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Optimization/Linear Programming 2

- Simplex method

- Duality

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Optimization

DEFINITION

Given $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ in $\mathbb{R}^m$, $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ in $\mathbb{R}^n$, and an $m \times n$ matrix $A = [a_{ij}]$, the **canonical linear programming problem** is the following:

Find an n-tuple $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ to maximize

$$f(x_1, \dots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

and

$$x_j \geq 0 \quad \text{for } j = 1, \dots, n$$

This may be restated in vector-matrix notation as follows:

$$\text{Maximize } f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \qquad (1)$$

$$\text{subject to the constraints } A\mathbf{x} \leq \mathbf{b} \qquad (2)$$

$$\text{and } \mathbf{x} \geq \mathbf{0} \qquad (3)$$

where an inequality between two vectors applies to each of their coordinates.

Any vector $\mathbf{x}$ that satisfies (2) and (3) is called a **feasible solution**, and the set of all feasible solutions, denoted by $\mathcal{F}$, is called the **feasible set**. A vector $\overline{\mathbf{x}}$ in $\mathcal{F}$ is an **optimal solution** if $f(\overline{\mathbf{x}}) = \max_{\mathbf{x} \in \mathcal{F}} f(\mathbf{x})$.

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Optimization

THE SIMPLEX ALGORITHM FOR A CANONICAL LINEAR PROGRAMMING PROBLEM

1. Change the inequality constraints into equalities by adding slack variables. Let M be a variable equal to the objective function, and below the constraint equations write an equation of the form

$$\text{(objective function)} - M = 0$$

2. Set up the initial simplex tableau. The slack variables (and M) provide the initial basic feasible solution.
3. Check the bottom row of the tableau for optimality. If all the entries to the left of the vertical line are nonnegative, then the solution is optimal. If some are negative, then choose the variable x_k for which the entry in the bottom row is as negative as possible.[3]
4. Bring the variable x_k into the solution. Do this by pivoting on the positive entry a_{pk} for which the nonnegative ratio b_i/a_{ik} is the smallest. The new basic feasible solution includes an increased value for M.
5. Repeat the process, beginning at step 3, until all the entries in the bottom row are nonnegative.

Optimization

Primal and dual problem

Primal Problem P	Dual Problem P^*
Maximize $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$	Minimize $g(\mathbf{y}) = \mathbf{b}^T\mathbf{y}$
subject to $A\mathbf{x} \leq \mathbf{b}$	subject to $A^T\mathbf{y} \geq \mathbf{c}$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{y} \geq \mathbf{0}$

THEOREM 7 THE DUALITY THEOREM

Let P be a (primal) linear programming problem with feasible set $\mathscr{F}$, and let P^* be the dual problem with feasible set $\mathscr{F}^*$.

a. If $\mathscr{F}$ and $\mathscr{F}^*$ are both nonempty, then P and P^* both have optimal solutions, say $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$, respectively, and $f(\bar{\mathbf{x}}) = g(\bar{\mathbf{y}})$.

b. If one of the problems P or P^* has an optimal solution $\bar{\mathbf{x}}$ or $\bar{\mathbf{y}}$, respectively, then so does the other, and $f(\bar{\mathbf{x}}) = g(\bar{\mathbf{y}})$.

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Example 2 (Lay. p.42)

Primal problem

$$\begin{array}{ll} \text{Maximize} & f(x_1, x_2) = 2x_1 + 3x_2 \\ \text{subject to} & x_1 \leq 30 \\ & x_2 \leq 20 \\ & x_1 + 2x_2 \leq 54 \end{array}$$

and $x_1 \geq 0, x_2 \geq 0.$

(x_1, x_2)	$2x_1 + 3x_2$
(0, 0)	0
(30, 0)	60
(30, 12)	96 ←
(14, 20)	88
(0, 20)	60

