Projective Geometry

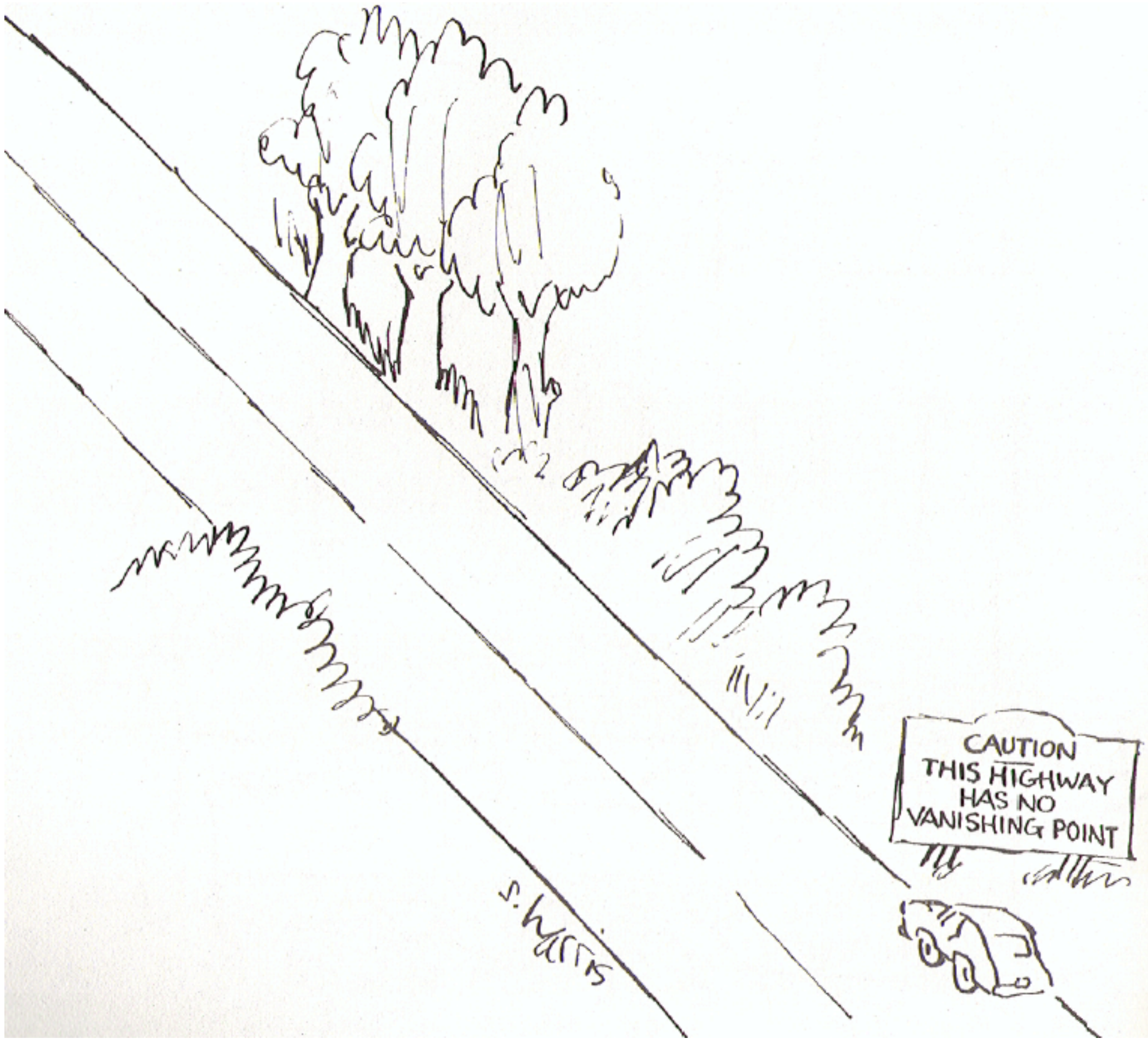
Slides by

Jianbo Shi,

