Homework 5

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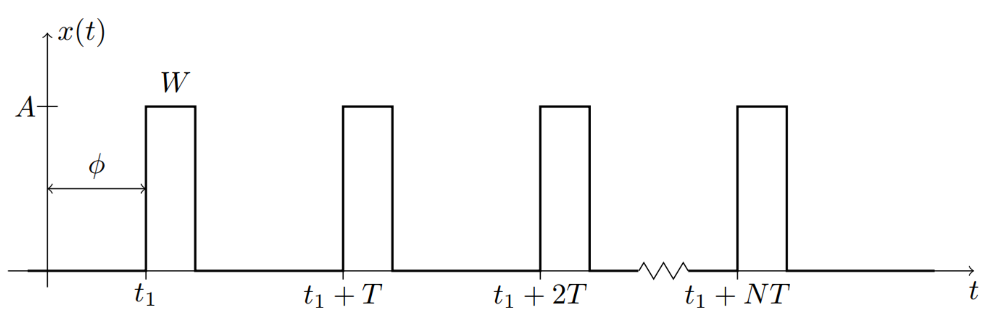
ADSI Problem 4.2: Autocorrelation functions from plot

The outcome of a random process is a pulse train with period T and pulse width W as shown in the graph. Let the phase ϕ be uniformly distributed between 0 and T so that the random process is stationary.

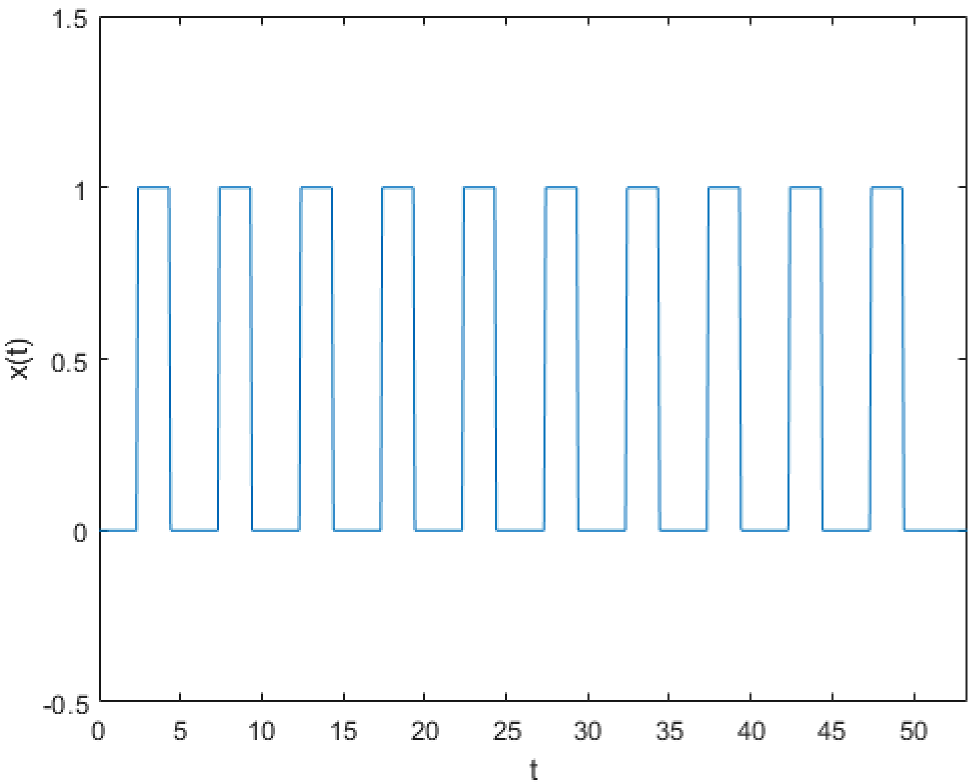


1) Sketch and explain what the autocorrelation looks like

Determine, using only simple arguments and drawings, what the autocorrelation function $R_{XX}(\tau)$ will look like.

