Hilbert Transform

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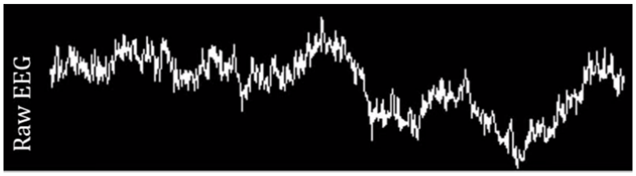
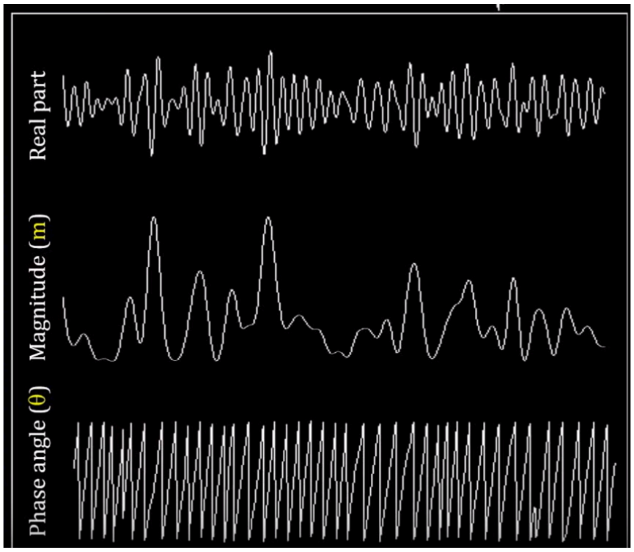
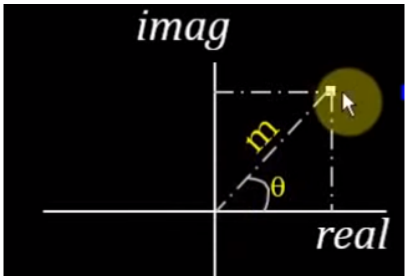
What is Hilbert Transform?

The discrete Hilbert transform is a process used to generate complex-valued signals from real-valued signals.

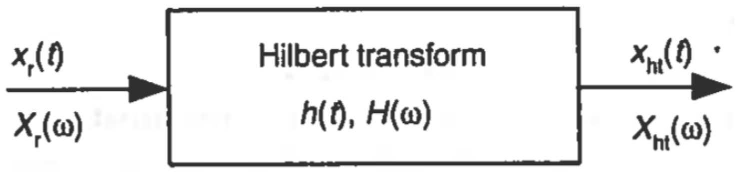
It is a method for extracting the magnitude and phase information from a real-valued signal like a EEG data:







Notation used to define the continuous Hilbert transform:



$x_r(t)$ = a real continuous time-domain input signal
$h(t)$ = the time impulse response of a Hilbert transformer
$x_{ht}(t)$ = the HT of $x_r(t)$, ($x_{ht}(t)$ is also a real time-domain signal)
$X_r(\omega)$ = the Fourier transform of real input $x_r(t)$
$H(\omega)$ = the frequency response (complex) of a Hilbert transformer
$X_{ht}(\omega)$ = the Fourier transform of output $x_{ht}(t)$
ω = continuous frequency measured in radians/second
t = continuous time measured in seconds.

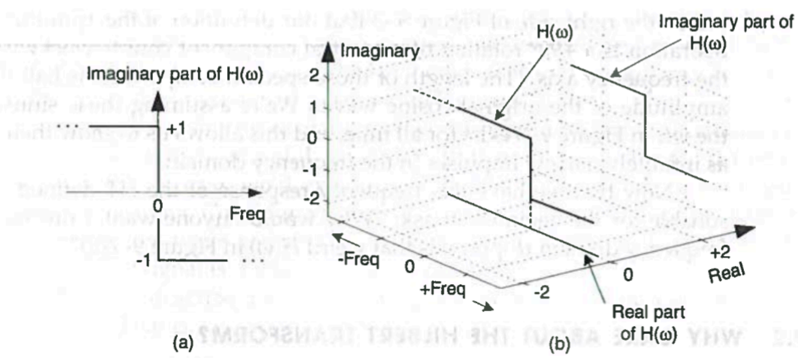


Figure 9-2 The complex frequency response of $H(\omega)$.

