Probability

Table of Contents

Statistics.......2
Parameter vs. Statistic.......2
Estimators.......2
Estimator Evaluation.......2
Bias.......3
Variance.......3
Mean Square Error.......3
Bias-variance tradeoff.......3
Consistency.......4
Probability.......4
Random variables.......4
Statistical averages.......5
Expectation.......5
Expection rules.......5
Variance.......5
Variance Rules.......6
Standard Deviation.......6
Probability Distributions (Probability Density Functions).......6
Useful Distribution: Gaussian or Normal Distribution.......7
Generating a realisation in MATLAB.......8
Properties of normal random variables.......8
Useful Distribution: Uniform Distribution.......9
Generate realisation in MATLAB.......9
Jointly distributed random variables.......10
Convariance.......10
Correlation.......11
[✔] Quiz: sketch the probability density function of a sine wave.......11
[✔] Quiz: sketch a realisation of a discrete-time random process.......13
[✔] Quiz: compute the expectation of a random variable given probability mass function.......13
[✔] Quiz: find the probability density function from a realisation signal.......14
Problem 13.13: Decompose expectation (MSE objective function).......16
a) Express the objective function in terms of its parameters.......16
b) Using partial derivatives to determine the values of parameters.......17
Problem 13.22: Marginal and conditional distributions.......18
a) Determine marginal distributions and conditional probabilities.......18
b) Determine if two random variables are independent.......20
ADSI Problem 4.1: Discrete distribution (Poisson).......21
1) Sketch the density function.......21
2) Calculate the probability of measuring 4 or 5 events in an experiment.......22
3) Calculate the probability of measuring 3 or more events in an experiment.......22
ADSI Problem 4.3: Realisation a discrete random process.......23
1) Draw a realisation of a random process.......23
2) Is the random process is deterministic or non-deterministic?.......24
3) Plot realisations of the random process in MATLAB.......25
Exam 2018 Problem 5: Given PDS function, marginal, correlations.......26
[✔] 1) Show that the probability density function is valid.......26
[✔] 2) Calculate the marginal probabiltiy density function.......27
[✔] 3) Without calculations, argue whether var(X) or var(Y) is larger.......28
[✔] 4) Show that X and Y are uncorrelated.......28
Exam 2018 Problem 5: Continuous Probability Density Function.......30
[✔] 1) Find a valid probability density function.......30
[✔] 2) Compute the probability given a probability density function?.......31

[✔] 3) Compute the expected value a function: 32
Exam 2016 Problem 4: Discrete Random Process 32
1) Sketch a realization of a signal with this density function 33
2) Compute marginal density function 34
3) Determine whether signal is deterministic or random by calculating the correlation 36
Functions 36

Statistics

Parameter vs. Statistic

In statistics, a **parameter** denoted θ is a numerical value that states something about the entire population. Since a parameter is difficult to obtain, we use a statistic. A **statistic** is a numerical value that states something about a small sample of the entire population. The value of a parameter is a fixed number whereas a statistic can vary because it depends on the given sample.

Estimators

An **estimator** denoted $\hat{\theta}$ is a rule or a formula that can approximate some parameter θ given a sample of N observations.

For example, to estimate the mean value m of a random variable X, we can use the arithmetic average as an estimate:

$$\hat{m} = \frac{1}{N}\sum_{k=1}^{N} x_k.$$

This is called the **sample mean** estimator.

Sample variance

$$\hat{\sigma}^2 \triangleq \frac{1}{N}\sum_{k=1}^{N}(x_k - \hat{m})^2$$

Sample covariance

$$\hat{\sigma}_{xy} \triangleq \frac{1}{N}\sum_{k=1}^{N}(x_k - \hat{m}_x)(y_k - \hat{m}_y)$$

Estimator Evaluation

The quality of an estimator $\hat{\theta}$ of a parameter θ is evaluated based on three properties:

1. Bias
2. Variance
3. Mean square error

Bias

$$B(\hat{\theta}) \triangleq \mathrm{E}(\hat{\theta}) - \theta. \qquad (14.3)$$

When $B(\hat{\theta}) = 0$ then the estimator yields the correct value on average when used a large number of time. An estimator with $B = 0$ is called **unbiased** estimator.

Variance

$$\mathrm{var}(\hat{\theta}) = \mathrm{E}\{[\hat{\theta} - \mathrm{E}(\hat{\theta})]^2\} \qquad (14.4)$$

The variance describes the spread of the estimates about its expected value.

Mean Square Error

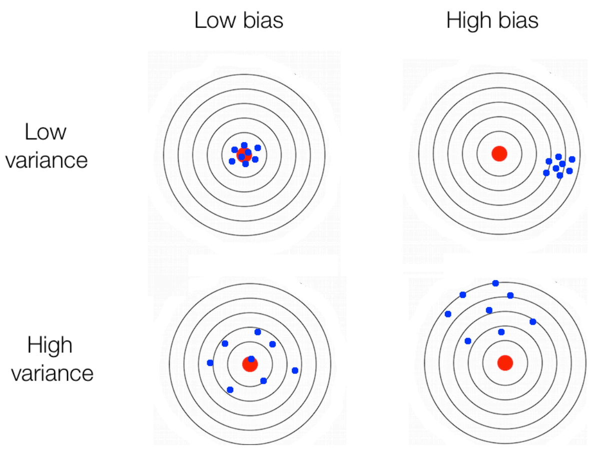
measures the average deviation of the estimator $\hat{\theta}$ from the true value θ:

$$\mathrm{mse}(\hat{\theta}) = \mathrm{E}[(\hat{\theta} - \theta)^2]. \qquad (14.5)$$

By rewrite Eq. (14.5) we see that the mean square error takes into consideration both ***the variance and the bias*** of the estimator:

$$\mathrm{mse}(\hat{\theta}) = \mathrm{var}(\hat{\theta}) + B^2(\hat{\theta}). \qquad (14.6)$$

Bias-variance tradeoff



A good estimator should have zero bias and the smallest possible variance. However, typically these two quantities conflict with each other. This is know as the bias-variance tradeoff.

Consistency

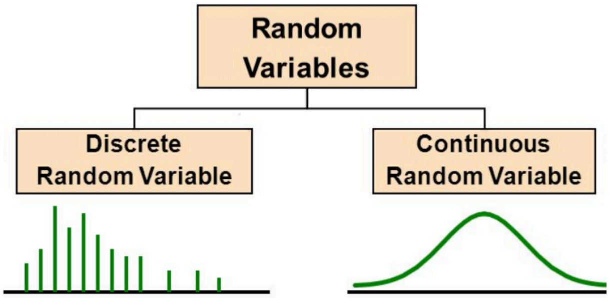
An estimator is said to be **consistent** if its estimate converges to the true value θ as the number of observations N approaches infinity.

From Eq. (14.6), we see that if variance and the bias of the estimator is zero then the estimator converges to the true value.

$$\mathrm{mse}(\hat{\theta}) = \mathrm{var}(\hat{\theta}) + B^2(\hat{\theta}). \qquad (14.6)$$

Therefore, a sufficient condition for a consistent estimator is that boths its variance and bias converge to zero as the number of observations becomes very large.

Probability



Random variables

A random variable is a variable that can take on a numerical value determined by the outcome of a random experiment.

A random variable is defined on a sample space S.

A random variable $X(\zeta)$ is a function that assigns a real number to any outcome ζ i.e. $X : S \to \mathbb{R}$. For simplicity, we often drop the dependence on the outcome and write X.

We use capital letters to denote random variables and lower-case letters for values they may take. For example, $X = x$ means that X takes on the value x.

A ***discrete*** random variable can take values from a finite or countably infinite set of numbers.

A ***continuous*** random variable can take values from any real number in a specified range.

Statistical averages

Expectation

The ***mean value*** or ***expectation*** of a random variable X is defined as:

$$m_x \triangleq \mathrm{E}(X) \triangleq \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x. \tag{13.7}$$

The mean value is an average of the values that a random variable takes on, where each value is weighted by the probability that the random variable is equal to that value.

The mean value is a measure of where the values of the random variable X are centered.

Expection rules

Let X and Y be two random variables and a and b be two constants. Following expectation properties apply:

1. $E[a] = a$ e.g. $E(42) = 42$
2. $E[a\,X] = a\,E[X]$ e.g. if you multiply every value by 2, the expectation doubles
3. $E[a \pm X] = a \pm E[X]$ e.g. if you add 42 to every case, the expectation increases by 42
4. $E[X + Y] = E[X] + E[Y]$
5. If X and Y are independent, then $E[\mathrm{XY}] = E[X]E[Y]$
6. $E[a \pm \mathrm{bX}] = a \pm E[b\,X] = a \pm b\,E[X]$
7. $E[b(a \pm \mathrm{X})] = b\,E[a \pm \mathrm{X}] = b(a \pm E[X])$
8. $E[a\,X + b\,Y] = a\,E[X] + b\,E[Y]$ (using rule 2 and 4)

Variance

The ***variance*** of a random variable X is defined as:

$$\sigma_x^2 \triangleq \mathrm{var}(X) \triangleq \mathrm{E}[(x - m_x)^2] = \int_{-\infty}^{\infty} (x - m_x)^2 f_X(x)\mathrm{d}x, \tag{13.8}$$

Another way to compute the variance is:

(13.11) $$\mathrm{var}(X) = E[X^2] - (E[X])^2 = E[X^2] - m_x^2$$

The variance measures the spread of the distribution about its mean value.

