

# Homework 3

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## [✓] ADSI Problem 2.1: Implementation of All-zero lattice filters

Work you way through the MATLAB code in Figure 9.26 and make sure you understand what is going on.

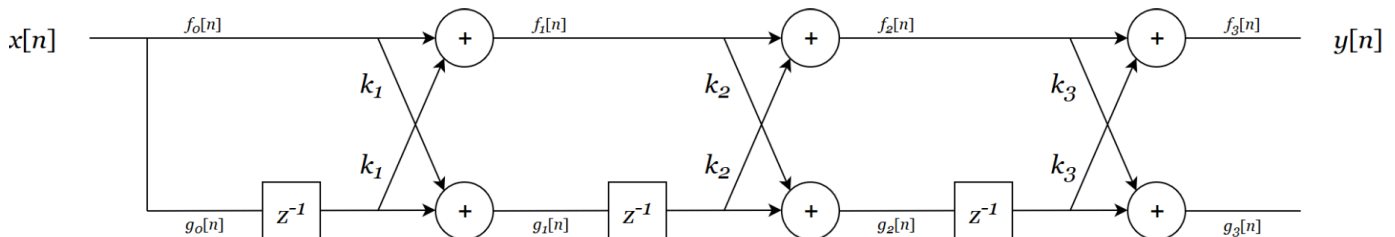
Code can be found in azlatfilt.m.

## [✓] ADSI Problem 2.2: Find impulse response of an all-zero lattice filter

Consider an all-zero (FIR) lattice filter with  $k_1 = 0.65$ ,  $k_2 = -0.34$  and  $k_3=0.8$ .

1. Find the impulse response of the filter by tracing an impulse  $x(n) = \delta(n)$  through the filter.

Since the lattice filter has three reflection coefficients, it is a 3-stage all-zero lattice filter.



Here are the formulas needed to trace through the filter.

$$x[n] = f_0[n] = g_0[n],$$

$$y[n] = f_M[n].$$

$$f_m[n] = f_{m-1}[n] + k_m g_{m-1}[n-1], \quad m = 1, 2, \dots, M$$

$$g_m[n] = k_m f_{m-1}[n] + g_{m-1}[n-1]. \quad m = 1, 2, \dots, M$$

The input is  $x[n] = \delta[n] = [1, 0, 0, 0]$

$n = 1$

$$f_0(n) = g_0(n) = x(1) = 1$$

$$f_1(n) = f_0(n) + k_1 g_0(n-1) = 1 + 0 = 1$$

$$g_1(n) = k_1 f_0(n) + g_0(n-1) = 0.65 + 0 = 0.65$$

$$f_2(n) = f_1(n) + k_2 g_1(n-1) = 1 - 0.34 \cdot 0 = 1$$

$$g_2(n) = k_2 f_1(n) + g_1(n-1) = -0.34 \cdot 1 + 0 = -0.34$$

$$f_3(n) = f_2(n) + k_3 g_2(n-1) = 1 + 0.8 \cdot 0.0 = \underline{1}$$

$$g_3(n) = k_3 f_2(n) + g_2(n-1) = 0.8 \cdot 1 + 0 = 0.8$$

$n = 2$

$$f_0(n) = g_0(n) = x(2) = 0$$

$$f_1(n) = 0 + 0.65 \cdot 1 = 0.65$$

$$g_1(n) = 0 + 1 = 1$$

$$f_2(n) = 0.65 - 0.34 \cdot 0.65 = 0.429$$

$$g_2(n) = -0.34 \cdot 0.65 + 0.65 = 0.429$$

$$f_3(n) = 0.429 + 0.8 \cdot -0.34 = \mathbf{0.157}$$

$$g_3(n) = 0.8 \cdot 0.429 - 0.34 = 0.0032$$

$n = 4$

$$f_0(n) = g_0(n) = x(4) = 0$$

$$f_1(n) = 0$$

$$g_1(n) = 0$$

$$f_2(n) = 0$$

$$g_2(n) = 0$$

$$f_3(n) = \mathbf{0.8}$$

$$g_3(n) = 1$$

We can also use the MATLAB function **azlatfilt** given in Figure 9.26 in the book.

```
k = [0.65, -0.34, 0.8];
x = [1, 0, 0, 0];
G = 1;
y = azlatfilt(k, x, G)
```

```
y = 1x4
    1.0000    0.1570    0.0032    0.8000
```