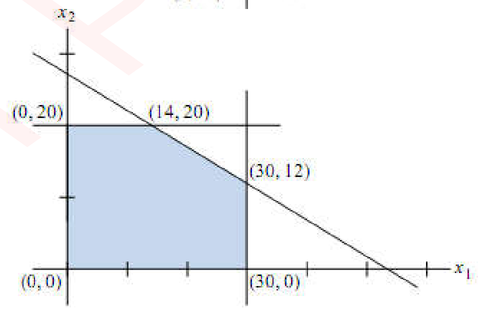
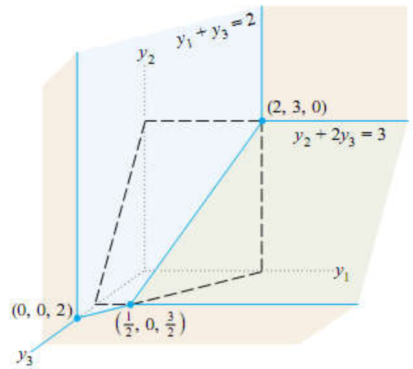


FIGURE 1

Dual problem

$$\begin{array}{ll} \text{Minimize} & g(y_1, y_2, y_3) = 30y_1 + 20y_2 + 54y_3 \\ \text{subject to} & y_1 + y_3 \geq 2 \\ & y_2 + 2y_3 \geq 3 \end{array}$$

and $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$



y	g(y)
(0, 0, 2)	108
$(\frac{1}{2}, 0, \frac{3}{2})$	96 ←
(2, 3, 0)	120

FIGURE 1 The minimum of $g(y_1, y_2, y_3) = 30y_1 + 20y_2 + 54y_3$.

Optimization

Example 2 (Lay. p.42)

Primal problem

$$\begin{array}{ll} \text{Maximize} & f(x_1, x_2) = 2x_1 + 3x_2 \\ \text{subject to} & x_1 \qquad\quad \leq 30 \\ & \qquad\quad x_2 \leq 20 \\ & x_1 + 2x_2 \leq 54 \\ \multicolumn{2}{l}{\text{and } x_1 \geq 0,\ x_2 \geq 0.} \end{array}$$

Dual problem

$$\begin{array}{ll} \text{Minimize} & g(y_1, y_2, y_3) = 30y_1 + 20y_2 + 54y_3 \\ \text{subject to} & y_1 \qquad + \ y_3 \geq 2 \\ & \quad y_2 + 2y_3 \geq 3 \\ \multicolumn{2}{l}{\text{and } y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.} \end{array}$$

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & 8 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 12 \\ 1 & 0 & 1 & 0 & 0 & 0 & 30 \\ \hline 0 & 0 & \frac{1}{2} & 0 & \frac{3}{2} & 1 & 96 \end{array}\right] \end{array}$$

Solution, Dual problem

Optimization

THEOREM 7 THE DUALITY THEOREM (CONTINUED)

Let P be a (primal) linear programming problem and let P^* be its dual problem. Suppose P (or P^*) has an optimal solution.

c. If either P or P^* is solved by the simplex method, then the solution of its dual is displayed in the bottom row of the final tableau in the columns associated with the slack variables.

Example 4 Lay. p.44 (example 6, sec. 9.3 cont')

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & \end{array} \\
\left[\begin{array}{cccccc|c}
0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & 8 \\
0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 12 \\
1 & 0 & 1 & 0 & 0 & 0 & 30 \\
\hline
0 & 0 & \frac{1}{2} & 0 & \frac{3}{2} & 1 & 96
\end{array}\right]
\end{array}$$

Note the marginal value for cashews is ½, marginal value for filberts 0 and mv. for peanuts is 3/2, corresponding to only 12 pounds of 20 pounds od filberts is used.

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Example 3.1. Suppose you are the boss of a factory that makes silicon chips. Your factory makes two kinds of chips: memory chips and microprocessors. To make each sort of chip requires the services of three machines, called Machine A, Machine B and Machine C. To make a memory chip requires

1 minute on Machine A

2 minutes on Machine B

1 minute on Machine C.

To make a microprocessor requires

1 minute on Machine A

3 minutes on Machine B

4 minutes on Machine C.

The profit from making a memory chip is \$1, while the profit from making a microprocessor is \$2. How many of each sort of chip should your factory make per hour?

Primal problem

Maximize $f = x_1 + 2x_2$

$$x_1 + x_2 \leqslant 60$$

$$2x_1 + 3x_2 \leqslant 60$$

$$x_1 + 4x_2 \leqslant 60$$

$$x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0.$$

Optimization

In examples like the one above, the *dual problem* of the LP-problem corresponds to asking the question: "How much is a minute on each of my three machines worth to me?" These values are called the *shadow prices*, and can be described by supposing a hypothetical person knocks on your door and asks to rent an hour on each of your machines. What is the lowest price you could accept?[1] Let u_1 be the price you need to charge for a minute on Machine A, u_2 the price for a minute on Machine B, and u_3 the price for a minute on Machine C.