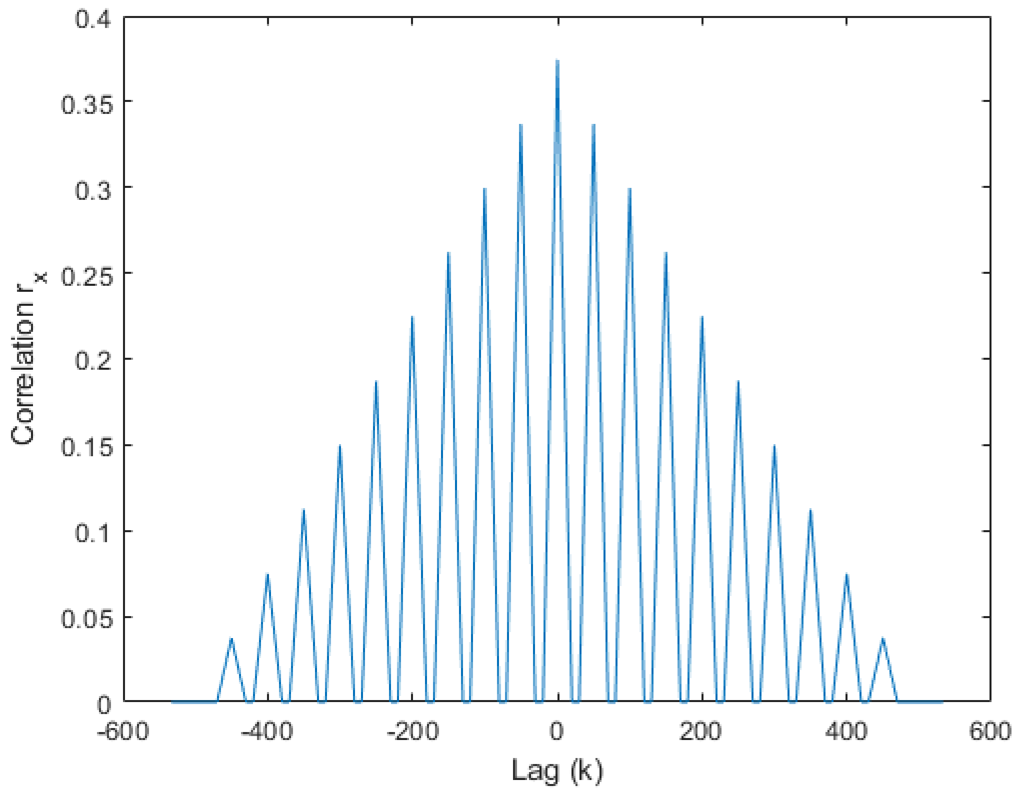
[✔] 2) Use computer simulations to verify the above result

```
A = 1;
W = 2;  % Pulse width
T = 5; % Period width
N = 10;

% Random number from a uniform distribution in the open interval (a, b)
% a = 0; b = T; random_number = (b-a)*rand + a
phi = T*rand(1);

maxT = phi+W/2 + T*N;

t=0:.1:maxT;   % Time vector
d= phi+W/2:T:phi+W/2+T*N-1; % Delay vector
y = pulstran(t,d,'rectpuls', W);
plot(t,y)

ylabel('x(t)');
xlabel('t');
ylim([-0.5, A+0.5]);
xlim([0, max(t)]);
```



Autocorrelation is symmetrical around zero.

```
[r_xx, lags] = xcorr(y, 'biased');
plot(lags, r_xx);
ylabel('Correlation r_x');
xlabel('Lag (k)')
```



ADSI Problem 4.4: MA(q) processes

In this problem we will investigate the $\mathrm{MA}(q)$ process as defined by Eq. (13.132) by excluding the feedback part

$$y[n] = \sum_{k=0}^{q} b_k x[n-k]$$

where the input is a zero-mean Gaussian white noise process with unit variance.

[✔] 1) Write out the full expressions

Write out the full expressions for $\mathrm{MA}(0)$, $\mathrm{MA}(1)$, $\mathrm{MA}(2)$ and $\mathrm{MA}(3)$ processes.

