Homework 4

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Problem 13.9: Determine the mean, ACVS and stationarity of a random process

9. A random process $x[n]$ is characterized by

$$x[n] = A(\zeta)\cos\left[\Omega(\zeta)n + \Theta(\zeta)\right],$$

where random variables $A(\zeta)$, $\Omega(\zeta)$, and $\Theta(\zeta)$ are mutually independent. Random variables $A(\zeta) \sim U(0,1)$ and $\Theta(\zeta) \sim U(-\pi,\pi)$ are of continuous type while $\Omega(\zeta)$ is of discrete type taking values 10 and 20 radians with equal probability.

[✔] a) Determine the mean sequence $m_x[n]$

We need to compute:

$$E[x[n]] = E[A\cos(\Omega n + \Theta)]$$

Since the three random variables are independent then we can write:

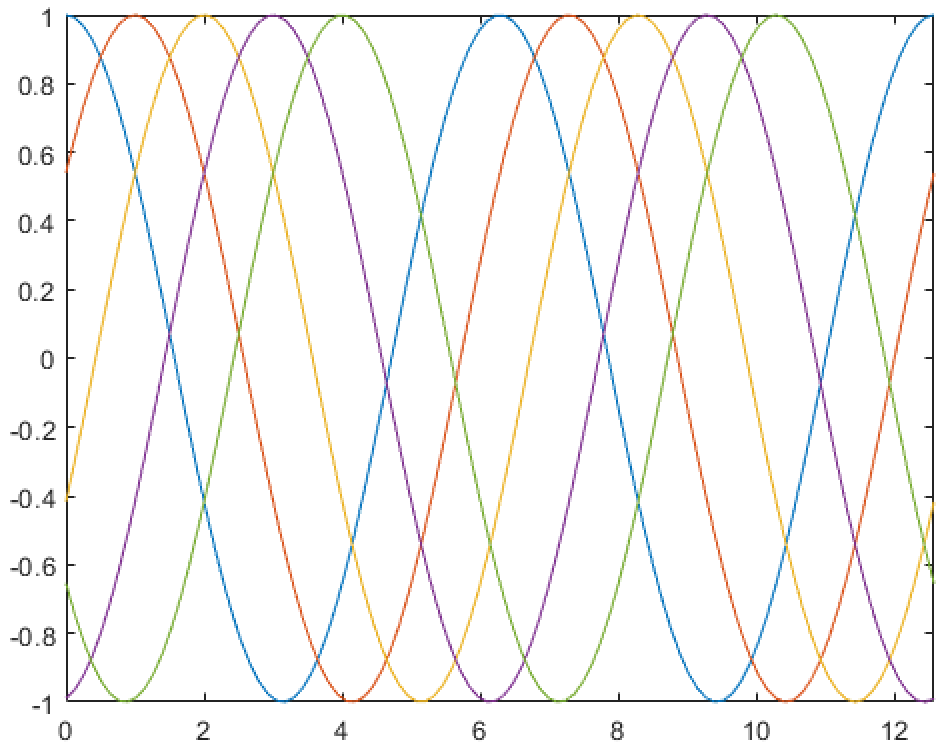
$$E[x[n]] = E[A]E[\cos(\Omega n + \Theta)]$$

We know that $E[A] = \frac{1}{2}$ since $A \sim U(0,1)$.

Computing the expression $E[\sin(\Theta)]$ requires a bit of an explanation.

Suppose we want to compute $E[\cos(\Omega n+\Theta)]$ where $\Theta \sim U(-\pi,\pi)$. Let us pick one frequency ω (realise one value of Ω). Then let us pick a lot of realisations of Θ. Now if we plot the function $\cos(\omega n+\theta)$ for different values of θ then we will see something like this:

```
n = linspace(0, 4*pi);
plot(n, cos(n), n, cos(n-1), n, cos(n-2), n, cos(n-3), n, cos(n-4));
xlim([0, 4*pi]);
```



If we plot hundreds of cosine functions shifted slightly, we get a large blob of points from -1 to 1. For this reason, the quality $E[\cos(\Omega n+\Theta)]$ will be zero because the mean value is 0. Formally, we can write:

$$E[\cos(\Omega n+\Theta)] = \int_{-\pi}^{\pi} f_{\Theta}(\Theta)\cdot\cos(\Omega n+\Theta)\,d\Theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\Omega n+\Theta)\,d\Theta$$

So we are integrating cosine function over 2π which is zero.

