Sheila Kiprotich.

A ***sequence*** is an ordered list of infinitely many numbers. A sequence can be characterized as a function whose domain is the set of natural numbers, and whose outputs are the terms of the sequence.

Example 1: $\{1, 2, 3, 4, 5, 6, \ldots\}$

Example 2: $\{1, 2, 1, 2, 1, 2, \ldots\}$

Example 3: $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\}$

Example 4: $\{8, -4, 2, -1, \frac{1}{2}, -\frac{1}{4}, \ldots\}$

Example 5: $\{1, 1, 2, 3, 5, 8, \ldots\}$ Fibonacci

General Form: $\{a_1, a_2, a_3, a_4, a_5, a_6, \ldots\}$

$a_n =$ n^th^ term

$a(n)$ is the n^th^ term using classic function notation quence can be defined ***explicitly***, meaning a rule is provided for directly lculating any term.

Example: Find the first five terms of the sequence defined by the following rule. Then graph the sequence.

$$a_n = \frac{3n-1}{n}$$

$a_1 = 2$

$a_2 = 5/2$

$a_3 = 8/3$

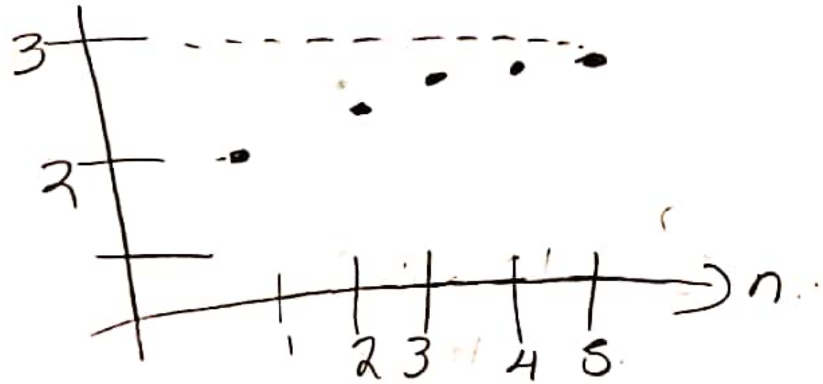
$a_4 = 11/4$

$a_5 = 14/5$

n	a_n
1	$2 = 2$
2	$5/2 = 2\,1/2$
3	$8/3 = 2\,2/3$
4	$11/4 = 2\,3/4$
5	$14/5 = 2\,\frac{4}{6}$

$\downarrow$

3.



uence can be defined ***recursively***, meaning a rule is provided for calculated a m by using one or more previous terms. In such cases, at least one term should e known to establish the rest of the terms.

Example: Find the first five terms of the sequence defined by the following rule. Use 2 as the first term, and then repeat the task using 4 as the first term.

$$a_{n+1} = 2 + \frac{1}{2}a_n \quad n \geq 1 \qquad a_1 = 2 \qquad a_1 = 4$$

$\ldots, a_{n-1}, a_n, a_{n+1}, \ldots$

$n=1: \; a_2 = 2 + \frac{1}{2}a_1 = 2 + \frac{1}{2}(2) = 3$

$n=2: \; a_3 = 2 + \frac{1}{2}a_2 = \frac{7}{2}$

$n=3: \; a_4 = 2 + \frac{1}{2}a_3 = \frac{15}{4}$

$a_5 = 2 + \frac{1}{2}a_4 = \frac{31}{8}$

$a_2 = 4$

$a_3 = 4$

$a_4 = 4$

$a_5 = 4$

Sequences with Desmos

$N = 100$

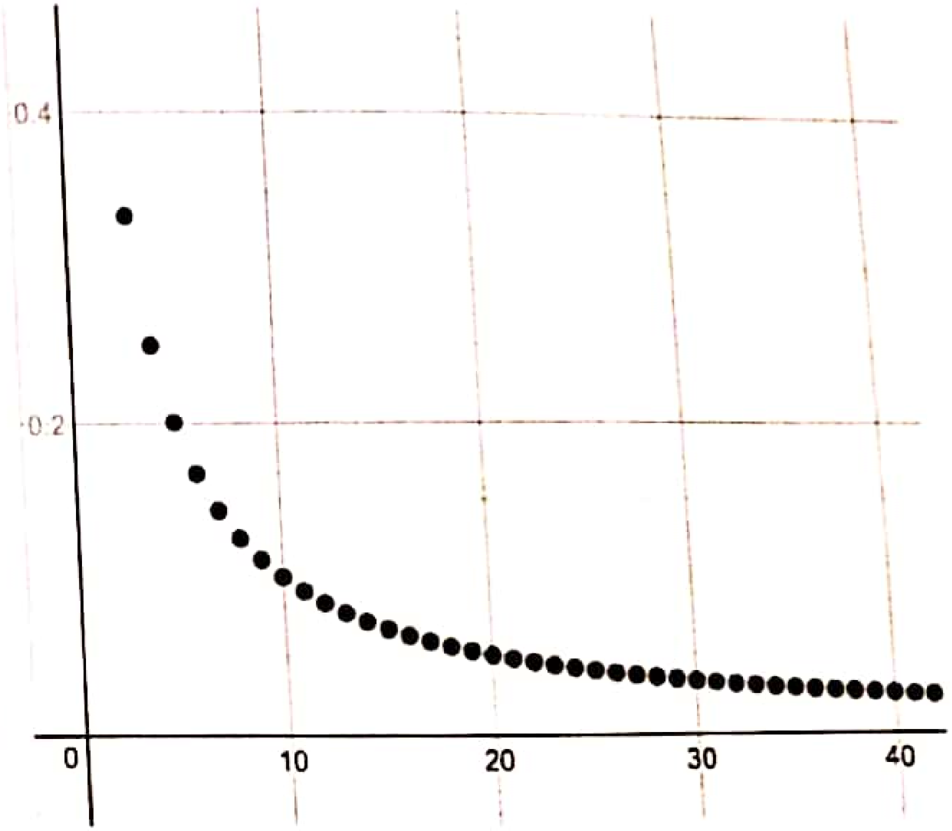
$1 \leq N \leq 100$ step: 1

$n = [1, \ldots, N]$

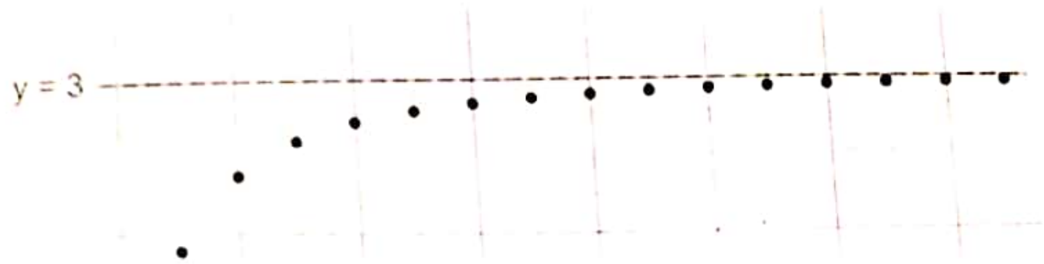
$a(x) = \frac{1}{x}$

$(n, a(n))$

Label

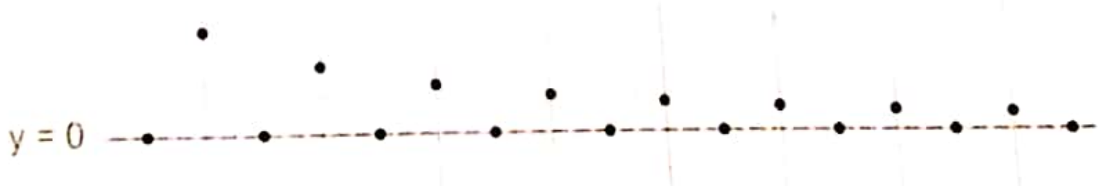


uence is ***convergent*** if its terms approach a single finite number as n proaches infinity.



