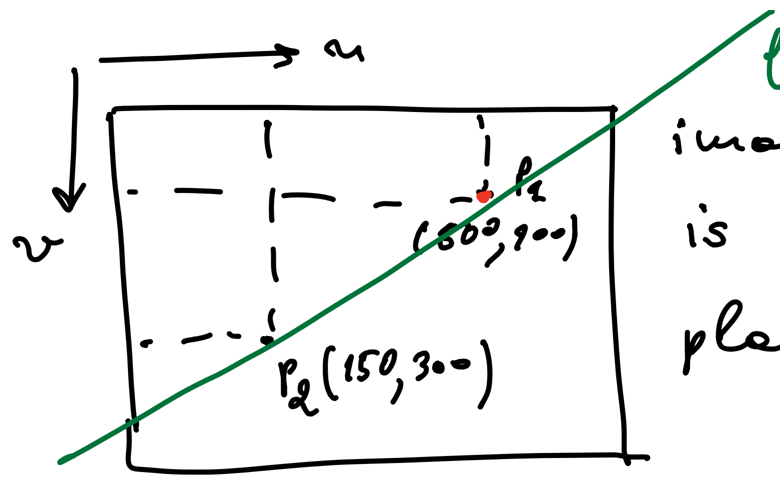


image plane is a projective plane $\mathbb{P}^2$

If a point in the image plane has coordinates (u,v) then the point in $\mathbb{P}^2$ will be $\begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$.

$$p_1 = \begin{pmatrix} 500 \\ 100 \\ 1 \end{pmatrix} \text{ same as } \begin{pmatrix} 1000 \\ 200 \\ 2 \end{pmatrix} \text{ or } \begin{pmatrix} -500 \\ -100 \\ -1 \end{pmatrix}$$

because
$$\frac{p_{11}}{p_{13}} = 500$$
$$\frac{p_{12}}{p_{13}} = 100$$

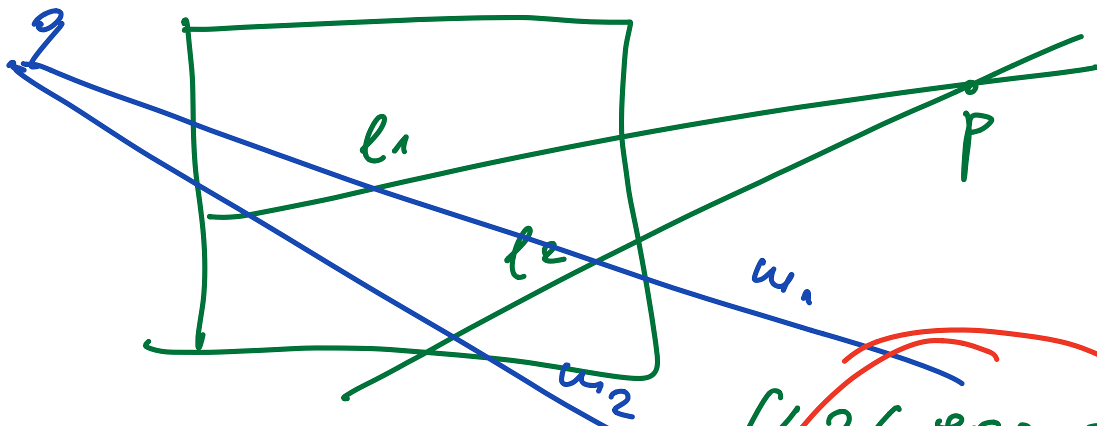
$$\sim \begin{pmatrix} 500 \\ 100 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1000 \\ 200 \\ 2 \end{pmatrix}$$

"projectively equivalent"

we will not write $\begin{pmatrix} 500 \\ 100 \\ 1 \end{pmatrix} = \begin{pmatrix} 1000 \\ 200 \\ 2 \end{pmatrix}$

$$l \sim p_1 \times p_2 \sim \begin{pmatrix} 150 \\ 300 \\ 1 \end{pmatrix} \times \begin{pmatrix} 500 \\ 100 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 200 \\ 350 \\ -135000 \end{pmatrix}$$

$$\sim \begin{pmatrix} -0.0015 \\ -0.0026 \\ 1 \end{pmatrix} \qquad \boxed{-15u - 26v + 10000 = 0}$$