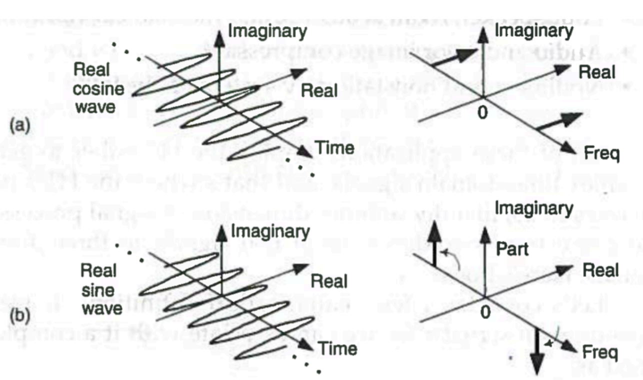


Figure 9-3 The Hilbert transform: (a) $\cos(\omega t)$; (b) its transform is $\sin(\omega t)$.

Why do we need it?

Typically, we start with a real-valued signal which can be conceptualised as:

$$x_r(t) = A\cos(\omega\, t)$$

Given a real-value signal, we cannot estimate or extract the phase angle and the power. What we need to represent the signal as a complex sinusoid using Euler's formula:

$$x_c(t) = A\, e^{j\omega t}$$

$$x_c(t) = A\cos(\omega\, t) + j\, A\sin(\omega t)$$

$$x_c(t) = x_r(t) + j\, x_i(t)$$

where

- $x_c(t)$ is known as an *analytic signal* because it has no negative-frequency spectral components. This means that $X_c(e^{j\omega})$ is zero over the negative frequency range. This is also known as a *one-sided spectrum*.
- $j\, x_i(t)$ is called the *phase-quadrature component*.

Using complex signals of the real signals simplifies and improves the performance of many signal processing operations.

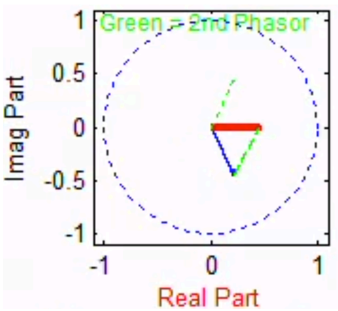
The answer is: we need to understand the HT because it's useful in so many complex-signal (quadrature) processing applications. A brief search on the Internet reveals HT-related signal processing techniques being used in the following applications:

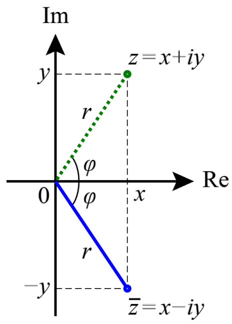
- Quadrature modulation and demodulation (communications)
- Automatic gain control (AGC)
- Analysis of two- and three-dimensional complex signals
- Medical imaging, seismic data and ocean wave analysis
- Instantaneous frequency estimation
- Radar/sonar signal processing, and time-domain signal analysis using wavelets
- Time difference of arrival (TDOA) measurements
- High definition television (HDTV) receivers
- Loudspeaker, room acoustics, and mechanical vibration analysis
- Audio and color image compression
- Nonlinear and nonstationary system analysis.

Real Signals as Complex Exponentials

A real-valued signal can be seen as a a single frequency phasor (red) that is composed of the sum of two phasors:

- a positive frequency phasor (blue) rotates counter-clockwise
- a negative frequency phasor (green) rotates clockwise





If we project the sum of the two phasors in the real part, we get the red phasor. However, if we project the sum into the imaginary part, the two phasors cancel out one another and we get zero imaginary component.

