Invertibility, minimum-phase and allpass

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Allpass systems

The definition of allpass is that the magnitude of its frequency response is constant G (usually $G = 1$) for all frequencies.

$$|H(e^{j\omega})| = G. \tag{5.155}$$

We can say that:

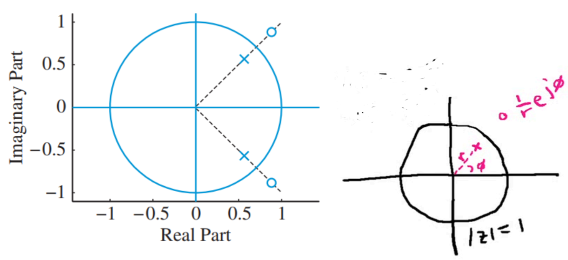
- an allpass system treats all frequencies identically with respect to gain
- an allpass system preserves the power or energy of their input signals

The system function of an allpass system has the poles and zeros occur in conjugate reciprocal pairs

$$H_{ap}(z) = e^{j\beta} \prod_{k=1}^{N} \frac{z^{-1} - p_k^*}{1 - p_k z^{-1}}, \tag{5.158}$$

where β is constant that is usually set to $\beta = 0$.

So if we have a pole $p_k = re^{j\phi}$ then the zero is located at $\frac{1}{p_k^*} = \frac{1}{r} e^{j\phi}$

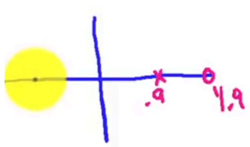


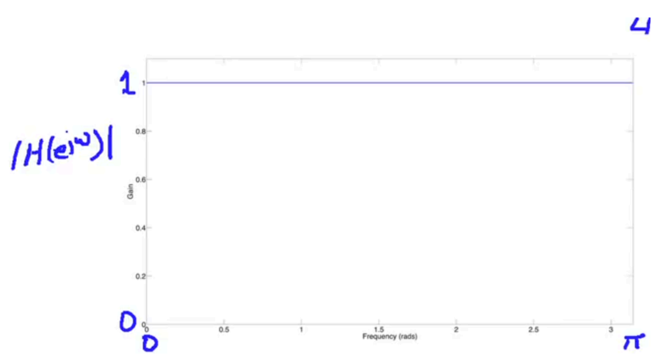
A second-order allpass filter is given by:

$$H_{ap}(z) = \frac{a_2^* + a_1^* z^{-1} + z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = z^{-2} \frac{1 + a_1^* z + a_2^* z^2}{1 + a_1 z^{-1} + a_2 z^{-2}}, \tag{5.164}$$

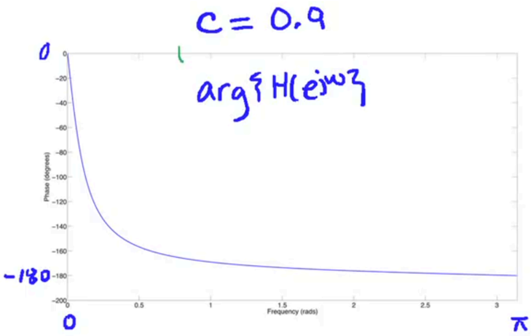
Example: All-pass System

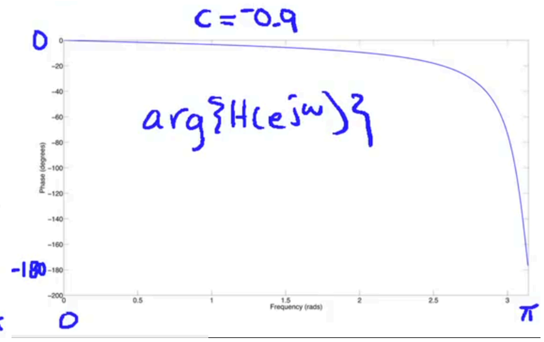
$$H(z) = \frac{z^{-1} - c^*}{1 - c z^{-1}}$$





Although the magnitude response is the same, the phase response changes depending on the pole.





Minimum-phase systems

A stable, causal system has a stable, causal inverse
if and only if
all poles and zeros are inside $|z|=1$

called: Minimum phase system

A causal and stable LTI system with a causal and stable inverse is known as a ***minimum-phase*** system

If poles and zeros are outside the unit circle, we call it **maximum-phase**.

If some are outside and others inside the unit circle, it is known as **mixed-phase**.

Invertibility

5.16 Invertibility and minimum-phase systems

$$x(n) \rightarrow \boxed{h(n)} \xrightarrow{y(n)} \boxed{h_{inv}(n)} \xrightarrow{x(n)}$$

$$h(n) * h_{inv}(n) = \delta(n) \quad = \left(z^{-n_d}\right)$$

z

$$H(z)\,H_{inv}(z) = 1 \iff H_{inv}(z) = \frac{1}{H(z)}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

$$H_{inv}(z) = \frac{A(z)}{B(z)}$$

zeroes ⇔ poles

When is a filter invertible?

A filter $H(z) = \frac{B(z)}{A(z)}$ is said to be stable and causal if all the poles of $H(z)$ are inside the unit circle.

If the inverse filter $H_{\text{inv}}(z) = \frac{A(z)}{B(z)}$ has to be stable and causal, then all the poles of $H_{\text{inv}}(z)$ must be inside the unit circle or equivalently all zeros of $H(z)$ must be inside the unit circle.

In practice, we say that a system is only invertible if its zeros and poles are inside the unit circle.

For a filter to be invertible, all its zeros and poles must be inside the unit circle. In other words, the filter must be minimum-phase.

Filter Decomposition

Any rational system function $H(z)$

$$H(z) = \underbrace{H_{min}(z)}_{\text{minimum phase}} \underbrace{H_{ap}(z)}_{\text{all pass}}$$

Any rationional system function can be factorised into two system functions:

$$H(z) = H_{\text{min}}(z) \cdot H_{\text{ap}}(z)$$

By decomposing $H(z)$ we made a new polynomial that consists of two parts:

- Minimum-phase system: all zeros are inside the unit circle
- All-pass system: which will have zeros outside the unit circle. We are basically storing all the phase in the all-pass filter instead

To factorise or decompose, we take following steps:

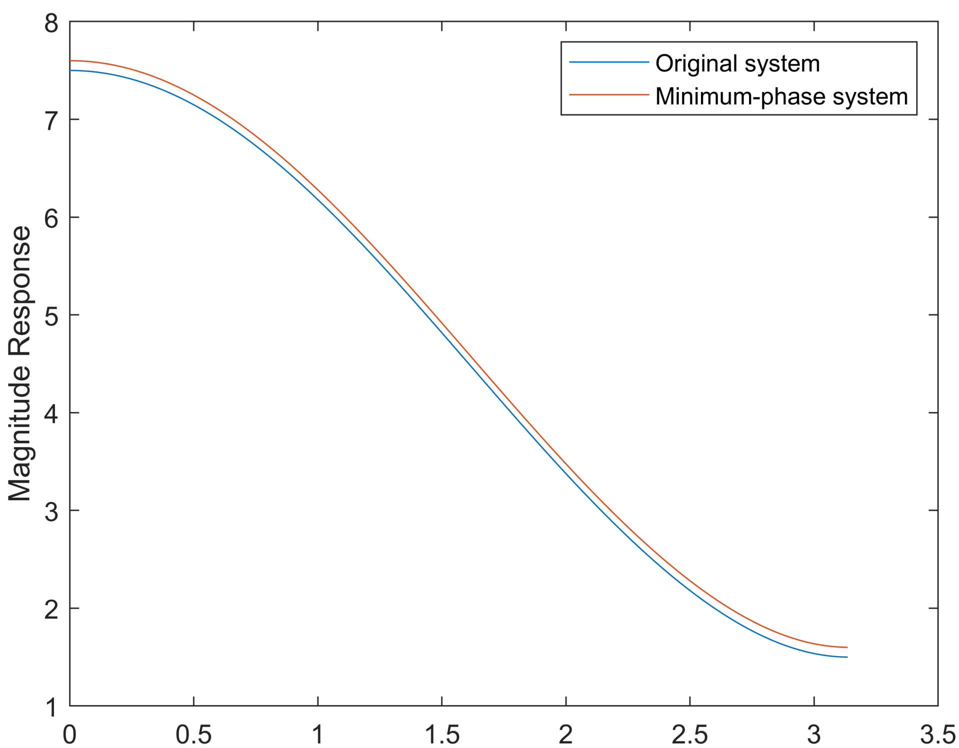
1. Take the zeros of $H(z)$ that lie outside the unit circle and move them to the allpass system function $H_{\text{ap}}(z)$
2. Add poles to the allpass system in conjugate reciprocal locations of the zeros
3. To keep the equation balanced, we add zeros in the minimum-phase system function to even out the poles that we added to the allpass system function

Since the poles that we added to the allpass system were all inside the unit circle, the zeros that we added to the minimim-phase system are also inside the unit circle.