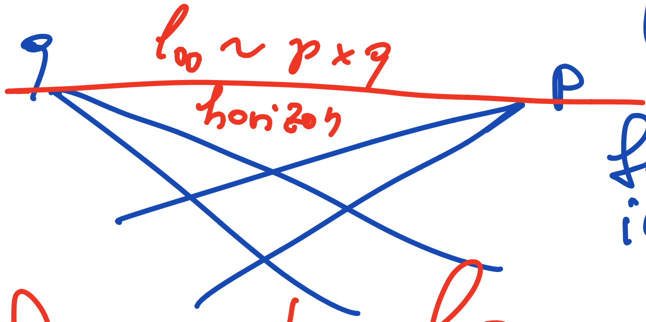


$$p \sim l_1 \times l_2 = \begin{pmatrix} 436989672 \\ 725725604 \\ 76860 \end{pmatrix} \sim \begin{pmatrix} 5685 \\ 944 \\ 1 \end{pmatrix}$$

$$q \sim w_1 \times w_2 \sim \begin{pmatrix} 599 \\ 963 \\ 1 \end{pmatrix}$$

finite but outside image plane.

$l_\infty \sim p \times q$



Do not forget $\sim$!

In $\mathbb{P}^2$ addition and subtraction do not make sense:

$$\cancel{\begin{pmatrix}583\\963\\1\end{pmatrix} \overset{\longleftrightarrow}{+} \begin{pmatrix}100\\200\\1\end{pmatrix}}$$

but matrix vector multiplication makes sense as well as inner and cross products:

line l

point p

$$l^T p = 0$$

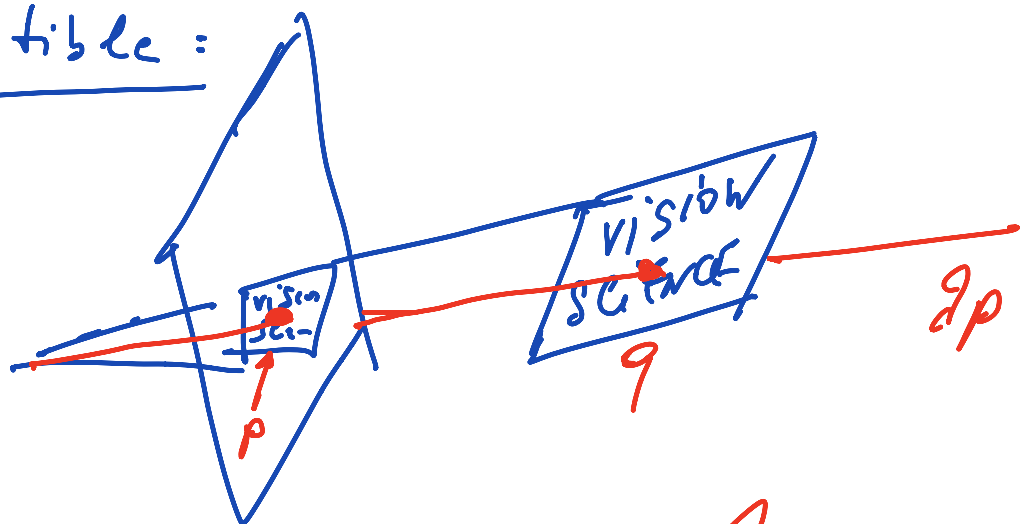
$$l \sim p_1 \times p_2$$

$$p \sim l_1 \times l_2$$

Projective Transformation

examples: picture of a plane
is a proj. transformation

(1) **invertible:**



in general: $p \Rightarrow$ ray λp
but if I know that it comes from
a specific plane then the
inverse is the intersection with
this plane