The impulse response of the given filter is:

$$h[n] = [1, 0.157, 0.0032, 0.8]$$

## ✓ ADSI Problem 2.3: Find the reflection coefficients for an all-zero lattice filter

An all-zero lattice filter has the following system function

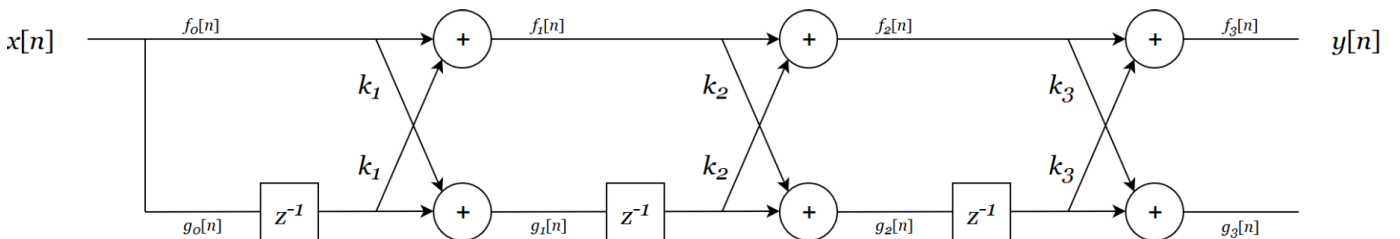
$$H(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$$

1. Find the reflection coefficients  $k_1$ ,  $k_2$  and  $k_3$  for the corresponding lattice filter.

The recursive algorithm works as follows:

1. Compute  $A_M(z) = \frac{H(z)}{h[0]}$
2. Compute  $k_M = a_M$  where  $a_M$  is the last coefficients of  $A_M(z)$ . For example,  
if  $A_M(z) = 1 + 0.06z^{-1} - 0.42z^{-2} + 0.5z^{-3}$  then  $k_M = 0.5$
3. Compute  $B_M(z)$  by flipping the coefficients of  $A_M(z)$
4. Set  $m = M$
5. Compute  $A_{m-1}(z) = \frac{1}{1 - k_m^2} [A_m(z) - k_m B_m(z)]$ . Notice that this is where the algorithm fails because  
if  $k_m = 1 \rightarrow k_m^2 = 1$  then we will have division by zero.
6. Compute  $k_{m-1} = a_{m-1}$  where  $a_{m-1}$  is the last coefficients of  $A_{m-1}(z)$
7. Compute  $B_{m-1}(z)$  by flipping the coefficients of  $A_{m-1}(z)$ . Alternatively, compute  $B_{m-1}(z) = z^{-m-1}A_{m-1}\left(\frac{1}{z}\right)$
8. Set  $m = m - 1$
9. Go to step 5 if  $m \neq 0$
10. We know that  $A_0(z) = B_0(z) = 1$

In this problem, we have a 3-stage all-zero lattice filter:



We are given its transfer function:

$$H(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$$

Since this is a FIR system of the form:

$$H(z) = \sum_{k=0}^M h[k]z^{-k}$$

We know that  $M = 3$ ,  $h[k] = \left[1, \frac{13}{24}, \frac{5}{8}, \frac{1}{8}\right]$  and  $h[0] = 1$ .

**Step 1: Compute  $A_3(z)$**

$$A_M(z) = A_3(z) = \frac{H(z)}{h[0]} = \frac{H(z)}{1} = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{8}z^{-3}$$

**Step 2: Compute  $k_3$**

The last coefficients of  $A_3(z)$  is  $\frac{1}{8}$  so  $k_3 = \frac{1}{8}$

**Step 3: Compute  $B_3(z)$**

Once we have  $A_M(z)$ , we can compute  $B_M(z)$  by simply flipping the coefficients of  $A_M(z)$ .

$$B_M(z) = B_3(z) = \frac{1}{8} + \frac{5}{8}z^{-1} + \frac{13}{24}z^{-2} + z^{-3}$$