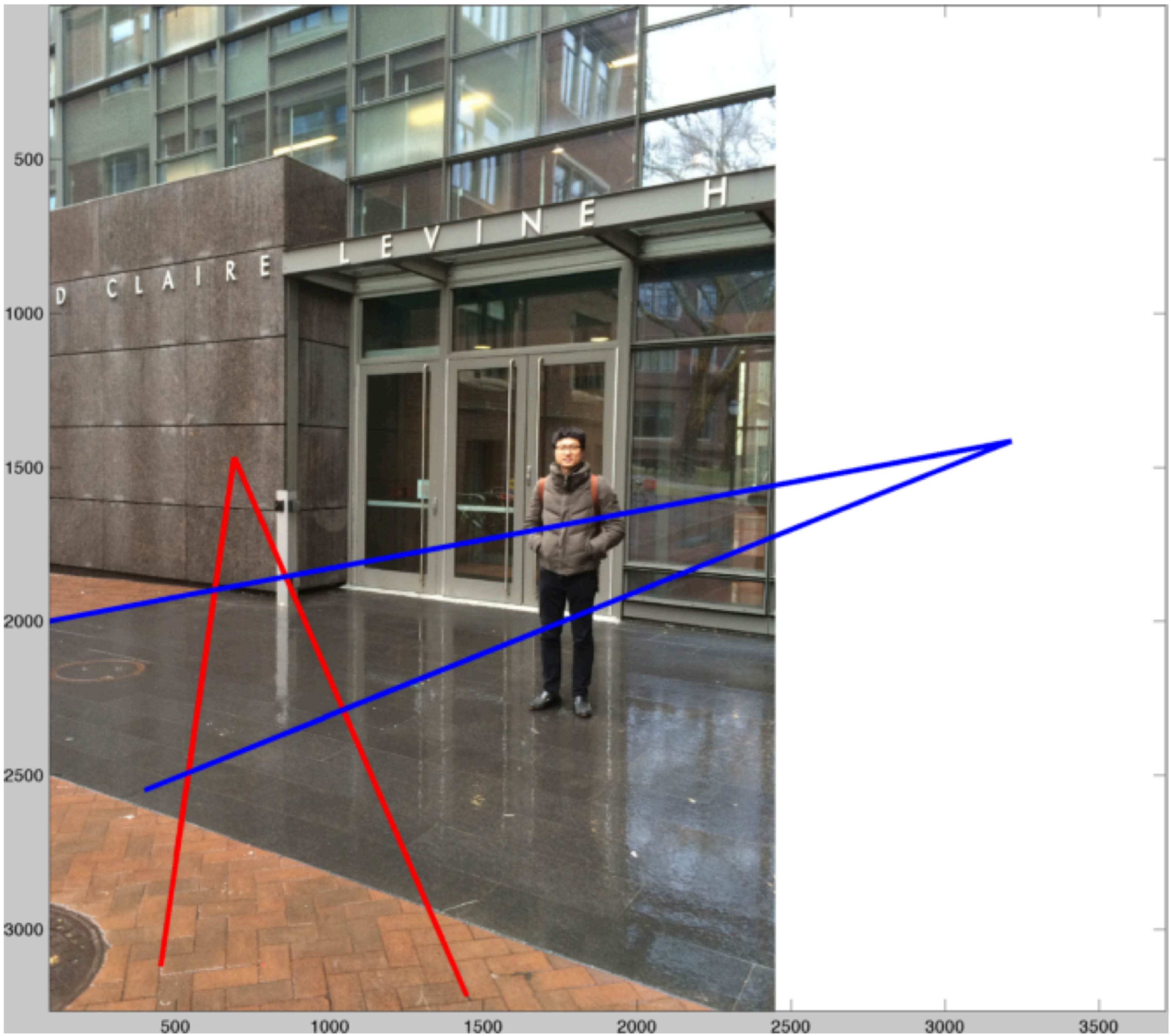
Hyun Soo Park,

Kostas Daniilidis

Vanishing point



