Autocorrelation Functions

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Summary

The autocorrelation of the **complex sinusoid** $z(n) = A\, e^{j(\omega n+\phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is given by:

$$r_{zz}(\ell) = A^2\, e^{j\omega\ell}$$

The relation between $r_{zz}(\ell)$ and $r_{zz}(-\ell)$ is:

$$r_{zz}(-\ell) = r_{zz}^*(\ell)$$

The autocorrelation function of a real **cosine signal** $z(n) = A\cos(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is:

$$r_{zz}(\ell) = \frac{A^2}{2}\cos(\omega\ell)$$

The autocorrelation function of a real **sine signal** $z(n) = A\sin(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is:

$$r_{zz}(\ell) = -\frac{A^2}{2}\cos(\omega\ell)$$

The autocorrelation of white noise $w(x)$ is

$$r_{\mathrm{ww}}(\ell) = \sigma_w^2 \delta(\ell)$$

Let $v(n)$ and $w(n)$ be **two uncorrelated white noise** processes with variance $\sigma_v^2 = 0.49$ and $\sigma_w^2 = 1$. The cross-correlation between these processes is:

$$r_{\mathrm{vw}}(\ell) = 0$$

Details

Autocorrelation and cross-correlation of complex signals

The autocorrelation function of a complex signal is given by:

$$r_{\mathrm{xx}}(\ell) = E[x(n)x^*(n-\ell)]$$

The cross-correlation function of a complex signal is given by:

$$r_{\mathrm{yx}}(\ell) = E[y(n)x^*(n-\ell)]$$

Relations between r(ell) and r(-ell)

Let us compute the autocorrelation for $-\ell$:

$$r_{\mathrm{xx}}(-\ell) = E[x(n)x^*(n+\ell)]$$

Suppose $m = n + \ell$. Then we can write $n = m - \ell$. Let us substitute all n with m in the above expression:

$$r_{\mathrm{xx}}(-\ell) = E[x(m-\ell)x^*(m)]$$

Computing the complex conjucate of the expectation we get:

$$r_{\mathrm{xx}}(-\ell) = E[x^*(m-\ell)x(m)]^*$$

$$r_{\mathrm{xx}}(-\ell) = E[x(m)x^*(m-\ell)]^*$$

Since $r^*_{xx}(\ell) = E[x(m)x^*(m-\ell)]^*$, we know that:

$$r_{xx}(-\ell) = r^*_{xx}(\ell)$$

Autocorrelation of complex signal

Problem:

What is the autocorrelation function of the complex sinusoid $x(n) = A\,e^{j(\omega n+\phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$

Solution:

The autocorrelation function for complex signals can be computed as:

$$r_{xx}(\ell) = E[z(n)z^*(n-\ell)]$$

Plugging the given complex sinusiod into the formula, we get:

$$r_{xx}(\ell) = E[A\,e^{j(\omega n+\phi)}A\,e^{-j(\omega(n-\ell)+\phi)}]$$

Since A is a constant, we can move it outside the expected value:

$$r_{xx}(\ell) = A^2E[\,e^{j(\omega n+\phi)}\,e^{-j(\omega(n-\ell)+\phi)}]$$

$$r_{xx}(\ell) = A^2E[\,e^{j\omega n+j\phi}\,e^{-j\omega n+j\omega\ell-j\phi}]$$

$$r_{xx}(\ell) = A^2E[\,e^{j\omega n+j\phi+(-j\omega n+j\omega\ell-j\phi)}\,]$$

$$r_{xx}(\ell) = A^2E[\,e^{j\omega n+j\phi-j\omega n+j\omega\ell-j\phi}\,]$$

$$r_{xx}(\ell) = A^2E[\,e^{j\omega\ell}\,]$$

We know that $E[e^{j\omega\ell}] = e^{j\omega\ell}$ because the expected value of a constant is just the constant itself. Notice that ϕ is no longer in the expression $e^{j\omega\ell}$. Therefore, the autocorrelation of the complex sinusoid is:

$$r_{xx}(\ell) = A^2\,e^{j\omega\ell}$$

Thus, the autocorrelation of the complex sinusoid $z(n) = A\,e^{j(\omega n+\phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is given by:

$$r_{zz}(\ell) = A^2\,e^{j\omega\ell}$$

Autocorrelation of a real cosine signal

Problem:

What is the autocorrelation function of a real signal $x(n) = A\cos(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$?

Solution:

In ADSI Problem 4.4.1, we know that the autocorrelation of the complex sinusoid $z(n) = A\, e^{j(\omega n + \phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is given by:

$$r_{zz}(\ell) = A^2\, e^{j\omega\ell}$$

Since the result from 1) uses Euler, we need to convert the signal to complex exponential.

