Lecture 2: Invertibility and minimum-phase systems

Table of Contents

Allpass systems....1
Invertibility....2
When is a filter invertible?....3
Quiz....3
Filter Decomposition....4
Example Problem 1....5
Example Problem 2....6
Magnitude Response of Decomposition....8
Phase response of the Decomposition....9
Why is the decomposition useful?....10
Minimum Delay Property....10
Quiz....12
Quiz....13

Allpass systems

The definition of allpass is that the magnitude of its frequency response is constant G (usually $G = 1$) for all frequencies.

$$|H(e^{j\omega})| = G. \qquad (5.155)$$

We can say that:

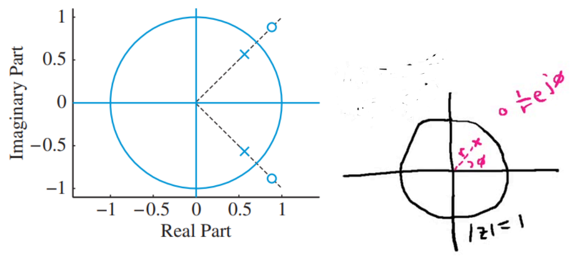
- an allpass system treats all frequencies identically with respect to gain
- an allpass system preserves the power or energy of their input signals

The system function of an allpass system has the poles and zeros occur in conjugate reciprocal pairs

$$H_{ap}(z) = e^{j\beta} \prod_{k=1}^{N} \frac{z^{-1} - p_k^*}{1 - p_k z^{-1}}, \qquad (5.158)$$

where β is constant that is usually set to $\beta = 0$.

So if we have a pole $p_k = re^{j\phi}$ then the zero is located at $\frac{1}{p_k^*} = \frac{1}{r}e^{j\phi}$

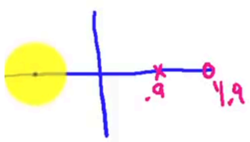


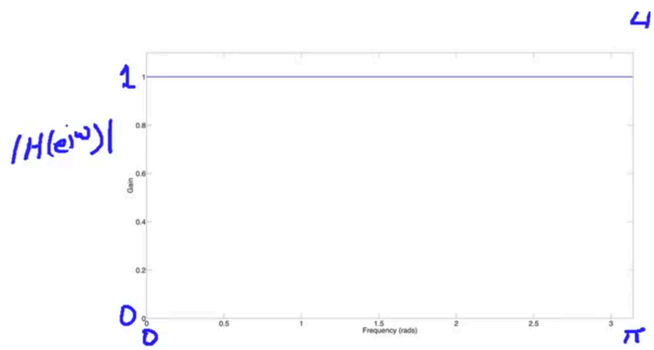
A second-order allpass filter is given by:

$$H_{ap}(z) = \frac{a_2^* + a_1^* z^{-1} + z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = z^{-2} \frac{1 + a_1^* z + a_2^* z^2}{1 + a_1 z^{-1} + a_2 z^{-2}}, \tag{5.164}$$

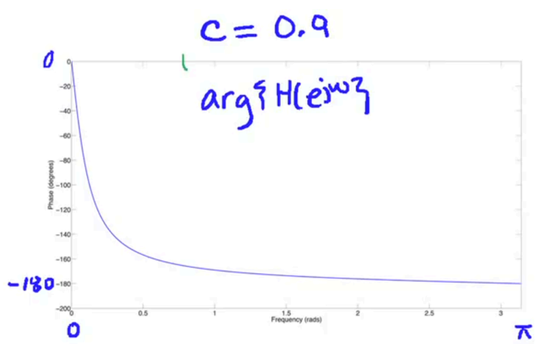
Example: All-pass System

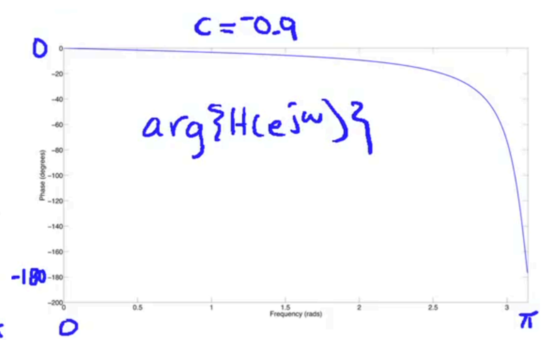
$$H(z) = \frac{z^{-1} - c^*}{1 - c z^{-1}}$$





Although the magnitude response is the same, the phase response changes depending on the pole.





Invertibility

5.16 Invertibility and minimum-phase systems

$$x(n) \longrightarrow \boxed{h(n)} \xrightarrow{y(n)} \boxed{h_{inv}(n)} \longrightarrow x(n)$$

?

$$h(n) * h_{inv}(n) = \delta(n) \quad = \left(z^{-n_d}\right)$$

z

$$H(z)\,H_{inv}(z) = 1 \iff H_{inv}(z) = \frac{1}{H(z)}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

$$H_{inv}(z) = \frac{A(z)}{B(z)}$$

zeroes $\leftrightarrow$ poles

When is a filter invertible?

