Homework 8

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Problem 5.1: Frequency estimation using Pisarenko's method

Find the frequency and amplitude of a single real sinusoidal signal in white noise, $y(n) = A\cos(2\pi f n + \phi) + w(n)$ when the first few values of the autocorrelation function are given by

$$r_y(l) = \begin{cases} 3, & l = 0 \\ 0, & l = \pm 1 \\ -2, & l = \pm 2 \end{cases}$$

Problem 5.2: Pisarenko

Let a signal be given by

$$x(n) = A_1 \sin(\omega_1 n + \phi_1) + A_2 \sin(\omega_2 n + \phi_2) + w(n)$$

Where the phases ϕ_1 and ϕ_2 are uncorrelated and uniformly distributed from 0 to 2π, $w(n)$ is zero mean gaussian white noise.

1) Calculate the autocorrelation function

2) Choose appropriate values for the amplitudes and frequencies and for the noise power

3) Calculate the eigenvalues of the autocorrelation matrix as a function of its size and compare with your expectation

4) Use the Pisarenko method to calculate the spectrum and compare with the expected results

Problem 5.3: Frequency resolution of the Pisarenko method

In this problem the goal is to investigate the frequency resolution of the Pisarenko method. We will use your two-sinusoids-in-white-noise model from the previous problem:

Let a signal be given by

$$x(n) = A_1 \sin(\omega_1 n + \phi_1) + A_2 \sin(\omega_2 n + \phi_2) + w(n)$$

Where the phases ϕ_1 and ϕ_2 are uncorrelated and uniformly distributed from 0 to 2π, $w(n)$ is zero mean gaussian white noise.

1) Create two realisations, use the Pisarenko method to compute the spectrum and comment on the stability of the method

a) Create a number of short realizations, i.e. N=64 or N=128 with different initial phases.

b) Calculate the autocorrelation matrices of the realizations.

c) Comment on the stability of the Pisarenko method

2) Can shifts in frequency be measured?

Create a number of short realizations where one of the frequencies is shifted 1% up or down. Can this shift in frequency be measured?

3) How small a frequency shift can you measure with a sequence length of *N*=128?

Problem 5.4: Pisarenko and coloured noise

In the derivation of the harmonic methods it was assumed that the signal consisted of a sinusoids or complex exponentials in white noise. In real life, the assumption of white noise is likely to be wrong in a lot of situations.

1) Create a MATLAB model of a sinusoidal signal in white, slightly coloured and very coloured noise.

2) Compare the eigenvalues of the autocorrelation matrix for the three different scenarios.

3) Calculate the Pisarenko spectra and discuss whether Pisarenko is useful when the noise is coloured.

Problem 5.5: Pisarenko, wrong choise of eigenvector

In example 14.5.1 from the note the autocorrelation matrix for a process consisting of a single sinusoid in additive white noise is given as

$$\boldsymbol{R}_y = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

The example proceeds to find $f_1 = \frac{1}{8}$

1) How does the use of a wrong eigenvalue influence the solution?

Repeat the example, but use the two other eigenvalues as starting points. How does the use of a wrong eigenvalue influence the solution?

Step 1: Compute the autocorrelation matrix $\mathbf{R}_{yy}$

```
r_yy = [3, 1, 0];
R_yy = toeplitz(r_yy)
```

```
R_yy = 3×3
     3     1     0
     1     3     1
     0     1     3
```

```
% Choose which eigenvalues is used for the calculations.
% The eigenvalues are sorted in increasing order, so
% the first eigenvalue is the smallest.
chosen_eig_idx = 1;
```

Step 2: Find the minimum eigenvalue and the corresponding eigenvector. The elements of this eigenvector is the parameters of the $\mathrm{ARMA}(2p, 2p)$ model

```
[eig_vecs, D] = eig(R_yy);
eig_vals = diag(D);

% The variance of the noise corresponds to the minimum eigenvalue
eig_val_min = eig_vals(chosen_eig_idx)
```

```
eig_val_min = 1.5858
```

Using the smallest eigenvalue, we should get:

$$\sigma_w^2 = \lambda_{\min} = 3 - \sqrt{2} \quad = 1.5858$$

```
% Get the eigenvector corresponding to the minimum eigenvalue
eig_vec_min = eig_vecs(:,chosen_eig_idx);

% Ensure that a_0 = 1 (this is by definition)
eig_vec_min = eig_vec_min / eig_vec_min(1)
```

```
eig_vec_min = 3×1
    1.0000
   -1.4142
    1.0000
```

If the smallest eigenvalue is used we would get $a_0 = 1$, $a_1 = -1.4142$, $a_2 = 1$

Step 3: Find the frequencies $\{f_i\}$ of the sinusoids. This can be done by computing the roots of the polynomial $A(z)$ in (14.5.4). This polynomial has $2p$ poles on the unit circle which correspond to the frequencies of the system.