The general $\mathrm{ARMA}(p,q)$ is given by the difference equation:

$$y[n] = -\sum_{k=1}^{p} a_k y[n-k] + \sum_{k=0}^{q} b_k x[n-k], \qquad (13.132)$$

When the feedback part is excluded, all values of a_k is set to zero, we are left with:

$$y[n] = \sum_{k=0}^{q} b_k\, x[n-k]$$

We can write out the full expressions for the difference processes as follows:

$$\mathrm{MA}(0) \;\rightarrow\; y[n] = b_0\, x[n]$$

$$\mathrm{MA}(1) \;\rightarrow\; y[n] = b_0\, x[n] + b_1\, x[n-1]$$

$$\mathrm{MA}(2) \;\rightarrow\; y[n] = b_0\, x[n] + b_1\, x[n-1] + b_2\, x[n-2]$$

$$\mathrm{MA}(3) \;\rightarrow\; y[n] = b_0\, x[n] + b_1\, x[n-1] + b_2\, x[n-2] + b_3\, x[n-3]$$

2) Calcute the autocorrelation for the MA processes

Calculate the autocorrelation $r_{yy}[l]$ for the $\mathrm{MA}(0)$ to $\mathrm{MA}(3)$ processes.

The autocorrelation function of a random process is defined as:

$$r_{\mathrm{yy}}(\ell) = E\left[y(n) \cdot y(n-\ell)\right]$$

To compute the autocorrelation for the $\mathrm{MA}(0)$, just plug its difference equation into this equation:

$$r_{\mathrm{yy}}(\ell) = E\left[b_0\, x(n) \cdot b_0\, x(n-\ell)\right]$$

$$r_{\mathrm{yy}}(\ell) = b_0^2 E\left[x(n) \cdot x(n-\ell)\right]$$

$$r_{\mathrm{yy}}(\ell) = b_0^2\, r_{\mathrm{xx}}(\ell)$$

Since $r_{\mathrm{xx}}(\ell) = \sigma^2\delta(\ell) = \delta(\ell)$. ***Why?***

$$r_{\mathrm{yy}}(\ell) = b_0^2\, \delta(\ell)$$

The autocorrelation for the $\mathrm{MA}(1)$ process is:

$$r_{\mathrm{yy}}(\ell) = E\left[(b_0\, x[n] + b_1\, x[n-1]) \cdot (b_0\, x[n-\ell] + b_1\, x[n-\ell-1])\right]$$

$$r_{\mathrm{yy}}(\ell) = E\left[b_0^2\, x[n]x[n-\ell] + b_0 b_1 x[n]x[n-\ell-1] + b_0 b_1\, x[n-1]\, x[n-\ell] + b_1^2 x[n-1]x[n-\ell-1]\right]$$

$$r_{\mathrm{yy}}(\ell) = E\left[b_0^2\, x[n]x[n-\ell] + b_0 b_1 (x[n]x[n-\ell-1] + x[n-1]\, x[n-\ell]) + b_1^2 x[n-1]x[n-\ell-1]\right]$$

$$r_{\mathrm{yy}}(\ell) = b_0^2\, E\left[x[n]x[n-\ell]\right] + b_0 b_1\, E\left[(x[n]x[n-\ell-1] + x[n-1]\, x[n-\ell])\right] + b_1^2\, E\left[x[n-1]x[n-\ell-1]\right]$$

$$r_{\mathrm{yy}}(\ell) = b_0^2\, E\left[x[n]x[n-\ell]\right] + b_0 b_1\, (E\left[x[n]x[n-\ell-1]\right] + E\left[x[n-1]\, x[n-\ell]\right]) + b_1^2\, E\left[x[n-1]x[n-\ell-1]\right]$$

Since

- $E[x[n]x[n-\ell]] = r_{xx}(\ell)$
- $E[x[n]x[n-\ell-1]] = r_{xx}(\ell-1)$
- $E[x[n-1]x[n-\ell]] = r_{xx}(\ell+1)$

3) Calculate power density spectra

Calculate the general expressions for power density spectra of MA(0), MA(1) and MA(2) processes using Eq. (13.112).

