Table of Contents

Homework 2

ADSI Problem 1.4: Filter decomposition

1) Decompose a FIR filter with one zero outside the unit circle

Let the system function for a FIR filter be given by

$$H(z) = 1 - 3z^{-1} + \frac{5}{2}z^{-2} - z^{-3}$$

1. Decompose the system function into a product of a minimum-phase filter and an all-pass filter, $H(z) = H_{min}(z)H_{ap}(z)$.

Any system funcrtion can be decomposed into a product of a minimum-phase filter and an all-pass filter using the following formula:

$$H(z) = H_{\text{min}}(z)H_{\text{ap}}(z)$$

The minimum phase filter can be computed as follows:

$$H_{\text{min}}(z) = \prod_i -\frac{1}{a_i^*}H_i(z)\left(1 - a_i z^{-1}\right)$$

where $H_i(z)$ corresponds to the part of the transfer function where the i'th zero is inside the unit circle and $a_i = \frac{1}{z_i^*}$ and z_i is the zero outside the unit circle.

The allpass filter can be calculated:

$$H_{\text{ap}}(z) = \prod_i \frac{z^{-1} - a_i^*}{1 - a_i z^{-1}}$$

The filter decomposition algorithm has following steps:

1. convert transfer function into pole-zero representation in order to find the zeros that are outside the unit circle

2. compute a and its conjugate a^*
3. find $H_1(z)$ which corresponds to the part of the transfer function where zeros are inside the unit circle
4. plugin the numbers for the formula for the minimum-phase filter
5. plugin the numbers for the formula for the allpass filter
6. put everything together:

Step 1: convert transfer function into pole-zero representation in order to find the zeros that are outside the unit circle

```
syms z;
rts = roots([1, -3, 5/2, -1])
```

```
rts = 3×1 complex
   2.0000 + 0.0000i
   0.5000 + 0.5000i
   0.5000 - 0.5000i
```

```
H_outside = 1; % Represents part of H where zeros are outside the unit circle
H_inside = 1;  % Represents part of H where zeros are inside the unit circle
zeros_outside = [];
for i = 1:numel(rts)
    root = rts(i);
    if abs(root) > 1
        H_outside = H_outside * (1 - root*z^-1);
        zeros_outside = [zeros_outside; root];
    else
        H_inside = H_inside * (1 - root*z^-1);
    end
end

z0 = 0;
if numel(zeros_outside) == 1
    z0 = zeros_outside(1);
else
    disp('Something is wrong! The transfer function has more than one zero outside the unit cir
end

H_outside = expand(H_outside);
H_inside = expand(H_inside);

% Sanity check
H = expand(H_inside * H_outside)
```

H =

$$\frac{5}{2\,z^2} - \frac{3}{z} - \frac{1}{z^3} + 1$$

The zero-pole representation of the transfer function is:

```
H_outside
```

```
H_outside =
```

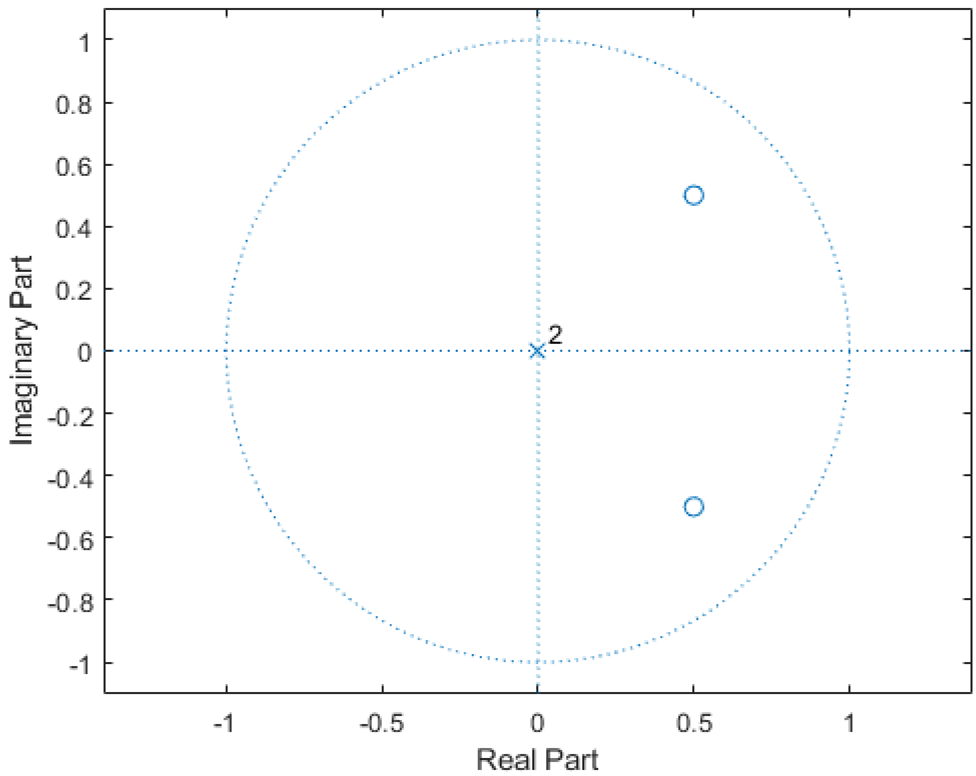
$$1-\frac{2}{z}$$

```
H_inside
```

```
H_inside =
```

$$\frac{1}{2\,z^2}-\frac{1}{z}+1$$

```
zplane([1, -1, 1/2]) % Should all have zeros inside the unit circle
```