**Step 4: Set  $m = M$**

$$m = 3$$

**Step 5: Compute  $A_2(z)$**

Using the formula  $A_{m-1}(z) = \frac{1}{1 - k_m^2} [A_m(z) - k_m B_m(z)]$  we can compute  $A_2(z)$

$$A_2(z) = \frac{1}{1 - k_3^2} [A_3(z) - k_3 B_3(z)]$$

$$A_2(z) = \frac{1}{1 - \left(\frac{1}{8}\right)^2} \left[ \left(1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{8}z^{-3}\right) - \frac{1}{8} \left(\frac{1}{8} + \frac{5}{8}z^{-1} + \frac{13}{24}z^{-2} + z^{-3}\right) \right]$$

$$A_2(z) = \frac{9}{8} \left[ 1 - \frac{1}{8} \cdot \frac{1}{8} + \left(\frac{13}{24} - \frac{1}{8} \cdot \frac{5}{8}\right)z^{-1} + \left(\frac{5}{8} - \frac{1}{8} \cdot \frac{13}{24}\right)z^{-2} + \left(\frac{1}{8} - \frac{1}{8}\right)z^{-3} \right]$$

$$A_2(z) = \frac{9}{8} \left[ \frac{8}{8} + \frac{1}{8}z^{-1} + \frac{4}{9}z^{-2} + 0z^{-3} \right]$$

$$A_2(z) = 1 + \frac{3}{8}z^{-1} + \frac{1}{2}z^{-2}$$

**Step 6: Compute  $k_2$**

We can read of the last coefficient of  $A_2(z)$  which is  $k_2 = \frac{1}{2}$

**Step 7: Compute  $B_2(z)$** 

Once we have  $A_2(z)$ , we can compute  $B_2(z)$  by simply flipping the coefficients of  $A_2(z)$ .

$$B_2(z) = \frac{1}{2} + \frac{3}{8}z^{-1} + z^{-2}$$

**Step 8: Set  $m = 2$** 

Step 9: Go to step 5

**Step 5: Compute  $A_1(z)$** 

We can compute  $A_1(z)$  as follows

$$A_1(z) = \frac{1}{1 - k_2^2} [A_2(z) - k_2 B_2(z)]$$

$$A_1(z) = \frac{1}{1 - \left(\frac{1}{2}\right)^2} \left[ \left(1 + \frac{3}{8}z^{-1} + \frac{1}{2}z^{-2}\right) - \frac{1}{2} \left(\frac{1}{2} + \frac{3}{8}z^{-1} + z^{-2}\right) \right]$$

$$A_1(z) = \frac{4}{3} \left[ 1 - \frac{1}{2} \cdot \frac{1}{2} + \left(\frac{3}{8} - \frac{1}{2} \cdot \frac{3}{8}\right)z^{-1} + \left(\frac{1}{2} - \frac{1}{2}\right)z^{-2} \right]$$

$$A_1(z) = \frac{4}{3} \left[ \frac{3}{4} + \frac{3}{16}z^{-1} + 0z^{-2} \right]$$

$$A_1(z) = 1 + \frac{1}{4}z^{-1}$$

**Step 6: Compute  $k_1$** 

We can read of the last coefficient of  $A_1(z)$  which is  $k_1 = \frac{1}{4}$

We are done! The result is  $k_1 = \frac{1}{4}$ ,  $k_2 = \frac{1}{2}$  and  $k_3 = \frac{1}{3}$

Let us verify our result using the MATLAB function **tf2latc** to convert transfer function filter parameters to lattice filter form

```
b = [1, 13/24, 5/8, 1/3];
a = 1;
k = tf2latc(b, a)
```

```
k = 3x1
    0.2500
    0.5000
    0.3333
```

We can also try the algorithm (Figure 9.24) from the book

```
k = fir2lat(b)
```

```
k = 1x3
    0.2500    0.5000    0.3333
```

Both functions agree!

The reflection coefficients are  $k_1 = \frac{1}{4}, k_2 = \frac{1}{2}, k_3 = \frac{1}{3}$

## [✓] ADSI Problem 2.5: All-zero lattice filter, find coefficients from system function

A filter has the system function

$$H(z) = 1 + 2z^{-1} + z^{-2}$$

1. Calculate  $k_1$  and  $k_2$  for the corresponding lattice filter and draw the lattice structure.

We cannot use a recursive algorithm if one of the reflection coefficients is equal to 1. In this problem,  $k_2 = 1$  so if we attempt to compute  $A_1(z)$ , it will lead to division by zero:

$$A_1(z) = \frac{1}{1 - k_2^2} [A_2(z) - k_2 B_2(z)] = \frac{1}{1 - 1} [A_2(z) - k_2 B_2(z)]$$

We can try out it out in MATLAB function:

```
b = [1, 2, 1];
a = 1;
% tf2latc(b) % This fails because one coeff. is 1
```

Instead we can use a cascade of two single-stage filters.

