CIS 580, Machine Perception, Spring 2021
Homework 6 Solutions: Image Processing Basics

Deliverable is a pdf report including your calculations, plots, and explanations. No code submission but you can include code in your pdf if it helps in the explanation. You are allowed to use the Fourier Table (link in the homework website).

1 Convolution of image with a Gaussian

Consider the the grey-value version of the image `statue.jpg`. Using the equation

$$g(x,y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{1}$$

with $\sigma = 1$ construct a 5x5 Gaussian filter and convolve it with your image. What happens to the image? Do this again with $\sigma = 0.1$. Plot the images and explain the difference.

Solution

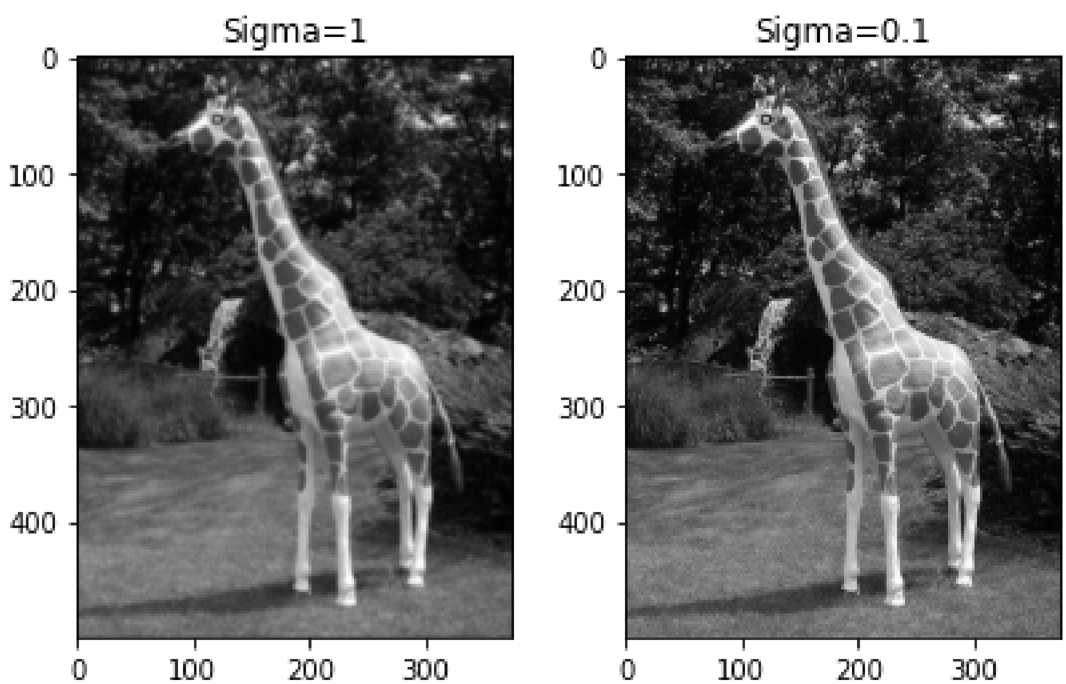


Figure 1: Caption

We can observe that convolving with Guassian filter with $\sigma = 1$ introduce some blur in the image while convolving with Gaussian filter with = 0.1 results in an image very similar to the original

2 Convolution of greyscale image with different Gaussian kernels

Assume the 1D Gaussian:

$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

Prove that the convolution of two Gaussians with σ_1, σ_2 is a Gaussian. Find its σ in terms of σ_1, σ_2. You can use all Fourier pairs in the Fourier table without proving them.

Solution

For a 1D Gaussian we know that

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \circ\!\!-\!\!\bullet\; e^{-\sigma^2\omega^2/2}$$

This means that

$$g_{\sigma_1} * g_{\sigma_2} \circ\!\!-\!\!\bullet\; G_{\sigma_1} G_{\sigma_2} = e^{-\sigma_1^2\omega^2/2} e^{-\sigma_2^2\omega^2/2} = e^{-\left(\sqrt{\sigma_1^2+\sigma_2^2}\right)^2 \omega^2/2} \bullet\!\!-\!\!\circ\; \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-\frac{t^2}{2(\sigma_1^2+\sigma_2^2)}}$$

So the result of the convolution is a Gaussian with $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

3 Convolution of Step Edge with Gaussian derivative

Compute ("by hand") the convolution of a step edge

$$h(t) = \begin{cases} H/2 & \text{if } t \geq 0, \\ -H/2 & \text{if } t < 0 \end{cases}.$$

with the first derivative of a Gaussian with standard deviation σ. Write a program that performs this convolution. Use H and σ as arguments. Keep H fixed equal to 1 and vary σ from 0.5 to 5 with step 0.5. Plot the 10 curves arising from convolving the step edge with these 10 Gaussian derivatives. What do you observe?

