrt{u}\sigma}\right)^{-8}\left(-\frac{1}{2\sqrt{2}\sigma u^{\frac{3}{2}}}\right)e^{-u}du$$

$$=-\frac{2^4}{2\sqrt{\pi}}\sigma^{-9-1}\int_\infty^0 \left(\sqrt{u}\sigma\right)^8 u^{-\frac{3}{2}}e^{-u}du$$

$=\frac{2^3}{\sqrt{\pi}}\sigma^{-10+8}\int_0^\infty u^{-\frac{3}{2}+\frac{8}{2}}e^{-u}du$

$=\frac{8}{\sqrt{\pi}}\sigma^{-2}\int_0^\infty u^{\frac{5}{2}}e^{-u}du$

$=\frac{8}{\sqrt{\pi}}\sigma^{-2}\int_0^\infty u^{\frac{7}{2}-1}e^{-u}du$

$=\frac{8}{\sqrt{\pi}}\sigma^{-2}\Gamma\left(\frac{7}{2}\right)$

$=\frac{8}{\sqrt{\pi}}\sigma^{-2}\frac{15\sqrt{\pi}}{8}$

$=15\sigma^{-2}$

Similarly, $I_2 = 18\sigma^{-2}$ and $I_3 = 9\sigma^{-2}$. Now, putting all these values into equation (4.10), we get

$I(\sigma) = I_1 - I_2 + I_3$

$=15\sigma^{-2} - 18\sigma^{-2} + 9\sigma^{-2}$

$=(15 - 18 + 9)\sigma^{-2}$

$=(24 - 18)\sigma^{-2}$

$=6\sigma^{-2}$

So, the Fisher Information of the inverse *Maxwell* distribution is $6\sigma^{-2}$.

4.4 The bounds of the variance estimate of the inverse *Maxwell* parameter

Let, $R_1, R_2, \dots\dots R_n$ be a random sample of size n follows inverse *Maxwell* distribution having pdf $f(r_1,\sigma), f(r_2,\sigma), \dots\dots f(r_n,\sigma)$, where the value of the parameter σ is unknown. By using Fisher information and Cauchy-Schwartz inequality we can determine the lower bound for the variance of an estimator of the parameter σ. Finally, the lower bound of variance of an arbitrary estimator $\hat{\sigma}$ is given by

$$V(\hat{\sigma}) \geq \frac{\left(\frac{\partial}{\partial\sigma}E[\hat{\sigma}]\right)^2}{n \times E\left(\frac{\partial \log f(r,\sigma)}{\partial\sigma}\right)^2}. \qquad (4.11)$$

The inequality (4.11) is called the Cramer-Rao inequality in honor of the Sweden statistician Cramer and Indian statistician C. R. Rao who independently developed this inequality during 1940.

From equation (4.11) we can write,

$$V(\hat{\sigma}) \geq \frac{\left(\frac{\partial}{\partial\sigma} E[\hat{\sigma}]\right)^2}{n \times I(\sigma)}$$

$$\geq \frac{\sigma^2 \left(\frac{\partial}{\partial\sigma} E[\hat{\sigma}]\right)^2}{6n}.$$

This inequality shows that when the fisher information increases then the variance of the estimator decreases.

4.5 Summary of the parameter estimation of inverse *Maxwell* distribution

Estimation is a crucial issue in any statistical analysis. Here we estimate the parameter of the inverse *Maxwell* distribution by MLE and method of moments. We also calculate the fisher information and Cramer-Rao lower bound to realize the information obtained by the sample and the information limit to the variance of estimator.

As it is always anticipated to have application for any method proposed. In the next chapter, the application of inverse *Maxwell* distribution has been discussed in the field named statistical process control.

CHAPTER FIVE

APPLICATION OF THE INVERSE MAXWELL DISTRIBUTION

In this chapter we discuss possible application of the inverse *Maxwell* distribution. The results which have been derived in the previous chapter also have been used in this regard. Some real-life phenomena are presented along with simulation study as well.

5.1 Application of inverse *Maxwell* distribution in process monitoring

From section 3.2 we know that the inverse *Maxwell* distribution is derived from *Maxwell* distribution. The *Maxwell* distribution has immense application in statistical mechanics, chemistry, lifetime modelling as well as in statistical process control [63], [64], [65], [66]. Although the inverse *Maxwell* distribution is also a non-normal skewed distribution like *Maxwell* distribution but according to our knowledge from the literature review the inverse *Maxwell* distribution has not applied in any field. In this chapter we will discuss its application in the process monitoring. Statistical process control (SPC) is a method of quality control which is widely used in industry to monitor the process by using statistical tools. The quality control concept in manufacturing was first conceived by Walter A. Shewhart in 1920. While working in Bell Telephone Laboratories, he conducted research on methods to improve quality and came up with the SPC [67]. By considering the hypothesis whether the process is in statistical control or not, SPC guides the decision about the quality of the process. For this purpose, different tools are used. Its seven major tools are Histogram and Stem and leaf plot, Check sheet, Pareto chart, Cause and effect diagram, Defect concentration diagram, Scatter diagram and control chart [68]. Among all of the above tools, the Shewhart control chart is the most technically sophisticated. In the literature, Shewhart control charts are widely used to monitor variation/shift in process quality characteristics of location as well as dispersion. [69] introduced *improved R chart* (IRC) and *improved S chart* (ISC) for monitoring the process variance. The typical Shewhart control charts are constructed based on the assumption that observed data to monitor the quality characteristics of the process is from normal or near-normal distribution [70].

But in real life there are many situations that do not meet the normality assumption rather that follow skewed distribution. A number of researchers studied quality monitoring in this context. [71], observed the quality characteristics of the compressive strength (kgf/cm2) of concrete under *log-normal* distribution. [72], studied the control chart for skewed distribution including *Weibull, Gamma* and *log-normal* distribution. [73], studied control chart for location parameter of the log-normal process. In lifetime data applications, [74] studied the *Weibull* distribution under type II censoring for monitoring shape parameter. The application of the *Maxwell* distribution in the field of SPC is first discussed by Hossain et. al. [75].

The inverse *Maxwell* distribution has only one parameter which is the scale parameter given in equation (3.17) in Chapter three.

5.1.1 Derivation of the distribution of V (the estimate of scale parameter)

For the construction of the Statistical Process Control (SPC) we need to exhibit a theory. In this section we will introduce the theory that will help us to construct the SPC for inverse *Maxwell* distribution.