First, we factorise the original FIR filter to two cascade filters. This can be done by finding the roots and using the formula:

$$H(z) = (1 - r_1 z^{-1})(1 - r_2 z^{-1}) \cdots (1 - r_M z^{-1}) \text{ where } r_i \text{ are the roots of the transfer function.}$$

Find the roots:

```
roots(b)
```

```
ans = 2x1
    -1
    -1
```

Factorise the original FIR filter into two cascading filters:

$$H(z) = (1 - (-1)z^{-1})(1 - (-1)z^{-1}) = (1 + z^{-1})(1 + z^{-1})$$

Recall that that a single stage all-zero filter has following difference equation:

$$y[n] = x[n] + k_1 x[n-1]$$

Taking the z-transform we get:

$$Y_{az}(z) = X_{az}(z) + k_1 X_{az}(z)z^{-1}$$

The system function of a single stage all-zero lattice filter is therefore:

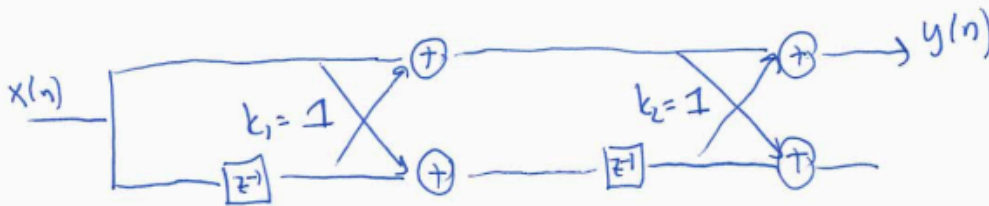
$$H_{az}(z) = \frac{Y_{az}(z)}{X_{az}(z)} = 1 + k_1 z^{-1}$$

It turns out that we can cascade two single stage all-zero lattice filters the same way as we cascade FIR filters.

instead we can factorize  $H(z) = A_z(z)$

$$H(z) = (1 + z^{-1})(1 + z^{-1})$$

so we can instead implement as a cascade of two single stages with  $k_1=1$  and it turns out that it also works in a single two-stage lattice



So in our case, we have  $k_1 = 1$  and  $k_2 = 1$ .

Let us check if we can get the impulse response of the original filter.

```
k = [1, 1];
lat2fir(k, 1)
```

```
ans = 1x3
      1      2      1
```

```
latc2tf(k) % Same as lat2fir
```

```
ans = 1x3
      1      2      1
```

Success!

**[✓] ADSI Problem 2.7: Find system function an all-pole filter from reflection coefficients**

Determine the system function for an all-pole filter with the reflection coefficients  $k_1 = 0.6$ ,  $k_2 = 0.3$ ,  $k_3 = 0.5$  and  $k_4 = 0.9$ .

We can do this by hand using following two formulas:

$$f_{m-1}[n] = f_m[n] - k_m g_{m-1}[n-1]$$

$$g_m[n] = k_m f_{m-1}[n] + g_{m-1}[n-1]$$

We start computing the difference equations of the last section:

$$f_0[n] = f_1[n] - k_1 g_0[n-1]$$

$$g_1[n] = k_1 f_0[n] + g_0[n-1]$$

Then we compute the next stage:

$$f_1[n] = f_2[n] - k_2 g_1[n-1]$$

$$g_2[n] = k_2 f_1[n] + g_1[n-1]$$

Next, we can substitute  $f_1[n]$  into the expression  $f_0[n]$ :

$$f_0[n] = f_2[n] - k_2 g_1[n-1] - k_1 g_0[n-1]$$

$$f_0[n] = f_2[n] - k_2(k_1 f_0[n-1] + g_0[n-2]) - k_1 g_0[n-1]$$

And so on.

Finally, we will get a large expression in which we replace:

- $g_0[n]$  and  $f_0[n]$  with  $y[n]$
- $f_4[n]$  with  $x[n]$

Alternatively, we can use MATLAB to compute the transfer function:

```
k = [0.6, 0.3, 0.5, 0.9];
[num, dem] = latc2tf(k, 'allpole')
```

```
num = 1x5
      1      0      0      0      0
dem = 1x5
      1.0000      1.3800      1.3110      1.3370      0.9000
```