This sequence converges to 3

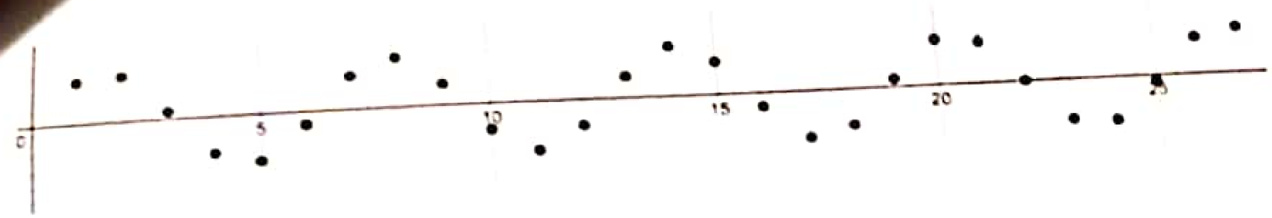
This sequence converges to 1



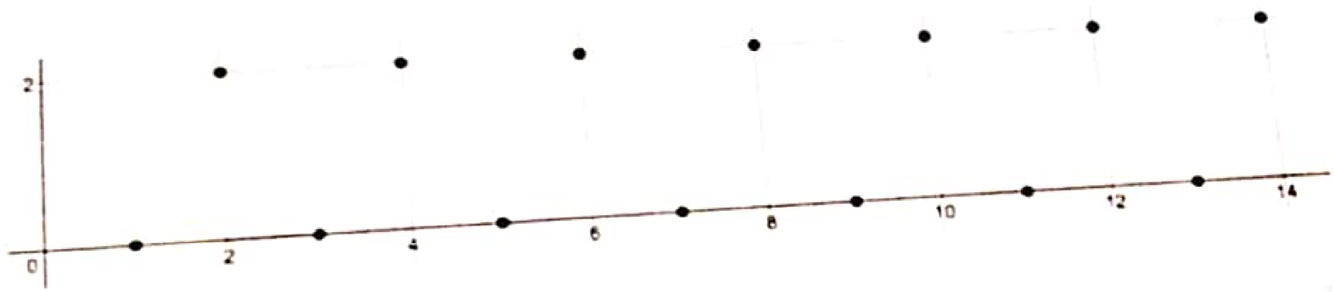
This sequence converges to 0

This is a constant sequence and it converges to 0.5

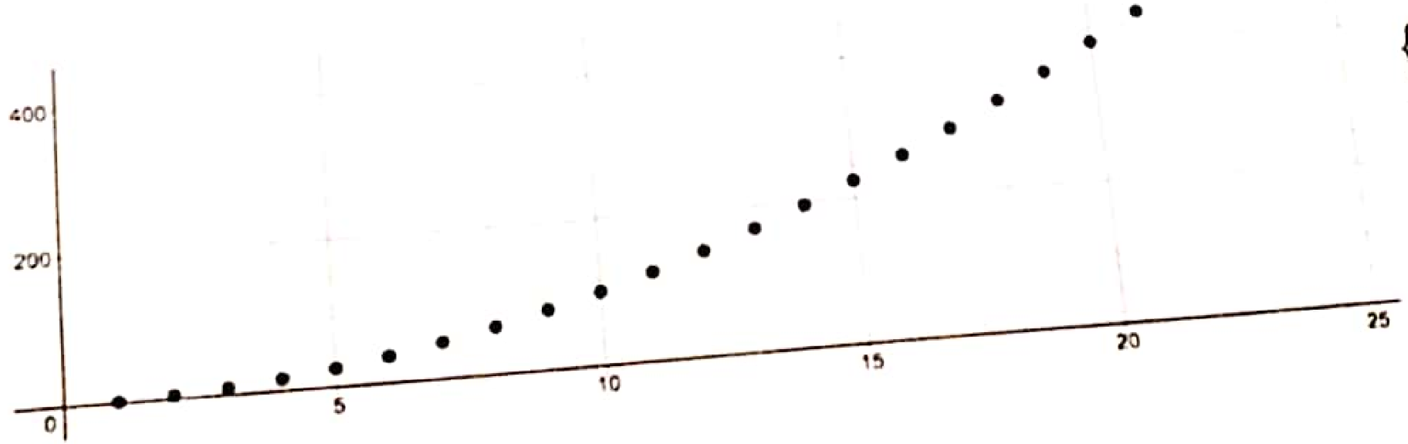
ence that is not convergent is called ***divergent***.



{sin(n)} does not approach a single number, and is divergent

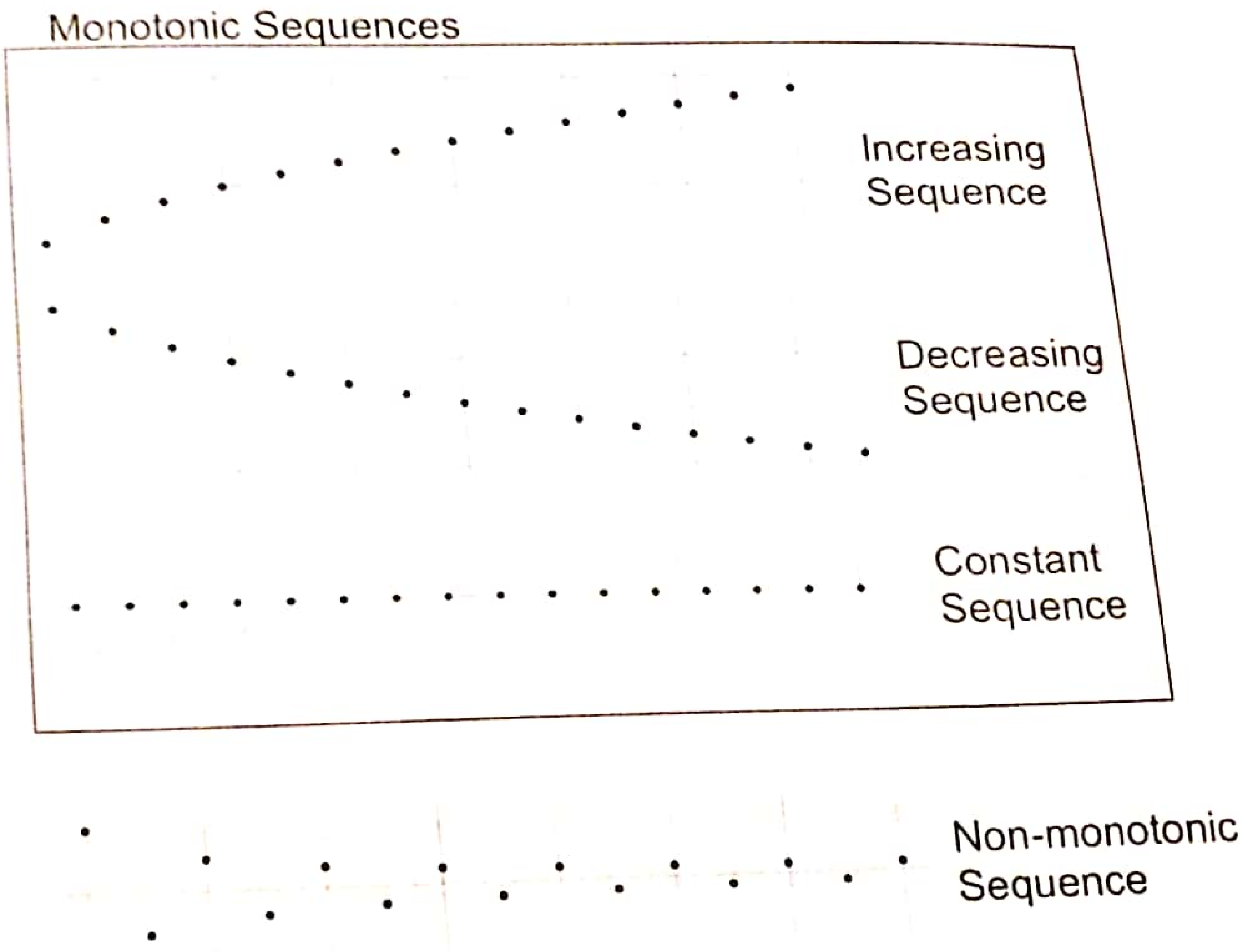


{0,2,0,2,0,2…} is an oscillating sequence, and is divergent

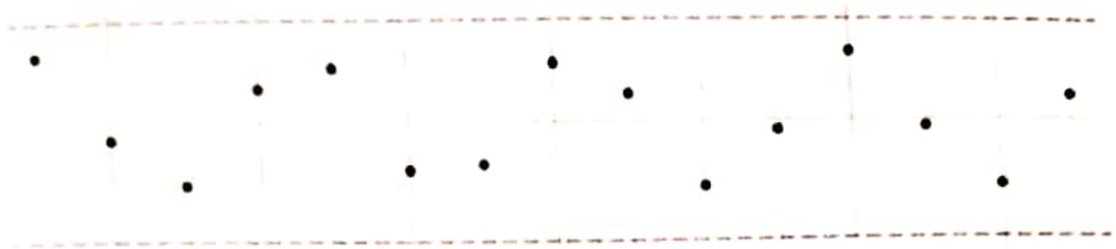


$\{n^2\}$ increases without bound, and is divergent

uence is ***monotonic*** if it eventually becomes an increasing, decreasing, non-
reasing, non-decreasing, or constant sequence.



uence is ***bounded*** if all of its terms lie within some window of finite width.
erwise, the sequence is ***unbounded***.



A sequence is “bounded” if ALL of its terms lie within some window of finite width

A sequence is “unbounded” if NO window of finite width can contain ALL of the terms of the sequence

Sequence Theorems

Indicate whether each statement is true or false. If you claim false, provide a counter example.

1. Convergent → Bounded True

2. Bounded → Convergent False $a_n = \sin n$

3. Monotone → Convergent False $a_n = n^2$

4. Convergent → Monotone False $a_n = \left(\frac{-1}{n}\right)^n$

5. Bounded and Monotone → Convergent True

L'Hopital's Rule

Suppose $\lim\limits_{n\to\infty} \dfrac{a(n)}{b(n)}$ has the form $\dfrac{\infty}{\infty}$ or $\dfrac{0}{0}$ (these are indeterminate forms)

Then $\lim\limits_{n\to\infty} \dfrac{a(n)}{b(n)} = \lim\limits_{n\to\infty} \dfrac{a'(n)}{b'(n)}$ provided this new limit exists, or is $\pm\infty$

For example, find the limit.

$$\lim_{n\to\infty} \frac{\overset{\to\infty}{4n} + \overset{\to\infty}{\ln n}}{\underset{\downarrow\infty}{\sqrt{n}} + \underset{\downarrow\infty}{3n}} \to \frac{\infty+\infty}{\infty+\infty} \to \frac{\infty}{\infty} \to \frac{4}{3}$$

$$\lim_{n\to\infty} \frac{4+\frac{1}{n}}{\frac{1}{2\sqrt{n}}+3} = \frac{4+0}{0+3} = \frac{4}{3}$$

$$\lim_{n\to\infty} \left\{ \frac{\ln(4n)}{\ln(12n)} \right\}$$

L'Hopital's Rule

Find the following limits.

