Homework 4

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Problem 13.9: Determine the mean, ACVS and stationarity of a random process

9. A random process $x[n]$ is characterized by

$$x[n] = A(\zeta)\cos\left[\Omega(\zeta)n + \Theta(\zeta)\right],$$

where random variables $A(\zeta)$, $\Omega(\zeta)$, and $\Theta(\zeta)$ are mutually independent. Random variables $A(\zeta) \sim U(0, 1)$ and $\Theta(\zeta) \sim U(-\pi, \pi)$ are of continuous type while $\Omega(\zeta)$ is of discrete type taking values 10 and 20 radians with equal probability.

[✔] a) Determine the mean sequence $m_x[n]$

We need to compute:

$E[x[n]] = E[A\cos(\Omega n + \Theta)]$

Since the three random variables are independent then we can write:

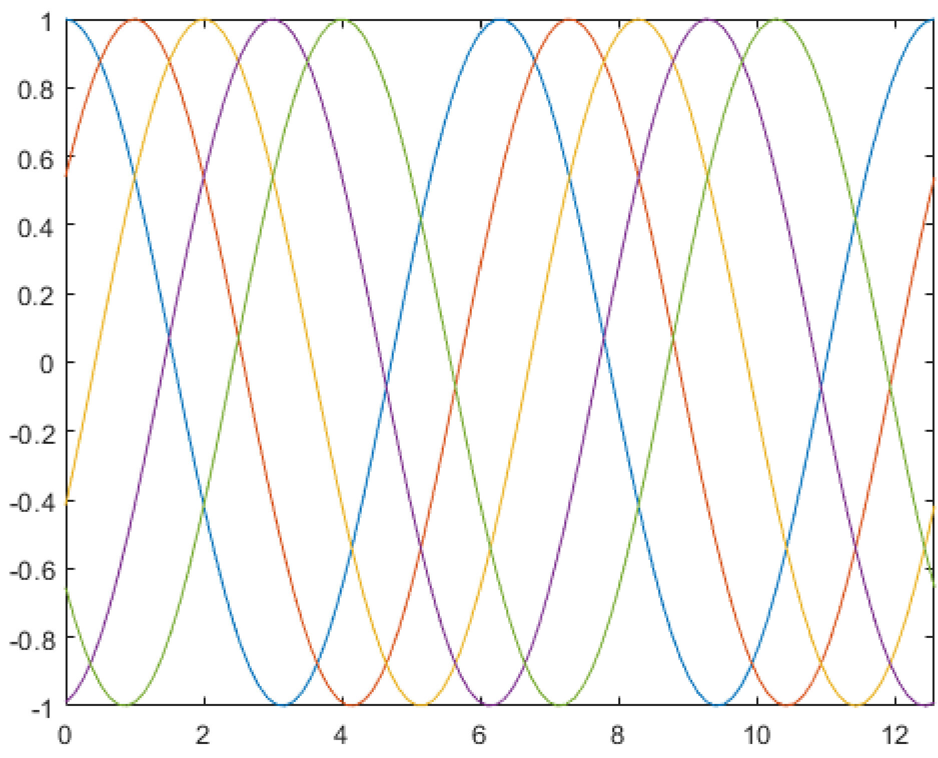
$E[x[n]] = E[A]E[\cos(\Omega n + \Theta)]$

We know that $E[A] = \frac{1}{2}$ since $A \sim U(0,1)$.

Computing the expression $E[\sin(\Theta)]$ requires a bit of an explanation.

Suppose we want to compute $E[\cos(\Omega n + \Theta)]$ where $\Theta \sim U(-\pi, \pi)$. Let us pick one frequency ω (realise one value of Ω). Then let us pick a lot of realisations of Θ. Now if we plot the function $\cos(\omega n + \theta)$ for different values of θ then we will see something like this:

```
n = linspace(0, 4*pi);
plot(n, cos(n), n, cos(n-1), n, cos(n-2), n, cos(n-3), n, cos(n-4));
xlim([0, 4*pi]);
```



If we plot hundreds of cosine functions shifted slightly, we get a large blob of points from -1 to 1. For this reason, the quality $E[\cos(\Omega n + \Theta)]$ will be zero because the mean value is 0. Formally, we can write:

$$E[\cos(\Omega n + \Theta)] = \int_{-\pi}^{\pi} f_\Theta(\Theta) \cdot \cos(\Omega n + \Theta)\, d\Theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\Omega n + \Theta)\, d\Theta$$

So we are integrating cosine function over 2π which is zero.

In signal processing, we like to add random shifts ala $\Theta \sim U(0, 2\pi)$ to avoid that the expected value or the mean value becomes dependent on time.

Alternatively, we can use

Using the rule $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, we rewrite the expression to:

$$E[x[n]] = E[A(\cos(\Omega n)\cos(\Theta) - \sin(\Omega n)\sin(\Theta))]$$
$$E[x[n]] = E[A\cos(\Omega n)\cos(\Theta) - A\sin(\Omega n)\sin(\Theta)]$$

At this point, we need to employ some expection rules to decompose the expression.