The positive value in the dual problem means that how much machine A, B and C are. In this case, machine A is worth more than machine B and C.

$$\mathbf{x}^* = \begin{bmatrix} 12 \\ 12 \end{bmatrix}, \qquad \mathbf{x}_s^* = \begin{bmatrix} 36 \\ 0 \\ 0 \end{bmatrix}.$$

Hence $12u_4^* + 12u_5^* = 0$ and $36u_1^* + 0u_2^* + 0u_3^* = 0$, so $u_1^* = u_4^* = u_5^* = 0$. Now plugging these values into (3.1) we get $u_2^* = 2/5$ and $u_3^* = 1/5$. This means that each minute on Machine B is worth \$0.40 to us, each minute on Machine C is worth \$0.20 to us while a minute on Machine A is worth nothing to us. (This value of \$0 reflects the fact that Machine A is idle for 36 minutes out of every hour, hence we could give away a minute on Machine A without losing any money.)

Optimization

Network flow (example)

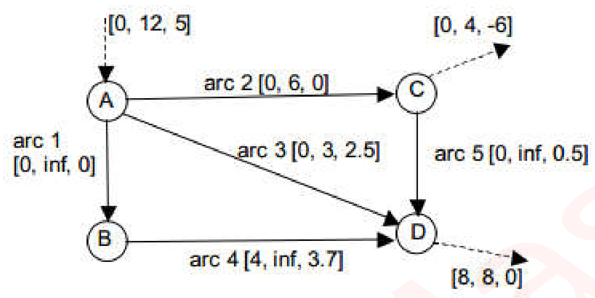

Figure 10.2: A fully labeled network diagram.

There are three parameters associated with each arc: the lower flow bound, the upper flow bound, and the cost per unit of flow. The arc labeling convention that we will use shows a triple of numbers in square brackets, [*l*, *u*, *c*]. *l* is the lower bound on the flow in the arc, with a default value of zero if not explicitly specified; *u* is the upper bound on the flow in the arc, with a default value of infinity if not explicitly specified; *c* is the cost per unit of flow in the arc, with a default value of zero if not explicitly specified. For example, an arc having a lower flow bound of zero, and upper flow bound of 25, and a cost per unit of flow of $6 would be labeled [0, 25, 6].

Source and sink node behavior is controlled by the label on the phantom arc associated with the node. If the upper and lower flow bounds on the phantom arc are identical, then the node relationship is an equation, but if the upper and flow bounds on the arc differ, then the node relationship is an inequality.

Consider the network diagram in Figure 10.2 for example. The phantom arcs on the 3 source and sink nodes are fully labeled. Node A is a source of up to 12 units of flow at a cost of $5 per unit of flow. Node C is a sink of up to 4 units of flow at an *income* of $6 per unit of flow – the negative cost per unit of flow means income. Node D is a sink of exactly 8 units of flow, but with no cost or income associated with that flow. The remaining arcs are also labeled following the convention. Note that arc 4 has a positive lower bound.

node A:	$x_1 + x_2 + x_3 \leq 12$
node B:	$x_4 - x_1 = 0$
node C lower bound:	$x_5 - x_2 \geq -4$
node C upper bound:	$x_5 - x_2 \leq 0$
node D:	$-x_3 - x_4 - x_5 = -8$
flow bound arc 2:	$x_2 \leq 6$
flow bound arc 3:	$x_3 \leq 3$
flow bound arc 4:	$x_4 \geq 4$
nonnegativity:	$x_1, x_2, x_3, x_4, x_5 \geq 0$

The minimum cost objective function can be written as:

$$\text{minimize } 5A - 6C + 2.5x_3 + 3.7x_4 + 0.5x_5$$

$$A = x_1 + x_2 + x_3 \text{ and } C = x_2 - x_5$$

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Optimization

Matrix Games, Example 5 ([Lay] p. 45)

EXAMPLE 5 Solve the game whose payoff matrix is $A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}$.