$$\lim_{n\to\infty} \frac{n^2+n}{2n^3-1} \to \frac{\infty}{\infty} \to \boxed{0}$$

$$\lim_{n\to\infty} \frac{2n+1}{6n^2} \to \frac{\infty}{\infty} \to \boxed{0}$$

$$\lim_{n\to\infty} \frac{2}{12n} \to \frac{2}{\infty} \to 0.$$

$$\lim_{n\to\infty} \frac{2n+1}{6n^2}$$

$$= \lim_{n\to\infty} \frac{2n}{6n^2} + \frac{1}{6n^2} = 0$$

$$= \lim_{n\to\infty} \frac{2}{6n^2} + \frac{1}{6n^2} = 0$$

$$= \lim_{n\to\infty} \underbrace{\frac{1}{3n}}_{0} + \underbrace{\cancel{\frac{1}{6n^2}}}_{0} = 0$$

$$\lim_{n\to\infty} \frac{n^2+n}{2n^2-1} \to \frac{\infty}{\infty} = \boxed{\frac{1}{2}}$$

$$\lim_{n\to\infty} \frac{2n+1}{4n} \to \frac{\infty}{\infty} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{2}{4} \to \frac{1}{2}.$$

$$\lim_{n\to\infty} \frac{2n+1}{4n}.$$

$$\lim_{n\to\infty} \frac{1}{2} + \underbrace{\frac{1}{4n}}_{0} \to \frac{1}{2}.$$

divergent:

$$\lim_{n\to\infty} \frac{n^2+n}{2n-1} \to \infty.$$

$$\lim_{n\to\infty} \frac{2n+1}{2} \to \frac{\infty}{2} \to \infty.$$

Other Indeterminate Forms

Quotient Indeterminate Forms: $\frac{\infty}{\infty}$ $\frac{0}{0}$

Product Indeterminate Form: $0 \cdot \infty$

Difference Indeterminate Form: $\infty - \infty$

Exponential Indeterminate Form: 1^{∞} ∞^{0} 0^{0}

The following are not indeterminate forms

$\frac{\infty}{0} \to \infty$ $\quad 1 \cdot \infty \to \infty$ $\quad 0^{\infty} \to 0$

$\frac{0}{\infty} \to 0$ $\quad \infty + \infty = \infty$ $\quad \infty^{1} \to \infty$

Product Indeterminate

Find the following limit.

$$\lim_{n\to\infty} \underbrace{3n}_{\infty} \underbrace{\sin\frac{2}{n}}_{0} \to \infty \cdot 0 \to \boxed{6}$$

$$\lim_{n\to\infty} \frac{\sin 2/n}{1/3n} \to \frac{0}{0} \quad L'Hop.$$

$$\lim_{n\to\infty} \frac{\left[\cos\frac{2}{n}\right] \cdot \frac{-2}{n^2}}{\frac{1}{3}\left(\frac{-1}{n^2}\right)}$$

$$\lim_{n\to\infty} 6\cos \underbrace{\frac{2}{n}}_{\to 0} \to 6\cos 0 = \boxed{6}$$

Difference Indeterminate

Find the following limit.

$$\lim_{n\to\infty} \underset{\downarrow \atop \infty}{ne^{2/n}} - \underset{\downarrow \atop \infty}{n} \to \infty - \infty \to \boxed{2}$$

$$\lim_{n\to\infty} n\left(e^{2/n}-1\right) \to \infty \cdot 0.$$

$$\lim_{n\to\infty} \frac{e^{2/n}-1}{1/n} \to \frac{0}{0} \quad \text{L' Hop.}$$

$$\lim_{n\to\infty} \frac{e^{2/n} \cdot -2/n^2}{-1/n^2}$$

$$\lim_{n\to\infty} 2e^{\cancel{2/n}\to 0} \to 2e^0 = 2.$$

Exponential Indeterminate

...nd the following limit.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \to 1^\infty \to \boxed{e}$$

$\frac{1}{n} \to 0$

$\ln e^a = a$

$e^{\ln a} = a$

$\ln a^b = b \ln a$

$$\lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} e^{n \ln\left(1+\frac{1}{n}\right)} \to e^{\infty \cdot 0}$$

$n \to \infty$, $\ln\left(1+\frac{1}{n}\right) \to \ln 1 = 0$

$$\lim_{n\to\infty} e^{\frac{\ln\left(1+\frac{1}{n}\right)}{1/n}} \to e^{\frac{0}{0}} \quad L'Hop.$$

$$\lim_{n\to\infty} e^{\left(\frac{\frac{1}{1+1/n}\cdot \cancel{-1/n^2}}{\cancel{-1/n^2}}\right)} \to e^1 = e$$

$\frac{1}{1+1/n} \to 1$

Domination Chain
(of non-decreasing sequences)

Constants	Logarithms	Powers of n	Exponentials	Factorials	

$$k \ll \log_b n \ll n^k \ll a^n \ll n! \ll n^n$$

$b > 1 \qquad k > 0 \qquad a > 1$

I use this notation to mean that the greater (or dominating) sequence will eventually overtake the lesser (or dominated) sequence.

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 120$

$6! = 720$

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Summation Notation

$$\sum_{k=2}^{5}(2k-1) = 3+5+7+9 = 24.$$

$2(2)-1$
$4-1=3.$
$2(3)-1$
$6-1=5$
$2(4)-1$
$8-1=7.$
$2(5)-1$
$10-1=9.$

$$\sum_{m=1}^{4}(2m+1) = 3+5+7+9 = 24.$$

$231-22+1 = 210$ terms

$$\sum_{j=22}^{231} 2 = 2+2+2+\cdots+2 = 2(210).$$
$$=420.$$

Sequences of Partial Sums

n[th] Partial Sum: $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$

Example: $a_k = (-2)^{k-1}$

$a_1 = 1$) $\times -2$
$a_2 = -2$
$a_3 = 4$) $\times -2$
$a_4 = -8$) $\times -2$.
$a_5 = 16$.
$a_6 = -32$.

Find the first 5 partial sums.

$S_1 = 1 = a_1$
$S_2 = -1 = a_1 + a_2$
$S_3 = 3 = a_1 + a_2 + a_3$
$S_4 = S_3 + a_4 = -5$
$S_5 = S_4 + a_5 = 11$

Series

A *series* is the sum of all the terms of a sequence:

$$s_\infty = \sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots$$

If this sum exists, then we say the series is ***convergent.*** Otherwise we say the series is ***divergent***

Example 1:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$

Convergent

Example 2:

$$0.3 + 0.03 + 0.003 + \cdots$$

$$\begin{array}{r} .3 \\ +.03 \\ +.003 \\ +.0003 \\ \vdots \\ \hline .3333\cdots = \frac{1}{3} \end{array}$$

Convergent

Example 3:

$$1 + 2 + 3 + 4 + \cdots = \infty$$

divergent

Preview 10/14/19

Geometric Series

Geometric Sequence:

$\{a, ar, ar^2, ar^3, \cdots\}$

$a_n = ar^{n-1}$

$a_{20} = a \cdot r^{19}$

Geometric Partial Sum:

$$s_n = \frac{a(1-r^n)}{1-r}$$

100

Geometric Series:

If $-1 < r < 1$ then

$$s_\infty = \frac{a}{1-r}$$

otherwise S_∞ diverges

Example:

$$\sum_{n=2}^{\infty} \frac{5}{3^n} = \frac{5}{9} + \frac{5}{27} + \frac{5}{81} + \frac{5}{243} + \cdots = \frac{5/9}{1-1/3} = \frac{5/9}{2/3} = \frac{5}{9} \cdot \frac{3}{2} = \boxed{\frac{5}{6}}$$

$\times 1/3$ $\times 1/3$

a

$r = \frac{1}{3} \in (-1,1)$ ✓

$-1 < \frac{1}{3} < 1$ ✓

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1/2}{1-1/2} = 1$$

$r = 1/2 \in (-1,1)$ ✓

$a = 1/2$

$\times 1/10$ $\times 1/10$

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = \frac{3/10}{1-1/10} = \frac{3/10}{9/10}$$

$r = \frac{1}{10} \in (-1,1)$ ✓

$= \frac{3}{9} = \frac{1}{3}$

$a = 3/10$

Geometric Series

If convergent, find the sum of each geometric series.

$$\times 4/3 \quad \times 4/3$$

$$r = \frac{4}{3} \notin (-1,1)$$

$$\sum_{k=1}^{\infty} \frac{5 \cdot 4^k}{3^{k+2}} = \frac{20}{27} + \frac{80}{81} + \frac{320}{243} + \cdots = \infty \quad \text{divergent}$$

$$\sum_{k=1}^{\infty} \frac{5 \cdot 4^k}{3^k \cdot 3^2} = \sum_{k=1}^{\infty} \frac{5}{9}\left(\frac{4}{3}\right)^k = \frac{20}{27} + \cdots$$

$$\times -2/5 \quad \times -2/5$$

$$\sum_{m=1}^{\infty} \frac{7 \cdot 2^m}{(-5)^{m+1}} = \sum_{m=1}^{\infty} \frac{7 \cdot 2^m}{(-5)^1(-5)^m} = \sum_{m=1}^{\infty} \left(-\frac{7}{5}\right)\left(-\frac{2}{5}\right)^m = \frac{14}{25} - \frac{28}{125} + \frac{56}{625}$$

$$r = \frac{-2}{5} \in (-1,1)\checkmark \qquad \frac{14/25}{1-(-2/5)} = \frac{14/25}{7/5}$$

$$a = \frac{14}{25} \qquad = \frac{\cancel{14}^{2}}{\cancel{25}_{5}} \cdot \frac{\cancel{5}^{1}}{\cancel{7}} = \boxed{\frac{2}{5}}$$

Sequence Convergence vs Series Convergence

If convergent, find the limit of the sequence as n approaches infinity.

$$\left\{\frac{12}{3^n}\right\} = \left\{\frac{12}{3}, \frac{12}{9}, \frac{12}{27}, \cdots\cdots, \sim 0, \sim 0, \cdots\right\} \xrightarrow[\text{as } n \to \infty]{} 0 \quad \text{Convergent.}$$

$$= \left\{4, \frac{4}{3}, \frac{4}{9}, \cdots\right\} \to 0$$

If convergent, find the sum of the series.

$$\sum_{n=1}^{\infty} \frac{12}{3^n} = 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \cdots = \frac{4}{1-1/3} = \frac{4}{2/3} = \boxed{6.}$$

$r = 1/3 \in (-1,1)$ ✓

$a = 4$

Convergent.

