Homework 7

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Problem 14.42

Consider the harmonic process signal model

$$x[n] = \cos(0.44\pi n + \phi_1) + \cos(0.46\pi + \phi_2) + v[n], \quad 0 \le n \le 256$$

where ϕ_1 and ϕ_2 are IID random variables uniformly distributed over $[-\pi, \pi]$ and $v[n] \sim \mathrm{WGN}(0, 2)$.

The autocorrelation of real sinusoid $z(n) = A\cos(\omega_0 n + \phi)$ is $r_{zz}(\ell) = \frac{A^2}{2}\cos(\omega_0 \ell)$.

Since the PSD is the Fourier transform of the autocorrelation, we get:

$$S_{zz}(\omega) = \frac{A^2}{4}\delta(\omega + \omega_0) + \frac{A^2}{4}\delta(\omega - \omega_0)$$

This means that power is concentrated at $\pm\omega_0$. In the PSD, we will therefore see two peaks $\pm\omega_0$

The autocorrelation of white Gaussian noise is $r_{vv}(\ell) = \sigma_v^2\delta(\ell) = 2\delta(\ell)$. The PSD of Gaussian noise is constant around σ_v^2.

So the true PSD should be something like:

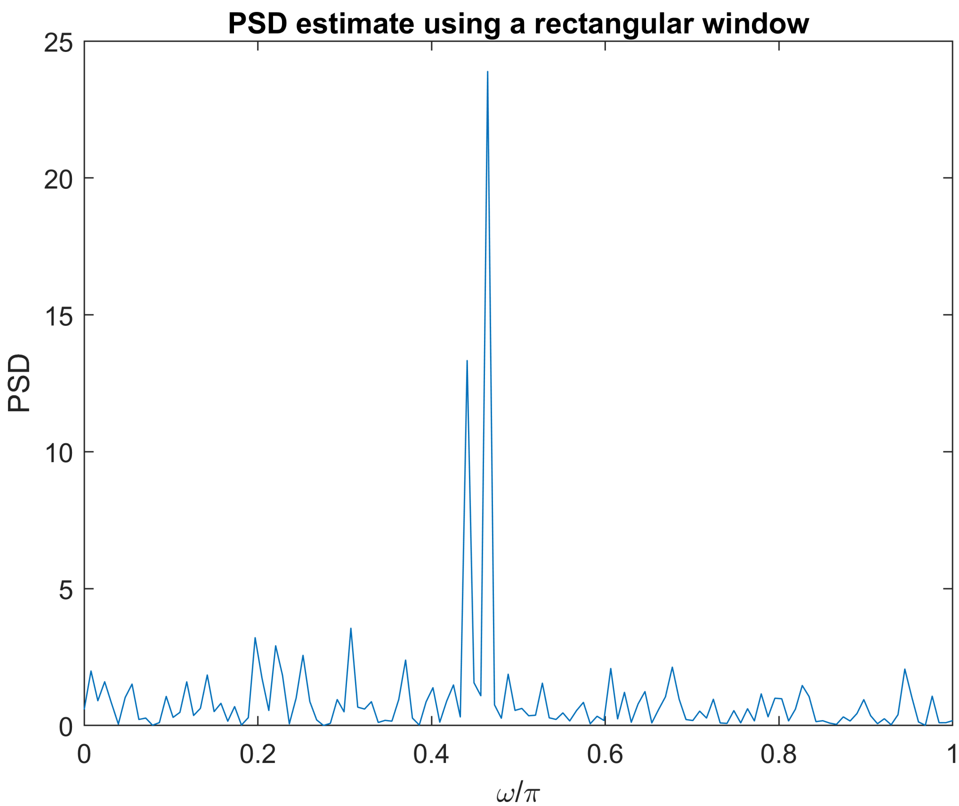
$$S_{xx}(\omega) = \frac{1}{4}\delta(\omega - 0.44\pi) + \frac{1}{4}\delta(\omega - 0.46\pi) + 2$$