Theorem 5.1 The PDF of the inverse *Maxwell* distribution given in (3.17) and by considering the transformation $P = 1/2R^2\sigma^2$ then PDF follows $gamma\left(\frac{3}{2}, 1\right)$.

Proof. From the PDF of the inverse *Maxwell* distribution that is given in equation (3.17), let, $p = \frac{1}{2r^2\sigma^2}$, and by simplification we get, $r = \sqrt{\frac{1}{2\sigma^2 p}}$. Then the Jacobian transformation becomes, $J = \frac{dr}{dt} = \frac{1}{2\sqrt{2}\sigma p^{\frac{3}{2}}}$. Now, the distribution of p can be written as,

$$f_P(p) = f_R(t) \times |J|$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\sigma^{-3}\left(\sqrt{\frac{1}{2\sigma^2 p}}\right)^{-4} e^{-p} \times \frac{1}{2\sqrt{2}\sigma p^{\frac{3}{2}}}$$

$$= \frac{1}{\sqrt{\pi}}\sigma^{-3}\left(\frac{1}{2\sigma^2 p}\right)^{-2} e^{-p}2^{-1}\sigma^{-1}p^{-\frac{3}{2}}$$

$$= \frac{1}{\sqrt{\pi}}\sigma^{-4}(2\sigma^2 p)^2 e^{-p}2^{-1}p^{-\frac{3}{2}}$$

$$= \frac{1}{\sqrt{\pi}}\sigma^{-4}4\sigma^4 p^2 e^{-p}2^{-1}p^{-\frac{3}{2}}$$

$$= \frac{2}{\sqrt{\pi}}p^{2-\frac{3}{2}}e^{-p}$$

$$= \frac{2}{\sqrt{\pi}}p^{\frac{1}{2}}e^{-p}$$

Finally,

$$f_P(p) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} p^{\frac{3}{2}-1} e^{-p}, \tag{5.1}$$

which is the PDF of *gamma* distribution with parameter $3/2$ and 1.

Hence, the theorem proved. □

Symbolically, we can write, $P \sim gamma\left(\frac{3}{2}, 1\right)$, or $\frac{1}{2R^2\sigma^2} \sim gamma\left(\frac{3}{2}, 1\right)$.

Theorem 5.2 The PDF of the inverse *Maxwell* distribution given in (3.17) and by considering the transformation $P = 1/2R^2\sigma^2$ and taking n samples from the PDF, if we let the random variable $V = \hat{\sigma}^2$ and the random variable $Q = 3nV/2\sigma^2$ then PDF of Q follows $gamma\left(\frac{3n}{2}, 1\right)$.

Proof. According to the additive property of the *gamma* distribution we know that if $R_1 \sim gamma(\alpha_1, \beta)$ and $R_2 \sim gamma(\alpha_2, \beta)$ then $R_1 + R_2 \sim gamma\ (\alpha_1 + \alpha_2, \beta)$. Now considering the general case we can write if $R_i \sim gamma(\alpha_i, \beta)$ then for n samples, $\sum_{i=1}^{n} r_i \sim gamma\ (\sum_{i=1}^{n} \alpha_i, \beta)$. For our case, $p = \frac{1}{2r^2\sigma^2} \sim gamma(3/2,1)$ then $\sum_{i=1}^{n} p_i \sim gamma\left(\frac{3n}{2}, 1\right)$. Now remember the MLE of scale parameter derived in section (4.1.1) and let $\hat{\sigma}^2 = V$. So, we can write, $V = (3n)^{-1} \sum_{i=1}^{n} \frac{1}{r_i^2}$

$$\Rightarrow 3nV = \sum_{i=1}^{n} \frac{1}{r_i^2}$$

$$\Rightarrow \frac{3nV}{2\sigma^2} = \sum_{i=1}^{n} \frac{1}{2r_i^2\sigma^2}$$

$$\Rightarrow \frac{3nV}{2\sigma^2} = \sum_{i=1}^{n} p_i.$$

From the statement of the theorem we can write $Q = 3nV/2\sigma^2$ and according to the additive property of the *gamma* distribution we can consider Q as a pivotal quantity and the PDF of the Q follows *gamma* distribution with parameter $3n/2$ and 1.

Hence the theorem proved. □

Here, Q is a *gamma* random variable so its' mean is $E[Q] = E\left[\frac{3nV}{2\sigma^2}\right] = \frac{3n}{2}$.

So, we can write $E\left[\frac{V}{\sigma^2}\right] = 1 \implies E[V] = \sigma^2$. Therefore,

$$E[V] = \sigma^2 \tag{5.2}$$

By equation (5.2) we can say that V is an unbiased estimator of σ^2. Following the above procedure first we find the variance of Q and from this we will calculate the variance of V.

The variance of Q is $Var(Q) = Var\left[\frac{3nV}{2\sigma^2}\right] = \frac{3n}{2}$. This can be written as $\left[\frac{3n}{2\sigma^4}\right] Var(V) = 1$

Hence,

$$Var(V) = \frac{2\sigma^4}{3n}. \tag{5.3}$$

From [66] we know that the CDF of R follows *gamma* distribution with parameter α and β, i.e. $F(r) = \frac{1}{\Gamma\alpha}\gamma\left(\alpha, \frac{r}{\beta}\right)$. In our case the CDF of the pivotal quantity, Q follows *gamma* distribution with parameter $3n/2$ and 1, i.e. $F(q) = \frac{1}{\Gamma\left(\frac{3n}{2}\right)}\gamma\left(\frac{3n}{2}, q\right)$, where $\gamma(.,.)$ denote the incomplete *gamma* function. So, the $\alpha\ th$ quantile can be derived as, $F(q) = \alpha$, $\Rightarrow Q = F^{-1}(\alpha)$, $\Rightarrow \frac{3nV}{2\sigma^2} = F^{-1}(\alpha)$, $\Rightarrow V = \left(\frac{2\sigma^2}{3n}\right)F^{-1}(\alpha)$. So, we can write

$$V_\alpha = \left(\frac{2\sigma^2}{3n}\right)F^{-1}(\alpha). \tag{5.4}$$

5.1.2 Process Monitoring

To detect the variation of the scale parameter σ^2 in process monitoring we adopt the estimate of V. Here, we use two techniques to monitor the process first one based on probability limits and the next on based on $L -$ sigma limits. In case of probability limits the $lower\ probability\ limits$ and $upper\ probability\ limits$ are denoted by LPL and UPL. Similarly, in case of the second case the $lower\ control\ limit$, the $center\ line$ and the $upper\ control\ limit$ are denoted by LCL, CL and UCL.Now have to test whether any shift presents in the process or not. To do so we need to set the following hypothesis.