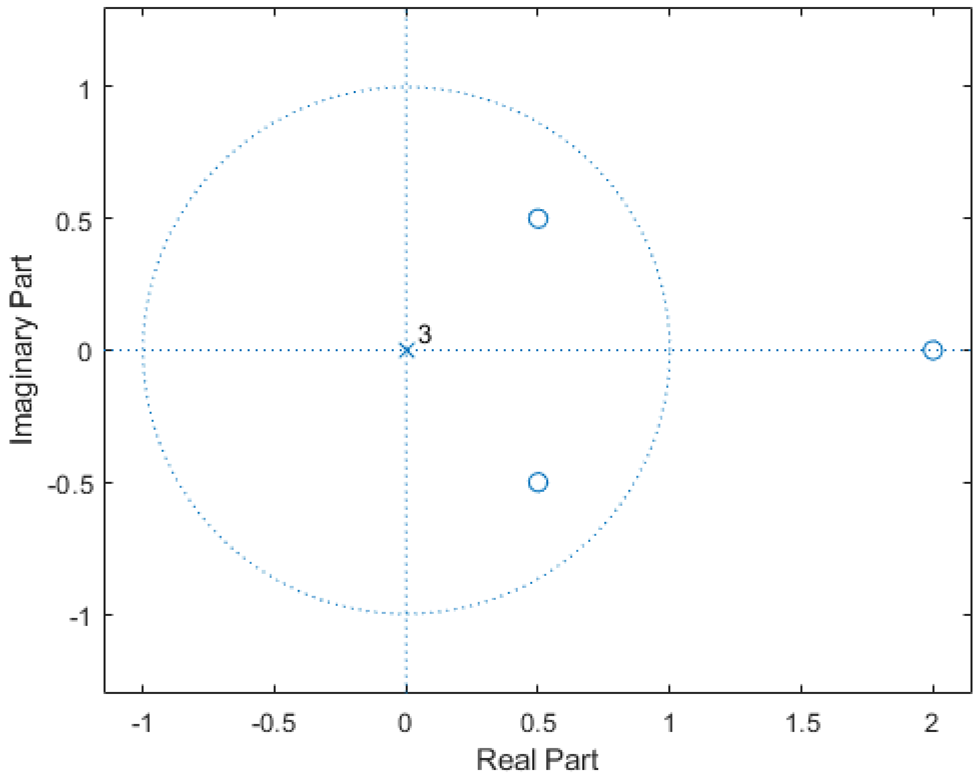
$$H(z)=(1-2z^{-1})\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)$$

```
% Sanity check
conv([1, -2], [1, -1, 1/2])
```

```
ans = 1×4
    1.0000   -3.0000    2.5000   -1.0000
```

To find the zero z_0 that is outside the unit circle, we can plot zplane:

```
zplane([1, -3, 5/2, -1]);
```



In this exercise, the zero outside the unit circle is $z_0 = 2$.

```
z0
```

```
z0 = 2
```

Step 2: compute a and its conjugate a^*

```
a = 1/z0
```

```
a = 0.5000
```

```
a_conj = conj(a)
```

```
a_conj = 0.5000
```

$$a = \frac{1}{z_0} = \frac{1}{2}$$

$$a^* = \left(\frac{1}{z_0}\right)^* = \left(\frac{1}{2}\right)^* = \frac{1}{2}$$

Step 3: find $H_1(z)$ which corresponds to the part of the transfer function where zeros are inside the unit circle

```
H_inside
```

```
H_inside =
```

$$\frac{1}{2\,z^2} - \frac{1}{z} + 1$$

The zero-pole representation of the transfer function is:

$$H(z) = (1 - 2z^{-1})\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)$$

therefore $H_1(z)$ is

$$H_1(z) = 1 - z^{-1} + \frac{1}{2}z^{-2}$$

Step 4: plugin the numbers for the formula for minimum-phase filter

$$H_{\min}(z) = -\frac{1}{a^*}H_1(z)(1 - az^{-1})$$

$$H_{\min}(z) = -\frac{1}{\left(\frac{1}{2}\right)}\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)(1 - 0.5z^{-1})$$

$$H_{\min}(z) = -2\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)(1 - 0.5z^{-1})$$

```
-2 * conv([1, -1, 1/2], [1, -0.5])
```

```
ans = 1×4
   -2.0000    3.0000   -2.0000    0.5000
```

$$H_{\min}(z) = -2 + 3z^{-1} - 2z^{-2} + \frac{1}{2}z^{-3}$$

```
H_min = expand((-1/a_conj) * H_inside * (1 - a*z^-1))
```

```
H_min =
```

$$\frac{3}{z} - \frac{2}{z^2} + \frac{1}{2\,z^3} - 2$$

```
H_min_b = [-2, 3, -2, 1/2];
H_min_rts = roots(H_min_b)
```