Sequence Convergence vs Series Convergence

If convergent, find the limit of the sequence as n approaches infinity.

$$\left\{\frac{n+1}{2n+3}\right\} = \left\{\frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \ldots\right\} \to \frac{1}{2} \text{ convergent.}$$

$\frac{1}{2}$

If convergent, find the sum of the series.

$$\sum_{n=1}^{\infty} \frac{n+1}{2n+3} = \frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \frac{5}{11} + \cdots + \underbrace{\sim\frac{1}{2} + \sim\frac{1}{2} + \sim\frac{1}{2} + \cdots}_{\infty} = \infty$$

div.

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Sequence Convergence vs Series Convergence

If convergent, find the limit of the sequence as n approaches infinity.

$\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\} \to 0$ conv.

If convergent, find the sum of the series.

$\sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}}_{\text{harmonic}} + \cdots = \infty$ div

$a_n \to 0 \Leftarrow \sum a_n$ conv

The Divergence Test

If the terms of a sequence do not approach zero, then the corresponding series is divergent. The symbolic version of this statement is below.

$$\boxed{a_n \nrightarrow 0 \Rightarrow \Sigma a_n \text{ diverges}}$$

If the terms of a sequence do approach zero, then we can make no conclusion about the convergence of the series (consider the harmonic series).

Identify the series below that must be divergent.

$\sum_{k=1}^{\infty} \frac{k}{5k+1}$

$\left\{\frac{k}{5k+1}\right\} \to \frac{1}{5} \neq 0$: $\Rightarrow \sum_{k=1}^{\infty} \frac{k}{5k+1}$ div.

$\sum_{k=1}^{\infty} \sin k$

$\{\sin k\}$ div. $\{\sin k\} \nrightarrow 0$: $\sum_{k=1}^{\infty} \sin k$ div

$\sum_{k=1}^{\infty} \frac{2^k}{k!}$

$\left\{\frac{2^k}{k!}\right\} \to 0$ $\sum_{k=1}^{\infty} \frac{2^k}{k!} = ?$ inconclusive:

$\sum_{k=2}^{\infty} \frac{\sqrt{k}}{\ln k}$

$\left\{\frac{\sqrt{k}}{\ln k}\right\} \to \infty$ div. thus $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{\ln k} = \infty$ div:

The "Tail" of a Series

Find the sum of each convergent geometric series.

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \frac{1/2}{1-1/2} = 1$$

$$\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = \frac{1/8}{1-1/2} = \frac{1}{4}$$

The convergence of a series is determined by the behavior of the "tail" of the series. If a series is convergent, then it will be convergent no matter the starting value of the index. For convergent series, changing the starting value of the index will only affect the sum of the series, not whether the series is convergent.

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The Integral Test

Suppose that $f(x)$ is an integrable function that eventually decreases to zero on the interval $[k, \infty)$, such that $f(n) = a_n$ on this interval. Then the following holds.

$$\boxed{\int_k^\infty f(x)dx \text{ converges} \Leftrightarrow \sum_{n=k}^{\infty} a_n \text{ converges}}$$

Example:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$a_n = \frac{1}{n}$ $\quad f(x) = \frac{1}{x}$

$$\int_1^\infty \frac{1}{x}\,dx = \ln|x|\Big|_1^\infty$$

$$= \infty - 0$$

$$= \infty \text{ div.}$$

thus $\sum \frac{1}{n}$ diverges by the integral test.

...the integral test to determine the convergence of each series.

$\sum \frac{2}{\sqrt{n}}$

$\int_1^\infty \frac{2}{\sqrt{x}}\,dx = \int_1^\infty 2x^{-1/2}\,dx = \frac{2x^{1/2}}{1/2}\Big|_1^\infty = 4\sqrt{x}\Big|_1^\infty = \infty$ div

thus $\sum \frac{2}{\sqrt{n}}$ div by integral test.

$\sum \frac{3}{n^2}$

$\int_1^\infty \frac{3}{x^2}\,dx = \int_1^\infty 3x^{-2}\,dx = \frac{-3}{x}\Big|_1^\infty = 0 - 3(-3)$
$= 3$ conv

thus $\sum \frac{3}{n^2}$ conv by Integral test.

$\sum \frac{4}{n^2+1}$

$\int_0^\infty \frac{4}{x^2+1}\,dx = 4\tan^{-1}x\Big|_0^\infty = 4\frac{\pi}{2} = 2\pi$ conv

thus $\sum \frac{4}{n^2+1}$ conv by integral test.

The P-Series Test

Recall that the improper integral $\int_k^{\infty} \frac{a}{x^p}\,dx$ converges if and only if $p > 1$

Thus by the integral test, the series $\boxed{\sum_{n=k}^{\infty} \frac{a}{n^p} \text{ converges if and only if } p > 1}$

For example, determine the convergence of the three series below:

$\sum \frac{1}{n}$

p=1

thus Series div by p-test:

$\sum \frac{10}{\sqrt{n}}$ p=1/2

the series div by P-test.

$\sum \frac{3}{4n^2}$ P=2. 3/4 / n^2

the Series Conv by p-test.

$\sum \frac{2}{\sqrt[3]{n^2}}$ $\frac{2}{n^{2/3}}$ = P $\frac{2}{3}$

P=2/3.

the Series div by P-test.

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The Limit Comparison Test

Suppose that the sequences a_n and b_n have only positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$

If L is positive and finite, then the two series, $\sum a_n$ and $\sum b_n$, converge or diverge together

Use the limit comparison test to determine the convergence of the series below.

$$\sum \frac{2n+1}{n^2+\sqrt{n}}$$

$a_n = \frac{2n+1}{n^2+\sqrt{n}}$ Let $b_n = \frac{2n}{n^2} = \frac{2}{n}$.

$\frac{a_n}{b_n} = \frac{\frac{2n+1}{n^2+\sqrt{n}}}{\frac{2}{n}} = \frac{2n+1}{n^2+\sqrt{n}} \cdot \frac{n}{2} = \frac{2n^2+n}{2n^2+2\sqrt{n}} \to \frac{2}{2} = \frac{1}{1} = L$

pos. finite.

Also note $\sum b_n = \sum \frac{2}{n}$ div by the p-test.

thus $\sum a_n = \sum \frac{2n+1}{n^2+\sqrt{n}}$ div by the Limit Comp. test.

he limit comparison test to determine the convergence of the series below.

$$\sum \frac{\sqrt{n+10}}{n^2+7n}$$

$a_n = \frac{\sqrt{n+10}}{n^2+7n}$ Let $b_n = \frac{\sqrt{n}}{n^2} = \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$

$$\frac{a_n}{b_n} = \frac{\sqrt{n+10}}{n^2+7n} \cdot \frac{n^{3/2}}{1} = \frac{\sqrt{n(1+\frac{10}{n})}\, n^{3/2}}{n^2(1+\frac{7}{n})} = \frac{\sqrt{n}\sqrt{1+\frac{10}{n}}\, n^{3/2}}{n^2(1+\frac{7}{n})}.$$

$$= \frac{n^2\sqrt{1+\frac{10}{n}}}{n^2(1+\frac{7}{n})} = \frac{\sqrt{1+\frac{10}{n}}}{1+\frac{7}{n}} \to \frac{\sqrt{1}}{1} = 1 = L.$$

Positive Fin

Note $\sum b_n = \sum \frac{1}{n^{3/2}}$ conv by p-series test.

Thus $\sum a_n = \sum \frac{\sqrt{n+10}}{n^2+7n}$ conv by lim. comp test.

The Basic Comparison Tests

Suppose that the sequence a_n has only positive terms.

One way to show whether the series $\sum a_n$ converges is to find a positive sequence b_n such that $a_n \ll b_n$ and such that $\sum b_n$ converges.

One way to show whether the series $\sum a_n$ diverges is to find a positive sequence b_n such that $a_n \gg b_n$ and such that $\sum b_n$ diverges.

If the total area under the red function is finite, then the total area under the green function is finite.

If the total area under the green function is infinite, then the total area under the red function is infinite.

Practice Finding Dominating/Dominated Sequences

To safely increase the value of a fraction, you can increase the numerator, decrease the denominator, or do both.

$\frac{4}{10} < \frac{5}{10} \qquad \frac{4}{10} < \frac{4}{8} \qquad \frac{4}{10} < \frac{5}{8}$.

To safely decrease the value of a fraction, you can decrease the numerator, increase the denominator, or do both.

$\frac{4}{10} > \frac{3}{10} \qquad \frac{4}{10} > \frac{4}{12} \qquad \frac{4}{10} > \frac{3}{12}$.

It is not conclusive the effect of increasing (or decreasing) both the numerator and denominator.

$\frac{3}{4} < \frac{4}{5} \qquad \frac{4}{3} > \frac{5}{4}$.

the basic comparison test to determine whether the given series converges or diverges.

$$\sum \frac{1}{n^2 + \sqrt{n}}$$

Guess: Series conv

Goal: $a_n = \frac{1}{n^2+\sqrt{n}}$ find b_n such that $a_n \le b_n$ and $\sum b_n$ conv.

