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Optimization
Solving Linear Equations

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Solving Linear Equations

system of linear equations

$$\boldsymbol{Ax} = \boldsymbol{b},$$

EXAMPLE 1 Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

Solution Let A be the matrix of coefficients of the system and row reduce the augmented matrix $[\, A \quad \mathbf{0}\,]$ to echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example from: D.C. Lay, “Linear Algebra”

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$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \quad - \tfrac{4}{3}x_3 &= 0 \\ x_2 \quad &= 0 \\ 0 &= 0 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v}, \quad \text{where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

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Solve the following system:

$$\begin{aligned} 2.0000x_1 + 2.0000x_2 &= 6.0000 \\ 2.0000x_1 + 2.0005x_2 &= 6.0010 \end{aligned} \qquad \mathbf{x} = (1, 2)^T$$

$$\begin{aligned} 2.000x_1 + 2.000x_2 &= 6.000 \\ 2.000x_1 + 2.001x_2 &= 6.001 \end{aligned} \qquad \mathbf{x}' = (2, 1)^T$$

relative residual

$$\frac{\|\mathbf{b} - A\mathbf{x}'\|}{\|\mathbf{b}\|} = \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$$

relative error.

$$\|\mathbf{e}\|/\|\mathbf{x}\|$$

$$\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty} = \frac{0.0005}{6.0010} \approx 0.000083 \qquad \frac{\|\mathbf{e}\|_\infty}{\|\mathbf{x}\|_\infty} = 0.5$$

Can the relative error be esitimated by the residual error?

$$\frac{\|\mathbf{r}\|}{\|A\|} \le \|\mathbf{e}\| \le \|A^{-1}\|\,\|\mathbf{r}\|$$

$$\frac{\|\mathbf{b}\|}{\|A\|} \le \|\mathbf{x}\| \le \|A^{-1}\|\,\|\mathbf{b}\|$$

$$\frac{1}{\|A\|\,\|A^{-1}\|}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \le \frac{\|\mathbf{e}\|}{\|\mathbf{x}\|} \le \|A\|\,\|A^{-1}\|\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$$

The number $\|A\|\,\|A^{-1}\|$ is called the *condition number* of A and will be denoted by cond(A). Thus

(10) $$\frac{1}{\mathrm{cond}(A)}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \le \frac{\|\mathbf{e}\|}{\|\mathbf{x}\|} \le \mathrm{cond}(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$$

$$\|A\|_\infty = \max_{1\le i\le m}\left(\sum_{j=1}^{n}|a_{ij}|\right)$$

Cond(A)~16000

Example from S. J. Leon, "Linear Algebra", sec. 7.4

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Try this:

- Calculate the relative residual for: $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} \mathbf{5} \\ \mathbf{2} \end{bmatrix}$ when the computed solution is x'= $\begin{bmatrix} \mathbf{1.1} \\ \mathbf{0.88} \end{bmatrix}$
- Calculate $\boldsymbol{cond_{\infty}(A)}$
- The correct solution x'= $\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$ does this comply with the bounds on the relative error: $\frac{\|\boldsymbol{e}\|}{\|\boldsymbol{x}\|}$

Example from S. J. Leon, "Linear Algebra", sec. 7.4

Solving Linear Equations

Let $\boldsymbol{x}^*$ be a vector that minimizes $\|\boldsymbol{Ax} - \boldsymbol{b}\|^2$; that is, for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$\|\boldsymbol{Ax} - \boldsymbol{b}\|^2 \geq \|\boldsymbol{Ax}^* - \boldsymbol{b}\|^2.$$

We refer to the vector $\boldsymbol{x}^*$ as a *least-squares solution* to $\boldsymbol{Ax} = \boldsymbol{b}$.

Theorem 12.1 *The unique vector $\boldsymbol{x}^*$ that minimizes $\|\boldsymbol{Ax} - \boldsymbol{b}\|^2$ is given by the solution to the equation $\boldsymbol{A}^\top \boldsymbol{Ax} = \boldsymbol{A}^\top \boldsymbol{b}$; that is, $\boldsymbol{x}^* = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{b}$.*