For a filter to be invertible, zeros and poles must be inside the unit circle.

A causal and stable LTI system with a causal and stable inverse is known as a *minimum-phase* system

If poles and zeros are outside the unit circle, we call it maximum-phase.

If some are outside and others inside the unit circle, it is known as mixed-phase.

A stable, causal system has a stable, causal inverse
if and only if
all poles and zeros are inside $|z|=1$

called: Minimum phase system

Quiz

Consider a FIR filter with a number of zeros. Most of them are located inside the unit circle but a number of them is located exactly on the unit circle. There are no zeros outside the unit circle.

$$x(n) \longrightarrow \boxed{H(z)} \longrightarrow y(n)$$

Can the action of the filter be undone by an inverse filter ?

A: Yes, the filtering process can be undone
B: No, the filtering process can't be undone
C: We need to know more about $H(z)$

The answer is NO. If we have a FIR where the zero is exactly at 1 kHz. Given a signal, the filter will remove frequency component at 1 kHz. The problem is that the inverse filter cannot reconstruct the original signal because we have deleted information from the signal i.e., the missing frequency component cannot be recovered.

Filter Decomposition

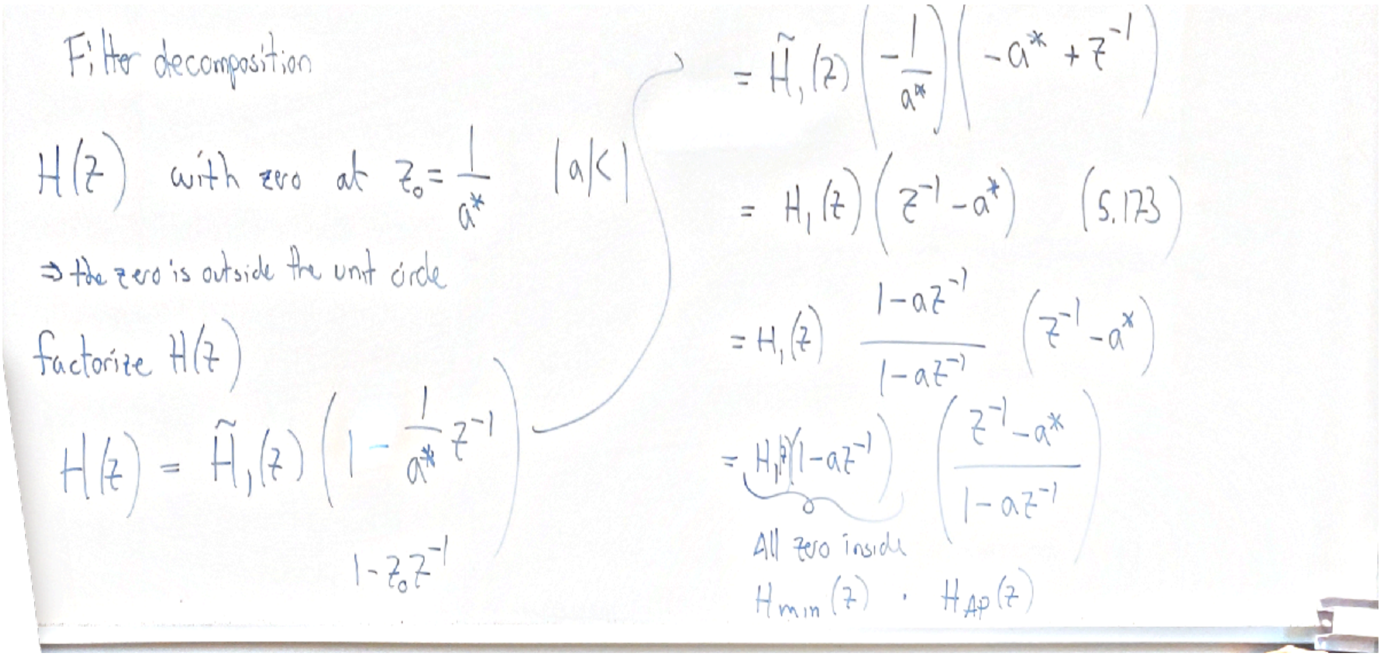
Any rational system function $H(z)$

$$H(z) = \underbrace{H_{min}(z)}_{\text{minimum phase}} \underbrace{H_{ap}(z)}_{\text{all pass}}$$

Any rationional system function can be factorised into two system functions:

- A minimum-phase system function
- An allpass system function.

$$H(z) = H_{\text{min}}(z) \cdot H_{\text{ap}}(z)$$



Notice that $H(z)$ has a zero outside the unit circle because $|a| < 1$.