$\frac{1}{n^2+\sqrt{n}} \le \underbrace{\frac{1}{n^2}}_{b_n}$ and $\sum b_n = \sum \frac{1}{n^2}$ conv by p-test.

thus $\sum \frac{1}{n^2+\sqrt{n}}$ conv by basic comp.

mparison Tests

the basic comparison test to determine whether the given series converges or diverges.

$$\sum \underbrace{\frac{1}{n+\sqrt{n}}}_{a_n}$$

Guess: Series div.

Goal: Find b_n such that $a_n >> b_n$ and $\sum b_n$ div

$$\frac{1}{n+\sqrt{n}} >> \frac{1}{n+n} = \underbrace{\frac{1}{2n}}_{b_n}$$

and $\sum \frac{1}{2n} = \sum \frac{1/2}{n}$ div by p-test.

... the basic comparison test to determine whether the given series converges or diverges.

$\underbrace{\sum \frac{\ln n}{n}}_{a_n}$ Guess: Series div.

Goal: Find b_n s.t. $a_n >> b_n$ and $\sum b_n$ div.

$\frac{\ln n}{n} >> \frac{1}{n}$ and $\sum \frac{1}{n}$ div by p-test.

thus $\sum \frac{\ln n}{n}$ div by basic comp test.

mparison Tests

e the basic comparison test to determine whether the given series converges or diverges.

$$\sum \frac{\ln n}{n^2}$$

a_n

Guess: Series conv.

Goal: Find b_n s.t. $a_n << b_n$ and $\sum b_n$ conv.

$$\frac{\ln n}{n^2} << \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$$

b_n

$\sum 1/n^{3/2}$ conv by p-test.

... the basic comparison test to determine whether the given series converges or diverges.

$$\sum \underbrace{\frac{2n+1}{n^2+\sqrt{n}}}_{a_n}$$

Guess: Series div.

Goal: Find b_n s.t. $a_n >> b_n$ and $\sum b_n$ div.

$$\frac{2n+1}{n^2+\sqrt{n}} >> \frac{2n}{n^2+n^2} = \frac{2n}{2n^2} = \underbrace{\frac{1}{n}}_{b_n}$$

$\sum \frac{1}{n}$ div by p-test.

thus $\sum \frac{2n+1}{n^2+\sqrt{n}}$. div by basic Comp test.

…e the basic comparison test to determine whether the given series converges or diverges.

$$\sum \underbrace{\frac{\sqrt{n+10}}{n^2+7n}}_{a_n}$$

Guess: Series conv.

Goal: Find b_n s.t. $a_n \leq b_n$ and $\sum b_n$ conv.

$$\frac{\sqrt{n+10}}{n^2+7n} \leq \frac{\sqrt{n+n}}{n^2} = \frac{\sqrt{2}\sqrt{n}}{n^2} = \underbrace{\frac{\sqrt{2}}{n^{3/2}}}_{b_n}$$

$\sum \frac{\sqrt{2}}{n^{3/2}}$ conv by p-test.

$\sum \frac{\sqrt{n+10}}{n^2+7n}$ conv by basic comp. test!!!

Sheila Kiprotich.

The Alternating Series Test

Suppose that the sequence a_n eventually decreases to zero.

From this information alone we cannot conclude whether the series $\sum a_n$ converges or diverges. However if we then make the series alternate in sign, then the resulting series $\sum(-1)^n a_n$ converges.

Use the alternating series test to determine the convergence of the series below.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \cdots$$

$\approx -.69$

$$\sum (-1)^n \frac{1}{n}$$

$a_n = \frac{1}{n}$ decreases to 0.

thus $\sum \frac{(-1)^n}{n}$ converges by alt. series test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \approx -.69$$

$$\sum(-1)^{n+1}\sin\frac{1}{n}$$

$a_n = \sin(\frac{1}{n})$. decreases to zero (as $n \to \infty$).

thus $\sum(-1)^{n+1}\sin\frac{1}{n}$ conv. by A.S.T.

Determine whether the series below converge or diverge.

$\sum \frac{(-1)^n n}{2n+1}$

$a_n = \frac{n}{2n+1} \to \frac{1}{2} \neq 0$

$(-1)^n n$ DO

thus $\sum \frac{(-1)^n n}{2n+1}$ div.

by the div. test.

$\sum \frac{\cos(n\pi)}{\sqrt{n+1}} = \sum \frac{(-1)^n}{\sqrt{n+1}}$

$\{\cos(n\pi)\}_{n=1} = \{-1, 1, -1, 1 \ldots\}$

$= \{(-1)^n\}$

$a_n = \frac{1}{\sqrt{n+1}}$ decr. to zero.

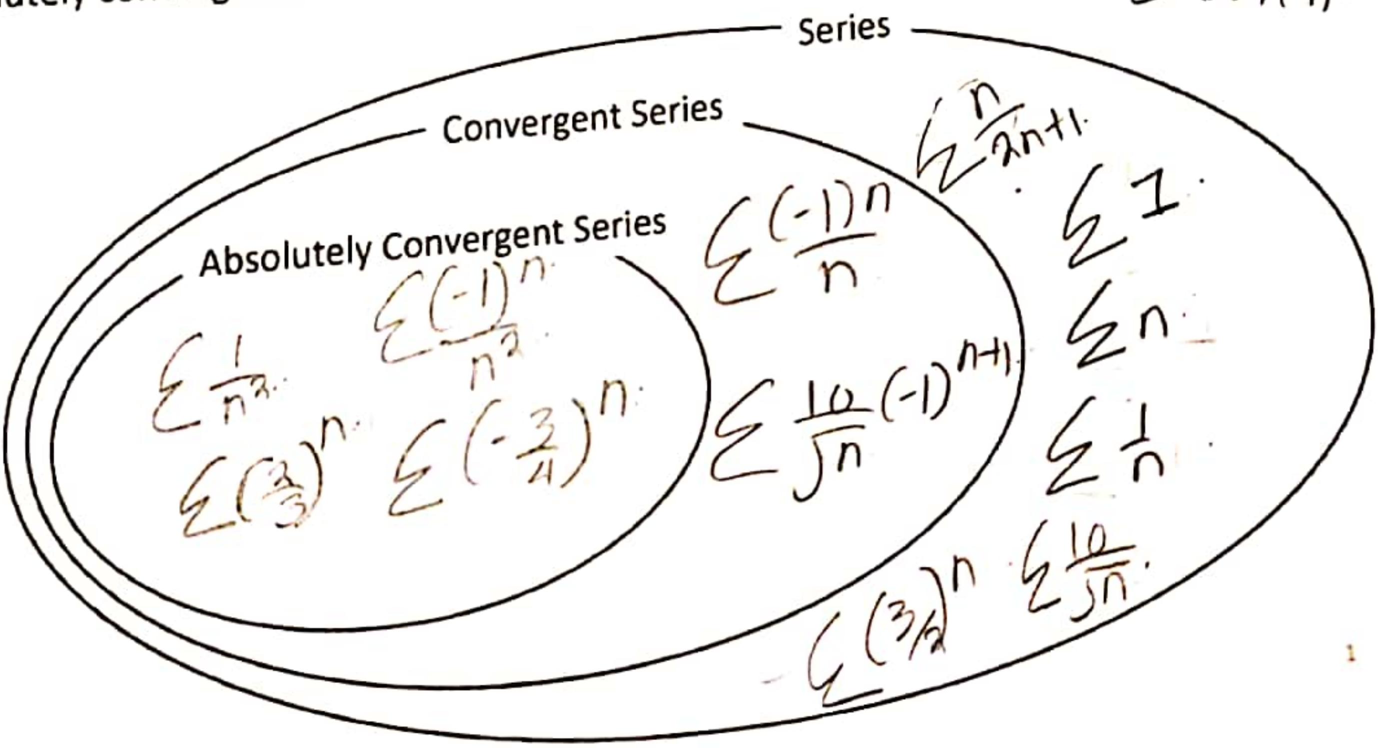
$\left\{\frac{(-1)^n}{\sqrt{n+1}}\right.$ conv. by A.S.T.

11.6 Absolute vs Conditional Convergence and the Ratio and Root Tests

If a series $\sum a_n$ has positive and negative terms and it converges, but $\sum |a_n|$ diverges, then the series $\sum a_n$ is ***conditionally convergent.***

If a series $\sum |a_n|$ converges, then $\sum a_n$ also converges and $\sum a_n$ is ***absolutely convergent.***
Note, if a series only has positive terms and it is convergent, then it is automatically absolutely convergent.

S brick



11.6 Absolute vs Conditional Convergence and the Ratio and Root Tests

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum \frac{(-1)^n}{2n}$$

$\sum \frac{1}{2n}$ div by Ptest.

$\sum (-1)^n \frac{1}{2n}$ conv by alt. series test.

$\{\frac{1}{2n}\}$ decr to zero.

$\sum \frac{(-1)^n}{2n}$ is conditionally convergent.

11.6 Absolute vs Conditional Convergence and the Ratio and Root Tests

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$\sum \frac{(-1)^n}{3^n}$ $\sum \left(\frac{-1}{3}\right)^n$ $\sum \frac{1}{3^n}$ conv geom. series

$r = 1/3 \in (-1,1)$

$\sum \frac{(-1)^n}{3^n}$ conv geom. series.

$r = -1/3 \in (-1,1)$.

11.6 Absolute vs Conditional Convergence and the Ratio and Root Tests

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum \frac{(-1)^n n}{2n+1}$$

$\sum \frac{n}{2n+1}$ div by div test.

$\left\{\frac{n}{2n+1}\right\} \to \frac{1}{2} \neq 0$.

$\left\{\sum \frac{(-1)^n n}{2n+1}\right\}$ div by div test.

$\left\{\frac{(-1)^n n}{2n+1}\right\} \not\to 0$.

11.6 Absolute vs Conditional Convergence and the Ratio and Root Tests

Determine whether the series is absolutely convergent, ~~conditionally convergent, or~~ divergent.

