Week 8 Notes

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Difference Equations of Sinusoids

A signal consisting of p sinusoidal components has the difference equation:

$$x(n) = -\sum_{m=1}^{2p} a_m x(n-m) \qquad (14.5.2)$$

This corresponds to a system with the system function:

$$H(z) = \frac{1}{1 + \sum_{m=1}^{2p} a_m z^{-m}} \qquad (14.5.3)$$

From the polynomial $A(z)$, we observe that the system has $2p$ poles on the unit circle which correspond to the frequencies of the sinusoids

$$A(z) = 1 + \sum_{m=1}^{2p} a_m z^{-m} \qquad (14.5.4)$$

Now, suppose that the sinusoids are corrupted by a white noise sequence *w(n)* with zero mean and variance σ_w^2:

$$y(n) = x(n) + w(n) \qquad (14.5.5)$$

The difference equation for (14.5.5) is an $\mathrm{ARMA}(2p, 2p)$ process that can be expressed in matrix form:

$$\mathbf{Y}^t\mathbf{a} = \mathbf{W}^t\mathbf{a} \qquad (14.5.7)$$

where:

- $\boldsymbol{Y}^t = [y(n)\ \ y(n-1)\ \cdots\ \ y(n-2p)]$ is the observed data vector of size $2p+1$
- $\boldsymbol{W}^t = [w(n)\ \ w(n-1)\ \cdots\ \ w(n-2p)]$ is the noise vector of size $2p+1$
- $\boldsymbol{a} = [1\ \ a_1\ \cdots a_{2p}]$ is the coefficients vector

If we multiply (14.5.7) by $\boldsymbol{Y}$ and take the expected value, we obtain the following:

$$(\mathbf{\Gamma}_{yy} - \sigma_w^2\mathbf{I})\mathbf{a} = \mathbf{0} \qquad (14.5.9)$$

where:

- Γ_{yy} is the autocorrelation matrix
- σ_w^2 is an eigenvalue of the autocorrelation matrix
- $\boldsymbol{a}$ is the eigenvector associated with the eigenvalue σ_w^2

In MATLAB, we can compute the autocorrelation matrix as follows:

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R_yy = toeplitz(xcorr(y))
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The Pisarenko harmonic decomposition method

The Pisarenko method is used to estimate the power spectrum density of a random process.

The method makes two assumptions:

- The signal $x(n)$ consists of p sinusoids that has been corrupted by white noise.
- The autocorrelation matrix of size $(p+1)\times(p+1)$ is known or can be estimated

Given these assumptions, the Pisarenko method can recover the sinusoidal frequencies of the corrupted signal using the following steps:

Step 1: Compute the autocorrelation matrix $\mathbf{R}_{\text{yy}}$

Step 2: Find the minimum eigenvalue and the corresponding eigenvector. The elements of this eigenvector is the parameters of the $\text{ARMA}(2p, 2p)$ model

Step 3: Find the frequencies $\{f_i\}$ of the sinusoids. This can be done by computing the roots of the polynomial $A(z)$ in (14.5.4). This polynomial has $2p$ poles on the unit circle which correspond to the frequencies of the system.

$$A(z) = 1 + \sum_{m=1}^{2p} a_m z^{-m} \qquad (14.5.4)$$

Step 4: Solve Eq. (14.5.11) for the signal powers $\{P_i\}$

$$\begin{bmatrix} \cos 2\pi f_1 & \cos 2\pi f_2 & \cdots & \cos 2\pi f_p \\ \cos 4\pi f_1 & \cos 4\pi f_2 & \cdots & \cos 4\pi f_p \\ \vdots & \vdots & & \vdots \\ \cos 2\pi p f_1 & \cos 2\pi p f_2 & \cdots & \cos 2\pi p f_p \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_p \end{bmatrix} = \begin{bmatrix} \gamma_{yy}(1) \\ \gamma_{yy}(2) \\ \vdots \\ \gamma_{yy}(p) \end{bmatrix} \qquad (14.5.11)$$

where

- $\gamma_{yy}(1), \gamma_{yy}(2), \cdots, \gamma_{yy}(p)$ are the autocorrelation values
- $P_i = \frac{A_i^2}{2}$ is the average power of the ith sinusoid and A_i is the corresponding amplitude