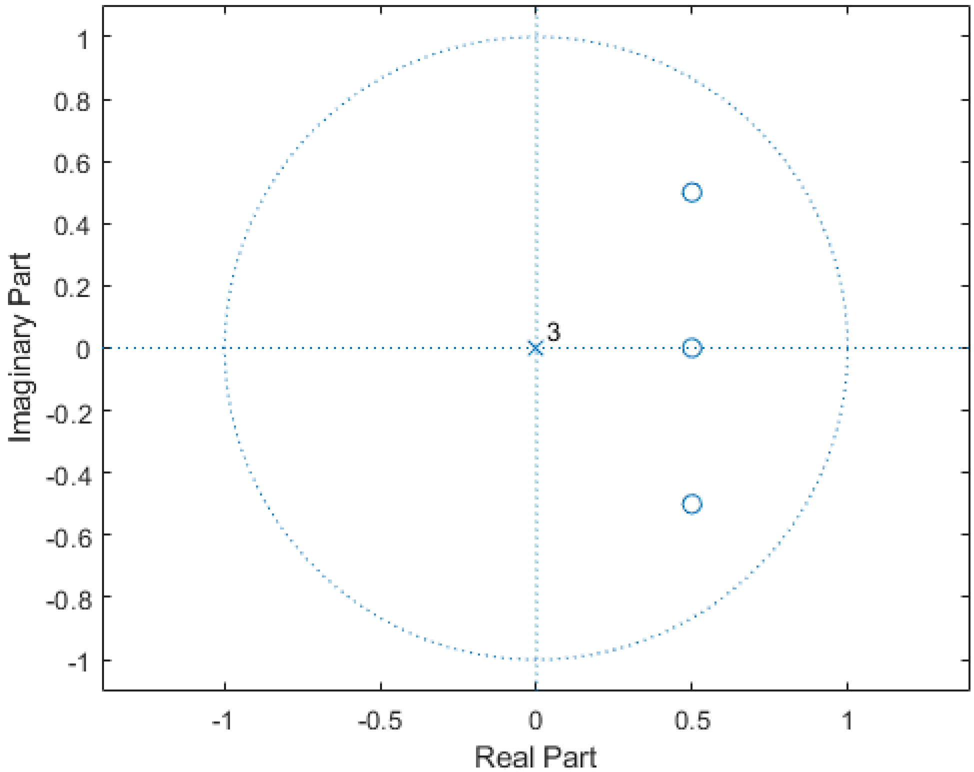
```
H_min_rts = 3×1 complex
   0.5000 + 0.5000i
```

```
    0.5000 - 0.5000i
    0.5000 + 0.0000i
```

```
isminphase(H_min_b)
```

```
ans = logical
   1
```

```
zplane(H_min_b)
```



Step 5: compute the allpass filter :

$$H_{\mathrm{ap}}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$H_{\mathrm{ap}}(z) = \frac{z^{-1} - 0.5}{1 - 0.5z^{-1}}$$

```
H_ap = (z^-1 - a_conj) / (1 - a*z^-1)
```

H_ap =

$$-\frac{\frac{1}{z} - \frac{1}{2}}{\frac{1}{2\,z} - 1}$$

Step 6: put everything together:

$$H_{\min}(z) = -2 + 3z^{-1} - 2z^{-2} + \frac{1}{2}z^{-3}$$

$$H_{\text{ap}}(z) = \frac{z^{-1} - 0.5}{1 - 0.5z^{-1}}$$

$$H(z) = H_{\min}(z)H_{\text{ap}}(z)$$

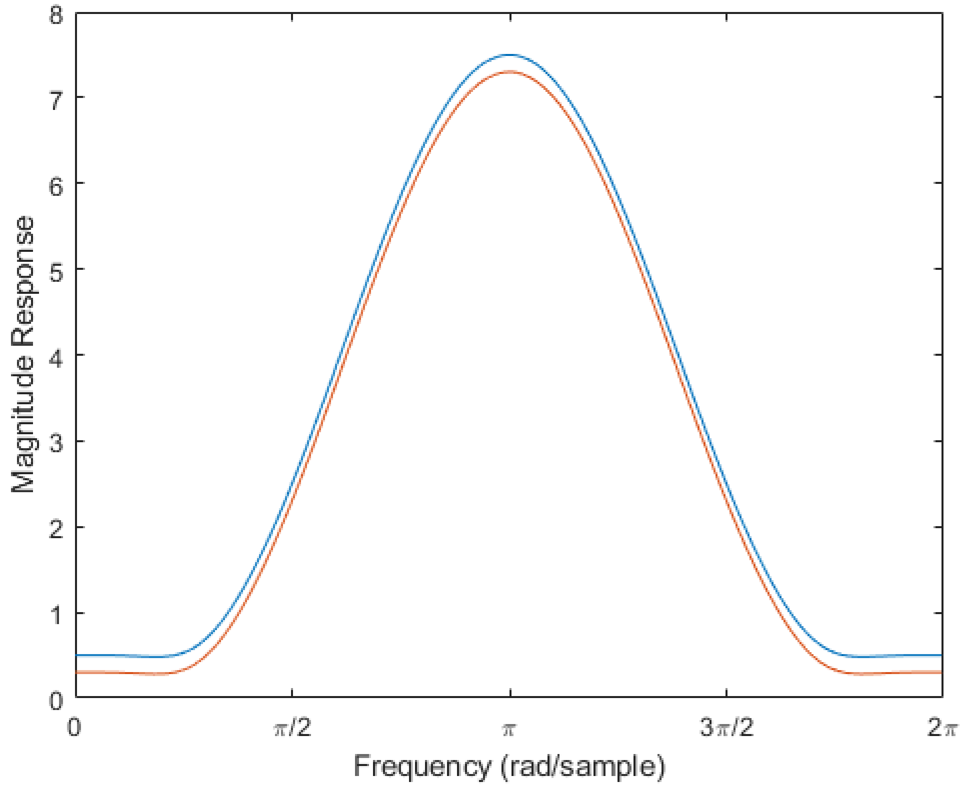
$$H(z) = \left(-2 + 3z^{-1} - 2z^{-2} + \frac{1}{2}z^{-3}\right)\left(\frac{z^{-1} - 0.5}{1 - 0.5z^{-1}}\right)$$