$\sum \frac{5}{n^2}$ Conv by P-test.

Review of Factorials

Simplify the following.

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$4! = 4 \cdot 3 \cdot 2 = 24$

$3! = 3 \cdot 2 = 6$

$2! = 2 \cdot 1 = 2$

$1! = 1$

$0! = 1$

$2^0 = 1$

$\frac{8!}{6!} = \frac{8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!}} = 56$

$\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)(n)\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{2}}{n\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{2}}$

$= (n+2)(n+1)$

$\frac{n!}{(n+1)!} = \frac{n!}{(n+1)(n)} = \frac{1}{n+1}$

The Ratio Test

Suppose that the sequence $\left|\frac{a_{n+1}}{a_n}\right| \to K$

If $K < 1$ then the series $\sum a_n$ is absolutely convergent.
If $K > 1$ then the series $\sum a_n$ is divergent.
If $K = 1$ then the ratio test is inconclusive.

Use the ratio test to determine the convergence of the series.

$$\sum \frac{(-2)^n}{n!} = 1 - \frac{2}{1!} + \frac{4}{2!} - \frac{8}{3!} + \frac{16}{4!} - - - - -$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-2)^{n+1}}{(n+1)!}}{\frac{(-2)^n}{n!}}\right| = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{\cancel{2^n} \cdot 2^1}{\cancel{2^n}} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \frac{2}{n+1} \to 0 = K.$$

thus $\sum \frac{(-2)^n}{n!}$ abs con by ratio test.

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11.6 Absolute vs Conditional Convergence and the Ratio and Root Tests

Use the ratio test to determine the convergence of the series.

$$\sum \frac{(-3)^n(n+2)}{2^n}$$

$$\left| \frac{(-3)^{n+1}((n+1)+2)}{2^{n+1}} \cdot \frac{2^n}{(-3)^n(n+2)} \right| = \frac{\cancel{3^n} \cdot 3 \cdot \cancel{2^n}}{\cancel{3^n}\ \cancel{2^n} \cdot 2} \frac{n+3}{n+2}$$

$$= \frac{3}{2}\left(\frac{n+3}{n+2}\right) \to \frac{3}{2}(1) = \frac{3}{2} = K.$$

thus $\sum \frac{(-3)^n(n+2)}{2^n}$ div by ratio te

11.6 Absolute vs Conditional Convergence and the Ratio and Root Tests

Use the ratio test to determine the convergence of the series.

$$\sum \frac{n+2}{4n^2+3n}$$

$$\left| \frac{n+3}{4(n+1)^2+3(n+1)} \cdot \frac{4n^2+3n}{n+2} \right| = \frac{n+3}{n+2} \cdot \frac{n(4n+3)}{(n+1)(4(n+1)+3)},$$

$$= \frac{n+3}{n+2} \cdot \frac{n}{n+1} \cdot \frac{4n+3}{4n+7} \to 1 \cdot 1 \cdot \frac{4}{4} = 1 = R.$$

$$a_n = \frac{n+2}{4n^2+3n} \qquad \frac{a_n}{b_n} = \frac{n+2}{4n^2+3n} \cdot \frac{4n}{1} = \frac{4n^2+8n}{4n^2+3n} \to \frac{4}{4} = 1 = L.$$

L is pos and finite.

$$b_n = \frac{n}{4n^2} = \frac{1}{4n}.$$

also $\sum b_n = \sum \frac{1}{4n}$ div by p-test.

thus $\sum a_n = \sum \frac{n+2}{4n^2+3n}$ div by Lim. Comp.-test.

11.6 Absolute vs Conditional Convergence and the Ratio and Root Tests

Use the ratio test to determine the convergence of the series.

$\sum \frac{4n!}{n^n}$ $\left|\frac{4(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{4n!}\right| = \frac{(n+1)!}{n!(n+1)^n(n+1)^1} n^n$

$= \cancel{(n+1)}\left(\frac{n}{\cancel{n+1}}\right)^n \cdot \frac{1}{n+1}$

$= \left(\frac{n}{n+1}\right)^n$

$a^b = e^{\ln a^b} = e^{b \ln a}$

$\left(\frac{n}{n+1}\right)^n = e^{n \ln\left(\frac{n}{n+1}\right)} = e^{\frac{\ln\left(\frac{n}{n+1}\right)}{1/n}} = e^{\frac{\ln a - \ln(n+1)}{1/n}}$

$\frac{\frac{(n+1)}{(n+1)}\frac{1}{n} \cdot \left(\frac{1}{n+1}\right)\frac{n}{n}}{e^{-1/n^2}} = e \cdot \frac{1}{n(n+1)} \cdot -n^2 = e^{\frac{-n^2}{n^2+n}} \to e^{-1} = 1/e = 15 \cdot 10$

Since $k < 1$ $\sum \frac{4n!}{n^n}$ is abs. conv. by ratio test.

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The Root Test

Suppose that the sequence $\left|(a_n)^{1/n}\right| \to K$

If $K < 1$ then the series $\sum a_n$ is absolutely convergent.
If $K > 1$ then the series $\sum a_n$ is divergent.
If $K = 1$ then the ratio test is inconclusive.

Use the root test to determine the convergence of the series.

$$\sum \frac{(-2)^n}{n^n} = \sum \left(\frac{-2}{n}\right)^n$$

$$\left| \left[\left(\frac{-2}{n}\right)^n \right]^{1/n} \right| = \left| \frac{-2}{n} \right| = \frac{2}{n} \to 0 = K$$

thus $\sum \frac{(-2)^n}{n^n}$ abs conv. by root test.

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11.6 Absolute vs Conditional Convergence and the Ratio and Root Tests

Use the root test to determine the convergence of the series.

$$\sum \left(\frac{3n}{2n+4}\right)^n$$

$$\left| \left[\left(\frac{3n}{2n+4} \right)^n \right]^{1/n} \right| = \frac{3n}{2n+4} \to 3/2 = K.$$

thus $\sum \left(\frac{3n}{2n+4}\right)^n$ div by root test.

11.6 Absolute vs Conditional Convergence and the Ratio and Root Tests

Use the root test to determine the convergence of the series.

$\sum\left(\frac{n}{n+4}\right)^n$ $\left|\left[\left(\frac{n}{n+4}\right)^n\right]^{1/n}\right| = \frac{n}{n+4} \to 1 = K$. inconclusive.

$\left(\frac{n}{n+4}\right)^n = e^{n \ln\left(\frac{n}{n+4}\right)} = e^{\frac{\ln\left(\frac{n}{n+4}\right)}{1/n}} = e^{\frac{\ln n - \ln(n+4)}{1/n}}$

$\frac{(n+4)}{(n+4)}\left(\frac{1}{n}\right) - \left(\frac{1}{n+4}\right)\frac{n}{n} = e^{\frac{4}{n(n+1)} \cdot -n^2} = e^{\frac{-4n^2}{n^2+n}} \to e^{-4} = \frac{1}{e^4} \neq 0$

thus $\sum\left(\frac{n}{n+4}\right)^n$ div by div test.

Introduction to Taylor Polynomials

Given a function $f(x)$, the n^{th} degree Taylor Polynomial centered at $x = a$ is a polynomial whose first n derivatives match the first n derivatives of the function at $x = a$.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$f^{(30)}(x)$

n^{th} der.

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Find the 3rd degree Taylor Polynomial centered at $x = 2$ for the given function.

$f(x) = \frac{4}{x}$

k	$f^{(k)}(x)$	$f^{(k)}(2)$
0	$4x^{-1}$	2
1	$-4x^{-2}$	-1
2	$8x^{-3}$	1
3	$-24x^{-4}$	$-24\left(\frac{1}{16}\right) = -\frac{3}{2}$

$T_3(x) = 2 - (x-2) + \frac{1}{2!}(x-2)^2$

$T_3(x) = 2 - (x-2) + \frac{1}{2!}(x-2)^2 + \frac{-3/2}{3!}(x-2)^3$

$T_3(x) = 2 - (x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{4}(x-2)^3$

11.11 Taylor Polynomials

Find the 4th degree Taylor Polynomial centered at $x = 1$ for the given function.
Use the Taylor Polynomial to predict the value of $\sqrt{1.2}$

$f(x) = \sqrt{x}$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}$

$$T_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$$T_4(x) = 1 + \frac{1}{2}(x-1) + \frac{-1/4}{2}(x-1)^2 + \frac{3/8}{6}(x-1)^3 + \frac{-15/16}{24}(x-1)^4$$

$$T_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

$$T_4(1.2) = 1 + \frac{1}{2}\left(\frac{1}{5}\right) - \frac{1}{8}\left(\frac{1}{5}\right)^2 + \frac{1}{16}\left(\frac{1}{5}\right)^3 - \frac{5}{128}\left(\frac{1}{5}\right)^4$$

$$\frac{17527}{16000}$$

$$\sqrt{1.2} = f(1.2) \approx T_4(1.2) = \frac{17527}{16000} \approx 1.0954375$$

11.11 Taylor Polynomials

Find the 3rd degree Taylor Polynomial centered at $x = 0$ for the given function. Use the Taylor Polynomial to predict the value of tan 0.5

$f(x) = \tan x$

K	$f^{(K)}(x)$	$f^{(K)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^2 x$	2

$T_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$

$T_3(x) = 0 + 1(x-0) + \frac{0}{2!}(x-0)^2 + \frac{2}{3!}(x-0)^3$

$T_3(x) = x + \frac{1}{3}x^3$

$T_3(.5) = \frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 = \frac{13}{24} = 0.5416 \approx \tan(0.5)$

Sheila kiprotich

11.8 Power Series

Introduction to Power Series

Consider the geometric series below using x as the ratio. Use the ratio test and other tests to confirm its interval of convergence. radius of convergence is 1