Magnitude Response of Decomposition

Plotting the maginatude response of the original system and the minimum-phase system, we observe that they are on top of each other

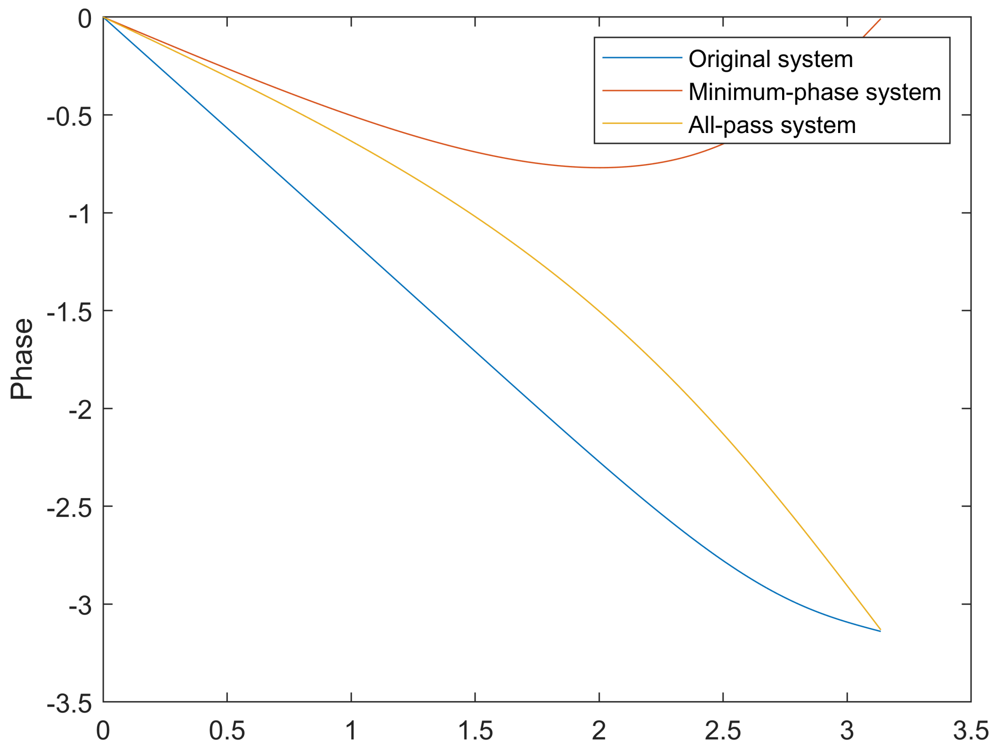
```
[H_orig, w] = freqz([1, 4.5, 2]); % Original
[H_min, w] = freqz([4, 3,   0.5]);
[H_ap, w] = freqz([1/4, 1], [1, 1/4]);

% Add 0.1 otherwise they will be on top of each other
plot(w, abs(H_orig), w, abs(H_min)+0.1)
legend('Original system', 'Minimum-phase system')
ylabel('Magnitude Response')
```



Phase response of the Decomposition

We can plot the phase of the three systems. The phase of the original system is hidden in the all-pass system. So by decomposing a filter into minimum-phase and all-pass systems, the excess phase is stored in the all-pass system. The phase lag of a system with poles and zeros inside the unit cicle is less than the original system which had the identical magnitude response.

```
plot(w, angle(H_orig), w, angle(H_min), w, angle(H_ap))
ylabel('Phase')
legend('Original system', 'Minimum-phase system', 'All-pass system')
```



Why is the decomposition useful?

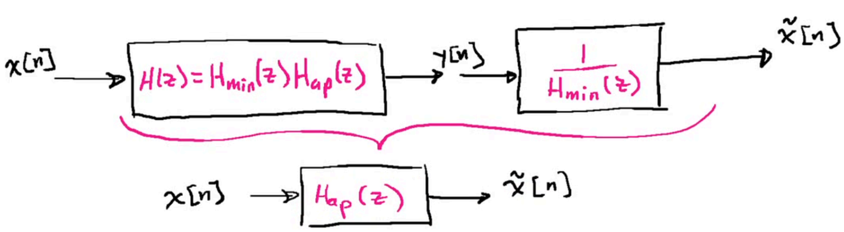
This decomposition is useful if we want to find an approximate inverse system for $H(z)$.

The minimum-phase component of any system can be inverted. This allows os to take any system, run a signal $x(n)$ through it and run the output $y(n)$ through the inverse of the minimum-phase part.

The end result is the same as passing the signal $x(n)$ through an allpass filter.

What we achieve is that:

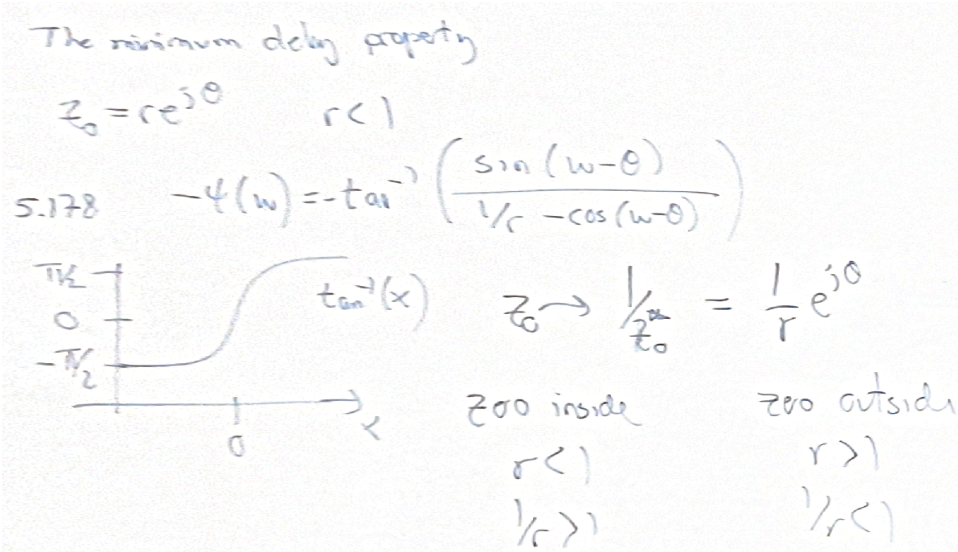
- no magnitude distortion meaning that the magnitude of each frequency is not altered by the process
- we only get phase distortion (which may not be important depending on the application)



The magnitude response of an allpass filter is constant for all frequencies, however its phase response may different. For this reason, allpass filters are used for **delay equalisation**.

Minimum Delay Property

What happens when we take a zero and flip it. We know that the magnitude response does not change.



By setting $r < 1$ we specifically place the zero inside the unit circle.

If we flip z_0 to $\frac{1}{z_0^*} = \frac{1}{r}e^{j\phi}$

If we have zero inside $r < 1$ then $\frac{1}{r} > 1$

If we have zero outside $r > 1$ then $\frac{1}{r} < 1$

The only thing that changes in the phase response when we flip z_0 to inside the unit circle is $\frac{1}{r}$.

$$-\Psi(\omega) = -\tan^{-1}\frac{\sin(\omega-\theta)}{1/r - \cos(\omega-\theta)}, \qquad (5.178)$$

If we go from zero inside to outside, the $\frac{1}{r}$ becomes smaller. The tangent will increase and we get a larger number. This means that we get more phase. So we add phase as the zero goes from inside the unit circle to outside the unit circle.

If we look at the group delay:

$$\tau(\omega) = \frac{r - \cos(\omega-\theta)}{(r + 1/r) - 2\cos(\omega-\theta)}. \qquad (5.179)$$

By taking a zero that is inside the unit circle and putting it outside, we increase the group delay.

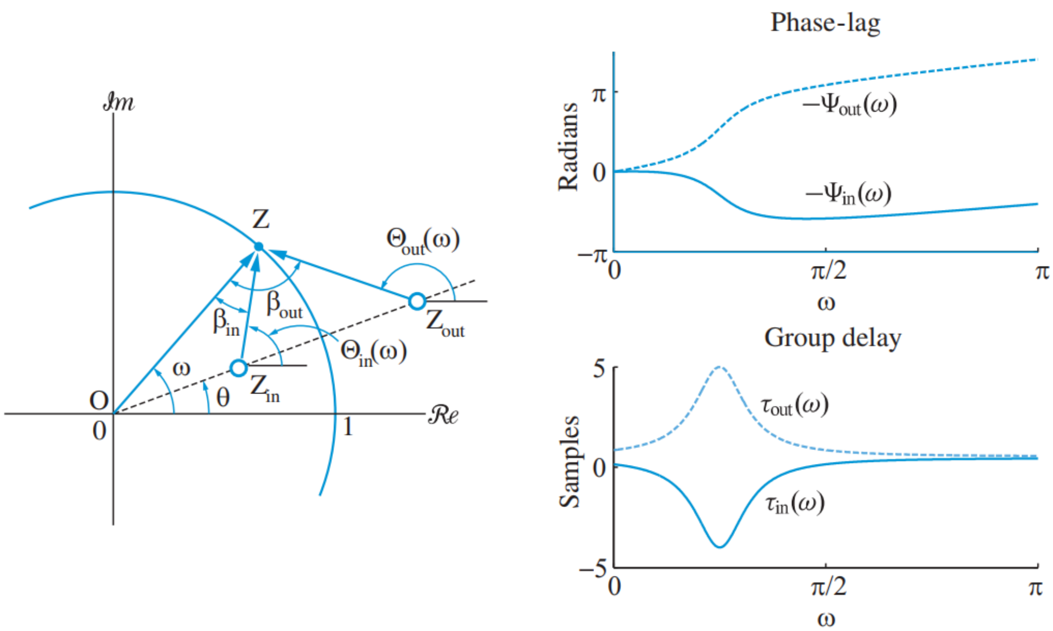


Figure 5.33 (a) Geometrical proof that a zero outside the unit circle introduces a larger phase-lag than a zero inside the unit circle. (b) Phase response and group delay for a zero at $z_{in} = 0.8e^{j\pi/4}$ and its conjugate reciprocal zero $z_{out} = (1/0.8)e^{j\pi/4}$.

Exam 2015 Problem 4: True/False

A serial connection of two all-pass filters is also an all-pass filter.

Answer: TRUE

The transfer function of a serial connection of two fiters is just the individual transfer functions multiplied. For the magnitude part this becomes:

$$|H_{tot}(\omega)| = |H_{AP_1}(\omega)| \cdot |H_{AP_2}(\omega)| = 1 \cdot 1 = 1$$

Is a system invertible?

1. A system with $H(z) = \frac{1+3z^{-1}}{1+\frac{1}{4}z^{-1}}$ is invertible.

```
clear variables;
```

Answer: FALSE!