The system function is:

$$H(z) = \frac{1}{1 + 1.38z^{-1} + 1.311z^{-2} + 1.337z^{-3} + 0.9z^{-4}}$$

## ADSI Problem 2.10: Linear prediction and lattice filters



The idea behind linear prediction is that the signal  $\hat{x}(n)$  can be estimated as a weighted linear combination of the  $p$  previous samples

$$\hat{x}(n) = - \sum_{k=1}^p a_p(k)x(n-k)$$

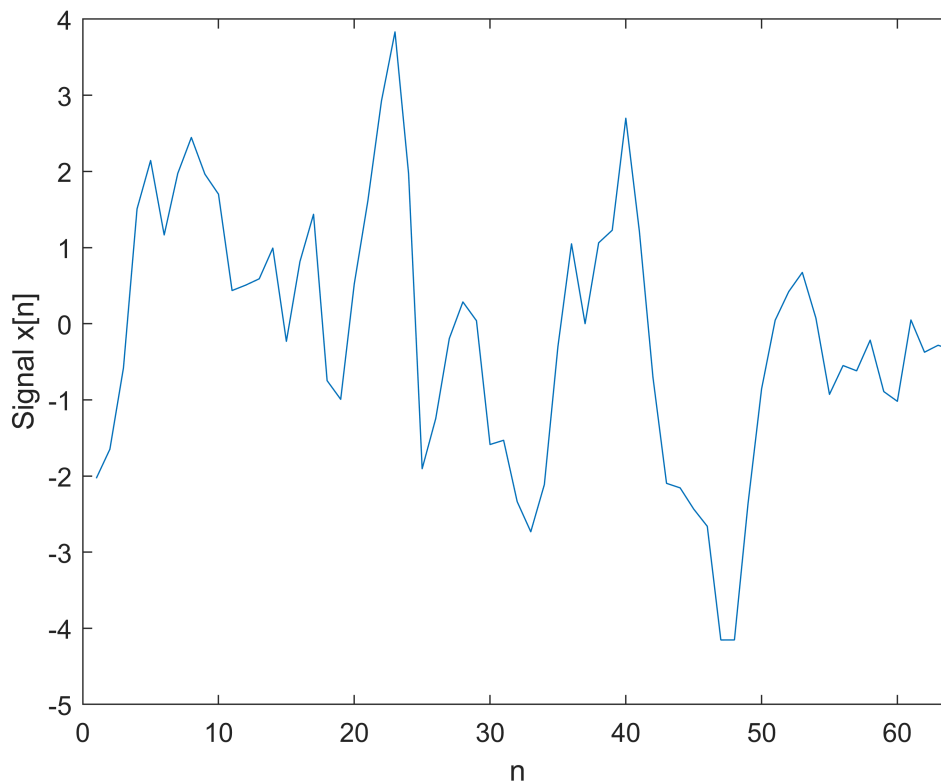
## 1) Compute autocorrelation from a signal

The file `signal1.dat` contains 64 samples from an artificial signal.

1. Load `signal1.dat` into MATLAB. Create an autocorrelation of the signal using `xcorr` and plot it. What does the autocorrelation tell you about the signal?

Autocorrelation means computing the cross-correlation between a signal  $x[n]$  with the same signal by shifting by some amount  $x[n-k]$

```
x = importdata('signal1.dat');  
plot(x);  
xlabel('n');  
ylabel('Signal x[n]');  
xlim([0, numel(x)]);
```

