Homework 6

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ADSI Problem 4.9: MA(q) processes

In this problem we will investigate the MA(q) process as defined by Eq. (13.132) by excluding the feedback part

$$y[n] = \sum_{k=0}^{q} b_k x[n-k]$$

where the input is a zero-mean Gaussian white noise process with unit variance.

[✔] 1) Write out the full expressions

Write out the full expressions for $\mathrm{MA}(0)$, $\mathrm{MA}(1)$, $\mathrm{MA}(2)$ and $\mathrm{MA}(3)$ processes.

The general $\mathrm{ARMA}(p, q)$ is given by the difference equation:

$$y[n] = -\sum_{k=1}^{p} a_k y[n-k] + \sum_{k=0}^{q} b_k x[n-k], \qquad (13.132)$$

When the feedback part is excluded, all values of a_k is set to zero, we are left with:

$$y[n] = \sum_{k=0}^{q} b_k\, x[n-k]$$

We can write out the full expressions for the difference processes as follows:

$\mathrm{MA}(0) \;\rightarrow\; y[n] = b_0\, x[n]$

$\mathrm{MA}(1) \;\rightarrow\; y[n] = b_0\, x[n] + b_1\, x[n-1]$

$\mathrm{MA}(2) \;\rightarrow\; y[n] = b_0\, x[n] + b_1\, x[n-1] + b_2\, x[n-2]$

$\mathrm{MA}(3) \;\rightarrow\; y[n] = b_0\, x[n] + b_1\, x[n-1] + b_2\, x[n-2] + b_3\, x[n-3]$

2) Calcute the autocorrelation for the MA processes

Calculate the autocorrelation $r_{yy}[l]$ for the $\mathrm{MA}(0)$ to $\mathrm{MA}(3)$ processes.

The autocorrelation function of a random process is defined as:

$$r_{\mathrm{yy}}(\ell) = E[y(n)y(n-\ell)]$$

To compute the autocorrelation for the $\mathrm{MA}(0)$, just plug its difference equation into this equation:

$$r_{\mathrm{yy}}(\ell) = E[b_0\, x(n) \cdot b_0\, x(n-\ell)]$$

$$r_{\mathrm{yy}}(\ell) = b_0^2 E[x(n) \cdot x(n-\ell)]$$

$$r_{\mathrm{yy}}(\ell) = b_0^2\, r_{\mathrm{xx}}(\ell)$$

Since $r_{\mathrm{xx}}(\ell) = \sigma^2\delta(\ell) = \delta(\ell)$. ***Why?***

$$r_{\mathrm{yy}}(\ell) = b_0^2\, \delta(\ell)$$

The autocorrelation for the $\mathrm{MA}(1)$ process is:

$$r_{\mathrm{yy}}(\ell) = E[(b_0\, x[n] + b_1\, x[n-1]) \cdot (b_0\, x[n-\ell] + b_1\, x[n-\ell-1])]$$

$$r_{yy}(\ell) = E\left[b_0^2\, x[n]x[n-\ell] + b_0 b_1 x[n]x[n-\ell-1] + b_0 b_1\, x[n-1]\, x[n-\ell] + b_1^2 x[n-1]x[n-\ell-1]\right]$$

$$r_{yy}(\ell) = E\left[b_0^2\, x[n]x[n-\ell] + b_0 b_1 (x[n]x[n-\ell-1] + x[n-1]\, x[n-\ell]) + b_1^2 x[n-1]x[n-\ell-1]\right]$$

$$r_{yy}(\ell) = b_0^2\, E[x[n]x[n-\ell]] + b_0 b_1\, E[(x[n]x[n-\ell-1] + x[n-1]\, x[n-\ell])] + b_1^2\, E[x[n-1]x[n-\ell-1]]$$

$$r_{yy}(\ell) = b_0^2\, E[x[n]x[n-\ell]] + b_0 b_1\, (E[x[n]x[n-\ell-1]] + E[x[n-1]\, x[n-\ell]]) + b_1^2\, E[x[n-1]x[n-\ell-1]]$$

Since

- $E[x[n]x[n-\ell]] = r_{xx}(\ell)$
- $E[x[n]x[n-\ell-1]] = r_{xx}(\ell-1)$
- $E[x[n-1]x[n-\ell]] = r_{xx}(\ell+1)$

3) Calculate power density spectra

Calculate the general expressions for power density spectra of MA(0), MA(1) and MA(2) processes using Eq. (13.112).

The power spectral density (PSD) is defined as:

$$S_{xx}(\omega) \triangleq \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell]\mathrm{e}^{-\mathrm{j}\ell\omega}. \qquad (13.112)$$

In 2) we found that the autocorrelation for $\mathrm{MA}(0)$ is $r_{yy}(\ell) = b_0^2\, \delta(\ell)$, so plug it in this PSD formula:

$$S_{yy}^{\mathrm{MA}(0)}(\omega) = \sum_{\ell=-\infty}^{\infty} r_{yy}(\ell)e^{-\mathrm{j}\ell\omega} = \sum_{\ell=-\infty}^{\infty} b_0^2\, \delta(\ell)e^{-\mathrm{j}\ell\omega}$$

We notice that $r_{yy}(\ell)$ has non-zero value only when $\ell = 0$. Therefore, the terms for when $\ell \neq 0$ are zero. We can, therefore, remove the infinite sum. Since $\delta(0) = 1$ and $e^{-\mathrm{j}0\omega} = 1$, we have:

$$S_{yy}^{\mathrm{MA}(0)}(\omega) = b_0^2\, \delta(0)e^{-\mathrm{j}0\omega} = b_0^2 \cdot 1 \cdot 1 = b_0^2$$