A small variance indicates that X is more likely to assume values close to its mean.

A large variance indicates that the values of X are spread over a wider interval about the mean.

The variance is used as a measure of variability of a random variable.

Variance Rules

Let X and Y be two random variables and a and b be two constants. Following variance identities apply:

1. $\mathrm{var}(a) = 0$
2. $\mathrm{var}(a \pm X) = \mathrm{var}(X)$
3. $\mathrm{var}(b\,X) = b^2 \cdot \mathrm{var}(X)$
4. $\mathrm{var}(a \pm b\,X) = b^2 \cdot \mathrm{var}(X)$
5. If X and Y are indepedent i.e. $\mathrm{cov}(X, Y) = 0$ then $\mathrm{var}(X \pm Y) = \mathrm{var}(X) + \mathrm{var}(Y)$
6. If X and Y are indepedent $\mathrm{var}(a\,X + b\,Y) = a^2\,\mathrm{var}(X) + b^2\,\mathrm{var}(Y)$

Standard Deviation

The ***standard deviation*** is a measure of spread which has the same units as the original observations:

(13.9) $$\sigma_x \triangleq \sqrt{\mathrm{var}(X)}.$$

Probability Distributions (Probability Density Functions)

We can study the nature of a random variable by its probability distribution.

A probability distribution can constructed by creating a *histogram* of a large set of observations.

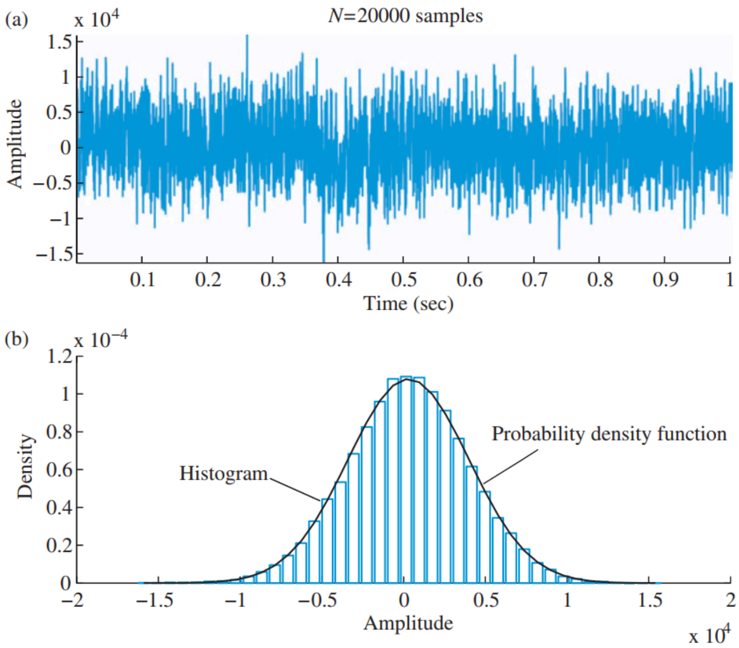


Figure 13.4 (a) Waveform of F-16 noise recorded at the co-pilot's seat with a sampling rate of 19.98 kHz using a 16 bit ADC. (b) Histogram and theoretical probability density function.

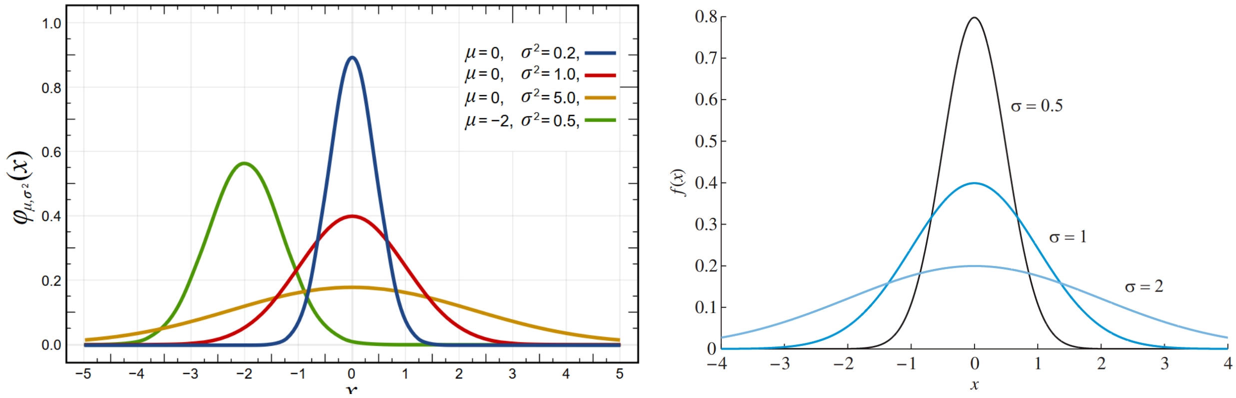
A probability distribution of a continuous random variable X denoted $f_X(x)$ is called the ***probability density function*** (PDF).

Note! that the function $f_X(x)$ does not represent the probability of any event. It is only when the function is integrated between two points that it yields a probability.

$$\Pr(a_1 < x < a_2) = \int_{a_1}^{a_2} f_X(x)\mathrm{d}x \tag{13.3}$$

Useful Distribution: Gaussian or Normal Distribution

A random variable X normally distributed with mean m and variance σ^2 is denoted $X \sim N(m, \sigma^2)$.



The probability density function of a normal distribution for various values of the standard deviation. The spread of the distribution increases with increasing σ; the height is reduced because the area is always equal to unity.

Its probability density function is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}. \quad -\infty < x < \infty$$

The mean is

$$E(X) = m$$

The variance is:

$$\text{var}(X) = \sigma^2$$

Generating a realisation in MATLAB

To generate N random numbers for a Gaussian random variable with mean μ and variance σ^2 we use following MATLAB functions

```
clear variables;

N = 1000;
variance = 25;
std_dev = sqrt(variance);
mu = 500;
w = std_dev.*randn(N,1) + mu;

% Calculate the sample mean, standard deviation, and variance.
stats = [mean(w) std(w) var(w)]
```

```
stats = 1×3
  500.1909    5.1244   26.2591
```

Properties of normal random variables

Any linear combination of normal random variables is itself a normal random variable.

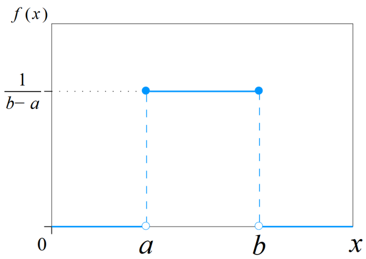
An important property of normal random variables is that if X is normal with mean m and variance σ^2, then $aX + b$ is normally distributed with mean $am + b$ and variance $a^2\sigma^2$ (see Tutorial Problem 4), that is

$$X \sim N(m, \sigma^2) \Rightarrow Y = aX + b \sim N(am + b, a^2\sigma^2). \tag{13.17}$$

Central Limit Theorem: the sum of a large number of independent random variables has approximately a normal distribution.

Useful Distribution: Uniform Distribution

A random variable X uniformly distributed over the interval (a, b) is denoted by $X \sim U(a, b)$.



The probability density function is given by:

$$f_X(x) = \begin{cases} 1/(b-a), & \text{if } a < x < b \\ 0. & \text{otherwise} \end{cases}$$

The mean is

$$E(X) = \frac{b + a}{2}$$

The mean of squared is:

$$E(X^2) = \frac{a^2 + b^2 + \mathrm{ab}}{3}$$

The variance is:

$$\mathrm{var}(X) = E(X^2) - [E(X)]^2 = \frac{1}{12}(b - a)^2$$

Generate realisation in MATLAB

The MATLAB function **rand** generates random numbers according to $X \sim U(0, 1)$

By default, `rand` returns normalized values (between 0 and 1) that are drawn from a uniform distribution. To change the range of the distribution to a new range, (*a*, *b*), multiply each value by the width of the new range, (*b* – *a*) and then shift every value by *a*.

```
a = 50;
b = 100;
r = (b-a).*rand(4, 1) + a
```

```
r = 4×1
    82.2744
    69.2663
    54.8620
    98.1518
```

Generate a random value for the uniform distribution $\phi \sim U(0, 2\pi)$:

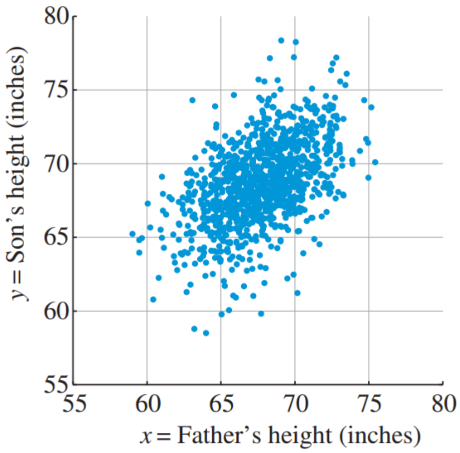
```
phi = 2*pi*rand()
```

```
phi = 5.5722
```