In signal processing, we like to add random shifts ala $\Theta \sim U(0,2\pi)$ to avoid that the expected value or the mean value becomes dependent on time.

Alternatively, we can use

Using the rule $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$, we rewrite the expression to:

$E[x[n]] = E[A(\cos(\Omega n)\cos(\Theta) - \sin(\Omega n)\sin(\Theta))]$
$E[x[n]] = E[A\cos(\Omega n)\cos(\Theta) - A\sin(\Omega n)\sin(\Theta)]$

At this point, we need to employ some expection rules to decompose the expression.

Let X and Y be two random variables and a and b be two constants. Following expectation identities apply:

1. $E[a] = a$ e.g. $E(42) = 42$
2. $E[aX] = aE[X]$ e.g. if you multiply every value by 2, the expectation doubles
3. $E[a \pm X] = a \pm E[X]$ e.g. if you add 42 to every case, the expectation increases by 42
4. $E[X + Y] = E[X] + E[Y]$
5. If X and Y are independent, then $E[\mathrm{XY}] = E[X]E[Y]$
6. $E[a \pm \mathrm{bX}] = a \pm E[bX] = a \pm bE[X]$
7. $E[b(a \pm \mathrm{X})] = bE[a \pm \mathrm{X}] = b(a \pm E[X])$

Use rule 4:

$E[x[n]] = E[A\cos(\Omega n)\cos(\Theta)] - E[A\sin(\Omega n)\sin(\Theta)]$

Use rule 5 multiple times:

$E[x[n]] = E[A]E[\cos(\Omega n)]E[\cos(\Theta)] - E[A]E[\sin(\Omega n)]E[\sin(\Theta)]$

But how can we continue from here?

We know that $\Theta \sim U(-\pi, \pi)$ so $E[\Theta] = \frac{\pi + (-\pi)}{2} = \frac{0}{2} = 0$

b) Determine the ACVS $c_X[m, n]$

c) Comment on the stationarity of the random process

[✔] Problem 13.13: MSE objective function

13. Consider the mse objective function (13.56)

$$J(a, b) = \mathrm{E}[(Y - aX - b)^2].$$

a) Express the objective function in terms of its parameters

(a) Express $J(a, b)$ in terms of the parameters a, b, and the moments of X and Y.

Use MATLAB to expand the expression inside the expected value:

```
syms a b X Y
expand((Y-a*X - b)^2)
```

ans = $X^2 a^2 - 2\,X\,Y\,a + 2\,X\,a\,b + Y^2 - 2\,Y\,b + b^2$

So we have:

$J(a,b) = E[X^2a^2 - 2\,X\,Y\,a + 2\,X\,a\,b + Y^2 - 2\,Y\,b + b^2]$

Let X and Y be two random variables and a and b be two constants. Following expectation identities apply:

1. $E[a] = a$ e.g. $E(42) = 42$
2. $E[a\,X] = a\,E[X]$ e.g. if you multiply every value by 2, the expectation doubles
3. $E[a \pm X] = a \pm E[X]$ e.g. if you add 42 to every case, the expectation increases by 42
4. $E[X + Y] = E[X] + E[Y]$
5. If X and Y are independent, then $E[\mathrm{XY}] = E[X]E[Y]$

Use rule 4:

$J(a,b) = E[X^2a^2 - 2\,X\,Y\,a + 2\,X\,a\,b + Y^2 - 2\,Y\,b + b^2]$
$J(a,b) = E[X^2a^2] - E[2\,X\,Y\,a] + E[2\,X\,a\,b] + E[Y^2] - E[2\,Y\,b] + E[b^2]$

Use rule 1 and rule 2:

$J(a,b) = a^2E[X^2] - 2\,a\,E[\,X\,Y] + 2\,a\,b\,E[X] + E[Y^2] - 2\,b\,E[Y] + b^2$

b) Using partial derivatives to determine the values of parameters

(b) Using partial derivatives $\dfrac{\partial J}{\partial a}$ and $\dfrac{\partial J}{\partial b}$, determine the values of a and b by solving the equations $\partial J/\partial a = 0$ and $\partial J/\partial b = 0$ that minimize $J(a, b)$ to obtain optimum values given in (13.58) and (13.62).

