

Table of Contents

Homework 2.....	1
ADSI Problem 1.4: Filter decomposition.....	1
1) Decompose a FIR filter with one zero outside the unit circle.....	1
2) Show that and have the same magnitude response.....	7
3) Decompose a filter that has two zeros outside the unit circle.....	8
ADSI Problem 1.5 Filter decomposition.....	13
1) Show that a FIR filter with a difference equation is not minimum-phase.....	13
2) Find the difference equation for the corresponding minimum-phase FIR filter.....	15
ADSI Problem 1.8: Designing all-pass filters.....	17
*) Use iirgrpdelay in MATLAB to design a 4th order all-pass filter. You can set edges=[0 1].....	18
*) What happens as you increase the order of the filter?.....	18
*) What order of the filter is necessary to bring the maximum group delay error below 0.5 samples?.....	18
*) Plot the impulse response of the filter. Does the length of the impulse response match your expectations.....	18

Homework 2

ADSI Problem 1.4: Filter decomposition

1) Decompose a FIR filter with one zero outside the unit circle

Let the system function for a FIR filter be given by

$$H(z) = 1 - 3z^{-1} + \frac{5}{2}z^{-2} - z^{-3}$$

1. Decompose the system function into a product of a minimum-phase filter and an all-pass filter,
 $H(z) = H_{\min}(z)H_{\text{ap}}(z)$.

Any system function can be decomposed into a product of a minimum-phase filter and an all-pass filter using the following formula:

$$H(z) = H_{\min}(z)H_{\text{ap}}(z)$$

The minimum phase filter can be computed as follows:

$$H_{\min}(z) = \prod_i -\frac{1}{a_i^*} H_i(z) (1 - a_i z^{-1})$$

where $H_i(z)$ corresponds to the part of the transfer function where the i 'th zero is inside the unit circle and $a_i = \frac{1}{z_i^*}$ and z_i is the zero outside the unit circle.

The allpass filter can be calculated:

$$H_{\text{ap}}(z) = \prod_i \frac{z^{-1} - a_i^*}{1 - a_i z^{-1}}$$

The filter decomposition algorithm has following steps:

1. convert transfer function into pole-zero representation in order to find the zeros that are outside the unit circle
2. compute a and its conjugate a^*
3. find $H_1(z)$ which corresponds to the part of the transfer function where zeros are inside the unit circle
4. plugin the numbers for the formula for the minimum-phase filter
5. plugin the numbers for the formula for the allpass filter
6. put everything together:

Step 1: convert transfer function into pole-zero representation in order to find the zeros that are outside the unit circle

```
syms z;
rts = roots([1, -3, 5/2, -1])
```

```
rts = 3x1 complex
    2.0000 + 0.0000i
    0.5000 + 0.5000i
    0.5000 - 0.5000i
```

```
H_outside = 1; % Represents part of H where zeros are outside the unit circle
H_inside = 1; % Represents part of H where zeros are inside the unit circle
zeros_outside = [];
for i = 1: numel(rts)
    root = rts(i);
    if abs(root) > 1
        H_outside = H_outside * (1 - root*z^-1);
        zeros_outside = [zeros_outside; root];
    else
        H_inside = H_inside * (1 - root*z^-1);
    end
end

z0 = 0;
if numel(zeros_outside) == 1
    z0 = zeros_outside(1);
else
    disp('Something is wrong! The transfer function has more than one zero outside the unit circle')
end

H_outside = expand(H_outside);
H_inside = expand(H_inside);

% Sanity check
H = expand(H_inside * H_outside)
```

H =

$$\frac{5}{2z^2} - \frac{3}{z} - \frac{1}{z^3} + 1$$

The zero-pole representation of the transfer function is:

H_outside

H_outside =

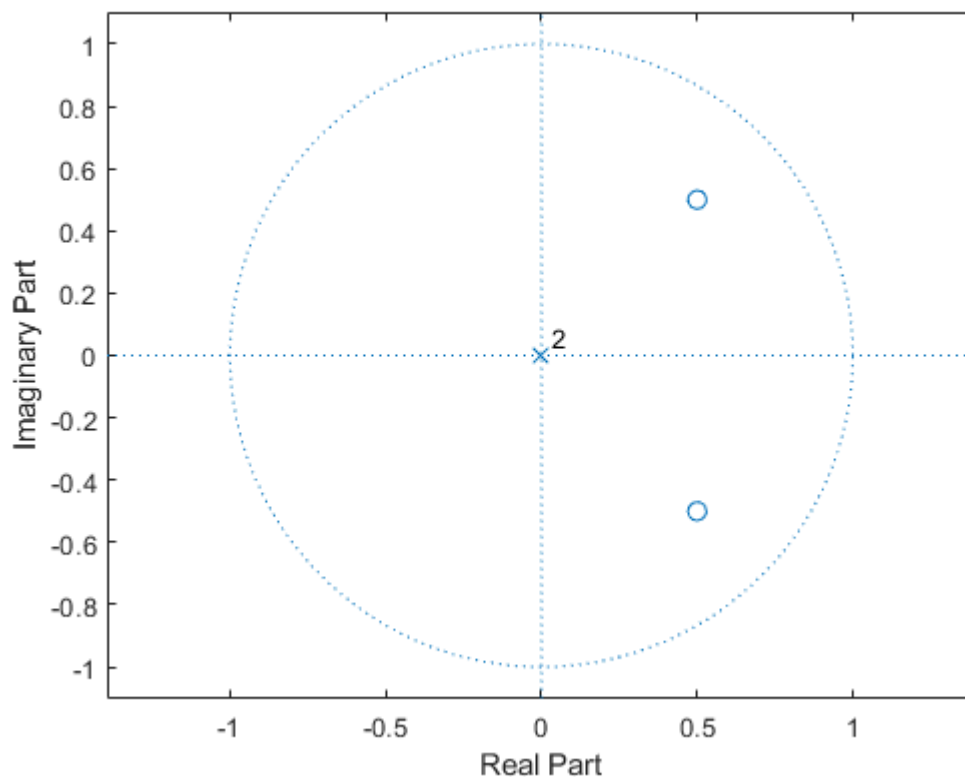
$$1 - \frac{2}{z}$$

H_inside

H_inside =

$$\frac{1}{2z^2} - \frac{1}{z} + 1$$

zplane([1, -1, 1/2]) % Should all have zeros inside the unit circle



$$H(z) = (1 - 2z^{-1})\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)$$

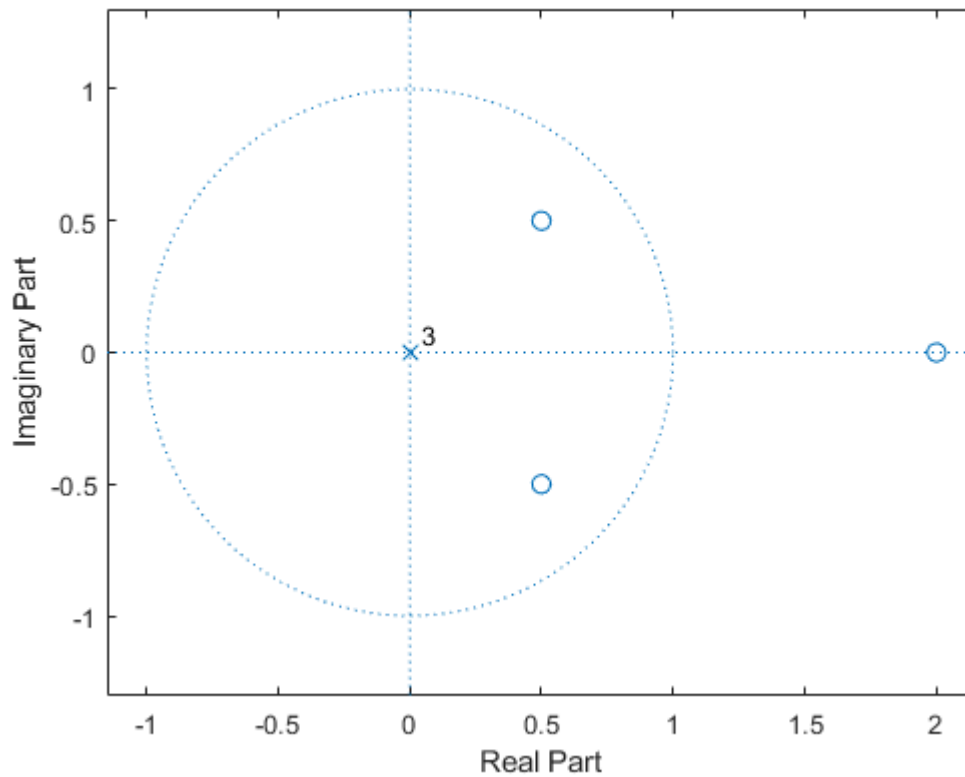
% Sanity check

```
conv([1, -2], [1, -1, 1/2])
```

```
ans = 1×4  
1.0000 -3.0000 2.5000 -1.0000
```

To find the zero z_0 that is outside the unit circle, we can plot zplane:

```
zplane([1, -3, 5/2, -1]);
```



In this exercise, the zero outside the unit circle is $z_0 = 2$.

```
z0
```

```
z0 = 2
```

Step 2: compute a and its conjugate a^*

```
a = 1/z0
```

```
a = 0.5000
```

```
a_conj = conj(a)
```

```
a_conj = 0.5000
```

$$a = \frac{1}{z_0} = \frac{1}{2}$$

$$a^* = \left(\frac{1}{z_0}\right)^* = \left(\frac{1}{2}\right)^* = \frac{1}{2}$$

Step 3: find $H_1(z)$ which corresponds to the part of the transfer function where zeros are inside the unit circle

H_inside

H_inside =

$$\frac{1}{2z^2} - \frac{1}{z} + 1$$

The zero-pole representation of the transfer function is:

$$H(z) = (1 - 2z^{-1})\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)$$

therefore $H_1(z)$ is

$$H_1(z) = 1 - z^{-1} + \frac{1}{2}z^{-2}$$

Step 4: plugin the numbers for the formula for minimum-phase filter

$$H_{\min}(z) = -\frac{1}{a^*}H_1(z)(1 - az^{-1})$$

$$H_{\min}(z) = -\frac{1}{\left(\frac{1}{2}\right)}\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)(1 - 0.5z^{-1})$$

$$H_{\min}(z) = -2\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)(1 - 0.5z^{-1})$$

-2 * conv([1, -1, 1/2], [1, -0.5])

ans = 1x4

-2.0000 3.0000 -2.0000 0.5000

$$H_{\min}(z) = -2 + 3z^{-1} - 2z^{-2} + \frac{1}{2}z^{-3}$$

H_min = expand((-1/a_conj) * H_inside * (1 - a*z^-1))

