Structure from Motion III

$$q_i^T(\underbrace{T\times}R p_i) = 0 \quad \text{epipolar constraint}$$

essential $E = \hat{T}R$ where

$$\hat{T}a = T\times a$$

$$q_i^T E p_i = 0$$

given: (p_i, q_i) unknown: E 3×3

$$\boxed{\text{fundamental} \quad q_{pix,i}^T F p_{pix,i} = 0 \qquad F = K_q^{-T} E K_p^{-1}}$$

$$\underbrace{\begin{pmatrix} p_x q^T & p_y q^T & p_z q^T \end{pmatrix}}_{n\times 9}^{A} \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}}_{9\times 1} = 0$$

$e_1 \rightarrow$ 1st col of E

$A = USV^T$ $\quad S = \mathrm{diag}(\sigma_i)$

solution $V(:9)$

better check if indeed

$\mathrm{rank}(A) = 8$: $\sigma_8 \gg \sigma_9 \approx 0$

next question : $E \overset{?}{\Rightarrow} T, R$

what can we do just with E ?

1. 8-point RANSAC

2. constrain the correspondence problem because if you know E then a point q is a match to a point p iff q on the epipolar line $q^T E p = 0$ $\left(\begin{array}{l}\text{line coefficients} \\ = Ep\end{array}\right)$

$$E \overset{?}{\Longrightarrow} T,\ R$$

remember $\|T\| = 1$

$E = \hat{T}R$ ← 5 unknowns

9 equations

$\Rightarrow$ There must be a constraint among the E-elements or some property that E must satisfy.

Properties of E-matrix

① $E^T T = (\hat{T}R)^T T = R^T \hat{T}^T T$

$(AB)^T = B^T A^T$ $\qquad = -R^T \hat{T} T = 0$

$\hat{A}^T = -\hat{A} \qquad \hat{A}A = A \times A = 0$

$\Rightarrow E^T$ singular because $\|T\| = 1$

$\Rightarrow E$ singular

$$\Rightarrow \boxed{\begin{array}{l} \det(E) = 0 \\ \text{rank}(E) < 3 \end{array}}$$

A is singular $\overset{\text{def}}{\Longleftrightarrow} \exists x \neq 0 \; Ax = 0$

(2) $EE^T = \hat{T} \underbrace{R R^T}_{I} \hat{T}^T = \hat{T} \hat{T}^T$

$= TT^T - \|T\|^2 I$

If $A = USV^T$ then

$\underbrace{AA^T}_{\text{symm. matrix}} = USV^T V S^T U^T \quad SS^T = S^2$

$= US^2U^T$

$\longrightarrow U \Lambda U^T \rightarrow$ diagonal of eigenvalues

$$\sigma(A) = \sqrt{\lambda(\underbrace{AA^T}_{})}$$

$$\lambda(\overset{\text{semi}}{\text{pos.def.}}) \geq 0$$

To compute by hand singular values we have to compute eigenvalues:

$$\det(EE^T - \lambda I) = 0$$

$$\lambda_1 = \lambda_2 = \|T\|^2 \Rightarrow \sigma_1 = \sigma_2 = \|T\| \neq 0$$

$$\lambda_3 = 0 \quad (\text{expected } \det(EE^T) = \det(\sigma)^2 = 0$$

$$\boxed{\begin{array}{l}\text{If a } 3\times 3 \text{ matrix is an } \underline{E\text{-matrix}} \text{ then} \\ \sigma_1 = \sigma_2 > 0 \text{ and } \sigma_3 = 0\end{array}}$$

If a 3×3 matrix can be decomposed as a product of an antisymmetric and orthogonal then $\sigma_1 = \sigma_2 > 0$ and $\sigma_3 = 0$

necessary condition only!

To find T and R we need ~~the~~ to prove sufficient condition!

We need to prove:

If $\sigma_1 = \sigma_2 > 0$ and $\sigma_3 = 0$ for a matrix E then

$\exists\, T \neq 0$ and R orthogonal :

$$E = \hat{T} R .$$

Proof: Q orthogonal 3×3

$$\widehat{\underbrace{Qa}_{3\times 1}} = Q\hat{a}Q^T$$

why?

$$\widehat{Qa}\, b = Qa \times b$$

$$\boxed{Q(c\times d) = Qc\times Qd}$$

$$= Q(a\times Q^T b)$$

$$= Q\hat{a}Q^T b$$

consider

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

antisymm ortho

$\sigma_1 = \sigma_2 = 1$

$\sigma_3 = 0$

canonical E-matrix

2nd possibility

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\widehat{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}} R_z\left(\frac{\pi}{2}\right)$$

$$\widehat{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} R_z\left(-\frac{\pi}{2}\right)$$

