Week 5: Random Processes

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Introduction

Why work with random processes? Because we need a mathematical description of the random nature of the process that generated the observed values of a given signal.

A random process is a collection signals with “some” probability assigned to each.

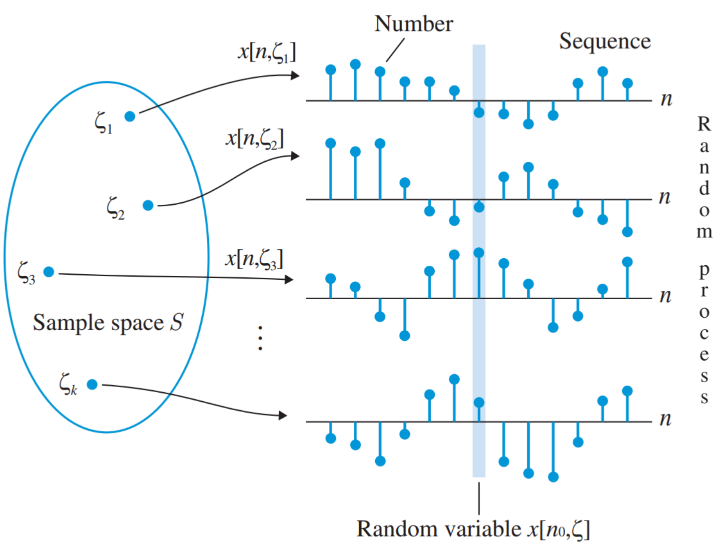


Figure 13.9 The concept of a random (stochastic) process as a mapping from the sample space of a random experiment to an ensemble of sequences.

A random process can be thought of a bin that contains multiple signals of infinite length

Every time, we perform a random experiment, we randomly pick one *realization* of the process.

Stationary Random Processes

A stationary random process is a process which characteristics that do not change over time i.e., for different values of n.

This implies:

a) The mean and the variance of a signal $x[n]$ randomly picked from the stationary process does not depend on time n but is constant:

$$E(x[n]) = m_x \quad \text{and} \quad \text{var}(x[n]) = \sigma_x^2, \text{ for all } n. \tag{13.72}$$

b) For two signals $x[n]$ and $x[m]$ randomly picked from the stationary process, the autocorrelations and autocovariance only depend on the lag ℓ and not time:

$$c_{xx}[n,m] \triangleq \text{cov}(x[n], x[m]) = c_{xx}[\ell]. \text{ for all } m, n \tag{13.73}$$

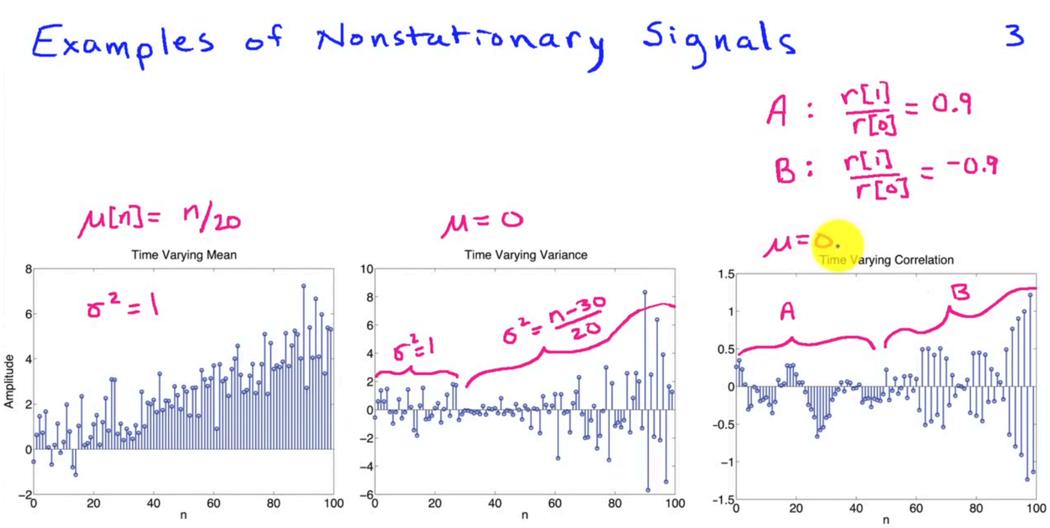
Wide-sense stationary

Wide-sense stationary (WSS): A random process that satisfies both a) and b) are called *wide-sense stationary* or *second-order stationary*.

The *autocorrelation sequence* (ACRS) of a wide-sense stationary process is

$$r_{xx}[m+\ell, m] \triangleq E(x[m+\ell]x[m]) = r_{xx}[\ell] = c_{xx}[\ell] + m_x^2 \tag{13.74}$$

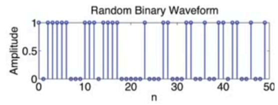
Examples of non-stationary signals



Examples of stationary signals

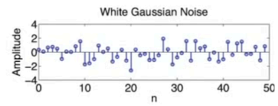
Examples of Stationary Signals -

1) Random Binary Waveform $x[n] = \begin{cases} 0 & p = 1/2 \\ 1 & p = 1/2 \end{cases}$



$\mu = 1/2$, $r[k] = \begin{cases} 1/2 & k=0 \quad E\{x^2[n]\} \\ 1/4 & k \neq 0 \quad E\{x[n]x[n-k]\} \end{cases}$