$H_0: \sigma^2 = \sigma_0^2$; or, $\delta = 1$, i.e. No shift occurs in the process.

$H_1: \sigma^2 = \sigma_1^2 = \delta\sigma_0^2$; or, $\delta \neq 1$, i.e. Shift occurs in the process.

Here, δ denotes the shift in the process. Recall equation (5.4) to drive the probability limits for V which can be written as

$LPL: V_{\frac{\alpha}{2}} = \left[\frac{2\sigma^2}{3n}\right]F^{-1}\left(\frac{\alpha}{2}\right)$, and

$$UPL: V_{1-\frac{\alpha}{2}} = \left[\frac{2\sigma^2}{3n}\right] F^{-1}\left(1-\frac{\alpha}{2}\right).$$

These can also be presented as

$$LPL: V_{\frac{\alpha}{2}} = L_1\sigma^2, \text{ and } \quad UPL: V_{1-\frac{\alpha}{2}} = L_2\sigma^2.$$

where $L_1 = \left[\frac{2}{3n}\right] F^{-1}\left(\frac{\alpha}{2}\right)$ and $L_2 = \left[\frac{2}{3n}\right] F^{-1}\left(1-\frac{\alpha}{2}\right)$. These coefficients are calculated from the *gamma* distribution and multiplied by some constants. For different sample size n and different false alarm rate α these coefficients vary. Table 5.1 shows an illustration of variations in quantiles.

Table 5.1: *Gamma* quantiles for different n and α.

Sample size (n)	**False alarm rate α**					
	0.005		0.0027		0.002	
	L_1	L_2	L_1	L_2	L_1	L_2
1	0.0150	4.7734	0.0099	5.0294	0.0081	5.4221
2	0.0878	3.3749	0.0706	3.6228	0.0635	3.7430
3	0.1611	2.8292	0.1380	3.0101	0.1280	3.0975
4	0.2218	2.5265	0.1959	2.6722	0.1845	2.7425
5	0.2713	2.3300	0.2442	2.4536	0.2322	2.5132
6	0.3124	2.1901	0.2848	2.2987	0.2725	2.3507
7	0.3471	2.0845	0.3194	2.1819	0.3070	2.2284
8	0.3768	2.0014	0.3493	2.0901	0.3369	2.1324
9	0.4027	1.9338	0.3753	2.0156	0.3630	2.0546
10	0.4254	1.8777	0.3984	1.9538	0.3862	1.9901

In real-life scenario two situations appear in process monitoring of the inverse *Maxwell* scale parameter: i) σ^2 known and ii) σ^2 unknown. In case of known σ^2, the limits can be written as

$$LPL = L_1\sigma_0^2; \quad CL = \sigma_0^2 \text{ and } \quad UPL = L_2\sigma_0^2. \tag{5.5}$$

But in case of unknown σ^2 , we have to estimate V as and use this estimate to monitor the next process

$$LPL = L_1\bar{V}; \quad CL = \bar{V} \text{ and } \quad UPL = L_2\bar{V}. \tag{5.6}$$

where, $\bar{V}$ is the arithmetic mean of estimated V obtained in each of the sample over time.

To inspect the ability of the control chart we may use different measures one such measure is power of a chart. The traditional definition of power of a test is the probability of rejecting null hypothesis (H_0) when alternative hypothesis (H_1) is true. That is, $Power = Pr(reject\ H_0|H_1)$. In our case, the null hypothesis is rejected when the position of the plotting statistic (V) is either below the lower probability limit or above the upper probability limit. Hence,

$$Power = \Pr(V_\alpha < LPL_0|H_1) + \Pr(V_\alpha > UPL_0|H_1)$$

which insinuate that,

$$Power = \Pr\left(V_\alpha < \left(\frac{2\sigma_0^2}{3n}\right)F^{-1}\left(\frac{\alpha}{2}\right)\middle|\delta \neq 1\right) + \Pr\left(V_\alpha > \left(\frac{2\sigma_0^2}{3n}\right)F^{-1}\left(1-\frac{\alpha}{2}\right)\middle|\delta \neq 1\right),$$

where, $F^{-1}(\alpha) = \frac{3nV}{2\sigma^2}$. Finally, power of the chart is derived as follows

$$\begin{aligned} Power = 1 + &\frac{1}{\Gamma\left(\frac{3n}{2}\right)}\gamma\left(\frac{3n}{2}, \delta^{-1}F^{-1}\left(\frac{\alpha}{2}\right)\right) \\ &- \frac{1}{\Gamma\left(\frac{3n}{2}\right)}\gamma\left(\frac{3n}{2}, \delta^{-1}F^{-1}\left(1-\frac{\alpha}{2}\right)\right) \end{aligned} \tag{5.7}$$

In equation (5.7) if $\delta = 1$, when there is no shift in the process, then the power is equivalent to false alarm rate α. Figure 5.1 below shows the power for different process shift. When the sample size increases then the power is also increasing. So from the figure we can conclude that our proposed chart is also work for larger shift and the larger the shifts the higher the power.

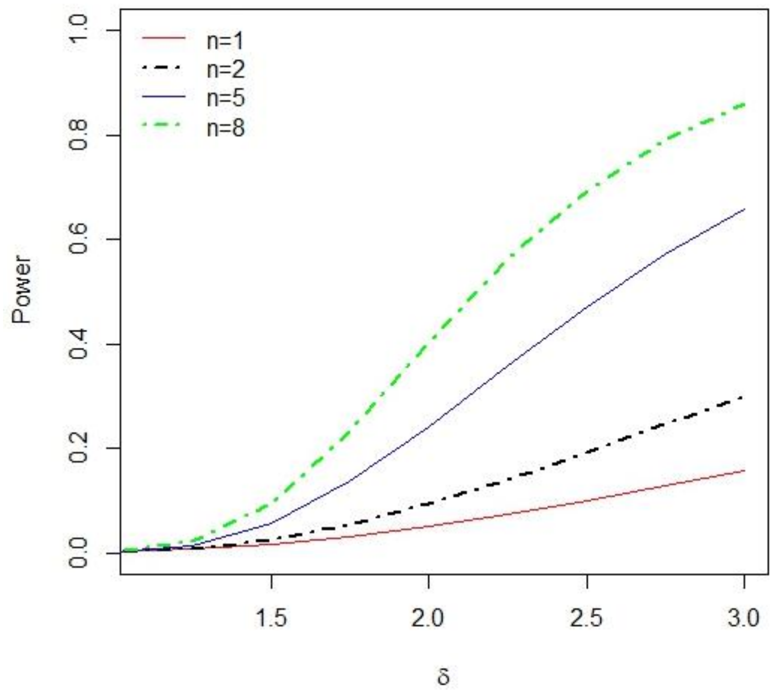

Figure 5.1: Power curves of control charts for different n at $\alpha = 0.027$.