By decomposing $H(z)$ we made a new polynomial that consists of two parts:

- Minimum-phase system: all zeros are inside the unit circle
- All-pass system: which will have zeros outside the unit circle. We are basically storing all the phase in the all-pass filter instead

To factorise or decompose, we take following steps:

1. Take the zeros of $H(z)$ that lie outside the unit circle and move them to the allpass system function $H_{\mathrm{ap}}(z)$
2. Add poles to the allpass system in conjugate reciprocal locations of the zeros
3. To keep the equation balanced, we add zeros in the minimum-phase system function to even out the poles that we added to the allpass system function

Since the poles that we added to the allpass system were all inside the unit circle, the zeros that we added to the minimim-phase system are also inside the unit circle.

Example Problem 1

Decompose the system $H(z) = H_1(z)(1 - \beta z^{-1})$ where $|\beta| > 1$ and $H_1(z)$ is a minimum-phase system

Step 1: We notice that $1 - \beta z^{-1}$ is not an allpass system. We want it to look like this:

$$H_{\mathrm{ap}}(z) = \mathrm{e}^{\mathrm{j}\beta} \prod_{k=1}^{N} \frac{z^{-1} - p_k^*}{1 - p_k z^{-1}}, \qquad (5.158)$$

So we start by factorising $-\beta$ out. By doing this, we put a zero into allpass system

$$H(z) = H_1(z)(-\beta)\left(-\frac{1}{\beta} + z^{-1}\right) = H_1(z)(-\beta)\left(z^{-1} - \frac{1}{\beta}\right)$$

Step 2: We need to put a pole into the allpass system that is in conjugate reciprocal locations of the zero

$$H(z) = H_1(z)(-\beta)\left(\frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*}z^{-1}}\right)$$

Step 3: To keep the equality, we have to introduce a zero to the minimum-phase system function

$$H(z) = H_1(z)(-\beta)\left(1 - \frac{1}{\beta^*}z^{-1}\right)\left(\frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*}z^{-1}}\right)$$

Now the decomposition is:

$$H(z) = H_{\min}(z)H_{\text{ap}}(z)$$

where:

$$H_{\min}(z) = H_1(z)(-\beta)\left(1 - \frac{1}{\beta^*}z^{-1}\right)$$

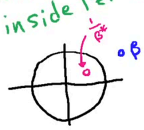
$$H_{\text{ap}}(z) = \frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*}z^{-1}}$$

Suppose $H(z) = H_1(z)(1 - \beta z^{-1})$; $|\beta| > 1$, $H_1(z)$ min phase

1) $H(z) = H_1(z)(-\beta)\left(z^{-1} - \frac{1}{\beta}\right)$

2+3) $H(z) = \underbrace{H_1(z)(-\beta)\left(1 - \frac{1}{\beta^*}z^{-1}\right)}_{H_{min}(z)} \underbrace{\frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*}z^{-1}}}_{H_{ap}(z)}$

zero at $z = \beta$ in $H(z)$ reflected inside $|z| = 1$



Example Problem 2

Decompose the system $H(z) = 1 + 4.5z^{-1} + 2z^{-2}$/

Step 1: Take the zeros of $H(z)$ that lie outside the unit circle and move them to the allpass system function

First, we need to write it as a product of two system functions. We can do that by finding the zeros:

```
roots([1, 4.5, 2])
```

```
ans = 2×1
   -4.0000
   -0.5000
```

Now we can write the system as:

$$H(z) = (1 - (-0.5)z^{-1})((1 - (-4)z^{-1}))$$

$$H(z) = (1 + 0.5z^{-1})(1 + 4z^{-1})$$

One zero $z = -0.5$ is inside the unit circle and the other is outside $z = -4$. We want to flip the zero outside the unit circle to be inside the unit circle.

So we start by factorising 4 out. By doing this, we put a zero into allpass system.

$$H(z) = (1 + 0.5z^{-1})(4)\left(\frac{1}{4} + z^{-1}\right) = 4(1 + 0.5z^{-1})\left(z^{-1} + \frac{1}{4}\right)$$

Step 2: We need to put a pole into the allpass system that is in conjugate reciprocal locations of the zero

$$H(z) = 4(1 + 0.5z^{-1})\left(\frac{z^{-1} + \frac{1}{4}}{1 + \frac{1}{4}z^{-1}}\right)$$

Step 3: To keep the equality, we have to introduce a zero to the minimum-phase system function

$$H(z) = 4(1 + 0.5z^{-1})\left(1 + \frac{1}{4}z^{-1}\right)\left(\frac{z^{-1} + \frac{1}{4}}{1 + \frac{1}{4}z^{-1}}\right)$$

To tidy up, we will multiply the first three products together.Multiplying two polynomials together is the same as doing convolution:

```
4 * conv([1, 0.5], [1, 1/4])
```

```
ans = 1×3
    4.0000    3.0000    0.5000
```