1) Estimate the PSD using the periodogram and plot the spectrum.

```
N = 256;
n = 0:N-1;
v = randn(1,N) * sqrt(2);
phi = 2*pi*(rand(1,2)-0.5);
x = cos(0.44*pi*n + phi(1)) + cos(0.46*pi*n + phi(2)) + v;

%plot(n, x); xlim([0, N]);
```

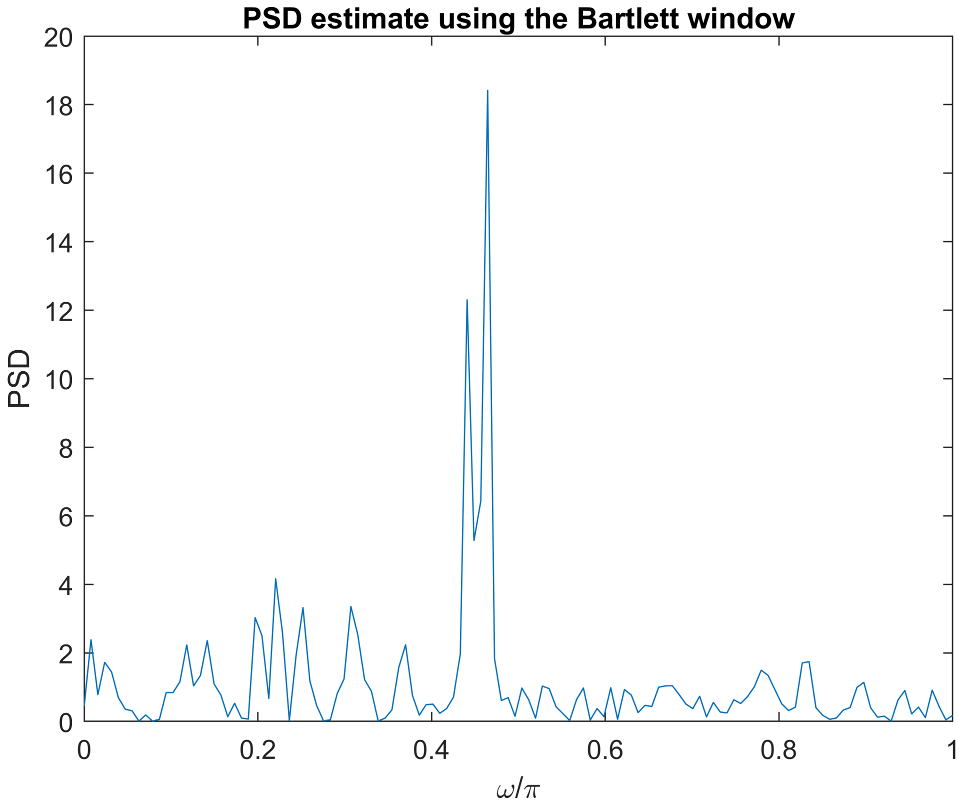
```
I = periodogram(x);
w = linspace(0,1,N/2)*pi;
IdB = db(I(1:N/2));
plot(w/pi, I(1:N/2));
xlabel('\omega/\pi')
ylabel('PSD')
title('PSD estimate using a rectangular window')
```



2) Estimate the PSD using the modified periodogram with Bartlett window

Estimate the PSD using the modified periodogram with Bartlett window and plot the spectrum.