Solution

The convolution of $h(t)$ with the derivative of the Gaussian $g'(t)$ is :

$$\begin{aligned}
\int_{-\infty}^{\infty} h(\tau) g'(t-\tau) d\tau &= \int_0^{\infty} \frac{H}{2} g'(t-\tau) d\tau + \int_{-\infty}^{0} \left(\frac{-H}{2}\right) g'(t-\tau) d\tau \\
&\overset{(u=t-\tau)}{=} \int_t^{-\infty} \left(\frac{-H}{2}\right) g'(u) du + \int_{\infty}^{t} \frac{H}{2} g'(u) du \\
&= \int_{\infty}^{t} H g'(u) du = H\left[g(t) - g(\infty)\right] = H g(t) = \frac{H}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}}
\end{aligned}$$

Running the same convolution in Python for different sigmas we get figure 2. We can see that all the results consist of a single Gaussian of different sigmas and zero mean, so they agree with the analytical results. Also for all of them the maximum is observed at point 0, which shows that we can use the maximum to find the position of the edge that also occurs at point 0.

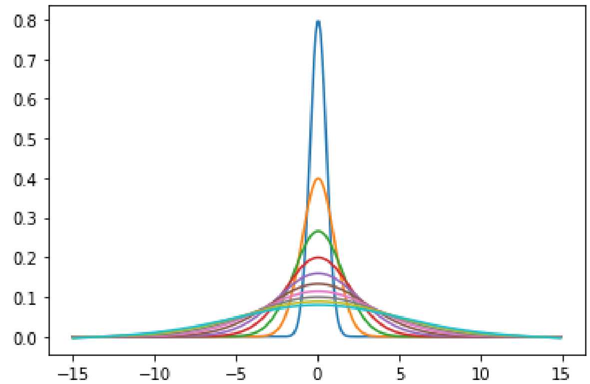


Figure 2: Convolution of step edge with derivative of Gaussian

4 Box Function

The 1D box function

$$h(t) = \begin{cases} 1/a, \text{if } |t| \leq a/2 \\ 0, \text{otherwise.} \end{cases}.$$

is the section of the image of a white line on black background. Compute ("by hand") its convolution with the 1st derivative of the Gaussian. Write a program that performs this convolution. Fix $a = 1$ and vary σ from 0.1 to 3.0 with step σ and plot all curves. What do you observe?

Solution

The convolution of the 1D box function $h(t)$ with the 1st derivative of Gaussian $g'(t)$ is the following:

$$\begin{aligned}\int_{-\infty}^{\infty} h(\tau)g'(t-\tau)d\tau &= \int_{-a/2}^{a/2} \frac{1}{a} g'(t-\tau)d\tau \\ &\overset{u=t-\tau}{=} \int_{t+a/2}^{t-a/2} -\frac{1}{a} g'(u)du \\ &= \int_{t-a/2}^{t+a/2} \frac{1}{a} g'(u)du = \frac{1}{a}\left[g(t+a/2) - g(t-a/2)\right] = \frac{1}{a\sigma\sqrt{2\pi}}\left(e^{-\frac{(t+a/2)^2}{2\sigma^2}} - e^{-\frac{(t-a/2)^2)}{2\sigma^2}}\right)\end{aligned}$$

So the analytical result indicates that we expect a Gaussian with mean $(-a/2)$ minus a Gaussian with mean $(a/2)$. In figure 3 we see the result we get when we run the convolution in Python. We can observe that indeed we get results that follow our expectation from the analytical computation of the convolution. Also the result of the convolution shows a positive peak close to $(-a/2)$ which is the point that we have a positive edge and a negative peak close to $(a/2)$ where we have a negative edge. So we can use the peaks of the convolution to find the locations of the edges in the signal. Finally from the result we can see that as σ increases the peaks become lower and the Gaussian are more spread out, making the localization of the exact position of the edge more difficult.