H_min =

$$\frac{3}{z} - \frac{2}{z^2} + \frac{1}{2z^3} - 2$$

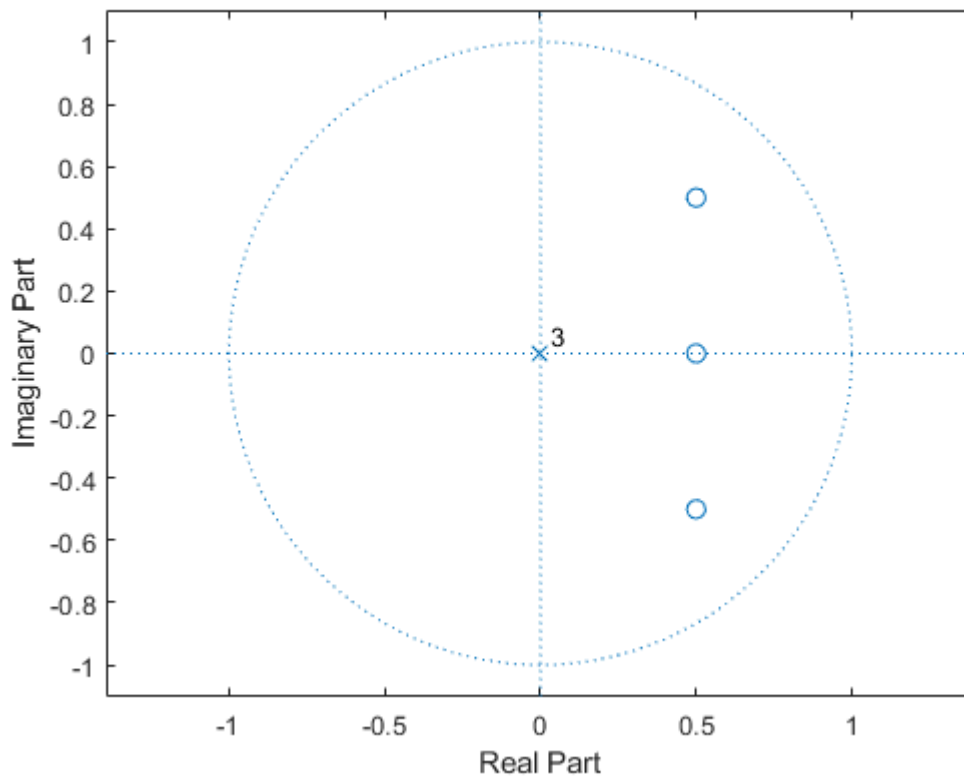
```
H_min_b = [-2, 3, -2, 1/2];
H_min_rts = roots(H_min_b)
```

```
H_min_rts = 3x1 complex
    0.5000 + 0.5000i
    0.5000 - 0.5000i
    0.5000 + 0.0000i
```

```
isminphase(H_min_b)
```

```
ans = logical
     1
```

```
zplane(H_min_b)
```



Step 5: compute the allpass filter :

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$H_{ap}(z) = \frac{z^{-1} - 0.5}{1 - 0.5z^{-1}}$$

$$H_{ap} = (z^{-1} - a_{conj}) / (1 - a*z^{-1})$$

$$H_{ap} =$$

$$-\frac{\frac{1}{z} - \frac{1}{2}}{\frac{1}{2z} - 1}$$

Step 6: put everything together:

$$H_{min}(z) = -2 + 3z^{-1} - 2z^{-2} + \frac{1}{2}z^{-3}$$

$$H_{ap}(z) = \frac{z^{-1} - 0.5}{1 - 0.5z^{-1}}$$

$$H(z) = H_{min}(z)H_{ap}(z)$$

$$H(z) = \left(-2 + 3z^{-1} - 2z^{-2} + \frac{1}{2}z^{-3}\right) \left(\frac{z^{-1} - 0.5}{1 - 0.5z^{-1}}\right)$$