So we can rewrite the system function as:

$$H(z) = (4 + 3z^{-1} + 0.5z^{-2})\left(\frac{z^{-1} + \frac{1}{4}}{1 + \frac{1}{4}z^{-1}}\right)$$

$$H(z) = 1 + 4.5z^{-1} + 2z^{-2}$$

$$H(z) = (1 + 0.5z^{-1})(1 + 4z^{-1})$$

$$z_o = \begin{cases} -0.5 \\ -4 \end{cases}$$

$$a = \frac{1}{z^*} = -\frac{1}{4} \rightarrow H^M(z) = (1 + 0.5z^{-1})\,4\left(z^{-1} + 1/4\right)$$

$$= 4(1 + 0.5z^{-1})\,\frac{1 + 1/4\,z^{-1}}{1 + 1/4\,z^{-1}}\,(z^{-1} + 1/4)$$

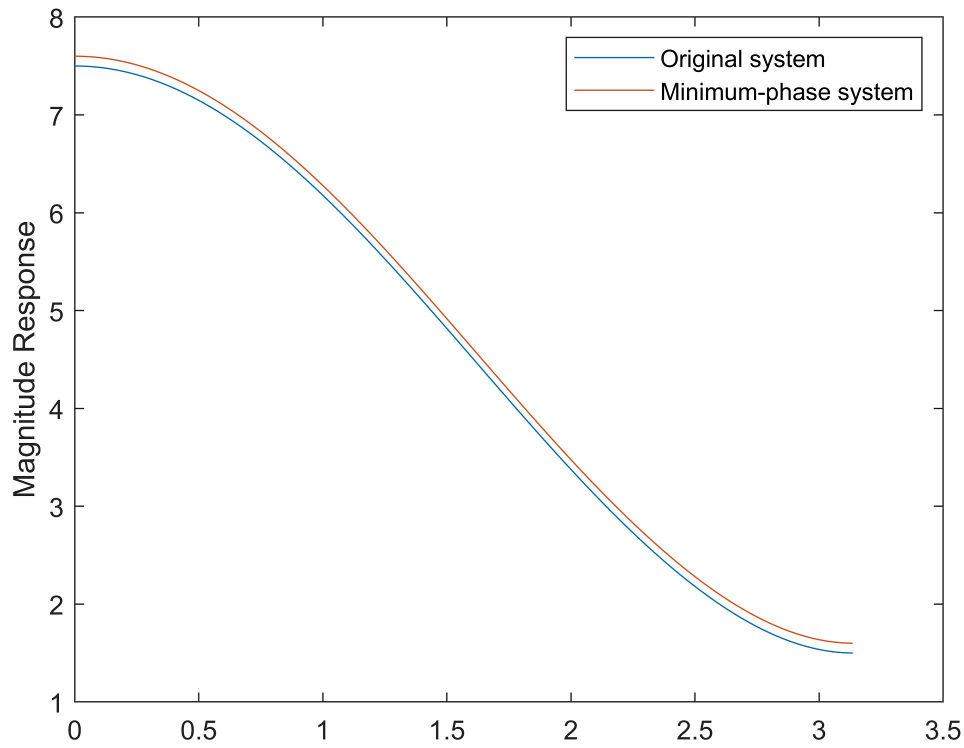
$$= 4(1 + 0.5z^{-1})(1 + 1/4\,z^{-1})\,\frac{1/4 + z^{-1}}{1 + 1/4\,z^{-1}}$$

$$= (4 + 3z^{-1} + 0.5z^{-2})\left(\frac{1/4 + z^{-1}}{1 + 1/4\,z^{-1}}\right)$$

Magnitude Response of Decomposition

Plotting the maginatude response of the original system and the minimum-phase system, we observe that they are on top of each other

