2.3 (a). A is hermitian $\Rightarrow A = A^*$.

$$Ax = \lambda x \Rightarrow (Ax)^* = (\lambda x)^* \Rightarrow x^* A^* = \lambda^* x^* \Rightarrow x^* A = \lambda^* x^* \Rightarrow x^* A x = \lambda^* x^* x$$

$$\Rightarrow x^* \lambda x = \lambda^* x^* x \Rightarrow \lambda x^* x = \lambda^* x^* x \Rightarrow (\lambda - \lambda^*)(x^* x) = 0.$$

Since x is nonzero vector, $x^* x$ is real and $x^* x \neq 0$.

Thus $\lambda - \lambda^* = 0$, which yields λ is real.

(b). Let $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$ and $\lambda_1 \neq \lambda_2$.

Then $x_1^* A x_2 = x_1^* \lambda_2 x_2 = \lambda_2 x_1^* x_2$,

also, $x_1^* A x_2 = x_1^* A^* x_2 = (A x_1)^* x_2 = (\lambda_1 x_1)^* x_2 = \lambda_1^* x_1^* x_2 = \lambda_1 x_1^* x_2$.

Thus $\lambda_2 x_1^* x_2 = \lambda_1 x_1^* x_2$

$\Rightarrow (\lambda_2 - \lambda_1)(x_1^* x_2) = 0$

$\lambda_2 \neq \lambda_1$

$\Rightarrow x_1^* x_2 = 0$, which means x_1 and x_2 are orthogonal.

2.4. The absolute of any eigenvalue of a unitary matrix is one.

Proof. Let A be the unitary matrix, i.e., $A^* A = A A^* = I$. Let $Ax = \lambda x$.

Then $x^* x = x^* I x = x^* A^* A x = (Ax)^* Ax = (\lambda x)^* (\lambda x) = (\lambda^* \lambda) x^* x = |\lambda|^2 x^* x$

(x is nonzero vector)

$\Rightarrow |\lambda|^2 = 1. \Rightarrow |\lambda| = 1$.

2.6. If A is nonsingular, we compute

$$A(I + \alpha u v^*) = (I + uv^*)(I + \alpha u v^*) = I + (1 + \alpha + \alpha v^* u) u v^* = I + (1 + \alpha(1 + v^* u)) u v^*.$$

If $1 + v^* u \neq 0$, we can set $\alpha = -\dfrac{1}{1 + v^* u}$, then we have $A(I + \alpha u v^*) = I$, i.e., $A^{-1} = I + \alpha u v^*$.

When $v^* u = -1$, A is singular. We can show this by proof by contradiction.

Assume A is nonsingular when $v^* u = -1$.

Then $A^2 = (I + uv^*)(I + uv^*) = I + 2uv^* + u(v^* u) v^* = I + 2uv^* + (v^* u) u v^* = I + uv^* = A$

This follows $A = I$, since A is nonsigular. Hence, $uv^* = 0$, which follows $u = 0$ or $v = 0$. This means $v^* u = 0$, which contradicts with $v^* u = -1$.

Thus, when $v^* u = -1$, A is sigular.

For any $\beta \in C$, $A(\beta u) = (I + uv^*)(\beta u) = \beta u + \beta u v^* u = \beta u + \beta u \cdot (-1) = 0$, $\Rightarrow$ span$\{u\} \subset$ null(A).

For any $x \in$ null(A), $Ax = 0 \Rightarrow (I + uv^*) x = 0 \Rightarrow x + u(v^* x) = 0$, $\Rightarrow x = -(v^* x) u$, $\Rightarrow x \in$ span$\{u\}$.

Thus, null$(A) =$ span$\{u\}$.

3.2 Proof: Let λ be an eigenvalue of matrix A, and x be the corresponding eigenvector.

Thus $Ax = \lambda x$.

Here we form a matrix $X = [x|x|\cdots|x] \in \mathbb{C}^{m\times m}$,

Then $AX = \lambda X$,

Thus $|\lambda|\|X\| = \|\lambda X\| = \|AX\| \leq \|A\|\,\|X\|$

Since $\|X\| > 0$, (because each column of X is nonzero vector).

So $|\lambda| \leq \|A\|$,

This holds for every eigenvalue of matrix A.

Thus, $\rho(A) \leq \|A\|$.

4.1 (a) $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ is a diagnal matrix,

$$\therefore A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $(A = IAI)$.

(c). $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

So, we have $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $AV = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$AV = U\Sigma \Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = [u_1, u_2, u_3]\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow U = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, $A = U\Sigma V^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A^*A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, Let $\det(A^*A - \lambda I) = 0$, $\Rightarrow (1-\lambda)^2 - 1 = 0 \Rightarrow \lambda = 2$ or 0.

$$\Rightarrow A^*A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \Rightarrow \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, AV = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}.$$

$$AV = U\Sigma \Rightarrow \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} = U\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, $A = U\Sigma V^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A^* = A$, so the eigenvectors of A form an othorgonal space.

Let $\det(A - \lambda I) = 0$, $\Rightarrow \lambda_1 = 2, \lambda_2 = 0$. with eigenvectors $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ and $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$. $P = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ is unitary

Then, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$. $\left(AP = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow A = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}P^*\right)$