We use the relation $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

$$x(n) = A\cos(\omega\, n + \phi)$$

$$x(n) = \frac{A}{2}\,(e^{j(\omega n + \phi)} + e^{-j(\omega n + \phi)})$$

$$x(n) = \frac{A}{2} e^{j(\omega n + \phi)} + \frac{A}{2} e^{-j(\omega n + \phi)}$$

In 1) we found that the autocorrelation of a complex sinusoid $x(n) = A\, e^{j(\omega\, n + \phi)}$ is $r_{xx}(\ell) = A^2 e^{j\omega\ell}$

Therefore, the autocorrelation of the real signal is:

$$r_{xx}(\ell) = \left(\frac{A}{2}\right)^2 e^{j\omega\ell} + \left(\frac{A}{2}\right)^2 e^{-j\omega\ell}$$

$$r_{xx}(\ell) = \frac{A^2}{4} e^{j\omega\ell} + \frac{A^2}{4} e^{-j\omega\ell}$$

$$r_{xx}(\ell) = \frac{A^2}{4}(e^{j\omega\ell} + e^{-j\omega\ell})$$

$$r_{xx}(\ell) = \frac{A^2}{2}\frac{1}{2}(e^{j\omega\ell} + e^{-j\omega\ell})$$

Using the relation $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ we can rewrite the autocorrelation to:

$$r_{xx}(\ell) = \frac{A^2}{2}\cos(\omega\ell)$$

Thus, the autocorrelation function of a real signal $z(n) = A\cos(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is:

$$r_{zz}(\ell) = \frac{A^2}{2}\cos(\omega\ell)$$

Autocorrelation of a sine signal

Problem:

What is the autocorrelation function of a real signal $z(n) = A\sin(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$?

Solution:

In ADSI Problem 4.4, we found that the autocorrelation of a complex sinusoid given by $y(n) = A\, e^{j(\omega n + \phi)}$ where A and ω are real constants and ϕ is a random variable with $\phi \sim U(0, 2\pi)$ is:

$$r_{yy}(\ell) = A^2\, e^{j\omega\ell}$$

To use this result, we need to convert the given signal in this problem to complex exponential signal.

A complex exponential signal is always formed by the sum of two real signals:

$$Ae^{j\omega n} = A\cos(\omega n) + j\, A\sin(\omega n)$$

Therefore, we know that:

$$\sin(\omega) = \frac{1}{2j}e^{j\omega} - \frac{1}{2j}e^{-j\omega}$$

Using this relation, we can rewrite a real signal $A\sin(\omega n + \phi)$ as:

$$z(n) = A\sin(\omega n + \phi)$$

$$z(n) = \frac{A}{2j}e^{j(\omega n + \phi)} - \frac{A}{2j}e^{-j(\omega n + \phi)}$$

To compute the autocorrelation function, we square the magnitude, remove the phase and replace n with ℓ:

$$r_{zz}(\ell) = \left(\frac{A}{2j}\right)^2 e^{j\omega\ell} - \left(\frac{A}{2j}\right)^2 e^{-j\omega\ell}$$

We know that $(2j)^2 = 2^2 \cdot j^2 = -4$ because $j = \sqrt{-1}$ so $j^2 = -1$

$$r_{zz}(\ell) = \frac{A^2}{-4}e^{j\omega\ell} - \frac{A^2}{-4}e^{-j\omega\ell}$$

$$r_{zz}(\ell) = -\frac{A^2}{4}e^{j\omega\ell} + \frac{A^2}{4}e^{-j\omega\ell}$$

We want to make the autocorrelation function in terms of $\cos(\cdot)$, we rewrite the expression as follows:

$$r_{zz}(\ell) = \left(-\frac{A^2}{2}\right) \cdot \frac{1}{2}(e^{j\omega\ell} + e^{-j\omega\ell})$$

Since $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$, we can rewrite the expression as:

$$r_{zz}(\ell) = -\frac{A^2}{2}\cos(\omega\ell)$$

Thus, the autocorrelation function of a real signal $z(n) = A\sin(\omega n + \phi)$ is

$$r_{zz}(\ell) = -\frac{A^2}{2}\cos(\omega\ell)$$

Autocorrelation of White noise

White noise is important for random signal modelling.

Suppose we have a random process that produces perfect random noise. Let $w(n)$ be a random signal from this process.

The **expected value** of the signal is zero because there are no patterns in white noise:

$$E[w(n)] = 0$$

The **autocorrelation of white noise** generates one peak at $\ell = 0$ because that is the only time when there is any correlation of the signal. One peak at $\ell = 0$ can be modelled by delta signal:

$$r_{ww}(\ell) = E[w(n)w(n-\ell)] = \sigma_w^2\delta(\ell)$$

where σ_w^2 is the variance of the signal which can be computed as follow:

$$\sigma_w^2 = E[w^2(n)] - E[w(n)]^2$$

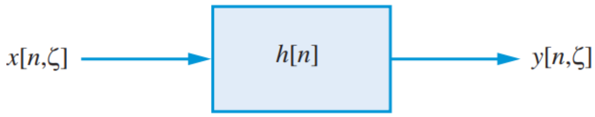
$$\sigma_w^2 = E[w^2(n)] - 0 \qquad \text{(by definition } E[w(n)] = 0\text{)}$$

$$\sigma_w^2 = E[w^2(n)]$$

System identification with white noise

INSIGHT*:* we can use white noise to find the impulse response of a system because the autocorrelation of white noise is the same the delta signal.