2) Show that $H(z)$ and $H_{min}(z)$ have the same magnitude response

2. Demonstrate that $H(z)$ and $H_{min}(z)$ have the same amplitude response.

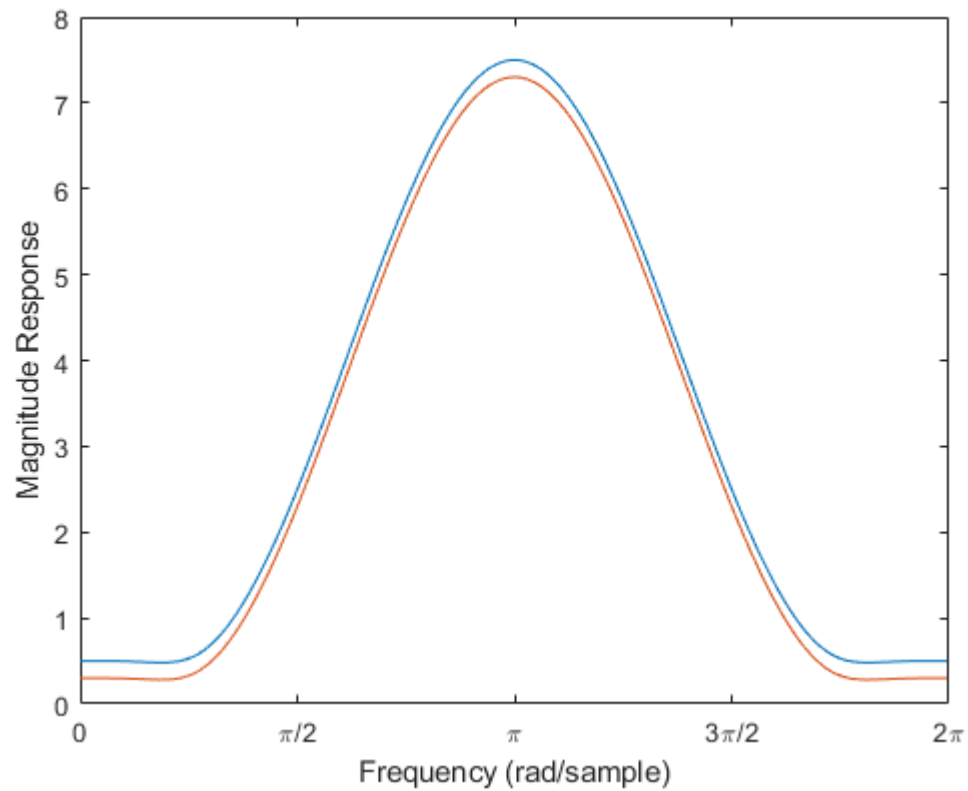
To show that $H(z)$ and $H_{min}(z)$ have the same magnitude response, we can plot them. If the two graphs are on top of each other then they have the same magnitude response.

```
% Coefficients for H(z)
b = [1, -3, 5/2, -1];
a = 1;
[H, w] = freqz(b,a,'whole');

% Coefficients for H_min(z)
H_min_b = [-2, 3, -2, 1/2];
H_min_a = 1;
[H_min_z, H_min_w] = freqz(H_min_b, H_min_a, 'whole');
% plot(H_min_w, log10(abs(H_min_z)));

% The offset ensures that the two graphs are not on top of each other
offset = -0.2;
plot(w, abs(H), H_min_w, abs(H_min_z) + offset);
set(gca,'XTick',0:pi/2:2*pi)
set(gca,'XTickLabel',{'0','\pi/2','\pi','3\pi/2','2\pi'})
xlabel('Frequency (rad/sample)')
ylabel('Magnitude Response')
```

```
xlim([0, 2*pi]);
```



3) Decompose a filter that has two zeros outside the unit circle

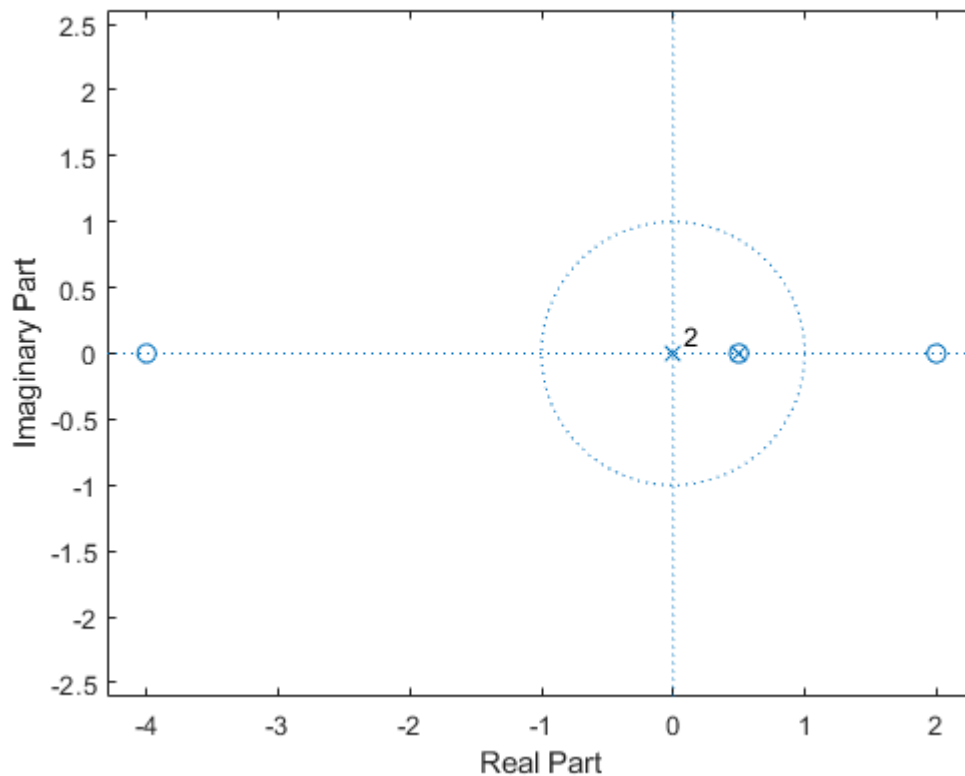
3. Repeat for the following filter

$$H(z) = \frac{1 + \frac{3}{2}z^{-1} - 9z^{-2} + 4z^{-3}}{1 - \frac{1}{2}z^{-1}}$$

```
b = [1, 3/2, -9, 4];  
a = [1, -1/2];
```

Plot the zeros and the poles:

```
zplane(b, a);
```

From the plot, we can see that there is a zero and a pole at 0.5. They will cancel each other.

```
% Find the zeros of H
rts = roots(b)
```

```
rts = 3x1
    -4.0000
     2.0000
     0.5000
```

Now, the transfer function can be rewritten as:

$$H(z) = \frac{(1 - (-4)z^{-1})(1 - 2z^{-1})\left(1 - \frac{1}{2}z^{-1}\right)}{1 - \frac{1}{2}z^{-1}} = (1 - (-4)z^{-1})(1 - 2z^{-1}) = (1 - (-4)z^{-1})(1 - 2z^{-1})$$

Let us find the coefficients of the rewritten transfer function:

```
H_new = expand((1+4*z^-1) * (1-2*z^-1))
```

```
H_new =
    2
  z - 8
    z^2 + 1
```

```
% Automatically extract the coefficients
b_new = coeffs(expand(H_new * z^2), 'all')
```

```
b_new = (1 2 -8)
```

```
% Find the zeros of H_new  
rts = roots(b_new)
```

```
rts =
```