Jointly distributed random variables

Convariance

Covariance provides a measure of the association between two random variables X and Y. The covariance can be used to summarise the information provided by the following scatter plot:

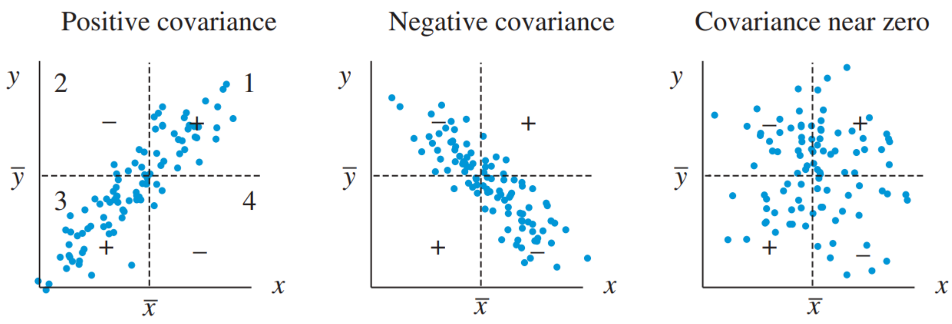


Covariance The *covariance* of two random variables X and Y is defined by

$$c_{xy} \triangleq \mathrm{cov}(X, Y) \triangleq \mathrm{E}[(X - m_x)(Y - m_y)] = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) \tag{13.25}$$

Two random variables with zero covariance are said to be *uncorrelated*. Thus,

$$X, Y \text{ uncorrelated } \Leftrightarrow \operatorname{cov}(X, Y) = 0 \text{ or } \mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y). \tag{13.27}$$



(a) (b) (c)

a) The relationship between *X* and *Y* is positive means that as *X* increases so does *Y*

b) The relationship between *X* and *Y* is negative meaning that as *X* increases, *Y* will decrease.

c) The random variables$X$ and Y are not related i.e. uncorrelated.

Correlation

The correlation between two random variables X and Y is defined as:

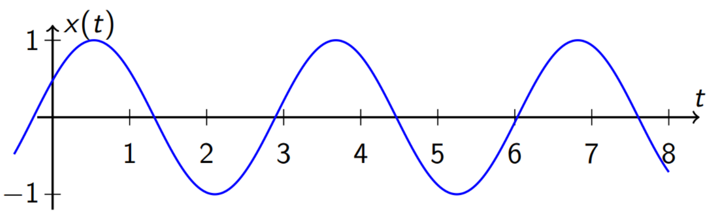
$$r_{xy} = E[\mathrm{XY}]$$

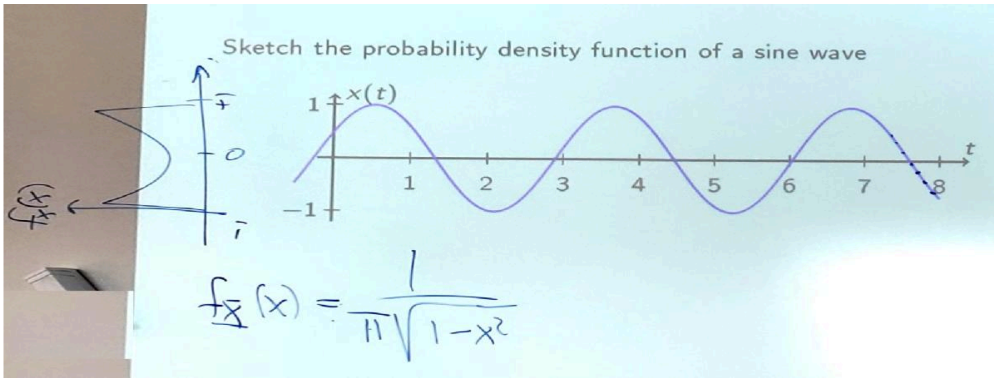
Correlation and covarance may create some confusion. Zero correlation implies that X and Y are orthogonal whereas zero covariance implies that X and Y are uncorrelated.

Table 13.1 Relationship between orthogonal and uncorrelated random variables.

Correlation:	$r_{xy} = \mathrm{E}(XY)$	$r_{xy} = 0 \Rightarrow X, Y$ orthogonal
Covariance:	$c_{xy} = \mathrm{E}[(X - m_x)(Y - m_y)]$	$c_{xy} = 0 \Rightarrow X, Y$ uncorrelated

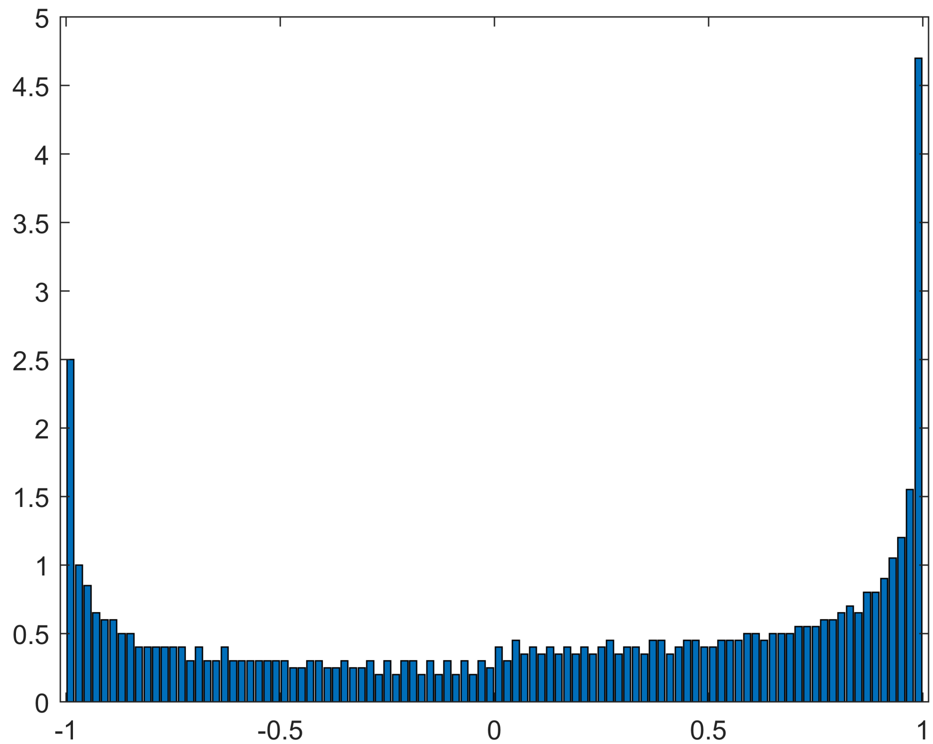
[✔] Quiz: sketch the probability density function of a sine wave





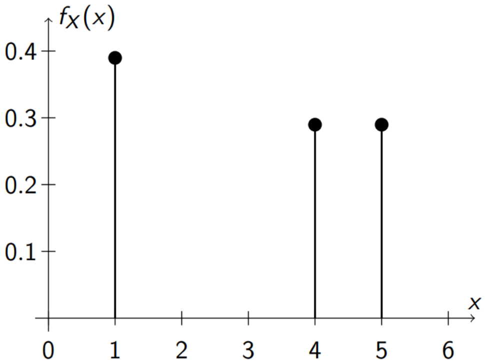
Try it out in MATLAB:

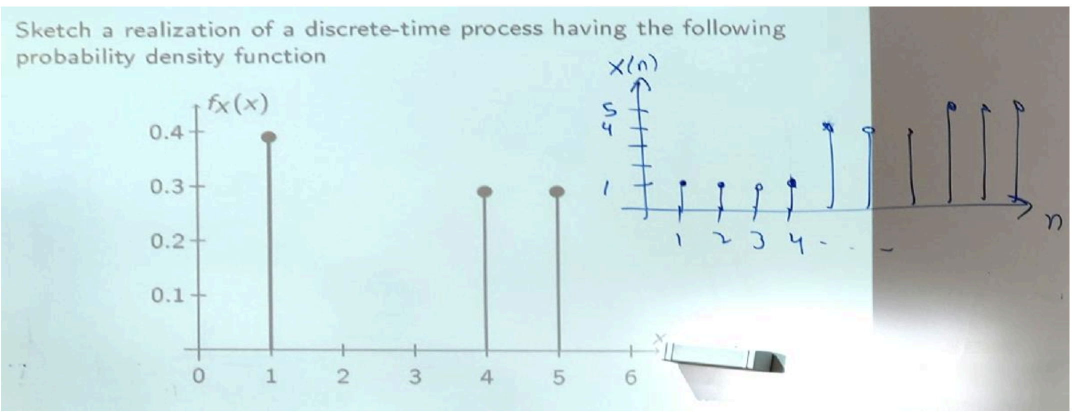
```
n = linspace(0, 8, 1000);
y = sin(n);
[xo, px] = epdf(y, 100);
bar(xo, px);
```