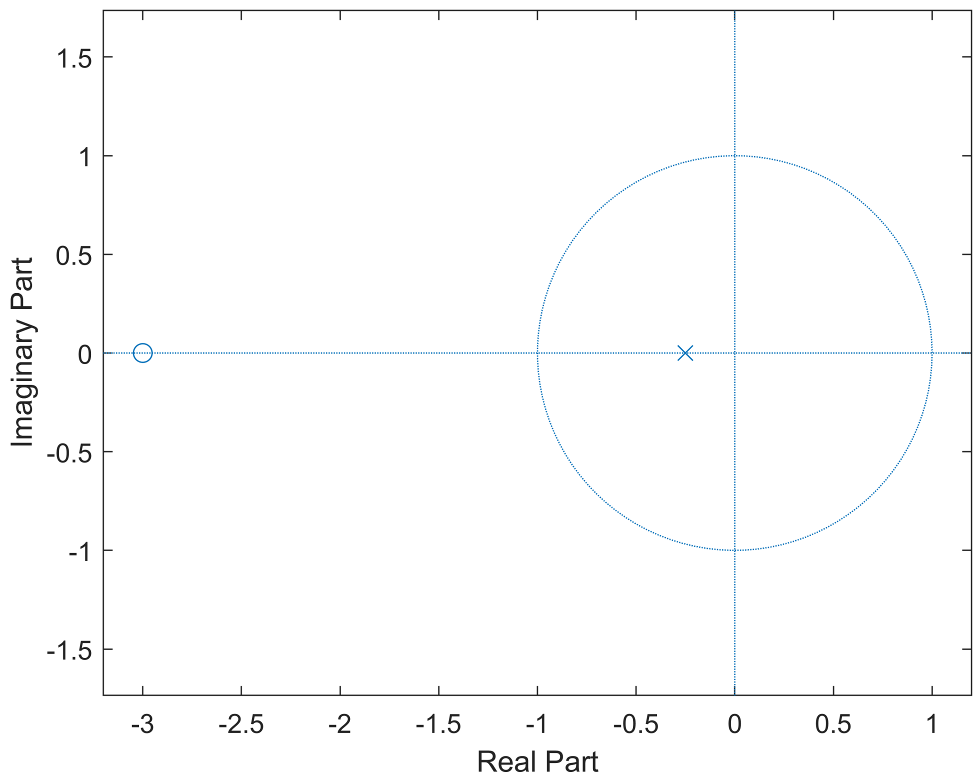
A filter $H(z) = \dfrac{B(z)}{A(z)}$ is said to be stable and causal if all the poles of $H(z)$ are inside the unit circle.

If the inverse filter $H_{\mathrm{inv}}(z) = \dfrac{A(z)}{B(z)}$ has to be stable and causal, then all the poles of $H_{\mathrm{inv}}(z)$ must be inside the unit circle or equivalently all zeros of $H(z)$ must be inside the unit circle.

In practice, we say that a system is only invertible if its zeros and poles are inside the unit circle (minimum phase).

Plotting the poles and zeros of the given system function, we observe that the zero is outside the unit circle.

```
b = [1, 3];
a = [1, 1/4];
zplane(b, a)
```



Is a filter invertible if the poles of the system function lies within the unit circle?

A filter with a rational system function $H(z) = B(z)/A(z)$ is invertible if the poles of the system function lies within the unit circle.

Answer: FALSE.

A filter $H(z) = \frac{B(z)}{A(z)}$ is said to be stable and causal if all the poles of $H(z)$ are inside the unit circle.

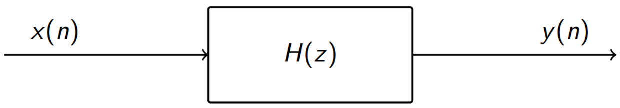
If the inverse filter $H_{\text{inv}}(z) = \frac{A(z)}{B(z)}$ has to be stable and causal, then all the poles of $H_{\text{inv}}(z)$ must be inside the unit circle or equivalently all zeros of $H(z)$ must be inside the unit circle.

So if we want the inverse filter to be stable, then the answer is false. Typically, we want an inversable filter to be minimum-phase which means that the filter *and* its inverse must causal and stable.

For a filter to be invertible, zeros and poles must be inside the unit circle.

Quiz: can a system with zeros located exactly on the unit circle be inverted?

Consider a FIR filter with a number of zeros. Most of them are located inside the unit circle but a number of them is located exactly on the unit circle. There are no zeros outside the unit circle.



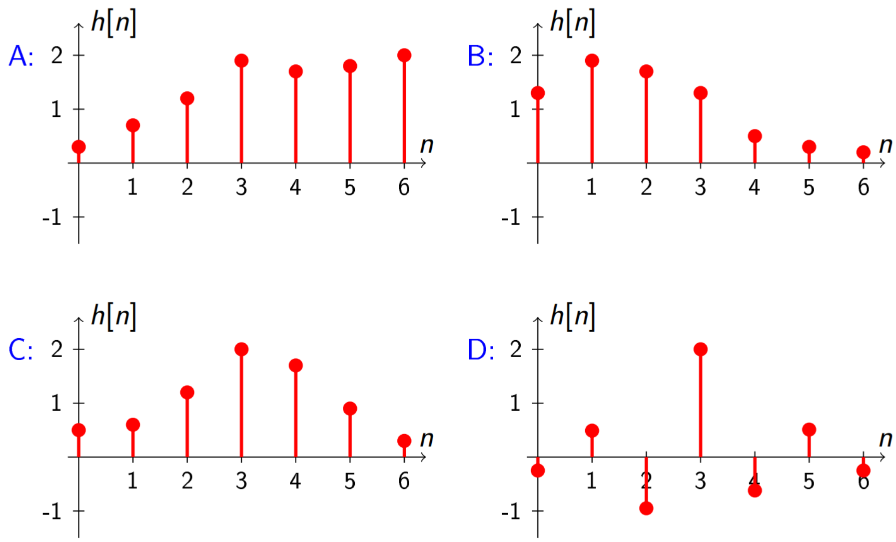
Can the action of the filter be undone by an inverse filter ?

A: Yes, the filtering process can be undone
B: No, the filtering process can't be undone
C: We need to know more about $H(z)$

The answer is NO. If we have a FIR where the zero is exactly at 1 kHz. Given a signal, the filter will remove frequency component at 1 kHz. The problem is that the inverse filter cannot reconstruct the original signal because we have deleted information from the signal i.e., the missing frequency component cannot be recovered.

Quiz: Determine a minimum-phase filter based on impulse response

The impulse response from 4 different FIR filters is shown below.
Which filter seems to be a minimum phase filter ?



The answer is B. Minimum-phase is the same thing as minimum group delay. If we put in energy any of these four filters, which one is the fastest to respond and pass the signal though. How fast does the system propagate the energy of a signal through the filter? If it is slow, then it is maximum-phase. If a filter is in-between then it is mixed-phase. If it is very phase, then it is minimum-phase.

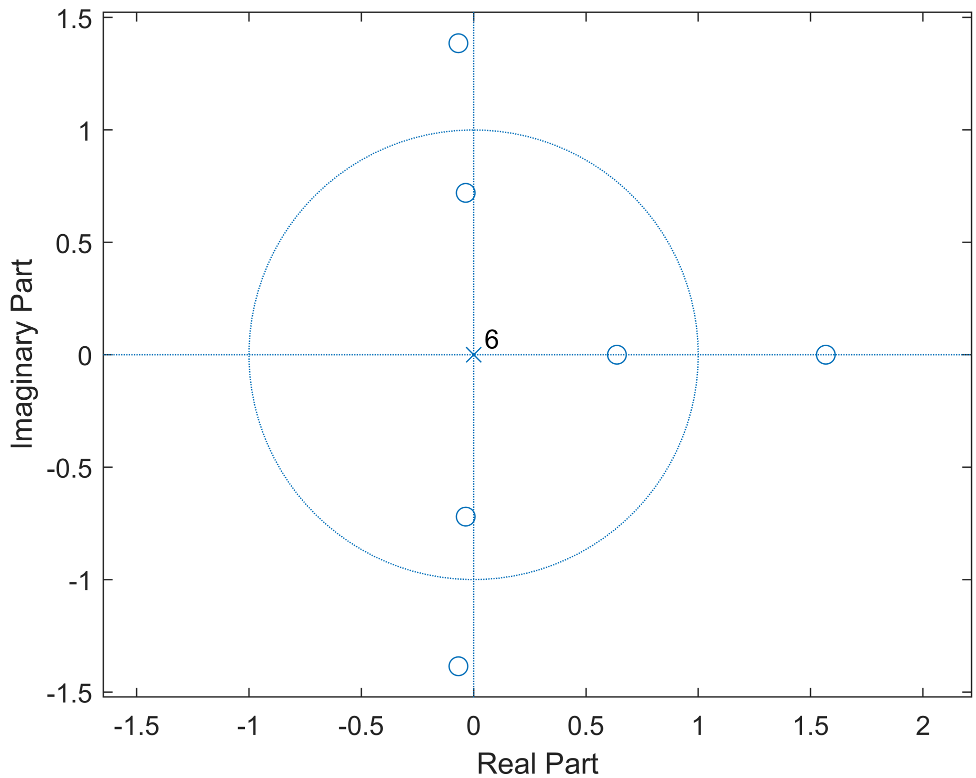
A and D has all the energy at the late end so it takes a while before all the energy passes through the system. Therefore, they has larger group delay than B, where the larger impulses are early.

Comparing B and C: B has energy at the begining, whereas C the energy is in the middle.

- A: Maximum-phase
- B: Minimum-phase
- C: Mixed-phase
- D: Mixed-hase

Let us simulate filter D in MATLAB:

```
zplane([-1, 2, -3, 5, -3, 2, -1])
```



Quiz: Has a linear phase filter a mixed group delay?

A FIR filter has a linear phase response if the coefficients fulfil the symmetry requirement

$$b(k) = b(M-k), \qquad \text{for } k = 0,\ 1, \ldots,\ M$$

What is the classification of such a filter?

