Homework 4

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Problem 13.9: Determine the mean, ACVS and stationarity of a random process

9. A random process $x[n]$ is characterized by

$$x[n] = A(\zeta)\cos\left[\Omega(\zeta)n + \Theta(\zeta)\right],$$

where random variables $A(\zeta)$, $\Omega(\zeta)$, and $\Theta(\zeta)$ are mutually independent. Random variables $A(\zeta) \sim U(0,1)$ and $\Theta(\zeta) \sim U(-\pi,\pi)$ are of continuous type while $\Omega(\zeta)$ is of discrete type taking values 10 and 20 radians with equal probability.

[✔] a) Determine the mean sequence $m_x[n]$

We need to compute:

$$E[x[n]] = E[A\cos(\Omega n + \Theta)]$$

Since the three random variables are independent then we can write:

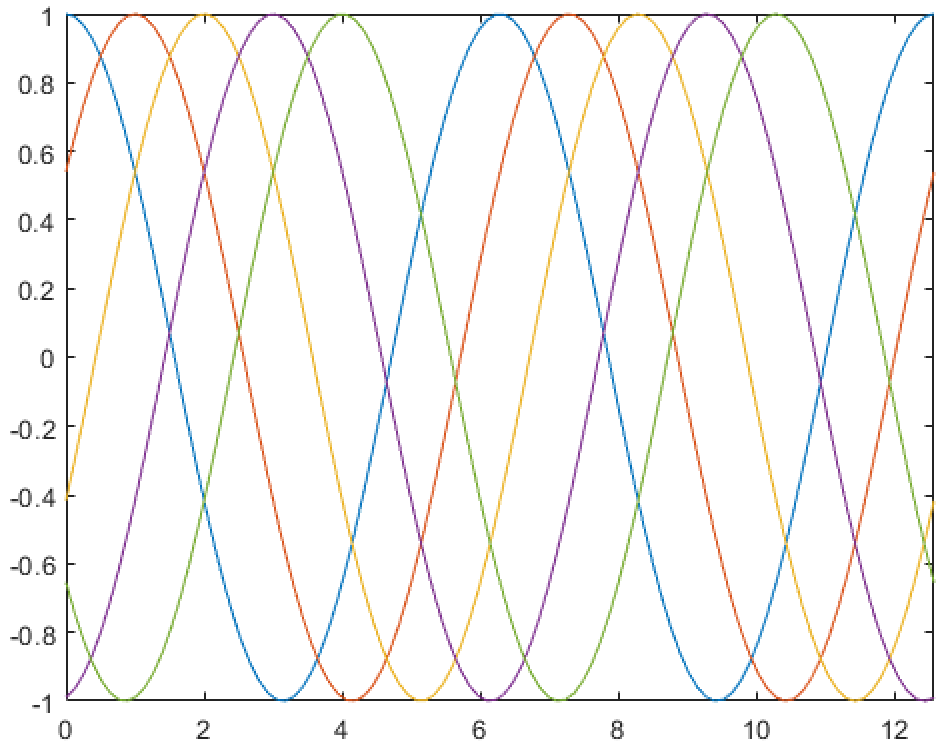
$$E[x[n]] = E[A]E[\cos(\Omega n + \Theta)]$$

We know that $E[A] = \frac{1}{2}$ since $A \sim U(0,1)$.

Computing the expression $E[\sin(\Theta)]$ requires a bit of an explanation.