[✔] Quiz: sketch a realisation of a discrete-time random process

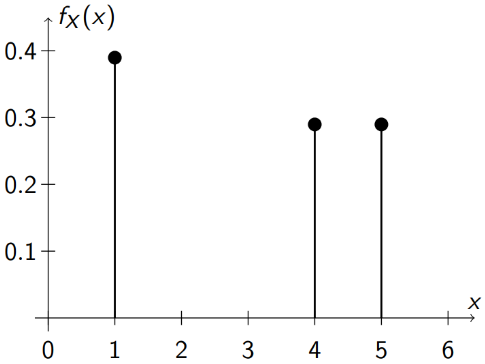
Sketch a realisation of a discrete-time process having the following probability density function





[✔] Quiz: compute the expectation of a random variable given probability mass function

Calculate the mean value of a random variable with the following probability density function:



We can write the probability density function as:

$$f_X(x) = \frac{4}{10}\delta[x-1] + \frac{0}{10}[x-2] + \frac{0}{10}[x-3] + \frac{3}{10}[x-4] + \frac{3}{10}[x-5]$$

$$= \frac{4}{10}\delta[x-1] + \frac{3}{10}[x-4] + \frac{3}{10}[x-5]$$

Mean value

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$E[X] = \mu = m_X = \bar{x} = \int_{-\infty}^{\infty} x f_X(x)dx$$

$$f_X(x) = 0.4\,\delta(x-1) + 0.3\,\delta(x-4) + 0.3\,\delta(x-5)$$

$$E[X] = \int_{-\infty}^{\infty} x\left(0.4\,\delta(x-1) + 0.3\,\delta(x-4) + 0.3\,\delta(x-5)\right)dx = 0.4\cdot 1 + 0.3\cdot 4 + 0.3\cdot 5 = 3.1$$

[✔] Quiz: find the probability density function from a realisation signal

For this problem we will use two random variales.

The random variable X_0 tells us the value of a signal at time t_0 whereas X_1 tells us the value of the same signal at time $t_0 + 1$ i.e., one time unit later. We are making two measurements at the same signal.

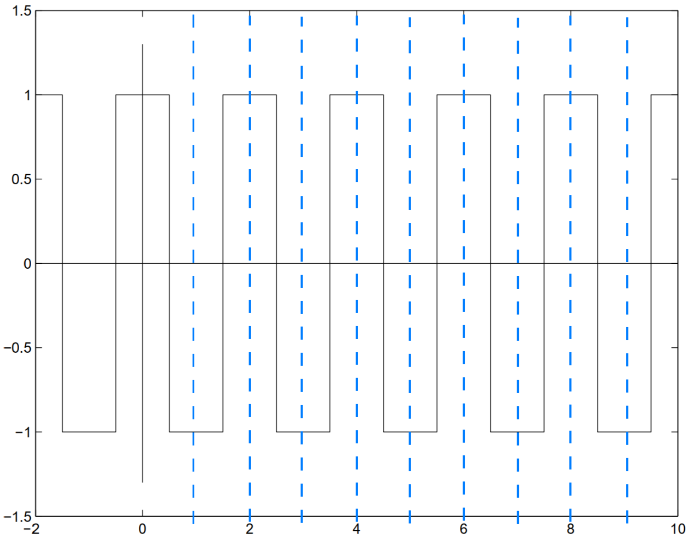
We use a generic formula for the joint PDF:

$$f_{X_0,X_1}(x_0, x_1) = p_0\delta[x_0+1]\delta[x_1+1] + p_1\delta[x_0+1]\delta[x_1-1] + p_2\delta[x_0-1]\delta[x_1+1] + p_3\delta[x_0-1]\delta[x_1-1]$$

where:

1. p_0 is the probability that if the signal is +1 at then it will remain +1 at $t_0 + 1$
2. p_1 is the probability that if the signal is +1 at then it will change -1 at $t_0 + 1$
3. p_2 is the probability that if the signal is -1 at then it will change +1 at $t_0 + 1$
4. p_3 is the probability that if the signal is -1 at then it will remain -1 at $t_0 + 1$

At each given signal, we make two measurements and see what is the likelihood that the signal will remain the same or change.



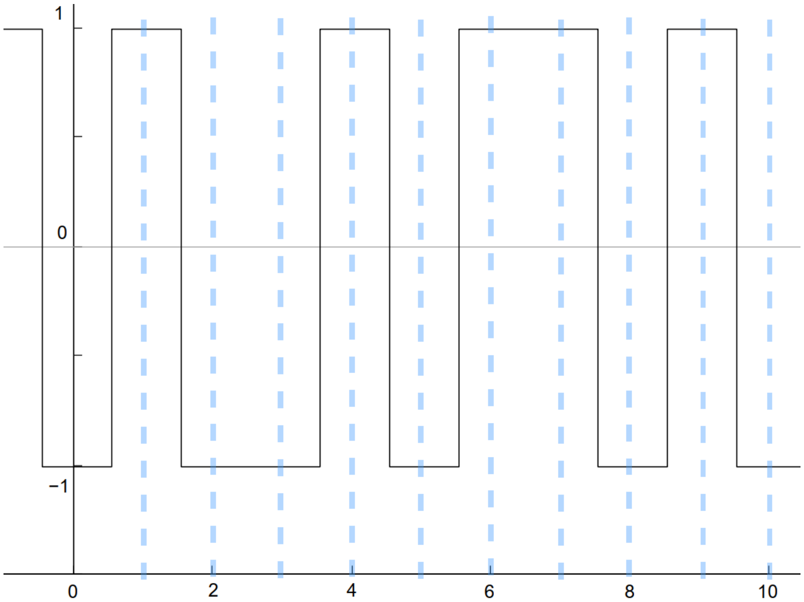
From the signal above, we make following observations:

1. $p_0 = 0$ because we never observe that when the signal is +1 at time t_0 then it will remain +1 at time $t_0 + 1$
2. $p_1 = 0.5$ because we observe that when the signal is +1 at time t_0 then it always changes to -1 at time $t_0 + 1$.
3. $p_2 = 0.5$ because when the signal is -1 at t_0 then it flips to +1 at $t_0 + 1$
4. $p_3 = 0$ because we never observe that when the signal is -1 at time t_0 then it will remain -1 at time $t_0 + 1$

Thus, the joint PDF for square signal is:

$$f_{X_0,X_1}(x_0, x_1) = \frac{1}{2}\delta[x_0 + 1]\delta[x_1 - 1] + \frac{1}{2}\delta[x_0 - 1]\delta[x_1 + 1]$$

Another more interesting problem is the signal below:



This is a short realisation so it is difficult to find the likelihoods by looking at the signal alone.

Problem 13.13: Decompose expectation (MSE objective function)

13. Consider the mse objective function (13.56)

$$J(a,b) = \mathrm{E}[(Y - aX - b)^2].$$

a) Express the objective function in terms of its parameters

(a) Express $J(a,b)$ in terms of the parameters a, b, and the moments of X and Y.

Use MATLAB to expand the expression inside the expected value:

```
syms a b X Y
expand((Y-a*X - b)^2)
```

ans = $X^2 a^2 - 2\,X\,Y\,a + 2\,X\,a\,b + Y^2 - 2\,Y\,b + b^2$

So we have:

$J(a,b) = E[X^2a^2 - 2\,X\,Y\,a + 2\,X\,a\,b + Y^2 - 2\,Y\,b + b^2]$

Let X and Y be two random variables and a and b be two constants. Following expectation identities apply:

1. $E[a] = a$ e.g. $E(42) = 42$
2. $E[a\,X] = a\,E[X]$ e.g. if you multiply every value by 2, the expectation doubles
3. $E[a \pm X] = a \pm E[X]$ e.g. if you add 42 to every case, the expectation increases by 42
4. $E[X + Y] = E[X] + E[Y]$
5. If X and Y are independent, then $E[\mathrm{XY}] = E[X]E[Y]$

Use rule 4:

$J(a,b) = E[X^2a^2 - 2\,X\,Y\,a + 2\,X\,a\,b + Y^2 - 2\,Y\,b + b^2]$
$J(a,b) = E[X^2a^2] - E[2\,X\,Y\,a] + E[2\,X\,a\,b] + E[Y^2] - E[2\,Y\,b] + E[b^2]$

Use rule 1 and rule 2:

$J(a,b) = a^2E[X^2] - 2\,a\,E[\,X\,Y] + 2\,a\,b\,E[X] + E[Y^2] - 2\,b\,E[Y] + b^2$

b) Using partial derivatives to determine the values of parameters

(b) Using partial derivatives $\dfrac{\partial J}{\partial a}$ and $\dfrac{\partial J}{\partial b}$, determine the values of a and b by solving the equations $\partial J/\partial a = 0$ and $\partial J/\partial b = 0$ that minimize $J(a,b)$ to obtain optimum values given in (13.58) and (13.62).

