Perception: Projective Transformations and Vanishing Points

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Projective Transformation

- Aka Homography or Collineation
- Represents the perspective projection from a ground plane to an image plane !
- It is an invertible 3x3 matrix but has 8 independent parameters
- For example if (X,Y) measured in meters on the ground and (u,v) in pixels

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \sim H \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}$$

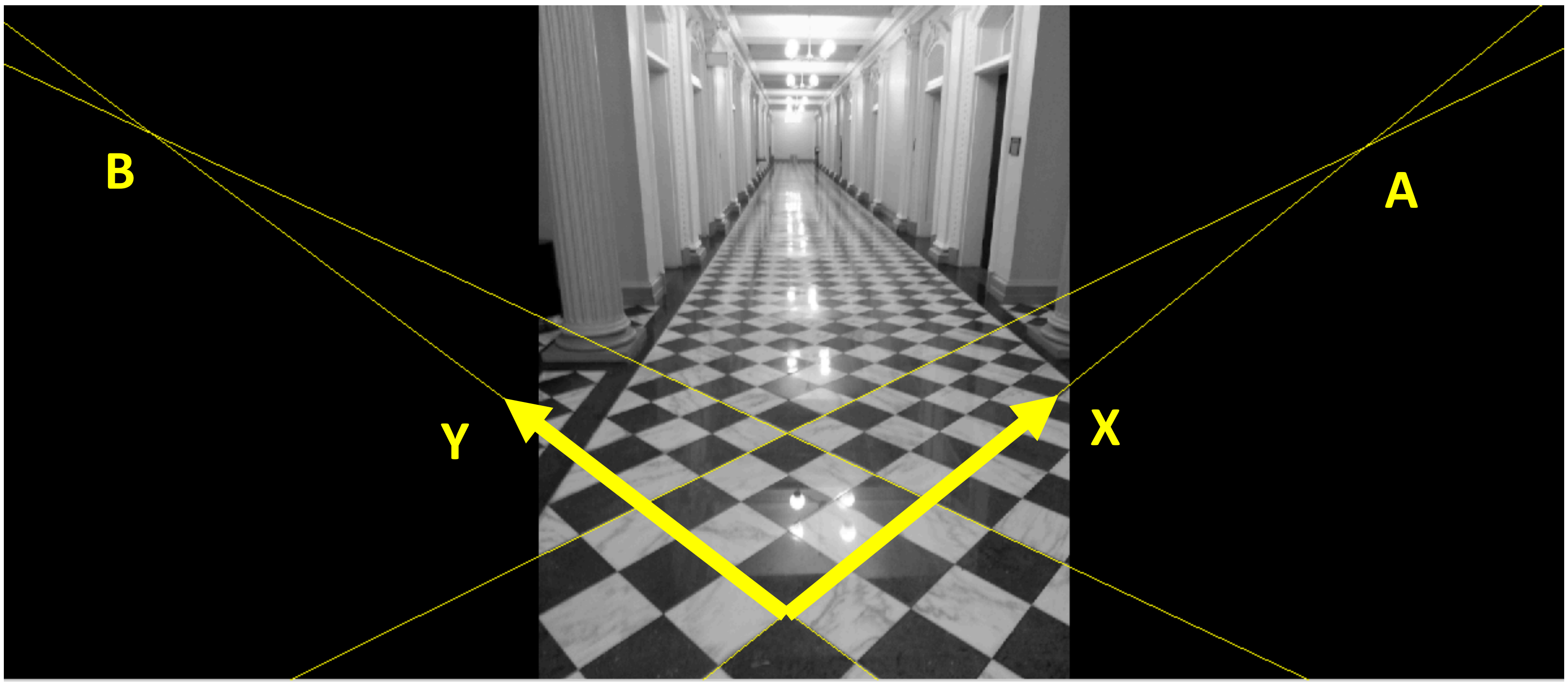
or

$$\begin{aligned} u &= \frac{H_{11}X + H_{12}Y + H_{13}}{H_{31}X + H_{32}Y + H_{33}} \\ v &= \frac{H_{21}X + H_{22}Y + H_{23}}{H_{31}X + H_{32}Y + H_{33}} \end{aligned}$$

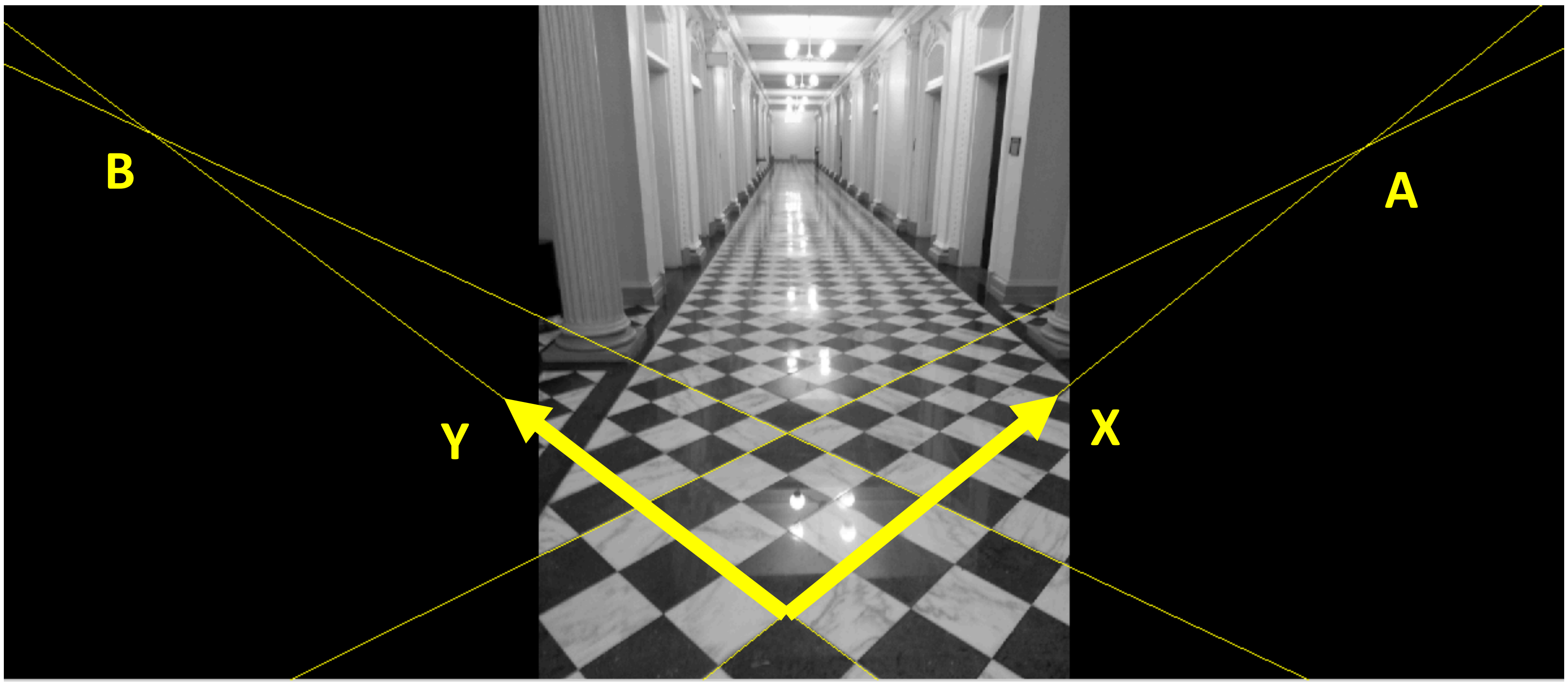
If we know H, we can make back and forth measurements:

10.2,2.3 in meters

Homography columns as vanishing points

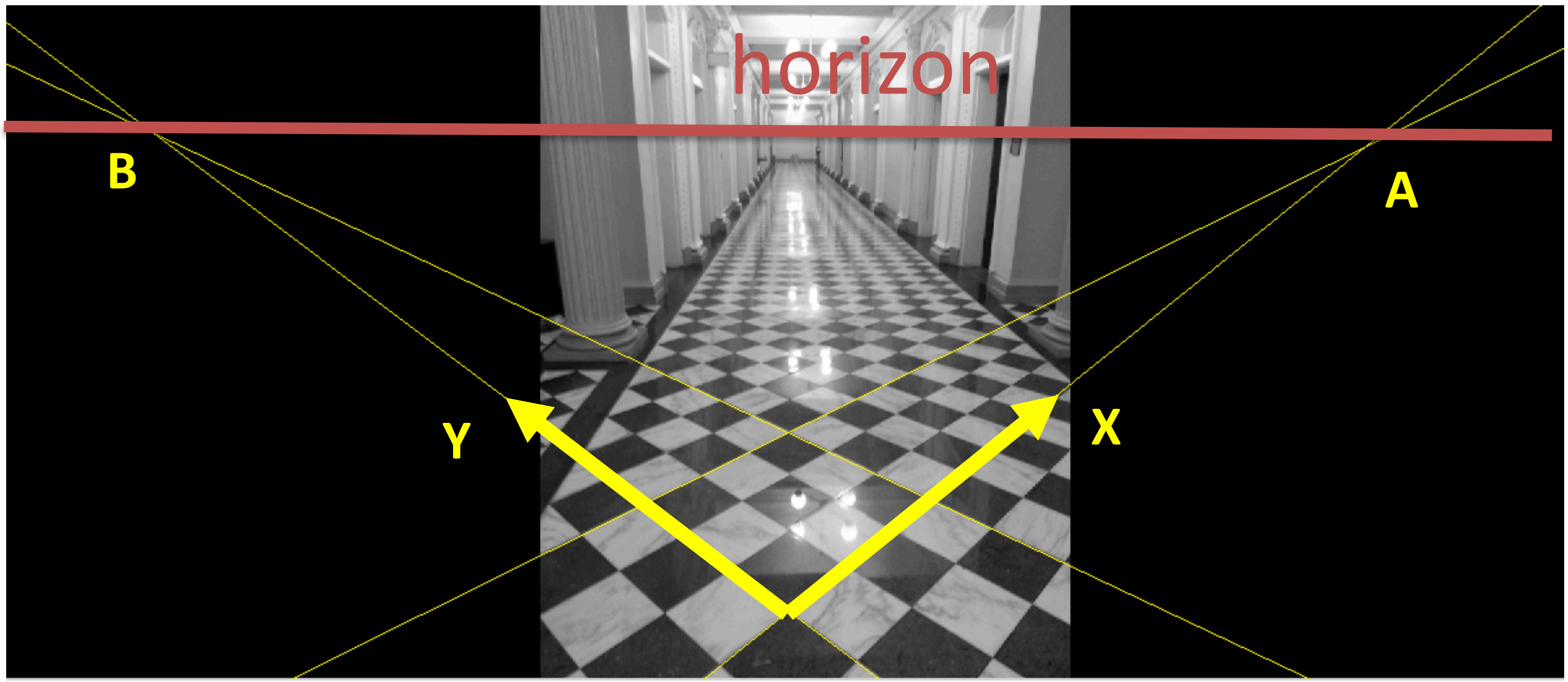


If $H = \begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix}$ then $h_1 \sim A$ and $h_2 \sim B$.

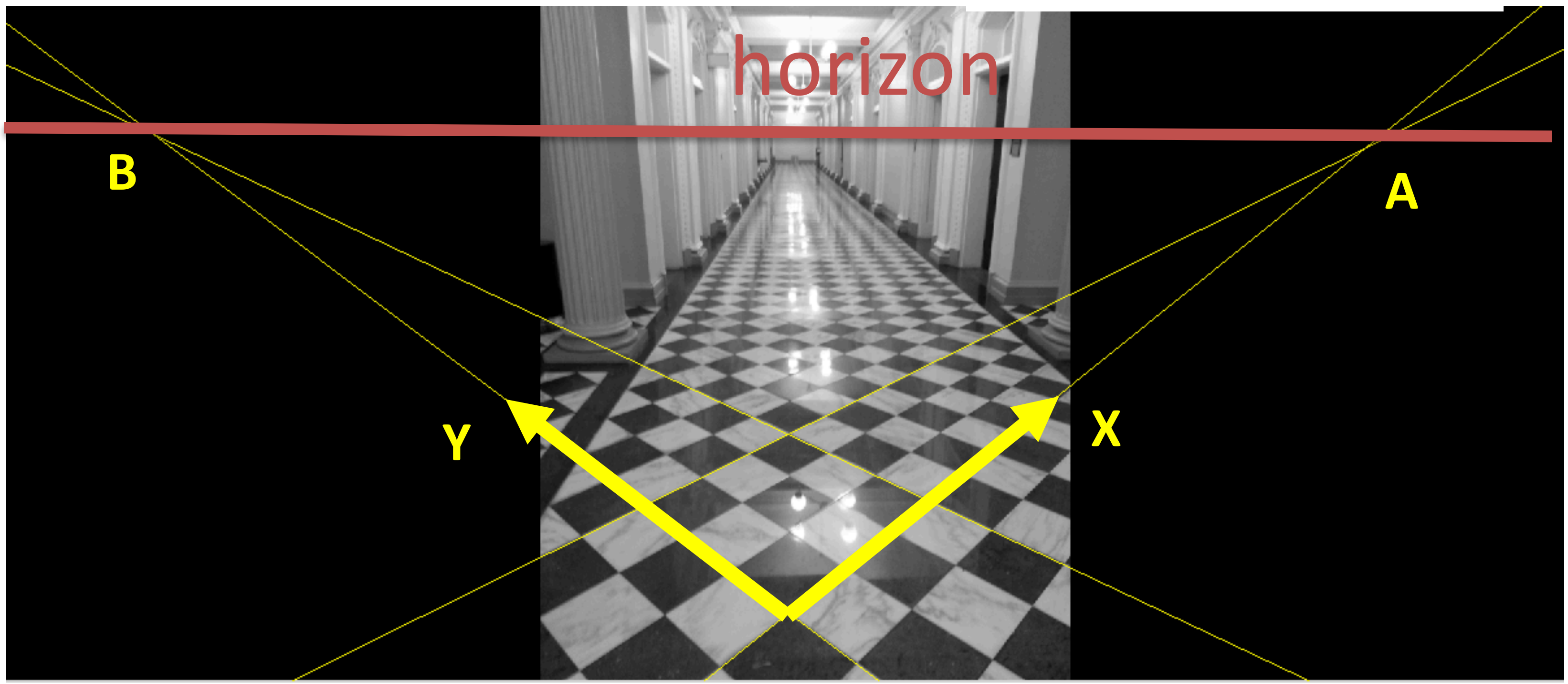


So the first two columns are two Orthogonal vanishing points

If we connect two vanishing points we obtain the horizon!

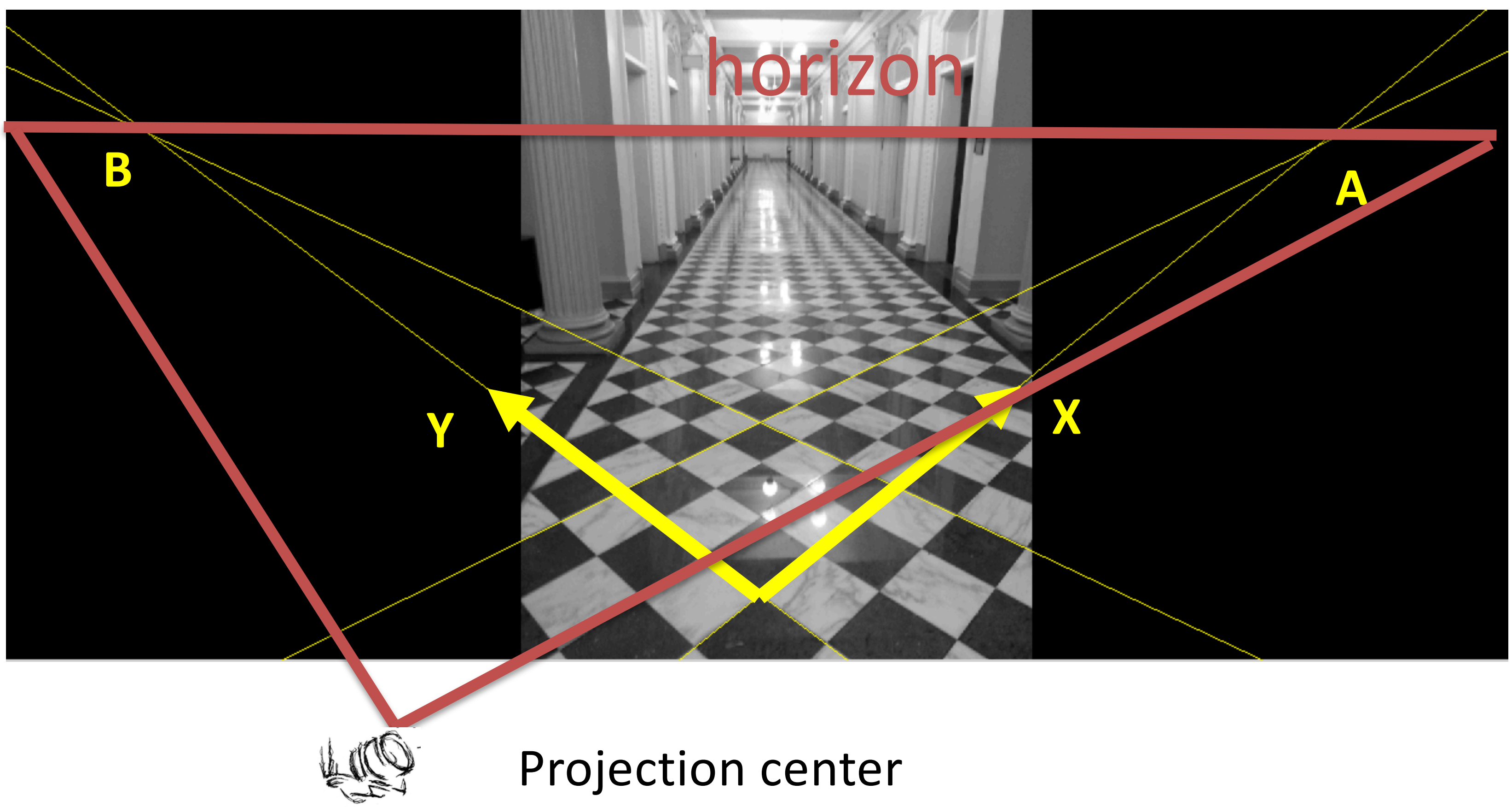


Equation of horizon: $$(h_1 \times h_2)^T \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$