Suppose we want to compute $E[\cos(\Omega n + \Theta)]$ where $\Theta \sim U(-\pi, \pi)$. Let us pick one frequency ω (realise one value of Ω). Then let us pick a lot of realisations of Θ. Now if we plot the function $\cos(\omega n + \theta)$ for different values of θ then we will see something like this:

```
n = linspace(0, 4*pi);
plot(n, cos(n), n, cos(n-1), n, cos(n-2), n, cos(n-3), n, cos(n-4));
xlim([0, 4*pi]);
```



If we plot hundreds of cosine functions shifted slightly, we get a large blob of points from -1 to 1. For this reason, the quality $E[\cos(\Omega n + \Theta)]$ will be zero because the mean value is 0. Formally, we can write:

$$E[\cos(\Omega n + \Theta)] = \int_{-\pi}^{\pi} f_\Theta(\Theta) \cdot \cos(\Omega n + \Theta)\, d\Theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\Omega n + \Theta)\, d\Theta$$

So we are integrating cosine function over 2π which is zero.

In signal processing, we like to add random shifts ala $\Theta \sim U(0, 2\pi)$ to avoid that the expected value or the mean value becomes dependent on time.

Alternatively, we can use

Using the rule $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, we rewrite the expression to:

$E[x[n]] = E[A(\cos(\Omega n)\cos(\Theta) - \sin(\Omega n)\sin(\Theta))]$
$E[x[n]] = E[A\cos(\Omega n)\cos(\Theta) - A\sin(\Omega n)\sin(\Theta)]$

At this point, we need to employ some expection rules to decompose the expression.

Let X and Y be two random variables and a and b be two constants. Following expectation identities apply:

1. $E[a] = a$ e.g. $E(42) = 42$
2. $E[a\,X] = a\,E[X]$ e.g. if you multiply every value by 2, the expectation doubles
3. $E[a \pm X] = a \pm E[X]$ e.g. if you add 42 to every case, the expectation increases by 42
4. $E[X + Y] = E[X] + E[Y]$
5. If X and Y are independent, then $E[\mathrm{XY}] = E[X]E[Y]$
6. $E[a \pm \mathrm{bX}] = a \pm E[b\,X] = a \pm b\,E[X]$
7. $E[b(a \pm \mathrm{X})] = b\,E[a \pm \mathrm{X}] = b(a \pm E[X])$

Use rule 4:

$E[x[n]] = E[A\cos(\Omega n)\cos(\Theta)] - E[A\sin(\Omega n)\sin(\Theta)]$

Use rule 5 multiple times:

$E[x[n]] = E[A]E[\cos(\Omega n)]E[\cos(\Theta)] - E[A]E[\sin(\Omega n)]E[\sin(\Theta)]$

But how can we continue from here?

We know that $\Theta \sim U(-\pi, \pi)$ so $E[\Theta] = \frac{\pi + (-\pi)}{2} = \frac{0}{2} = 0$

b) Determine the ACVS $c_X[m, n]$

c) Comment on the stationarity of the random process

[✔] Problem 13.13: MSE objective function

13. Consider the mse objective function (13.56)

$$J(a, b) = \mathrm{E}[(Y - aX - b)^2].$$

a) Express the objective function in terms of its parameters

(a) Express $J(a, b)$ in terms of the parameters a, b, and the moments of X and Y.

Use MATLAB to expand the expression inside the expected value:

```
syms a b X Y
expand((Y-a*X - b)^2)
```

ans = $X^2\, a^2 - 2\, X\, Y\, a + 2\, X\, a\, b + Y^2 - 2\, Y\, b + b^2$

So we have:

$J(a, b) = E[X^2 a^2 - 2\, X\, Y\, a + 2\, X\, a\, b + Y^2 - 2\, Y\, b + b^2]$

Let X and Y be two random variables and a and b be two constants. Following expectation identities apply:

1. $E[a] = a$ e.g. $E(42) = 42$
2. $E[a\,X] = a\,E[X]$ e.g. if you multiply every value by 2, the expectation doubles
3. $E[a \pm X] = a \pm E[X]$ e.g. if you add 42 to every case, the expectation increases by 42
4. $E[X + Y] = E[X] + E[Y]$
5. If X and Y are independent, then $E[\mathrm{XY}] = E[X]E[Y]$

Use rule 4:

