Week 5: Random Processes

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Introduction

Why work with random processes? Because we need a mathematical description of the random nature of the process that generated the observed values of a given signal.

A random process is a collection signals with “some” probability assigned to each.

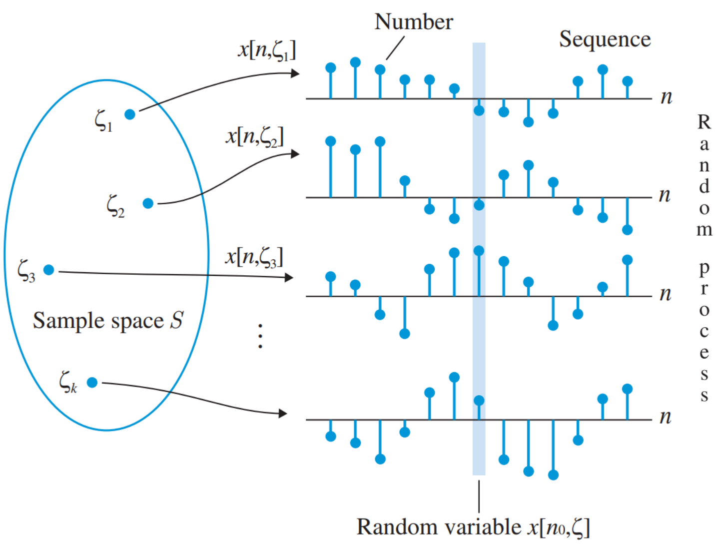


Figure 13.9 The concept of a random (stochastic) process as a mapping from the sample space of a random experiment to an ensemble of sequences.

A random process can be thought of a bin that contains multiple signals of infinite length

Every time, we perform a random experiment, we randomly pick one *realization* of the process.

Stationary Random Processes

A stationary random process is a process which characteristics that do not change over time i.e., for different values of n.

This implies:

a) The mean and the variance of a signal $x[n]$ randomly picked from the stationary process does not depend on time n but is constant:

$$E(x[n]) = m_x \quad \text{and} \quad \text{var}(x[n]) = \sigma_x^2, \text{ for all } n. \tag{13.72}$$

b) For two signals $x[n]$ and $x[m]$ randomly picked from the stationary process, the autocorrelations and autocovariance only depend on the lag ℓ and not time:

$$c_{xx}[n,m] \triangleq \text{cov}(x[n], x[m]) = c_{xx}[\ell]. \text{ for all } m, n \tag{13.73}$$

Wide-sense stationary

Wide-sense stationary (WSS): A random process that satisfies both a) and b) are called *wide-sense stationary* or *second-order stationary*.

The *autocorrelation sequence* (ACRS) of a wide-sense stationary process is

$$r_{xx}[m+\ell, m] \triangleq E(x[m+\ell]x[m]) = r_{xx}[\ell] = c_{xx}[\ell] + m_x^2 \tag{13.74}$$

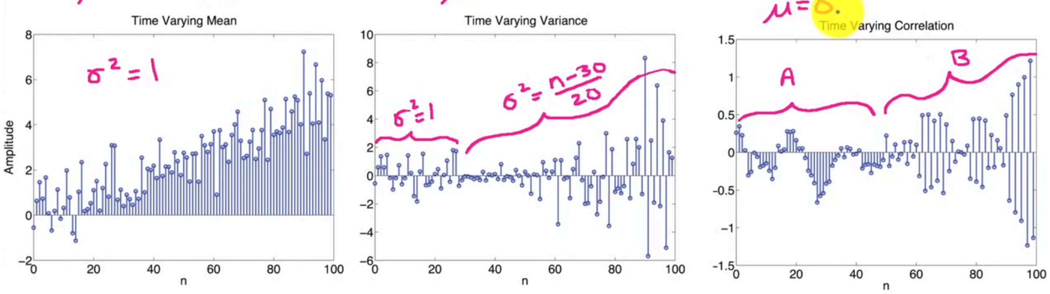
Examples of non-stationary signals

Examples of Nonstationary Signals 3

$A: \frac{r[1]}{r[0]} = 0.9$

$B: \frac{r[1]}{r[0]} = -0.9$

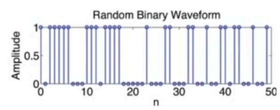
$\mu[n] = n/20$ $\mu = 0$ $\mu = 0.$



Examples of stationary signals

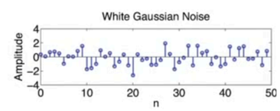
Examples of Stationary Signals -

1) Random Binary Waveform $x[n] = \begin{cases} 0 & P = 1/2 \\ 1 & P = 1/2 \end{cases}$



$\mu = 1/2$, $r[k] = \begin{cases} 1/2 & k=0 \quad E\{x^2[n]\} \\ 1/4 & k \neq 0 \quad E\{x[n]x[n-k]\} \end{cases}$