The power spectral density (PSD) is defined as:

$$S_{xx}(\omega) \triangleq \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell]\mathrm{e}^{-\mathrm{j}\ell\omega}. \qquad (13.112)$$

In 2) we found that the autocorrelation for $\mathrm{MA}(0)$ is $r_{yy}(\ell) = b_0^2\,\delta(\ell)$, so plug it in this PSD formula:

$$S_{yy}^{\mathrm{MA}(0)}(\omega) = \sum_{\ell=-\infty}^{\infty} r_{yy}(\ell)e^{-\mathrm{j}\ell\omega} = \sum_{\ell=-\infty}^{\infty} b_0^2\,\delta(\ell)e^{-\mathrm{j}\ell\omega}$$

We notice that $r_{yy}(\ell)$ has non-zero value only when $\ell = 0$. Therefore, the terms for when $\ell \neq 0$ are zero. We can, therefore, remove the infinite sum. Since $\delta(0) = 1$ and $e^{-\mathrm{j}0\omega} = 1$, we have:

$$S_{yy}^{\mathrm{MA}(0)}(\omega) = b_0^2\,\delta(0)e^{-\mathrm{j}0\omega} = b_0^2 \cdot 1 \cdot 1 = b_0^2$$

[✔] 4) Plot the power density spectra given coefficients

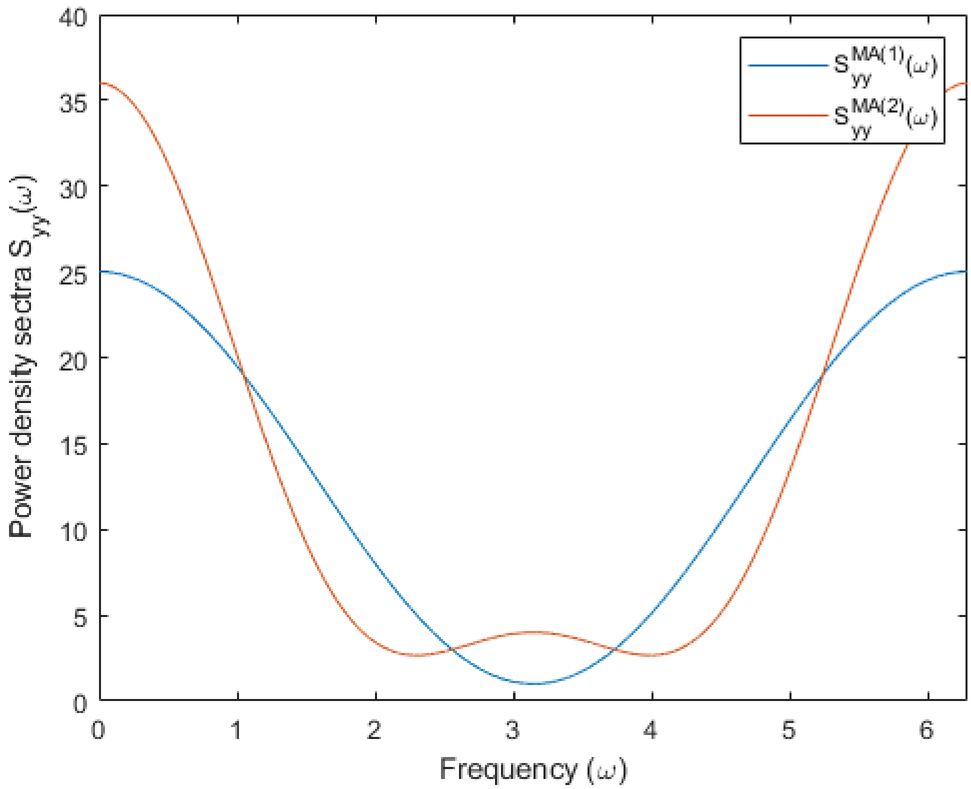
Plot the power density spectra for the two processes defined by

q	b_0	b_1	b_2
1	3	2	
2	3	2	1

We are given the coefficients for $S_{yy}^{\mathrm{MA}(1)}(\omega)$ and $S_{yy}^{\mathrm{MA}(2)}(\omega)$. We use results from 3) to plot the power density spectra for the two processes.

```
b0=3; b1=2; b2=1;
w=0:0.001:2*pi;
S1=(b0^2+b1^2)+2*b0*b1*cos(w);
S2=(b0^2+b1^2+b2^2)+2*(b0*b1+b1*b2)*cos(w)+2*b0*b2*cos(2*w);
plot(w,S1,w,S2)
legend('S_{yy}^{MA(1)}(\omega)','S_{yy}^{MA(2)}(\omega)')
```

```
xlabel('Frequency (\omega)')
ylabel('Power density sectra S_{yy}(\omega)')
xlim([0,2*pi])
```