http://pennpaint.blogspot.com/



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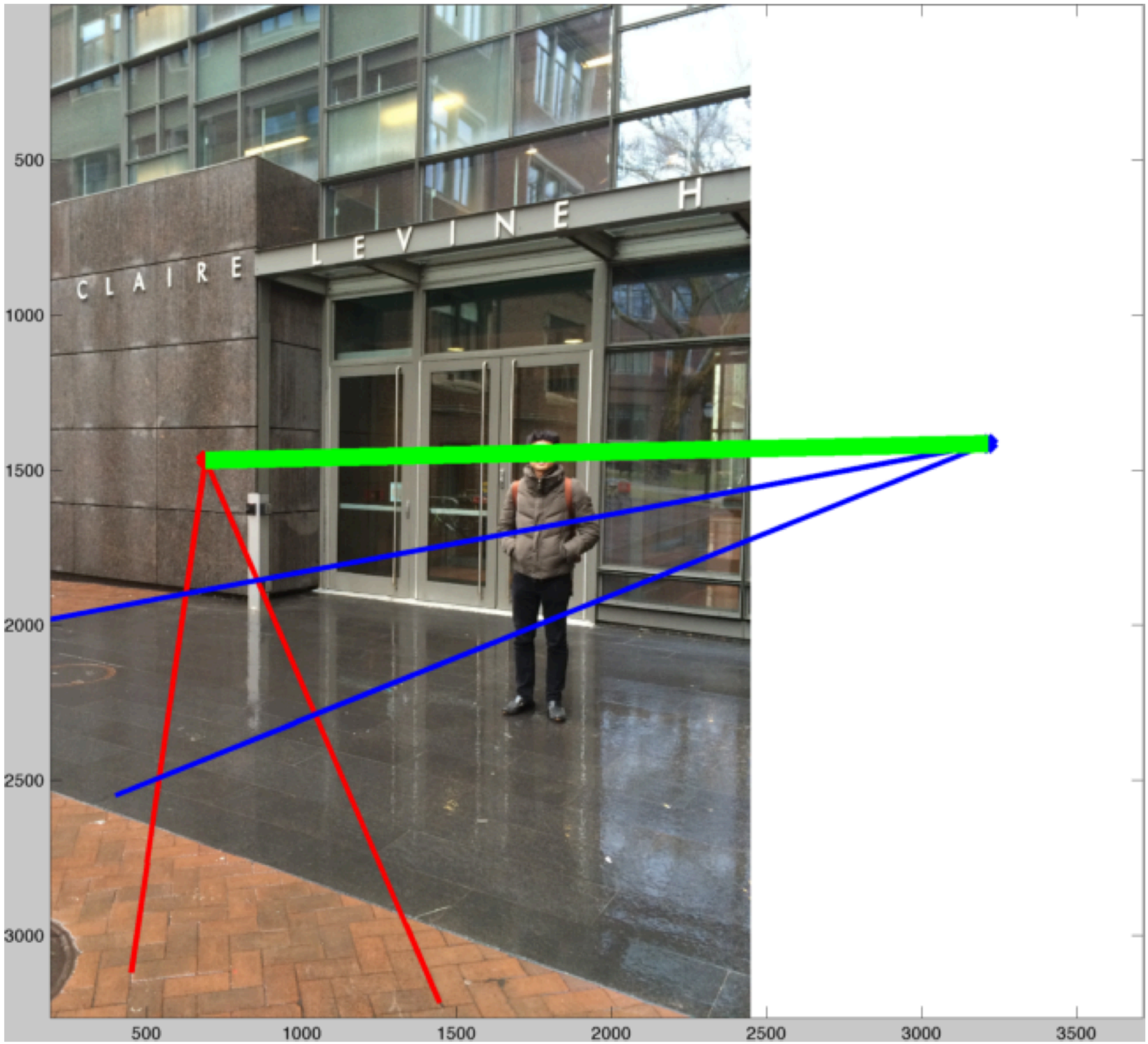