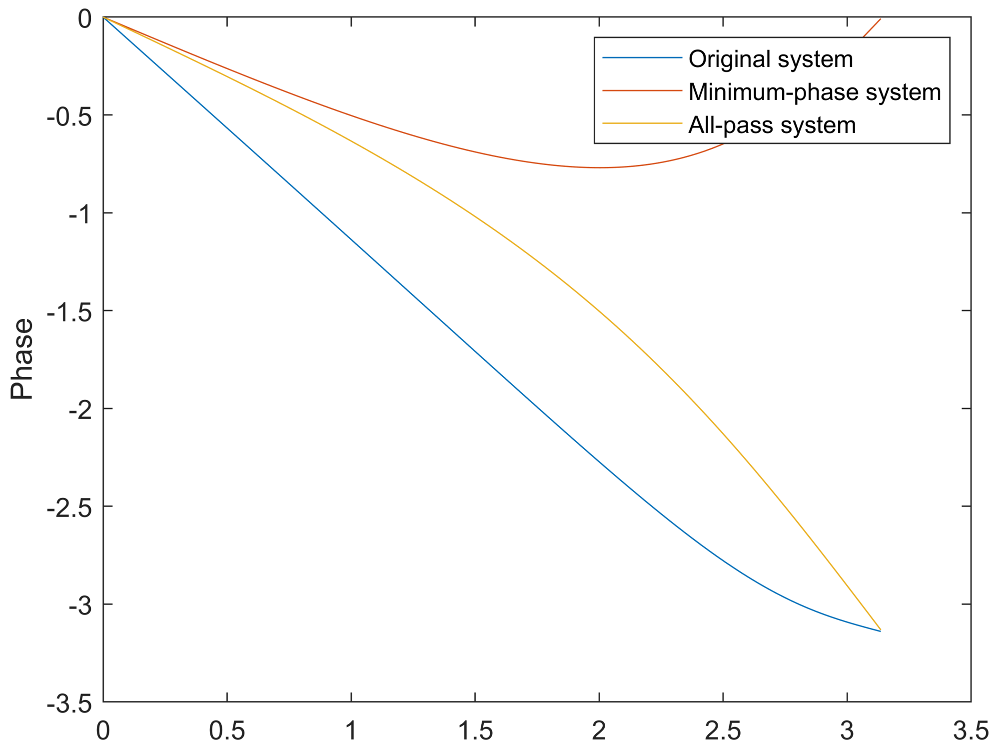
```
[H_orig, w] = freqz([1, 4.5, 2]); % Original
[H_min, w] = freqz([4, 3,   0.5]);
[H_ap, w] = freqz([1/4, 1], [1, 1/4]);

% Add 0.1 otherwise they will be on top of each other
plot(w, abs(H_orig), w, abs(H_min)+0.1)
legend('Original system', 'Minimum-phase system')
ylabel('Magnitude Response')
```



Phase response of the Decomposition

We can plot the phase of the three systems. The phase of the original system is hidden in the all-pass system. So by decomposing a filter into minimum-phase and all-pass systems, the excess phase is stored in the all-pass system. The phase lag of a system with poles and zeros inside the unit cicle is less than the original system which had the identical magnitude response.

```
plot(w, angle(H_orig), w, angle(H_min), w, angle(H_ap))
ylabel('Phase')
legend('Original system', 'Minimum-phase system', 'All-pass system')
```



Why is the decomposition useful?

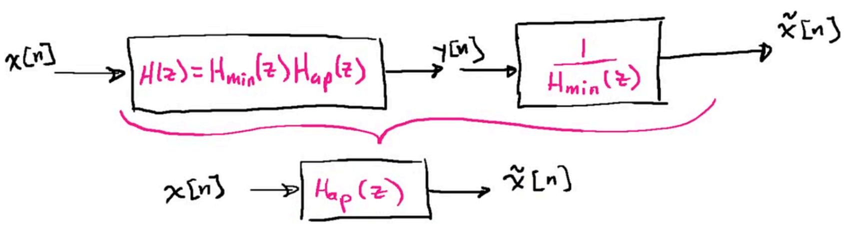
This decomposition is useful if we want to find an approximate inverse system for $H(z)$.

The minimum-phase component of any system can be inverted. This allows os to take any system, run a signal $x(n)$ through it and run the output $y(n)$ through the inverse of the minimum-phase part.

The end result is the same as passing the signal $x(n)$ through an allpass filter.

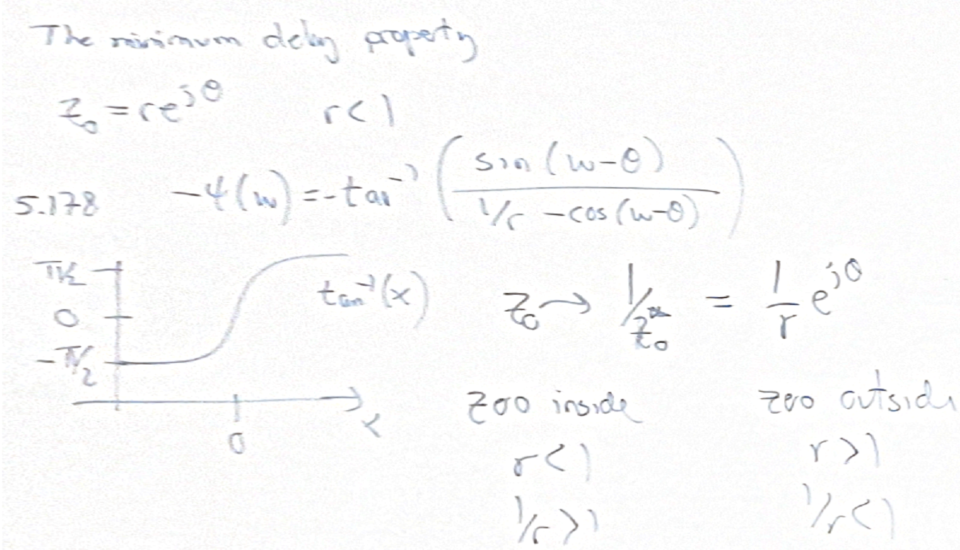
What we achieve is that:

- no magnitude distortion meaning that the magnitude of each frequency is not altered by the process
- we only get phase distortion (which may not be important depending on the application)



Minimum Delay Property

What happens when we take a zero and flip it. We know that the magnitude response does not change.



