Matched Filters

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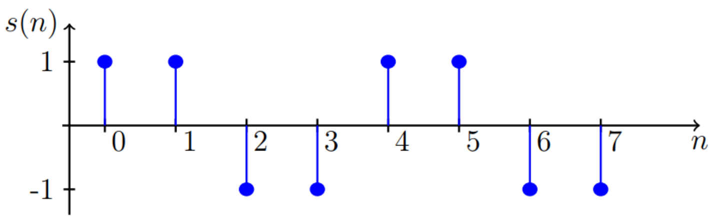
Exam 2017 Problem 3: Detect presence of signal using matched filter

Consider the deterministic signal, $s(n)$ shown below in blue. The signal is zero for all other values of n.



The signal is distorted by additive low frequency noise with autocorrelation $r_v(\ell) = 0.4^{|\ell|}$.

```
clear variables;
```

1) Design a matched filter and determine the improvement in SNR

Design a matched filter for detecting the presence of the signal and determine the improvement in signal to noise ratio.

The impulse response of the matched filter is given by:

$$\boldsymbol{h} = \kappa \boldsymbol{R}_v^{-1} \boldsymbol{s}. \qquad (14.97)$$

where R_v is autocorrelation matrix of noise and κ is the normalisation factor.

Although the maximum SNR can be obtained by any choise of constant κ, we choose the constant by requiring that:

(a) $\boldsymbol{h}^{\mathrm{T}}\boldsymbol{s} = 1$, which yields $\kappa = 1/\boldsymbol{s}^{\mathrm{T}}\boldsymbol{R}_v^{-1}\boldsymbol{s}$

(b) $\mathrm{E}(v_{\mathrm{o}}^2[n]) = \boldsymbol{h}^{\mathrm{T}}\boldsymbol{R}_v\boldsymbol{h} = 1$, which yields $\kappa = 1/\sqrt{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{R}_v^{-1}\boldsymbol{s}}$.

```
s = [1, 1, -1, -1, 1, 1, -1, -1]';

p = numel(s); % Signal length

% The autocorrelation matrix must be p x p since
% its inverse is multiplied by a p-tap signal s(n)
ell = 0:p-1;
r_vv = 0.4.^ell;
R_vv = toeplitz(r_vv);

% Compute normalisation factor (b)
k = 1/(s'*(R_vv\s));        % Using (a)
% k = 1/sqrt(s'*(R_vv\s)); % Using (b)

% Compute the filter
h = k*(R_vv\s); % Same as k*inv(R_vv)*s

% Print the matched filter coefficients
h
```

```
h = 8×1
    0.0735
    0.1422
   -0.1422
   -0.1422
    0.1422
    0.1422
   -0.1422
   -0.0735
```

The SNR at the input is given by:

$$\mathrm{SNR}_i = \frac{\text{power of signal}}{\text{power of noise}} = \frac{s^2(n = n_0)}{r_v(0)}$$

```
SNR_i = s(1)^2/r_vv(1)
```

```
SNR_i = 1
```

The optimum SNR at the output is given by:

$$\mathrm{SNR}_{\mathrm{o}} = a^2\tilde{\boldsymbol{s}}^{\mathrm{T}}\tilde{\boldsymbol{s}} = a^2\boldsymbol{s}^{\mathrm{T}}\boldsymbol{R}_v^{-1}\boldsymbol{s}. \tag{14.98}$$

Assuming the attenuation factor $a = 1$:

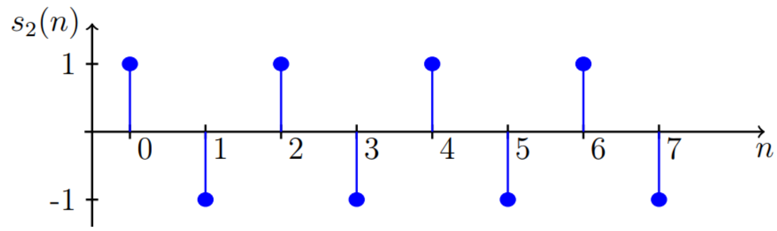
```
a = 1;
SNR_o = a^2 * s' * (R_vv\s)
```

```
SNR_o = 9.7143
```

The improvement in SNR is 9.7 (almost 10 times)