□

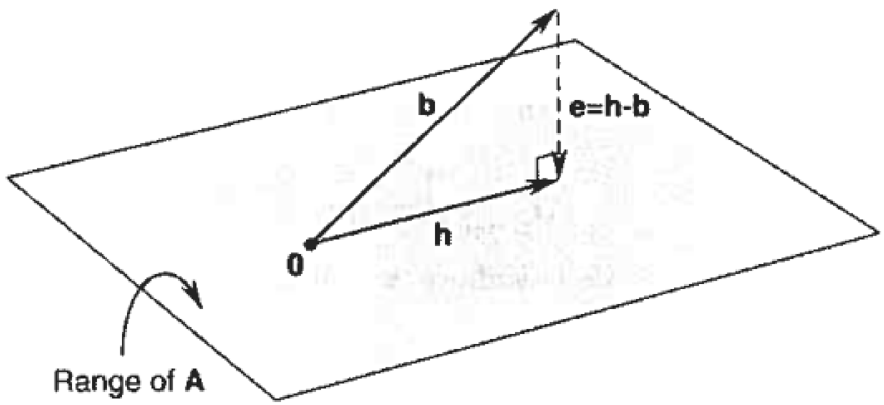

Figure 12.1 Orthogonal projection of $\boldsymbol{b}$ on the subspace $\mathcal{R}(\boldsymbol{A})$.

Solving Linear Equations

Let $\boldsymbol{x}^*$ be a vector that minimizes $\|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}\|^2$; that is, for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$\|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}\|^2 \geq \|\boldsymbol{A}\boldsymbol{x}^*-\boldsymbol{b}\|^2.$$

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Theorem 12.1 *The unique vector $\boldsymbol{x}^*$ that minimizes $\|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}\|^2$ is given by the solution to the equation $\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^\top\boldsymbol{b}$; that is, $\boldsymbol{x}^* = (\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top\boldsymbol{b}$.*

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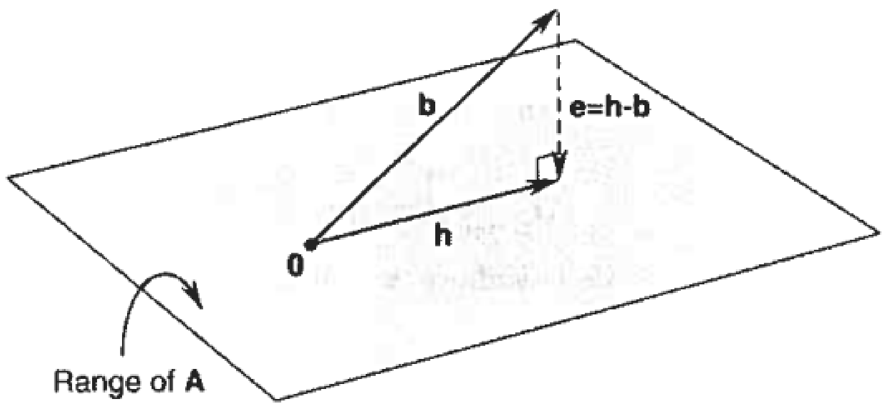

Figure 12.1 Orthogonal projection of $\boldsymbol{b}$ on the subspace $\mathcal{R}(\boldsymbol{A})$.

Solving Linear Equations

An alternative method of arriving at the least-squares solution is to proceed as follows. First, we write

$$\begin{aligned} f(\boldsymbol{x}) &= \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2 \\ &= (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})^\top (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) \\ &= \frac{1}{2}\boldsymbol{x}^\top (2\boldsymbol{A}^\top \boldsymbol{A})\boldsymbol{x} - \boldsymbol{x}^\top (2\boldsymbol{A}^\top \boldsymbol{b}) + \boldsymbol{b}^\top \boldsymbol{b}. \end{aligned}$$

Therefore, f is a quadratic function. The quadratic term is positive definite because rank $\boldsymbol{A} = n$. Thus, the unique minimizer of f is obtained by solving the FONC (see Exercise 6.31); that is,

$$\nabla f(\boldsymbol{x}) = 2\boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{x} - 2\boldsymbol{A}^\top \boldsymbol{b} = \boldsymbol{0}.$$

The only solution to the equation $\nabla f(\boldsymbol{x}) = \boldsymbol{0}$ is $\boldsymbol{x}^* = (\boldsymbol{A}^\top \boldsymbol{A})^{-1}\boldsymbol{A}^\top \boldsymbol{b}$.

Solving Linear Equations

Example 12.1 Suppose that you are given two different types of concrete. The first type contains 30% cement, 40% gravel, and 30% sand (all percentages of weight). The second type contains 10% cement, 20% gravel, and 70% sand. How many pounds of each type of concrete should you mix together so that you get a concrete mixture that has as close as possible to a total of 5 pounds of cement, 3 pounds of gravel, and 4 pounds of sand?

The problem can be formulated as a least-squares problem with

Solving Linear Equations

Example 12.2 *Line Fitting.* Suppose that a process has a single input $t \in \mathbb{R}$ and a single output $y \in \mathbb{R}$. Suppose that we perform an experiment on the

Table 12.1 Experimental Data for Example 12.2.

i	0	1	2
t_i	2	3	4
y_i	3	4	15

process, resulting in a number of measurements, as displayed in Table 12.1.

$$y = mt + c$$

$$\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2 = \sum_{i=0}^{2} (mt_i + c - y_i)^2 .$$

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Solving Linear Equations

Example 12.3 *Attenuation Estimation.* A wireless transmitter sends a discrete-time signal $\{s_0, s_1, s_2\}$ (of duration 3) to a receiver, as shown in Figure 12.3. The real number s_i is the value of the signal at time i.

