Autocorrelation Functions

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Summary

The autocorrelation of the **complex sinusoid** $z(n) = A\,e^{j(\omega n+\phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is given by:

$$r_{zz}(\ell) = A^2\,e^{j\omega\ell}$$

The relation between $r_{zz}(\ell)$ and $r_{zz}(-\ell)$ is:

$$r_{zz}(-\ell) = r^*_{zz}(\ell)$$

The autocorrelation function of a real **cosine signal** $z(n) = A\cos(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is:

$$r_{zz}(\ell) = \frac{A^2}{2}\cos(\omega\ell)$$

The autocorrelation function of a real **sine signal** $z(n) = A\sin(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is:

$$r_{zz}(\ell) = -\frac{A^2}{2}\cos(\omega\ell)$$

Details

Autocorrelation and cross-correlation of complex signals

The autocorrelation function of a complex signal is given by:

$$r_{\mathrm{xx}}(\ell) = E[x(n)x^*(n-\ell)]$$

The cross-correlation function of a complex signal is given by:

$$r_{\mathrm{yx}}(\ell) = E[y(n)x^*(n-\ell)]$$

Relations between r(ell) and r(-ell)

Let us compute the autocorrelation for $-\ell$:

$$r_{\mathrm{xx}}(-\ell) = E[x(n)x^*(n+\ell)]$$

Suppose $m = n + \ell$. Then we can write $n = m - \ell$. Let us substitute all n with m in the above expression:

$$r_{\mathrm{xx}}(-\ell) = E[x(m-\ell)x^*(m)]$$

Computing the complex conjucate of the expectation we get:

$$r_{\mathrm{xx}}(-\ell) = E[x^*(m-\ell)x(m)]^*$$

$$r_{\mathrm{xx}}(-\ell) = E[x(m)x^*(m-\ell)]^*$$

Since $r^*_{\mathrm{xx}}(\ell) = E[x(m)x^*(m-\ell)]^*$, we know that:

$$r_{\mathrm{xx}}(-\ell) = r^*_{\mathrm{xx}}(\ell)$$

Autocorrelation of complex signal

Problem:

What is the autocorrelation function of the complex sinusoid $x(n) = A\,e^{j(\omega n+\phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$

Solution:

The autocorrelation function for complex signals can be computed as:

$$r_{xx}(\ell) = E[z(n)z^*(n-\ell)]$$

Plugging the given complex sinusiod into the formula, we get:

$$r_{xx}(\ell) = E[A\, e^{j(\omega n+\phi)} A\, e^{-j(\omega(n-\ell)+\phi)}]$$

Since A is a constant, we can move it outside the expected value:

$$r_{xx}(\ell) = A^2 E[\, e^{j(\omega n+\phi)}\, e^{-j(\omega(n-\ell)+\phi)}]$$

$$r_{xx}(\ell) = A^2 E[\, e^{j\omega n+j\phi}\, e^{-j\omega n+j\omega\ell-j\phi}]$$

$$r_{xx}(\ell) = A^2 E[\, e^{j\omega n+j\phi+(-j\omega n+j\omega\ell-j\phi)}\,]$$

$$r_{xx}(\ell) = A^2 E[\, e^{j\omega n+j\phi-j\omega n+j\omega\ell-j\phi}\,]$$

$$r_{xx}(\ell) = A^2 E[\, e^{j\omega\ell}\,]$$

We know that $E[e^{j\omega\ell}] = e^{j\omega\ell}$ because the expected value of a constant is just the constant itself. Notice that ϕ is no longer in the expression $e^{j\omega\ell}$. Therefore, the autocorrelation of the complex sinusoid is:

$$r_{xx}(\ell) = A^2\, e^{j\omega\ell}$$

Thus, the autocorrelation of the complex sinusoid $z(n) = A\, e^{j(\omega n+\phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is given by:

$$r_{zz}(\ell) = A^2\, e^{j\omega\ell}$$

Autocorrelation of a real cosine signal

Problem:

What is the autocorrelation function of a real signal $x(n) = A\cos(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$?

Solution:

In ADSI Problem 4.4.1, we know that the autocorrelation of the complex sinusoid $z(n) = A\, e^{j(\omega n+\phi)}$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is given by:

$$r_{zz}(\ell) = A^2\, e^{j\omega\ell}$$

Since the result from 1) uses Euler, we need to convert the signal to complex exponential.