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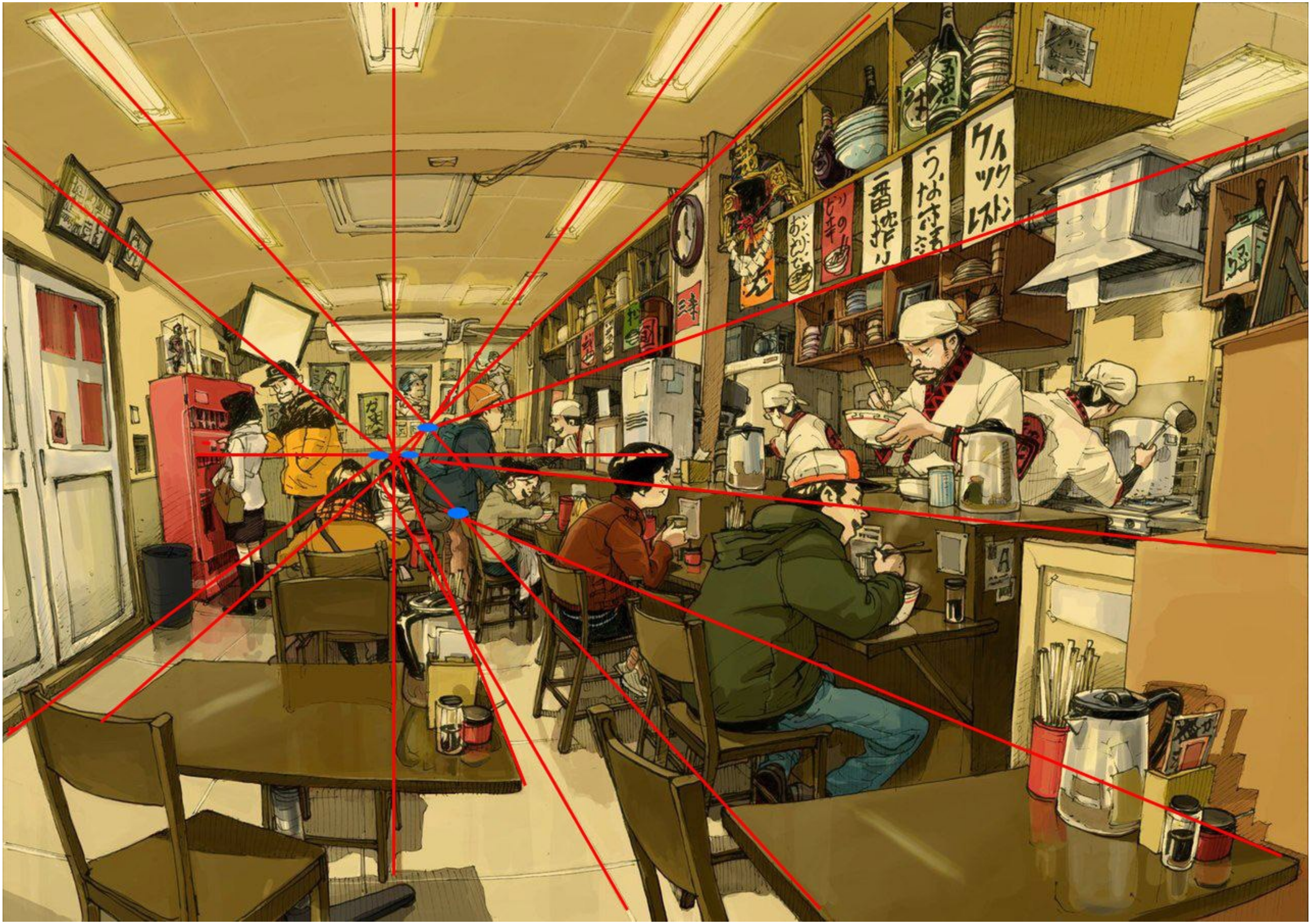