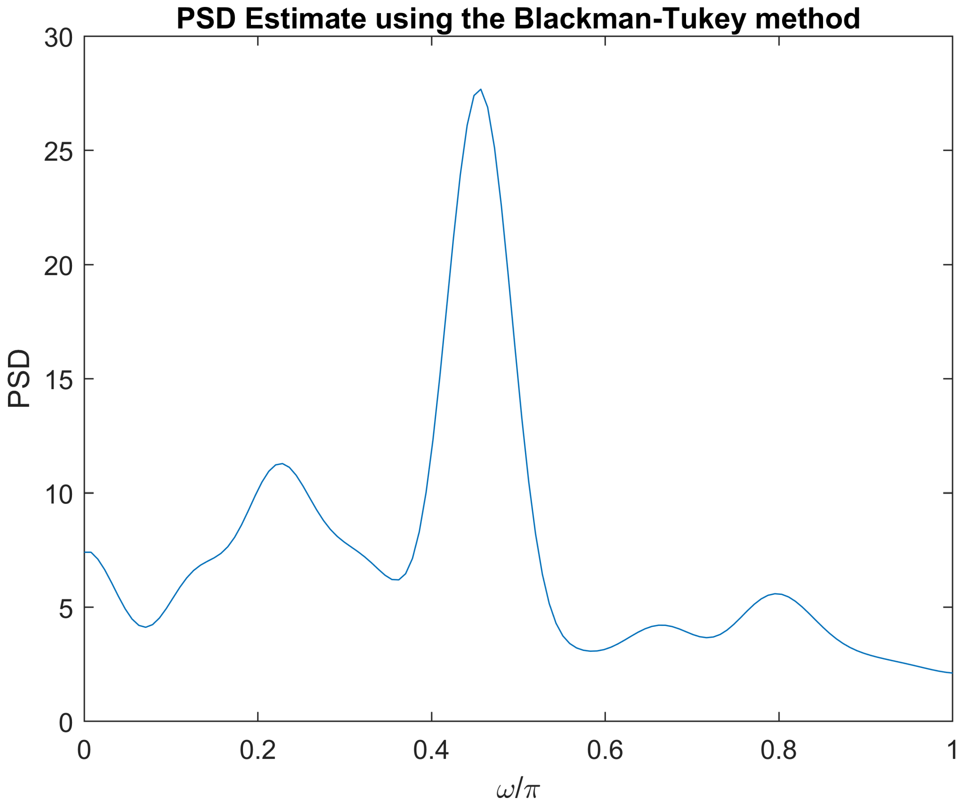
```
L = N;
I = periodogram(x, bartlett(L));
w = linspace(0,1,N/2)*pi;
IdB = db(I(1:N/2));
plot(w/pi, I(1:N/2));
xlabel('\omega/\pi')
ylabel('PSD')
title('PSD estimate using the Bartlett window')
```



3) Estimate the PSD using the Blackman–Tukey method

Estimate the PSD using the Blackman–Tukey method with Parzen window and *L*=32 and plot the spectrum.

```
L = 32;
K = N;
I = psdbt(x,L,K);
plot(w/pi, I(1:N/2));
xlabel('\omega/\pi')
ylabel('PSD')
title('PSD Estimate using the Blackman-Tukey method')
```



4) Which method performs best in terms of signal resolution?

Problem 14.57

This problem uses the signal file f16.mat that contains noise recorded at the copilot's seat of an F-16 airplane using a 16 bit A/D converter with $F_S = 19.98$ kHz.

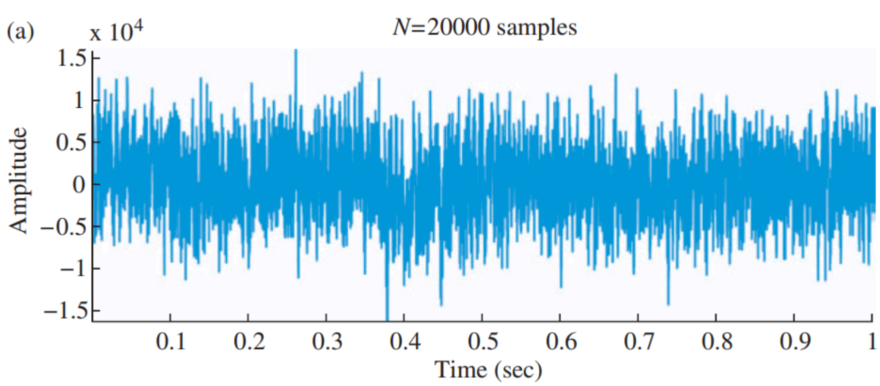


Figure (a) shows a waveform of F-16 noise recorded at the co-pilot's seat with a sampling rate of 19.98 kHz using a 16 bit ADC.