First, take the partial derivatives:

$$\frac{\partial\,J(a,b)}{\partial\,a} = 2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X]$$

$$\frac{\partial\,J(a,b)}{\partial\,b} = 2\,a\,E[X] - 2\,E[Y] + 2b$$

Next, solve the equations:

(Eq. 1) $2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X] = 0$

(Eq. 2) $2\,a\,E[X] - 2\,E[Y] + 2b = 0$

Isolate b in (Eq. 2):

$2b = -2\,a\,E[X] + 2\,E[Y]$
$b = -a\,E[X] + E[Y]$
$b = E[Y] - a\,E[X]$

This corresponds to (13.58) in the book:

(13.58) $\qquad b_{\mathrm{o}} = m_y - am_x$

Now, plug the expression for b into Eq. 1 in order to find an expression for a:

$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X] = 0$$
$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,(E[Y] - a\,E[X])\,E[X] = 0$$
$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,E[X]E[Y]\ - 2\,a\,E[X]E[X] = 0$$
$$2a\,E[X^2] - 2\,a\,E[X]E[X] - 2\,E[X\,Y] + 2\,E[X]E[Y]\ = 0$$
$$2a(\,E[X^2] - E[X]E[X]) - 2\,E[X\,Y] + 2\,E[X]E[Y]\ = 0$$
$$2a(\,E[X^2] - E[X]E[X]) = 2\,E[X\,Y] - 2\,E[X]E[Y]$$
$$a(\,E[X^2] - E[X]E[X]) = E[X\,Y] - E[X]E[Y]$$
$$a = \frac{E[X\,Y] - E[X]E[Y]}{E[X^2] - E[X]E[X]} = \frac{E[X\,Y] - E[X]E[Y]}{E[X^2] - E[X]^2}$$

We have found an expression for a. The numerator looks like it is the covariance:

Covariance The *covariance* of two random variables X and Y is defined by

$$c_{xy} \triangleq \mathrm{cov}(X, Y) \triangleq \mathrm{E}[(X - m_x)(Y - m_y)] = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) \quad (13.25)$$

The denominator looks like it is the variance:

$$\mathrm{var}(X) = E[X^2] - E[X]^2 = E[X^2] - m_x^2 \quad (13.11)$$

Therefore, the derived expression is the same as (13.62) in the book.

$$a_o = \frac{c_{xy}}{\sigma_x^2} = \rho_{xy}\frac{\sigma_y}{\sigma_x} \quad (13.62)$$

Problem 13.22: Marginal and conditional distributions

22. Consider two jointly distributed random variables X and Y with pdf

$$f(x, y) = \begin{cases} 8xy, & 0 \le x \le 1, 0 \le y \le x \\ 0. & \text{otherwise} \end{cases}$$

```
clear variables;
```

a) Determine marginal distributions and conditional probabilities

(a) Determine $f(x), f(y), f(x|y)$, and $f(y|x)$.

```
syms x y
```

```
fXY = 8*x*y;
```

The **marginal** distributions of random variables X and X are obtained by integration as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}y \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}x, \qquad (13.19)$$

```
fX = int(fXY, y, 0, x)
```

fX = $4\,x^3$

```
fY = int(fXY, x, 0, 1)
```

fY = $4\,y$

To compute $f(x|y)$ we use following relation:

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y). \qquad (13.23)$$

From Eq. 13.23, we know that:

$$f(x|y) = \frac{f(x,y)}{f(y)} \quad \text{and} \quad f(y|x) = \frac{f(x,y)}{f(x)}$$

```
fX_given_Y = fXY / fY
```

fX_given_Y = $2\,x$

```
fY_given_X = fXY / fX
```

fY_given_X =

$$\frac{2\,y}{x^2}$$

We can do it by hand too.

Compute the marginal distribution of X:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y$$

Since $f(x,y)$ has non-zero values for when $y \geq 0$ and $y \leq x$ i.e., y is between 0 and x, then we only need to find the integral w.r.t. y from 0 to x.

$$f(x) = \int_0^x 8xy\,\mathrm{d}y = \left[\frac{1}{2}8xy^2\right]_0^x = \frac{1}{2}8xx^2 - 0 = 4x^3$$

where $0 \leq x \leq 1$.

Compute the marginal distribution of Y:

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{dx}$$

We are given that $f(x, y)$ is non-zero when $x \geq 0$ and $x \leq 1$. Since we are trying to find the integral w.r.t. y, we integrate from y to 1.

$$f(y) = \int_{y}^{1} 8xy\,\mathrm{dx} = [4x^2y]_y^1 = 4y - 4y^3$$

where $0 \leq y \leq 1$

To compute $f(x|y)$ we use following relation:

$$f_{X,Y}(x, y) = f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y). \qquad (13.23)$$

From Eq. 13.23, we know that:

$$f(x|y) = \frac{f(x, y)}{f(y)} = \frac{8\mathrm{xy}}{4y - 4y^3} = \frac{4y \cdot 2x}{4y(1 - y^2)} = \frac{2x}{1 - y^2}$$

and

$$f(y|x) = \frac{f(x, y)}{f(x)} = \frac{8x\,y}{4x^3} = \frac{4x \cdot 2y}{4x \cdot x^2} = \frac{2y}{x^2}$$

b) Determine if two random variables are independent

(b) Are X and Y independent?

Random variables X and Y are statistically independent, if

$f(y|x) = f(y)$ or $f(x|y) = f(x)$.

In a) we have computed the following expressions:

- $f(x) = 4x^3$
- $f(y) = 4y - 4y^3$
- $f(y|x) = \dfrac{2y}{x^2}$
- $f(x|y) = \dfrac{2x}{1 - y^2}$

Clearly $f(y|x) \neq f(y)$ and $f(x|y) \neq f(x)$. Therefore, the answer is no! The random variables X and Y are not statistically independent.

ADSI Problem 4.1: Discrete distribution (Poisson)

The Poisson distribution is a discrete probability distribution that is used in model counting events i.e. pixel noise in CCD cameras and in particle detectors. The density function for the Poisson distribution is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{a^k e^{-a}}{k!} \delta(x - k).$$

Where a is a fixed positive constant. The density function gives the probability of observing a particular number of events (x) in a given experiment when these events occur with a known average number of events per experiment (a) and the events are independent of one another. Let $a = 3$.

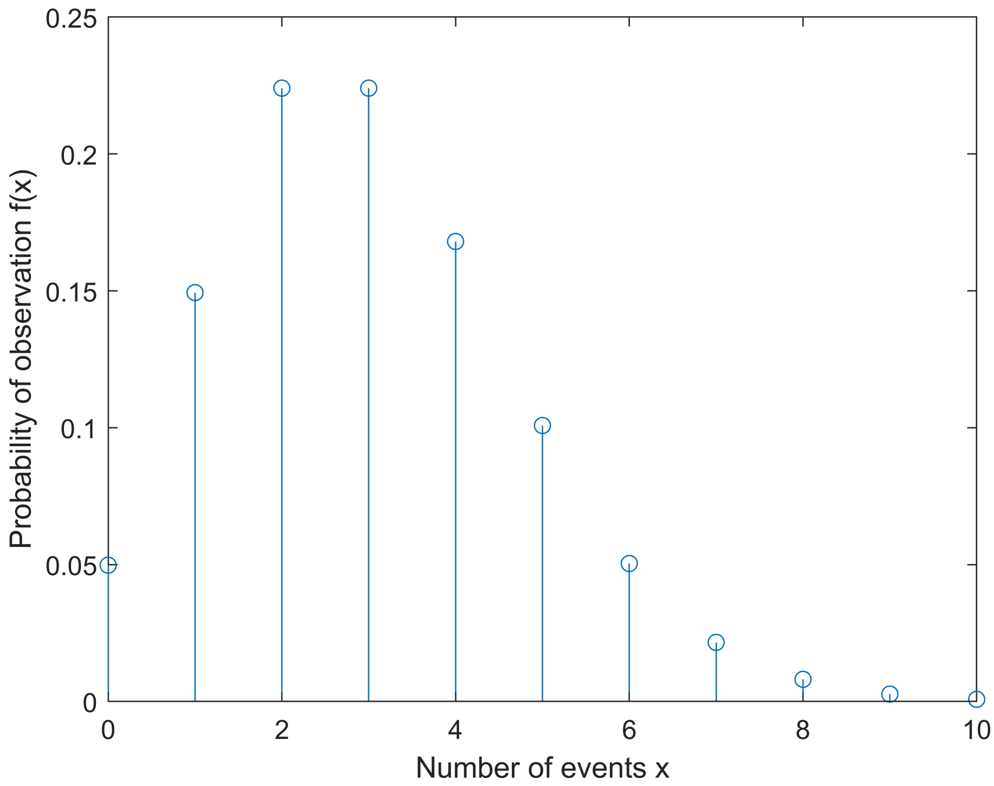
```
clear variables;
```

1) Sketch the density function

Use MATLAB to plot the density function. The probability mass function for the poisson distribution is:

$f(k) = \frac{a^k}{k!} \exp(-a)$

```
x = 0:10;
a = 3;
f = @(k) (exp(-a) * a.^k)/factorial(k);
%stem(x, f(x))
% applies the function f to the elements of x
y = arrayfun(f, x);
stem(x, y);
xlabel("Number of events x")
ylabel("Probability of observation f(x)")
```



2) Calculate the probability of measuring 4 or 5 events in an experiment.