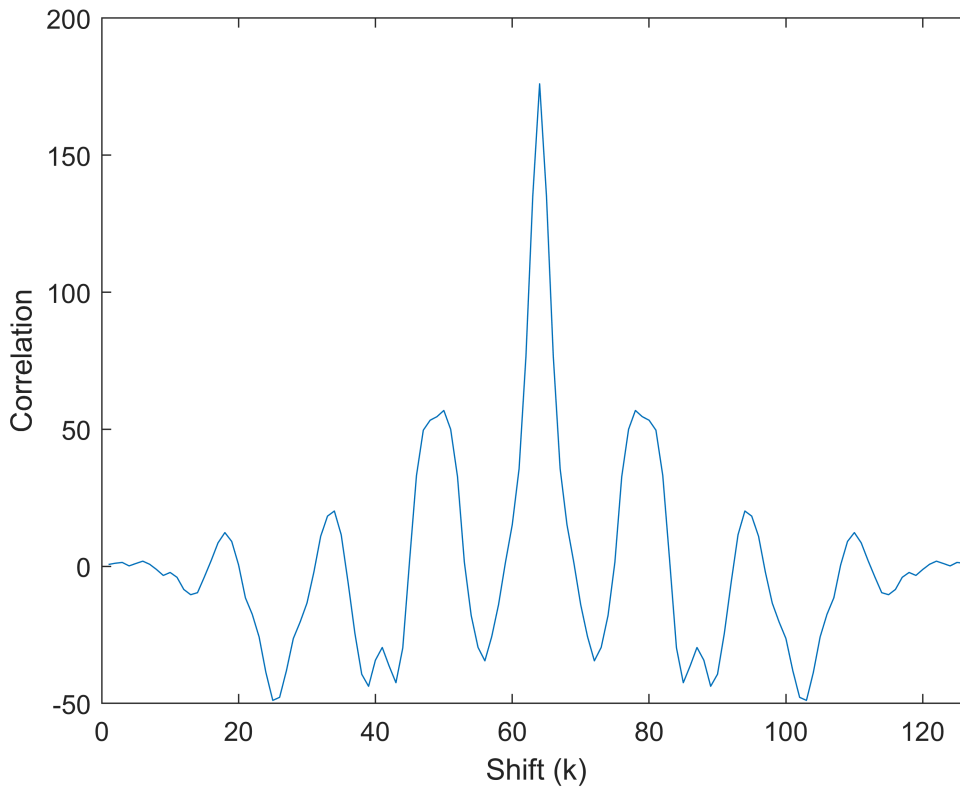


```
% r = xcorr(x) returns the autocorrelation sequence of x  
r = xcorr(x);
```

```

plot(r);
xlabel('Shift (k)');
ylabel('Correlation');
xlim([0, numel(r)]);

```



From the plot, we see that when the signal is shifted e.g. by 4, the correlation is low.

```
r(4)
```

```
ans = 0.1657
```

```
r(50)
```

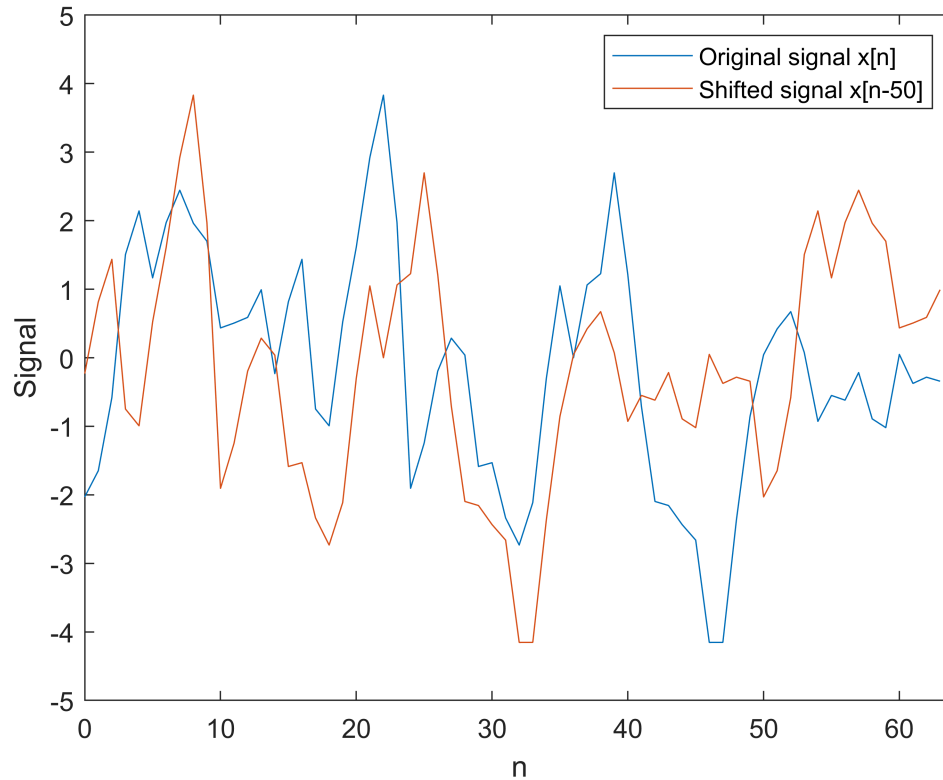
```
ans = 56.8712
```

This means that the original signal  $x[n]$  and the shifted signal  $x[n - 4]$  are not very similar. The correlation is a lot higher 56 when the signal is shifted by 50:

```

n = linspace(0, 63, 64)';
x_shifted_50 = circshift(x, 50);
plot(n, x, n, x_shifted_50);
legend('Original signal x[n]', 'Shifted signal x[n-50]')
xlabel('n');
ylabel('Signal');
xlim([0, numel(x)]);
ylim([-5, 5])

```



Shifting the signal by 64 yields the highest correlation. This makes sense because if we shift a signal of length 64 by 64 then we get the original signal  $x[n] = x[n - 64]$ .