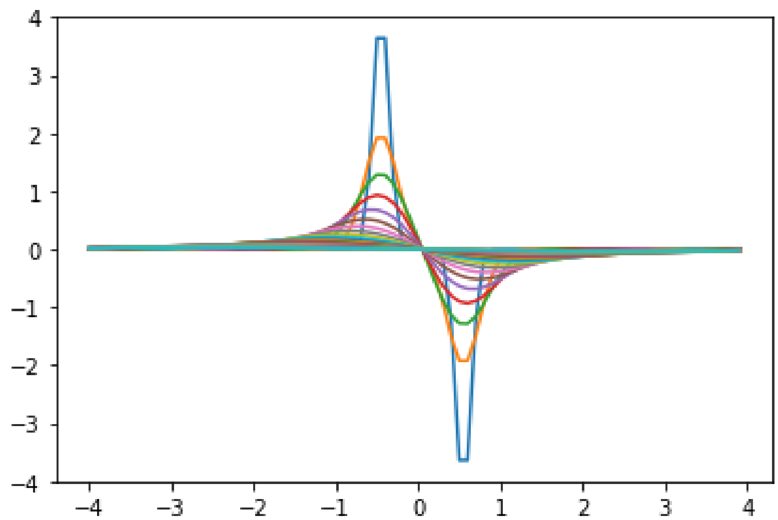


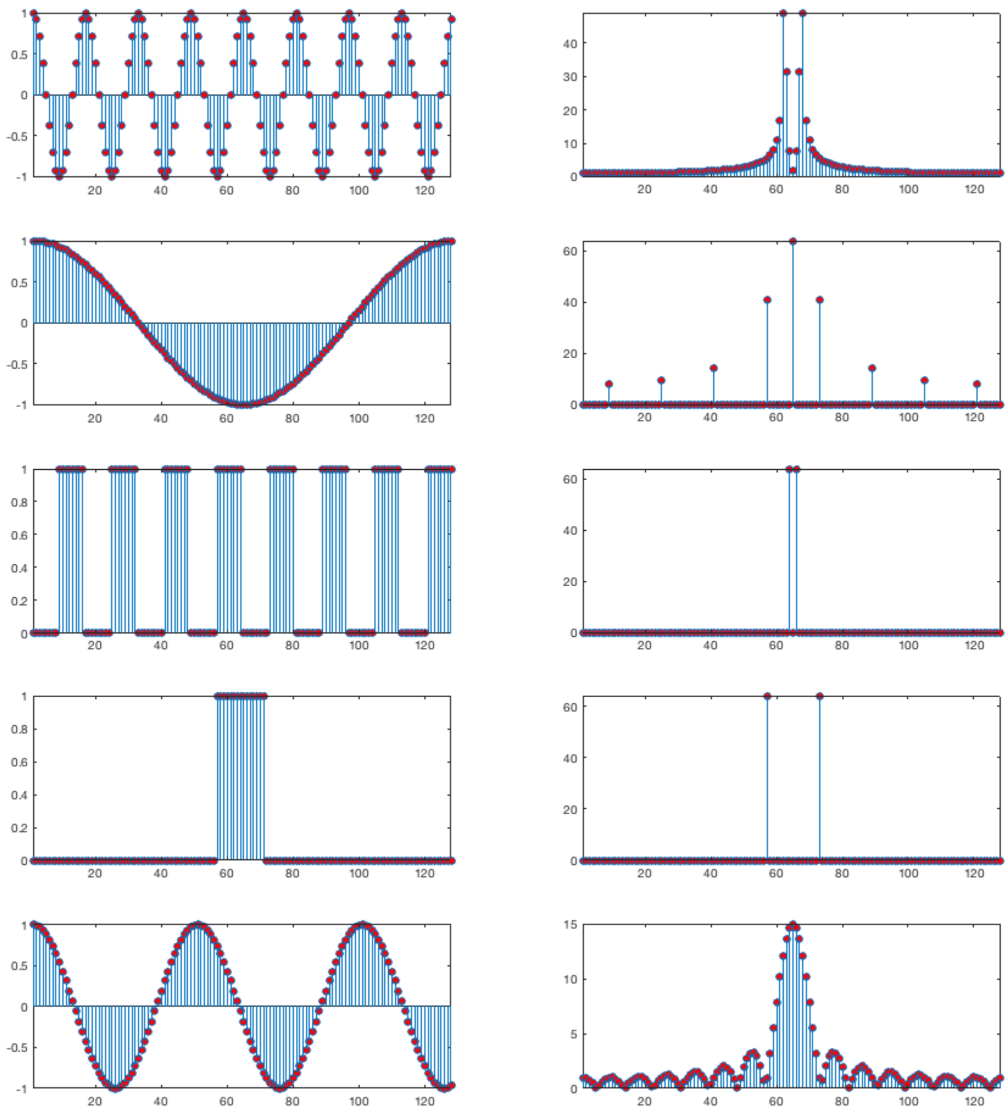
Figure 3: Convolution of box function with 1st derivative of a Gaussian kernel