The transmitted signal takes two paths to the receiver: a direct path, with delay 10 and attenuation factor a_1, and an indirect (reflected) path, with delay 12 and attenuation factor a_2. The received signal is the sum of the signals from these two paths, with their respective delays and attenuation factors.

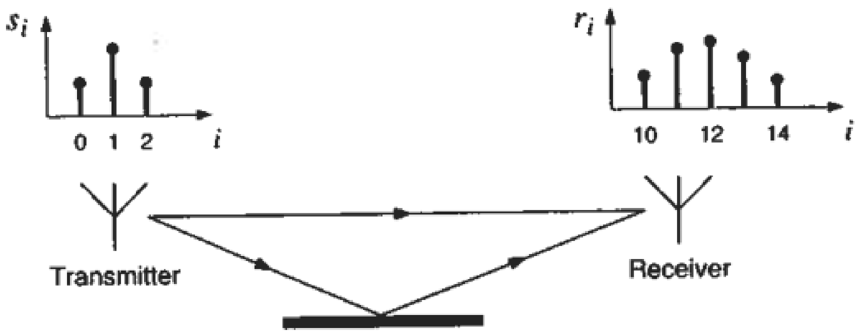

Figure 12.3 Wireless transmission in Example 12.3.

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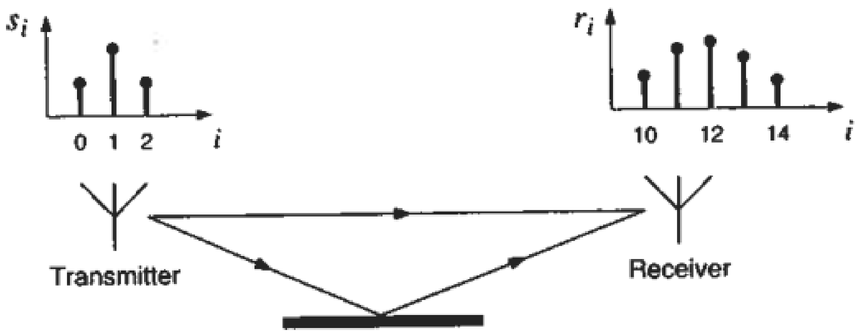

Figure 12.3 Wireless transmission in Example 12.3.

Solving Linear Equations

Suppose that the received signal is measured from times 10 through 14 as $r_{10}, r_{11}, \ldots, r_{14}$, as shown in the figure. We wish to compute the least-squares estimates of a_1 and a_2, based on the following values:

s_0	s_1	s_2	r_{10}	r_{11}	r_{12}	r_{13}	r_{14}
1	2	1	4	7	8	6	3

The problem can be posed as a least-squares problem with

$$\boldsymbol{A} = \begin{bmatrix} s_0 & 0 \\ s_1 & 0 \\ s_2 & s_0 \\ 0 & s_1 \\ 0 & s_2 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} r_{10} \\ r_{11} \\ r_{12} \\ r_{13} \\ r_{14} \end{bmatrix}.$$

The recursive least-mean sqaure

$$\left\| \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} x - \begin{bmatrix} b^{(0)} \\ b^{(1)} \end{bmatrix} \right\|^2 .$$

$$x^{(1)} = G_1^{-1} \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} b^{(0)} \\ b^{(1)} \end{bmatrix},$$

$$G_1 = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}.$$

$$\begin{aligned} x^{(1)} &= G_1^{-1} \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} b^{(0)} \\ b^{(1)} \end{bmatrix} \\ &= G_1^{-1} \left(G_1 x^{(0)} - A_1^{\mathsf{T}} A_1 x^{(0)} + A_1^{\mathsf{T}} b^{(1)} \right) \\ &= x^{(0)} + G_1^{-1} A_1^{\mathsf{T}} \left(b^{(1)} - A_1 x^{(0)} \right), \end{aligned}$$

where G_1 can be calculated using

$$G_1 = G_0 + A_1^{\mathsf{T}} A_1.$$

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The recursive least-mean sqaure

$$G_{k+1} = G_k + A_{k+1}^\top A_{k+1}$$
$$x^{(k+1)} = x^{(k)} + G_{k+1}^{-1} A_{k+1}^\top \left(b^{(k+1)} - A_{k+1} x^{(k)} \right).$$