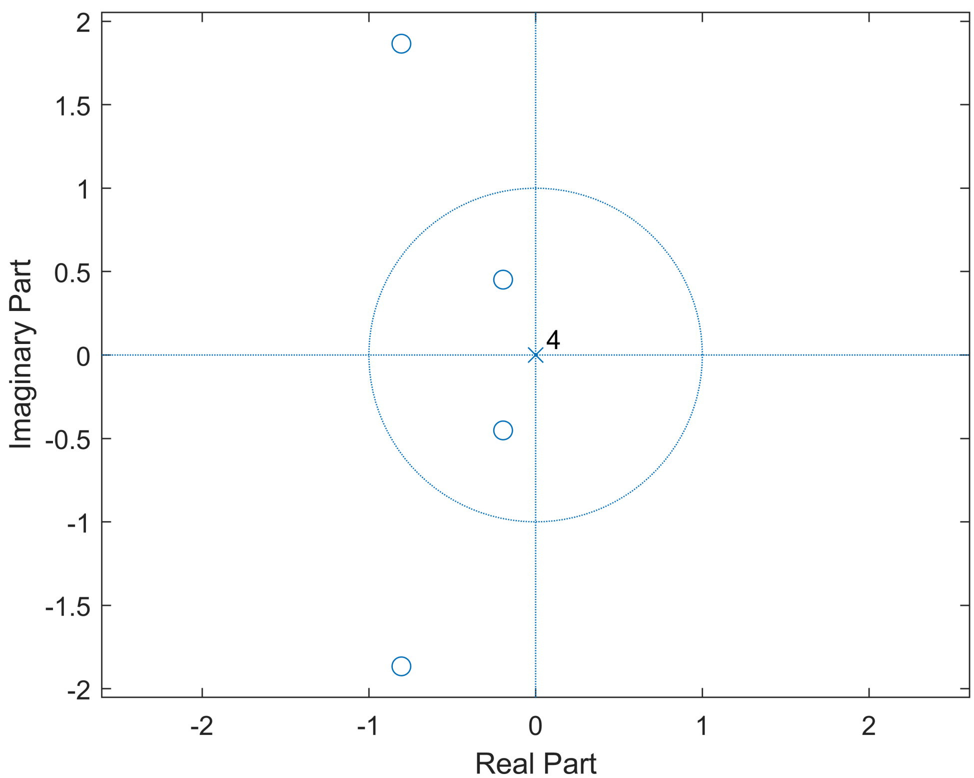
A: Minimum group delay
B: Mixed group delay
C: Maximum group delay
D: We need to know more about the $b(k)$'s

Recall that linear phase correspond to unity group delay.

The symmetry will be in the middle of the filter. The answer is B.

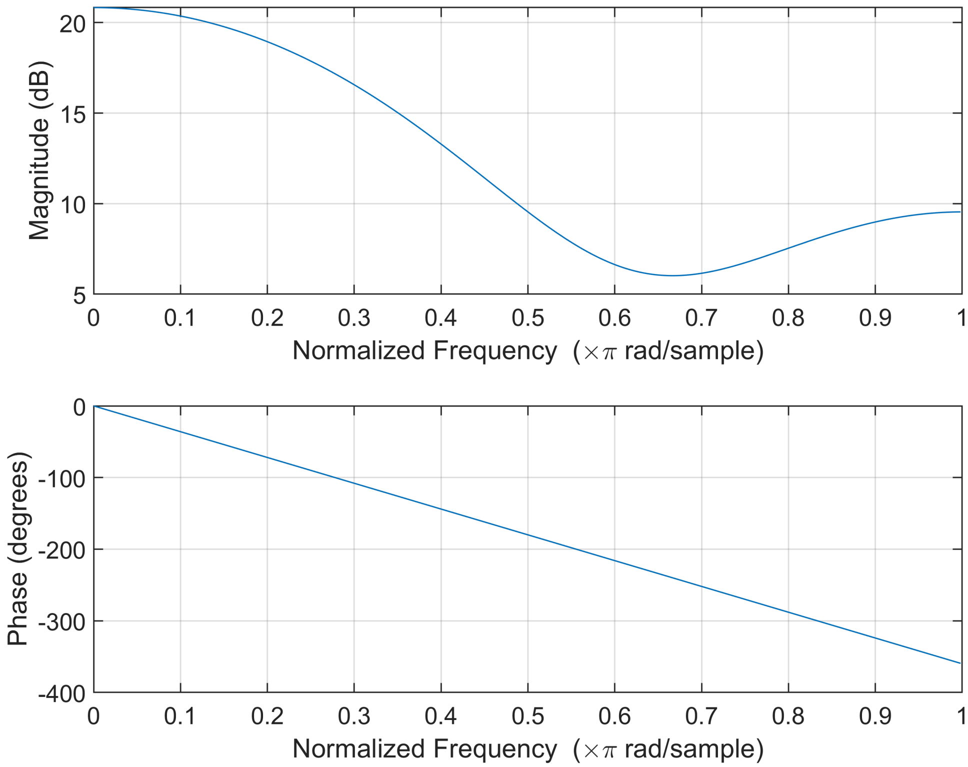
We can try it out in MATLAB. If we plot the a linear phase filter, we get two zeros inside and two zeros outside. Thefore, the filter is mixed-phase filter.

```
zplane([1, 2, 5, 2, 1])
```



The reason why the filter is called linear phase filter:

```
freqz([1, 2, 5, 2, 1])
```



Problem: Decompose a filter into minimum-phase and allpass

Decompose the system $H(z) = H_1(z)(1 - \beta z^{-1})$ where $|\beta| > 1$ and $H_1(z)$ is a minimum-phase system

Any system function can be decomposed into a product of a minimum-phase filter and an all-pass filter using the following formula:

$$H(z) = H_{\min}(z)H_{\mathrm{ap}}(z)$$

To factorise or decompose, we take following steps:

1. Take the zeros of $H(z)$ that lie outside the unit circle and move them to the allpass system function $H_{\mathrm{ap}}(z)$
2. Add poles to the allpass system in conjugate reciprocal locations of the zeros
3. To keep the equation balanced, we add zeros in the minimum-phase system function to even out the poles that we added to the allpass system function

Since the poles that we added to the allpass system were all inside the unit circle, the zeros that we added to the minimim-phase system are also inside the unit circle.

Step 1: We notice that $1 - \beta z^{-1}$ is not an allpass system. We want it to look like this:

$$H_{\rm ap}(z) = e^{j\beta} \prod_{k=1}^{N} \frac{z^{-1} - p_k^*}{1 - p_k z^{-1}}, \qquad (5.158)$$

So we start by factorising $-\beta$ out. By doing this, we put a zero into allpass system

$$H(z) = H_1(z)(-\beta)\left(-\frac{1}{\beta} + z^{-1}\right) = H_1(z)(-\beta)\left(z^{-1} - \frac{1}{\beta}\right)$$

Step 2: We need to put a pole into the allpass system that is in conjugate reciprocal locations of the zero

$$H(z) = H_1(z)(-\beta)\left(\frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*} z^{-1}}\right)$$

Step 3: To keep the equality, we have to introduce a zero to the minimum-phase system function

$$H(z) = H_1(z)(-\beta)\left(1 - \frac{1}{\beta^*} z^{-1}\right)\left(\frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*} z^{-1}}\right)$$

Now the decomposition is:

$$H(z) = H_{\min}(z) H_{\rm ap}(z)$$

where:

$$H_{\min}(z) = H_1(z)(-\beta)\left(1 - \frac{1}{\beta^*} z^{-1}\right)$$

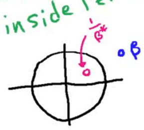
$$H_{\rm ap}(z) = \frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*} z^{-1}}$$

Suppose $H(z) = H_1(z)(1-\beta z^{-1})$; $|\beta|>1$, $H_1(z)$ min phase

1) $H(z) = H_1(z)(-\beta)\left(z^{-1} - \frac{1}{\beta}\right)$

2+3) $H(z) = \underbrace{H_1(z)(-\beta)\left(1 - \frac{1}{\beta^*} z^{-1}\right)}_{H_{min}(z)} \underbrace{\frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*} z^{-1}}}_{H_{ap}(z)}$

zero at $z = \beta$ in $H(z)$ reflected inside $|z|=1$



Problem: Decompose a filter into minimum-phase and allpass

Decompose the system $H(z) = 1 + 4.5z^{-1} + 2z^{-2}$

Any system function can be decomposed into a product of a minimum-phase filter and an all-pass filter using the following formula:

$$H(z) = H_{\min}(z)H_{\mathrm{ap}}(z)$$

To factorise or decompose, we take following steps:

1. Take the zeros of $H(z)$ that lie outside the unit circle and move them to the allpass system function $H_{\mathrm{ap}}(z)$
2. Add poles to the allpass system in conjugate reciprocal locations of the zeros
3. To keep the equation balanced, we add zeros in the minimum-phase system function to even out the poles that we added to the allpass system function

Since the poles that we added to the allpass system were all inside the unit circle, the zeros that we added to the minimim-phase system are also inside the unit circle.

Step 1: Take the zeros of $H(z)$ that lie outside the unit circle and move them to the allpass system function

First, we need to write it as a product of two system functions. We can do that by finding the zeros:

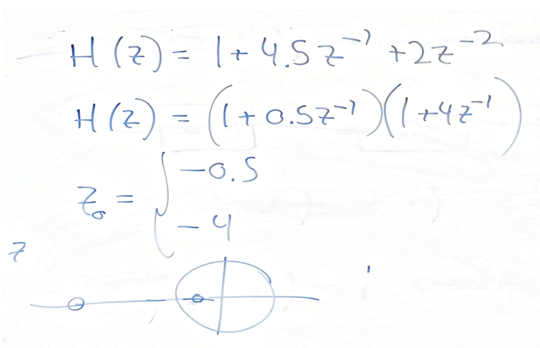
```
roots([1, 4.5, 2])
```

```
ans = 2×1
   -4.0000
   -0.5000
```

Now we can write the system as:

$$H(z) = (1 - (-0.5)z^{-1})((1 - (-4)z^{-1}))$$

$$H(z) = (1 + 0.5z^{-1})(1 + 4z^{-1})$$



One zero $z = -0.5$ is inside the unit circle and the other is outside $z = -4$. We want to flip the zero outside the unit circle to be inside the unit circle.