We can also use ARL (Average Run Length) to inspect the performance of the control chart. The traditional definition of ARL is,

$$ARL = \frac{1}{1-\beta} = \frac{1}{power}. \tag{5.8}$$

Besides these two-performance measure we have to discuss the run length of this control chart in details. Here we have used various run length behavior such as standard deviation of run length (SDRL), median of run length (MDRL), different percentiles of RL and RL curves in general. By adopting the Monte Carlo Simulations, we have estimated all this value. Here the ARL is estimated by equation (5.8) and Monte Carlo Simulations. The comparison between these two values are given in Table 5.2 (ARL_p and ARL_sdenote the values of average run length based on equation (5.8) and simulation respectively).

Table 5.2: ARL of RL of V −chart for different n at $\alpha = 0.0027$.

	Sample size (n)							
	1		3		6		10	
δ	ARL_p	ARL_s	ARL_p	ARL_s	ARL_p	ARL_s	ARL_p	ARL_s
1.00	370.37	370.14	370.37	370.55	370.37	369.29	370.37	369.85
1.25	146.87	146.15	95.09	96.07	60.58	60.40	39.01	39.37
1.50	62.31	62.37	28.80	28.76	14.55	14.47	8.06	8.16
1.75	32.47	32.32	12.71	12.73	5.99	6.04	3.32	3.31
2.00	19.81	19.95	7.17	7.37	3.39	3.36	1.99	2.02
2.25	13.51	13.44	4.74	4.71	2.32	2.33	1.49	1.48
2.50	9.97	9.96	3.48	3.47	1.80	1.82	1.25	1.26
2.75	7.79	7.80	2.76	2.77	1.52	1.52	1.13	1.14
3.00	6.36	6.39	2.30	2.31	1.35	1.35	1.07	1.08
5.00	2.68	2.70	1.26	1.26	1.03	1.03	1.00	1.00

From the above table we can see that the values of ARL_s that are calculated by the simulation are approximately equal to the values of ARL_p that are calculated from the traditional definition which reviles that the simulation is correct. From the table we can also state that as the shift and sample number increase then the average run length decreases. The values of $SDRL$ and $MDRL$ of RL that are generated by the Monte-Carlo simulation are given the following table.

From Table 5.3 we can say that the values of $SDRL$ and $MDRL$ are decreasing as the shift and sample number increases which is similar with the ARL. The various percentiles of RL are presented in the following table which is also estimated by the simulation.

Table 5.3: SDRL and MDRL of RL of V-chart for different n at $\alpha = 0.0027$.

	Sample size (n)							
	1		3		6		10	
δ	$SDRL$	$MDRL$	$SDRL$	$MDRL$	$SDRL$	$MDRL$	$SDRL$	$MDRL$
1.00	371.53	258	369.10	258	368.13	257	369.25	254
1.25	145.07	101	93.96	67	60.34	42	39.16	27
1.50	61.33	44	28.25	20	14.01	10	7.70	6
1.75	31.69	23	12.13	9	5.52	4	2.69	2
2.00	19.29	14	6.79	5	2.83	2	1.45	1
2.25	13.01	9	4.12	3	1.77	2	0.84	1
2.50	9.41	7	2.94	3	1.23	1	0.58	1
2.75	7.23	6	2.17	2	0.90	1	0.39	1
3	5.99	5	1.72	2	0.68	1	0.29	1
5	2.15	2	0.57	1	0.17	1	0.02	1

Based on the mean and variance of V given in equations (5.2) and (5.3), L-sigma limits of V are presented as follows:

$$LCL = E[V] - L \times SD(V) = \left[1 - L\sqrt{\frac{2}{3n}}\right]\sigma^2 = W_1\sigma^2, \tag{5.9}$$

$$CL = E[V] = \sigma^2, \text{ and} \tag{5.10}$$

$$UCL = E[V] + L \times SD(V) = \left[1 + L\sqrt{\frac{2}{3n}}\right]\sigma^2 = W_2\sigma^2, \tag{5.11}$$

where, $W_1 = \left[1 - L\sqrt{\frac{2}{3n}}\right]$ and $W_2 = \left[1 + L\sqrt{\frac{2}{3n}}\right]$.

The factor L is obtained using the *gamma* quantile in such a way that the expected false alarm rate (α) has been obtained.

In this case two situations again arise: i) σ^2 known and ii) σ^2 unknown. In case of known σ^2, the limits can be written as

$$LCL = W_1\sigma_0^2; \qquad CL = \sigma_0^2 \quad \text{and} \qquad UCL = W_2\sigma_0^2.$$

But in case of unknown σ^2, we have to estimate V and use it as follows:

$$LCL = W_1\bar{V}; \qquad CL = \bar{V} \quad \text{and} \qquad UCL = W_2\bar{V}.$$

Table 5.4 Percentiles of RL of V-chart for different n at α=0.0027

	Sample size (n)																			
	1					3					6					10				
δ	P_{10}	P_{25}	P_{50}	P_{75}	P_{95}	P_{10}	P_{25}	P_{50}	P_{75}	P_{95}	P_{10}	P_{25}	P_{50}	P_{75}	P_{95}	P_{10}	P_{25}	P_{50}	P_{75}	P_{95}
1.00	41	106	258	514	1089	41	107	258	513	855	39	105	257	517	841	39	106	254	514	853
1.25	17	42	101	202	433	11	28	67	134	221	7	18	42	84	138	4	12	27	54	91
1.50	7	19	44	86	184	4	9	20	39	65	2	4	10	20	33	1	3	6	11	18
1.75	4	10	23	44	95	2	4	9	17	28	1	2	4	8	13	1	1	2	4	7
2.00	2	6	14	28	58	1	2	5	10	16	1	1	2	4	7	1	1	1	3	4
2.25	2	4	9	18	39	1	2	3	6	10	1	1	2	3	5	1	1	1	2	3
2.50	2	3	7	14	29	1	1	3	5	7	1	1	1	2	3	1	1	1	1	2
2.75	1	3	6	11	23	1	1	2	4	6	1	1	1	2	3	1	1	1	1	2
3.00	1	2	5	8	18	1	1	2	3	5	1	1	1	2	2	1	1	1	1	1
5	1	1	2	3	7	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1