2) Discuss the improvement of SNR if a high-frequency signal is used

A second signal $s_2(n)$ is given by



2. Discuss whether the signal to noise ratio will be improved if the signal $s_2(n)$ is used instead of $s(n)$.

If we use $s_2(n)$ instead of $s(n)$, the output SNR becomes 17.3:

```
s = [1, -1, 1, -1, 1, -1, 1, -1]';
p = numel(s); % Signal length
ell = 0:p-1;
r_vv = 0.4.^ell;
R_vv = toeplitz(r_vv);
a = 1;
SNR_o = a^2 * s' * (R_vv\s)
```

```
SNR_o = 17.3333
```

The $s_2(n)$ is similar to $s(n)$ but it oscillates faster i.e., has higher frequency than the $s(n)$ signal. Since the noise is at low frequency, it is easier to separate the $s_2(n)$ signal from the noise. This means that a higher SNR can be expected from a signal with a higher frequency.

That is why the output SNR for $s_2(n)$ is better than that for $s(n)$.

Exam 2018 Problem 3: Matched Filters

A deterministic signal is given by

n	$s(n)$
0	1
1	-1
2	1
3	-1

The signal is distorted by additive low frequency noise with autocorrelation $r_v(l) = 0.8^{|l|}$.

```
clear variables;
```

1) Design a matched filter to improve the SNR

Design a matched filter to improve the signal to noise ratio and comment on the improvement.

The impulse response of the matched filter is given by:

$$\boldsymbol{h} = \kappa \boldsymbol{R}_v^{-1} \boldsymbol{s}. \qquad (14.97)$$

where R_v is autocorrelation matrix of noise and κ is the normalisation factor.

Although the maximum SNR can be obtained by any choise of constant κ, we choose the constant by requiring that:

(a) $\boldsymbol{h}^{\mathrm{T}}\boldsymbol{s} = 1$, which yields $\kappa = 1/\boldsymbol{s}^{\mathrm{T}}\boldsymbol{R}_v^{-1}\boldsymbol{s}$

(b) $\mathrm{E}(v_o^2[n]) = \boldsymbol{h}^{\mathrm{T}}\boldsymbol{R}_v\boldsymbol{h} = 1$, which yields $\kappa = 1/\sqrt{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{R}_v^{-1}\boldsymbol{s}}$.

```
s = [1, -1, 1, -1]';

p = numel(s); % Signal length

% The autocorrelation matrix must be p x p since
% its inverse is multiplied by a p-tap signal s(n)
ell = 0:p-1;
r_vv = 0.8.^ell;
R_vv = toeplitz(r_vv);

% Compute normalisation factor (b)
k = 1/sqrt(s'*(R_vv\s));

% Compute the filter
h = k*(R_vv\s);
```

```
% Print the matched filter coefficients
h
```

```
h = 4×1
    0.9449
   -1.7008
    1.7008
   -0.9449
```

The SNR at the input is given by:

$$\mathrm{SNR}_i = \frac{\text{power of signal}}{\text{power of noise}} = \frac{s^2(n = n_0)}{r_v(0)}$$

```
SNR_i = s(1)^2/r_vv(1)
```

```
SNR_i = 1
```

The optimum SNR is given by:

$$\mathrm{SNR}_o = a^2 \tilde{s}^T \tilde{s} = a^2 \boldsymbol{s}^T \boldsymbol{R}_v^{-1} \boldsymbol{s}. \qquad (14.98)$$

Assuming the attenuation factor $a = 1$:

```
a = 1;
SNR = a^2 * s' * (R_vv\s)
```

```
SNR = 28.0000
```

The improvement in SNR is 28 times.

2) Can SNR be improved by using a longer signal?

2. The $s(n)$ signal consists of two blocks each containing 1 and -1. Can the signal to noise ratio be improved by using more than two blocks?

Yes, the SNR can be improved. With three blocks, the SNR is increased to 46:

```
s = [1, -1, 1, -1, 1, -1]';
p = numel(s);
ell = 0:p-1;
r_vv = 0.8.^ell;
R_vv = toeplitz(r_vv);
k = 1/sqrt(s'*(R_vv\s));
h = k*(R_vv\s);

a = 1;
SNR = a^2 * s' * (R_vv\s)
```

```
SNR = 46.0000
```