Let X and Y be two random variables and a and b be two constants. Following expectation identities apply:

1. $E[a] = a$ e.g. $E(42) = 42$
2. $E[a\,X] = a\,E[X]$ e.g. if you multiply every value by 2, the expectation doubles
3. $E[a \pm X] = a \pm E[X]$ e.g. if you add 42 to every case, the expectation increases by 42
4. $E[X + Y] = E[X] + E[Y]$
5. If X and Y are independent, then $E[\mathrm{XY}] = E[X]E[Y]$
6. $E[a \pm \mathrm{bX}] = a \pm E[b\,X] = a \pm b\,E[X]$
7. $E[b(a \pm \mathrm{X})] = b\,E[a \pm \mathrm{X}] = b(a \pm E[X])$

Use rule 4:

$$E[x[n]] = E[A\cos(\Omega n)\cos(\Theta)] - E[A\sin(\Omega n)\sin(\Theta)]$$

Use rule 5 multiple times:

$$E[x[n]] = E[A]E[\cos(\Omega n)]E[\cos(\Theta)] - E[A]E[\sin(\Omega n)]E[\sin(\Theta)]$$

But how can we continue from here?

We know that $\Theta \sim U(-\pi, \pi)$ so $E[\Theta] = \frac{\pi + (-\pi)}{2} = \frac{0}{2} = 0$

b) Determine the ACVS $c_X[m, n]$

c) Comment on the stationarity of the random process

[✔] Problem 13.13: MSE objective function

13. Consider the mse objective function (13.56)

$$J(a,b) = \mathrm{E}[(Y - aX - b)^2].$$

a) Express the objective function in terms of its parameters

(a) Express $J(a,b)$ in terms of the parameters a, b, and the moments of X and Y.

Use MATLAB to expand the expression inside the expected value:

```
syms a b X Y
expand((Y-a*X - b)^2)
```

ans = $X^2\,a^2 - 2\,X\,Y\,a + 2\,X\,a\,b + Y^2 - 2\,Y\,b + b^2$

So we have:

$J(a,b) = E[X^2a^2 - 2\,X\,Y\,a + 2\,X\,a\,b + Y^2 - 2\,Y\,b + b^2]$

Let X and Y be two random variables and a and b be two constants. Following expectation identities apply:

1. $E[a] = a$ e.g. $E(42) = 42$
2. $E[a\,X] = a\,E[X]$ e.g. if you multiply every value by 2, the expectation doubles
3. $E[a \pm X] = a \pm E[X]$ e.g. if you add 42 to every case, the expectation increases by 42
4. $E[X + Y] = E[X] + E[Y]$
5. If X and Y are independent, then $E[\mathrm{XY}] = E[X]E[Y]$

Use rule 4:

$J(a,b) = E[X^2a^2 - 2\,X\,Y\,a + 2\,X\,a\,b + Y^2 - 2\,Y\,b + b^2]$
$J(a,b) = E[X^2a^2] - E[2\,X\,Y\,a] + E[2\,X\,a\,b] + E[Y^2] - E[2\,Y\,b] + E[b^2]$

Use rule 1 and rule 2:

$J(a,b) = a^2E[X^2] - 2\,a\,E[\,X\,Y] + 2\,a\,b\,E[X] + E[Y^2] - 2\,b\,E[Y] + b^2$

b) Using partial derivatives to determine the values of parameters

(b) Using partial derivatives $\dfrac{\partial J}{\partial a}$ and $\dfrac{\partial J}{\partial b}$, determine the values of a and b by solving the equations $\partial J/\partial a = 0$ and $\partial J/\partial b = 0$ that minimize $J(a, b)$ to obtain optimum values given in (13.58) and (13.62).

First, take the partial derivatives:

$$\frac{\partial J(a,b)}{\partial a} = 2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X]$$

$$\frac{\partial J(a,b)}{\partial b} = 2\,a\,E[X] - 2\,E[Y] + 2b$$

Next, solve the equations:

(Eq. 1) $2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X] = 0$

(Eq. 2) $2\,a\,E[X] - 2\,E[Y] + 2b = 0$

Isolate b in (Eq. 2):

$$2b = -2\,a\,E[X] + 2\,E[Y]$$
$$b = -a\,E[X] + E[Y]$$
$$b = E[Y] - a\,E[X]$$

This corresponds to (13.58) in the book:

(13.58) $\quad b_{\mathrm{o}} = m_y - am_x$

Now, plug the expression for b into Eq. 1 in order to find an expression for a:

$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X] = 0$$
$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,(E[Y] - a\,E[X])\,E[X] = 0$$
$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,E[X]E[Y] - 2\,a\,E[X]E[X] = 0$$
$$2a\,E[X^2] - 2\,a\,E[X]E[X] - 2\,E[X\,Y] + 2\,E[X]E[Y] = 0$$
$$2a(\,E[X^2] - E[X]E[X]) - 2\,E[X\,Y] + 2\,E[X]E[Y] = 0$$
$$2a(\,E[X^2] - E[X]E[X]) = 2\,E[X\,Y] - 2\,E[X]E[Y]$$
$$a(\,E[X^2] - E[X]E[X]) = E[X\,Y] - E[X]E[Y]$$
$$a = \frac{E[X\,Y] - E[X]E[Y]}{E[X^2] - E[X]E[X]} = \frac{E[X\,Y] - E[X]E[Y]}{E[X^2] - E[X]^2}$$

