Homework 4

Table of Contents

Problem 13.9: Determine the mean, ACVS and stationarity of a random process

9. A random process $x[n]$ is characterized by

$$x[n] = A(\zeta) \cos [\Omega(\zeta)n + \Theta(\zeta)],$$

where random variables $A(\zeta)$, $\Omega(\zeta)$, and $\Theta(\zeta)$ are mutually independent. Random variables $A(\zeta) \sim U(0,1)$ and $\Theta(\zeta) \sim U(-\pi, \pi)$ are of continuous type while $\Omega(\zeta)$ is of discrete type taking values 10 and 20 radians with equal probability.

[✓] a) Determine the mean sequence $m_x[n]$

We need to compute:

$E[x[n]] = E[A \cos(\Omega n + \Theta)]$

Since the three random variables are independent then we can write:

$E[x[n]] = E[A]E[\cos(\Omega n + \Theta)]$

We know that $E[A] = \frac{1}{2}$ since $A \sim U(0,1)$.

Computing the expression $E[\sin(\Theta)]$ requires a bit of an explanation.

Suppose we want to compute $E[\cos(\Omega n + \Theta)]$ where $\Theta \sim U(-\pi, \pi)$. Let us pick one frequency ω (realise one value of Ω). Then let us pick a lot of realisations of Θ. Now if we plot the function $\cos(\omega n + \theta)$ for different values of θ then we will see something like this:

```
n = linspace(0, 4*pi);
plot(n, cos(n), n, cos(n-1), n, cos(n-2), n, cos(n-3), n, cos(n-4));
xlim([0, 4*pi]);
```
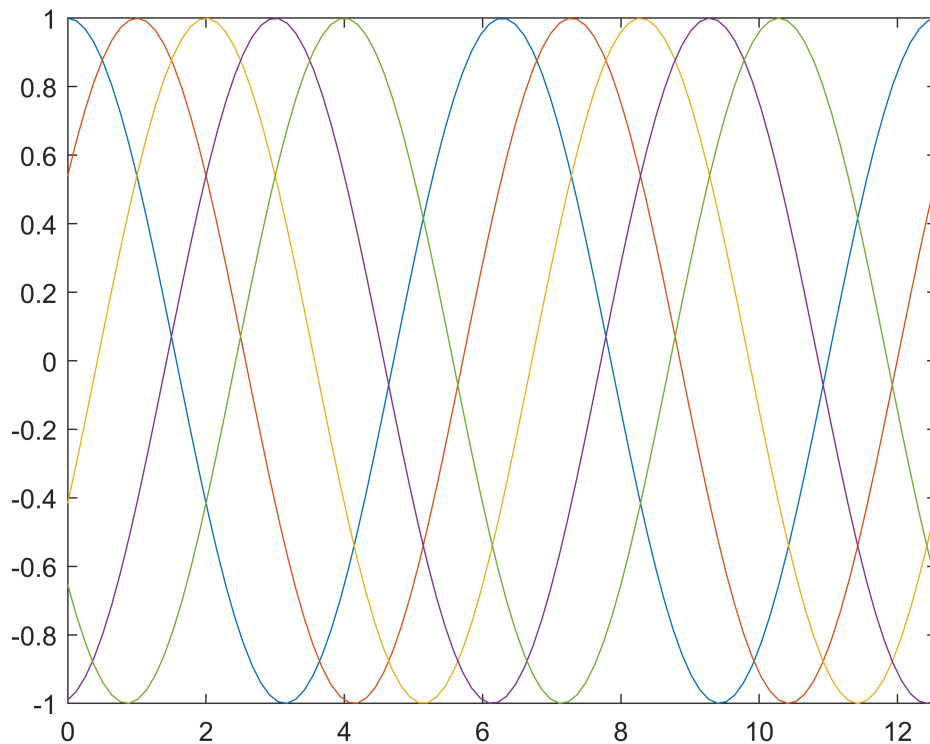

If we plot hundreds of cosine functions shifted slightly, we get a large blob of points from -1 to 1. For this reason, the quality $E[\cos(\Omega n + \Theta)]$ will be zero because the mean value is 0. Formally, we can write:

$$E[\cos(\Omega n + \Theta)] = \int_{-\pi}^{\pi} f_\Theta(\Theta) \cdot \cos(\Omega n + \Theta) \, d\Theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\Omega n + \Theta) \, d\Theta$$

So we are integrating cosine function over 2π which is zero.

In signal processing, we like to add random shifts ala $\Theta \sim U(0, 2\pi)$ to avoid that the expected value or the mean value becomes dependent on time.

Alternatively, we can use

Using the rule $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, we rewrite the expression to:

2

$$E[x[n]] = E[A(\cos(\Omega n)\cos(\Theta) - \sin(\Omega n)\sin(\Theta))]$$
$$E[x[n]] = E[A\cos(\Omega n)\cos(\Theta) - A\sin(\Omega n)\sin(\Theta)]$$

At this point, we need to employ some expection rules to decompose the expression.

Let X and Y be two random variables and a and b be two constants. Following expectation identities apply:

1. $E[a] = a$ e.g. $E(42) = 42$
2. $E[aX] = aE[X]$ e.g. if you multiply every value by 2, the expectation doubles
3. $E[a \pm X] = a \pm E[X]$ e.g. if you add 42 to every case, the expectation increases by 42
4. $E[X + Y] = E[X] + E[Y]$
5. If X and Y are independent, then $E[XY] = E[X]E[Y]$
6. $E[a \pm bX] = a \pm E[bX] = a \pm bE[X]$
7. $E[b(a \pm X)] = bE[a \pm X] = b(a \pm E[X])$

Use rule 4:

$$E[x[n]] = E[A\cos(\Omega n)\cos(\Theta)] - E[A\sin(\Omega n)\sin(\Theta)]$$

Use rule 5 multiple times:

$$E[x[n]] = E[A]E[\cos(\Omega n)]E[\cos(\Theta)] - E[A]E[\sin(\Omega n)]E[\sin(\Theta)]$$

But how can we continue from here?

We know that $\Theta \sim U(-\pi, \pi)$ so $E[\Theta] = \dfrac{\pi + (-\pi)}{2} = \dfrac{0}{2} = 0$

b) Determine the ACVS $c_X[m, n]$

c) Comment on the stationarity of the random process

Problem 13.13: MSE objective function

13. Consider the mse objective function (13.56)

$$J(a, b) = E[(Y - aX - b)^2].$$

a) Express the objective function in terms of its parameters

(a) Express $J(a, b)$ in terms of the parameters a, b, and the moments of X and Y.

b) Using partial derivatives to determine the values of parameters

(b) Using partial derivatives $\dfrac{\partial J}{\partial a}$ and $\dfrac{\partial J}{\partial b}$, determine the values of a and b by solving the equations $\partial J/\partial a = 0$ and $\partial J/\partial b = 0$ that minimize $J(a, b)$ to obtain optimum values given in (13.58) and (13.62).

Problem 13.22: Computing probabilities

22. Consider two jointly distributed random variables X and Y with pdf

$$f(x, y) = \begin{cases} 8xy, & 0 \le x \le 1, 0 \le y \le x \\ 0. & \text{otherwise} \end{cases}$$

a) Determine prior and conditional probabilities

(a) Determine $f(x), f(y), f(x|y),$ and $f(y|x)$.

b) Are X and Y independent?

(b) Are X and Y independent?