

Autocorrelation Functions

Table of Contents

Summary.....	1
Details.....	2
Autocorrelation and cross-correlation of complex signals.....	2
Relations between $r(\ell)$ and $r(-\ell)$	2
Autocorrelation of complex signal.....	3
Autocorrelation of a real cosine signal.....	3
Autocorrelation of a sine signal.....	5
Autocorrelation of White noise.....	6

Summary

The autocorrelation of the **complex sinusoid** $z(n) = A e^{j(\omega n + \phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is given by:

$$r_{zz}(\ell) = A^2 e^{j\omega\ell}$$

The relation between $r_{zz}(\ell)$ and $r_{zz}(-\ell)$ is:

$$r_{zz}(-\ell) = r_{zz}^*(\ell)$$

The autocorrelation function of a real **cosine signal** $z(n) = A \cos(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is:

$$r_{zz}(\ell) = \frac{A^2}{2} \cos(\omega\ell)$$

The autocorrelation function of a real **sine signal** $z(n) = A \sin(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is:

$$r_{zz}(\ell) = -\frac{A^2}{2} \cos(\omega\ell)$$

The autocorrelation of white noise $w(x)$ is

$$r_{ww}(\ell) = \sigma_w^2 \delta(\ell)$$

Let $v(n)$ and $w(n)$ be **two uncorrelated white noise** processes with variance $\sigma_v^2 = 0.49$ and $\sigma_w^2 = 1$. The cross-correlation between these processes is:

$$r_{vw}(\ell) = 0$$

Details

Autocorrelation and cross-correlation of complex signals

The autocorrelation function of a complex signal is given by:

$$r_{xx}(\ell) = E[x(n)x^*(n - \ell)]$$

The cross-correlation function of a complex signal is given by:

$$r_{yx}(\ell) = E[y(n)x^*(n - \ell)]$$

Relations between $r(\ell)$ and $r(-\ell)$

Let us compute the autocorrelation for $-\ell$:

$$r_{xx}(-\ell) = E[x(n)x^*(n + \ell)]$$

Suppose $m = n + \ell$. Then we can write $n = m - \ell$. Let us substitute all n with m in the above expression:

$$r_{xx}(-\ell) = E[x(m - \ell)x^*(m)]$$

Computing the complex conjugate of the expectation we get:

$$r_{xx}(-\ell) = E[x^*(m - \ell)x(m)]^*$$

$$r_{xx}(-\ell) = E[x(m)x^*(m - \ell)]^*$$

Since $r_{xx}^*(\ell) = E[x(m)x^*(m - \ell)]^*$, we know that:

$$r_{xx}(-\ell) = r_{xx}^*(\ell)$$

Autocorrelation of complex signal

Problem:

What is the autocorrelation function of the complex sinusoid $x(n) = A e^{j(\omega n + \phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$

Solution:

The autocorrelation function for complex signals can be computed as:

$$r_{xx}(\ell) = E[z(n)z^*(n - \ell)]$$

Plugging the given complex sinusoid into the formula, we get:

$$r_{xx}(\ell) = E[A e^{j(\omega n + \phi)} A e^{-j(\omega(n - \ell) + \phi)}]$$

Since A is a constant, we can move it outside the expected value:

$$r_{xx}(\ell) = A^2 E[e^{j(\omega n + \phi)} e^{-j(\omega(n - \ell) + \phi)}]$$

$$r_{xx}(\ell) = A^2 E[e^{j\omega n + j\phi} e^{-j\omega n + j\omega\ell - j\phi}]$$

$$r_{xx}(\ell) = A^2 E[e^{j\omega n + j\phi + (-j\omega n + j\omega\ell - j\phi)}]$$

$$r_{xx}(\ell) = A^2 E[e^{j\omega n + j\phi - j\omega n + j\omega\ell - j\phi}]$$

$$r_{xx}(\ell) = A^2 E[e^{j\omega\ell}]$$

We know that $E[e^{j\omega\ell}] = e^{j\omega\ell}$ because the expected value of a constant is just the constant itself. Notice that ϕ is no longer in the expression $e^{j\omega\ell}$. Therefore, the autocorrelation of the complex sinusoid is:

$$r_{xx}(\ell) = A^2 e^{j\omega\ell}$$

Thus, the autocorrelation of the complex sinusoid $z(n) = A e^{j(\omega n + \phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is given by:

$$r_{zz}(\ell) = A^2 e^{j\omega\ell}$$

Autocorrelation of a real cosine signal

Problem:

What is the autocorrelation function of a real signal $x(n) = A \cos(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$?

Solution:

In ADSI Problem 4.4.1, we know that the autocorrelation of the complex sinusoid $z(n) = A e^{j(\omega n + \phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is given by:

$$r_{zz}(\ell) = A^2 e^{j\omega\ell}$$

Since the result from 1) uses Euler, we need to convert the signal to complex exponential.