2) Show that $H(z)$ and $H_{\min}(z)$ have the same magnitude response

2. Demonstrate that $H(z)$ and $H_{min}(z)$ have the same amplitude response.

To show that $H(z)$ and $H_{\min}(z)$ have the same magnitude response, we can plot them. If the two graphs are on top of each other then then they have the same magnitude response.

```
% Coefficients for H(z)
b = [1, -3, 5/2, -1];
a = 1;
[H, w] = freqz(b,a,'whole');

% Coefficients for H_min(z)
H_min_b = [-2, 3, -2, 1/2];
H_min_a = 1;
[H_min_z, H_min_w] = freqz(H_min_b, H_min_a, 'whole');
% plot(H_min_w, log10(abs(H_min_z)));

% The offset ensures that the two graphs are not on top of each other
offset = -0.2;
plot(w, abs(H), H_min_w, abs(H_min_z) + offset);
set(gca,'XTick',0:pi/2:2*pi)
set(gca,'XTickLabel',{'0','\pi/2','\pi','3\pi/2','2\pi'})
xlabel('Frequency (rad/sample)')
ylabel('Magnitude Response')
xlim([0, 2*pi]);
```



3) Decompose a filter that has two zeros outside the unit circle

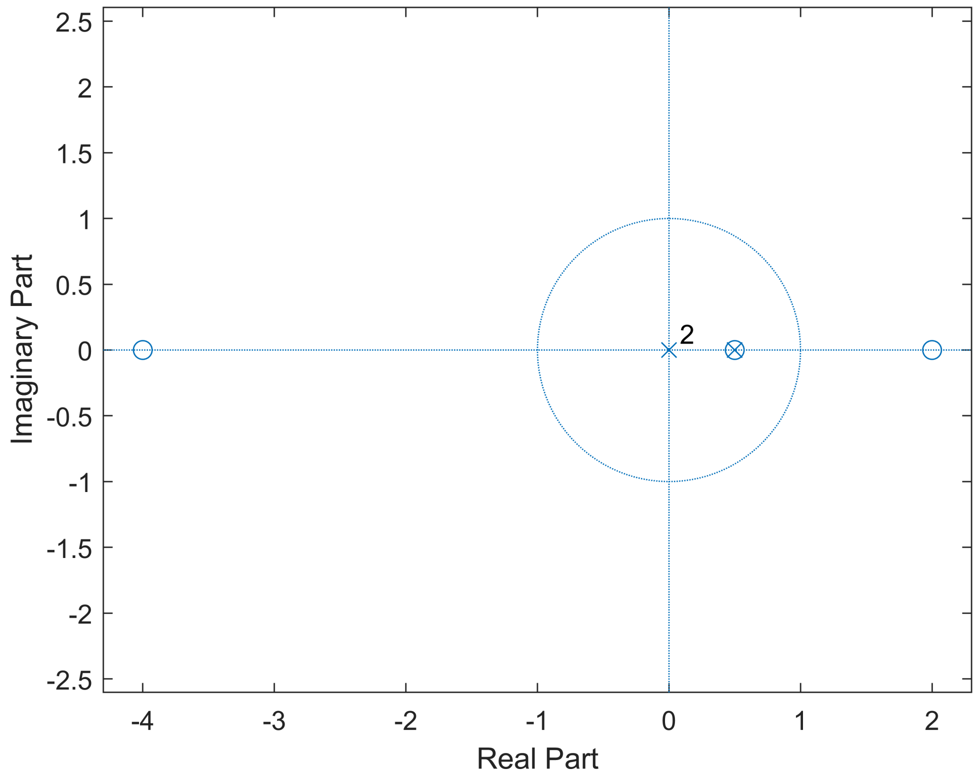
3. Repeat for the following filter

$$H(z) = \frac{1 + \frac{3}{2}z^{-1} - 9z^{-2} + 4z^{-3}}{1 - \frac{1}{2}z^{-1}}$$

```
b = [1, 3/2, -9, 4];
a = [1, -1/2];
```