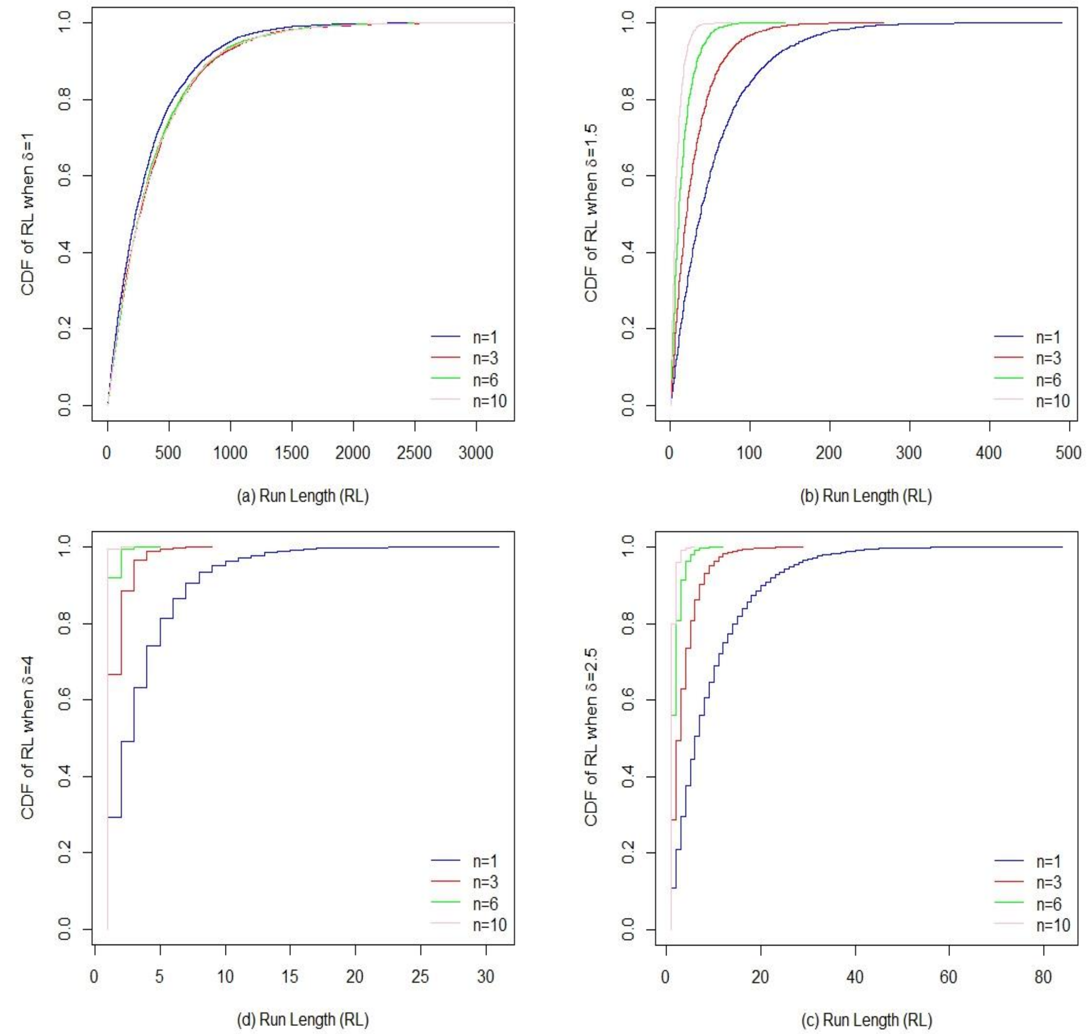

Figure 5.2: RL curves for different shifts δ and sample size n.

Figure 5.2 indicates that the RL distribution of V chart is positively skewed and its CDF is monotonic increasing.

Table 5.4: L coefficients.

Sample size (n)	False Alarm Rate (α)		
	0.005	0.0027	0.002
	L	L	L
2	0.3379	0.2706	0.2432
3	0.8675	0.7394	0.6851
4	1.5369	1.3517	1.2715
5	2.3004	2.0623	1.9579
6	3.1324	2.8450	2.7180
7	4.0168	3.6833	3.5351
8	4.9431	4.5661	4.3979
9	5.9038	5.4856	5.2984
10	6.8934	6.4360	6.2307

Table 5.5 below represents L coefficients which is used to calculate the factors W_1 and W_2. The L coefficients are chosen from the quantile relationship given in equation (5.5) such that we can get the fixed false alarm rate.

Table 5.5: Factor for constructing Control Chart for inverse *Maxwell* parameter.

Sample size (n)	False Alarm Rate (α)					
	0.005		0.0027		0.002	
	W_1	W_2	W_1	W_2	W_1	W_2
2	0.8049	1.1951	0.8438	1.1562	0.8596	1.1404
3	0.5911	1.4089	0.6514	1.3486	0.6770	1.3229
4	0.3726	1.6274	0.4482	1.5518	0.4809	1.5191
5	0.1600	1.8400	0.2470	1.7531	0.2851	1.7149
6	0.0000	2.0441	0.0518	1.9483	0.0939	1.9060
7	0.0000	2.2396	0.0000	2.1367	0.0000	2.0909
8	0.0000	2.4270	0.0000	2.3181	0.0000	2.2696
9	0.0000	2.6068	0.0000	2.4930	0.0000	2.4420
10	0.0000	2.7799	0.0000	2.6618	0.0000	2.6087

In addition, Table 5.6 displays the corresponding $W_1$and W_2 factors. we can use these factors directly to reduce the complexity of calculation involving the L factors. These W_1 and W_2 factors would be very helpful to construct control chart for inverse *Maxwell* parameter easily.

5.1.3 Simulation Study

Simulation is a liable practice, we can exploit it to inspect the performance of the control chart. In simulation we generate a sample of random data in such a way that imitates a real problem and recapitulates that sample in the similar approach. It is one of the most extensively used quantitative technique because it is so malleable and can yield so many effective results. In this section we discuss the performance of V chart in terms of simulated data. We modify algorithm that is proposed by [75] for generating data from inverse *Maxwell* distribution and use the results from the previous sections. The algorithm involves the following steps:

Step 1: Select a number for a random sample of size n.

Step 2: Propagate a random variable T of size n that follows *Gamma* distribution with shape parameter $3/2$ and scale parameter $2\sigma_0^2$.

Step 3: By taking the square root of T we will get *Maxwell* random variable X of size n.

Step 4: Calculate the inverse *Maxwell* random variable R of size n by setting $R = 1/X$.