We use the relation $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

$$x(n) = A\cos(\omega\, n + \phi)$$

$$x(n) = \frac{A}{2}\, (e^{j(\omega n+\phi)} + e^{-j(\omega n+\phi)})$$

$$x(n) = \frac{A}{2} e^{j(\omega n+\phi)} + \frac{A}{2} e^{-j(\omega n+\phi)}$$

In 1) we found that the autocorrelation of a complex sinusoid $x(n) = A\, e^{j(\omega\, n+\phi)}$ is $r_{xx}(\ell) = A^2 e^{j\omega\ell}$

Therefore, the autocorrelation of the real signal is:

$$r_{xx}(\ell) = \left(\frac{A}{2}\right)^2 e^{j\omega\ell} + \left(\frac{A}{2}\right)^2 e^{-j\omega\ell}$$

$$r_{xx}(\ell) = \frac{A^2}{4} e^{j\omega\ell} + \frac{A^2}{4} e^{-j\omega\ell}$$

$$r_{xx}(\ell) = \frac{A^2}{4} (e^{j\omega\ell} + e^{-j\omega\ell})$$

$$r_{xx}(\ell) = \frac{A^2}{2}\frac{1}{2} (e^{j\omega\ell} + e^{-j\omega\ell})$$

Using the relation $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ we can rewrite the autocorrelation to:

$$r_{xx}(\ell) = \frac{A^2}{2}\cos(\omega\ell)$$

Thus, the autocorrelation function of a real signal $z(n) = A\cos(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$ is:

$$r_{zz}(\ell) = \frac{A^2}{2}\cos(\omega\ell)$$

Autocorrelation of a sine signal

Problem:

What is the autocorrelation function of a real signal $z(n) = A\sin(\omega n + \phi)$ where A and ω are real constants and $\phi \sim U(0, 2\pi)$?

Solution:

In ADSI Problem 4.4, we found that the autocorrelation of a complex sinusoid given by $y(n) = A\, e^{j(\omega n+\phi)}$ where A and ω are real constants and ϕ is a random variable with $\phi \sim U(0, 2\pi)$ is:

$$r_{yy}(\ell) = A^2\, e^{j\omega\ell}$$

To use this result, we need to convert the given signal in this problem to complex exponential signal.

A complex exponential signal is always formed by the sum of two real signals:

$$Ae^{j\omega n} = A\cos(\omega n) + j\, A\sin(\omega n)$$

Therefore, we know that:

$$\sin(\omega) = \frac{1}{2j}e^{j\omega} - \frac{1}{2j}e^{-j\omega}$$

Using this relation, we can rewrite a real signal $A\sin(\omega n + \phi)$ as:

$$z(n) = A\sin(\omega n + \phi)$$

$$z(n) = \frac{A}{2j}e^{j(\omega n+\phi)} - \frac{A}{2j}e^{-j(\omega n+\phi)}$$

To compute the autocorrelation function, we square the magnitude, remove the phase and replace n with ℓ:

$$r_{zz}(\ell) = \left(\frac{A}{2j}\right)^2 e^{j\omega\ell} - \left(\frac{A}{2j}\right)^2 e^{-j\omega\ell}$$

We know that $(2j)^2 = 2^2 \cdot j^2 = -4$ because $j = \sqrt{-1}$ so $j^2 = -1$

$$r_{zz}(\ell) = \frac{A^2}{-4}e^{j\omega\ell} - \frac{A^2}{-4}e^{-j\omega\ell}$$

$$r_{zz}(\ell) = -\frac{A^2}{4}e^{j\omega\ell} + \frac{A^2}{4}e^{-j\omega\ell}$$

We want to make the autocorrelation function in terms of $\cos(\cdot)$, we rewrite the expression as follows:

$$r_{zz}(\ell) = \left(-\frac{A^2}{2}\right) \cdot \frac{1}{2}(e^{j\omega\ell} + e^{-j\omega\ell})$$

Since $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$, we can rewrite the expression as:

$$r_{zz}(\ell) = -\frac{A^2}{2}\cos(\omega\ell)$$

Thus, the autocorrelation function of a real signal $z(n) = A\sin(\omega n + \phi)$ is

$$r_{zz}(\ell) = -\frac{A^2}{2}\cos(\omega\ell)$$