Plot the zeros and the poles:

```
zplane(b, a);
```



From the plot, we can see that there is a zero and a pole at 0.5. They will cancel each other.

```
% Find the zeros of H
rts = roots(b)
```

```
rts = 3×1
   -4.0000
    2.0000
    0.5000
```

Now, the transfer function can be rewritten as:

$$H(z) = \frac{(1-(-4)z^{-1})(1-2z^{-1})\left(1-\frac{1}{2}z^{-1}\right)}{1-\frac{1}{2}z^{-1}} = (1-(-4)z^{-1})(1-2z^{-1}) = (1-(-4)z^{-1})(1-2z^{-1})$$

Let us find the coefficients of the rewritten transfer function:

```
H_new = expand((1+4*z^-1) * (1-2*z^-1))
```

H_new =

$$\frac{2}{z} - \frac{8}{z^2} + 1$$

```
% Automatically extract the coefficients
b_new = coeffs(expand(H_new * z^2), 'all')
```

b_new = $\begin{pmatrix} 1 & 2 & -8 \end{pmatrix}$

```
% Find the zeros of H_new
rts = roots(b_new)
```

rts =

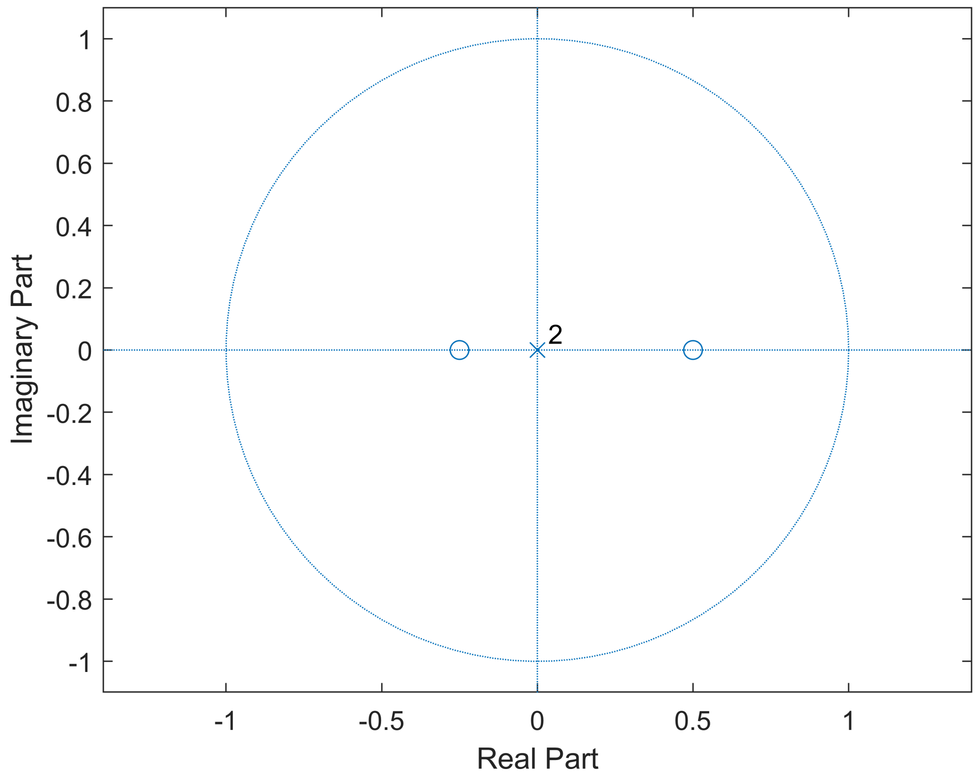
$$\begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

```
syms z;
H_outside = 1; % Represents part of H where zeros are outside the unit circle
H_inside = 1;  % Represents part of H where zeros are inside the unit circle
zeros_outside = [];
for i = 1:numel(rts)
    root = rts(i);
    if abs(root) > 1
        H_outside = H_outside * (1 - root*z^-1);
        zeros_outside = [zeros_outside; root];
    else
        H_inside = H_inside * (1 - root*z^-1);
    end
end

% Sanity check
H = expand(H_inside * H_outside)
```

H =

$$\frac{2}{z} - \frac{8}{z^2} + 1$$

```
% Finally compute the H_min and H_ap
H_min = 1;
H_ap = 1;
for i = 1:numel(zeros_outside)
    a = 1/zeros_outside(i);
    a_conj = conj(a);
    H_min = H_min * (-1/a_conj) * H_inside * (1 - a*z^-1);
    H_ap = H_ap * ((z^-1 - a_conj) / (1 - a*z^-1));
end

% Check that H_min is in fact minimum-phase
% Check that all zeros are within the unit circle!
H_min_b = coeffs(expand(H_min * z^10), 'all');
zplane(H_min_b)
```




```
% Display H_min
expand(H_min)
```

ans =

$$\frac{2}{z}+\frac{1}{z^2}-8$$

We can write $H_{\min}(z) = -8 + 2z^{-1} + z^{-2}$

```
% Display H_ap in order to rewrite it
H_ap
```