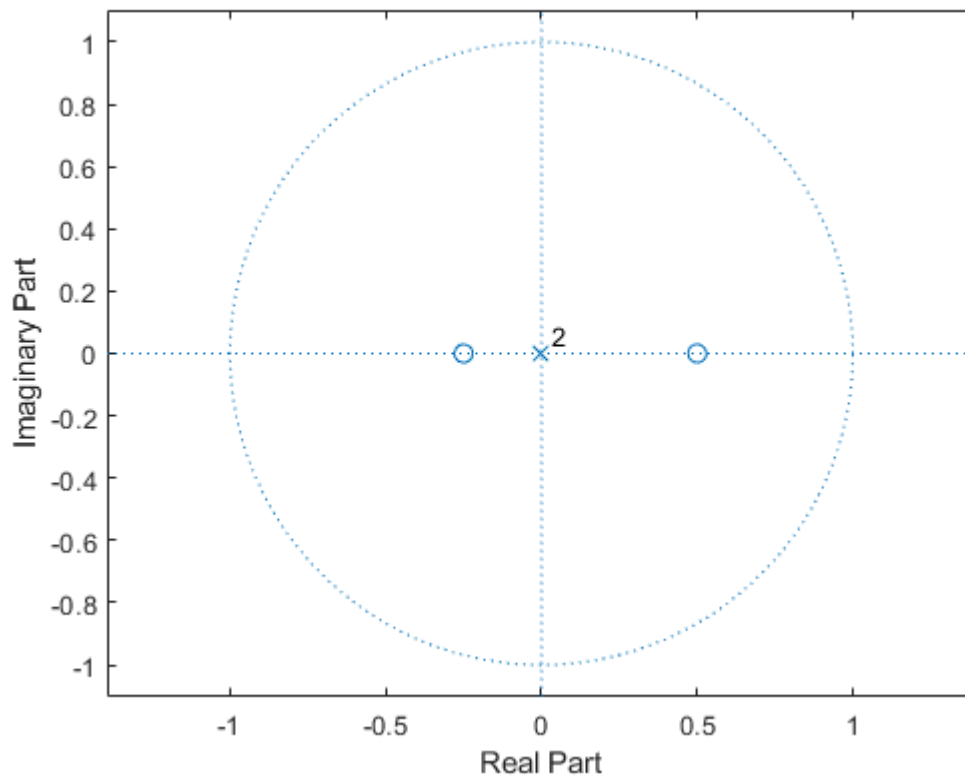
$$\begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

```
syms z;  
H_outside = 1; % Represents part of H where zeros are outside the unit circle  
H_inside = 1; % Represents part of H where zeros are inside the unit circle  
zeros_outside = [];  
for i = 1:numel(rts)  
    root = rts(i);  
    if abs(root) > 1  
        H_outside = H_outside * (1 - root*z^-1);  
        zeros_outside = [zeros_outside; root];  
    else  
        H_inside = H_inside * (1 - root*z^-1);  
    end  
end  
  
% Sanity check  
H = expand(H_inside * H_outside)
```

```
H =
```

$$\frac{2}{z} - \frac{8}{z^2} + 1$$

```
% Finally compute the H_min and H_ap  
H_min = 1;  
H_ap = 1;  
for i = 1:numel(zeros_outside)  
    a = 1/zeros_outside(i);  
    a_conj = conj(a);  
    H_min = H_min * (-1/a_conj) * H_inside * (1 - a*z^-1);  
    H_ap = H_ap * ((z^-1 - a_conj) / (1 - a*z^-1));  
end  
  
% Check that H_min is in fact minimum-phase  
% Check that all zeros are within the unit circle!  
H_min_b = coeffs(expand(H_min * z^10), 'all');  
zplane(H_min_b)
```



```
% Display H_min
expand(H_min)
```

ans =

$$\frac{2}{z} + \frac{1}{z^2} - 8$$

We can write $H_{\min}(z) = -8 + 2z^{-1} + z^{-2}$

```
% Display H_ap in order to rewrite it
H_ap
```

H_ap =

$$-\frac{\left(\frac{1}{z} - \frac{1}{2}\right) \left(\frac{1}{z} + \frac{1}{4}\right)}{\left(\frac{1}{2z} - 1\right) \left(\frac{1}{4z} + 1\right)}$$

Expanding the numerator, we get:

```
H_ap_num = expand(-1*(1/z - 1/2)*(1/z + 1/4))
```

H_ap_num =

$$\frac{1}{4z} - \frac{1}{z^2} + \frac{1}{8}$$

```
H_ap_den= expand((1/(2*z) - 1)*(1/(4*z) + 1))
```

H_ap_den =

$$\frac{1}{4z} + \frac{1}{8z^2} - 1$$

We can multiply both the numerator and denominator with 8, to get nice numbers:

```
(H_ap_num * 8) / (H_ap_den * 8)
```

ans =

$$\frac{\frac{2}{z} - \frac{8}{z^2} + 1}{\frac{2}{z} + \frac{1}{z^2} - 8}$$

Now, we can write the transfer function for the allpass filter:

$$H_{\text{ap}}(z) = \frac{1 + 2z^{-1} - 8z^{-2}}{-8 + 2z^{-1} + z^{-2}}$$

So we have:

$$H(z) = H_{\text{min}}(z)H_{\text{ap}}(z)$$

$$\text{where } H_{\text{min}}(z) = -8 + 2z^{-1} + z^{-2} \text{ and } H_{\text{ap}}(z) = \frac{1 + 2z^{-1} - 8z^{-2}}{-8 + 2z^{-1} + z^{-2}}$$