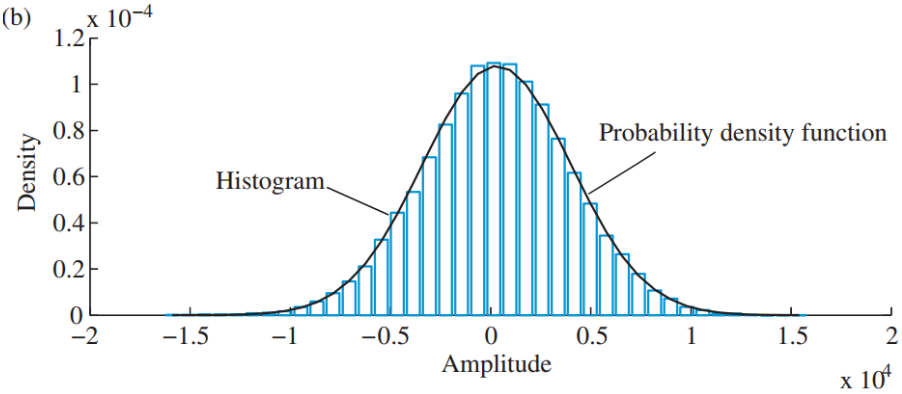
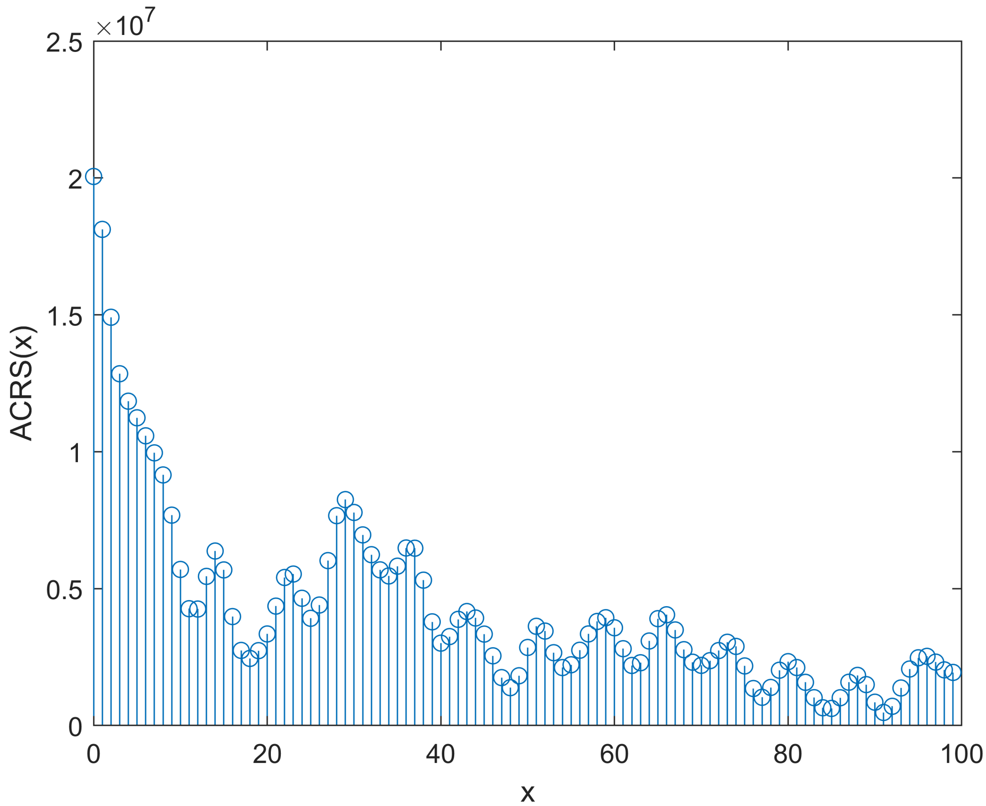


Figure (b) shows the histogram and theoretical probability density function of the F-16 noise.