This means that a real-valued signal has two complex exponentials (complex conjugate phasors) rotating in oppsite directions. The real parts are in phase and sum together, the imaginary parts are opposite polarity, and cancel out, which leaves only a real sinusoid as a result. The negative and positive frequency components ***are*** both necessary to produce the real signal.

A real cosine signal $x_r(t) = A\cos(\omega_0 t + \phi)$ can be expressed in terms of two complex exponentials using Eulers formula:

$$x_r(t) = A\cos(\omega_0 t + \phi)$$

$$x_r(t) = A\left(\frac{e^{j(\omega t+\phi)} + e^{-j(\omega+\phi)}}{2}\right)$$

$$x_r(t) = A\left(\frac{1}{2}e^{j(\omega t+\phi)} + \frac{1}{2}e^{-j(\omega t+\phi)}\right)$$

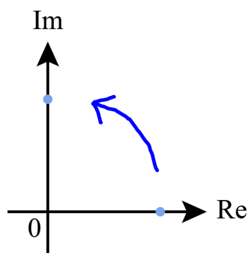
$$x_r(t) = A\left(\frac{1}{2}e^{j\phi}e^{j\omega t} + \frac{1}{2}e^{-j\phi}e^{-j\omega t}\right)$$

$x_r(t) = \frac{1}{2}Xe^{j\omega t} + \frac{1}{2}X^* e^{-j\omega t}$ where $X = Ae^{j\phi}$ and $X^* = Ae^{-j\phi}$

where X^* is complex conjucate of the phasor X. This shows that a real-valued signal with frequency ω is actually composed for a positive frequency $+\omega$ complex exponential and a negative frequency $-\omega$ complex exponential.

The phase-quadrature component can be obtained by a 90 degree (*quarter-cycle*) shift on the complex plan.

The Hilbert transform of $\cos(\theta)$ is $\sin(\theta)$ i.e., a quarter-cycle shift.



The Hilbert transform rotates the Fourier coefficients in the complex space which converts the real components into the imaginary component. The quarter-cycle shifted signal is then added to the original signal.

In other words, a Hilbert transform takes each frequency component present in the original signal and shifts its phase by $-\frac{\pi}{2}$.

For a continous time signal $x_r(t)$, the Hilbert transform is defined as:

$x_{\mathrm{ht}}(t) = x_r(t) * \frac{1}{\pi t}$ where $*$ is the convolution operation

We can see the Hilbert transform as a FIR filter with the impulse response $h(t) = \frac{1}{\pi t}$