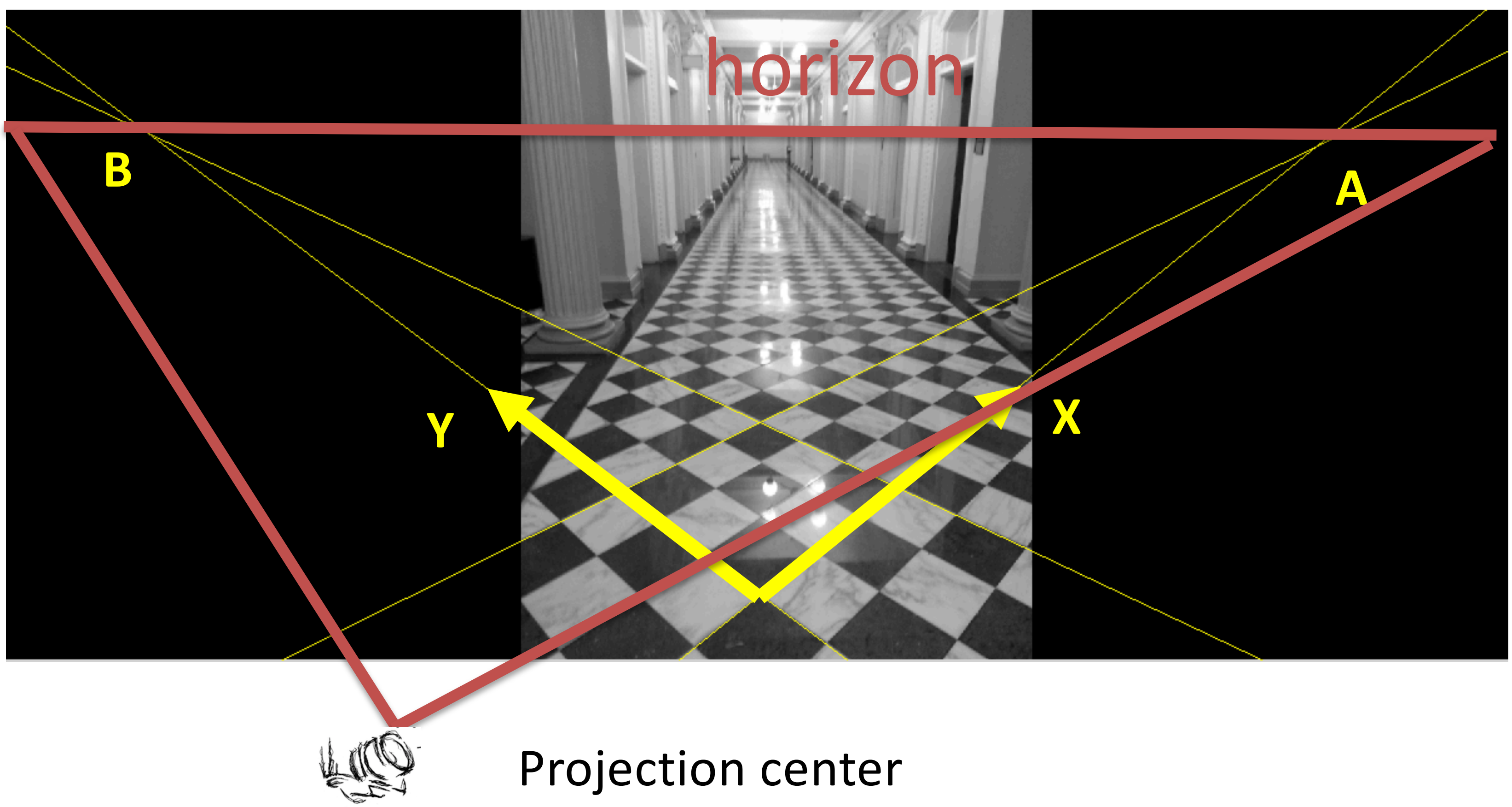
Horizon with projection center build a horizon plane with normal

$$(h_1 \times h_2)$$



The horizon plane is parallel to the ground plane and hence

$(h_1 \times h_2)$ is the normal to the ground plane expressed via pixels!



Horizon gives complete info about how ground plane is oriented!
If horizon is horizontal the center it means that optical axis is parallel to groundplane!

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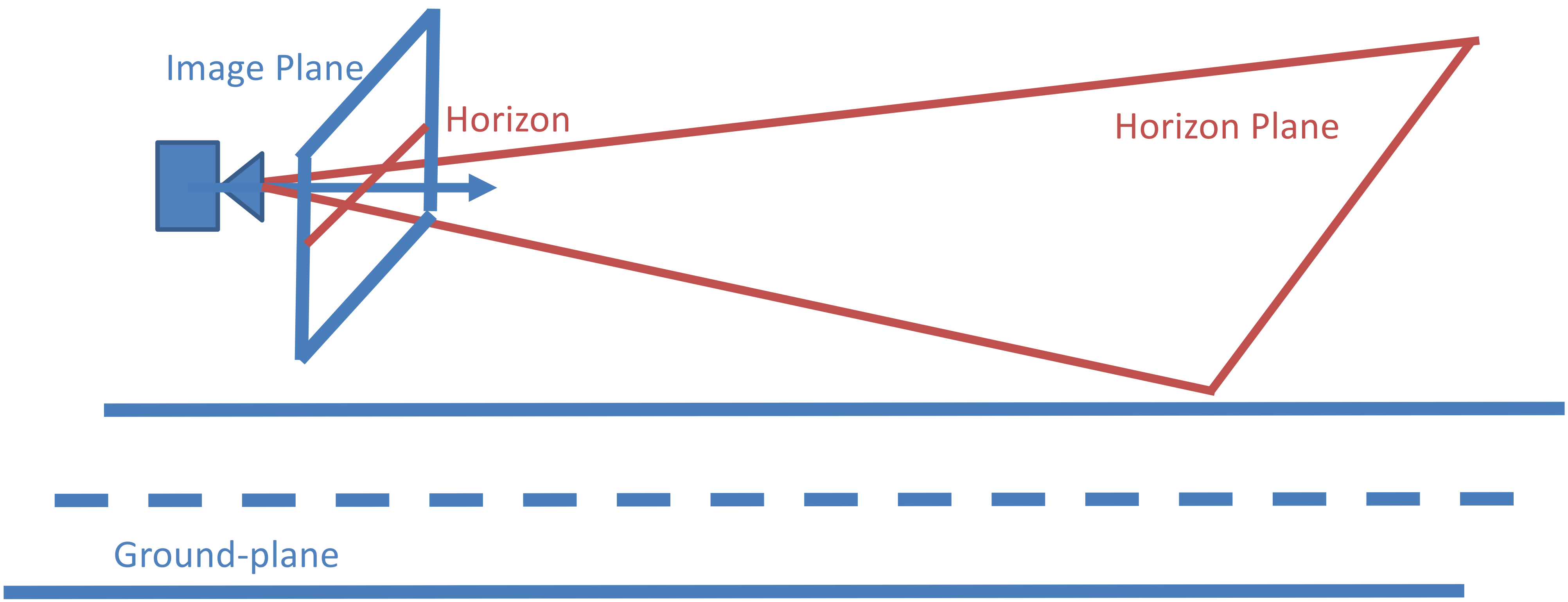
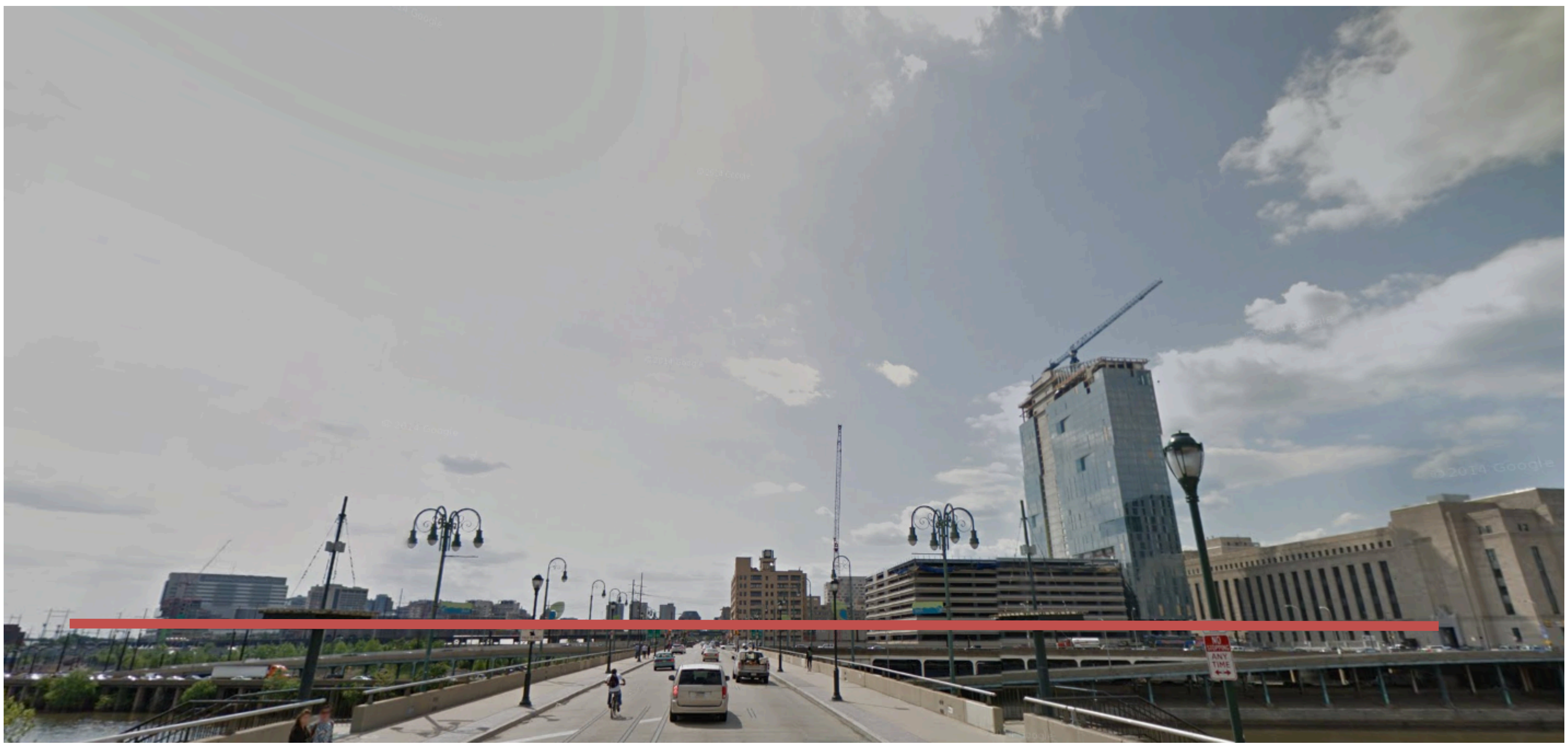