When we measure, we cannot get both 4 and 5 at the same time. This means that the event of measuring 4 and the event of measuring 5 are mutually exclusive since these two events cannot happen at the same time.

Therefore, the probability that one of the mutually exclusive events occur is the sum of their individual probabilities.

$$\Pr(x = 4 \text{ or } x = 5) = \Pr(x = 4) + \Pr(x = 5) = f(4) + f(5)$$

```
f(4)+f(5)
```

```
ans = 0.2689
```

Final answer is: $\Pr(x = 4 \text{ or } x = 5) = 0.2689$

3) Calculate the probability of measuring 3 or more events in an experiment.

To compute the probability of measuring 3, or more events:

$$\Pr(x \geq 3) = \sum_{k=3}^{\infty} f(k)$$

Since we have an infinite sum, it is a bit difficult to compute. Instead, we can use on the fact that the sum over the mass function is 1. This means that we only need to compute the probabilites of $1 - (f(0) + f(1) + f(2))$ is the same as computing the infinite sum:

$$\Pr(x \geq 3) = 1 - \sum_{k=0}^{2} f(k)$$

Let us do it in MATLAB:

```
1-(f(0) + f(1) + f(2))
```

```
ans = 0.5768
```

The final answer is: $\Pr(x \geq 3) = 0.5768$

ADSI Problem 4.3: Realisation a discrete random process

Let two random variables, X_1 and X_2 be defined from a random process $X(t)$ as $X_1 = X(t_1)$ and $X_2 = X(t_1 + 1)$. The joint density function for the two random variables is given by

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{8}\delta(x_1 - 1)\delta(x_2 - 1) + \frac{3}{8}\delta(x_1 - 1)\delta(x_2 + 1) + \frac{3}{8}\delta(x_1 + 1)\delta(x_2 - 1) + \frac{1}{8}\delta(x_1 + 1)\delta(x_2 + 1).$$

The random variable X_1 tells us the value of a signal at time t_1 whereas X_2 tells us the value of the same signal at time $t_1 + 1$ i.e., one time unit later. We are making two measurements at the same signal.

```
clear variables;
```

1) Draw a realisation of a random process

1. Draw a realization of $X(t)$ and explain the considerations that you made in the preparation of the realization.

From the δ functions, we see that the signal can only take on two values: -1 and +1. This means that the realisation of the random process $X(t)$ flips between -1 and +1 i.e., $X(t) \in \{-1, 1\}$.

From each term we can read off the likelihood of the behaviour of the signal.

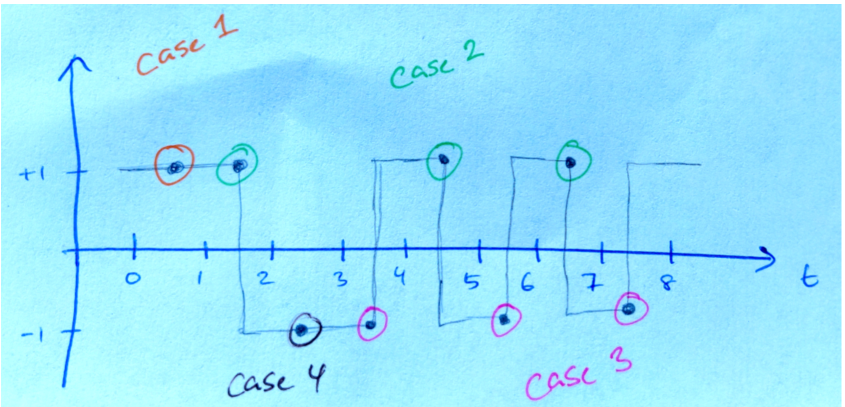
1. If the signal is +1 at t_1 then there is 1/8 probability that the signal will remain +1 at $t_1 + 1$
2. If the signal is +1 at t_1 then there is 3/8 probability that the signal will change to -1 at $t_1 + 1$
3. If the signal is -1 at t_1 then there is 3/8 probability that the signal will change to +1 at $t_1 + 1$
4. If the signal is -1 at t_1 then there is 1/8 probability that the signal will remain at -1 at $t_1 + 1$

This means that during a time window of 8 time units, the signal will be behave as follows:

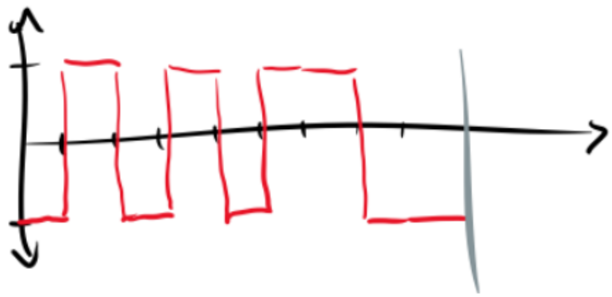
1. Given time t_0 when signal is +1, only once will the signal remain +1 at next time unit
2. Given time t_0 when signal is +1, three of the times the signal changes to -1 at next time unit

3. Given time t_0 when signal is -1, three of the times the signal changes to +1 at next time unit
4. Given time t_0 when signal is -1, only once will the signal remain -1 at next time unit

Thus, one realisation of the random process is as follows:



Another realisation of the random process is as follows:



2) Is the random process is deterministic or non-deterministic?

2. Discuss, using only the joint density function, whether the random process is deterministic or nondeterministic and state your arguments. Calculations are not called upon in this question.

At each given time instant t_0 the likelihood of the signal value changing sign at next time unit (either from +1 to -1 or from -1 to +1) is $\frac{6}{8} = \frac{3}{4}$.

And the likelihood of the signal value remaining the same is $\frac{2}{8} = \frac{1}{4}$.

So the joint probability density function only tells us that the signal will change sign or stay the same with some probability.

However, it does not say at what time instant the signal remain the same and at what time instants the signal will change.

Therefore, the random process is non-deterministic.

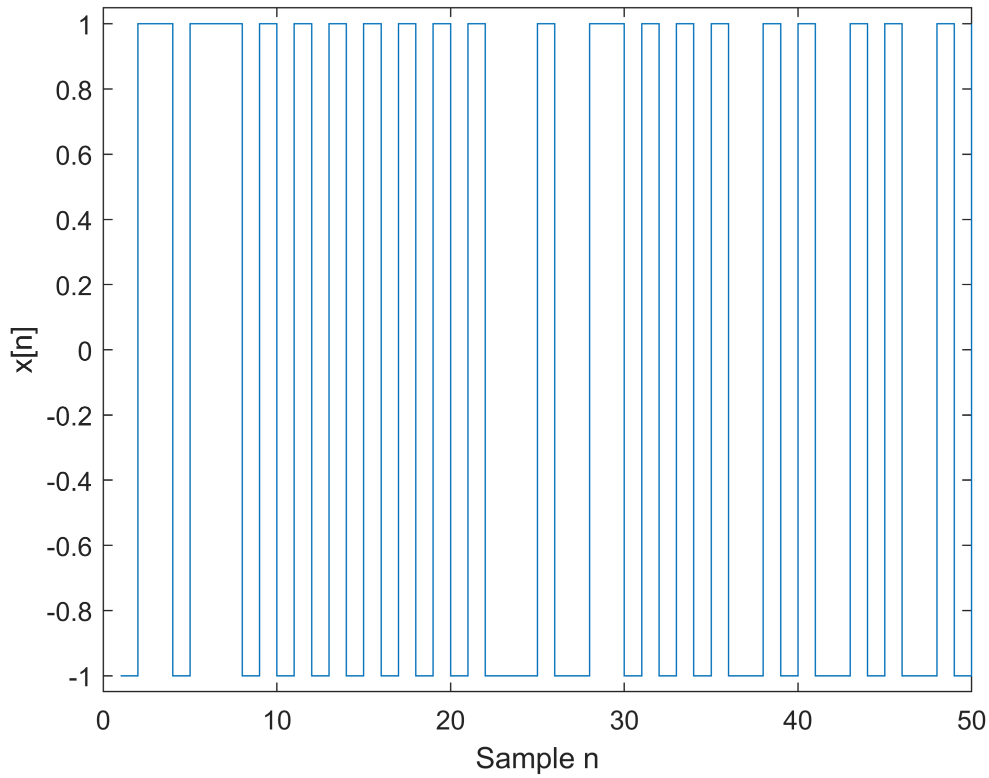
NOTE! We can make rather educated guesses for the next value of the signal so it wouldn't be completely unjustified to denote the signal as 'semi'-deterministic.

3) Plot realisations of the random process in MATLAB

3. Write a MATLAB script that can create and plot realizations of the random process.

At each given time instant t_0 the likelihood of the signal value changing sign at next time unit (either from +1 to -1 or from -1 to +1) is $\frac{6}{8} = \frac{3}{4}$. We will use this fact to implement a realisation of the random process.

```
% Setup
N = 50;
x=zeros(N,1);

%% equal probability of -1 or 1 in x(1)
if randn(1) >= 0
    x(1)=1;
else
    x(1)=-1;
end
%% change sign with 75% probability
for n=2:N
    if rand(1)<0.75
        x(n)=-x(n-1);
    else
        x(n)=x(n-1);
    end
end
%% plot result
stairs(x)
ylim([-1.05 1.05])
xlabel('Sample n')
ylabel('x[n]')
```



Exam 2018 Problem 5: Given PDS function, marginal, correlations

Consider the following joint probability density function.