We use the relation $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

$$x(n) = A \cos(\omega n + \phi)$$

$$x(n) = \frac{A}{2} (e^{j(\omega n + \phi)} + e^{-j(\omega n + \phi)})$$

$$x(n) = \frac{A}{2} e^{j(\omega n + \phi)} + \frac{A}{2} e^{-j(\omega n + \phi)}$$

In 1) we found that the autocorrelation of a complex sinusoid $x(n) = A e^{j(\omega n + \phi)}$ is $r_{xx}(\ell) = A^2 e^{j\omega\ell}$

Therefore, the autocorrelation of the real signal is:

$$r_{xx}(\ell) = \left(\frac{A}{2}\right)^2 e^{j\omega\ell} + \left(\frac{A}{2}\right)^2 e^{-j\omega\ell}$$

$$r_{xx}(\ell) = \frac{A^2}{4} e^{j\omega\ell} + \frac{A^2}{4} e^{-j\omega\ell}$$

$$r_{xx}(\ell) = \frac{A^2}{4} (e^{j\omega\ell} + e^{-j\omega\ell})$$

$$r_{xx}(\ell) = \frac{A^2}{2} \frac{1}{2} (e^{j\omega\ell} + e^{-j\omega\ell})$$

Using the relation $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ we can rewrite the autocorrelation to:

$$r_{xx}(\ell) = \frac{A^2}{2} \cos(\omega\ell)$$

Thus, the autocorrelation function of a real signal $z(n) = A \cos(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is:

$$r_{zz}(\ell) = \frac{A^2}{2} \cos(\omega\ell)$$

Autocorrelation of a sine signal

Problem:

What is the autocorrelation function of a real signal $z(n) = A \sin(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$?

Solution:

In ADSI Problem 4.4, we found that the autocorrelation of a complex sinusoid given by $y(n) = A e^{j(\omega n + \phi)}$ where A and ω are real constants and ϕ is a random variable with $\phi \sim U(0, 2\pi)$ is:

$$r_{yy}(\ell) = A^2 e^{j\omega\ell}$$

To use this result, we need to convert the given signal in this problem to complex exponential signal.

A complex exponential signal is always formed by the sum of two real signals:

$$A e^{j\omega n} = A \cos(\omega n) + j A \sin(\omega n)$$

Therefore, we know that:

$$\sin(\omega) = \frac{1}{2j} e^{j\omega} - \frac{1}{2j} e^{-j\omega}$$

Using this relation, we can rewrite a real signal $A \sin(\omega n + \phi)$ as:

$$z(n) = A \sin(\omega n + \phi)$$

$$z(n) = \frac{A}{2j} e^{j(\omega n + \phi)} - \frac{A}{2j} e^{-j(\omega n + \phi)}$$

To compute the autocorrelation function, we square the magnitude, remove the phase and replace n with ℓ :

$$r_{zz}(\ell) = \left(\frac{A}{2j}\right)^2 e^{j\omega\ell} - \left(\frac{A}{2j}\right)^2 e^{-j\omega\ell}$$

We know that $(2j)^2 = 2^2 \cdot j^2 = -4$ because $j = \sqrt{-1}$ so $j^2 = -1$

$$r_{zz}(\ell) = \frac{A^2}{-4} e^{j\omega\ell} - \frac{A^2}{-4} e^{-j\omega\ell}$$

$$r_{zz}(\ell) = -\frac{A^2}{4} e^{j\omega\ell} + \frac{A^2}{4} e^{-j\omega\ell}$$

We want to make the autocorrelation function in terms of $\cos(\cdot)$, we rewrite the expression as follows:

$$r_{zz}(\ell) = \left(-\frac{A^2}{2}\right) \cdot \frac{1}{2} (e^{j\omega\ell} + e^{-j\omega\ell})$$

Since $\cos(\theta) = \frac{1}{2} (e^{j\theta} + e^{-j\theta})$, we can rewrite the expression as:

$$r_{zz}(\ell) = -\frac{A^2}{2}\cos(\omega\ell)$$

Thus, the autocorrelation function of a real signal $z(n) = A \sin(\omega n + \phi)$ is

$$r_{zz}(\ell) = -\frac{A^2}{2}\cos(\omega\ell)$$

Autocorrelation of White noise

White noise is important for random signal modelling.

Suppose we have a random process that produces perfect random noise. Let $w(n)$ be a random signal from this process.

The **expected value** of the signal is zero because there are no patterns in white noise:

$$E[w(n)] = 0$$

The **autocorrelation of white noise** generates one peak at $\ell = 0$ because that is the only time when there is any correlation of the signal. One peak at $\ell = 0$ can be modelled by delta signal:

$$r_{ww}(\ell) = E[w(n)w(n - \ell)] = \sigma_w^2 \delta(\ell)$$

where σ_w^2 is the variance of the signal which can be computed as follow:

$$\begin{aligned}\sigma_w^2 &= E[w^2(n)] - E[w(n)]^2 \\ \sigma_w^2 &= E[w^2(n)] - 0 \quad (\text{by definition } E[w(n)] = 0) \\ \sigma_w^2 &= E[w^2(n)]\end{aligned}$$

INSIGHT: we can use white noise to find the impulse response of a system because the autocorrelation of white noise is the same the delta signal.