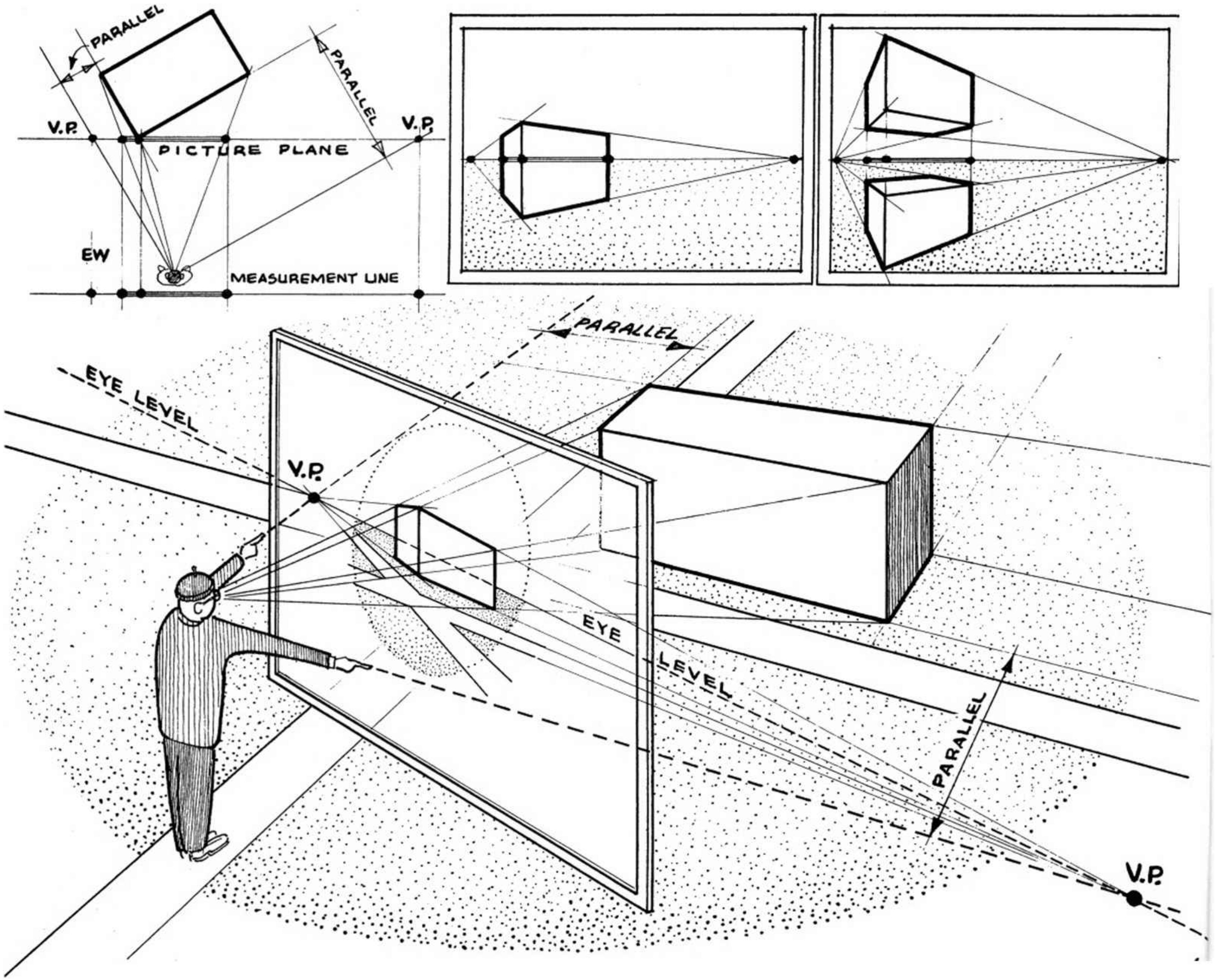
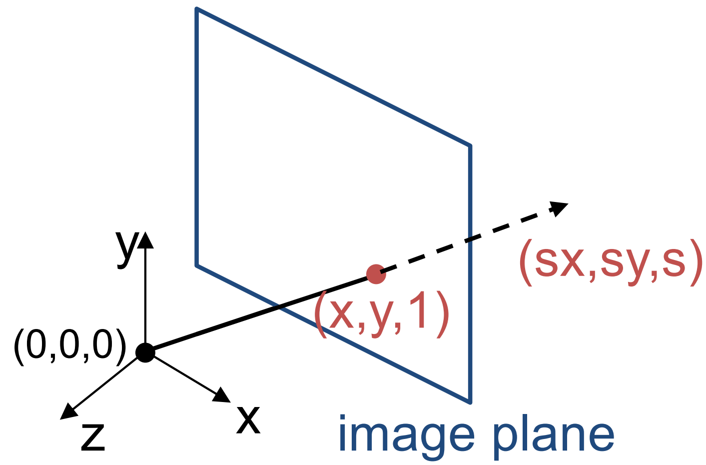