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & 1 \leq y \leq 2 \text{ and } 1 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

[✔] 1) Show that the probability density function is valid

1. Show that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx\, dy = 1$.

Integrate using MATLAB:

```
syms x y f(x,y) fX(x) fY(y)
f(x,y) = 1/2;
inner = int(f(x,y), x, 1, 3);
outer = int(inner, y, 1, 2)
```

outer = 1

```
% Alternative method is to numerically evaluate double integral
fun = @(x,y) 1/2 + 0*(x+y); % Hack: 0*(x+y) is added for MATLAB
```

```
integral2(fun, 1, 3, 1, 2)
```

```
ans = 1
```

[✔] 2) Calculate the marginal probabiltiy density function

2. Calculate the marginal probability density function $f_X(x)$.

The marginal probability density function is given as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\mathrm{d}y$$

The marginal probability for X is given as:

$$f_X(x) = \int_1^2 \frac{1}{2}\,\mathrm{dy} = \left[\frac{1}{2}y\right]_1^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

Check it in MATLAB:

```
fX(x) = int(f(x,y), y, 1, 2)
```

fX(x) =

$$\frac{1}{2}$$

```
% Alternative method
fun = @(y) 1/2 + 0*y; % Hack!
integral(fun, 1, 2)
```

```
ans = 0.5000
```

The marginal probability for Y is given as:

$$f_Y(y) = \int_1^3 \frac{1}{2}\,\mathrm{dx} = \left[\frac{1}{2}x\right]_1^3 = \frac{3}{2} - \frac{1}{2} = 1$$

```
fY(y) = int(f(x,y), x, 1, 3)
```

fY(y) = 1

```
% Alternative method
fun = @(x) 1/2 + 0*x; % Hack!
integral(fun, 1, 3)
```

```
ans = 1.0000
```

[✔] 3) Without calculations, argue whether var(X) or var(Y) is larger

The variance is a measure of the spread of the distribution about its mean value. A large variance indicates that the values of the random variable are spread over a wider interval about the mean. In this problem, X has a wider span of possible values. Therefore, $\mathrm{var}(X) > \mathrm{var}(Y)$.

[✔] 4) Show that X and Y are uncorrelated

Two random variables are uncorrelated if

$$\mathrm{cov}(X, Y) = E(\mathrm{XY}) - E[X]E[Y] = 0 \quad \text{or} \quad E[\mathrm{XY}] = E[X]E[Y]$$

The simplest way to show that X and Y are uncorrelated is if the two random variables are independent.

Two random variables X and Y are said to be independent if:

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

The joint probability is given as:

$$f_{X,Y}(x, y) = \frac{1}{2}$$

The marginal probability for X is given as:

$$f_X(x) = \int_1^2 \frac{1}{2}\,\mathrm{dy} = \left[\frac{1}{2}y\right]_1^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

The marginal probability for Y is given as:

$$f_Y(y) = \int_1^3 \frac{1}{2}\,\mathrm{dx} = \left[\frac{1}{2}x\right]_1^3 = \frac{3}{2} - \frac{1}{2} = 1$$

Since they are the random variables X and Y are indepedent they are also uncorrelated.

Another approach involves computing the covariance and showing that

$$\mathrm{cov}(X, Y) = E(\mathrm{XY}) - E[X]E[Y] = 0$$

We compute $E[X]$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{dx} = \int_1^3 \frac{1}{2}x\ \mathrm{dx} = \left[\frac{1}{4}x^2\right]_1^3 = \frac{1}{4}(3)^2 - \frac{1}{4}(1)^2 = 2$$

Check in MATLAB:

```
EX = int(f(x,y)*x, x, 1, 3)
```

EX = 2

```
% Alternative method
fun = @(x) 1/2*x;
integral(fun, 1, 3)
```

ans = 2

We compute $E[Y]$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\ \mathrm{dy} = \int_1^2 y\ \mathrm{dy} = \left[\frac{1}{2}y^2\right]_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2} = \frac{3}{2}$$

```
EY = int(fY(y)*y, y, 1, 2)
```

EY =

$$\frac{3}{2}$$

```
% Alternative method
fun = @(y) y;
integral(fun, 1, 2)
```

ans = 1.5000

We compute $E[\mathrm{XY}]$:

$$E[\mathrm{XY}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\ \mathrm{dy}\ \mathrm{dx}$$

$$E[\mathrm{XY}] = \int_1^3 \left(\int_1^2 \frac{1}{2}xy\ \mathrm{dy}\right) \mathrm{dx}$$

First we compute the inner integral:

$$\int_1^2 \frac{1}{2}xy\ \mathrm{dy} = \left[\frac{1}{4}x\,y^2\right]_1^2 = \frac{1}{4}x(2)^2 - \frac{1}{4}x = \frac{3}{4}x$$

```
inner = int(x*y*f(x,y), y, 1, 2)
```

inner =

$$\frac{3\,x}{4}$$

Next, we compute the outer integral:

$$\int_1^3 \frac{3}{4}x \text{ dx} = \left[\frac{3}{8}x^2\right]_1^3 = \frac{3}{8}(3)^2 - \frac{3}{8} = \frac{27}{8} - \frac{3}{8} = \frac{24}{8} = 3$$

```
EXY = int(inner, x, 1, 3)
```

EXY = 3

Two random variables are uncorrelated if the covariance $\text{cov}(X, Y) = E[\text{XY}] - E[X]E[Y] = 0$.

```
EXY - EX * EY
```

ans = 0

Since $\text{cov}(X, Y) = E[\text{XY}] - E[X]E[Y] = 0$ then X and Y are uncorrelated.

Exam 2018 Problem 5: Continuous Probability Density Function

Consider the following probability density function

$$f_X(x) = \begin{cases} \alpha x^2 + \frac{1}{4} & \text{for} - 1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

where α is a real number.

[✔] 1) Find a valid probability density function

1. Determine α so that $f_X(x)$ is a valid probability density function.

A valid probability density function must be non-negative everywhere and the integral from minus infinity to positive infinity must equal 1

A valid probability density function is given by:

$$\int_{-\infty}^{\infty} f_X(x) \text{ dx} = 1$$

We need to compute:

$$\int_{-1}^{1} \alpha x^2 + \frac{1}{4} \text{dx}$$

For convenience and to avoid silly mistakes, use MATLAB:

```
syms x a
int(a*x^2 + 1/4, x, -1, 1)
```

ans =

$$\frac{2a}{3} + \frac{1}{2}$$

Solve the equation for a in MATLAB:

```
solve(int(a*x^2 + 1/4, x, -1, 1) - 1)
```

ans =

$$\frac{3}{4}$$

Let us check the results:

```
int(3/4 * x^2 + 1/4, x, -1, 1)
```

ans = 1

For $f_X(x)$ to be a valid probability density function, a must be:

$$a = \frac{3}{4}$$

As α is positive, the probability density function is a upward curving parabola and therefore positive in its entire domain.

[✔] 2) Compute the probability given a probability density function?

2. What is the probability that $1/4 < X \leq 3/4$?

```
p = int(3/4 * x^2 + 1/4, x, 1/4, 3/4)
```

p =

$$\frac{29}{128}$$

```
vpa(p)
```

ans = 0.2265625

The answer is:

$$\Pr\left(\frac{1}{4} < X \leq \frac{3}{4}\right) = \frac{29}{128} = 0.2265625$$

[✔] 3) Compute the expected value a function:

Let a function be given by $g(x) = e^{-|x|}$.

3. Compute $E[g(x)]$.

We need to compute:

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,\mathrm{dx} = \int_{-1}^{1} e^{-|x|} \cdot \frac{3}{4}x^2 + \frac{1}{4}\,\mathrm{dx}$$

```
syms x
f = exp(-abs(x)) * 3/4 * x^2 + 1/4
```

f =

$$\frac{3\,x^2\,e^{-|x|}}{4} + \frac{1}{4}$$

```
int(f , x, -1, 1 )
```

ans =

$$\frac{7}{2} - \frac{15\,e^{-1}}{2}$$

```
vpa(int(f , x, -1, 1 ))
```

ans = $0.74090419121418258803357172378904$

The answer is:

$$E[g(x)] \approx 0.74$$

```
fxgx=@(x) (3/4*x.^2+1/4).*exp(-abs(x));
integral(fxgx,-1,1)
```

ans = 0.5570

Exam 2016 Problem 4: Discrete Random Process

Let two random variables, X_0 and X_1 be defined from a discrete time random process $X(n)$ as $X_0 = X(n)$ and $X_1 = X(n+1)$. The joint density function for the process is given by

$$f_{X_0,X_1}(x_0,x_1) = \frac{1}{6}\delta(x_0-2)\delta(x_1+1) + \frac{2}{6}\delta(x_0-2)\delta(x_1-2) + \frac{2}{6}\delta(x_0+1)\delta(x_1+1) + \frac{1}{6}\delta(x_0+1)\delta(x_1-2).$$

The given joint density function can put in a table where the green cells are the joint probabilities and the blue cells are the marginal probabilities:

	$x_1 = -1$	$x_1 = 2$	
$x_0 = -1$	$\frac{2}{6}$	$\frac{1}{6}$	$P(X_0 = -1) = \frac{3}{6}$
$x_0 = 2$	$\frac{1}{6}$	$\frac{2}{6}$	$P(X_0 = 2) = \frac{3}{6}$
	$P(X_1 = -1) = \frac{3}{6}$	$P(X_1 = 2) = \frac{3}{6}$	

```
clear variables;
```

1) Sketch a realization of a signal with this density function.