$$\begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \\ \hline 1 & 4 & 5 & 1 & 0 & 0 & 1 \\ 6 & 5 & 3 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 1 & 0 \end{array}$$

We need to add something in this case 3 to make all the values in A non-negative to generate the initial simplex tableau.

$$\begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \\ \hline 0 & \frac{19}{27} & 1 & \frac{2}{9} & -\frac{1}{27} & 0 & \frac{5}{27} \\ 1 & \frac{13}{27} & 0 & -\frac{1}{9} & \frac{5}{27} & 0 & \frac{2}{27} \\ \hline 0 & \frac{5}{27} & 0 & \frac{1}{9} & \frac{4}{27} & 1 & \frac{7}{27} \end{array}$$

To find the value of the game, we need to substract lambda from 3.

$$\bar{y}_1 = \frac{2}{27}, \quad \bar{y}_2 = 0, \quad \bar{y}_3 = \frac{5}{27}, \quad \text{with } \lambda = \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = \frac{7}{27} \qquad \hat{\mathbf{y}} = \bar{\mathbf{y}}/\lambda = \begin{bmatrix} \frac{2}{7} \\ 0 \\ \frac{5}{7} \end{bmatrix}$$

Here, we normalise to get probabilities

$$\bar{x}_1 = \frac{1}{9} = \frac{3}{27} \quad \text{and} \quad \bar{x}_2 = \frac{4}{27}, \quad \text{with } \lambda = \bar{x}_1 + \bar{x}_2 = \frac{7}{27} \qquad \hat{\mathbf{x}} = \bar{\mathbf{x}}/\lambda = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}$$

Optimization

Matrix Games revised ([Lay] p. 3)

EXAMPLE 1 Each player has a supply of pennies, nickels, and dimes. At a given signal, both players display (or "play") one coin. If the displayed coins are not the same, then the player showing the higher-valued coin gets to keep both. If they are both pennies or both nickels, then player C keeps both; but if they are both dimes, then player R keeps them. Construct a payoff matrix, using p for display of a penny, n for a nickel, and d for a dime.

$$\begin{array}{cc} & \text{Player } C \\ & \begin{array}{ccc} p & n & d \end{array} \\ \text{Player } R \quad \begin{array}{c} p \\ n \\ d \end{array} & \begin{bmatrix} -1 & -1 & -1 \\ 1 & -5 & -5 \\ 1 & 5 & 10 \end{bmatrix} \end{array}$$

$y^T = 1/(\frac{1}{7})(1/7 \quad 0 \quad 0) = (1 \quad 0 \quad 0)$ $\qquad$ $x^T = 1/(\frac{1}{7})(0 \quad 2/35 \quad 3/35) = (0 \quad 2/5 \quad 3/5)$

$$v = \frac{1}{\frac{1}{\frac{1}{7}}} \quad 6 = 1$$

Optimization

Matrix games (Cont.)

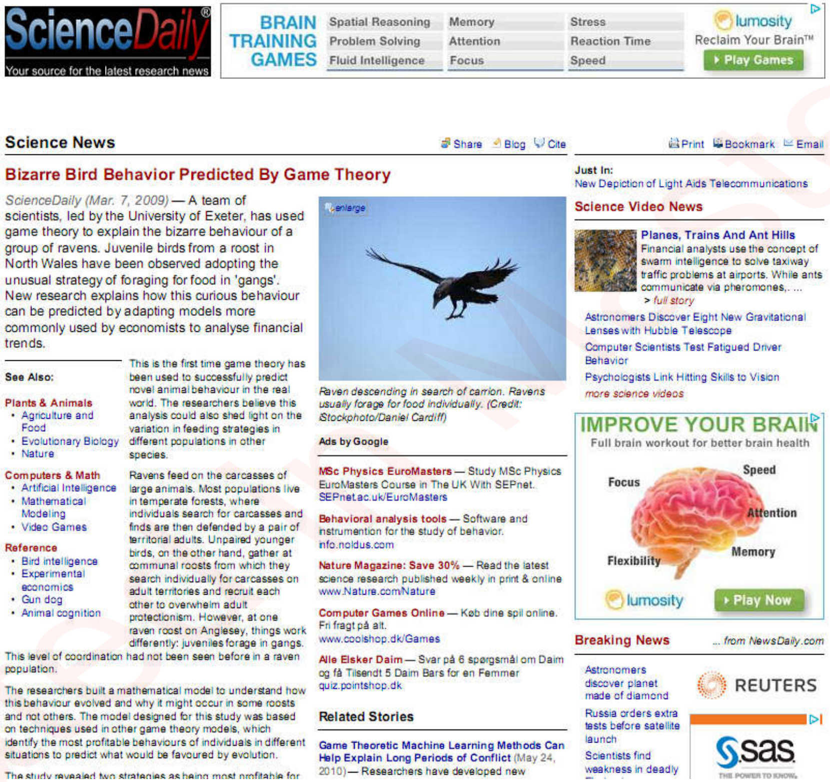
A Minimax Optimal strategy for a player is a (possibly randomized) strategy with the best guarantee on its expected gain over strategies the opponent could play in response — i.e., it is the strategy you would want to play if you imagine that your opponent knows you well.