So we start by factorising 4 out. By doing this, we put a zero into allpass system.

$$H(z) = (1 + 0.5z^{-1})(4)\left(\frac{1}{4} + z^{-1}\right) = 4(1 + 0.5z^{-1})\left(z^{-1} + \frac{1}{4}\right)$$

Step 2: We need to put a pole into the allpass system that is in conjugate reciprocal locations of the zero

$$H(z) = 4(1 + 0.5z^{-1})\left(\frac{z^{-1} + \frac{1}{4}}{1 + \frac{1}{4}z^{-1}}\right)$$

Step 3: To keep the equality, we have to introduce a zero to the minimum-phase system function

$$H(z) = 4(1 + 0.5z^{-1})\left(1 + \frac{1}{4}z^{-1}\right)\left(\frac{z^{-1} + \frac{1}{4}}{1 + \frac{1}{4}z^{-1}}\right)$$

To tidy up, we will multiply the first three products together.Multiplying two polynomials together is the same as doing convolution:

```
4 * conv([1, 0.5], [1, 1/4])
```

```
ans = 1×3
    4.0000    3.0000    0.5000
```

So we can rewrite the system function as:

$$H(z) = (4 + 3z^{-1} + 0.5z^{-2})\left(\frac{z^{-1} + \frac{1}{4}}{1 + \frac{1}{4}z^{-1}}\right)$$

Exam 2017 Problem 1: Decompose a Linear Phase Filter

Linear phase filters with the symmetry property are commonly applied in audio signal processing due to their frequency independent group delay.

Linear phase filters are FIR filters $H(z) = b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}$ with the symmetry property of the coefficients $b_k = b_{M-k}$ for $k = 0, 1 \ldots M$. Such filters are commonly applied in e.g. audio signal processing due to their frequency independent group delay.

Consider the following linear phase filter

$$H(z) = 1 + 2.5z^{-1} + z^{-2}.$$

```
h = [1, 2.5, 1];
```

1) Rewrite the system function as a product of a minimum phase filter and an all-pass filter

1. Rewrite $H(z)$ as a product of a minimum phase filter and an all-pass filter, i.e. $H(z) = H_{min}(z)H_{ap}(z)$.

Any system function can be decomposed into a product of a minimum-phase filter and an all-pass filter using the following formula:

$$H(z) = H_{\min}(z)H_{\mathrm{ap}}(z)$$

To factorise or decompose, we take following steps:

1. Take the zeros of $H(z)$ that lie outside the unit circle and move them to the allpass system function $H_{\mathrm{ap}}(z)$
2. Add poles to the allpass system in conjugate reciprocal locations of the zeros
3. To keep the equation balanced, we add zeros in the minimum-phase system function to even out the poles that we added to the allpass system function

Since the poles that we added to the allpass system were all inside the unit circle, the zeros that we added to the minimim-phase system are also inside the unit circle.

These steps are coded in MATLAB function (see at the end of the document):

```
[H_min, H_ap] = decompose_min_ap(h)
```

```
H_min =
```

$$\frac{2}{z} + \frac{1}{2\,z^2} + 2$$

```
H_ap =
```

$$\frac{\frac{1}{z} + \frac{1}{2}}{\frac{1}{2\,z} + 1}$$

So we have:

$$H_{\min}(z) = 2 + 2z^{-1} + \frac{1}{2}z^{-2} \quad \text{and} \quad H_{\mathrm{ap}}(z) = \frac{0.5 + z^{-1}}{1 + 0.5z^{-1}}$$

2) Discuss what happens to a signal if it is filtered using only $H_{\min}(z)$ instead of $H(z)$?

The two system functions $H(z)$ and $H_{\min}(z)$ have the same magnitude response, but their phase properties differ. In this particular case the $H(z)$ is a linear phase filter and the group delay through the filter is constant and independent of frequency. For the minimum phase system function the phase is not linear and the group delay becomes frequency dependent. A signal passing through $H_{\min}(z)$ instead of $H(z)$ will thus become phase distorted.

ADSI Problem 1.2

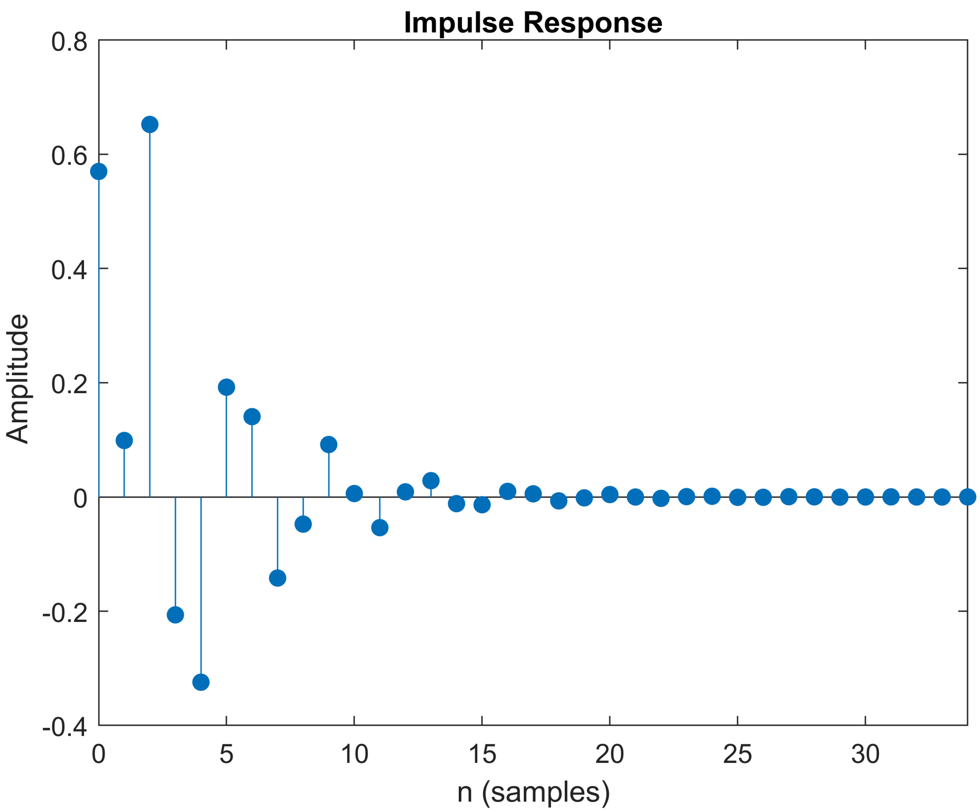
1) Compute the impulse response of allpass filter

An All-Pass filter has the following system function

$$H(z) = \frac{0.57 + 0.23z^{-1} + z^{-2}}{1 + 0.23z^{-1} + 0.57z^{-2}}$$

1. First, without performing calculations, try to predict what the impulse response of the filter will look like. Second, plot the impulse response and compare with your prediction.

```
b = [0.57, 0.23, 1];
a = [1, 0.23, 0.57];
impz(b, a);
```



2) Quantisation errors

Next, assume that the filter is to be implemented on a fixed point 16 bits signal processor, e.g. an Analog Devices Blackfin. This implies that the coefficients must be quantized to (1.15) format.

2. Is the filter still an all-pass filter when the coefficients have been quantized ?

3. If you recall how the calculation works, quantize the filter coefficients to 16 bits in (1.15) format.

```
b = [0.57, 0.23, 1];
a = [1, 0.23, 0.57];
[h, t] = impz(b, a);

bits = 16;
maxVal = (2^(bits-1))-1;
```

```
minVal = (2^(bits-1));
```

Normalise the inpulse response between -1 and +1 assuming that the maximum value is positive.

```
normalisedH = h./max(h);
```

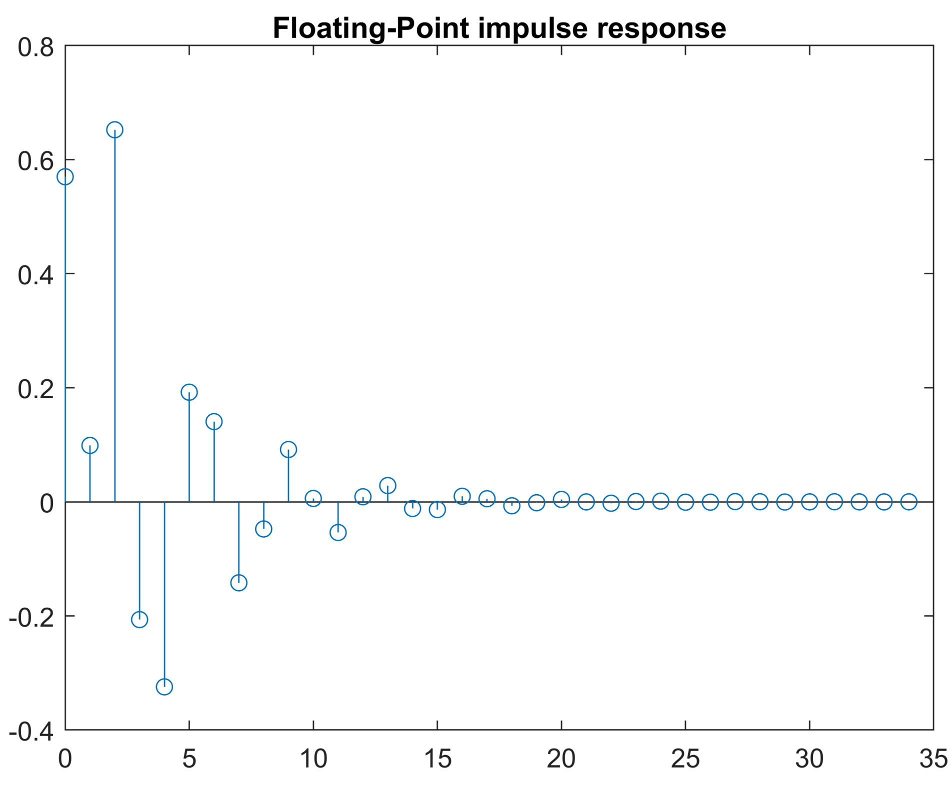
Quantise the positive values

```
quantisedH = normalisedH.* maxVal;
```