Step 5: Estimate the plotting statistics V.

Step 6: Repeat step 1 to step 5 until the expected amount of sample parcels are obtained.

Step 7: Develop the control limits as proposed in section 5.1.2.

Step 8: Plot all the values of V statistic in contrast to the control limits.

Now, according to the above steps by using statistical software R 3.4.2 we simulate data from inverse *Maxwell* distribution and cross checked the generated data by K-S test to ensure that the data set follows inverse *Maxwell* distribution. In our case we consider $\sigma_0 = 0.0094$ and generate 144 sample observations then by adopting K-S test, we failed to reject the null hypothesis of the data follow inverse *Maxwell* distribution. Then we divide the data set into 24 samples each of size 6.

Now from Table 5.1 for fixed false alarm rate $\alpha = 0.0027$ and $n = 6$, $L_1 = 0.2848$ and $L_2 = 2.2987$, and known $\sigma_0 = 0.0094$, the probability limits are

LPL =2.50E-05, CL =8.78E-05 and UPL =2.02E-04.

The visual representation of a V control chart based on these limits is given in Figure 5.3.

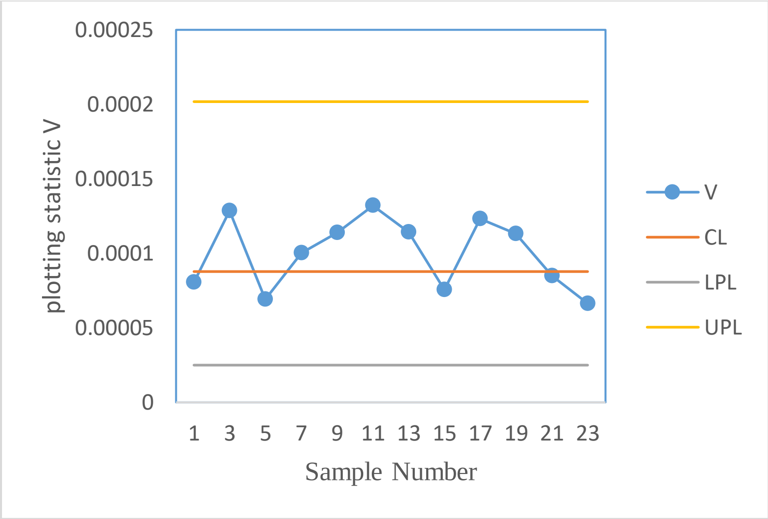

Figure 5.3: V-chart for inverse *Maxwell* parameter using probability limits (In control situation).

In case of L sigma limit for the same false alarm rate, sample size and σ_0, the values of W_1 and W_2 are 0.0518 and 1.9483 respectively. So, the L sigma limits are

LCL =4.55E-06, CL =8.78E-05 and UCL =1.71E-04.

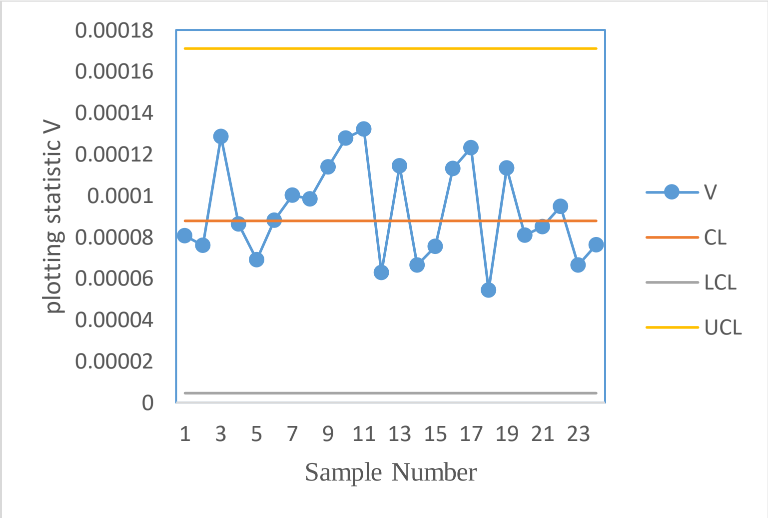

Figure 5.4: V-chart for inverse *Maxwell* parameter using L sigma limits (In control situation).

Figure 5.4 represents the corresponding control chart for V based on these L sigma limits. In Figure 5.3 and 5.4, we didn't see any kind of criterion that is given in [76] for out of control. So, we can state that the process is in control for both cases.

To inspect the ability of the chart for detecting the process when it is out of control, we consider a shift in the process scale parameter. We suspect that the process scale parameter σ^2 has been shifted to a new level such that $\delta = 2.75$ (after the 18th sample).

For fixed false alarm rate $\alpha = 0.0027$ and $n = 6$ and known $\sigma_0 = 0.0094$, the control charts are developed for out of control scenario using both probability and L sigma limits. Figure 5.5 and 5.6 describe these results.

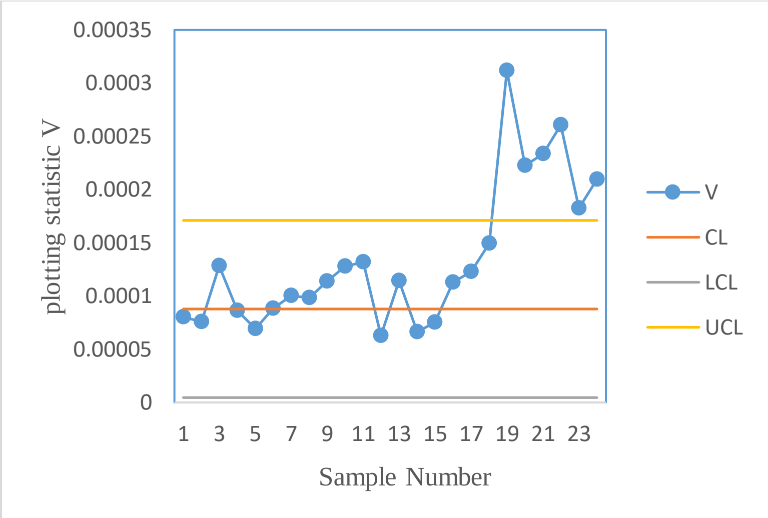

Figure 5. 5: V-chart for inverse *Maxwell* parameter using L sigma limits (Out of control situation).