## 2) Compute the optimum second-order linear predictor

Based on the autocorrelation values we can calculate the coefficients for the optimum  $p$ th order linear predictor with the normal equations (which we will derive later in the course)

$$r_x(l) = -\sum_{k=1}^p a_p(k)r_x(l-k), \quad l = 1, \dots, p$$

Let  $p = 2$ .

2. Compute  $a_2(1)$  and  $a_2(2)$ .

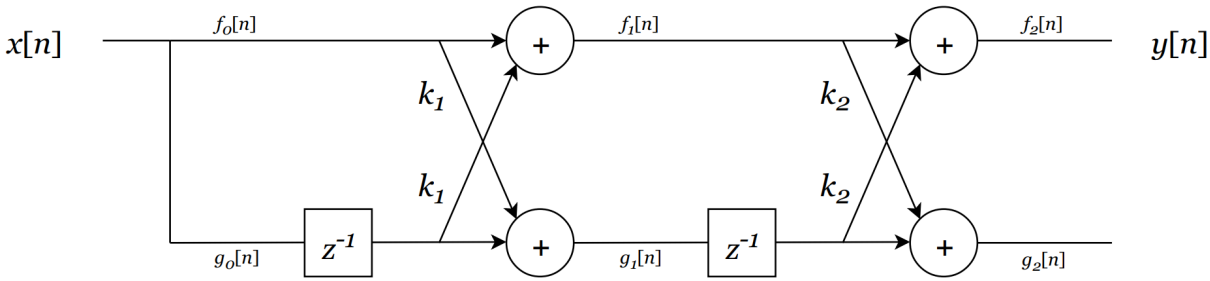
When  $p = 2$  then we have:

$$r_x(l) = -a_2(1)r_x(l-1) - a_2(2)r_x(l-2)$$

## 3) Compare the linear predictor with a two-stage all-zero lattice filter

Next, consider a two-stage all-zero lattice filter.

3. What is the relation between the output of the filter,  $f_2(n)$ , and the above linear prediction?



We know that the output of a two-stage all-zero lattice filter is:

$$y[n] = x[n] + k_1(1 + k_2)x[n - 1] + k_2x[n - 2]$$

Suppose  $x[n] = 0$ . By pattern matching, we can say that:

$$k_1(1 + k_2) = -a_2(1) \text{ and } k_2 = -a_2(2)$$

#### 4) Compute reflection coefficients from the linear predictor

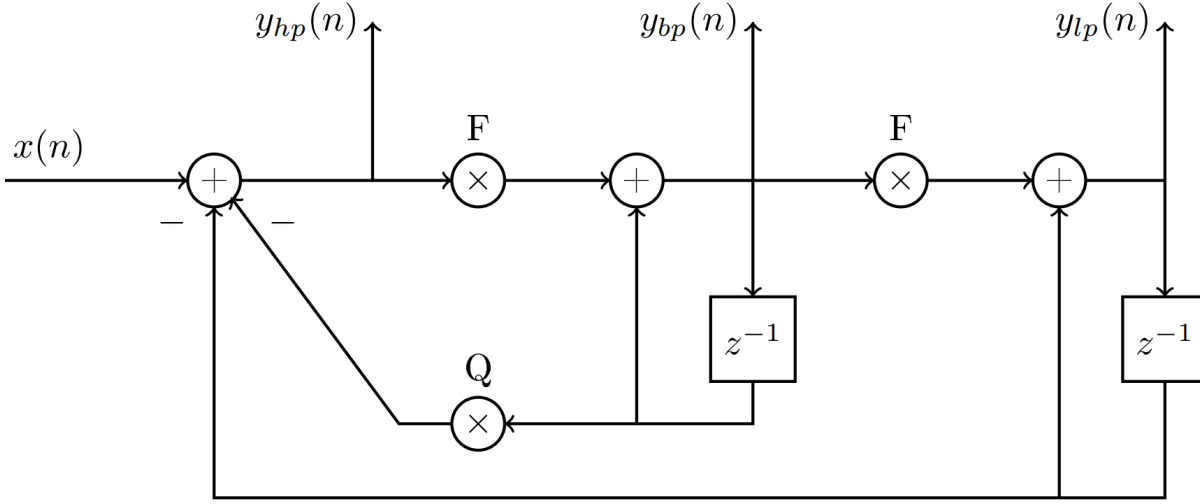
4. Compute the reflection coefficients  $k_1$  and  $k_2$  from the  $a_2(1)$  and  $a_2(2)$  values found above.