First, take the partial derivatives:

$$\frac{\partial J(a,b)}{\partial a} = 2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X]$$

$$\frac{\partial J(a,b)}{\partial b} = 2\,a\,E[X] - 2\,E[Y] + 2b$$

Next, solve the equations:

(Eq. 1) $2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X] = 0$

(Eq. 2) $2\,a\,E[X] - 2\,E[Y] + 2b = 0$

Isolate b in (Eq. 2):

$$2b = -2\,a\,E[X] + 2\,E[Y]$$
$$b = -a\,E[X] + E[Y]$$
$$b = E[Y] - a\,E[X]$$

This corresponds to (13.58) in the book:

$$(13.58) \qquad b_{\mathrm{o}} = m_y - am_x$$

Now, plug the expression for b into Eq. 1 in order to find an expression for a:

$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,b\,E[X] = 0$$
$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,(E[Y] - a\,E[X])\,E[X] = 0$$
$$2a\,E[X^2] - 2\,E[X\,Y] + 2\,E[X]E[Y] \; - 2\,a\,E[X]E[X] = 0$$
$$2a\,E[X^2] - 2\,a\,E[X]E[X] - 2\,E[X\,Y] + 2\,E[X]E[Y] \; = 0$$
$$2a(\,E[X^2] - E[X]E[X]) - 2\,E[X\,Y] + 2\,E[X]E[Y] \; = 0$$
$$2a(\,E[X^2] - E[X]E[X]) = 2\,E[X\,Y] - 2\,E[X]E[Y]$$
$$a(\,E[X^2] - E[X]E[X]) = E[X\,Y] - E[X]E[Y]$$
$$a = \frac{E[X\,Y] - E[X]E[Y]}{E[X^2] - E[X]E[X]} = \frac{E[X\,Y] - E[X]E[Y]}{E[X^2] - E[X]^2}$$

We have found an expression for a. The numerator looks like it is the covariance:

Covariance The *covariance* of two random variables X and Y is defined by

$$(13.25) \qquad c_{xy} \triangleq \mathrm{cov}(X,Y) \triangleq \mathrm{E}[(X - m_x)(Y - m_y)] \; = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$$

The denominator looks like it is the variance:

$$(13.11) \qquad \mathrm{var}(X) = E[X^2] - E[X]^2 = E[X^2] - m_x^2$$

Therefore, the derived expression is the same as (13.62) in the book.

$$(13.62) \qquad a_{\mathrm{o}} = \frac{c_{xy}}{\sigma_x^2} = \rho_{xy}\frac{\sigma_y}{\sigma_x}$$

Problem 13.22: Computing distributions

22. Consider two jointly distributed random variables X and Y with pdf

$$f(x, y) = \begin{cases} 8xy, & 0 \le x \le 1, 0 \le y \le x \\ 0. & \text{otherwise} \end{cases}$$

a) Determine marginal distributions and conditional probabilities

(a) Determine $f(x)$, $f(y)$, $f(x|y)$, and $f(y|x)$.

The **marginal** distributions of random variables X and X are obtained by integration as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\mathrm{d}y \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\mathrm{d}x, \qquad (13.19)$$

Compute the marginal distribution of X:

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y = \int_0^x 8xy\,\mathrm{d}y = \left[\frac{1}{2}8xy^2\right]_0^x = \frac{1}{2}8xx^2 - 0 = 4x^3 \text{ where } 0 \le x \le 1$$

Compute the marginal distribution of Y:

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}x = \int_y^1 8xy\,\mathrm{d}x = [4yx^2]_y^1 = 4y - 4y^3 \text{ where } 0 \le y \le 1$$

To compute $f(x|y)$ we use following relation:

$$f_{X,Y}(x, y) = f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y). \qquad (13.23)$$

From Eq. 13.23, we know that:

$$f(x|y) = \frac{f(x, y)}{f(y)} = \frac{8\text{xy}}{4y - 4y^3} = \frac{4y \cdot 2x}{4y(1 - y^2)} = \frac{2x}{1 - y^2}$$

and

$$f(y|x) = \frac{f(x, y)}{f(x)} = \frac{8x\,y}{4x^3} = \frac{4x \cdot 2y}{4x \cdot x^2} = \frac{2y}{x^2}$$

b) Are X and Y independent?

(b) Are X and Y independent?

Random variables X and Y are statistically independent, if $f(y|x) = f(y)$ or $f(x|y) = f(x)$.

In a) we have computed the following expressions:

- $f(x) = 4x^3$
- $f(y) = 4y - 4y^3$
- $f(y|x) = \frac{2y}{x^2}$
- $f(x|y) = \frac{2x}{1 - y^2}$

Clearly $f(y|x) \neq f(y)$ and $f(x|y) \neq f(x)$. Therefore, the answer is no! The random variables X and Y are not statistically independent.