H_ap =

$$-\frac{\left(\frac{1}{z}-\frac{1}{2}\right)\,\left(\frac{1}{z}+\frac{1}{4}\right)}{\left(\frac{1}{2\,z}-1\right)\,\left(\frac{1}{4\,z}+1\right)}$$

Expanding the numerator, we get:

```
H_ap_num = expand(-1*(1/z - 1/2)*(1/z + 1/4))
```

```
H_ap_num =
```

$$\frac{1}{4z} - \frac{1}{z^2} + \frac{1}{8}$$

```
H_ap_den= expand((1/(2*z) - 1)*(1/(4*z) + 1))
```

```
H_ap_den =
```

$$\frac{1}{4z} + \frac{1}{8z^2} - 1$$

We can multiply both the numerator and denominator with 8, to get nice numbers:

```
(H_ap_num * 8) / (H_ap_den * 8)
```

```
ans =
```

$$\frac{\frac{2}{z} - \frac{8}{z^2} + 1}{\frac{2}{z} + \frac{1}{z^2} - 8}$$

Now, we can write the transfer function for the allpass filter:

$$H_{\text{ap}}(z) = \frac{1 + 2z^{-1} - 8z^{-2}}{-8 + 2z^{-1} + z^{-2}}$$

So we have:

$$H(z) = H_{\text{min}}(z)H_{\text{ap}}(z)$$

where $H_{\text{min}}(z) = -8 + 2z^{-1} + z^{-2}$ and $H_{\text{ap}}(z) = \frac{1 + 2z^{-1} - 8z^{-2}}{-8 + 2z^{-1} + z^{-2}}$

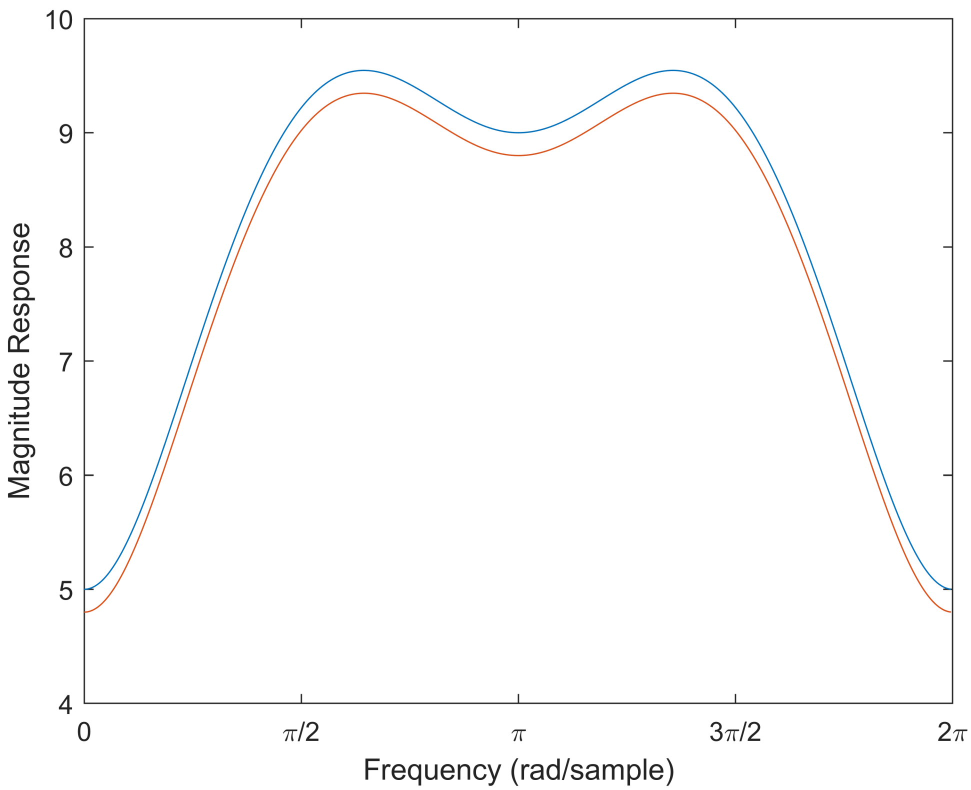
To show that $H(z)$ and $H_{\text{min}}(z)$ have the same magnitude response, we can plot them. If the two graphs are on top of each other then then they have the same magnitude response.

$$H(z) = \frac{1 + \frac{3}{2}z^{-1} - 9z^{-2} + 4z^{-3}}{1 - \frac{1}{2}z^{-1}}$$

$$H_{\text{min}}(z) = -8 + 2z^{-1} + z^{-2}$$

```
% Coefficients for H(z)
b = [1, 3/2, -9, 4];
a = [1, -1/2];
[H, w] = freqz(b,a,'whole');
```

```
% Coefficients for H_min(z)
H_min_b = [-8, 2, 1];
H_min_a = 1;
[H_min_z, H_min_w] = freqz(H_min_b, H_min_a, 'whole');
% plot(H_min_w, log10(abs(H_min_z)));

% The offset ensures that the two graphs are not on top of each other
offset = -0.2;
plot(w, abs(H), H_min_w, abs(H_min_z) + offset);
set(gca,'XTick',0:pi/2:2*pi)
set(gca,'XTickLabel',{'0','\pi/2','\pi','3\pi/2','2\pi'})
xlabel('Frequency (rad/sample)')
ylabel('Magnitude Response')
xlim([0, 2*pi]);
```