We want to analyze this signal in terms of its ACRS and its spectral characteristics.

1) Compute and plot the ACRS estimate of the noise process.

```
load('f16.mat')
x = f16;
L = 100;
r_xx = acrsfft(x, L);
stem(0:L-1, r_xx)
xlabel('x')
ylabel('ACRS(x)')
```



2) Estimate model parameters for an *AR(2) and AR(4)* models.

3) Compute and plot the PSD estimate using the AR(2) and AR(4) models.

4) Compute and plot the periodogram PSD estimate of the noise process.

5) Compute the Bartlett PSD estimate

Compute the Bartlett PSD estimate using L = 32, 64, and 128. Plot your result of the averaged estimate.

6) Compute the Blackman–Tukey PSD estimate

Compute the Blackman-Tukey PSD estimate using L = 32, 64, and 128 using the Bartlett lag window. Plot your result of the averaged estimate.

7) Compute the Welch PSD estimate

Compute the Welch PSD estimate using 50% overlap, Hamming window, and L = 32, 64, and 128. Plot your result of the averaged estimate.

8) Compare the plots in the above four parts and comment on your observation.

ADSI Problem 4.21: Minimum variance spectral estimation

The discussion of the improved filter bank method taught us to choose filters, that for a given center frequency reduced the influence of spectral leakage by selectively moving the sidelobes around to attenuate specific frequency regions of high spectral density. This problem takes a closer look at this interpretation.

1) Create a stable signal model

Create a signal model with a suitable behaviour i.e. a couple of peaks in the spectrum. It is unimportant whether the model is AR, MA or ARMA as long as the model is stable.

2) Use the signal model to plot the true spectrum.

3) Create a realization of the signal and determine the autocorrelation matrix *Rx* in an appropriate size.

4) Calculate the spectrum using Eq. (14.4.12) from the note.

$$P_{xx}^{\mathrm{MV}}(f) = \frac{1}{\mathbf{E}^H(f)\mathbf{R}_{xx}^{-1}\mathbf{E}(f)} \qquad (14.4.12)$$

5) Calculate the optimum filters

Choose a few representative frequencies in your spectrum. Calculate the optimum filters for estimation of these frequencies and plot the magnitude response of these. Are the filter magnitude responses in concordance with your intuition?

ADSI Problem 4.22: Minimum variance and spectral resolution

On page 472 of the book "Statistical and adaptive signal processing - Spectral estimation, signal modeling, adaptive filtering and array processing" by Manolakis, Ingle and Kogon it is stated that *In addition, minimum-variance spectral estimation provides improved resolution - better than the the $\delta f \approx 1/N$ associated with the DFT methods.* In this problem this claim is investigated. Let an $AR(2)$ process be given by

$$x(n) = 0.75x(n-1) - 0.5x(n-2) + w(n)$$

Where $w(n)$ is Gaussian zero-mean unit-variance white noise. Two sinusoidal signals are added to the $AR(2)$ process. They are given by

$$s_1(n) = 5\cos(2\pi\frac{1}{7}n + \phi_1) \quad \text{and} \quad s_2(n) = 4\cos(2\pi\left(\frac{1}{7} + \delta f\right)n + \phi_2)$$

Where δf is a small frequency offset and ϕ_1 and ϕ_2 are random phases uniformly distributed in the range from 0 to 2π.

1) Create a realisation of the signal

Create a 4096 samples long realization of the signal where $\delta f = \frac{1}{50}$.

2) Plot the spectrum of the process using minimum variance and the periodogram

Eventhough the periodogram is generally a poor spectral estimator we will use it in this exercise for simplicity.

2. Plot the spectrum of the process using the minimum variance method and the periodogram. Can the two sinusoids be resolved with both methods?

3) Verify if the book's claim is justified

Adjust δf and find the minimum value where the two sinusoids can be resolved with the two techniques. Is the claim from the book justified?

Problem 2 from Exam 2012: Construct an AR(2) model using a signal from a random process

A sequence of data from a random process is given by

$$\{1,\ -1,\ 0,\ 2,\ 3,\ 2,\ -4,\ 1\}$$

Use the data to construct an $AR(2)$ model of the random process and account for any assumptions you make.