5 1D FFT Quiz

Match the original signals with the corresponding Fourier transform (DFT). On the right we show the magnitude of the `fftshift` of the transform. Explain your choices.

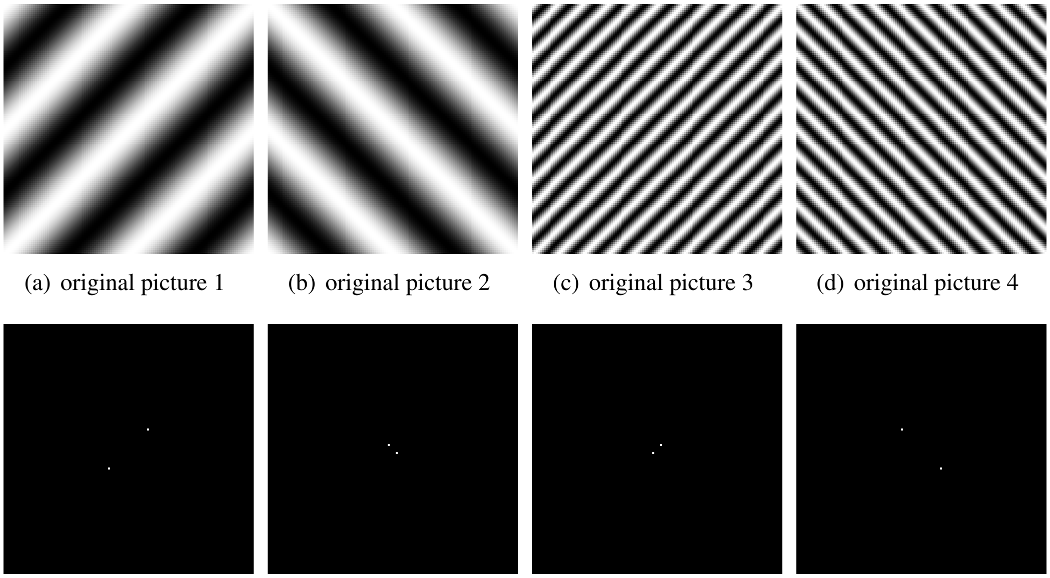
Solution

- **Left Column: Row 1 corresponds to Right Column: Row 4**. On the time domain we have a cosine signal with higher frequency compared to the cosine signal in row 2, so we expect the DFT to have 2 diracs that are located in higher frequencies.
- **Left Column: Row 2 corresponds to Right Column: Row 3**. On the time domain we have a cosine signal with lower frequency, so we epxect the DFT to have 2 diracs that are located in the lower frequencies.
- **Left Column: Row 3 correspond to Right Column: Row 2**. On the time domain we have a box function periodically repeated through time. This repetition corresponds to sampling in the frequency domains so we expect the DFT to have sparse samples of the sinc function which is the fourier transform of the box function
- **Left Column: Row 4 correspond to Right Column: Row 5**. On the time domain we have the box function so we expect the fourier transform to be the sinc function.
- **Left Column: Row 5 corresponds to Right Column: Row 1**. On the time domain we have a cosine signal that ends in the middle of its period. This means that when we continue the signal by repeating it through time (which is an assumption of the DFT) we will not get exactly a cosine signal. As a result we expect a fourier transform that will be similar to the fourier transform of a cosine signal but with some leakage in the higher frequencies.