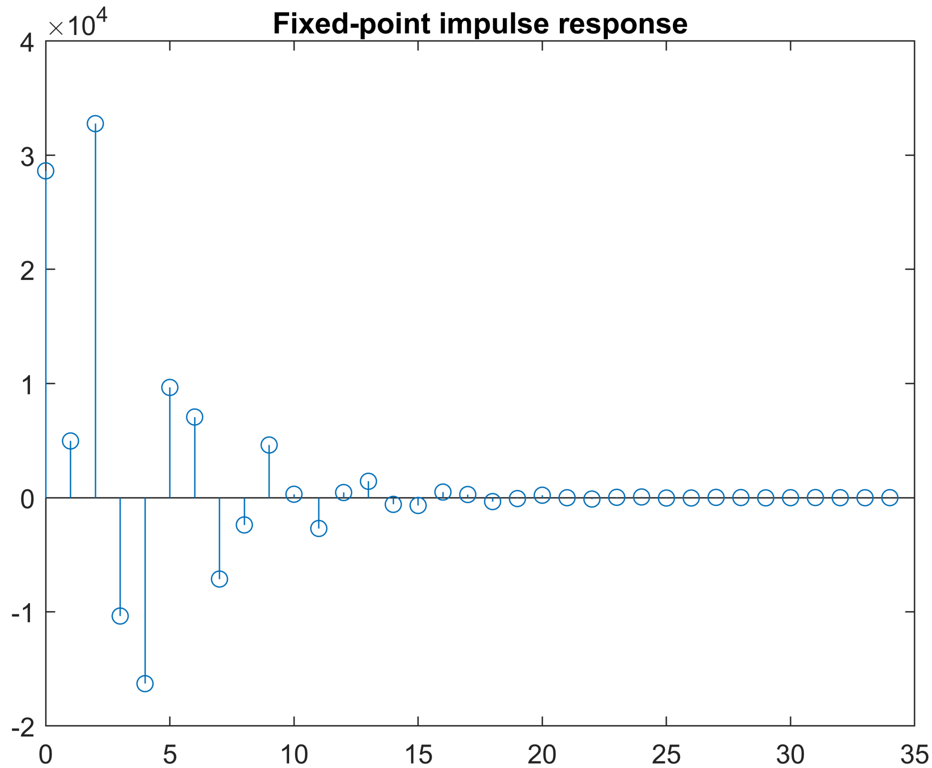
Find the index of negative values:

```
negIndices = find(normalisedH < 0);
quantisedH(negIndices) = normalisedH(negIndices).* minVal;
```

```
stem([0:34], h);
title('Floating-Point impulse response');
```




```
stem([0:34], quantisedH);
title('Fixed-point impulse response');
```



ADSI Problem 1.4: Filter decomposition

```
clear variables;
```

1) Decompose a FIR filter with one zero outside the unit circle

Let the system function for a FIR filter be given by

$$H(z) = 1 - 3z^{-1} + \frac{5}{2}z^{-2} - z^{-3}$$

1. Decompose the system function into a product of a minimum-phase filter and an all-pass filter, $H(z) = H_{min}(z)H_{ap}(z)$.

To decompose the system function, we take following steps:

1. Take the zeros of $H(z)$ that lie outside the unit circle and move them to the allpass system function $H_{\text{ap}}(z)$
2. Add poles to the allpass system in conjugate reciprocal locations of the zeros
3. To keep the equation balanced, we add zeros in the minimum-phase system function. This will even out the poles that we added to the allpass system function.

Since the poles that we added to the allpass system were all inside the unit circle, the zeros that we added to the minimim-phase system are also inside the unit circle.

This is coded in a MATLAB function:

```
h = [1, -3, 5/2, -1];
[H_min, H_ap] = decompose_min_ap(h)
```

H_min =

$$\frac{3}{z}-\frac{2}{z^2}+\frac{1}{2\,z^3}-2$$

H_ap =

$$-\frac{\frac{1}{z}-\frac{1}{2}}{\frac{1}{2\,z}-1}$$

We put everything together:

$$H(z) = H_{\min}(z)H_{\text{ap}}(z)$$

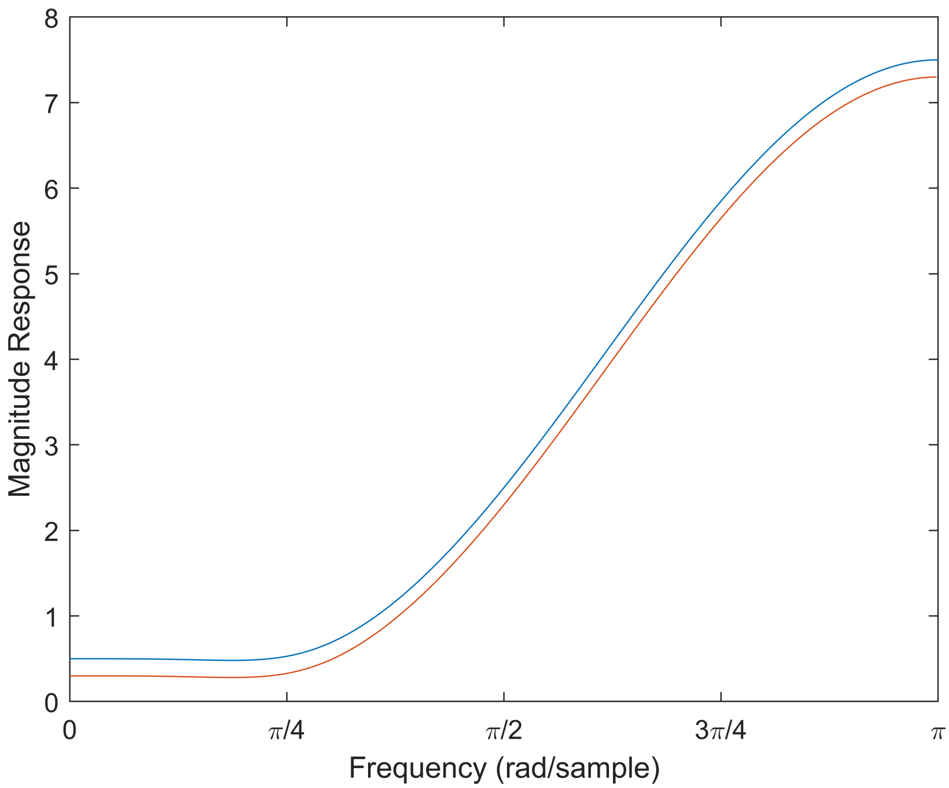
$$= \left(-2+3z^{-1}-2z^{-2}+\frac{1}{2}z^{-3}\right)\left(\frac{z^{-1}-0.5}{1-0.5z^{-1}}\right)$$

2) Show that the system and its correspond minimum-phase have the same magnitude response

2. Demonstrate that $H(z)$ and $H_{min}(z)$ have the same amplitude response.

To show that $H(z)$ and $H_{\min}(z)$ have the same magnitude response, we can plot them. If the two graphs are on top of each other then then they have the same magnitude response.

```
% Coefficients for H(z)
b = [1, -3, 5/2, -1];
a = 1;
[H, w] = freqz(b,a,'whole');

% Coefficients for H_min(z)
H_min_b = [-2, 3, -2, 1/2];
H_min_a = 1;
[H_min_z, H_min_w] = freqz(H_min_b, H_min_a);
% plot(H_min_w, log10(abs(H_min_z)));

% The offset ensures that the two graphs are not on top of each other
offset = -0.2;
plot(w, abs(H), H_min_w, abs(H_min_z) + offset);
set(gca,'XTick',0:pi/4:pi)
set(gca,'XTickLabel',{'0','\pi/4','\pi/2','3\pi/4','\pi'})
xlabel('Frequency (rad/sample)')
ylabel('Magnitude Response')
xlim([0, pi]);
```



3) Decompose a filter that has two zeros outside the unit circle

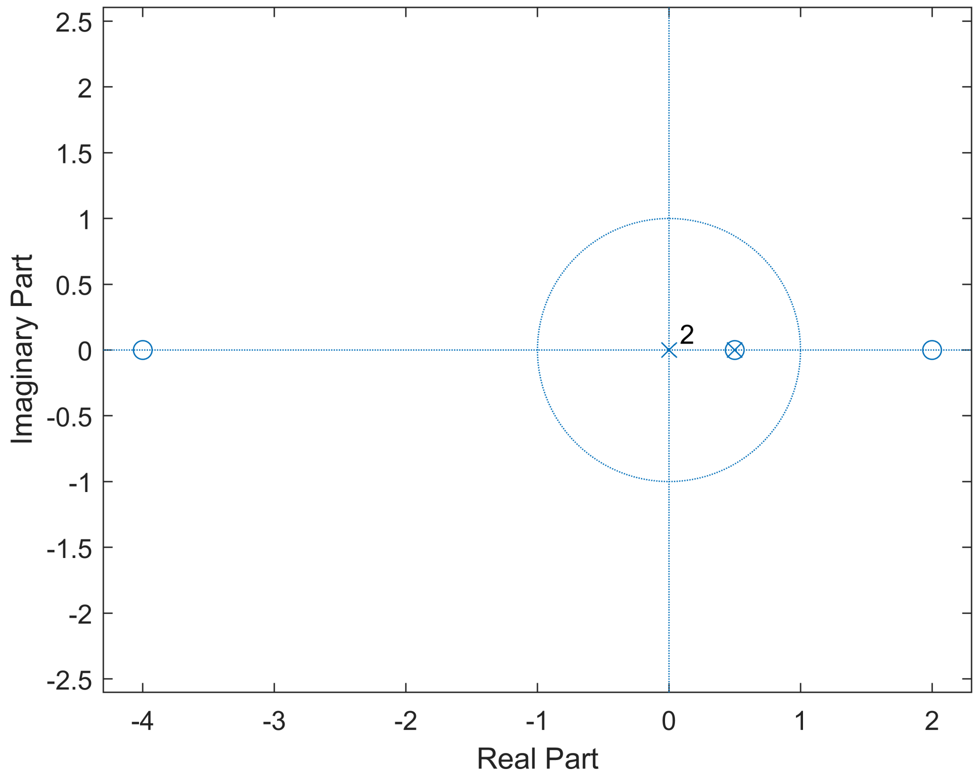
3. Repeat for the following filter

$$H(z) = \frac{1 + \frac{3}{2}z^{-1} - 9z^{-2} + 4z^{-3}}{1 - \frac{1}{2}z^{-1}}$$

```
b = [1, 3/2, -9, 4];
a = [1, -1/2];
```