$$p = \text{proj.}(q)$$

$$q = \text{proj}^{-1}(p)$$

(2) lines will remain lines

$\Rightarrow$ proj. transf. will be linear transformation

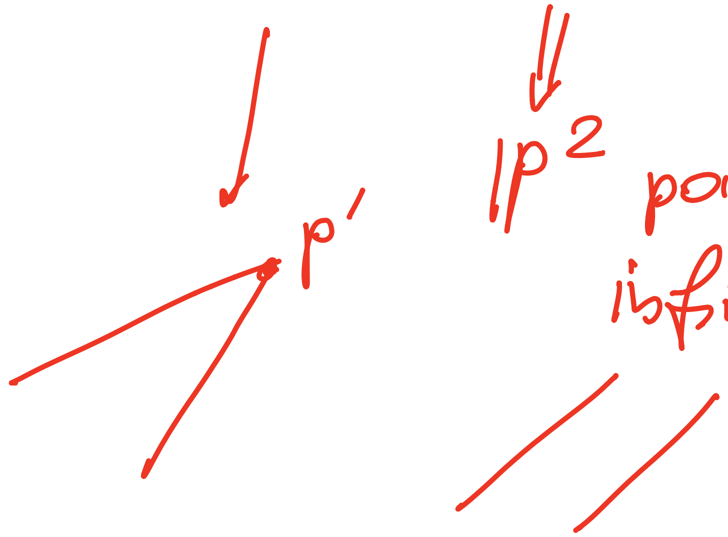
(1) and (2) $\Rightarrow$ invertible matrix

$\boxed{\text{A real } 3\times 3 \text{ invertible matrix } A \text{ is a proj. transformation} \quad p' \sim A\, p}$

image $\qquad$ floor

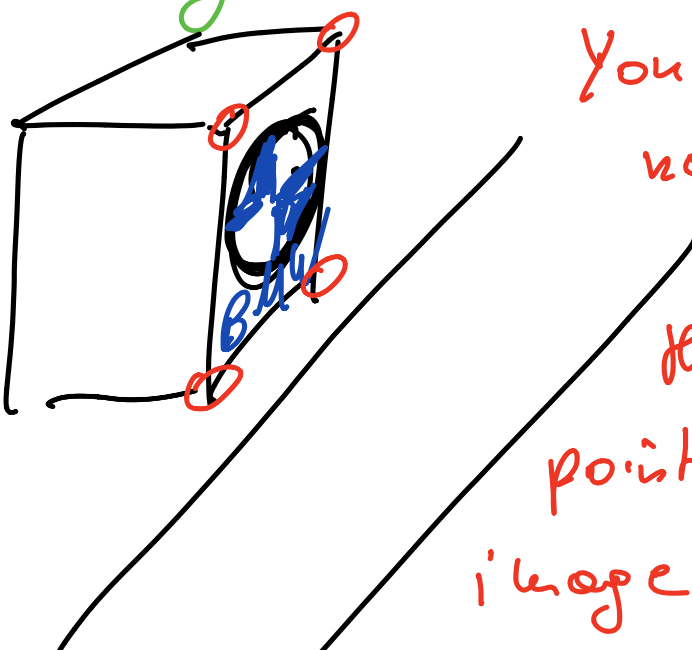
$\Downarrow$

$\mathbb{P}^2$ point at infinity



$\boxed{\text{We can compute a projective transformation from 4 point correspondences}}$

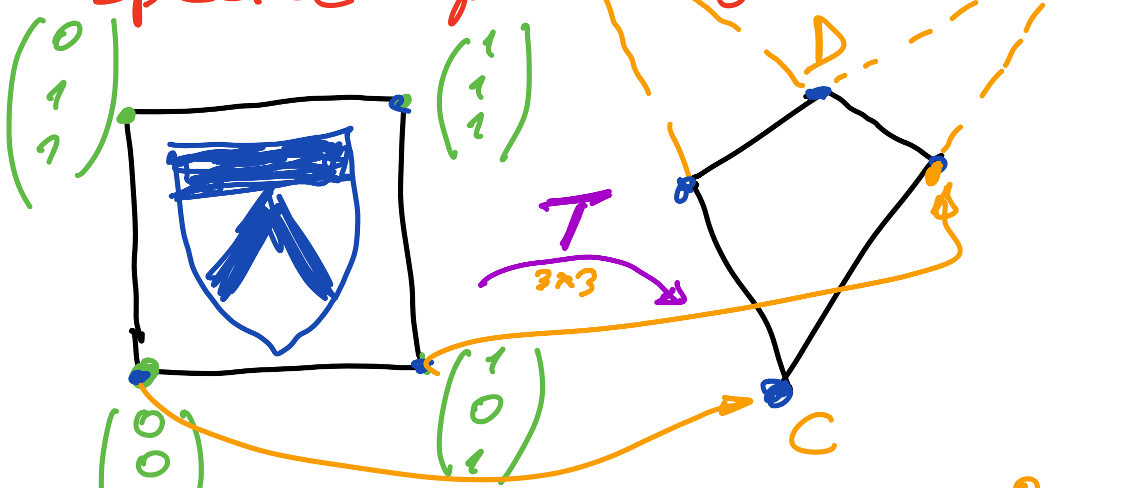
↳ revolution in applications including virtual advertising