ADSI Problem 1.5 Filter decomposition

1) Show that a FIR filter with a difference equation is not minimum-phase

Consider a FIR filter with the following difference equation

$$y(n) = x(n) + 2x(n-1) + 2x(n-2)$$

1. Show that the FIR filter is not minimum-phase.

To determine whether the filter is a minimum-phase or not, we need get the transfer function of this FIR filter $H(z)$ which is defined as:

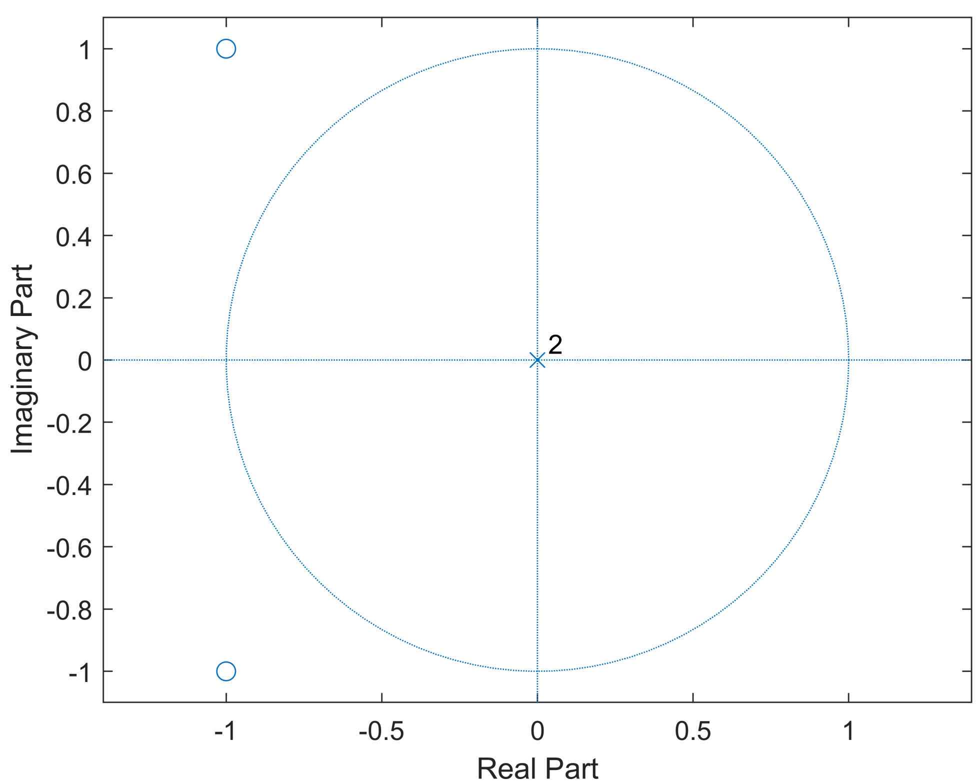
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} \cdots + a_N z^{-N}}$$

This means that we must take the z-transform of difference equation:

$$Y(z) = X(z) + 2X(z)z^{-1} + 2X(z)z^{-2}$$
$$Y(z) = X(z)(1 + 2z^{-1} + 2z^{-2})$$
$$\frac{Y(z)}{X(z)} = 1 + 2z^{-1} + 2z^{-2}$$
$$H(z) = 1 + 2z^{-1} + 2z^{-2}$$

Once we have the transfer function, we can determine if the filter is a minimum-phase by looking at the zplane plot:

```
zplane([1, 2, 2])
```



Since all the zeros are outside the unit cirlce, clearly the given FIR filter is not minimum-phase. It is a **maximum-phase**.

2) Find the difference equation for the corresponding minimum-phase FIR filter

We found that the transfer function for the difference equation is:

$$H(z) = 1 + 2z^{-1} + 2z^{-2}$$

```
% Find the zeros of H
rts = roots([1, 2, 2])
```

```
rts = 2×1 complex
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
```