Plot the zeros and the poles:

```
zplane(b, a);
```




```
[H_min, H_ap] = decompose_min_ap(b)
```

H_min =

$$\frac{10}{z} - \frac{3}{z^2} - \frac{1}{2\,z^3} + \frac{1}{4\,z^4} - 8$$

H_ap =

$$-\frac{\left(\frac{1}{z} - \frac{1}{2}\right)\,\left(\frac{1}{z} + \frac{1}{4}\right)}{\left(\frac{1}{2\,z} - 1\right)\,\left(\frac{1}{4\,z} + 1\right)}$$

Expanding the numerator, we get:

```
syms z;
H_ap_num = expand(-1*(1/z - 1/2)*(1/z + 1/4))
```

H_ap_num =

$$\frac{1}{4\,z} - \frac{1}{z^2} + \frac{1}{8}$$

```
H_ap_den = expand((1/(2*z) - 1)*(1/(4*z) + 1))
```

H_ap_den =

$$\frac{1}{4z} + \frac{1}{8z^2} - 1$$

We can multiply both the numerator and denominator with 8, to get nice numbers:

```
(H_ap_num * 8) / (H_ap_den * 8)
```

```
ans =
```

$$\frac{\frac{2}{z} - \frac{8}{z^2} + 1}{\frac{2}{z} + \frac{1}{z^2} - 8}$$

Now, we can write the transfer function for the allpass filter:

$$H_{\text{ap}}(z) = \frac{1 + 2z^{-1} - 8z^{-2}}{-8 + 2z^{-1} + z^{-2}}$$

So we have:

$$\begin{aligned} H(z) &= H_{\min}(z)H_{\text{ap}}(z) \\ &= (-8 + 2z^{-1} + z^{-2})\left(\frac{1 + 2z^{-1} - 8z^{-2}}{-8 + 2z^{-1} + z^{-2}}\right) \end{aligned}$$

[✔] ADSI Problem 1.5 Filter decomposition

1) Show that a FIR filter with a difference equation is not minimum-phase

Consider a FIR filter with the following difference equation

$$y(n) = x(n) + 2x(n-1) + 2x(n-2)$$

1. Show that the FIR filter is not minimum-phase.

To determine whether the filter is a minimum-phase or not, we need get the transfer function of this FIR filter $H(z)$ which is defined as:

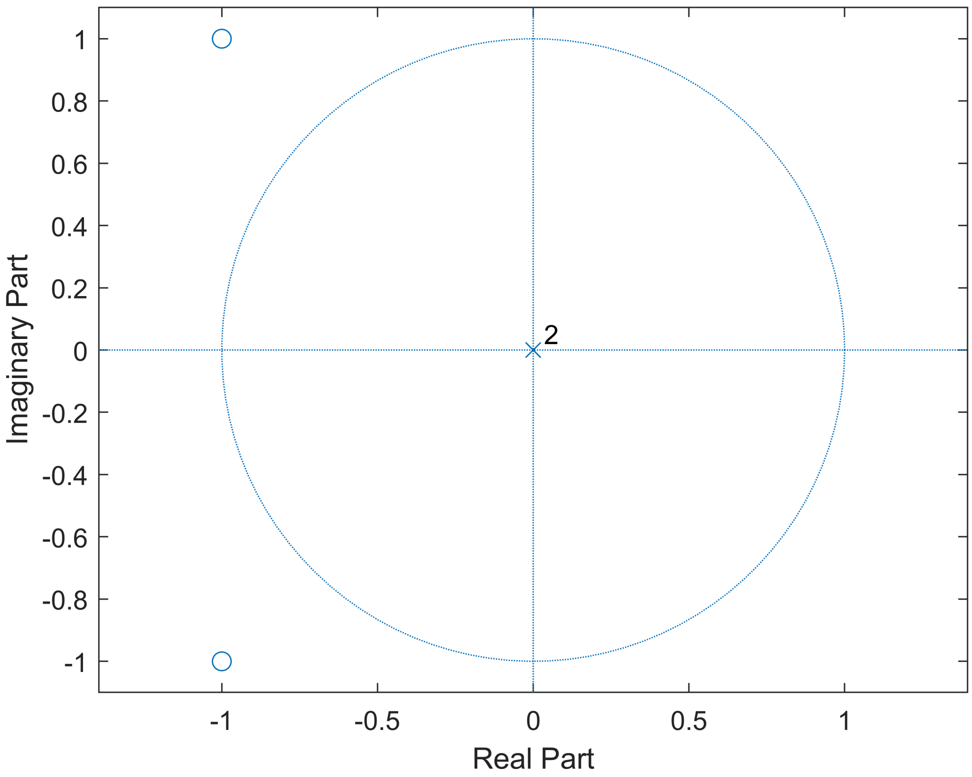
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} \cdots + a_N z^{-N}}$$

This means that we must take the z-transform of difference equation:

$$
\begin{aligned}
Y(z) &= X(z) + 2X(z)z^{-1} + 2X(z)z^{-2} \\
Y(z) &= X(z)(1 + 2z^{-1} + 2z^{-2}) \\
\frac{Y(z)}{X(z)} &= 1 + 2z^{-1} + 2z^{-2} \\
H(z) &= 1 + 2z^{-1} + 2z^{-2}
\end{aligned}
$$

Once we have the transfer function, we can determine if the filter is a minimum-phase by looking at the zplane plot:

```
zplane([1, 2, 2])
```



Since all the zeros are outside the unit cirlce, clearly the given FIR filter is not minimum-phase. It is a **maximum-phase**.

2) Find the difference equation for the corresponding minimum-phase FIR filter

To decompose the system function, we take following steps:

1. Take the zeros of $H(z)$ that lie outside the unit circle and move them to the allpass system function $H_{\text{ap}}(z)$
2. Add poles to the allpass system in conjugate reciprocal locations of the zeros
3. To keep the equation balanced, we add zeros in the minimum-phase system function. This will even out the poles that we added to the allpass system function.

Since the poles that we added to the allpass system were all inside the unit circle, the zeros that we added to the minimim-phase system are also inside the unit circle.

This is coded in a MATLAB function.

We found that the transfer function for the difference equation is:

$$H(z) = 1 + 2z^{-1} + 2z^{-2}$$

```
h = [1, 2, 2];
[H_min, H_ap] = decompose_min_ap(h)
```

H_min =

$$\frac{2}{z} + \frac{1}{z^2} + 2$$

H_ap =

$$\frac{\left(\frac{1}{z} + \frac{1}{2} - \frac{1}{2}\,\mathrm{i}\right)\left(\frac{1}{z} + \frac{1}{2} + \frac{1}{2}\,\mathrm{i}\right)}{\left(1 + \frac{\frac{1}{2} - \frac{1}{2}\,\mathrm{i}}{z}\right)\left(1 + \frac{\frac{1}{2} + \frac{1}{2}\,\mathrm{i}}{z}\right)}$$

Let us simply this expression:

```
H_ap_num = expand((1/z + 1/2 - 1/2i) * (1/z + 1/2 + 1/2i))
```

H_ap_num =

$$\frac{1}{z} + \frac{1}{z^2} + \frac{1}{2}$$

```
H_ap_den = expand((1+ (1/2 -1/2i)*z^-1) * (1 + (1/2+1/2i)*z^-1 ))
```

H_ap_den =

$$\frac{1}{z} + \frac{1}{2\,z^2} + 1$$

So the allpass system function is:

$$H_{\text{ap}}(z) = \frac{\frac{1}{2} + z^{-1} + z^{-2}}{1 + z^{-1} + \frac{1}{2}z^{-2}}$$

And the decomposition is:

$$H(z) = H_{\min}(z)H_{\text{ap}}(z)$$

$$= (2 + 2z^{-1} + z^{-2})\left(\frac{\frac{1}{2} + z^{-1} + z^{-2}}{1 + z^{-1} + \frac{1}{2}z^{-2}}\right)$$

