CS206 Problem Set 2

1. NLA 6.1

   SOLUTION:

$$(I - 2P)(I - 2P) = I - 4P + 4P^2 = I$$

2. NLA 6.4

   SOLUTION: Given a matrix A, the orthogonal projector onto the range of A can be expressed by the formula:

$$P = A(A^*A)^{-1}A^*$$

   Using this formula, we have:

   (a) The orthogonal projector onto range(A) is

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

   The image under P of the vector $(1, 2, 3)^*$ is $(2, 2, 2)^*$.

   (b) The orthogonal projector onto range(B) is

$$P = \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}$$

   The image under P of the vector $(1, 2, 3)^*$ is $(2, 0, 2)^*$.

3. NLA 7.1

   SOLUTION:

(a) The QR factorization of A is

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(a) The QR factorization of B is

$$B = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

4. Determine (on paper) classical and modified Gram-Schmidt orthogonalization for the vectors

$$a_1 = (1, \epsilon, 0, 0)^T, \quad a_2 = (1, 0, \epsilon, 0)^T, \quad a_3 = (1, 0, 0, \epsilon)^T$$

During your calculation, make the approximation $1 + \epsilon^2 \approx 1$.

SOLUTION:

Assuming $\epsilon \to 0$:

(a) Classical GS: The resulting Q matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ \epsilon & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

(b) Modified GS: The resulting Q matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ \epsilon & -1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

Check orthogonality:

(a) Classical GS: $< q_2, q_3 >= 1/2$

(b) Modified GS: $< q_2, q_3 >= 0$

5. NLA 7.3

   SOLUTION:

   Based on full QR factorization of A, $|det(A)| = |det(Q)det(R)| = |det(R)| = \prod_{j=1}^{m} r_{jj}$.

   In addition, we note that $a_j = \sum_{i=1}^{j} r_{ij} q_i$. Therefore, $\|a_j\|_2 \geq r_{jj}$, by triangular inequality, for all $j = 1, \cdots, m$. And consequently, we have $|det(A)| = \prod_{j=1}^{m} r_{jj} \leq \prod_{j=1}^{m} \|a_j\|_2$.

6. NLA 8.2

7. Apply the `[Q,R]=mgs(A)` function you have written in the previous problem to the following matrix

   $$A = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$$

   Check the orthogonality of `Q` matrix by calculating `norm(Q'*Q-eye(2))`. Compare the value returned by `mgs` *vs* the one returned by the `qr` function in Matlab.

8. NLA 10.1

   SOLUTION: The householder reflector $F = I - 2qq^*$ with $q = v/\|v\|_2 \in C^m$. The eigenvalues of F are: 1 with multiplicity $m-1$ - the corresponding eigenspace is the hyperplance orthogonal to q, and -1 with eigenvector q. The determinent of F is -1. F has m singular values, all being 1.

9. NLA 10.2

10. NLA 10.3