We have found an expression for a. The numerator looks like it is the covariance:

Covariance The *covariance* of two random variables X and Y is defined by

(13.25) $\quad c_{xy} \triangleq \mathrm{cov}(X, Y) \triangleq \mathrm{E}[(X - m_x)(Y - m_y)] = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$

The denominator looks like it is the variance:

(13.11) $\quad \mathrm{var}(X) = E[X^2] - E[X]^2 = E[X^2] - m_x^2$

Therefore, the derived expression is the same as (13.62) in the book.

$$(13.62) \qquad a_o = \frac{c_{xy}}{\sigma_x^2} = \rho_{xy}\frac{\sigma_y}{\sigma_x}$$

Problem 13.22: Computing distributions

22. Consider two jointly distributed random variables X and Y with pdf

$$f(x, y) = \begin{cases} 8xy, & 0 \le x \le 1, 0 \le y \le x \\ 0. & \text{otherwise} \end{cases}$$

a) Determine marginal distributions and conditional probabilities

(a) Determine $f(x), f(y), f(x|y)$, and $f(y|x)$.

The **marginal** distributions of random variables X and X are obtained by integration as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\mathrm{d}y \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\mathrm{d}x, \qquad (13.19)$$

Compute the marginal distribution of X:

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y = \int_0^x 8xy\,\mathrm{d}y = \left[\frac{1}{2}8xy^2\right]_0^x = \frac{1}{2}8xx^2 - 0 = 4x^3 \text{ where } 0 \le x \le 1$$

Compute the marginal distribution of Y:

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}x = \int_y^1 8xy\,\mathrm{d}x = [4yx^2]_y^1 = 4y - 4y^3 \text{ where } 0 \le y \le 1$$

To compute $f(x|y)$ we use following relation:

$$f_{X,Y}(x, y) = f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y). \qquad (13.23)$$

From Eq. 13.23, we know that:

$$f(x|y) = \frac{f(x, y)}{f(y)} = \frac{8\mathrm{xy}}{4y - 4y^3} = \frac{4y \cdot 2x}{4y(1 - y^2)} = \frac{2x}{1 - y^2}$$

and

$$f(y|x) = \frac{f(x, y)}{f(x)} = \frac{8x\,y}{4x^3} = \frac{4x \cdot 2y}{4x \cdot x^2} = \frac{2y}{x^2}$$

b) Are X and Y independent?

(b) Are X and Y independent?

Random variables X and Y are statistically independent, if $f(y|x) = f(y)$ or $f(x|y) = f(x)$.

In a) we have computed the following expressions:

- $f(x) = 4x^3$
- $f(y) = 4y - 4y^3$
- $f(y|x) = \dfrac{2y}{x^2}$
- $f(x|y) = \dfrac{2x}{1 - y^2}$

Clearly $f(y|x) \neq f(y)$ and $f(x|y) \neq f(x)$. Therefore, the answer is no! The random variables X and Y are not statistically independent.

ADSI Problem 4.1: Poisson distribution

The Poisson distribution is a discrete probability distribution that is used in model counting events i.e. pixel noise in CCD cameras and in particle detectors. The density function for the Poisson distribution is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{a^k e^{-a}}{k!} \delta(x - k).$$

Where a is a fixed positive constant. The density function gives the probability of observing a particular number of events (x) in a given experiment when these events occur with a known average number of events per experiment (a) and the events are independent of one another. Let $a = 3$.

1) Sketch the density function

2) Calculate the probability of measuring 4 or 5 events in an experiment.

3) Calculate the probability of measuring 3 or more events in an experiment.

ADSI Problem 4.3: Random process

Let two random variables, X_1 and X_2 be defined from a random process $X(t)$ as $X_1 = X(t_1)$ and $X_2 = X(t_1 + 1)$. The joint density function for the two random variables is given by

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{8}\delta(x_1 - 1)\delta(x_2 - 1) + \frac{3}{8}\delta(x_1 - 1)\delta(x_2 + 1) + \frac{3}{8}\delta(x_1 + 1)\delta(x_2 - 1) + \frac{1}{8}\delta(x_1 + 1)\delta(x_2 + 1).$$

1) Draw a realisation of a random process

1. Draw a realization of $X(t)$ and explain the considerations that you made in the preparation of the realization.

2) Determine whether the random process is deterministic or non-deterministic

2. Discuss, using only the joint density function, whether the random process is deterministic or nondeterministic and state your arguments. Calculations are not called upon in this question.

3) Plot realisations of the random process in MATLAB

3. Write a MATLAB script that can create and plot realizations of the random process.