If horizon moves to the bottom it means we look upwards!



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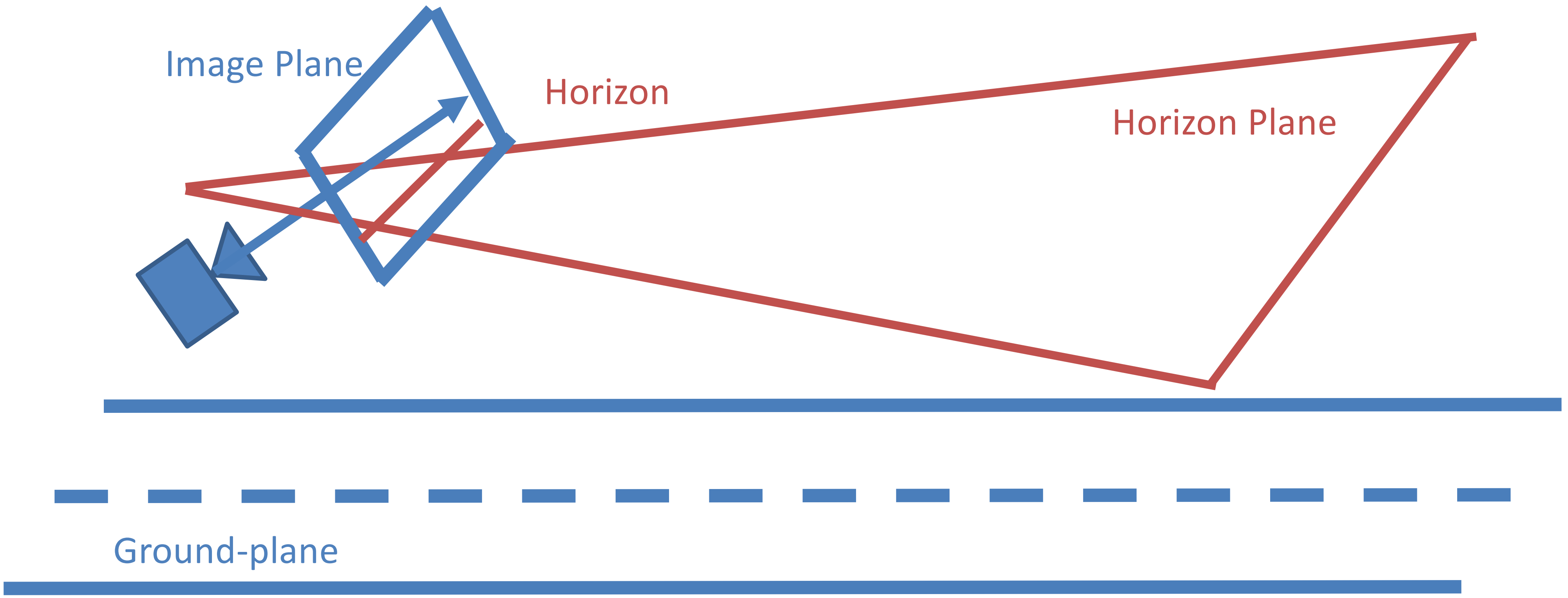
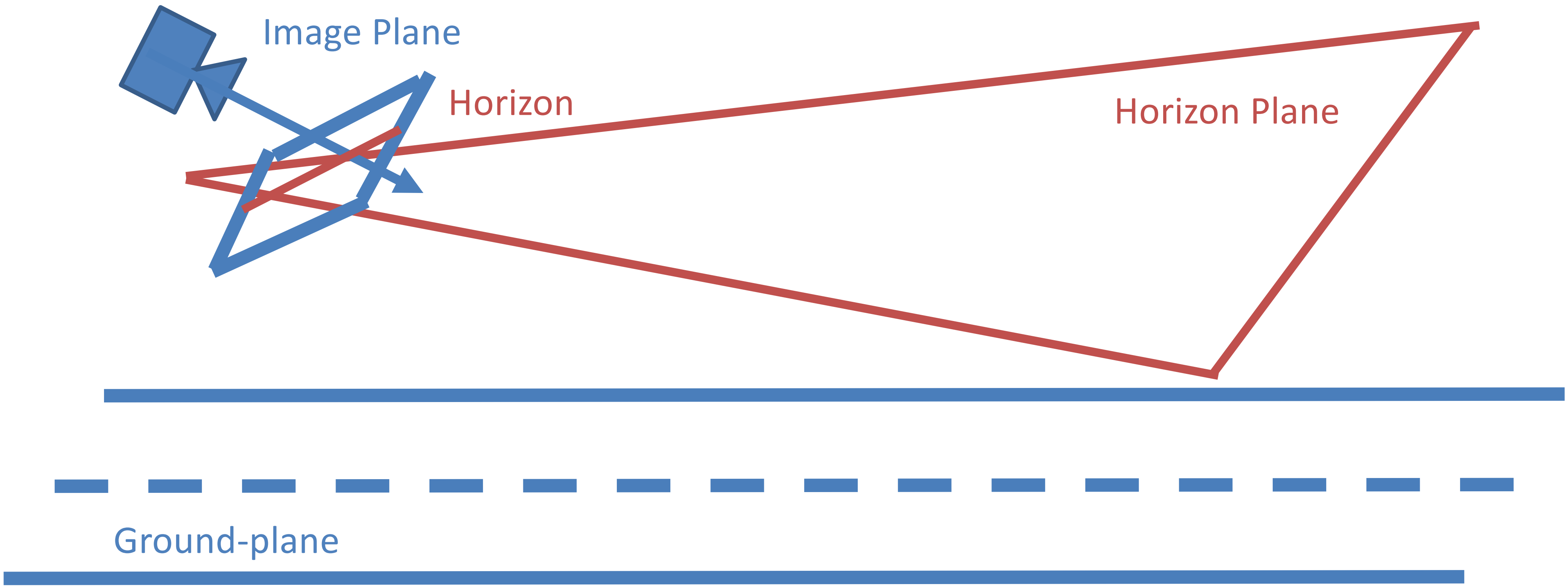
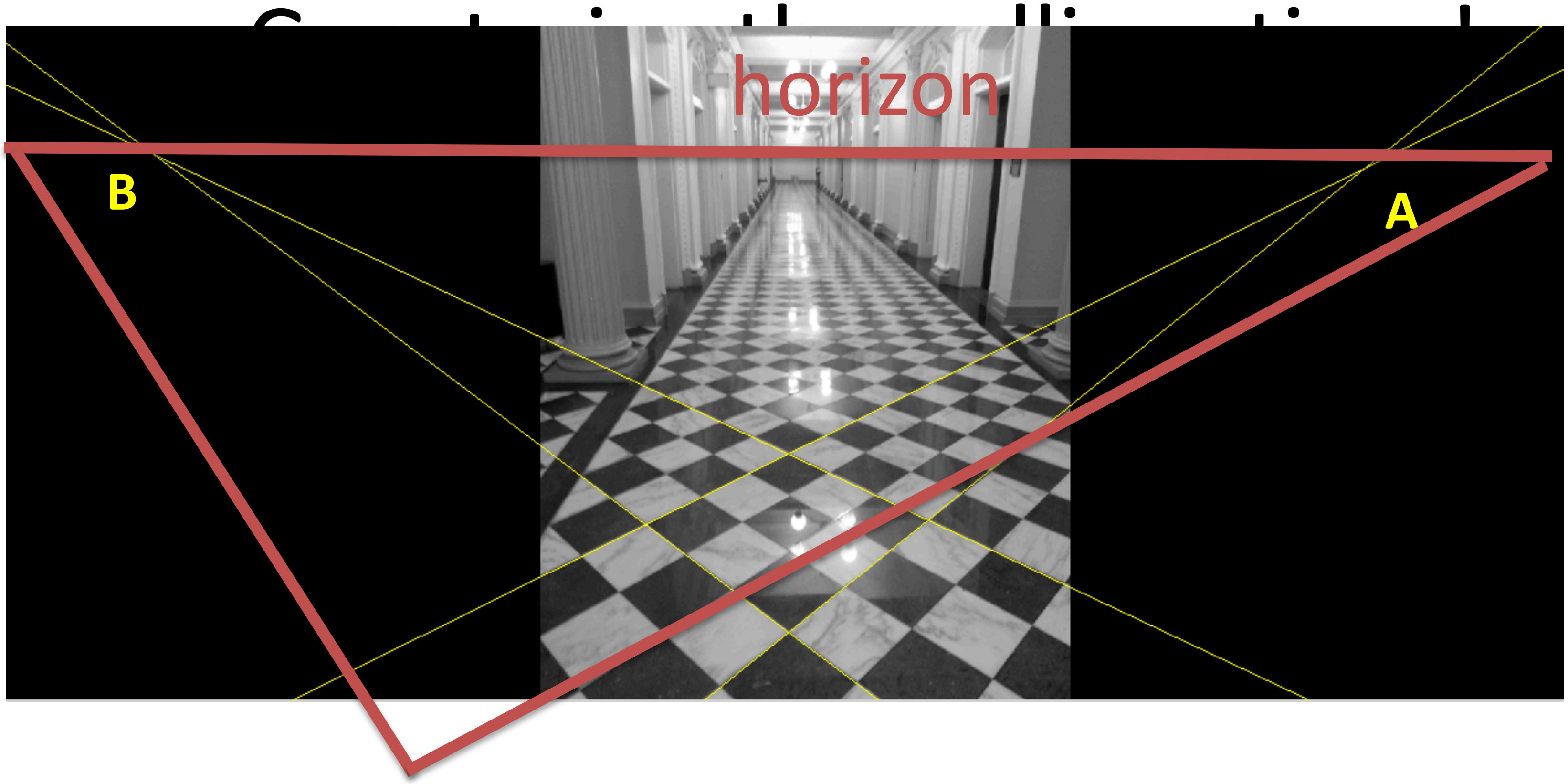