[✔] 4) Plot the power density spectra given coefficients

Plot the power density spectra for the two processes defined by

q	b_0	b_1	b_2
1	3	2	
2	3	2	1

We are given the coefficients for $S_{\mathrm{yy}}^{\mathrm{MA}(1)}(\omega)$ and $S_{\mathrm{yy}}^{\mathrm{MA}(2)}(\omega)$. We use results from 3) to plot the power density spectra for the two processes.

```
b0=3; b1=2; b2=1;
w=0:0.001:2*pi;
S1=(b0^2+b1^2)+2*b0*b1*cos(w);
S2=(b0^2+b1^2+b2^2)+2*(b0*b1+b1*b2)*cos(w)+2*b0*b2*cos(2*w);
plot(w,S1,w,S2)
legend('S_{yy}^{MA(1)}(\omega)','S_{yy}^{MA(2)}(\omega)')
xlabel('Frequency (\omega)')
ylabel('Power density sectra S_{yy}(\omega)')
xlim([0,2*pi])
```

ADSI Problem 4.10: MA processes and phase properties

Consider the following two MA(q) systems and assume that they are excited by zero-mean white Gaussian noise with unit variance.

$$y_1[n] = 2x[n] - x[n-2] \quad \text{and} \quad y_2[n] = x[n] - 2x[n-2]$$

[✔] 1) What is the order q of the processes

The order q of an $\mathrm{MA}(q)$ process is given by the longest delay, i.e., $x[n-q]$. Therefore, both systems are of order 2.

[✔] 2) Compare the phase properties of the two systems

To determine the phase properties of a system, we need to plot where its the zeros are located in relation to the unit circle in a zero-pole plot. If all zeros are located within the unit circle, then we have a minimum-phase system.

```
h1 = [2, 0, -1]; % Impulse response for y1[n]
zplane(h1);
```

From the zero-pole plot, we observe that all the zeros are within the unit circle. Therefore, we can conclude that the system $y_1[n]$ is a minimum-phase.

Let us make the zero-plot for the system $y_2[n] = x[n] - 2x[n-2]$

```
h2 = [1, 0, -2]; % Impulse response for y2[n]
zplane(h2);
```

Since both zeros are outside the unit circle, we can conclude the system $y_2[n]$ is a maximum-phase system.

ADSI Problem 4.12: MA(q) spectral estimation

The autocorrelation function for a MA(1) process has been found to be

$\|l\|$	$r_{yy}(l)$
0	2
1	1
$\|l\| \geq 2$	0

1) Plot the power density spectrum

2) Calculate the model parameters

ADSI Problem 4.15: AR(p) signal modelling

Consider the following AR(2) process

$$y[n] = \frac{5}{6}y[n-1] - \frac{1}{6}y[n-2] + x[n]$$

where $x[n]$ is a zero-mean WGN process with variance equal to 2.

1) Calculate and plot the theoretical power density spectrum.

2) Create a 1024 samples long realization of the process and calculate the autocorrelation

Create a 1024 samples long realization of the process and calculate the autocorrelation with `xcorr`.

3) Fit AR(4) and AR(6) models

Fit AR(4) and AR(6) models to the experimentally derived autocorrelation function values. Are the retrieved model parameters in concordance with your expectations.

4) Compare the power density spectra of the AR(4) and AR(6) model with the result from question 1.

Compare the power density spectra of the AR(4) and AR(6) model with the result from question 1.

ADSI Problem 4.17: Signal modelling of speech

One of the reasons for modelling signals is that the model gives us the ability to describe a given signal with a few parameters rather than being forced to deal with the entire signal. This is used in e.g. compression of speech signals and recognition of speech. In the problem we will create all-pole models of speech signals. The hypothesis is that a signal $x(n)$ can be modelled as Gaussian white noise $w(n)$ sent through a Pth order all-pole filter i.e.

$$x(n) = -\sum_{k=1}^{P} a_k x(n-k) + w(n).$$

We can not expect to have an exact match between the signal and the model. Instead, the model and the signal should have the same statistical properties.

1) Record your voice

Use the microphone in your pc to record a few sequences where you use both voiced (i.e. 'a', 'b' or 'd') and unvoiced (i.e. 'f', 's' or 't') sounds. Try also to create (white) noise with your voice.

2) Can a short sequence of your signal be described with an AR(1) model?

The model is built on the assumption that the signals are stationary. Obviously, a speech signal is non-stationary, but if we only consider short sequences of 20-25 ms duration these sequences are quite close to being stationary.

An AR(1) model driven by zero mean white noise with variance σ_w^2 is described by

$$x(n) = -ax(n-1) + w(n)$$

The autocorrelation function of the AR(1) process is given by

$$r_{xx}(l) = \frac{\sigma_w^2}{1-a^2}(-a)^{|l|}$$

Use Matlabs **`xcorr`** command and the above equation to decide whether a short sequence of your signal can be described with an AR(1) model.

3) Create an AR(2) model using Yule-Walter equations (13.149)

4) Whiten your speech signal

One way of testing whether the assumed model is good is to use the model coefficients to whiten the signal. If the result of the whitening is white noise, the model is good.

$$w(n) = \sum_{k=0}^{P} a_k x(n-k) \quad \text{with } a_0 \equiv 1$$

Whiten your speech signal with the AR(2) coefficients and test whether the output is white.

5) Increase the order of the all-pole filter

Increase the order of the all-pole filter by using (13.142) and see if you can get a good model of the signal.

6) Repeat the above questions for voiced, unvoiced and noisy speech signals.

Repeat the above questions for voiced, unvoiced and noisy speech signals.