http://www.joshuanava.biz/perspective/in-other-words-the-observer-simply-points-in-the-same-direction-as-the-lines-in-order-to-find-their-vanishing-point.html



The projective plane

- Why do we need homogeneous coordinates?
  – represent points at infinity, homographies, perspective projection, multi-view relationships
- What is the geometric intuition?
  – a point in the image is a *ray* in projective space

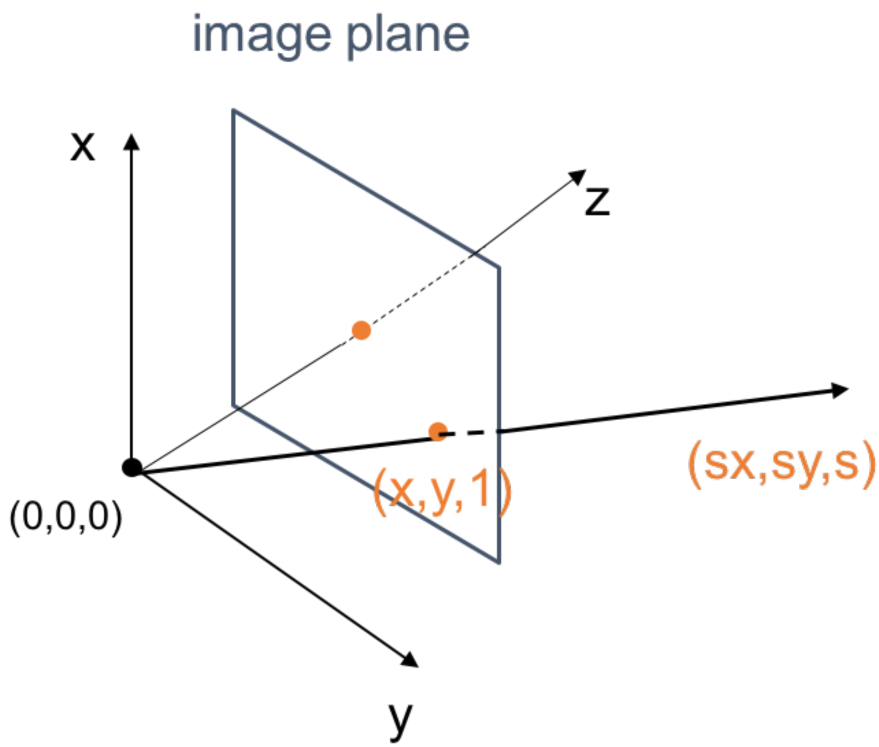


- Each *point* (x,y) on the plane is represented by a *ray* (sx,sy,s)
  – all points on the ray are equivalent: (x, y, 1) ≅ (sx, sy, s)

Point

Homogeneous coordinates

- represent coordinates in 2 dimensions with a 3-vector



$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projective equivalence

Definition 1. *Given a set X, a relation $R \subset X \times X$ is called* **equivalence** *(and denoted with $x \sim y$ for $(x, y) \in R$) if it is reflexive ($x \sim x$), symmetric ($x \sim y \Rightarrow y \sim x$), and transitive ($x \sim y,\ y \sim z \ \Rightarrow x \sim z$).*
An **equivalence class** *with representative $a \in X$ is the subset of all elements of X which are equivalent to a.*