From the joint density function we observe that the signal can take on the two values; +2 or -1.

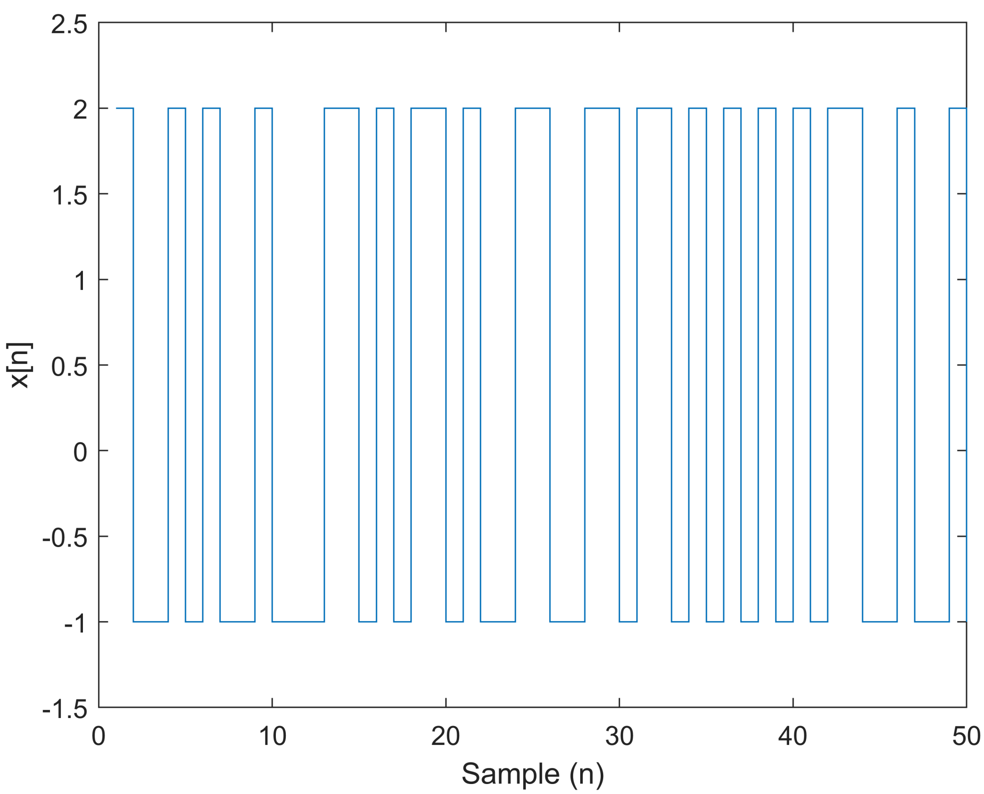
1. If the signal is +2 at t_0 then it will go to -1 at $t_0 + 1$ with probablity $\frac{1}{6}$
2. If the signal is +2 at t_0 then it will remain -2 at $t_0 + 1$ with probablity $\frac{2}{6}$
3. If the signal is -1 at t_0 then it will remain -1 at $t_0 + 1$ with probablity $\frac{2}{6}$
4. If the signal is -1 at t_0 then it will go to +2 at $t_0 + 1$ with probablity $\frac{1}{6}$

At given time, the value of the signal will remain with probability $\frac{2}{3}$,

A realisation coded in MATLAB:

```
pos_val = 2;
neg_val = -1;
change_proba = 2/3; % Probability for change

N = 50;
x = zeros(N,1);

if rand(1) < 0.5
    x(1)=pos_val;
```

```
else
    x(1)=neg_val;
end

for n=2:N
    if rand(1) < change_proba
        if x(n-1) == pos_val
            x(n) = neg_val;
        else
            x(n) = pos_val;
        end
    else
        x(n)=x(n-1);
    end
end

stairs(x)
ylim([neg_val - 0.5, pos_val + 0.5])
xlabel('Sample (n)')
ylabel('x[n]')
```



2) Compute marginal density function

2. Show that the marginal density function for X_0 is $f_{X_0}(x_0) = \frac{1}{2}\delta(x_0 - 2) + \frac{1}{2}\delta(x_0 + 1)$ and calculate the mean value of the signal.

The given joint density function can put in a table where the green cells are the joint probabilities and the blue cells are the marginal probabilities:

	$x_1 = -1$	$x_1 = 2$	
$x_0 = -1$	$\frac{2}{6}$	$\frac{1}{6}$	$P(X_0 = -1) = \frac{3}{6}$
$x_0 = 2$	$\frac{1}{6}$	$\frac{2}{6}$	$P(X_0 = 2) = \frac{3}{6}$
	$P(X_1 = -1) = \frac{3}{6}$	$P(X_1 = 2) = \frac{3}{6}$	

The marginal density function is given as:

$$f_{X_0}(x_0) = \int_{-\infty}^{\infty} f_{X_0,X_1}(x_0, x_1)\,\mathrm{dx}_1$$

Since the random variables are discrete, the above integral amounts to summing out X_1. Doing that we get the required margin mass function:

$$f_{X_0}(x_0) = \frac{3}{6}\delta(x_0 - 2) + \frac{3}{6}\delta(x_0 + 1)$$

Alternatively, we can compute:

$$\begin{aligned}
f_{X_0}(x_0) &= \int_{-\infty}^{\infty} f_{X_0,X_1}(x_0, x_1)dx_1 \\
&= \int_{-\infty}^{\infty} \Big(\frac{1}{6}\delta(x_0 - 2)\delta(x_1 + 1) + \frac{2}{6}\delta(x_0 - 2)\delta(x_1 - 2) \\
&\qquad + \frac{2}{6}\delta(x_0 + 1)\delta(x_1 + 1) + \frac{1}{6}\delta(x_0 + 1)\delta(x_1 - 2)\Big)dx_1 \\
&= \frac{1}{6}\delta(x_0 - 2) + \frac{2}{6}\delta(x_0 - 2) + \frac{2}{6}\delta(x_0 + 1) + \frac{1}{6}\delta(x_0 + 1) \\
&= \frac{1}{2}\delta(x_0 - 2) + \frac{1}{2}\delta(x_0 + 1).
\end{aligned}$$

The expected value of a discrete random variable X with outcomes $x_1, x_2, \cdots, x_k$ is given by:

$$E(X) = \sum_{i=1}^{k} x_i P(X = x_i)$$

In this problem, we have

$$E(X_0) = -1 \cdot P(X_0 = -1) + 2 \cdot P(X_0 = 2) = -1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{1}{2}$$

Alternative way:

$$
\begin{aligned}
E[X_0] &= \int_{-\infty}^{\infty} x_0 f_{X_0}(x_0) dx_0 \\
&= \int_{-\infty}^{\infty} x_0 \left(\frac{1}{2}\delta(x_0 - 2) + \frac{1}{2}\delta(x_0 + 1) \right) dx_0 \\
&= 2\frac{1}{2} + (-1)\frac{1}{2} = \frac{1}{2}.
\end{aligned}
$$

3) Determine whether signal is deterministic or random by calculating the correlation.

3. Calculate the correlation $E[X_0 X_1]$ and use this correlation to classify the signal as deterministic or random.

The correlation can be calculated as:

$$
\begin{aligned}
E[X_0 X_1] &= \int_{-\infty}^{\infty} x_0 x_1 f_{X_0,X_1}(x_0, x_1) dx_0 dx_1 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_0 x_1 \Big(\frac{1}{6}\delta(x_0 - 2)\delta(x_1 + 1) + \frac{2}{6}\delta(x_0 - 2)\delta(x_1 - 2) \\
&\qquad + \frac{2}{6}\delta(x_0 + 1)\delta(x_1 + 1) + \frac{1}{6}\delta(x_0 + 1)\delta(x_1 - 2) \Big) dx_0 dx_1 \\
&= \frac{1}{6}(2)(-1) + \frac{2}{6}(2)(2) + \frac{2}{6}(-1)(-1) + \frac{1}{6}(-1)(2) \\
&= 1.
\end{aligned}
$$

Since there is a non-zero correlation between the signal at time *n* and *n* + 1 this means that the signal is predictable. The signal is only partially and not perfectly predictable as evident from the above discussion, i.e. the signal is twice as likely to stay at the same value as to switch value. The full classification is thus **partly deterministic**.

Functions

```
function [xo, px]=epdf(x, bins)
    [nx,xo]=hist(x,bins);
    dx=xo(3)-xo(2);
    px=nx/(dx*length(x));
```

```
end
```