You only need to identify these 4 points in your image.

You do not need to know f, u_0, v_0, R, T!

We will prove here that we will be able to compute the transformation from 4 "Special" points



origin

(real application)

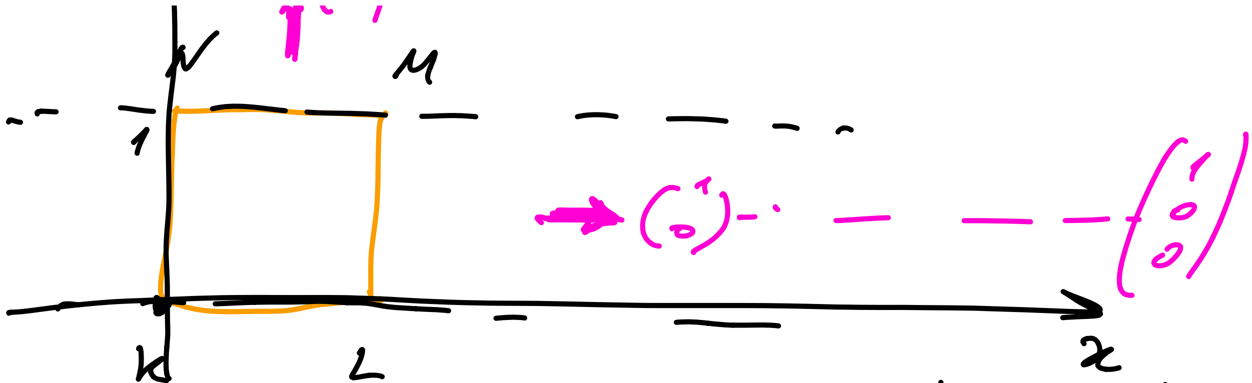
$$C \sim T_{3\times3}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$D \sim T_{3\times3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

vanishing point $A \sim T_{3\times3}\begin{pmatrix}1\\0\\0\end{pmatrix}$

vanishing point $B \sim T_{3\times3}\begin{pmatrix}0\\1\\0\end{pmatrix}$

4Y $\begin{pmatrix}0\\1\end{pmatrix}$



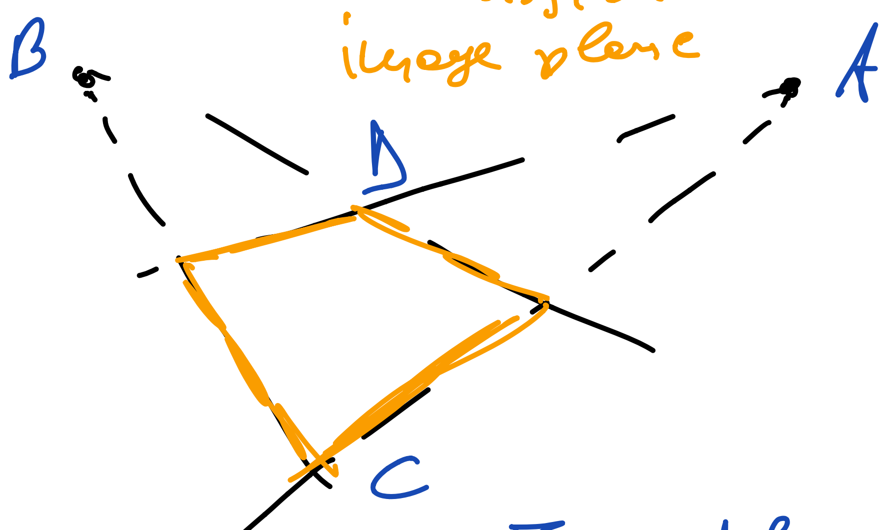
$$KL \cap NM = \begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix} \sim \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$KL \to y=0$ $\qquad NM \to y=1$ $\quad y-1=0$

$$\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix}=0 \qquad \begin{pmatrix}0 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix}=0$$