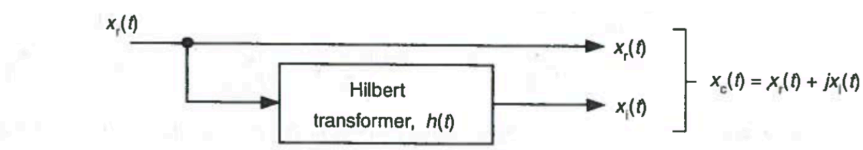


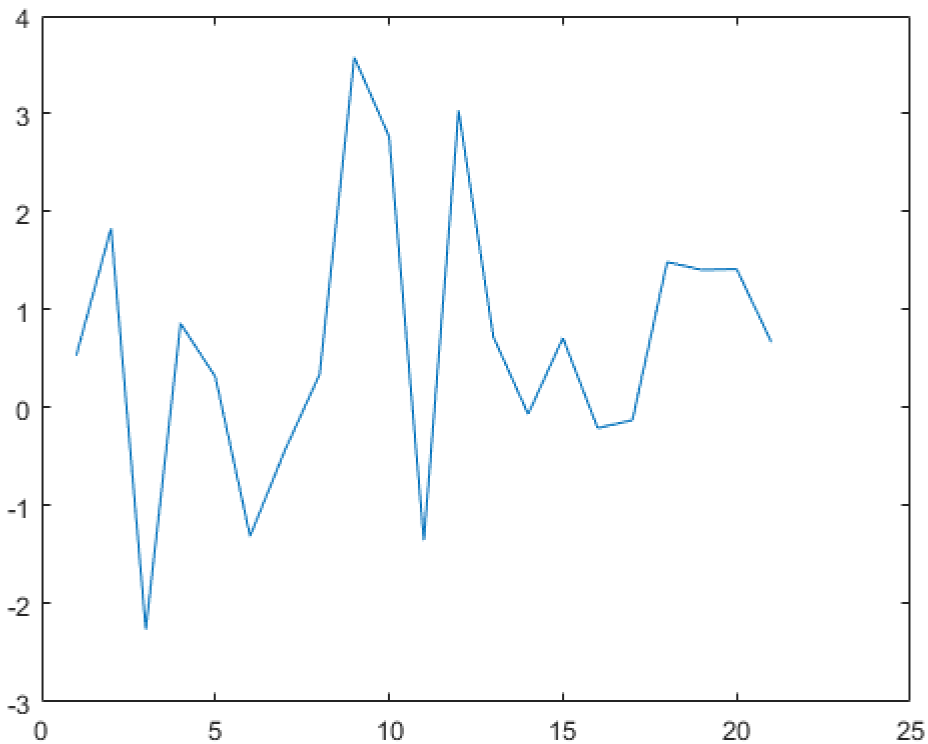
Figure 9-4 Functional relationship between the $x_c(t)$ and $x_r(t)$ signals.

FFT-based Hilbert transform

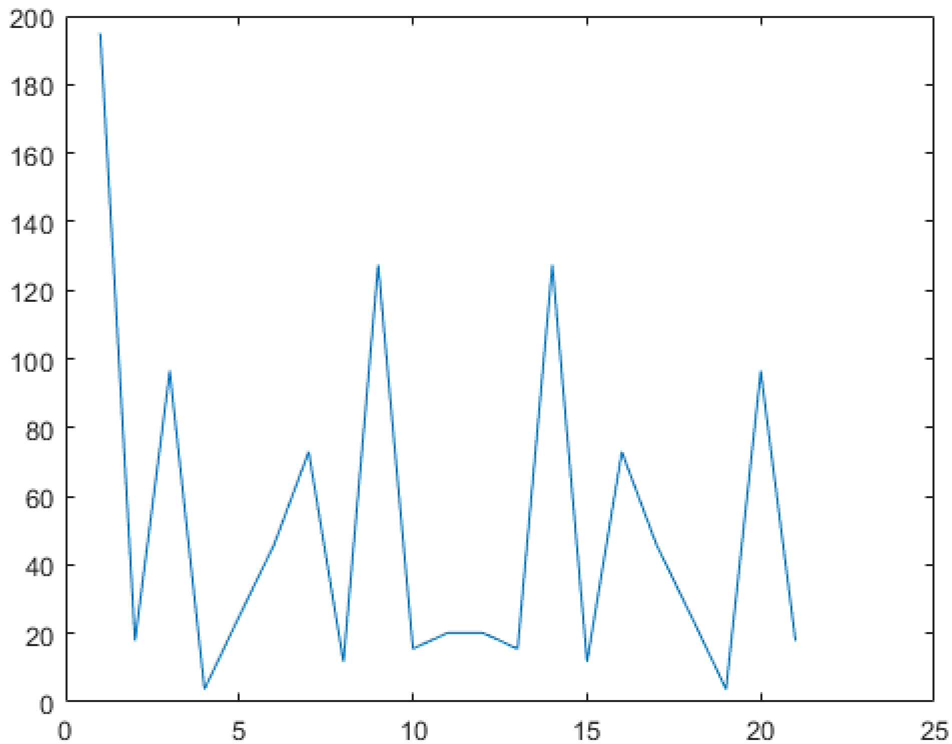
The FFT-based Hilbert transform can be implemented as three steps:

1. Take the FFT of a time-series signal
2. Rotate the Fourier coefficients
3. Take the inverse FFT of the rotated Fourier coefficients to get a time-series signal

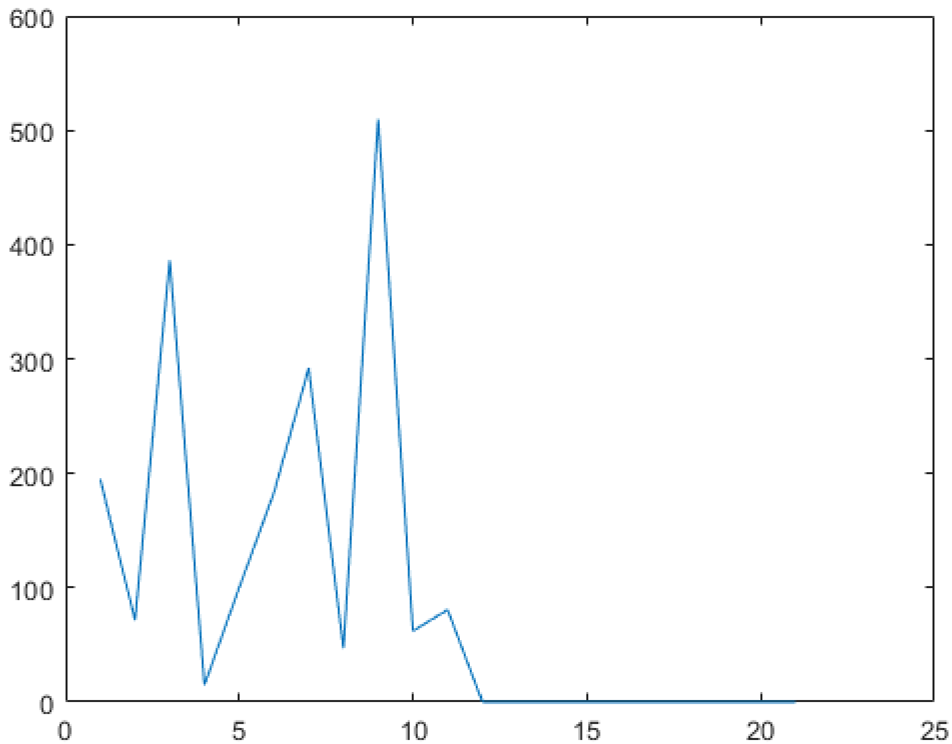
```
clear variables;

n = 21;

% Generate a signal
x = randn(n, 1);
plot(x)
```




```
% Step 1: Take the FFT
f = fft(x);

plot(f.*conj(f))
```




```
% Create a copy of the Fourier coefficients
complexf = 1i*f;

% Identify the indices for the positive and negative frequencies
% because they are rotated in different ways.
pos_freq_idx = 2:floor(n/2) + mod(n, 2);
neg_freq_idx = ceil(n/2) + 1 + ~mod(n, 2):n;

% Rotate the Fourier coefficients by computing i*A*sin(wt) component.
% Note! Positive frequencies are rotated counter-clockwise while
% negative frequencies are rotated clockwise.
f(pos_freq_idx) = f(pos_freq_idx) + -1i*complexf(pos_freq_idx);
f(neg_freq_idx) = f(neg_freq_idx) +  1i*complexf(neg_freq_idx);

plot(f.*conj(f))
```




```
hilbert_x = ifft(f);

plot(abs(hilbert_x))
```