ADSI Problem 4.5: MA processes and phase properties

Consider the following two MA(q) systems and assume that they are excited by zero-mean white Gaussian noise with unit variance.

$$y_1[n] = 2x[n] - x[n-2] \quad \text{and} \quad y_2[n] = x[n] - 2x[n-2]$$

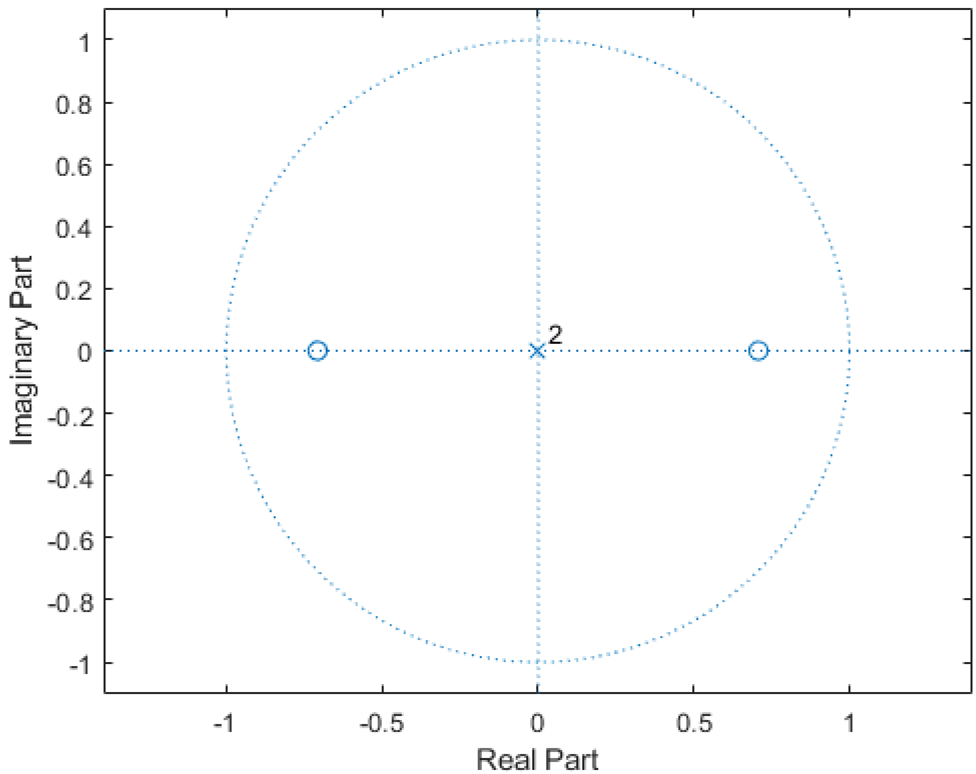
[✔] 1) What is the order q of the processes

The order q of an $\mathrm{MA}(q)$ process is given by the longest delay, i.e., $x[n-q]$. Therefore, both systems are of order 2.

[✔] 2) Compare the phase properties of the two systems

To determine the phase properties of a system, we need to plot where its the zeros are located in relation to the unit circle in a zero-pole plot. If all zeros are located within the unit circle, then we have a minimum-phase system.

