Signal Modelling

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Signal Modelling

Signal modelling is about gaining some understanding of certain signals so we can model them. We do not need to have a full suite of samples but we want to understand from a statistical perspective.

White noise a building blocks that we can use.

Signal modelling is about taking some unknown signal $y(n)$ and coming up with an LTI system that generates $y(n)$ given some white noise input.

All the correlation that was between samples before are stored in the filter coefficients. This becomes a model of our signal. If we know the filter, we know the signal. The samples of the signal will not be the same but the signal will have the same statistical properties.

We cannot generate the exact same $y(n)$ using different white noise inputs. Instead we generate signals with the **same statistical properties**.

Regular processes

A regular process is just a minimum-phase LTI system given by the difference equation:

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] \qquad (13.129)$$

This is just a FIR filter with infinitely many coefficients.

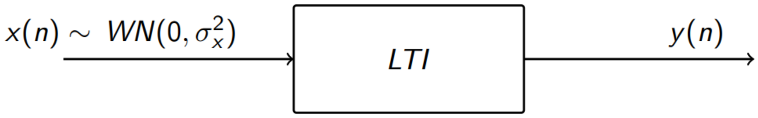
The filter $h[n]$ is known as a ***synthesis*** or ***colouring*** filter.

Since the LTI system is minimum-phase, it is causal, stable and therefore also invertible.

$$x[n] = \sum_{k=0}^{\infty} h_{\text{inv}}[k]y[n-k] \qquad (13.130)$$

This system is known as an ***analysis*** or ***whitening*** filter.

If feed this minimum-phase LTI system a white noise signal, we can compute the autocorrelation and PSD of the resulting process:



The autocorrelation of the input signal (white noise) is:

$r_{xx}(\ell) = \sigma_x^2 \delta(\ell)$, where $\sigma_x^2 = E[x^2(n)]$ is the variance

The power spectral density of the input signal is:

$$S_{xx}(\omega) = \sigma_x^2$$

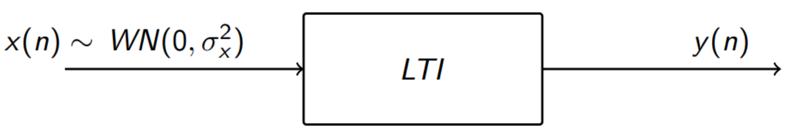
The autocorrelation of the output signal is:

$$r_{yy}(\ell) = \sigma_x^2 \, r_{hh}(\ell)$$

The power spectral density of the output signal is power transfer function of the LTI system:

$$S_{yy}(\omega) = \sigma_x^2 \left| H(e^{j\omega}) \right|^2$$

Finding the synthesis filter $h[m]$ from the autocorrelation or PSD is known as ***spectral factorization***. This is essentially what signal modelling is about; taking some unknown signal $y[n]$ and coming up with a minimum-phase LTI system that generates $y(n)$ given some white noise input.



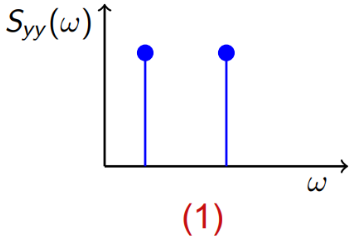
Paley-Wiener condition

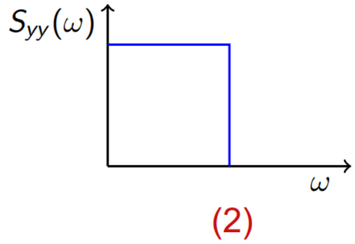
A process is regular if its PSD satisfies the Paley–Wiener condition:

$$\int_{-\pi}^{\pi} |\ln S_{yy}(\omega)| \mathrm{d}\omega < \infty. \qquad (13.131)$$

There are two kind of signals that cannot be modelled by an LTI system driven by white noise:

1. signal with line spectra e.g. sinusoids $\cos(\omega)$
2. bandwidth limited spectra





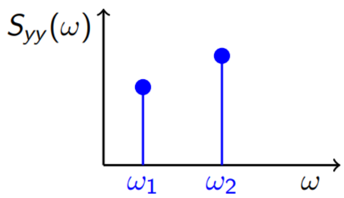
We cannot model these types of signals because it does not fulfill the Paley-Wiener condition:

$$\int_{-\pi}^{\pi} |\ln S_{yy}(\omega)| d\omega < \infty$$

Modelling signals with line spectra

Signals with line spectra are easy to model.

Suppose we want to model following signal:



We can just use a **harmonic process** to model the signal:

$$x[n] = \sum_{k=1}^{p} A_k \cos(\omega_k n + \phi_k)$$

where

- $\omega_k \neq 0$
- p is a constant denoting the number of frequency components
- $A_1, \cdots, A_p$ are constants denoting amplitudes
- $\omega_1, \cdots, \omega_p$ are constants denoting frequencies
- $\phi_1, \cdots, \phi_p$ are pairwise independent random variables uniformly distributed in the interval $(0, 2\pi)$

This model has $3p$ parameters A_k, ω_k and ϕ_k that we need to estimate. We can find these parameters using the least-squares.

The harmonic process is wide-sense stationary with mean value zero:

$$E[x(n)] = 0$$

The autocorrelation of the harmonic process is:

$$r_{xx}(\ell) = \frac{1}{2}\sum_{k=1}^{p} A_k^2 \cos(\omega_k \ell)$$

Take the Fourier Transform, we get the power spectral density:

$$S_{xx}(\omega) = \sum_{k=1}^{p} 2\pi \frac{A_k^2}{4}\left[\delta(\omega - \omega_k) + \delta(\omega + \omega_k)\right]$$

Modelling signals with bandwidth limited spectra

If we want to compute a filter with the brick-wall shape (2), we need an infinitely long non-causal filter. However, we said that we only wanted minimum-phase filters i.e. causal and stable filter. Therefore, this is not going to work.

Wold decomposition

The Wold decomposition says: *any stationary discrete-time process can be expressed as the sum of two* ***uncorrelated*** *processes, one regular process and one harmonic process.*