If horizon moves to the top it means we look downwards !

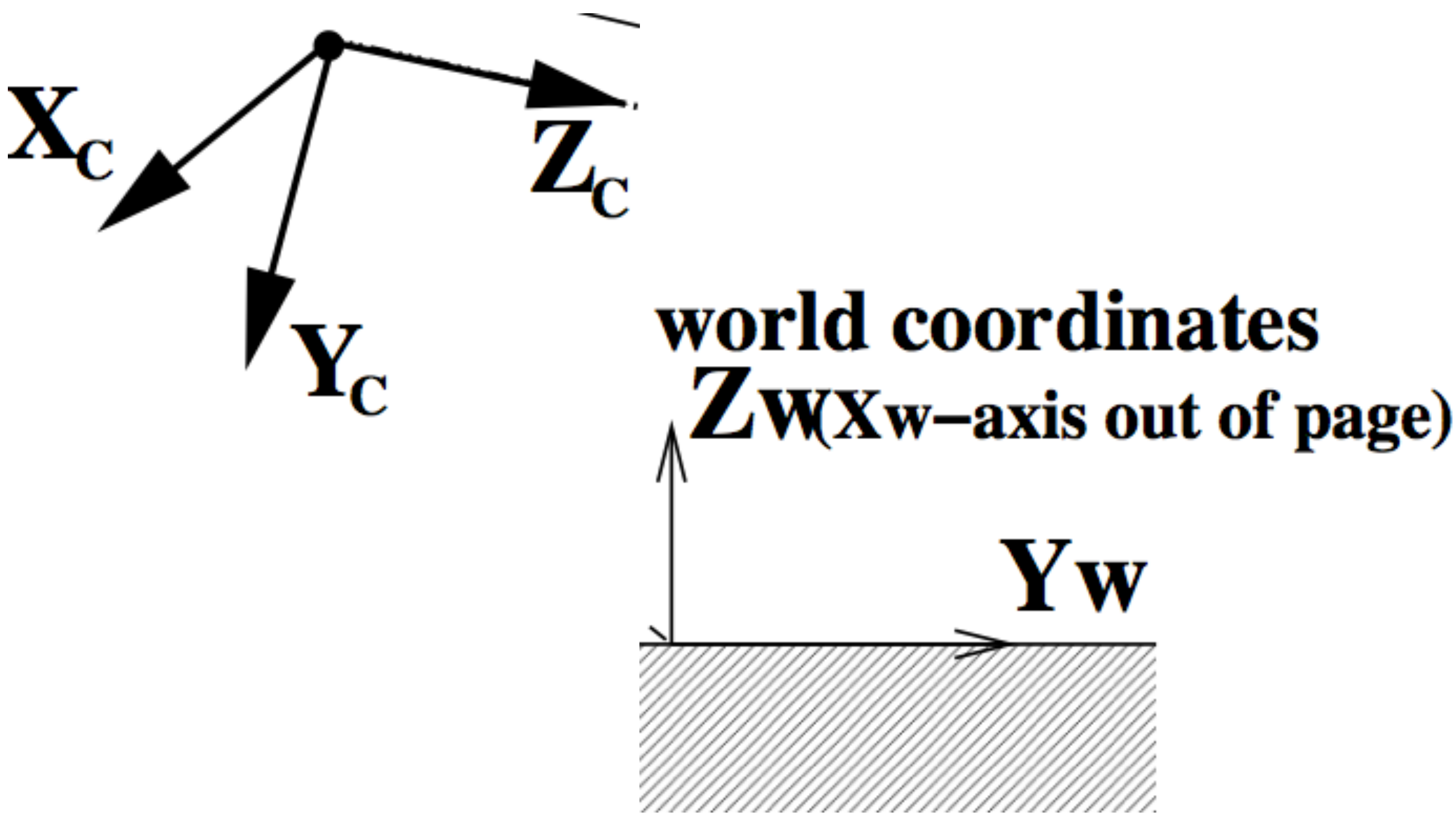
If horizon moves to the bottom it means we look upwards!



Horizon tells us how camera is oriented.



Collineation from plane Z_w=0



Pose from Projective Transformation

Recall the projection from world to camera

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim K \begin{pmatrix} r_1 & r_2 & r_3 & T \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}$$

and assume that all points in the world lie in the ground plane $Z = 0$.

Then the transformation reads

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim K \begin{pmatrix} r_1 & r_2 & T \end{pmatrix} \begin{pmatrix} X \\ Y \\ W \end{pmatrix}$$

What information does H reveal about VP?

H is a transformation from $\mathbb{P}^2$ to $\mathbb{P}^2$:

$$H \sim K \begin{pmatrix} r_1 & r_2 & T \end{pmatrix}$$

- First column is projection of world (1,0,0) ~ Kr_1
- Second column is projection of world (0,1,0) ~ Kr_2
- The line connecting them is $Kr_1 \times Kr_2$

Projective transformation of lines

$\boxed{\text{If } A \text{ maps a point to } Ap\text{, then where does a line } l \text{ map to?}}$

Line equation in original plane

$$l^T p = 0$$

Line equation in image plane $p' \sim Ap$

$$l^T A^{-1} p' = 0$$

implies that $l' = A^{-T} l$.

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Projection of W=0

Recall projections of lines are $H^{-T} l = H^{-T} (0,0,1)'$

If H = (h1 h2 h3) then H^{-T} is (h2xh3 h3xh1 h1xh2)

so horizon line has coefficients is h1xh2

which was expected because it connects the two vanishing points h1 and h2.