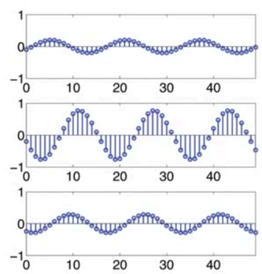
2) White Gaussian Noise $w[n] \sim N(0, \sigma_w^2)$
independent samples



$\mu = 0$, $r[k] = \begin{cases} \sigma_w^2 & k=0 \\ 0 & k \neq 0 \end{cases}$

3) Random Sinusoid $y[n] = A\cos(\omega_0 n + \phi)$

$A \sim N(0, \sigma_A^2)$; $\phi \sim U(0, 2\pi)$



$\mu = 0$, $r[k] = E\{A^2 \cos(\omega_0 n + \phi)\cos(\omega_0(n-k) + \phi)\}$

$= \frac{\sigma_A^2}{2}\cos(\omega_0 k)$

Properties of autocorrelation sequence

Properties of autocorrelation functions:

1. $r_{xx}(0) = \overline{X^2}$
2. $r_{xx}(l) = r_{xx}(-l)$
3. $r_{xx}(0) \geq |r_{xx}(l)|$
4. If $X(k) = \overline{X} + N(k)$ then $r_{xx}(l) = \overline{X}^2 + r_{NN}(l)$
5. If $X(k) = A\cos(\omega k + \theta) + N(k)$ then
$r_{xx}(l) = \frac{A^2}{2}\cos(\omega l) + r_{NN}(l)$
6. $\lim_{|\tau| \to \infty} r_{xx}(l) = 0$ for ergodic, zero-mean processes with no periodic components
7. $\mathscr{F}[r_{xx}(l)] \geq 0 \qquad \forall \omega$

Power Spectral Density (PSD)

Power spectral density is used to characterise stationary random processes in the frequency domain. It is the discrete-time Fourier Transform of the auto-correlation sequence $r_{xx}(\ell)$ for the process.

It is defined as follows:

$$S_{xx}(\omega) = R_{xx}(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} r_{xx}(\ell) e^{-j\ell\omega}$$

Given the PSD, we can compute the auto-correlation sequence:

$$r_{xx}(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{xx}(\omega) e^{j\omega\ell} \, d\omega$$

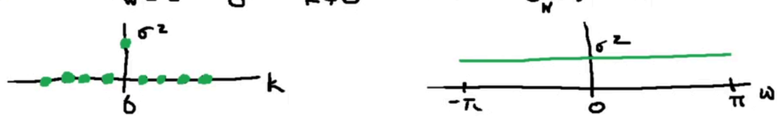
Examples

Examples

1) white noise

$r_w[k] = \begin{cases} \sigma^2 & k = 0 \\ 0 & k \neq 0 \end{cases} \xleftrightarrow{\text{DTFT}} S_w(\omega) = \sigma^2$

3



The power spectral density is constant which means that we have the same power across all frequencies. This signal is called white noise because the power is equally distributed across the entire spectrum.

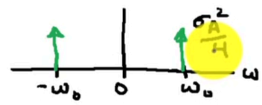
2) Random Sinusoid

$s[n] = A\cos(\omega_0 n + \phi)$

ϕ: uniform $[0, 2\pi]$

A: Gaussian, $E\{A\}=0$, $E\{A^2\}=\sigma_A^2$

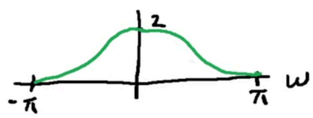
$$r_s[k] = \frac{\sigma_A^2}{2}\cos(\omega_0 k) \xleftrightarrow{DTFT} S_s(\omega) = \frac{\sigma_A^2}{4}\delta(\omega+\omega_0) + \frac{\sigma_A^2}{4}\delta(\omega-\omega_0)$$



The power spectral density shows that the power is concentrated at $\pm\omega_0$. The area under these concentrations is $\frac{1}{4}\sigma A^2$.

3) Colored Noise

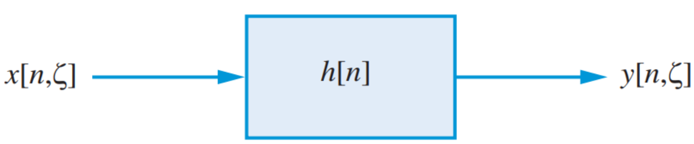
$$r_c[k] = \begin{cases} 1 & k=0 \\ 1/2 & k=\pm 1 \\ 0 & \text{otherwise} \end{cases} \xleftrightarrow{DTFT} S_c(\omega) = 1 + \cos(\omega)$$



Power Spectral Density and LTI systems

Suppose $x[n, \zeta]$ is a signal from a random process and $h[n]$ is an impulse response of a Linear Time-Invariant system.

If we pass the signal $x[n, \zeta]$ through the system, we get $y[n, \zeta]$.