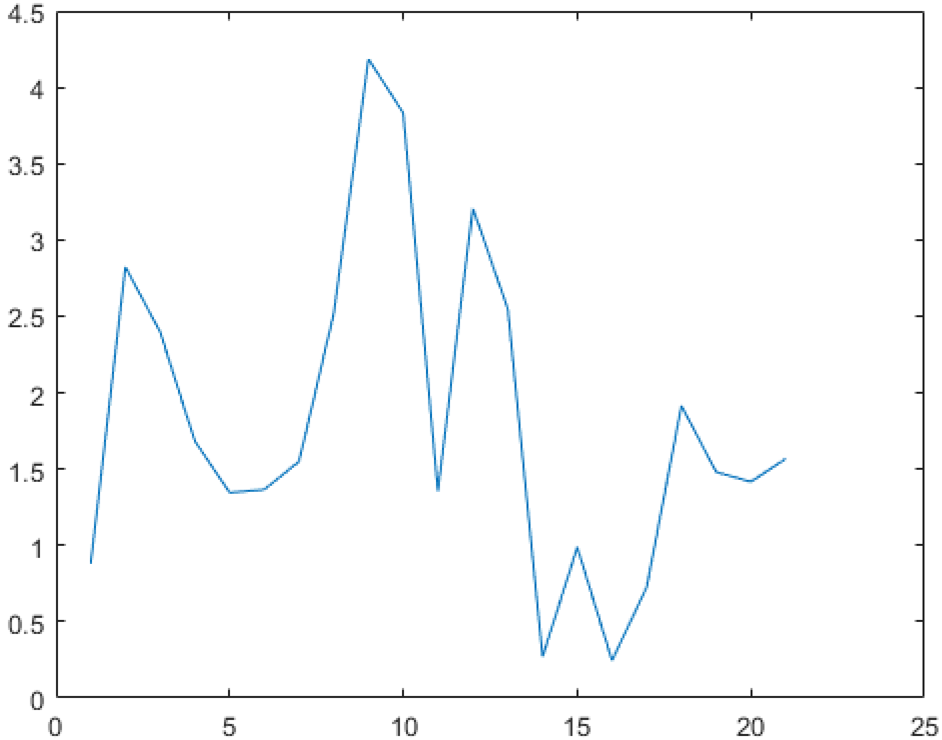


Table of selected Hilbert transforms

In the following table, the frequency parameter ω is real.

Signal $u(t)$	Hilbert transform[fn 1] $H(u)(t)$
$\sin(\omega t)$ [fn 2]	$\operatorname{sgn}(\omega)\sin\left(\omega t-\frac{\pi}{2}\right)=-\operatorname{sgn}(\omega)\cos(\omega t)$
$\cos(\omega t)$ [fn 2]	$\operatorname{sgn}(\omega)\cos\left(\omega t-\frac{\pi}{2}\right)=\operatorname{sgn}(\omega)\sin(\omega t)$
$e^{i\omega t}$	$\operatorname{sgn}(\omega)e^{i\left(\omega t-\frac{\pi}{2}\right)}=-i\cdot\operatorname{sgn}(\omega)e^{i\omega t}$
$\dfrac{1}{t^2+1}$	$\dfrac{t}{t^2+1}$
e^{-t^2}	$2\pi^{-1/2}F(t)$ (see Dawson function)
Sinc function $\dfrac{\sin(t)}{t}$	$\dfrac{1-\cos(t)}{t}$
Rectangular function $\operatorname{rect}(t)=\Pi(t)=\begin{cases}0, & \text{if } \lvert t\rvert>\frac{1}{2}\\ \frac{1}{2}, & \text{if } \lvert t\rvert=\frac{1}{2}\\ 1, & \text{if } \lvert t\rvert<\frac{1}{2}.\end{cases}$	$\dfrac{1}{\pi}\ln\left\lvert\dfrac{t+\frac{1}{2}}{t-\frac{1}{2}}\right\rvert$
Dirac delta function $\delta(x)=\begin{cases}+\infty, & x=0\\ 0, & x\neq 0\end{cases}$	$\dfrac{1}{\pi t}$
Characteristic Function $\chi_{[a,b]}(t)$	$\dfrac{1}{\pi}\ln\left\lvert\dfrac{t-a}{t-b}\right\rvert$