Perception:
How to compute intrinsics from vanishing points

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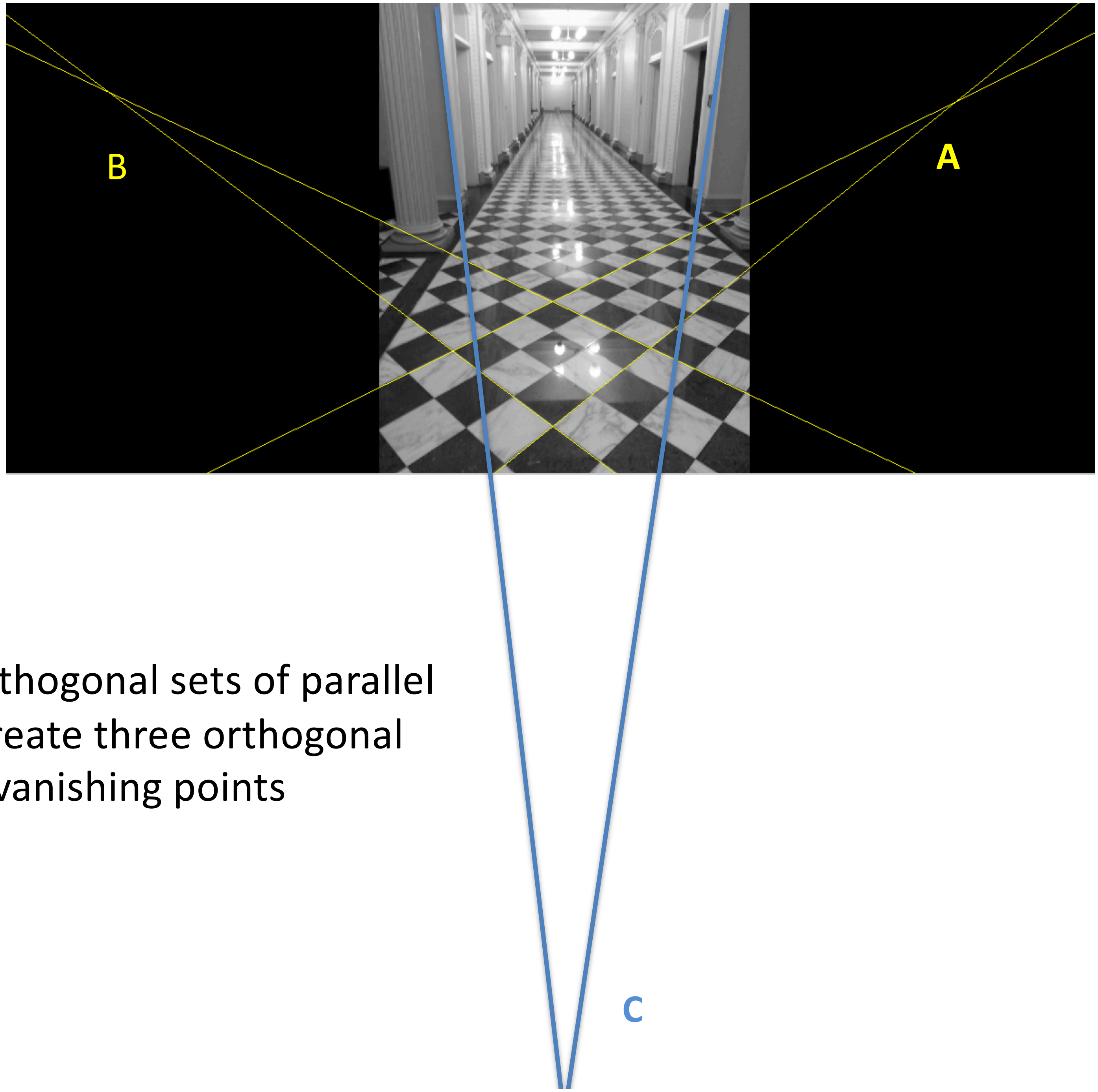
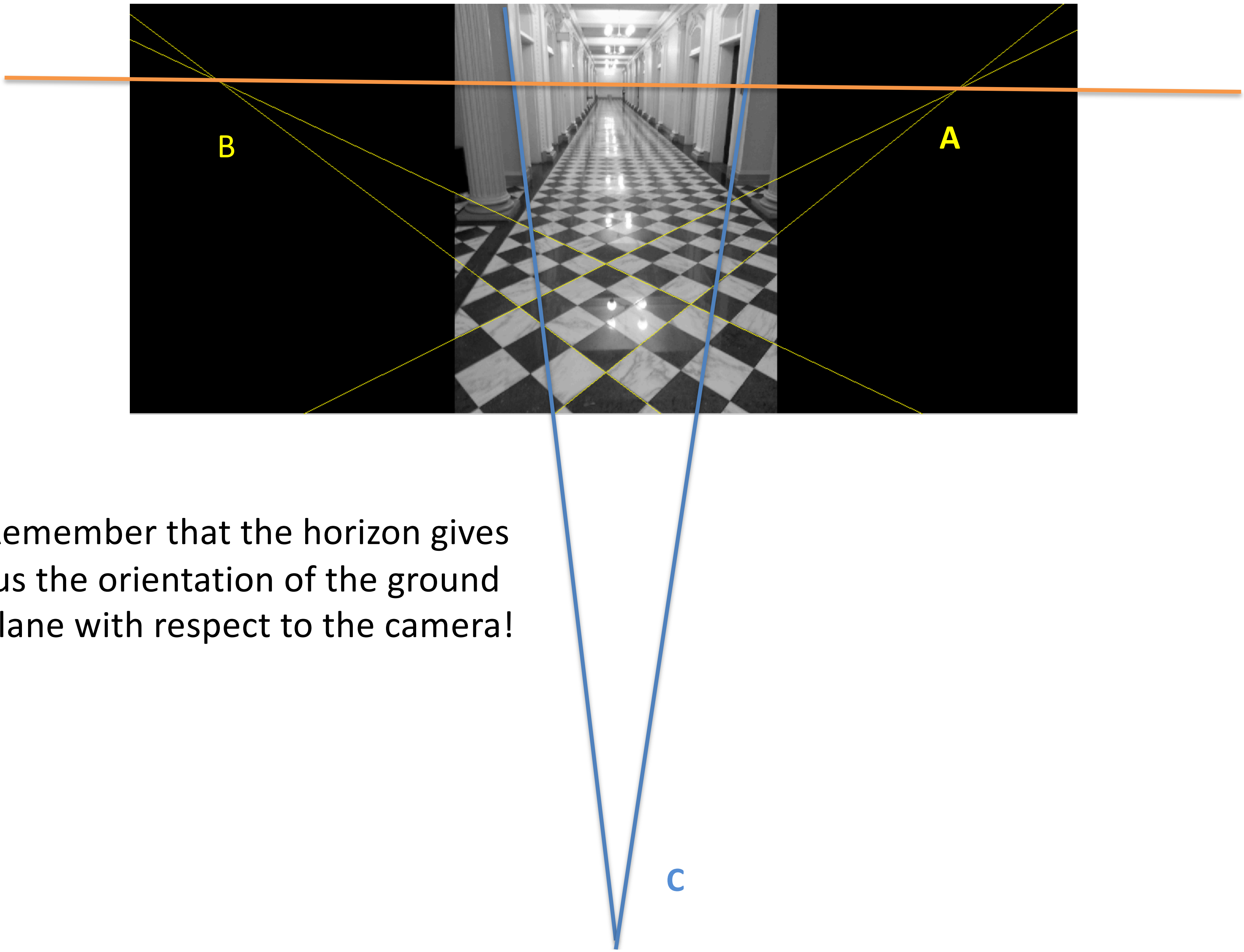
Manhattan world: A scene with three orthogonal sets of parallel lines



