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Optimization/Matrix Games

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Optimization/Matrix Games

Matrix games

EXAMPLE 1 Each player has a supply of pennies, nickels, and dimes. At a given signal, both players display (or "play") one coin. If the displayed coins are not the same, then the player showing the higher-valued coin gets to keep both. If they are both pennies or both nickels, then player C keeps both; but if they are both dimes, then player R keeps them. Construct a payoff matrix, using p for display of a penny, n for a nickel, and d for a dime.

R plays safe:

Examine each row for the minimal value, now choose row where this value is max

C plays safe:

Examine each column for the maximal value, now choose column where this value is min

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DEFINITION

The number v_R, defined by

$$v_R = \max_{\mathbf{x}\in X} v(\mathbf{x}) = \max_{\mathbf{x}\in X} \min_{\mathbf{y}\in Y} E(\mathbf{x},\mathbf{y}) = \max_{\mathbf{x}\in X} \min_{j} \mathbf{x}\cdot\mathbf{a}_j$$

with the notation as described above, is called the **value of the game to row player** R. A strategy $\hat{\mathbf{x}}$ for R is called **optimal** if $v(\hat{\mathbf{x}}) = v_R$.

THEOREM 1

In any matrix game, $v_R \leq v_C$.

THEOREM 2

Minimax Theorem

In any matrix game, $v_R = v_C$. That is,

$$\max_{\mathbf{x}\in X} \min_{\mathbf{y}\in Y} E(\mathbf{x},\mathbf{y}) = \min_{\mathbf{y}\in Y} \max_{\mathbf{x}\in X} E(\mathbf{x},\mathbf{y})$$

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DEFINITION

The common value $v = v_R = v_C$ is called the **value of the game**. Any pair of optimal strategies $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is called a **solution** to the game.

THEOREM 3

Fundamental Theorem for Matrix Games

In any matrix game, there are always optimal strategies. That is, every matrix game has a solution.

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EXAMPLE 4 Consider the game whose payoff matrix is

$$A = \begin{bmatrix} 1 & 5 & 3 & 6 \\ 4 & 0 & 1 & 2 \end{bmatrix}$$

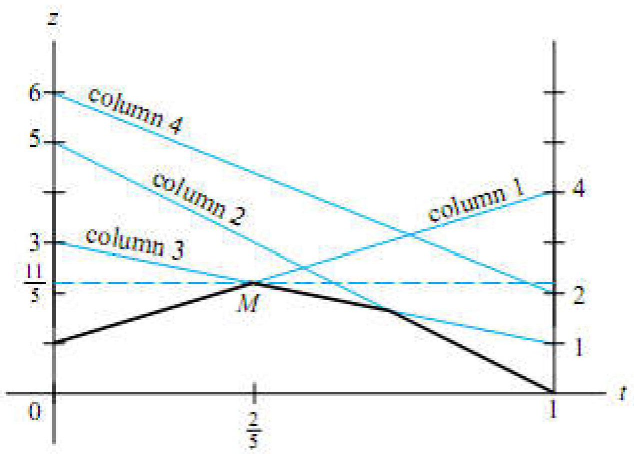

FIGURE 2

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THEOREM 4 Let $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ be optimal strategies for an $m \times n$ matrix game whose value is v, and suppose that

$$\hat{\mathbf{x}} = \hat{x}_1 \mathbf{e}_1 + \cdots + \hat{x}_m \mathbf{e}_m \quad \text{in } \mathbb{R}^m \tag{5}$$

Then $\hat{\mathbf{y}}$ is a convex combination of the pure strategies $\mathbf{e}_j$ in $\mathbb{R}^n$ for which $E(\hat{\mathbf{x}}, \mathbf{e}_j) = v$. In addition, $\hat{\mathbf{y}}$ satisfies the equation

$$E(\mathbf{e}_i, \hat{\mathbf{y}}) = v \tag{6}$$

for each i such that $\hat{x}_i \neq 0$.

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Matrix games (Extra)

A Minimax Optimal strategy for a player is a (possibly randomized) strategy with the best guarantee on its expected gain over strategies the opponent could play in response — i.e., it is the strategy you would want to play if you imagine that your opponent knows you well.

Here is another game: Suppose a kicker is shooting a penalty kick against a goalie who is a bit weaker on one side. Let's say the kicker can kick left or right, the goalie can dive left or right, and the payoff matrix for the kicker (the chance of getting a goal) looks as follows:

Kicker \ Goalie	*left*	*right*
left	0	1
right	1	0.5

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Matrix games (Extra)

When two animals play hawk, the cost of losing is given by the cost, C, since one of the opponents is going to be injured. Either contestant has a 1/2 probability of losing so the average cost is C/2. Likewise, each hawk gains an average resource value of V/2. The net payoff for each individual hawk would be V/2-C/2 or (V-C)/2. When two animals play dove, there is no cost to the doves. Each dove has a 1/2 probability of winning so they divide the resource and the net payoff is V/2. When a hawk meets a dove, the hawk always wins at no cost, so the net payoff for the hawk is V. Conversely, the dove that engages the hawk gains nothing, but experiences no cost so the net payoff is 0.

Table 8.1 Payoff matrix for Hawk-Dove.		**Common type**	
		Hawk	**Dove**
Rare type	**Hawk**	$\frac{V-C}{2}$	V
	Dove	0	$\frac{V}{2}$