1 0.5 0 -0.5 -1
20 40 60 80 100 120
40 30 20 10 0
20 40 60 80 100 120
1 0.5 0 -0.5 -1
20 40 60 80 100 120
60 40 20 0
20 40 60 80 100 120
1 0.8 0.6 0.4 0.2 0
20 40 60 80 100 120
60 40 20 0
20 40 60 80 100 120
1 0.8 0.6 0.4 0.2 0
20 40 60 80 100 120
60 40 20 0
20 40 60 80 100 120
1 0.5 0 -0.5 -1
20 40 60 80 100 120
15 10 5 0
20 40 60 80 100 120

6 2D Fourier Transform [10 pts]

Match the following pictures and 2D Fourier transform shown in figure 4. Explain your choices.



(a) original picture 1 (b) original picture 2 (c) original picture 3 (d) original picture 4

(e) Fourier 1 (f) Fourier 2 (g) Fourier 3 (h) Fourier 4

Figure 4: images and corresponding Fourier transform

Solutions

Here we want the Fourier transform to have the same orientation as the periodic signal in the image domain. Also higher frequencies correspond to points further away from the center of the fourier images. Given that the correspondence is the following:

- (a)-> (f)
- (b)-> (g)
- (c)-> (h)
- (d)-> (e)

7 Filter Design

Suppose you are given signal $y(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$ ~~with $\omega_1 = 2\pi/4$, $\omega_2 = 2\pi/8$~~. Define a Gaussian filter $g(t) = 2\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{t^2}{2\sigma^2}}$.

1. Compute "by hand" the output signal $s(t) = (y * g)(t)$, after we convolve y with g. You can use the Fourier tables.

2. Compute σ such that the output of the convolution is

$$s(t) = \frac{1}{2}cos(\omega_1 t) + a * cos(\omega_2 t)$$

for some $a \geq 0$. In your solution what is the value of a in terms of ω_1 and ω_2?

Solution
(a)

$$\begin{aligned}(y * g) \circ\!\!-\!\!\bullet\, & 2e^{-\sigma^2\omega^2/2}\left(\pi\delta(\omega-\omega_1)+\pi\delta(\omega+\omega_1)+\pi\delta(\omega-\omega_2)+\pi\delta(\omega+\omega_2)\right) \\ &= 2\pi e^{-\sigma^2\omega_1^2/2}(\delta(\omega-\omega_1)+\delta(\omega+\omega_1))+2\pi e^{-\sigma^2\omega_2^2/2}(\delta(\omega-\omega_2)+\delta(\omega+\omega_2))\end{aligned}$$

Now when we get from the frequencie domain to the time domain we get that the result of the covolution is

$$\begin{aligned}2\pi e^{-\sigma^2\omega_1^2/2}(\delta(\omega-\omega_1)+\delta(\omega+\omega_1))+2\pi e^{-\sigma^2\omega_2^2/2}(\delta(\omega-\omega_2)+\delta(\omega+\omega_2)) \bullet\!\!-\!\!\circ\, 2e^{-\sigma^2\omega_1^2/2}\cos(\omega_1 t)+2e^{-\sigma^2\omega_2^2/2}\cos(\omega_2 t) \\ \implies (y*g)(t) = 2e^{-\sigma^2\omega_1^2/2}\cos(\omega_1 t)+2e^{-\sigma^2\omega_2^2/2}\cos(\omega_2 t)\end{aligned}$$

(b) We first solve for the coefficient of $cos(\omega_1 t)$ to be $1/2$. So we get

$$2e^{-\sigma^2\omega_1^2/2} = 1/2 \implies \sigma^2 = \frac{2\ln(4)}{\omega_1^2} \implies = \sqrt{\frac{2\ln(4)}{\omega_1^2}}$$

Replacing σ in the coefficient of $\cos(\omega_2 t)$ we get

$$a = 2e^{-\sigma^2\omega_2^2/2} = 2e^{\left(-\ln(4)\frac{\omega_2^2}{\omega_1^2}\right)} = 2\, 4^{-\left(\frac{\omega_2}{\omega_1}\right)^2}$$