$$A(z) = 1 + \sum_{m=1}^{2p} a_m z^{-m} \qquad (14.5.4)$$

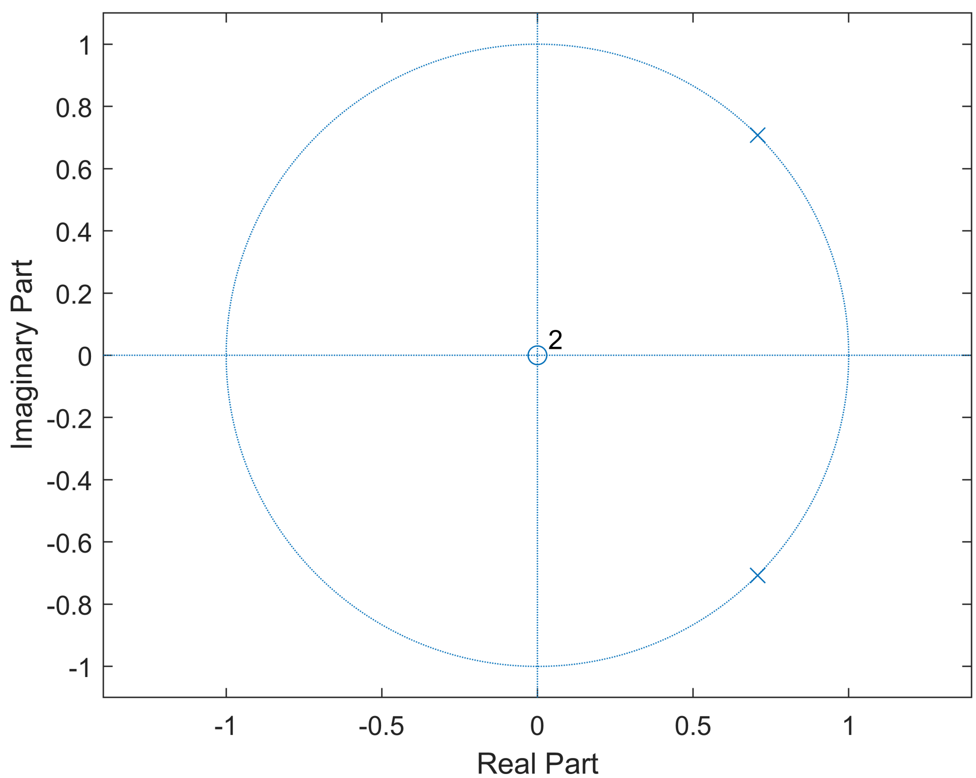
```
z = roots(eig_vec_min)
```

```
z = 2×1 complex
   0.7071 + 0.7071i
   0.7071 - 0.7071i
```

So we have $z_1 = 0.7071 + 0.7071j$ and $z_2 = 0.7071 - 0.7071j$

Note that $|z_1| = |z_2| = 1$ so poles are on the unit circle:

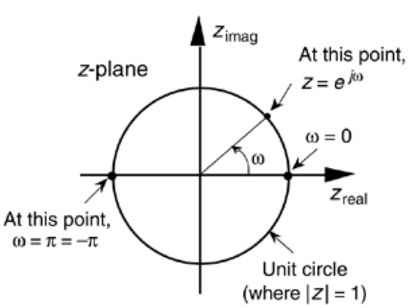
```
% norm(z(1)), norm(z(2))
% Verify that the poles are on the unit circle
zplane(1, eig_vec_min')
```



In general, the z is defined as follows:

$$z = Ae^{j2\pi f} = A[\cos(2\pi f) + j\sin(2\pi f)]$$

Recall that $\angle z = 2\pi f$ and $f = \frac{\angle z}{2\pi}$ because of the definition of the complex number z:



Since the two z quantities that we found are on the unit circle, their magnitude $A = 1$.

We also found that two pair of zs are complex conjucates of one another. So we only use one of them to compute the frequency:

```
f1 = abs(angle(z(2))) / (2*pi)
```

```
f1 = 0.1250
```

Step 4: Solve Eq. (14.5.11) for the signal powers $\{P_i\}$

$$\begin{bmatrix} \cos 2\pi f_1 & \cos 2\pi f_2 & \cdots & \cos 2\pi f_p \\ \cos 4\pi f_1 & \cos 4\pi f_2 & \cdots & \cos 4\pi f_p \\ \vdots & \vdots & & \vdots \\ \cos 2\pi p f_1 & \cos 2\pi p f_2 & \cdots & \cos 2\pi p f_p \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_p \end{bmatrix} = \begin{bmatrix} \gamma_{yy}(1) \\ \gamma_{yy}(2) \\ \vdots \\ \gamma_{yy}(p) \end{bmatrix} \qquad (14.5.11)$$

Since we only have one frequency, the equation becomes:

$\cos(2\pi f_1)P_1 = \gamma_{\text{yy}}(1)$ where $\gamma_{\text{yy}}(1)$ is the second autocorrelation value:

We want to find P_1:

$$P_1 = \frac{\gamma_{\text{yy}}(1)}{\cos(2\pi f_1)}$$

```
P1 = r_yy(2) / cos(2*pi*f1)
```

```
P1 = 1.4142
```

From section 14.5.1, we are given:

$$P_i = \frac{A_i^2}{2}$$

This allows us to estimate the amplitude A_i given P_i:

$$A_i = \sqrt{2P_i}$$

```
A1 = sqrt(2*P1)
```

```
A1 = 1.6818
```

Exam 2012, Problem 3: PSD Estimation

For a given random process $\{x(n)\}$ the autocorrelation has been estimated and is given by

$\lvert m \rvert$	$r_x(m)$
0	4
1	2
2	-1

1) Estimate PSD assuming sinusoidal white noise

Estimate the power density spectrum under the assumption that $\{x(n)\}$ consist of a single sinusoidal signal in additive white noise.

2) Estimate PSD assuming MA(2) process

Estimate the power density spectrum under the assumption that $\{x(n)\}$ can be described as a $MA(2)$ process.