Take any matrix E (not $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$)

$$E = USV^T = U \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T$$

$$= \sigma U \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T$$

$$= \sigma U \widehat{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}} R_z\left(\frac{\pi}{2}\right) V^T$$

$$= \sigma U \widehat{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}} \underbrace{U^T U}_{I} R_z\left(\frac{\pi}{2}\right) V^T$$

$$= \sigma \; \widehat{\mathcal{U}\begin{pmatrix}0\\0\\-1\end{pmatrix}} \mathcal{U}^T \quad \mathcal{U} R_z\left(\tfrac{\pi}{2}\right) V^T$$

$$= \sigma \; \widehat{\underbrace{\mathcal{U}\begin{pmatrix}0\\0\\-1\end{pmatrix}}_{-u_3}} \quad \mathcal{U} R_z\left(\tfrac{\pi}{2}\right) V^T$$

$$= -\sigma \; \hat{u}_3 \quad \underbrace{\mathcal{U} R_z\left(\tfrac{\pi}{2}\right) V^T}_{\text{orthogonal}}$$

? $\quad \|T\| \quad$ antisymmetric

orthogonal because product of 3 orthogonals

1st sol $= \boxed{- \; \widehat{\sigma u_3} \; \mathcal{U} R_z\left(\tfrac{\pi}{2}\right) V^T}$

alternative decomposition

2nd sol $= \boxed{\sigma \; \hat{u}_3 \; \mathcal{U} R_z\left(-\tfrac{\pi}{2}\right) V^T}$

Is $-$ a problem?

No, because the E-matrix

can be determined only up to a scale! $p^T E q = -p^T E q = 0$

$$\boxed{\begin{array}{l} 3^{rd}\ sol: \sigma\ \hat{u}_3\ U R_z(\tfrac{\pi}{2}) V^T \\ \hline 4^{th}\ sol: -\sigma\ \hat{u}_3\ U R_z(-\tfrac{\pi}{2}) V^T \end{array}}$$

$$R_z(-\tfrac{\pi}{2}) \neq -R_z(\tfrac{\pi}{2})$$

4 solutions

	$R_z(\frac{\pi}{2})$	$R_z(-\frac{\pi}{2})$
u_3	$\hat{u}_3\ U R_z(\frac{\pi}{2}) V^T$	$\hat{u}_3\ U R_z(-\frac{\pi}{2}) V^T$
$-u_3$	$\widehat{-u_3}\ U R_z(\frac{\pi}{2}) V^T$	$\widehat{-u_3}\ U R_z(-\frac{\pi}{2}) V^T$

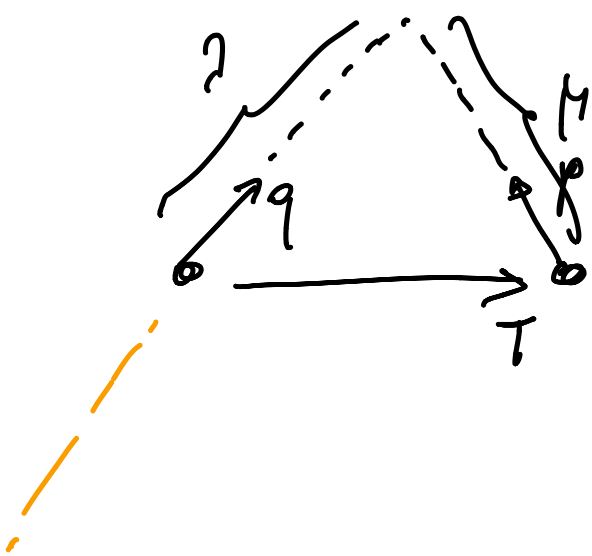
$$\boxed{\text{Why 4 solutions?}}$$

(1) mirror $\hat{u}_3$ and $\widehat{-u_3}$

$$q^T(T \times Rp) = 0$$

$$q^T(-T \times Rp) = 0$$

$$\lambda q = R\mu p + T$$



$$\underset{\uparrow}{(-\lambda)} q = R \underset{\uparrow}{(-\mu)} p - T$$

negative depth

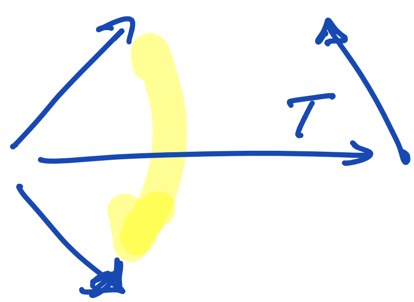
(2) $R_{z,\frac{\pi}{2}}$ vs. $R_z(-\frac{\pi}{2})$

for the same T

twisted pair ambiguity

$$R = U R_{z,\frac{\pi}{2}} V^T$$

then $R' = U R_{z,-\frac{\pi}{2}} V^T = R_{T,\pi} R$



$$q^T(T \times R_{T,\pi} R_p)$$

$$= q^T(T \times R_p)$$

In this case only

the λ are negative depths.

Given R, T, $\lambda q = R \mu p + T$
find depths λ, μ !

TRIANGULATION

$$\lambda q - R \mu p = T$$

$$\underbrace{\begin{pmatrix} q & -Rp \end{pmatrix}}_{3\times 2} \underbrace{\begin{pmatrix} \lambda \\ \mu \end{pmatrix}}_{2\times 1} = \underbrace{T}_{3\times 1}$$

$\boxed{\text{From the four solutions only one will produce } \lambda > 0 \ \mu > 0.}$