By setting $r < 1$ we specifically place the zero inside the unit circle.

If we flip z_0 to $\frac{1}{z_0^*} = \frac{1}{r}e^{j\phi}$

If we have zero inside $r < 1$ then $\frac{1}{r} > 1$

If we have zero outside $r > 1$ then $\frac{1}{r} < 1$

The only thing that changes in the phase response when we flip z_0 to inside the unit circle is $\frac{1}{r}$.

$$-\Psi(\omega) = -\tan^{-1}\frac{\sin(\omega-\theta)}{1/r-\cos(\omega-\theta)}, \qquad (5.178)$$

If we go from zero inside to outside, the $\frac{1}{r}$ becomes smaller. The tangent will increase and we get a larger number. This means that we get more phase. So we add phase as the zero goes from inside the unit circle to outside the unit circle.

If we look at the group delay:

$$\tau(\omega) = \frac{r-\cos(\omega-\theta)}{(r+1/r)-2\cos(\omega-\theta)}. \qquad (5.179)$$

By taking a zero that is inside the unit circle and putting it outside, we increase the group delay.

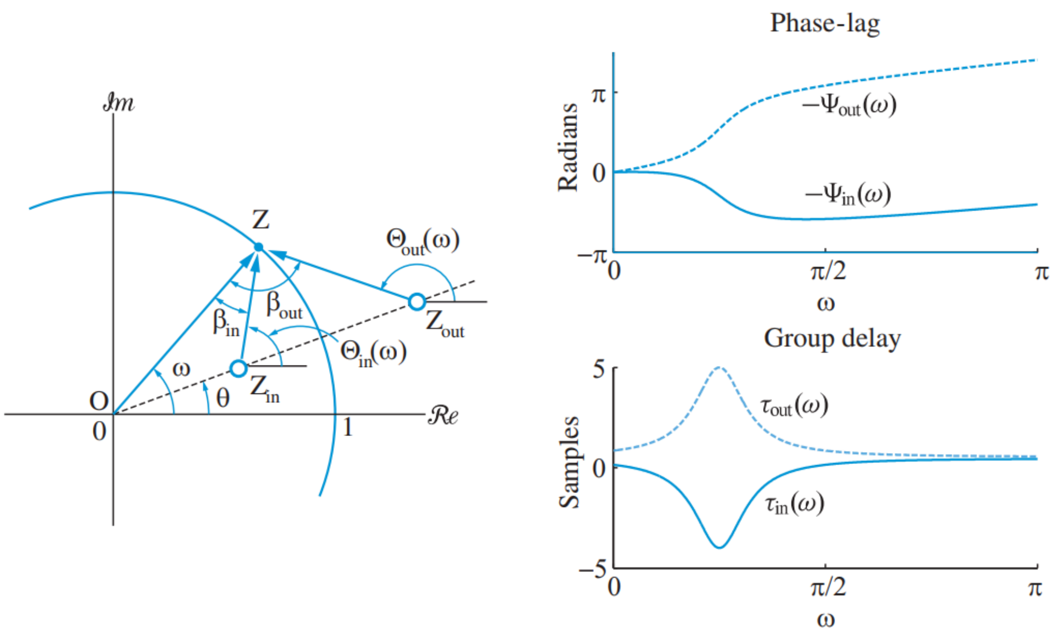
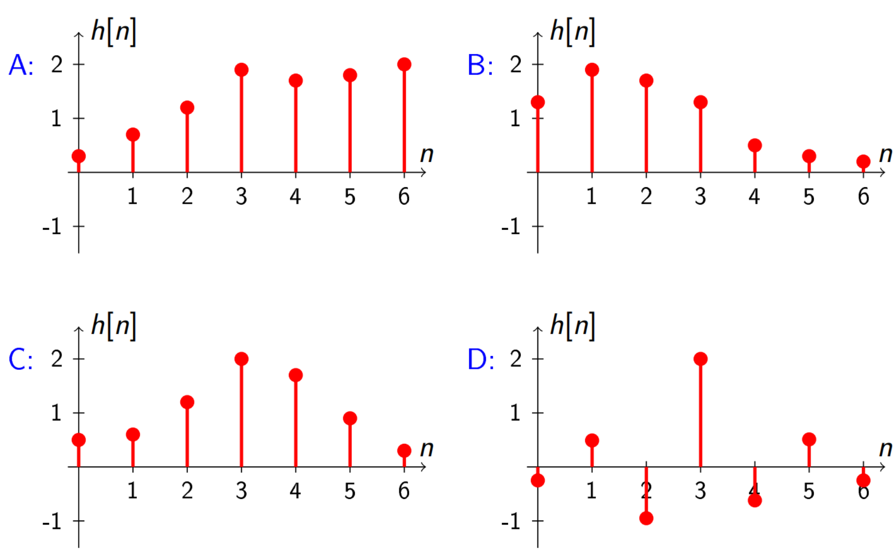


Figure 5.33 (a) Geometrical proof that a zero outside the unit circle introduces a larger phase-lag than a zero inside the unit circle. (b) Phase response and group delay for a zero at $z_{in} = 0.8e^{j\pi/4}$ and its conjugate reciprocal zero $z_{out} = (1/0.8)e^{j\pi/4}$.