To show that $H(z)$ and $H_{\text{min}}(z)$ have the same magnitude response, we can plot them. If the two graphs are on top of each other then they have the same magnitude response.

$$H(z) = \frac{1 + \frac{3}{2}z^{-1} - 9z^{-2} + 4z^{-3}}{1 - \frac{1}{2}z^{-1}}$$

$$H_{\text{min}}(z) = -8 + 2z^{-1} + z^{-2}$$

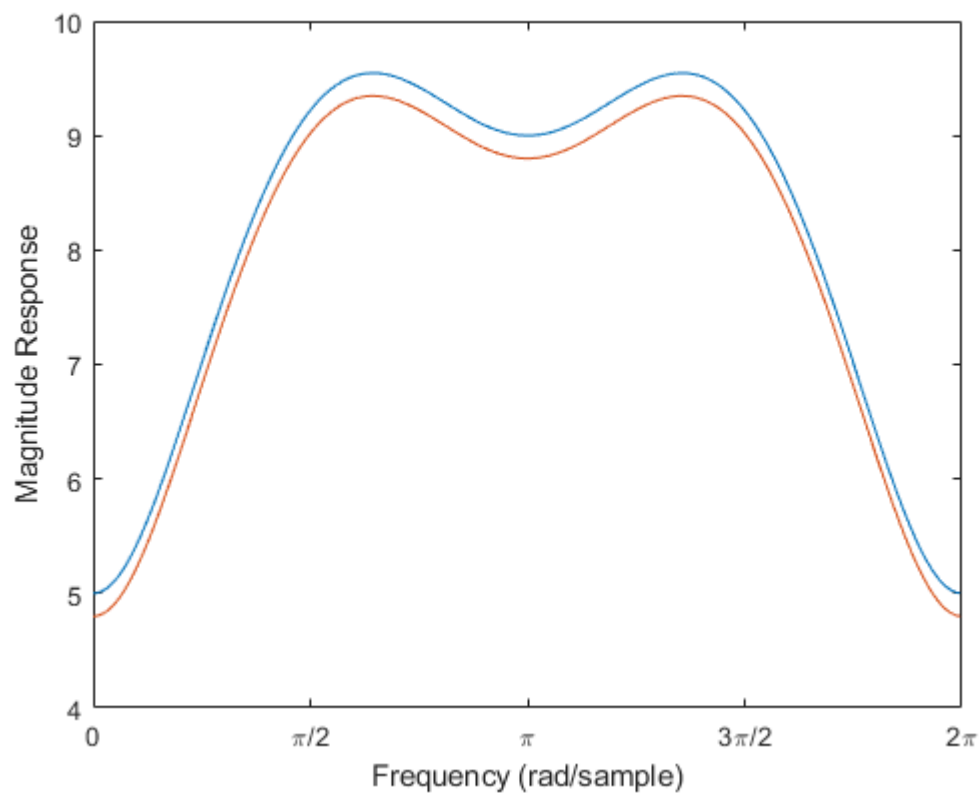
```
% Coefficients for H(z)
b = [1, 3/2, -9, 4];
a = [1, -1/2];
[H, w] = freqz(b,a,'whole');
```

```

% Coefficients for H_min(z)
H_min_b = [-8, 2, 1];
H_min_a = 1;
[H_min_z, H_min_w] = freqz(H_min_b, H_min_a, 'whole');
% plot(H_min_w, log10(abs(H_min_z)));

% The offset ensures that the two graphs are not on top of each other
offset = -0.2;
plot(w, abs(H), H_min_w, abs(H_min_z) + offset);
set(gca, 'XTick', 0:pi/2:2*pi)
set(gca, 'XTickLabel', {'0', '\pi/2', '\pi', '3\pi/2', '2\pi'})
xlabel('Frequency (rad/sample)')
ylabel('Magnitude Response')
xlim([0, 2*pi]);

```



ADSI Problem 1.5 Filter decomposition

1) Show that a FIR filter with a difference equation is not minimum-phase

Consider a FIR filter with the following difference equation

$$y(n) = x(n) + 2x(n-1) + 2x(n-2)$$

1. Show that the FIR filter is not minimum-phase.

To determine whether the filter is a minimum-phase or not, we need get the transfer function of this FIR filter $H(z)$ which is defined as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1} + b_2z^{-2} \dots + b_Mz^{-M}}{1 + a_1z^{-1} + a_2z^{-2} \dots + a_Nz^{-N}}$$

This means that we must take the z-transform of difference equation:

$$Y(z) = X(z) + 2X(z)z^{-1} + 2X(z)z^{-2}$$

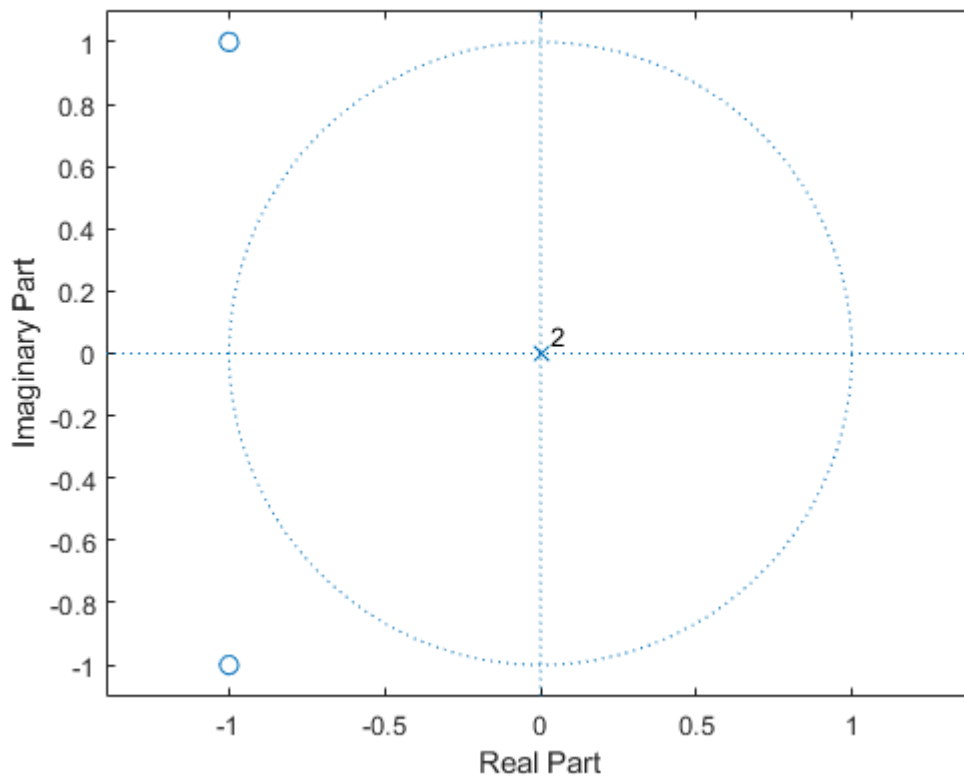
$$Y(z) = X(z)(1 + 2z^{-1} + 2z^{-2})$$

$$\frac{Y(z)}{X(z)} = 1 + 2z^{-1} + 2z^{-2}$$

$$H(z) = 1 + 2z^{-1} + 2z^{-2}$$

Once we have the transfer function, we can determine if the filter is a minimum-phase by looking at the zplane plot:

```
zplane([1, 2, 2])
```



Since all the zeros are outside the unit circle, clearly the given FIR filter is not minimum-phase. It is a **maximum-phase**.

2) Find the difference equation for the corresponding minimum-phase FIR filter

We found that the transfer function for the difference equation is:

$$H(z) = 1 + 2z^{-1} + 2z^{-2}$$

```
% Find the zeros of H
rts = roots([1, 2, 2])
```

```
rts = 2x1 complex
-1.0000 + 1.0000i
-1.0000 - 1.0000i
```

```
syms z;
H_outside = 1; % Represents part of H where zeros are outside the unit circle
H_inside = 1; % Represents part of H where zeros are inside the unit circle
zeros_outside = [];
for i = 1:numel(rts)
    root = rts(i);
    if abs(root) > 1
        H_outside = H_outside * (1 - root*z^-1);
        zeros_outside = [zeros_outside; root];
    else
        H_inside = H_inside * (1 - root*z^-1);
    end
end
% Sanity check
H = expand(H_inside * H_outside)
```

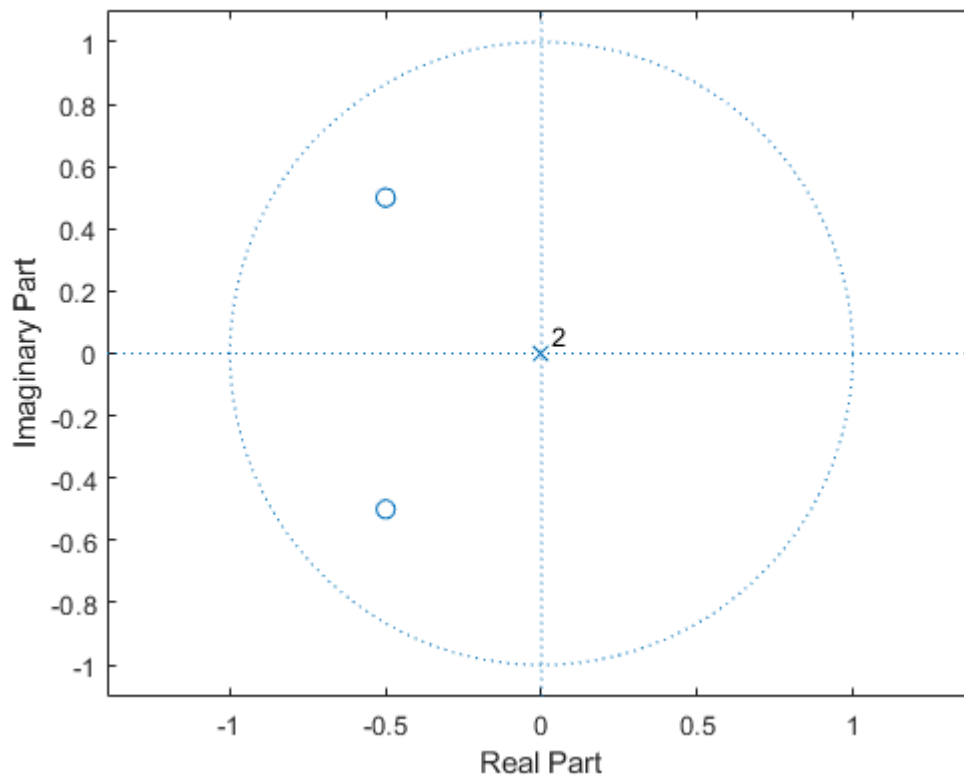
H =

$$\frac{2}{z} + \frac{2}{z^2} + 1$$

```
% Finally compute the H_min and H_ap
H_min = 1;
H_ap = 1;
for i = 1:numel(zeros_outside)
    a = 1/zeros_outside(i);
    a_conj = conj(a);
    H_min = H_min * (-1/a_conj) * H_inside * (1 - a*z^-1);
    H_ap = H_ap * ((z^-1 - a_conj) / (1 - a*z^-1));
end
```

```
% Check that H_min is in fact minimum-phase
% Check that all zeros are within the unit circle!
```

```
H_min_b = coeffs(expand(H_min * z^10), 'all');
zplane(H_min_b)
```



From the zero-pole plot we can see that we have a minimum-phase filter.