In the special case where the new data at each step are such that A_{k+1} is a matrix consisting of a single row, $A_{k+1} = a_{k+1}^\top$, and $b^{(k+1)}$ is a scalar,

$$P_{k+1} = P_k - \frac{P_k a_{k+1} a_{k+1}^\top P_k}{1 + a_{k+1}^\top P_k a_{k+1}},$$
$$x^{(k+1)} = x^{(k)} + P_{k+1} a_{k+1} \left(b_{k+1} - a_{k+1}^\top x^{(k)} \right).$$

The recursive least-mean sqaure

Example 12.6 Let

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} b^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$A_1 = a_1^\top = [2 \quad 1] b^{(1)} = b_1 = [3],$$

$$A_2 = a_2^\top = [3 \quad 1] b^{(2)} = b_2 = [4].$$

First compute the vector $x^{(0)}$ minimizing $\|A_0 x - b^{(0)}\|^2$. Then, use the RLS algorithm to find $x^{(2)}$ minimizing

$$\left\| \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} x - \begin{bmatrix} b^{(0)} \\ b^{(1)} \\ b^{(2)} \end{bmatrix} \right\|^2.$$

We have

$$P_0 = (A_0^\top A_0)^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix},$$

$$x^{(0)} = P_0 A_0^\top b^{(0)} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}.$$

Applying the RLS algorithm twice, we get:

$$P_1 = P_0 - \frac{P_0 a_1 a_1^\top P_0}{1 + a_1^\top P_0 a_1} = \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix},$$

$$x^{(1)} = x^{(0)} + P_1 a_1 \left(b_1 - a_1^\top x^{(0)} \right) = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix},$$

$$P_2 = P_1 - \frac{P_1 a_2 a_2^\top P_1}{1 + a_2^\top P_1 a_2} = \begin{bmatrix} 1/6 & -1/4 \\ -1/4 & 5/8 \end{bmatrix},$$

$$x^{(2)} = x^{(1)} + P_2 a_2 \left(b_2 - a_2^\top x^{(1)} \right) = \begin{bmatrix} 13/12 \\ 5/8 \end{bmatrix}.$$

Solution with minimum norm

$$\begin{aligned} \text{minimize} \quad & \|\boldsymbol{x}\| \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \end{aligned}$$

Theorem 12.2 *The unique solution $\boldsymbol{x}^*$ to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ that minimizes the norm $\|\boldsymbol{x}\|$ is given by*

$$\boldsymbol{x}^* = \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{A}^\top)^{-1}\boldsymbol{b}.$$

KACZMARZ'S ALGORITHM

Let $\boldsymbol{a}_j^\top$ denote the jth row of $\boldsymbol{A}$, and b_j the jth component of $\boldsymbol{b}$, and μ a positive scalar, $0 < \mu < 2$. With this notation, Kaczmarz's algorithm is:

1. Set $i := 0$, initial condition $\boldsymbol{x}^{(0)}$.
2. For $j = 1, \ldots, m$, set
$$\boldsymbol{x}^{(im+j)} = \boldsymbol{x}^{(im+j-1)} + \mu\left(b_j - \boldsymbol{a}_j^\top \boldsymbol{x}^{(im+j-1)}\right)\frac{\boldsymbol{a}_j}{\boldsymbol{a}_j^\top \boldsymbol{a}_j}.$$
3. Set $i := i + 1$; go to step 2.

Theorem 12.3 *In Kaczmarz's algorithm, if $\boldsymbol{x}^{(0)} = \boldsymbol{0}$, then $\boldsymbol{x}^{(k)} \to \boldsymbol{x}^* = \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{A}^\top)^{-1}\boldsymbol{b}$ as $k \to \infty$.* □

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Moore-Penrose invers

Theorem 12.6 *Consider a system of linear equations* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, rank $\boldsymbol{A} = r$. *The vector* $\boldsymbol{x}^* = \boldsymbol{A}^\dagger \boldsymbol{b}$ *minimizes* $\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2$ *on* $\mathbb{R}^n$. *Furthermore, among all vectors in* $\mathbb{R}^n$ *that minimize* $\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2$, *the vector* $\boldsymbol{x}^* = \boldsymbol{A}^\dagger \boldsymbol{b}$ *is the unique vector with minimal norm.* □

For the case in which a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with $m \geq n$ and rank $\boldsymbol{A} = n$, we can easily check that the following is a pseudoinverse of $\boldsymbol{A}$:

$$\boldsymbol{A}^\dagger = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top.$$

For the case in which a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with $m \leq n$ and rank $\boldsymbol{A} = m$, we can easily check, as we did in the previous case, that the following is a pseudoinverse of $\boldsymbol{A}$:

$$\boldsymbol{A}^\dagger = \boldsymbol{A}^\top (\boldsymbol{A}\boldsymbol{A}^\top)^{-1}.$$