The output of a stable LTI system to the input $x[n, \zeta]$ is another sequence

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k],$$

For simplicity we drop the dependence on ζ. This is the simplest way to treat LTI systems with random process inputs.

We cannot predict the effects of an LTI system on any specific realization of the input process. But we can accurately predict its effect on **the average properties**.

a) Computing the mean: If $x[n]$ is stationary then the mean of the output is constant

$$\mathrm{E}(y[n]) = m_x \sum_{k=-\infty}^{\infty} h[k] \triangleq m_y.$$

b) Computing the auto-correlation sequence of the output and the power spectral density:

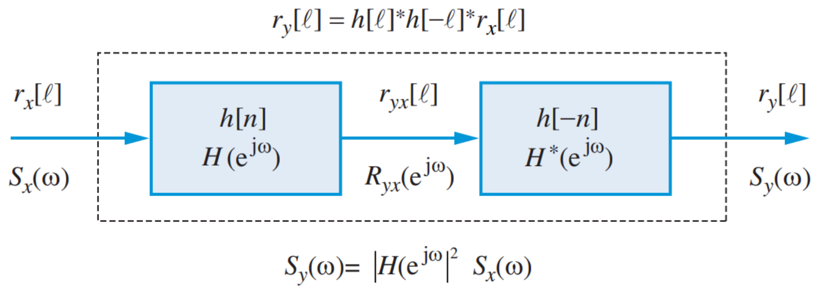


Figure 13.12 The ACRS and the PSD of the output process can be thought of as "filtered" by an LTI system with impulse response $r_{hh}[n] = h[n] * h[-n]$. Therefore, although LTI systems process individual sequences, they have the same effect on all sequences with the same mean and ACRS.

c) The average power of the output: $E\{y^2[n]\}$

$$\mathrm{E}(y^2[n]) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})\right|^2 S_{xx}(\omega)\mathrm{d}\omega. \tag{13.116}$$

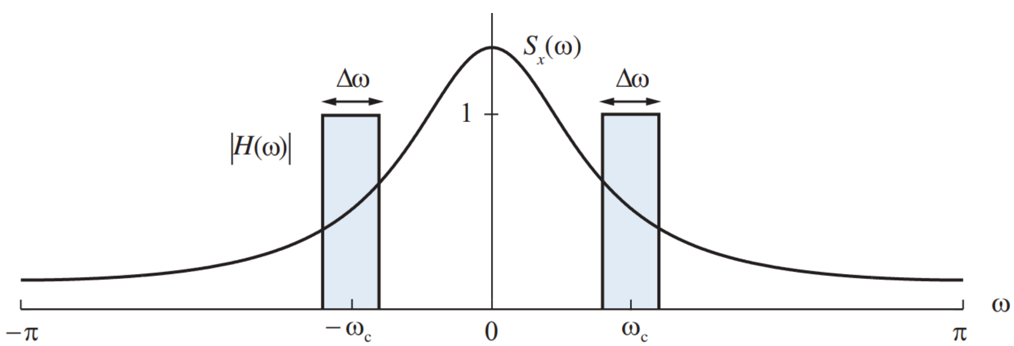
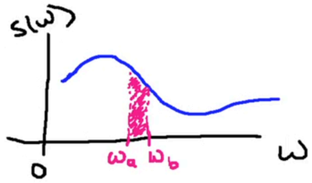


Figure 13.13 Physical interpretation of power spectrum density as power at the output of a narrowband LTI system.

Let P_{ab} be a normalised integration of the PSD from the interval ω_a to ω_b. The quantity P_{ab} is the expected (or average) contribution to the total power (or variance) that is due to the components of the random process between ω_a and ω_b. In other words, the area under the curve between ω_a and ω_b is the power that that portion of the spectrum is expected to contribute to the random process. It tells us how power is distrubuted in a frequency spectrum.



$$P_{ab} = \frac{1}{2\pi}\int_{\omega_a}^{\omega_b} S(\omega)\,d\omega$$

avg contribution to total power (variance) due to components of the random process between ω_a and ω_b

avg $\Leftrightarrow$ expected

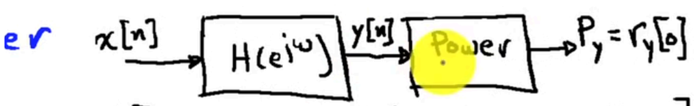
Power Spectrum Analyser

The idea is that if we want to know the power spectrum of some input signal $x[n]$, we can pass that signal to an ideal bandpass filter with very narrow band and compute the power at the output.

We define the bandpass filter to have the unit response in the vecinity of ω_c and $\Delta\omega$ is very small

ideal bandpass filter

If we look at the output of that system



$$P_y = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_y(\omega)\,d\omega \approx \frac{1}{2\pi}\left[S_x(-\omega_c)\Delta\omega + S_x(\omega_c)\Delta\omega\right]$$

$$= \frac{\Delta\omega}{\pi} S_x(\omega_c)$$