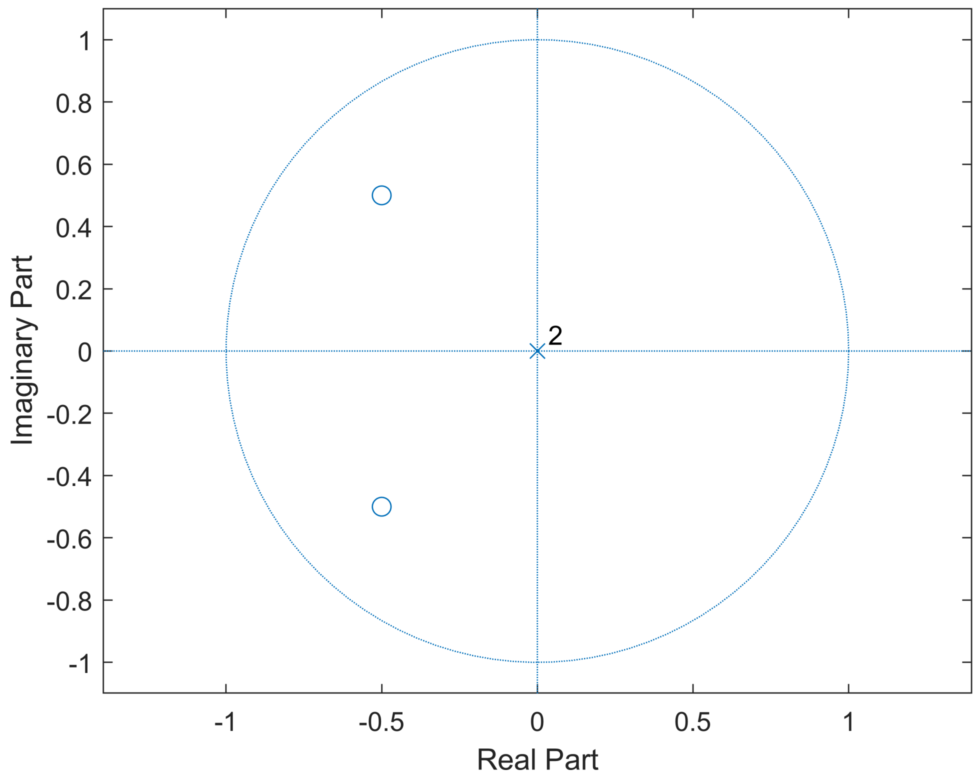
```
syms z;
H_outside = 1; % Represents part of H where zeros are outside the unit circle
H_inside = 1; % Represents part of H where zeros are inside the unit circle
zeros_outside = [];
for i = 1:numel(rts)
    root = rts(i);
    if abs(root) > 1
        H_outside = H_outside * (1 - root*z^-1);
        zeros_outside = [zeros_outside; root];
    else
        H_inside = H_inside * (1 - root*z^-1);
    end
end
% Sanity check
H = expand(H_inside * H_outside)
```

H =

$$\frac{2}{z} + \frac{2}{z^2} + 1$$

```
% Finally compute the H_min and H_ap
H_min = 1;
H_ap = 1;
for i = 1:numel(zeros_outside)
    a = 1/zeros_outside(i);
    a_conj = conj(a);
    H_min = H_min * (-1/a_conj) * H_inside * (1 - a*z^-1);
    H_ap = H_ap * ((z^-1 - a_conj) / (1 - a*z^-1));
end

% Check that H_min is in fact minimum-phase
% Check that all zeros are within the unit circle!
```

```
H_min_b = coeffs(expand(H_min * z^10), 'all');
zplane(H_min_b)
```



From the zero-pole plot we can see that we have a minimum-phase filter.

Finally, we can write out the transfer function of the minimum-phase part:

```
expand(H_min)
```

ans =

$$\frac{2}{z} + \frac{1}{z^2} + 2$$

$H_{\min}(z) = 2 + 2z^{-1} + z^{-2}$

```
H_ap
```

H_ap =

$$\frac{\left(\frac{1}{z} + \frac{1}{2} - \frac{1}{2}\,\mathrm{i}\right)\left(\frac{1}{z} + \frac{1}{2} + \frac{1}{2}\,\mathrm{i}\right)}{\left(1 + \frac{\frac{1}{2} - \frac{1}{2}\,\mathrm{i}}{z}\right)\left(1 + \frac{\frac{1}{2} + \frac{1}{2}\,\mathrm{i}}{z}\right)}$$

Let us simply this expression:

```
H_ap_num = expand((1/z + 1/2 - 1/2i) * (1/z + 1/2 + 1/2i))
```

H_ap_num =

$$\frac{1}{z} + \frac{1}{z^2} + \frac{1}{2}$$

```
H_ap_den = expand((1+ (1/2 -1/2i)*z^-1) * (1 + (1/2+1/2i)*z^-1 ))
```

H_ap_den =

$$\frac{1}{z} + \frac{1}{2\,z^2} + 1$$

$$H_{\mathrm{ap}}(z) = \frac{\frac{1}{2} + z^{-1} + z^{-2}}{1 + z^{-1} + \frac{1}{2}z^{-2}}$$

Hence:

$$H(z) = H_{\min}(z)H_{\mathrm{ap}}(z)$$

$$H(z) = (2 + 2z^{-1} + z^{-2})\left(\frac{\frac{1}{2} + z^{-1} + z^{-2}}{1 + z^{-1} + \frac{1}{2}z^{-2}}\right)$$

We can write $H_{\min}(z) = 2 + 2z^{-1} + z^{-2}$ as difference equation:

$$H_{\min}(z) = \frac{Y(z)}{X(z)}$$

$$Y(z) = X(z)(2 + 2z^{-1} + z^{-2})$$

Now, we take the inverse z-transform:

```
iztrans(2 + 2*z^-1 + z^-2)
```

ans = $2\,\delta_{n-1,0} + \delta_{n-2,0} + 2\,\delta_{n,0}$

We get:

$$y[n] = 2x[n] + 2x[n-1] + x[n-2]$$