$ax+by+c\cdot 1=0$

image plane



Which matrix $T_{3\times 3}$ satisfies

only way to convert $\sim$ to $=$

$$A \sim T\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \alpha\, a = T\begin{pmatrix}1\\0\\0\end{pmatrix}$$

unknown scalar

$$a \to \begin{pmatrix}3000\\2000\\?\end{pmatrix}$$

$$B \sim T\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \beta\, b = T\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$C \sim T\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \gamma\, c = T\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$D \sim T\begin{pmatrix}1\\1\\1\end{pmatrix} \qquad \delta\, d = T\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\boxed{\begin{pmatrix}1\\2\\3\end{pmatrix} \sim \begin{pmatrix}3\\6\\9\end{pmatrix}} \qquad \boxed{\alpha \begin{matrix}1\\2\\3\end{matrix} = \begin{matrix}3\\6\\9\end{matrix}}$$

T had already 9 unknown (3x3 matrix) and I introduced 4 more ?

$$T = \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix}$$

$3\times 1 \quad 3\times 1 \quad 3\times 1$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = t_1 \qquad T\begin{pmatrix}0\\1\\0\end{pmatrix} = t_2 \qquad T\begin{pmatrix}0\\0\\1\end{pmatrix} = t_3$$

$$\boxed{\begin{aligned} \alpha a &= T\begin{pmatrix}1\\0\\0\end{pmatrix} = t_1 \\ \beta b &= T\begin{pmatrix}0\\1\\0\end{pmatrix} = t_2 \\ \gamma c &= T\begin{pmatrix}0\\0\\1\end{pmatrix} = t_3 \\ \delta d &= T\begin{pmatrix}1\\1\\1\end{pmatrix} = t_1 + t_2 + t_3 \end{aligned}}$$

a, b, c, d are the 3×1 coordinates of A, B, C, D, respectively

$$\boxed{\delta d = \alpha a + \beta b + \gamma c}$$

only 4 unknowns and 3 equations.

$$\begin{pmatrix}\cdot\\\cdot\\\cdot\end{pmatrix}$$

WLOG we set $\delta = 1$

(why: $\delta \neq 0$)

$$d = \alpha a + \beta b + \gamma c$$

$$= \underset{3\times 3}{\begin{pmatrix} a & b & c \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}}$$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} a & b & c \end{pmatrix}^{-1} d$$

$$\boxed{T = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix}}$$

Why didn't I use the 4 corner:

$$\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

given

$\mu \quad m = T\begin{pmatrix}0\\0\\1\end{pmatrix} = t_3$

$\begin{pmatrix}1&0&0\\2&0&0\end{pmatrix}$

$T = (t_1 \; t_2 \; t_3)$

$\nu \quad n = T\begin{pmatrix}1\\0\\1\end{pmatrix} = t_1 + t_3$

$\lambda \quad l = T\begin{pmatrix}1\\1\\1\end{pmatrix} = t_1 + t_2 + t_3$

$\sigma \quad s = T\begin{pmatrix}0\\1\\1\end{pmatrix} = t_2 + t_3$

$\parallel$

1

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

$$m_1 = \frac{t_{13}}{t_{33}}$$

$$m_2 = \frac{t_{23}}{t_{33}}$$

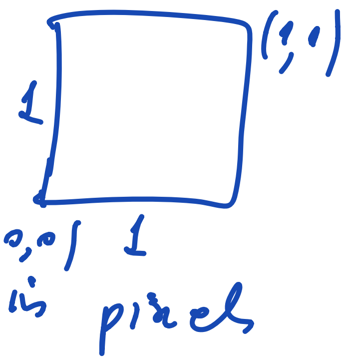
system with unknown t_{ij} ...