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \quad \text{if } -1 < x < 1$$

Interval of convergence

$$\left|\frac{x^{n+1}}{x^n}\right| = |x| \xrightarrow[n\to\infty]{} |x| = k$$

abs conv if $k < 1$

$|x| < 1$

$-1 < x < 1$

Incon if $k = 1$

$|x| = 1$

$x = 1, -1$

$x = 1$

$\sum_{n=0}^{\infty} 1^n = \sum_{n=0} 1 = \infty$ div

$x = -1$

$\sum_{n=0} (-1)^n = 1 - 1 + 1 - 1 + 1 - \ldots$ div

The general form of a power series is below. Like for the geometric power series, a power series might only converge for certain values of x.

$$\sum_{n=0}^{\infty} a_n (f(x))^n = a_0 + a_1 f(x) + a_2 (f(x))^2 + a_3 (f(x))^3 + \cdots$$

11.8 Power Series

Find the interval and radius of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{(4x)^n}{n} \quad \left| \frac{\frac{(4x)^{n+1}}{n+1}}{\frac{(4x)^n}{n}} \right| \quad \left| \frac{4^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{4^n x^n} \right| = \left| \frac{4^{n+1}}{4^n} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{n}{n+1} \right|$$

$$= \left| 4 \cdot x \cdot \frac{n}{n+1} \right| = |4| \cdot |x| \cdot \left| \frac{n}{n+1} \right|$$

$$= 4|x| \frac{n}{n+1} \xrightarrow[n\to\infty]{} \quad 4|x| \cdot 1 = 4|x| = k$$

test $x = 1/4$ abs conv is $k = \underline{4|x| < 1}$

$\sum_{n=1} \frac{(4 \cdot 1/4)^n}{n}$ $4|x| < 1$ $4|x| - 1 = 4|x| = k$

$|x| < \frac{1}{4}$ in cond.

$\sum \frac{1^n}{n} \sum \frac{1}{n}$ div $\quad k = 1 \quad -1/4 < x < \frac{1}{4}$

by p-test $-\frac{1}{4} < x < \frac{1}{4}$ $\quad 4|x| = 1$ $\quad \boxed{-1/4 \leq x < \frac{1}{4}}$

conv by alt. ser. test $|x| = 1/4$ $\quad \boxed{r = 1/4}$

$-\frac{1}{4} < x < \frac{1}{4}$

$x = 1/4, -1/4$

test $= x - 1/4$ $\sum \frac{(4(-1/4))^n}{n} = \sum \frac{(-1)^n}{n} \{\frac{1}{n}\}$ decr to zero.

Find the interval and radius of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{(4-x)^n}{n^2}$$

$$\left| \frac{(4-x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(4-x)^n} \right|$$

$$= \left| (4-x) \left(\frac{n}{n+1}\right)^2 \right|$$

$$= |4-x| \left(\frac{n}{n+1}\right)^2$$

$$\to |4-x|$$

$|4-x| < 1$

$-1 < 4-x < 1$

$-5 < -x < -3$

$5 > x > 3$

$3 < x < 5$

$3 \leq x \leq 5$

$r = 1$

test $x=3$

$\sum \frac{(4-3)^n}{n^2}$

$= \sum \frac{1}{n^2}$ conv p-series

test $5=x$

$\sum \frac{(4-5)^n}{n^2} = \sum \frac{(-1)^n}{n^2}$ conv by alt-ser-test

$\left\{\frac{1}{n^2}\right\}$ decr to zero

11.8 Power Series

Find the interval and radius of convergence of the power series.

$$\sum n!\,x^n$$

$$\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$=|x|\cdot\frac{1}{n+1}$$

$\rightarrow 0$ no matter what x is

$R = 0 < 1$

abs. conv.

$$\boxed{\begin{array}{c}(-\infty,\infty)\\ r=\infty\end{array}}$$

11.8 Power Series

Find the interval and radius of convergence of the power series.

$\sum \frac{2n!}{x^n}$ $x \neq 0$

div for all values of x

$\left| \frac{2(n+1)!}{x^{n+1}} \cdot \frac{x^n}{2n!} \right|$

$= \frac{n+1}{|x|}$

$\to \infty > k$

$k > 1$

div.

$\sum_{n=1}^{\infty} n!\, x^n = x + 2x^2 + 6x^3 \ldots$

$= 0 + 0 + 0 \ldots$

$= 0$

$\left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right|$

$= (n+1)|x| \to \infty > k$

if $x \neq 0$

div.

$\sum n!\, x^n$ conv iff $x = 0$

interval is $\{0\}$

$r = 0$

5

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Taylor Series

Given a function $f(x)$, its Taylor Series centered at $x = a$ is given below.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Maclaurin Series

Given a function $f(x)$, its Maclaurin Series is given below.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

11.10 Taylor and Maclaurin Series

Find the Taylor Series centered at $x = 2$ for the given function. Express it using summation notation and write down the first four non-zero terms. Also give its interval of convergence.

$f(x) = \frac{4}{x}$

k	$f^{(k)}(x)$	$f^{(k)}(2)$
0	$4x^{-1}$	2
1	$-4x^{-2}$	-1
2	$8x^{-3}$	1
3	$-24x^{-4}$	$-3/2$
4	$96x^{-5}$	$\frac{96}{32} = 3$

$\left| \frac{(-1)^{k+1}(x-2)^{k+1} 2 \left(\frac{1}{2}\right)^{k+1}}{(-1)^k (x-2)^k 2 \left(\frac{1}{2}\right)^k} \right| = \left| (x-2)\frac{1}{2} \right|$ $k \to \infty$

$\frac{1}{2}|x-2| < 1$

$|x-2| < 2$

$-2 < x-2 < 2$

$0 < x < 4$

$(0,4)$

$T_6(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3 + f^{(4)}_{(2)}(x-2)^4 + \ldots$

$= 2 - (x-2) + \frac{1}{2}(x-2)^2 + \frac{-3/2}{6}(x-2)^3 + \frac{3}{24}(x-2)^4 + \ldots$

$\underset{k=0}{=2} - \underset{k=1}{(x-2)} + \frac{1}{2}\underset{k=2}{(x-2)^2} - \frac{1}{4}\underset{k=3}{(x-2)^3} + \frac{1}{8}\underset{k=4}{(x-2)^4} + \ldots$

$= \sum_{k=0}^{\infty} (-1)^k (x-2)^k 2 \cdot \left(\frac{1}{2}\right)^k$

$x=0$ $\sum (-1)^k (-2)^k 2 \cdot \left(\frac{1}{2}\right)^k$

$\sum 2$ div by div test.

$x=4$ $\sum (-1)^k (2)^k 2 \cdot \left(\frac{1}{2}\right)^k$

$\sum (-1)^k 2$ div by div test.

$(0,4)$

11.10 Taylor and Maclaurin Series

Find the Maclaurin series for the given function. Express it using summation notation and write down the first four non-zero terms. Also give its interval of convergence.

$$f(x) = \frac{1}{1-x}$$

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$(1-x)^{-1}$	1
1	$(1-x)^{-2}$	1
2	$2(1-x)^{-3}$	2
3	$6(1-x)^{-4}$	6
4	$24(1-x)^{-5}$	24

$$M(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \cdots$$

$$= 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$= \sum_{k=0}^{\infty} x^k$$

int of conv. $(-1, 1)$

11.10 Taylor and Maclaurin Series

Find the Maclaurin series for the given function. Express it using summation notation and write down the first four non-zero terms. Also give its interval of convergence.

$f(x) = e^x$

k	$f^{k}(x)$	$f^{(k)}(0)$
0	e^x	1
1	e^x	1
2	e^x	1
3	e^x	1

$$m(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$ int. of conv $(-\infty, \infty)$

4

$$e^x = \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

for any x

11.10 Taylor and Maclaurin Series

Find the Maclaurin series for the given function. Express it using summation notation and write down the first four non-zero terms. Also give its interval of convergence.

$f(x) = \ln(x+1)$

K	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24

$$M(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \cdots$$

$$= 0 + x - \frac{x^2}{2} + \frac{2}{6}x^3 - \frac{6}{24}x^4$$

$K=0$ $K=1$ $K=2$ $K=3$ $K=4$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$$

Int. of conv $(-1, 1]$

5

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

true if $x \in (-1, 1]$

11.10 Taylor and Maclaurin Series

Find the Maclaurin series for the given function. Express it using summation notation and write down the first four non-zero terms. Also give its interval of convergence.

$f(x) = \sin x$

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0

$$m(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} \cdots$$

$k=0 \quad k=1 \quad k=2 \quad k=3$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

true for any x

11.10 Taylor and Maclaurin Series

Find the Maclaurin series for the given function. Express it using summation notation and write down the first four non-zero terms. Also give its interval of convergence.

$f(x) = \sqrt{x+1}$

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$(x+1)^{1/2}$	1
1	$\frac{1}{2}(x+1)^{-1/2}$	$\frac{1}{2}$
2	$\frac{1}{2}\left(-\frac{1}{2}\right)(x+1)^{-3/2}$	$\frac{1}{2}\left(-\frac{1}{2}\right)$
3	$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x+1)^{-5/2}$	$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$
4	$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(x+1)^{-7/2}$	$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)$

$$M(x) = f_0 + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \cdots$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)x^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^3}{3!} + \cdots$$

$$= \sum_{k=0}^{\infty} \quad = \binom{1/2}{0} + \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \binom{1/2}{k} x^k$$

11.10 Taylor and Maclaurin Series

Find the Maclaurin series for the given function. Express it using summation notation and write down the first four non-zero terms. Also give its interval of convergence.