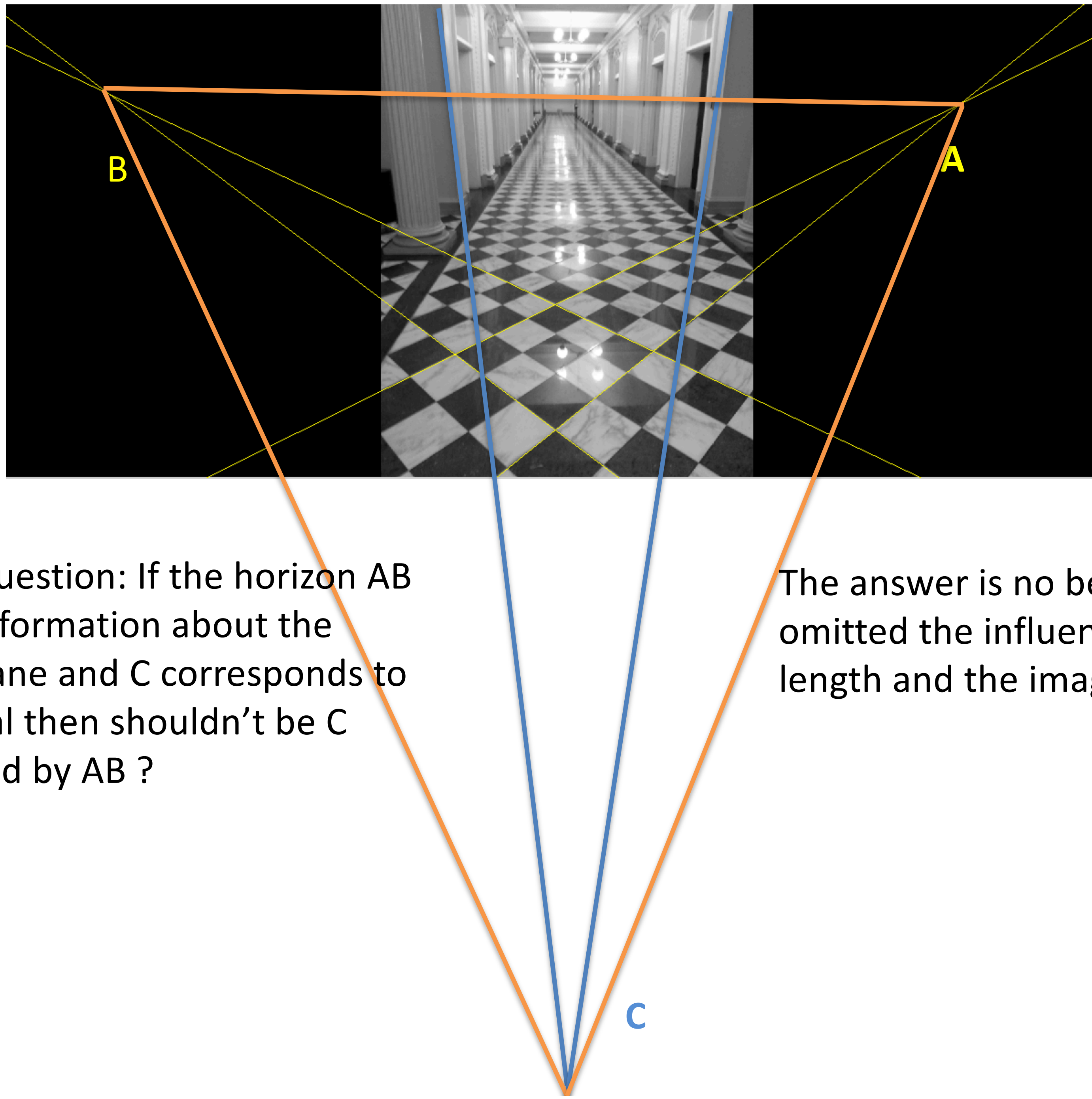
Three orthogonal sets of parallel lines create three orthogonal vanishing points

Line connecting AB is the horizon!



Remember that the horizon gives us the orientation of the ground plane with respect to the camera!

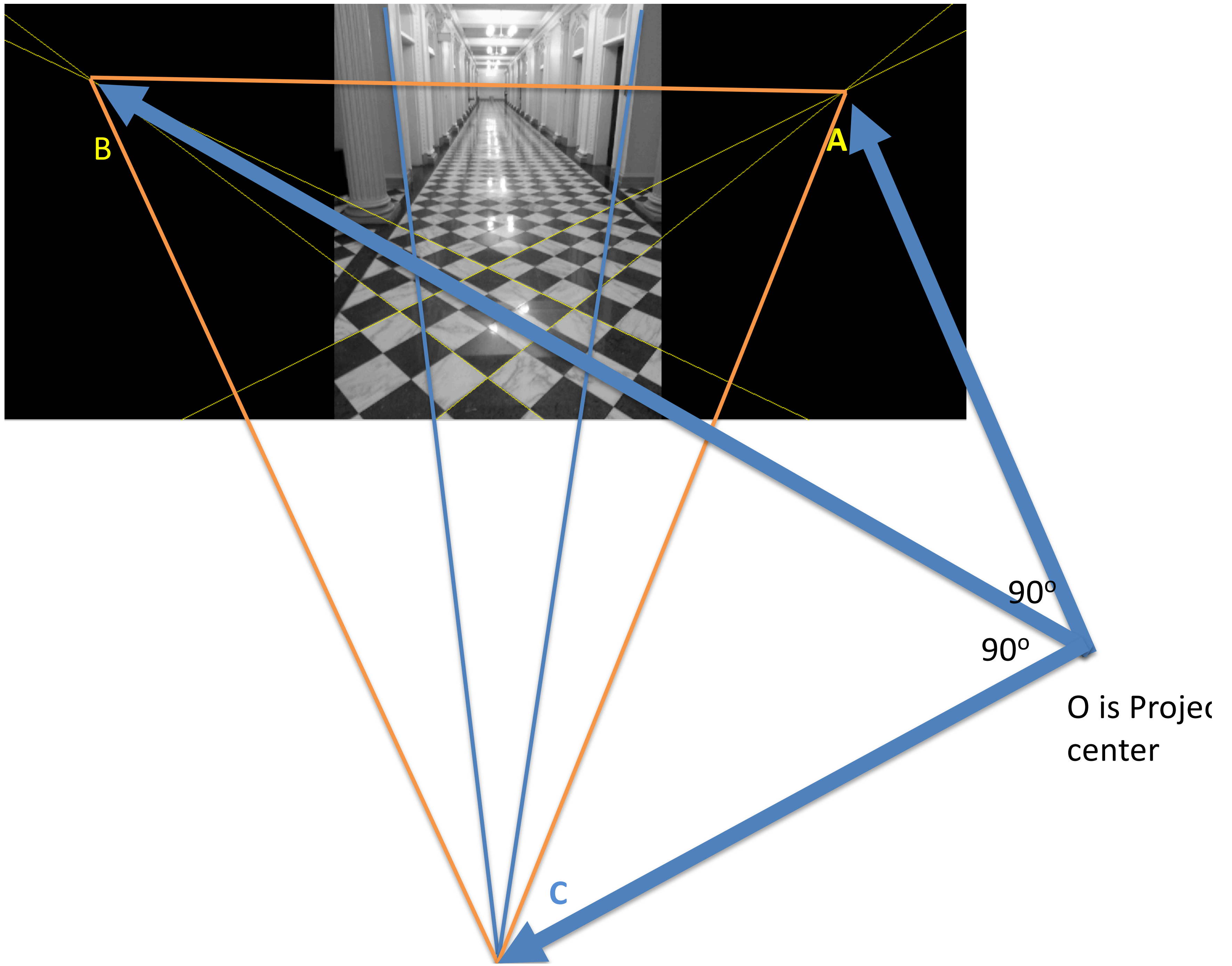
C is the vertical vanishing point!



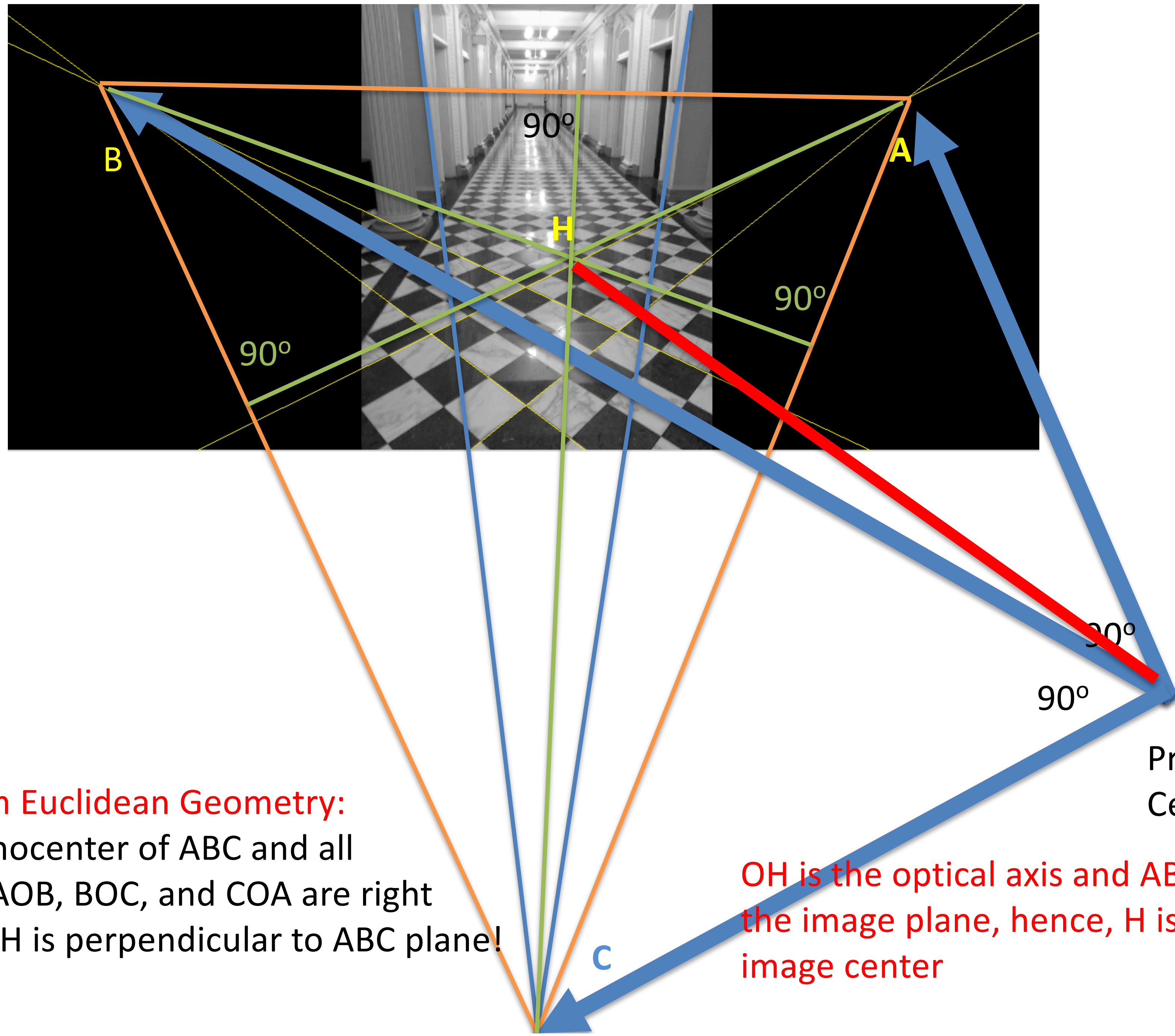
Obvious question: If the horizon AB gives us information about the ground plane and C corresponds to the vertical then shouldn't be C determined by AB ?