$$n_1 = \frac{t_{11} + t_{13}}{t_{31} + t_{33}}$$

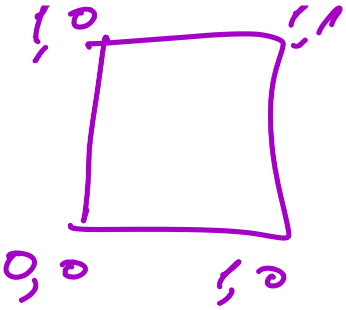
$(\quad) \sim T(\quad)$

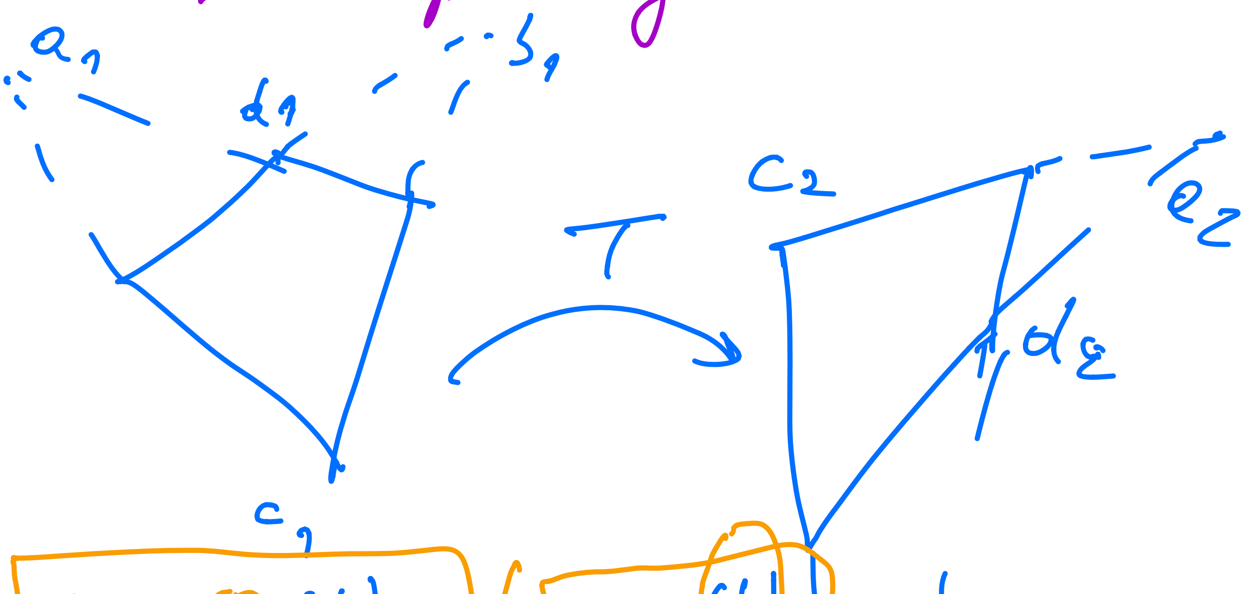
image

logo square



What happens if T is from a pixel plane to another pixel plane?

Answer: We use  as a proxy!



$$a_1 \sim T_1\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad a_2 \sim T_2\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$b_1 \sim T_1\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad b_2 \sim T_2\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$c_1 \sim T_1\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad c_2 \sim T_2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$d_1 \sim T_1\begin{pmatrix}1\\1\\1\end{pmatrix} \qquad d_2 \sim T_2\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$T_1 \qquad\qquad T_2$$

$$\Rightarrow \quad \boxed{a_2 \sim \underbrace{T_2 T_1^{-1}}_{T} a_1}$$

why not $\alpha_2 \sim T\alpha_1$

$$\begin{pmatrix} \; \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{21} \\ T_{22} \\ T_{23} \\ T_{31} \\ T_{32} \end{pmatrix} = 0$$

lines in $\mathbb{R}^2$ and $\mathbb{P}^2$

$$ax+by+c=0 \to \begin{pmatrix}a\\b\\c\end{pmatrix}^T\begin{pmatrix}x\\y\\1\end{pmatrix}=0$$

$$l \sim \begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$3x+4y+10=0 \leftarrow \boxed{3u+4v+10w=0}$$

nothing $\leftarrow$ $w=0$

$$(0\ 0\ 1)\begin{pmatrix}u\\v\\w\end{pmatrix}=0$$

$\downarrow$

l

line at infinity