$J(a, b) = E[X^2 a^2 - 2\, X\, Y\, a + 2\, X\, a\, b + Y^2 - 2\, Y\, b + b^2]$
$J(a, b) = E[X^2 a^2] - E[2\, X\, Y\, a] + E[2\, X\, a\, b] + E[Y^2] - E[2\, Y\, b] + E[b^2]$

Use rule 1 and rule 2:

$J(a, b) = a^2 E[X^2] - 2\, a\, E[\, X\, Y] + 2\, a\, b\, E[X] + E[Y^2] - 2\, b\, E[Y] + b^2$

b) Using partial derivatives to determine the values of parameters

(b) Using partial derivatives $\dfrac{\partial J}{\partial a}$ and $\dfrac{\partial J}{\partial b}$, determine the values of a and b by solving the equations $\partial J/\partial a = 0$ and $\partial J/\partial b = 0$ that minimize $J(a, b)$ to obtain optimum values given in (13.58) and (13.62).

First, take the partial derivatives:

$$\frac{\partial J(a,b)}{\partial a} = 2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X]$$

$$\frac{\partial J(a,b)}{\partial b} = 2\,a\,E[X] - 2\,E[Y] + 2b$$

Next, solve the equations:

(Eq. 1) $2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X] = 0$

(Eq. 2) $2\,a\,E[X] - 2\,E[Y] + 2b = 0$

Isolate b in (Eq. 2):

$$2b = -2\,a\,E[X] + 2\,E[Y]$$
$$b = -a\,E[X] + E[Y]$$
$$b = E[Y] - a\,E[X]$$

This corresponds to (13.58) in the book:

(13.58) $\quad b_o = m_y - am_x$

Now, plug the expression for b into Eq. 1 in order to find an expression for a:

$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X] = 0$$
$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,(E[Y] - a\,E[X])\,E[X] = 0$$
$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,E[X]E[Y]\ - 2\,a\,E[X]E[X] = 0$$
$$2a\,E[X^2] - 2\,a\,E[X]E[X] - 2\,E[X\,Y] + 2\,E[X]E[Y]\ = 0$$
$$2a(\,E[X^2] - E[X]E[X]) - 2\,E[X\,Y] + 2\,E[X]E[Y]\ = 0$$
$$2a(\,E[X^2] - E[X]E[X]) = 2\,E[X\,Y] - 2\,E[X]E[Y]$$
$$a(\,E[X^2] - E[X]E[X]) = E[X\,Y] - E[X]E[Y]$$
$$a = \frac{E[X\,Y] - E[X]E[Y]}{E[X^2] - E[X]E[X]} = \frac{E[X\,Y] - E[X]E[Y]}{E[X^2] - E[X]^2}$$

We have found an expression for a. The numerator looks like it is the covariance:

Covariance The *covariance* of two random variables X and Y is defined by

(13.25) $\quad c_{xy} \triangleq \text{cov}(X,Y) \triangleq \text{E}[(X - m_x)(Y - m_y)] \ = \text{E}(XY) - \text{E}(X)\text{E}(Y)$

The denominator looks like it is the variance:

(13.11) $\quad \text{var}(X) = E[X^2] - E[X]^2 = E[X^2] - m_x^2$

Therefore, the derived expression is the same as (13.62) in the book.

(13.62) $\quad a_o = \dfrac{c_{xy}}{\sigma_x^2} = \rho_{xy}\dfrac{\sigma_y}{\sigma_x}$

Problem 13.22: Computing probabilities

22. Consider two jointly distributed random variables X and Y with pdf

$$f(x,y) = \begin{cases} 8xy, & 0 \le x \le 1, 0 \le y \le x \\ 0. & \text{otherwise} \end{cases}$$

a) Determine prior and conditional probabilities

(a) Determine $f(x)$, $f(y)$, $f(x|y)$, and $f(y|x)$.

b) Are X and Y independent?

(b) Are X and Y independent?