$f(x) = (x+1)^r$

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$(x+1)^r$	1
1	$r(x+1)^{r-1}$	r
2	$r(r-1)(x+1)^{r-2}$	$r(r-1)$
3	$r(r-1)(r-2)(x+1)^{r-3}$	$r(r-1)(r-2)$

$m(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$

$= 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \frac{r(r-1)(r-2)(r-3)}{4!}x^4 + \cdots$

$= 1 + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \binom{r}{4}x^4$

$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$ where $\binom{r}{0} = 1$ e.g. $\binom{r}{5} = \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}$

True

Binomial Series

$\binom{r}{1} = \frac{r}{1!} = r$

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Altering a Known Series

For some functions, it can be complicated to find a series expansion by creating it from a table of derivatives. For some such cases, a series can be found by altering a known series. We discuss five ways this can be done.

1. Compositions (replacing x with something else)
2. Differentiation of a series
3. Integration of a series
4. Multiplying a series by constants or powers of x
5. Adding terms to a series

Altering a Known Series by Compositions

Find a Maclaurin series expansion for the given function. Write the first four non-zero terms and give the interval of convergence.

$f(x) = \ln(1 - 5x)$

$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$ for $x \in (-1, 1]$

$-1 < x \leq 1$

$\ln(1-5x) = \sum_{k=0}^{\infty} \frac{(-1)^k (-5x)^{k+1}}{k+1} = (-5x) - \frac{(-5x)^2}{2} + \frac{(-5x)^3}{3} \ldots$

replace x with $-5x$.

$\ln(1-5x) = \sum_{k=0}^{\infty} \frac{(-1)^k (-5x)^{k+1}}{k+1} = \frac{(-5x)}{2} - (-5)^2 + \frac{5x^3}{3}$ for $-1 < -5x \leq 1$:

$\ln(1-5x) = \sum_{k=0}^{\infty} \frac{(-1)^k (-5x)^{k+1}}{k+1} \cdot -5x - \frac{25x^2}{2} - \frac{125x^3}{3} \ldots$ for

$\frac{1}{3} > x \geq -\frac{1}{3}$

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Altering a Known Series by Differentiation

Find a Maclaurin series expansion for the given function. Write the first four non-zero terms and give the interval of convergence.

$f(x) = \frac{1}{(1-x)^2}$ $= \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$ for $-1 < x < 1$

$R = 1$.

↓ differentiate.

$k=0 \quad k=1 \quad k=2$

$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} k x^{k-1} = 0 + 1 + 2x + 3x^2 + \cdots$

for $-1 < x < 1$

$R = 1$.

$= \sum_{k=0}^{\infty} (k+1) x^k$.

Altering a Known Series by Integration

Find a Maclaurin series expansion for the given function. Write the first four non-zero terms and give the interval of convergence.

$f(x) = \cos x$

$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for $-\infty < x < \infty$

Integrate

$-\cos x = C + \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} - \cdots$

Find C: Let $x = 0$

$-1 = C + 0 - 0 + 0 - 0 \cdots$

$-\cos x = -1 + \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!}$

$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ $\quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \cdots$ for $-\infty < x < \infty$

$R = \infty$

Altering a Known Series by Multiplication

Find a Maclaurin series expansion for the given function. Write the first four non-zero terms and give the interval of convergence.

$f(x) = 3x^2e^x$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \text{ for } -\infty < x < \infty$$

$$3x^2e^x = \sum_{k=0}^{\infty} 3x^2\frac{x^k}{k!} = 3x^2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) \text{ for } -\infty < x < \infty$$

$$\sum_{k=0}^{\infty} \frac{3x^{k+2}}{k!}$$

Altering a Known Series by Addition

Find a Maclaurin series expansion for the given function. Write the first four non-zero terms and give the interval of convergence.

$f(x) = 3x^2 + 2 + \ln(x+1)$

$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$ for $x \in (-1, 1]$

$-1 < x \leq 1$

$+2+3x^2 \quad +2+3x^2 \quad +2+3x^2$

$3x^2 + 2 + \ln(x+1) = 3x^2 + 2 + \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$

$= 2 + x + \frac{5}{2}x^2 + \frac{x^3}{3} - \frac{x^4}{4} \ldots$ for $x \in (-1, 1]$

$-1 < x \leq 1$

11.9 Taylor and Maclaurin Series

Find the Maclaurin series for the given function. Express it using summation notation and write down the first four non-zero terms. Also give its interval of convergence.

$f(x) = \tan^{-1} x$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \text{ for } -1 < x < 1$$

$\frac{1}{1-x}$ $x \to -x^2$

replace x with $-x^2$ replace x with $-x^2$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

for $-1 < -x^2 < 1$

$1 > x^2 > -1$

$-1 < x^2 < 1$

$-1 < x < 1$

$\frac{1}{1+x^2}$ integrate

$\tan^{-1} x$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + x^8 \cdots \text{ for } -1 < x < 1$$

$R = 1$

integrate

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k x^{2k+1} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots$$

for $-1 < x < 1$

$R = 1$

11.9 Taylor and Maclaurin Series

Find the Maclaurin series for the given function. Express it using summation notation and write down the first four non-zero terms. Also give its interval of convergence.

$f(x) = \dfrac{3}{\sqrt[3]{8+x}}$ $= \dfrac{3}{\sqrt[3]{8(1+\frac{x}{8})}} = \dfrac{3}{2\sqrt[3]{1+\frac{x}{8}}} = \dfrac{3}{2}\left(1+\frac{x}{8}\right)^{-1/3}$

$(1+x)^k = \sum_{k=0}^{\infty} \binom{k}{n} x^n = 1 + \binom{k}{1}x + \binom{k}{2}x^2 + \binom{k}{3}x^3 + \cdots$

for $-1<x<1$

$\left(1+\frac{x}{8}\right)^{-1/3} = \sum_{k=0}^{\infty} \binom{-1/3}{k}\left(\frac{x}{8}\right)^k = 1 + \binom{-1/3}{1}\left(\frac{x}{8}\right) + \left(\frac{-1/3}{8}\right) + \left(\frac{-1/3}{2}\right)\left(\frac{x}{8}\right)^2 + \binom{-1/3}{3}\left(\frac{x}{8}\right)^3 \cdots$ for $-1<\frac{x}{8}<1$

$-8<x<8$

$\frac{3}{2}\left(1+\frac{x}{8}\right)^{-1/3} = \sum_{k=0}^{\infty} \frac{3}{2}\binom{-1/3}{k}\left(\frac{x}{8}\right)^k = \frac{3}{2} + \frac{3}{2}\binom{-1/3}{1}\left(\frac{x}{8}\right) + \frac{3}{2}\binom{-1/3}{2}\left(\frac{x}{8}\right)^2 + \frac{3}{2}\binom{-1/3}{3}\left(\frac{x}{8}\right)^3$ for $-8<x<8$

$\frac{3}{2}\left(1+\frac{x}{8}\right)^{-1/3} = \sum_{k=0}^{\infty} \frac{3}{2}\binom{-1/3}{k}\left(\frac{x}{8}\right)^k = \frac{3}{2} + \frac{3}{2}\left(-\frac{1}{3}\right)\left(\frac{x}{8}\right) + \frac{3\cdot(-1/3)}{2\cdot(24)}$

$\frac{3}{2}\left(1+\frac{x}{8}\right)^{-1/3} = \sum_{k=0}^{\infty} \frac{3}{2}\binom{-1/3}{k}\left(\frac{x}{8}\right)^k = \frac{3}{2} - \frac{1}{16}x + \frac{1}{192}x^2 \cdots$ for $-8<x<8$

Use an appropriate series to approximate the given value

e $\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

for $-1 \leq x \leq 1$.

Let $x = 1$

$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \cdots$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \cdots$

$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} \cdots$

11.9 Taylor and Maclaurin Series

Use an appropriate series to approximate the given value

π $\tan^{-1}x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $-1 \le x \le 1$

Let $x=1$

$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \cdots$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \cdots$

$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} \cdots$

Use an appropriate series to approximate the given value

$\sqrt{4.1}$

$\sqrt{4+1} = 2\sqrt{1+\frac{x}{4}} = 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$

$(1+x)^{1/2} = 1+\frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - - - \quad -1 < x < 1.$

$2\left(1+\frac{x}{4}\right)^{1/2} = 2+\frac{x}{4} - \frac{1}{4}\left(\frac{x}{4}\right)^2 + \frac{1}{8}\left(\frac{x}{4}\right)^3 + - - -$

$-1 < \frac{x}{4} < 1$

$\sqrt{4+x} = 2+\frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + - - - -$

$-4 < x < 4.$

$x = .1 \quad \sqrt{4.1} = 2 + \frac{1}{40} - \frac{1}{6400} + \frac{1}{512000} + - - - -$

Use an appropriate series to approximate the given value

$$\int_0^1 e^{x^2}\,dx$$

$e^x = \sum \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ for $-\infty < x < \infty$

$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120} + \cdots$

$\int_0^1 e^{x^2}\,dx = \int_0^1 1\,dx + \int_0^1 x^2\,dx + \int_0^1 \frac{x^4}{2}\,dx + \int_0^1 \frac{x^6}{6}\,dx + \cdots$

$= x\Big|_0^1 + \frac{x^3}{3}\Big|_0^1 + \frac{x^5}{10}\Big|_0^1 + \frac{x^7}{42}\Big|_0^1 + \cdots$

$= 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} + \cdots$

Find the sum of the series.

$$5+\frac{4}{2}-\frac{16}{24}+\frac{64}{720}-\ldots = 5+\frac{2^2}{2!}-\frac{2^4}{4!}+\frac{2^6}{6!}-\frac{2^8}{8!}----$$

$$4+\underbrace{1+\frac{2^2}{2!}-\frac{2^4}{4!}+\frac{2^6}{6!}-\frac{2^8}{8!}}----$$

Series for $\cos(2)$.

$4+\cos 2$.