Real signal	Hilbert transform
$a_1 g_1(t) + a_2 g_2(t);\ a_1, a_2 \in \mathbb{C}$	$a_1 \hat{g}_1(t) + a_2 \hat{g}_2(t)$
$h(t - t_0)$	$\hat{h}(t - t_0)$
$h(at);\ a \neq 0$	$\operatorname{sgn}(a)\hat{h}(at)$
$\frac{\mathrm{d}}{\mathrm{d}t} h(t)$	$\frac{\mathrm{d}}{\mathrm{d}t} \hat{h}(t)$
$\delta(t)$	$\frac{1}{\pi t}$
$e^{\mathrm{j}t}$	$-\mathrm{j}e^{\mathrm{j}t}$
$e^{-\mathrm{j}t}$	$\mathrm{j}e^{-\mathrm{j}t}$
$\cos(t)$	$\sin(t)$
$\operatorname{rect}(t)$	$\frac{1}{\pi} \ln \lvert (2t+1)/(2t-1) \rvert$
$\operatorname{sinc}(t)$	$\frac{\pi t}{2} \operatorname{sinc}^2(t/2) = \sin(\pi t/2) \operatorname{sinc}(t/2)$
$1/(1+t^2)$	$t/(1+t^2)$

Properties of the Hilbert transform

(1) A signal $x_r(t)$ and its Hilbert transform $x_i(t)$ have the **same power density spectrum**.

(2) A signal $x_r(t)$ and its Hilbert transform $x_i(t)$ have the **same autocorrelation function**.

(3) A signal $x_r(t)$ and its Hilbert transform $x_i(t)$ are mutually orthogonal so we can write:

$$\int_{-\infty}^{\infty} x_r(t) x_i(t)\,\mathrm{dt} = 0$$

(4) The effect of applying the Hilbert transform twice on a zero mean signal $x_r(t)$ is:

$$H[H[x_r(t)]] = -x_r(t)$$

For example, suppose $x_r(t) = \cos(t)$. Computing the Hilbert transform we get:

$$H[\cos(t)] = \cos\left(t - \frac{\pi}{2}\right)$$

Computing the Hilber transform again, we get:

$$H\left[\cos\left(t-\frac{\pi}{2}\right)\right] = \cos\left(t-\frac{\pi}{2}-\frac{\pi}{2}\right)$$

$$H\left[\cos\left(t-\frac{\pi}{2}\right)\right] = \cos(t-\pi)$$

$$H\left[\cos\left(t-\frac{\pi}{2}\right)\right] = -\cos(t)$$

(5) The energy of a Hilbert transform $x_i(t)$ is the same as the energy of the original signal $x_r(t)$. Hilbert Transform does not changes the magnitude of the input signal hence the energy is invariant

Assume that some zero-mean signal $x(k)$ with finite energy

$$\sum_{k=-\infty}^{\infty} x^2(k) = C$$

is Hilbert transformed to give $x_{HT}(k) = H[x(k)]$

Example:

$$x(t) \longrightarrow \boxed{h(t) = \frac{1}{\pi t}} \longrightarrow y(t)$$

- If $x(t) = 4\text{sinc}(2t)$, then evaluate $\int_{-\infty}^{\infty} |y(t)|^2 dt$.
- The integral is energy of $y(t)$.
- Also, $y(t)$ is Hilbert transform of $x(t)$.
- Hence, energy of $y(t)$ = energy of $x(t)$.
- Energy of $x(t) = 4\text{sinc}(2t)$, is, $E_x = \frac{4^2}{2} = 8$.

(6) The inverse Hilbert transform is $H^{-1}[x_r(t)] = -H[x_r(t)]$:

$$H(\omega) = \begin{cases} -j & \omega > 0 \\ 0 & \omega = 0 \\ j & \omega > 0 \end{cases} \quad \Rightarrow \quad H_{inv}(\omega) = -H(\omega) = \begin{cases} j & \omega > 0 \\ 0 & \omega = 0 \\ -j & \omega > 0 \end{cases}$$

Note: A DC term can't be recreated