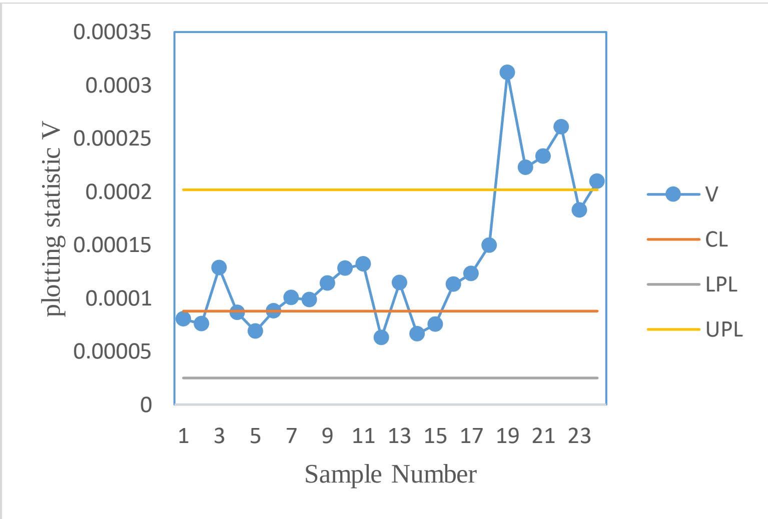

Figure 5.6: V-chart for inverse *Maxwell* parameter using probability limits (Out of control situation).

In this case we see that there some points that are out of the control limits. So, on the basis of [76] and Figure 5.5 and 5.6, we can state that the process is not in control.

5.1.4 Real Life Example

First of all, for any data set, we have to check the validity of the normality assumption before the beginning of analysis which helps us to know that whether the data comes from normal population or not. If the data does not follow the normal distribution, then we have to inspect which non-normal distribution it follows. We have different non-normal distributions but in case of our study we will consider the inverse *Maxwell* distribution. Now we have to check that whether the given data follows inverse *Maxwell* distribution or not. From [77], we know that various kinds of test are available to inspect the goodness of fit of data. Here we perform the Kolmogorov-Smirnov test to identify the validity of the inverse *Maxwell* assumption.

From the literature review we got different kinds of data such as lifetime data, time series data and genetic data that follows inverse *Maxwell* distribution. To expound the pertinence of these control charts with real data, we use the data which is found in [78]. To inspect whether the data set follows inverse *Maxwell* distribution or not according to the K-S test the null hypothesis is that the data set comes from inverse *Maxwell* distribution and the alternative hypothesis is that data set doesn't come from inverse *Maxwell* distribution. For our data set the K-S test statistic value is 0.105 where at 1% level of significance the critical value is 0.144. Clearly the calculated value is less than the tabulated value of K-S test statistic so we cannot reject the null hypothesis, i.e. the data follows inverse *Maxwell* distribution. We adopt this data to assemble control chart for inverse *Maxwell* parameter. After truncation the data consists of car brake pad's lifetime with 72 observations. We considered the data in the form of subgroups, each of size 6, that results into 12 subgroups. Then we use the subgroups further in the establishment of the control charts.

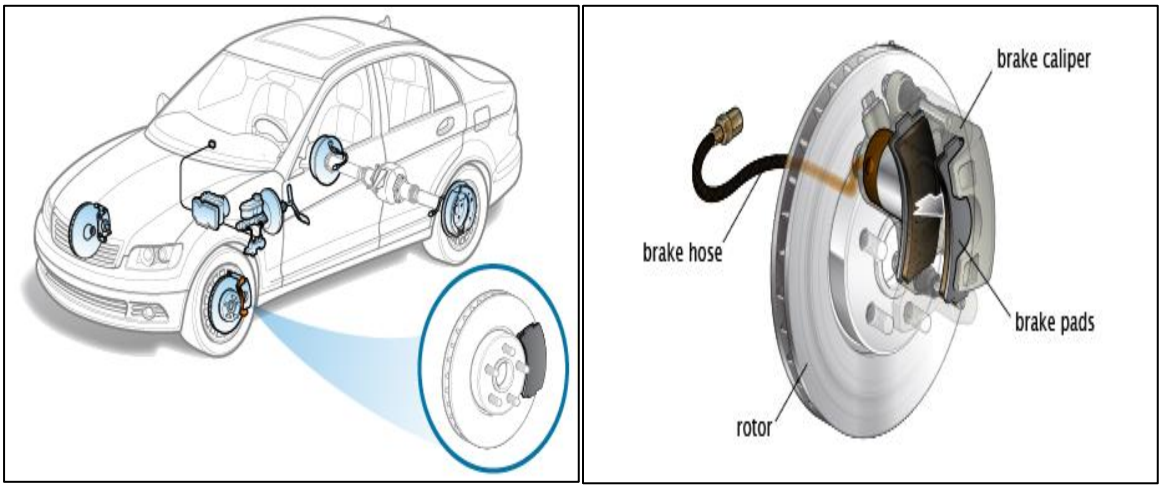

Figure 5.7: Car's brake pads.

The lifetime in kilometers are as follows:

Table 5.6: Lifetime of car's brake pad (in km)

Sample number	Observations					
	1st	2nd	3rd	4th	5th	6th
1	49.20	69.60	74.80	79.40	89.50	105.60
2	42.40	44.80	51.50	47.40	60.30	52.00
3	73.80	72.20	77.60	61.40	103.60	77.20
4	44.10	81.60	83.00	54.00	88.00	74.70
5	46.70	107.60	63.70	72.80	82.60	68.90
6	61.90	45.20	68.80	44.20	42.40	165.50
7	49.80	79.50	124.60	68.80	50.80	68.90
8	46.30	54.00	89.10	65.50	95.70	55.00
9	56.20	83.00	65.00	86.70	78.10	46.80
10	54.00	49.20	53.90	67.60	42.40	110.00
11	54.90	43.40	59.30	100.60	92.60	92.50
12	50.50	143.60	65.10	43.80	83.60	124.50

Now we fabricate the control chat for V using probability limits and L-sigma limits from an inverse Maxwell distribution. For our given data, the estimated value of the MLE is $\bar{V}$ =8.77915E-05. From the Table 5.1 for fixed false alarm rate $\alpha = 0.0027$ and n=6, we have L_1= 0.2848 and L_2= 2.2987. Hence, the probability limits are

LPL= 2.5003E-05; *CL*= 8.77915E-05 and *UPL*= 0.000202.