Finally, we can write out the transfer function of the minimum-phase part:

```
expand(H_min)
```

ans =

$$\frac{2}{z} + \frac{1}{z^2} + 2$$

$$H_{\min}(z) = 2 + 2z^{-1} + z^{-2}$$

H_ap

H_ap =

$$\frac{\left(\frac{1}{z} + \frac{1}{2} - \frac{1}{2}i\right) \left(\frac{1}{z} + \frac{1}{2} + \frac{1}{2}i\right)}{\left(1 + \frac{\frac{1}{2} - \frac{1}{2}i}{z}\right) \left(1 + \frac{\frac{1}{2} + \frac{1}{2}i}{z}\right)}$$

Let us simply this expression:

$$H_{ap_num} = \text{expand}((1/z + 1/2 - 1/2i) * (1/z + 1/2 + 1/2i))$$

$$H_{ap_num} =$$

$$\frac{1}{z} + \frac{1}{z^2} + \frac{1}{2}$$

$$H_{ap_den} = \text{expand}((1 + (1/2 - 1/2i)*z^{-1}) * (1 + (1/2 + 1/2i)*z^{-1}))$$

$$H_{ap_den} =$$

$$\frac{1}{z} + \frac{1}{2z^2} + 1$$

$$H_{ap}(z) = \frac{\frac{1}{2} + z^{-1} + z^{-2}}{1 + z^{-1} + \frac{1}{2}z^{-2}}$$

Hence:

$$H(z) = H_{min}(z)H_{ap}(z)$$

$$H(z) = (2 + 2z^{-1} + z^{-2}) \left(\frac{\frac{1}{2} + z^{-1} + z^{-2}}{1 + z^{-1} + \frac{1}{2}z^{-2}} \right)$$

We can write $H_{min}(z) = 2 + 2z^{-1} + z^{-2}$ as difference equation:

$$H_{min}(z) = \frac{Y(z)}{X(z)}$$

$$Y(z) = X(z)(2 + 2z^{-1} + z^{-2})$$

Now, we take the inverse z-transform:

$$\text{iztrans}(2 + 2*z^{-1} + z^{-2})$$

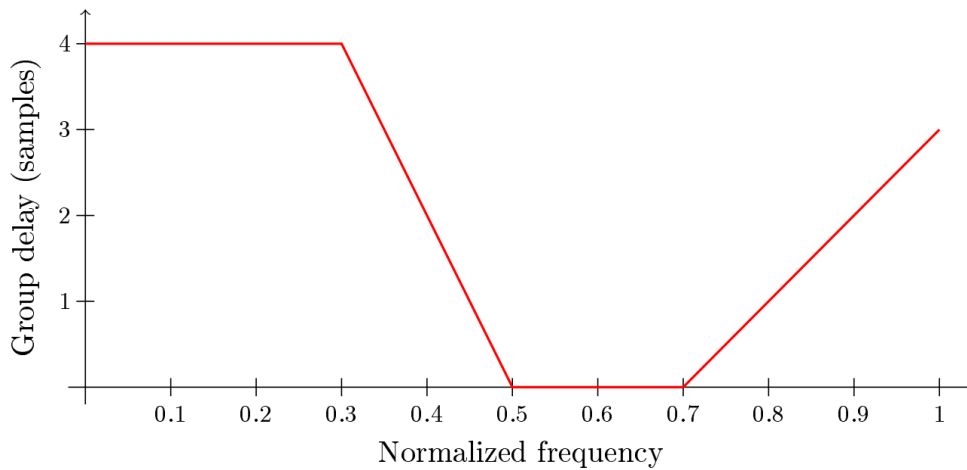
$$\text{ans} = 2\delta_{n-1,0} + \delta_{n-2,0} + 2\delta_{n,0}$$

We get:

$$y[n] = 2x[n] + 2x[n-1] + x[n-2]$$

ADSI Problem 1.8: Designing all-pass filters

Assume it is desired to create an all-pass filter with group delay versus frequency as shown in plot below. Due to the sharp edges in the group delay such a filter can hardly be designed but an approximation is feasible.



***) Use `iirgrpdelay` in MATLAB to design a 4th order all-pass filter. You can set `edges=[0 1]`**

A 4th order allpass filter has the following transfer function

$$H_{ap}(z) = \prod_i \frac{z^{-1} - p_i^*}{1 - p_i z^{-1}} = \frac{(z^{-1} - p_1^*)(z^{-1} - p_2^*)(z^{-1} - p_3^*)(z^{-1} - p_4^*)}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})(1 - p_3 z^{-1})(1 - p_4 z^{-1})}$$

where p_i corresponds to the i 'th pole.

The magnitude response of an allpass filter is constant for all frequencies, however its phase response may differ. For this reason, allpass filters are used for **delay equalisation**.

***) What happens as you increase the order of the filter?**

***) What order of the filter is necessary to bring the maximum group delay error below 0.5 samples?**

***) Plot the impulse response of the filter. Does the length of the impulse response match your expectations.**