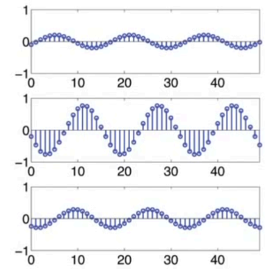
2) White Gaussian Noise $w[n] \sim N(0, \sigma_w^2)$ independent samples



$\mu = 0$, $r[k] = \begin{cases} \sigma_w^2 & k=0 \\ 0 & k \neq 0 \end{cases}$

3) Random Sinusoid $y[n] = A\cos(\omega_0 n + \phi)$

$A \sim N(0, \sigma_A^2)$; $\phi \sim U(0, 2\pi)$



$\mu = 0$, $r[k] = E\{A^2 \cos(\omega_0 n + \phi)\cos(\omega_0(n-k) + \phi)\}$

$= \frac{\sigma_A^2}{2}\cos(\omega_0 k)$

Properties of autocorrelation sequence

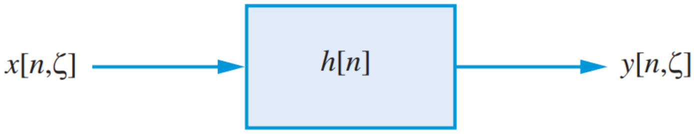
Properties of autocorrelation functions:

1. $r_{xx}(0) = \overline{X^2}$
2. $r_{xx}(l) = r_{xx}(-l)$
3. $r_{xx}(0) \geq |r_{xx}(l)|$
4. If $X(k) = \overline{X} + N(k)$ then $r_{xx}(l) = \overline{X}^2 + r_{NN}(l)$
5. If $X(k) = A\cos(\omega k + \theta) + N(k)$ then $r_{xx}(l) = \frac{A^2}{2}\cos(\omega l) + r_{NN}(l)$
6. $\lim_{|\tau| \to \infty} r_{xx}(l) = 0$ for ergodic, zero-mean processes with no periodic components
7. $\mathscr{F}[r_{xx}(l)] \geq 0 \qquad \forall \omega$

Power Spectral Density and LTI systems

Suppose $x[n, \zeta]$ is a signal from a random process and $h[n]$ is an impulse response of a Linear Time-Invariant system.

If we pass the signal $x[n, \zeta]$ through the system, we get $y[n, \zeta]$.



The output of a stable LTI system to the input $x[n,\zeta]$ is another sequence

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k],$$

For simplicity we drop the dependence on ζ. This is the simplest way to treat LTI systems with random process inputs.

We cannot predict the effects of an LTI system on any specific realization of the input process. But we can accurately predict its effect on **the average properties**. In other words, since we cannot have a mathematical model of random signals, we create a statistical model.

a) Computing the mean: If $x[n]$ is stationary then the mean of the output is constant

$$\mathrm{E}(y[n]) = m_x \sum_{k=-\infty}^{\infty} h[k] \triangleq m_y.$$

b) Computing the auto-correlation sequence of the output and the power spectral density:

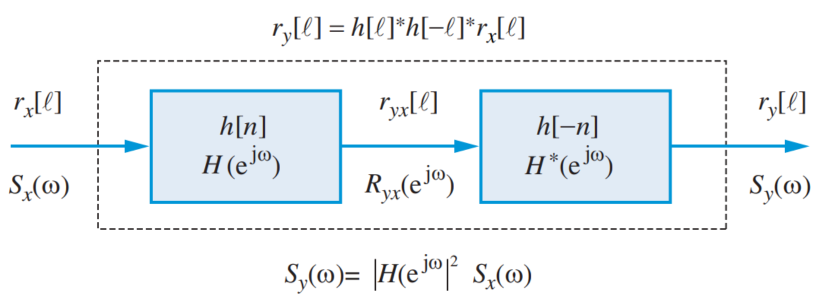


Figure 13.12 The ACRS and the PSD of the output process can be thought of as "filtered" by an LTI system with impulse response $r_{hh}[n] = h[n] * h[-n]$. Therefore, although LTI systems process individual sequences, they have the same effect on all sequences with the same mean and ACRS.

c) The average power of the output: $E\{y^2[n]\}$

$$\mathrm{E}(y^2[n]) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})\right|^2 S_{xx}(\omega)d\omega. \qquad (13.116)$$

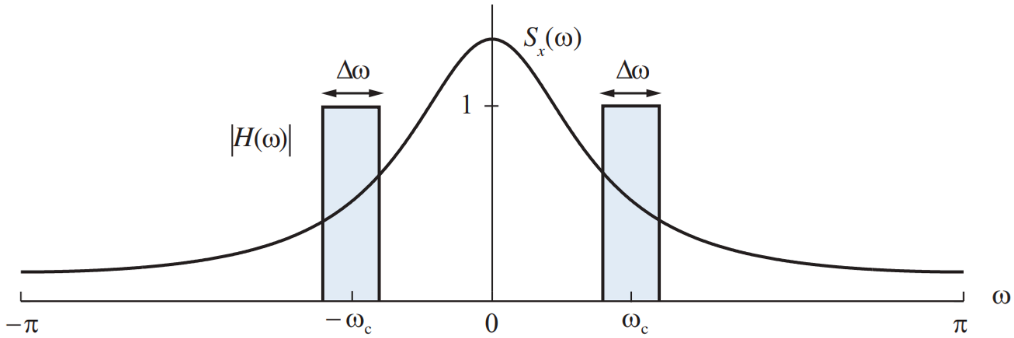


Figure 13.13 Physical interpretation of power spectrum density as power at the output of a narrowband LTI system.