Quiz: Detect

The impulse response from 4 different FIR filters is shown below.
Which filter seems to be a minimum phase filter ?



The answer is B. Minimum-phase is the same thing as minimum group delay. If we put in energy any of these four filters, which one is the fastest to respond and pass the signal though. How fast does the system propagate the energy of a signal through the filter? If it is slow, then it is maximum-phase. If a filter is in-between then it is mixed-phase. If it is very phase, then it is minimum-phase.

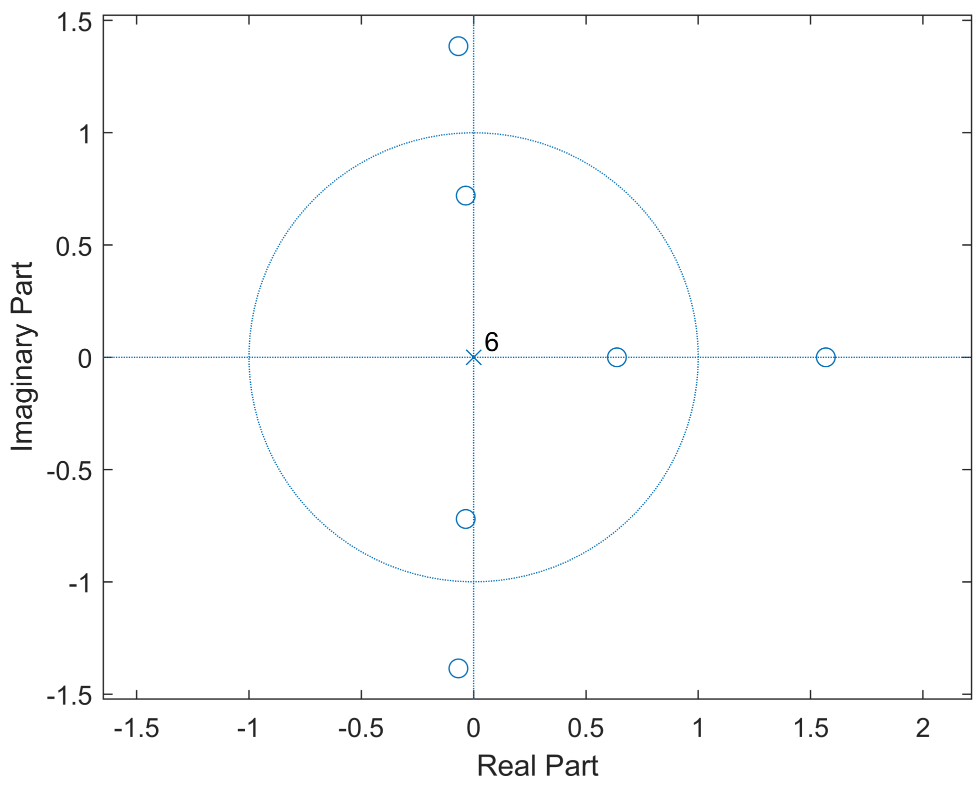
A and D has all the energy at the late end so it takes a while before all the energy passes through the system. Therefore, they has larger group delay than B, where the larger impulses are early.

Comparing B and C: B has energy at the begining, whereas C the energy is in the middle.

- A: Maximum-phase
- B: Minimum-phase
- C: Mixed-phase
- D: Mixed-hase

Let us simulate filter D in MATLAB:

```
zplane([-1, 2, -3, 5, -3, 2, -1])
```



Quiz

A FIR filter has a linear phase response if the coefficients fulfil the symmetry requirement

$$b(k) = b(M - k), \qquad \text{for } k = 0,\ 1, \ldots,\ M$$

What is the classification of such a filter?