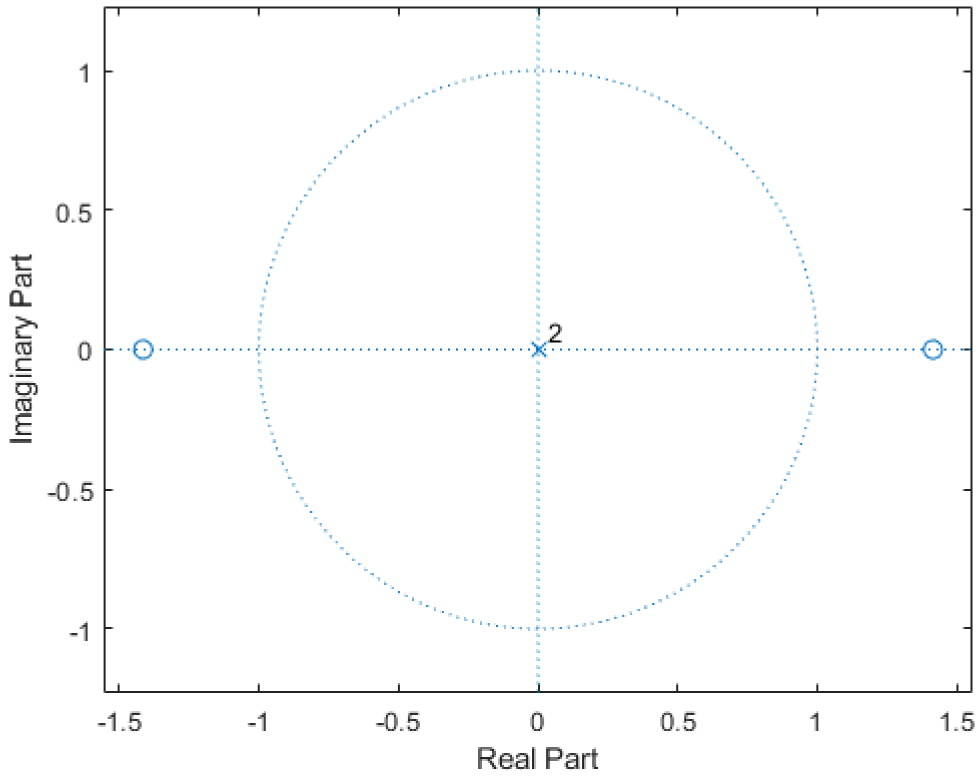
```
h1 = [2, 0, -1]; % Impulse response for y1[n]
zplane(h1);
```



From the zero-pole plot, we observe that all the zeros are within the unit circle. Therefore, we can conclude that the system $y_1[n]$ is a minimum-phase.

Let us make the zero-plot for the system $y_2[n] = x[n] - 2x[n-2]$

```
h2 = [1, 0, -2]; % Impulse response for y2[n]
zplane(h2);
```



Since both zeros are outside the unit circle, we can conclude the system $y_2[n]$ is a maximum-phase system.

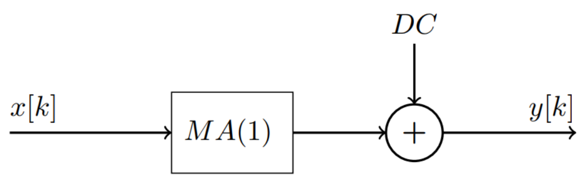
3) Calculate and plot the power density spectra of the two systems and comment on the result

ADSI Problem 4.6: MA processes, output corrupted

Consider a MA(1) process driven by Gaussian white noise sequence $w[n]$ with unit variance

$$y[n] = 3x[n] + x[n-1]$$

Assume that the output of the process is corrupted by the addition of a DC offset with a value of d before it is measured as shown in the figure

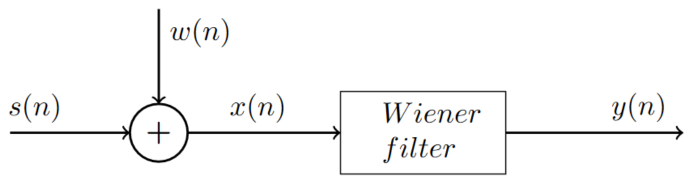


1) Calculate the autocorrelation of the corrupted signal

2) How is the power density spectrum of the *MA*(1) process affected by the DC-offset?

Exam 2012 Problem 4: Wiener filter for recovering corrupted signal

A periodic signal $s(n)$ is corrupted by uncorrelated, additive noise $w(n)$ as shown in the figure.



It is desired to recover the signal by a Wiener filter. The autocorrelation of the noise is given $r_w(l) = e^{-0.2\sqrt{|l|}}$. The first values of the autocorrelation of the signal is given in the table below.

$\lvert l \rvert$	$r_s(l)$
0	1
1	-0.4
2	0.2
3	0.1

1) Design a 3 tap Wiener filter to recover s(n)

2) Calculate the minimum mean square error

3) Discuss another Wiener filter with 2x noise amplitude

Consider a second, identical Wiener filtering problem, except that the noise amplitude is twice as big.

3. Discuss whether this second system will have the same minimum mean square error as the first system.