#### 5) Compute prediction error

5. Send the signal through a 2nd order all-zero lattice filter and analyze the prediction error. Is the signal predictable? Repeat for a higher order predictor.

### ADSI Problem 2.11 (Optional): The digital state variable filter

One interesting filter structure is the digital state variable filter which has its roots in the analog state variable filter. It can be considered as a standard second order filter, but with the valuable feature that the center frequency and Q-factor are independently tuneable. The structure of the filter is shown here



Where  $F = 2 \sin(\pi F_c / F_s)$  with  $F_c$  the filter corner frequency,  $F_s$  is the sampling frequency and  $Q$  is the  $Q$ -factor of the filter. From the structure, updating equations for the three state variables,  $y_{hp}(n)$ ,  $y_{bp}(n)$  and  $y_{lp}(n)$  can easily be written down. Index hp, bp and lp denotes highpass, bandpass and lowpass respectively. For  $y_{hp}(n)$  the equation becomes

$$y_{hp}(n) = x(n) - y_{lp}(n-1) - Qy_{bp}(n-1).$$

1. Write down the updating equations for the two other state variables.
2. Calculate the transfer function  $H_{LP}(z)$  from the input to the lowpass output.
3. Implement the state variable filter as a function in Matlab. Make sure the three equations are updated in the correct order.
4. Test the implementation by sending white noise sampled at 48000 Hz through the filter. Use  $F_c=1000$  Hz and  $Q=2$ . Plot the spectra of the three outputs on the same graph and compare with the expected spectra. Repeat for  $Q=10$ .

The filter is described in P. Dutilleux “Simple to operate digital time varying filters” AES 86th Convention, Hamburg (1989) In this paper Dutilleux shows that the digital state variable filter fails if  $F_c$  becomes too large. A rule of thumb is to keep  $F_c < F_s/10$ .

5. Demonstrate that this is true by using  $F_c=20000$  Hz in the above implementation.
6. A band reject output is also easily obtained with the digital state variable filter. Extend the above drawing with a  $y_{br}(n)$  output.

## Functions

```
function [y] = azlatfilt(k, x, G)
% AZLATFILT      Function for implementation of an all-zero FIR lattice filter
% From Figure 9.26 (p 515)

M = length(k);

% Create an array for the sequence f_m[n]
f = zeros(1,M);

% Create an array for the sequence g_m[n]
g = f;

% Create an array for the sequence g_m[n-1]
oldg = zeros(1,M);

% Keeps track of x[n-1]
oldx = 0;
x = G*x;
y = zeros(size(x));

for n=1:length(x)
    % Compute f1[n] = x[n] + k1x[n-1] -> Equation (9.57a)
    f(1) = x(n)+k(1)*oldx;

    % Compute g1[n] = k1x[n] + x[n-1] -> Equation (9.57b)
    g(1) = k(1)*x(n)+oldx;
    oldx = x(n);
    for m = 2:M
        % Compute: fm[n] = fm-1[n] + km gm-1[n-1] -> Equation (9.55a)
        f(m) = f(m-1)+k(m)*oldg(m-1);
```

```

        % Compute:  $g_m[n] = k_m f_{m-1}[n] + g_{m-1}[n-1]$  -> Equation (9.55b)
        g(m) = k(m)*f(m-1)+oldg(m-1);

        % Delay
        oldg(m-1) = g(m-1);
    end

    %  $y[n] = f_M[n]$  -> Equation (9.56b)
    y(n) = f(M);
end
end

```

```

function [k,G] = fir2lat(h)
% Converts FIR filter coefficients to lattice coefficients.
% From Figure 9.24 (p. 514)
    G = h(1);
    a = h/G;
    M = length(h)-1;
    k(M) = a(M+1);
    for m = M:-1:2
        b = fliplr(a);
        a = (a-k(m)*b)/(1-k(m)^2); a = a(1:m);
        k(m-1) = a(m);
    end
end

```

```

function h = lat2fir(k, G)
% Converts lattice coefficients to FIR filter coefficients.
% From Figure 9.25 (p. 515)
    a = 1;
    b = 1;
    M = length(k);
    for m = 1:1:M
        a = [a,0]+k(m)*[0,b];
        b = fliplr(a);
    end
    h = G*a;
end

```