The answer is no because we omitted the influence of the focal length and the image center.

Let's look at ABC as a tetrahedron OABC incl the projection center



Let H be the orthocenter of the triangle ABC



Theorem from Euclidean Geometry:
If H is the orthocenter of ABC and all three angles AOB, BOC, and COA are right angles , the OH is perpendicular to ABC plane!

OH is the optical axis and ABC is the image plane, hence, H is the image center

THEOREM: The image center (u_0, v_0) is the orthocenter of the triangle formed by the projections of three orthogonal vanishing points.

PROOF: See figure 2. Let $C = (u_0, v_0, 1)$ denote the homogeneous coordinates of the image center: it is defined as the intersection of image plane $V_1V_2V_3$ with the optical axis, which is the line through O and perpendicular to $V_1V_2V_3$.

$$OC \perp V_1V_2V_3 \Rightarrow OCV_1 \perp V_1V_2V_3$$
$$OV_1 \perp OV_2 \Rightarrow OCV_1 \perp \text{any line contained in } V_1V_2V_3$$

In particular $OCV_1 \perp V_2V_3$. Moreover, $OV_1 \perp OV_2$, therefore $\boxed{OCV_1 \perp OV_2V_3}$.

When two planes are perpendicular, their intersections with a third plane are also perpendicular. Therefore, the intersection of OCV_1 with $V_1V_2V_3$ is perpendicular to intersection of OV_2V_3 with $V_1V_2V_3$.

In other words $V_1C \perp V_2V_3$. A similar reasoning leads to $V_2C \perp V_3V_1$ and $V_3C \perp V_1V_2$, therefore C is the orthocenter of $V_1V_2V_3$.

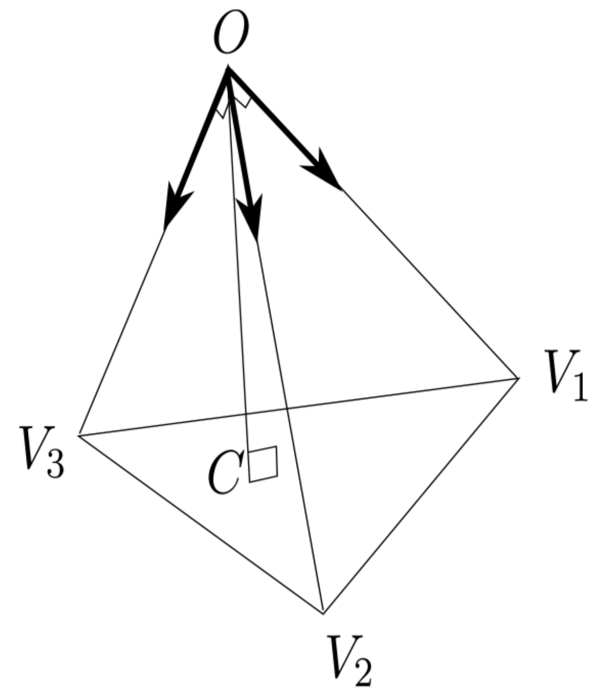
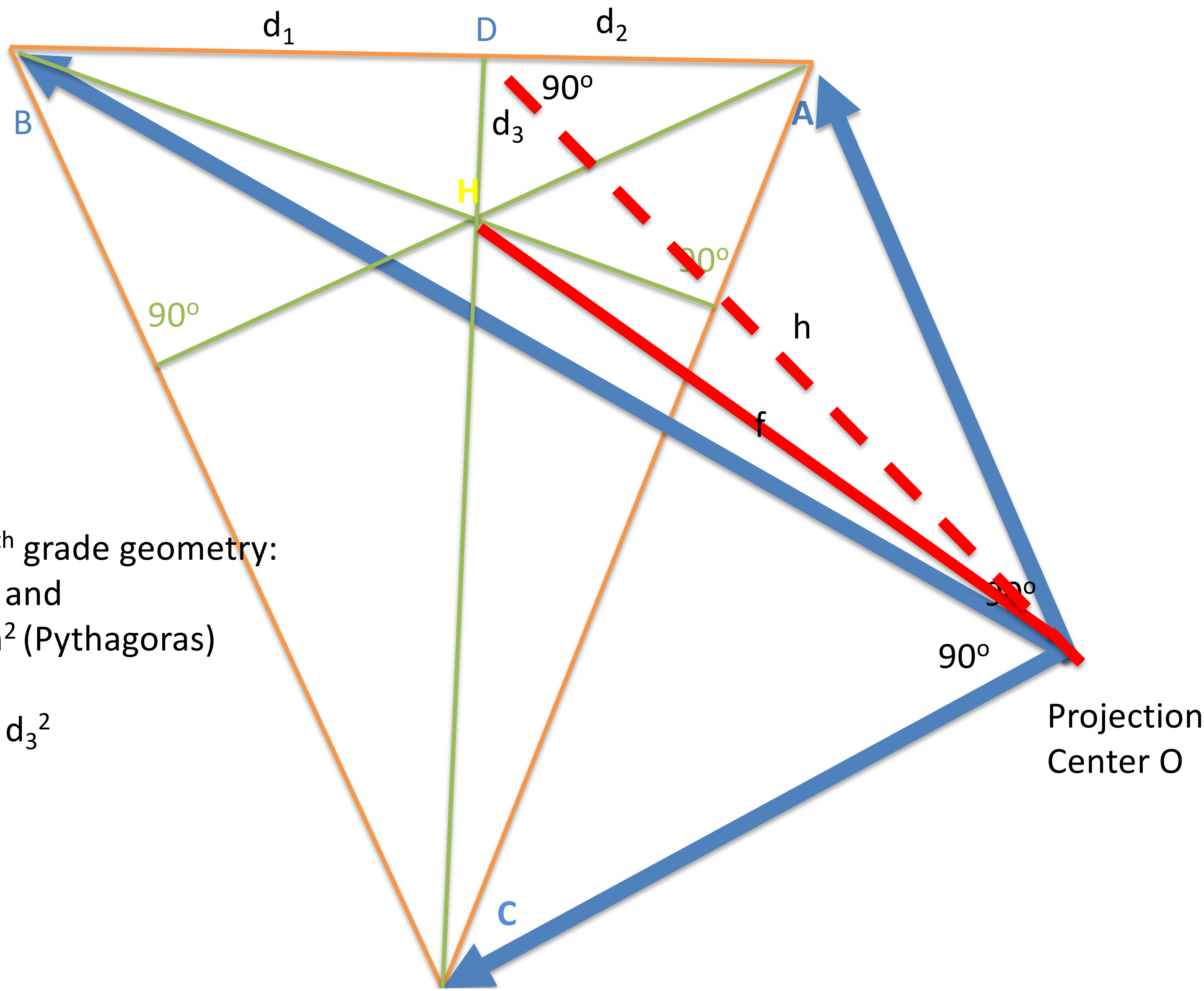


Figure 2: Three orthogonal vanishing points and image center.

We used the fact that the two non-parallel lines OV_2 and V_2V_3 contained in the plane OV_2V_3 are perpendicular to OCV_1, therefore $OV_2V_3 \perp OCV_1$.

We found the image center! What about the focal length (f=OH)?
Can it be computed from A,B, and C ?



Simple 9^{th} grade geometry:
$h^2 = d_1 d_2$ and
$f^2+d_3^2 = h^2$ (Pythagoras)
Hence
$f^2= d_1 d_2 - d_3^2$

Three orthogonal vanishing points allow computation of focal length and image center !