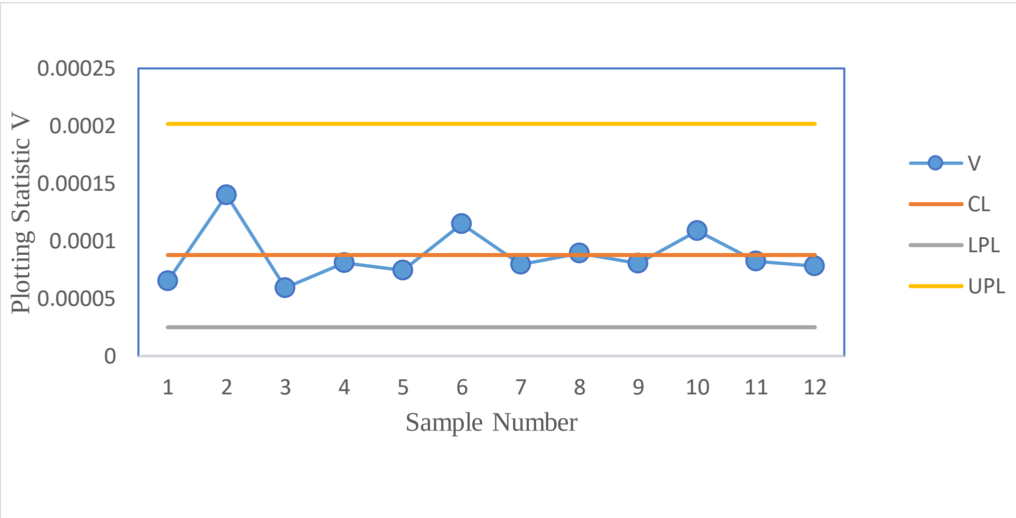

Figure 5.8: V-Chart for inverse *Maxwell* parameter using probability limits.

From [76], we know that there are several criterions for detecting the lack of control such as a point outside the control limits, a run of seven points or more, a run of points beyond some secondary limits, the points are too close to the central line, presence of trends and presence of cycles. None of the above criteria are present in Figure 5.8. so, we can say that the process is in control.

Again, from Table 5.4 for fixed fals e alarm rate $\alpha = 0.0027$ and n=6, we have $W_1 = 0.0518$ and $W_2 = 1.9483$. Hence, the probability limits are

LCL= 4.5476E-06; CL= 8.77915E-05 and UCL= 0.000171.

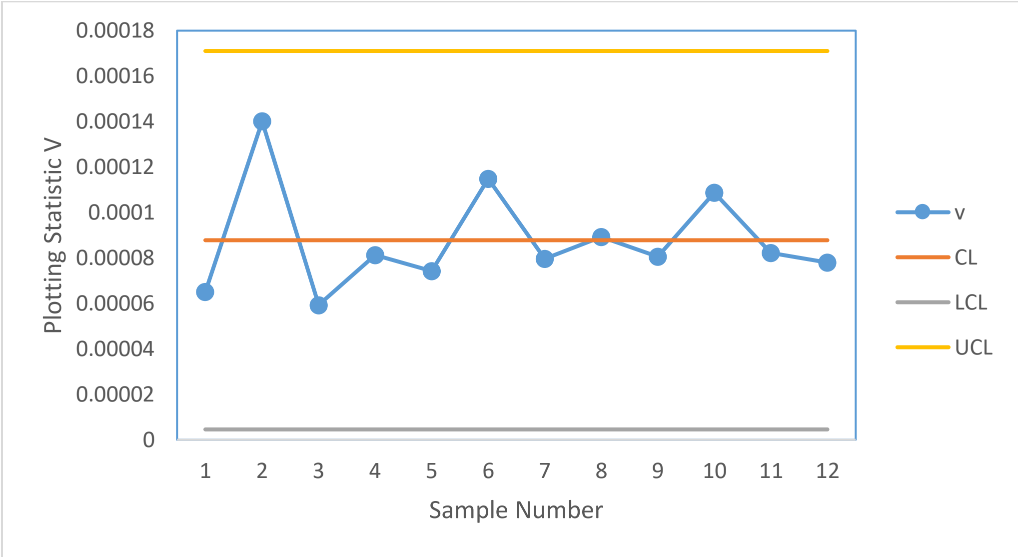

Figure 5.9: V-Chart for inverse *Maxwell* parameter using L-sigma limits.

Similarly following the criteria that is described above all of them are also absent in Figure 5.9. So also for in this case we can state that there is no shift in the process.

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CHAPTER SIX

CONCLUSIONS AND RECOMMENDATIONS

In this project we develop the inverse *Maxwell* distribution from *Maxwell* distribution and drive its various distributional properties such as skewness, kurtosis, mode, MGF, CF, entropy and fisher information along with the reliability and hazard curves. We also estimate the scale parameter of inverse *Maxwell* distribution by using Maximum Likelihood method and Method of moments. A theory is also introduced to monitor the process of the scale inverse *Maxwell* parameter. We use various measures to inspect the suitability of the V control chat under inverse *Maxwell* parameter. Finally, simulation study and real-life example have been demonstrated.

In the literature, the inverse *Maxwell* distribution is derived by the general from of the *Maxwell* distribution. In addition, different curves of the inverse *Maxwell* distribution are not well described. In the current project we develop inverse *Maxwell* distribution by using the standard from and derive all of its important curves correctly. As an application, in the field of process monitoring we studied inverse *Maxwell* distribution.

In Chapter three, we developed different properties of inverse distribution in general. We implemented these properties for inverse *Maxwell* distribution. The MGF and CF of the inverse *Maxwell* distribution are derived. First four raw and central moments of the inverse *Maxwell* distribution have been studied as well. We considered some special cases of the proposed distribution which may be implemented in some real-life scenarios.

In Chapter four we estimate the inverse *Maxwell* parameter by maximum likelihood method and method of moments. We also derive the fisher information function and the lower bound for the variance of the proposed distribution.

In Chapter five (the application chapter), we applied inverse *Maxwell* distribution in statistical process control. In this field of research many distributions have been used before, but inverse *Maxwell* distribution has not been applied yet. In addition to monitoring inverse *Maxwell* process parameter using typical Shewhart method.

6.1 Recommendations

In this section we would like to recommend the use of inverse *Maxwell* distribution for further future research. Some recommendations are given underneath.

1. We studied inverse *Maxwell* distribution with some specific properties. We haven't examined some other important properties such as limiting distribution etc.
2. We have looked at only MLE and method of moments. We may also consider in the future other methods of estimation such as the Bayesian estimation.
3. In the application part, we discussed control chart named Shewhart control chart. The EWMA and CUSMU structures of control chart can also be implemented for the inverse *Maxwell* distribution.
4. Can we apply this inverse *Maxwell* distribution in other field off research?

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