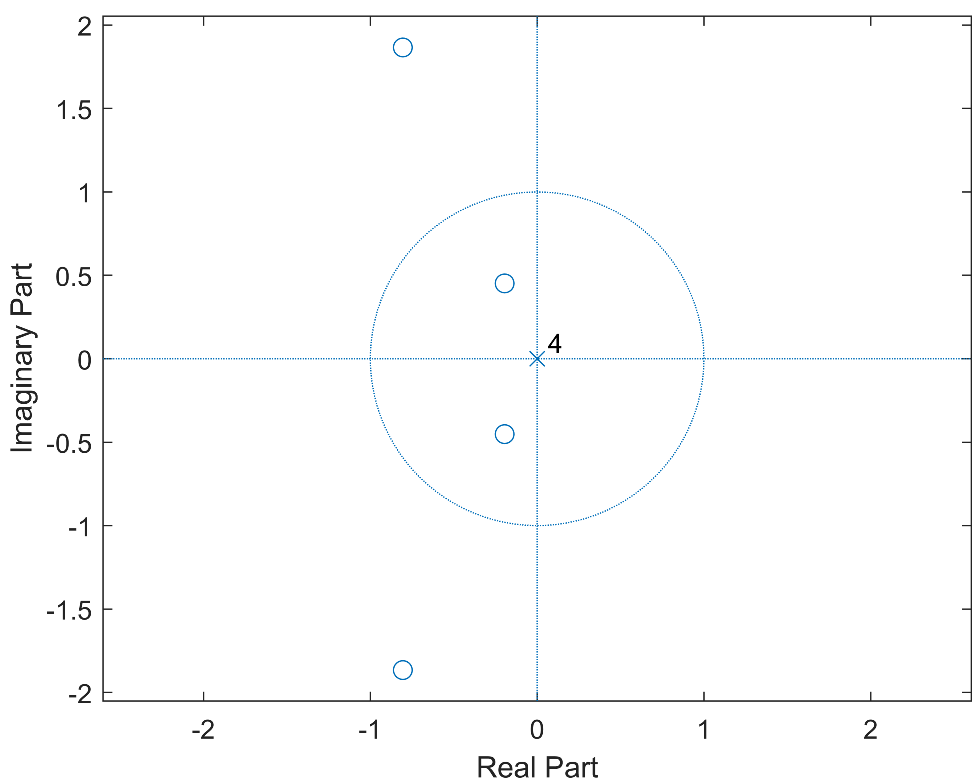
A: Minimum group delay
B: Mixed group delay
C: Maximum group delay
D: We need to know more about the $b(k)$'s

Recall that linear phase correspond to unity group delay.

The symmetry will be in the middle of the filter. The answer is B.

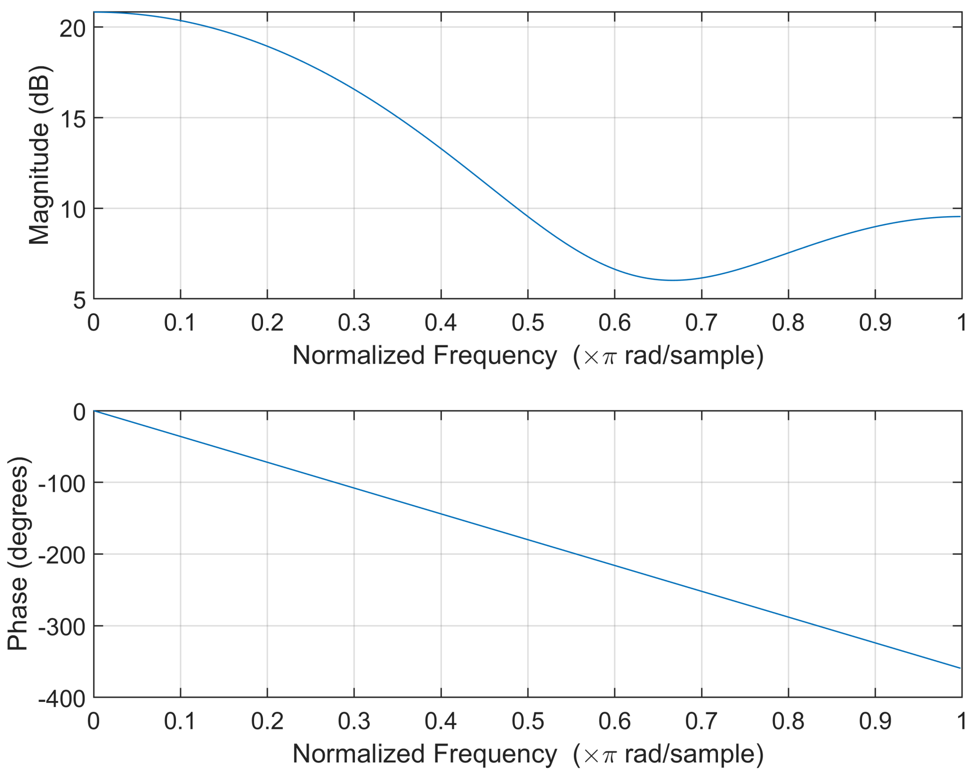
We can try it out in MATLAB. If we plot the a linear phase filter, we get two zeros inside and two zeros outside. Thefore, the filter is mixed-phase filter.

```
zplane([1, 2, 5, 2, 1])
```



The reason why the filter is called linear phase filter:

```
freqz([1, 2, 5, 2, 1])
```


Magnitude (dB)
20
15
10
5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)
Phase (degrees)
0
-100
-200
-300
-400
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)