We have an unknown system with an impulse response $h[n]$.



As input to this system, we give it a zero-mean white noise signal (a realisation of white noise random process):

$$x(n) \sim \mathrm{WN}\left(0, \sigma_x^2\right)$$

The cross-correlation between this white noise signal and output of the system is given by Eq. 13.100:

$$r_{yx}[\ell] = \sum_{k=-\infty}^{\infty} h[k] r_{xx}[\ell - k] = h[\ell] * r_{xx}[\ell]. \qquad (13.100)$$

Since $r_{\mathrm{xx}}(\ell) = \sigma_x^2 \delta(\ell)$ the convolution becomes:

$$r_{\mathrm{yx}}(\ell) = h(\ell) * \sigma_x^2 \delta(\ell)$$

$$r_{\mathrm{yx}}(\ell) = \sigma_x^2 h(\ell)$$

This means that the cross-correlation is just the impulse response.

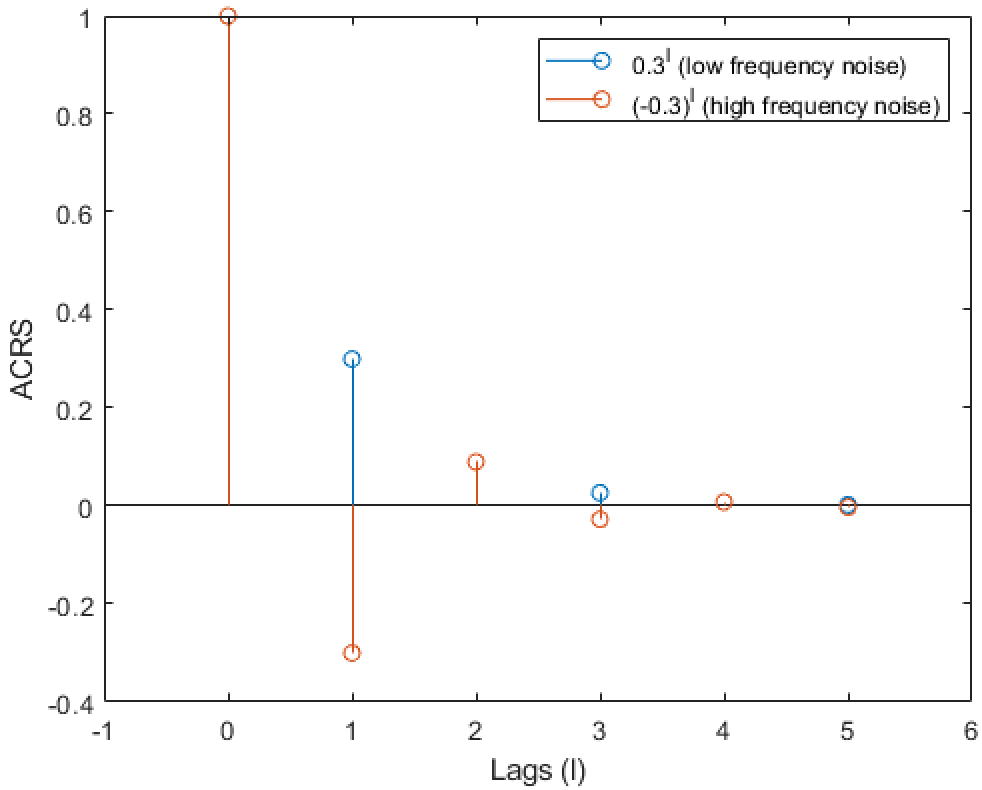
High frequency noise vs low frequency noise

We can use the autocorrelation of a noise to determine whether it is low-frequency vs high-frequency.

Plotting the ACRS, we see that the high-frequency noise oscillates whereas the low-frequency noise does not.

```
clear variables;
ell = 0:5;

stem(ell, 0.3.^ell)
hold on;
stem(ell, (-0.3).^ell)
hold off;
legend('0.3^l (low frequency noise)', '(-0.3)^l (high frequency noise)')
xlim([min(ell)-1, max(ell)+1])
hold off;
xlabel('Lags (l)')
ylabel('ACRS')
```


ACRS
Lags (l)
0.3^l (low frequency noise)
$(-0.3)^l$ (high frequency noise)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-1
0
1
2
3
4
5
6