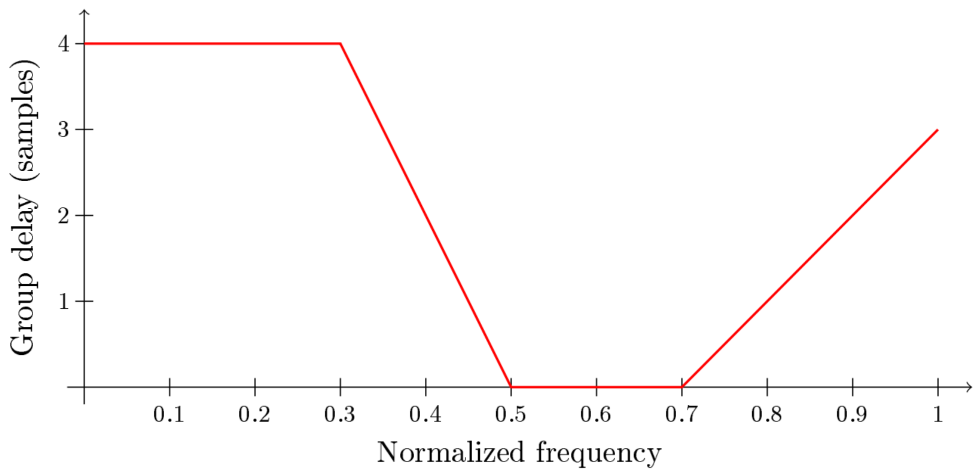
The difference equation for the minimim-phase system is:

$$y(n) = 2x(n) + 2x(n-1) + x(n-2)$$

ADSI Problem 1.8: Designing all-pass filters

```
clear variables;
```

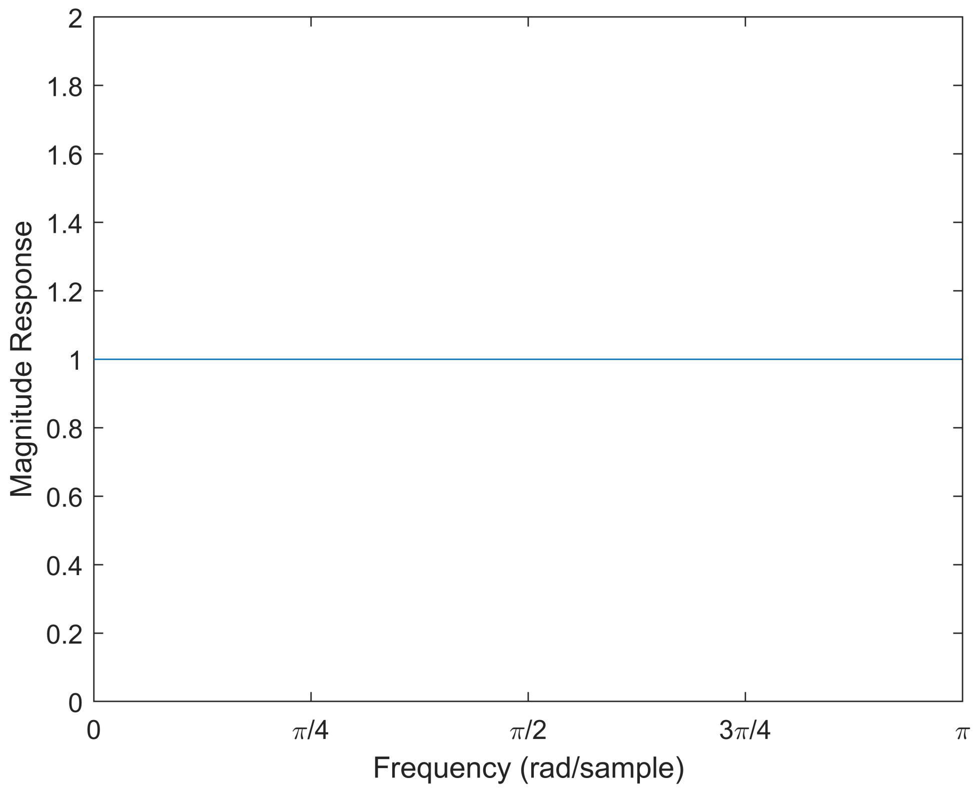
Assume it is desired to create an all-pass filter with group delay versus frequency as shown in plot below. Due to the sharp edges in the group delay such a filter can hardly be designed but an approximation is feasible.



1) Use iirgrpdelay in MATLAB to design a 4th order all-pass filter.

You can set edges=[0 1]

```
n = 4;                     % all-pass filter order
f = [0, 0.3, 0.5, 0.7, 1];
a = [4,   4,   0,   0, 3];
edges = [0, 1];

[b_ap, a_app] = iirgrpdelay(n, f, edges, a);


[H, w] = freqz(b_ap, a_app);
plot(w, H.*conj(H));
set(gca,'XTick', 0:pi/4:pi)
set(gca,'XTickLabel', {'0','\pi/4','\pi/2','3\pi/4','\pi'})
xlabel('Frequency (rad/sample)')
ylabel('Magnitude Response')
xlim([0, pi]);
ylim([0, 2])
```



2) What happens as you increase the order of the filter?

Problem 2: Allpass filters

Let a system function be given by:

$$H(z) = \frac{a + bz^{-1} + cz^{-2}}{c + bz^{-1} + az^{-2}}$$

where *a*, *b* and *c* are non-zero numbers.

1) Show that the system function H(z) is an allpass filter

An N-order allpass systems is given by

$$H_{ap}(z) = e^{j\beta} \prod_{k=1}^{N} \frac{z^{-1} - p_k^*}{1 - p_k z^{-1}}, \qquad (5.158)$$

where β is a constant (usually $\beta = 0$).

A second-order allpass system function can be expressed as:

$$H_{ap}(z) = \frac{a_2^* + a_1^* z^{-1} + z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = z^{-2} \frac{1 + a_1^* z + a_2^* z^2}{1 + a_1 z^{-1} + a_2 z^{-2}}, \qquad (5.164)$$

We can show that the given system function is an allpass filter if we can express it as in Eq. (5.164).

Since we want the denominator to contain a single 1 term, we scale both numerator and denominator by $\frac{1}{c}$:

$$H(z) = \frac{a + bz^{-1} + cz^{-2}}{c + bz^{-1} + az^{-2}}$$

$$H(z) = \frac{\frac{1}{c}(a + bz^{-1} + cz^{-2})}{\frac{1}{c}(c + bz^{-1} + az^{-2})}$$

$$H(z) = \frac{\frac{a}{c} + \frac{b}{c} z^{-1} + \frac{c}{c} z^{-2}}{\frac{c}{c} + \frac{b}{c} z^{-1} + \frac{a}{c} z^{-2}}$$

$$H(z) = \frac{\frac{a}{c} + \frac{b}{c} z^{-1} + z^{-2}}{1 + \frac{b}{c} z^{-1} + \frac{a}{c} z^{-2}}$$

2) Realise an allpass filter as an all-pole lattice filter

Allpass filters can be realized as all-pole lattice filters.

2. Find the two reflection coefficients as functions of the parameters, a, b, c and draw the corresponding lattice structure.

Based on Eq. 9.89 in the textbook we have that

$$A_2(z) = 1 + \frac{b}{c} z^{-1} + \frac{a}{c} z^{-2} \text{ and } B_2(z) = \frac{a}{c} + \frac{b}{c} z^{-1} + z^{-2}$$

From $A_2(z)$ the reflection coefficient $k_2 = \frac{a}{c}$ is directly evident. The other reflection coefficient is found using the step-down procedure 9.71

$$A_1(z) = \frac{1}{1 - k_2^2} \big(A_2(z) - k_2 B_2(z)\big)$$

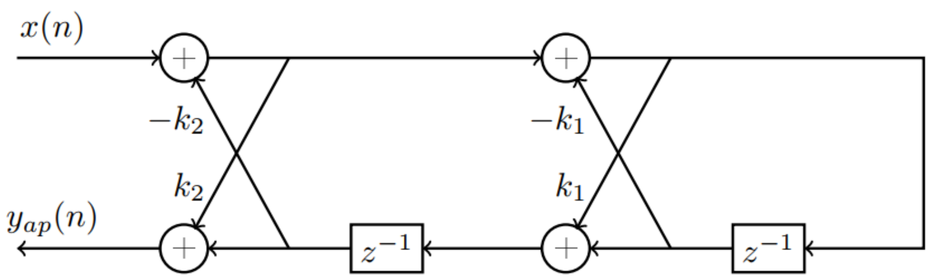
Therefore

$$A_1(z) = \frac{1}{1-\left(\frac{a}{c}\right)^2}\left(1+\frac{b}{c}z^{-1}+\frac{a}{c}z^{-2}-\frac{a}{c}\left(\frac{a}{c}+\frac{b}{c}z^{-1}+z^{-2}\right)\right)$$
$$= \frac{1}{1-\left(\frac{a}{c}\right)^2}\left(1-\left(\frac{a}{c}\right)^2+\frac{b}{c}\left(1-\frac{a}{c}\right)z^{-1}\right)$$

From which it is found that

$$k_1 = \frac{\frac{b}{c}\left(1-\frac{a}{c}\right)}{1-\left(\frac{a}{c}\right)^2} = \frac{\frac{b}{c}}{1+\frac{a}{c}}$$

The corresponding lattice structure is



Functions

```
function [H_min, H_ap]=decompose_min_ap(h)
    % Decomposes a filter into minimum-phase filter and all-pass filter
    %  h: the filter coefficients
    syms z;
    rts = roots(h);

    H_outside = 1; % Represents part of H where zeros are outside the unit circle
    H_inside = 1;  % Represents part of H where zeros are inside the unit circle
    zeros_outside = [];
    for i = 1:numel(rts)
        root = rts(i);
        if abs(root) > 1
            H_outside = H_outside * (1 - root*z^-1);
            zeros_outside = [zeros_outside; root];
```

```
        else
            H_inside = H_inside * (1 - root*z^-1);
        end
    end

    % Sanity check
    % H = expand(H_inside * H_outside)

    % Compute minimum-phase filter and all-pass filter
    H_min = 1;
    N = numel(zeros_outside);
    H_ap = 1;
    for i = 1:N
        z_i = zeros_outside(i);
        a = 1/z_i;
        a_conj = conj(a);
        H_min = H_min * (-(1/a_conj) * H_inside * (1-a * z^(-1)));
        H_ap = H_ap * (z^(-1) - a_conj) / (1 - a*z^(-1));
    end
    H_min = expand(H_min);
end
```