Here is another game: Suppose a kicker is shooting a penalty kick against a goalie who is a bit weaker on one side. Let's say the kicker can kick left or right, the goalie can dive left or right, and the payoff matrix for the kicker (the chance of getting a goal) looks as follows:

Kicker \ Goalie	*left*	*right*
left	0	1
right	1	0.5

Optimization

Matrix Games in behavioral science – something different



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Bizarre Bird Behavior Predicted By Game Theory

ScienceDaily (Mar. 7, 2009) — A team of scientists, led by the University of Exeter, has used game theory to explain the bizarre behaviour of a group of ravens. Juvenile birds from a roost in North Wales have been observed adopting the unusual strategy of foraging for food in 'gangs'. New research explains how this curious behaviour can be predicted by adapting models more commonly used by economists to analyse financial trends.

Raven descending in search of carrion. Ravens usually forage for food individually. (Credit: iStockphoto/Daniel Cardiff)

See Also:

Plants & Animals
- Agriculture and Food
- Evolutionary Biology
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- Artificial Intelligence
- Mathematical Modeling
- Video Games

Reference
- Bird intelligence
- Experimental economics
- Gun dog
- Animal cognition

This is the first time game theory has been used to successfully predict novel animal behaviour in the real world. The researchers believe this analysis could also shed light on the variation in feeding strategies in different populations in other species.

Ravens feed on the carcasses of large animals. Most populations live in temperate forests, where individuals search for carcasses and finds are then defended by a pair of territorial adults. Unpaired younger birds, on the other hand, gather at communal roosts from which they search individually for carcasses on adult territories and recruit each other to overwhelm adult protectionism. However, at one raven roost on Anglesey, things work differently: juveniles forage in gangs. This level of coordination had not been seen before in a raven population.

The researchers built a mathematical model to understand how this behaviour evolved and why it might occur in some roosts and not others. The model designed for this study was based on techniques used in other game theory models, which identify the most profitable behaviours of individuals in different situations to predict what would be favoured by evolution.

The study revealed two strategies as being most profitable for

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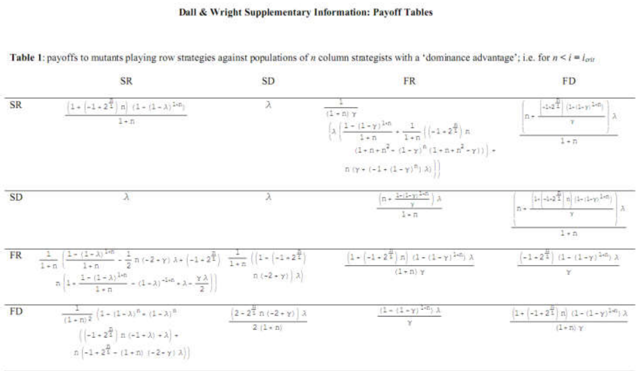

Dall & Wright Supplementary Information: Payoff Tables

Table 1: payoffs to mutants playing row strategies against populations of n column strategists with a 'dominance advantage'; i.e. for $n < i = i_{crit}$

	SR	SD	FR	FD
SR	[illegible]	λ	[illegible]	[illegible]
SD	λ	λ	[illegible]	[illegible]
FR	[illegible]	[illegible]	[illegible]	[illegible]
FD	[illegible]	[illegible]	[illegible]	[illegible]