Functions

```
function r=acrs(x, L)
    % Computes the ACRS r[m] for 0<= m <= L
    % r=acrs(x-mean(x),L) yields the ACVS
    N=length(x);
    x1=zeros(N+L-1,1);
    x2=x1;
    x1(1:N,1)=x;
    for m=1:L
        x2=zeros(N+L-1,1);
        x2(m:N+m-1,1)=x;
        r(m)=x1'*x2;
    end
    r=r(:)/N;
end

function I=psdper(x, K)
    % Compute periodogram I(ω) using the FFT.
    % K-point FFT >= N
    N=length(x);
    X=fft(x,K);
    I=X.*conj(X)/N;
    I(1)=I(2); % Avoid DC bias
    I=I(:);
end

function r=acrsfft(x, L)
    % Computate the autocorrelation sequence using the FFT.
    % r=acrsfft(x-mean(x),L) yields the ACVS
    N=length(x);
    Q=nextpow2(N+L);
    X=fft(x,2^Q);
    r0=real(ifft(X.*conj(X)));
    r=r0(1:L)/N;
end

function I=psdmodper(x, K)
    % Compute the modified periodogram PSD estimate.
    % K-point FFT >= N
    N=length(x);
    w=hann(N); % choose window
    w=w/(norm(w)/sqrt(N)); % sum wˆ2[k]=N
    X=fft(x(:).*w(:),K);
```

```
    I=X.*conj(X)/N;
    I(1)=I(2); % Avoid DC bias
    I=I(:);
end

function S=psdbt(x, L, K)
    % Computate the Blackman-Tukey PSD estimate.
    % Blackman-Tukey PSD estimator of S(2*pi*k/K)
    if size(x,1) < size(x,2)
        x = x';
    end
    N=length(x);
    w=hann(N); % Data window
    w=w/(norm(w)/sqrt(N)); % sum w^2[k]=N
    x=x.*w; % Data windowing
    r=acrsfft(x,L);
    wc=parzenwin(2*L-1); % Lag window
    rw=r.*wc(L:2*L-1); % Lag windowing
    g=zeros(K,1);
    g(1:L)=rw;
    g(K-L+2:K)=flipud(rw(2:L));
    G=fft(g,K); % f even => F real
    S=2*real(G(1:K/2));
    S(1)=S(2);
end

function S=psdwelch(x, L, K)
    % Compute the Welch PSD estimate.
    % Welch PSD estimator of S(2*pi*k/K)
    M=fix((length(x)-L/2)/(L/2)) % 50% overlap
    time=(1:L)';
    I=zeros(K,1);
    w=hanning(L); % Choose window
    w=w/(norm(w)/sqrt(L)); % sum w^2[k]=L
    for m=1:M
    %xw=w.*detrend(x(time)); % detrenting
    xw=w.*x(time); % data windowing
    X=fft(xw,K);
    I=I+X.*conj(X);
    time=time+L/2;
    end
    I=I/(M*L); S=2*I(1:K/2); S(1)=S(2);
end

function [a,v] = arfit(x,p)
    % fit AR(p) model from data
    % x: data
    % p: model order
    % a: a coefficients
    % v: variance
    rxx = xcorr(x(100:end),p,'biased');
    R = toeplitz(rxx(p+1:2*p));
    r = rxx(p+2:2*p+1);
```

```
    a = -R\r;
    v = rxx(p+1) + a'*r;
end

function [S] = ar2psd(a,w,v)
    % compute power spectral density from a coeffeicients in AR process
    % a: a coefficients (excluding initial 1)
    % w: omega (frequencies to compute at)
    % v: noise variance
    % S: power spectral density
    if nargin < 3
        v = 1;
    end
    H = ones(length(w),1);
    for k = 1:length(a)
        H = H + a(k)*exp(-1j*w*k);
    end
    H = 1./H;
    S = H.*conj(H)*v;
end
```