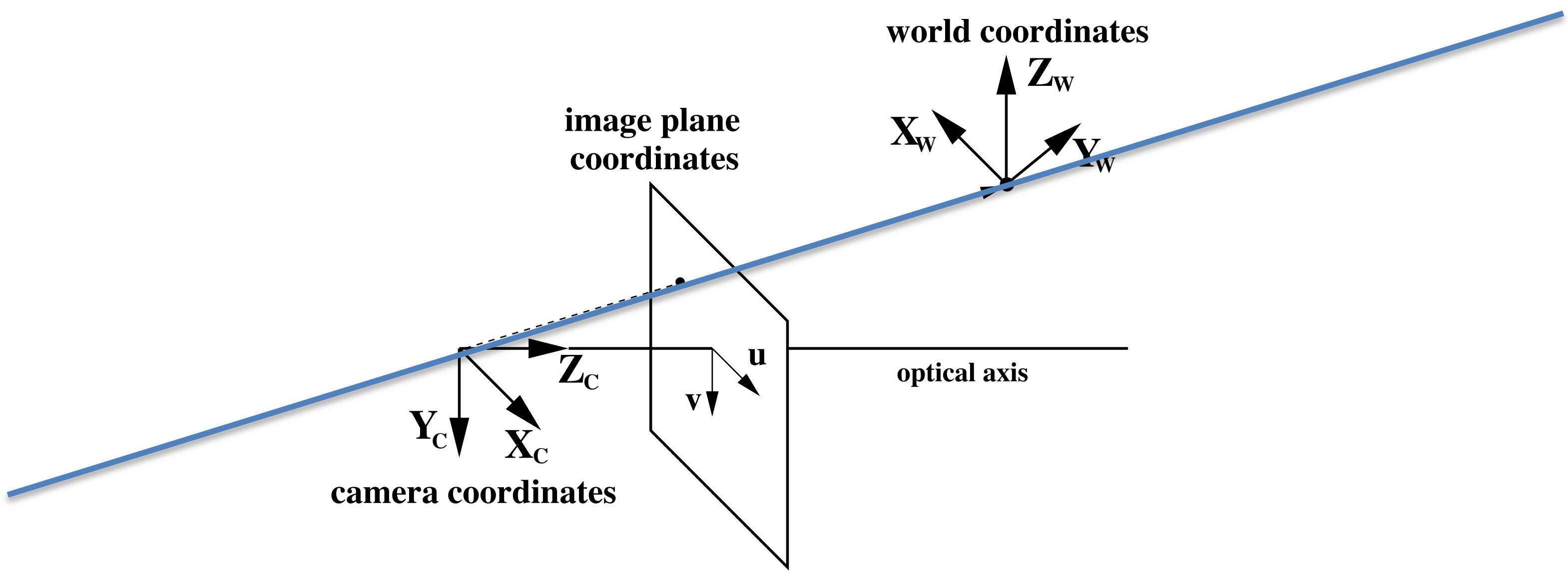
Definition 2. *Two elements of $\mathbb{R}^3 \backslash \{(0, 0, 0)\}$ are* **projectively equivalent**

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} \quad \textit{if} \quad \exists \lambda \in \mathbb{R} \setminus \{0\} : \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \lambda \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}$$

Projective Plane

Definition 3. *The* **projective plane** $\mathbb{P}^2$ *is the set of all projective equivalence classes of* $\mathbb{R}^3 \setminus \{(0,0,0)\}$.

Example: Points in $\mathbb{R}^3$ lying in the same line through the origin are projectively equivalent. A line through the origin is then an equivalence class. Its representative can be for example the pair of antipodal intersections with the unit sphere.



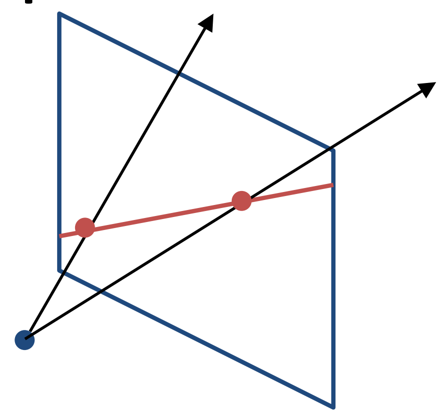
Injection of $\mathbb{R}^2$ in $\mathbb{P}^2$:

- A point $(x', y') \in \mathbb{R}^2$ is injected in $\mathbb{P}^2$ with the addition of the homogeneous coordinate 1: $(x', y', 1) \in \mathbb{P}^2$.

- Only the points (x, y, w) of $\mathbb{P}^2$ with $w \neq 0$ can be mapped to $\mathbb{R}^2$ as $(x/w, y/w)$. If projective equivalence can be visualized as lines through the origin in $\mathbb{R}^3$, then $\mathbb{R}^2$ is injected as the plane $w = 1$.

Projective lines

- What does a line in the image correspond to in projective space?

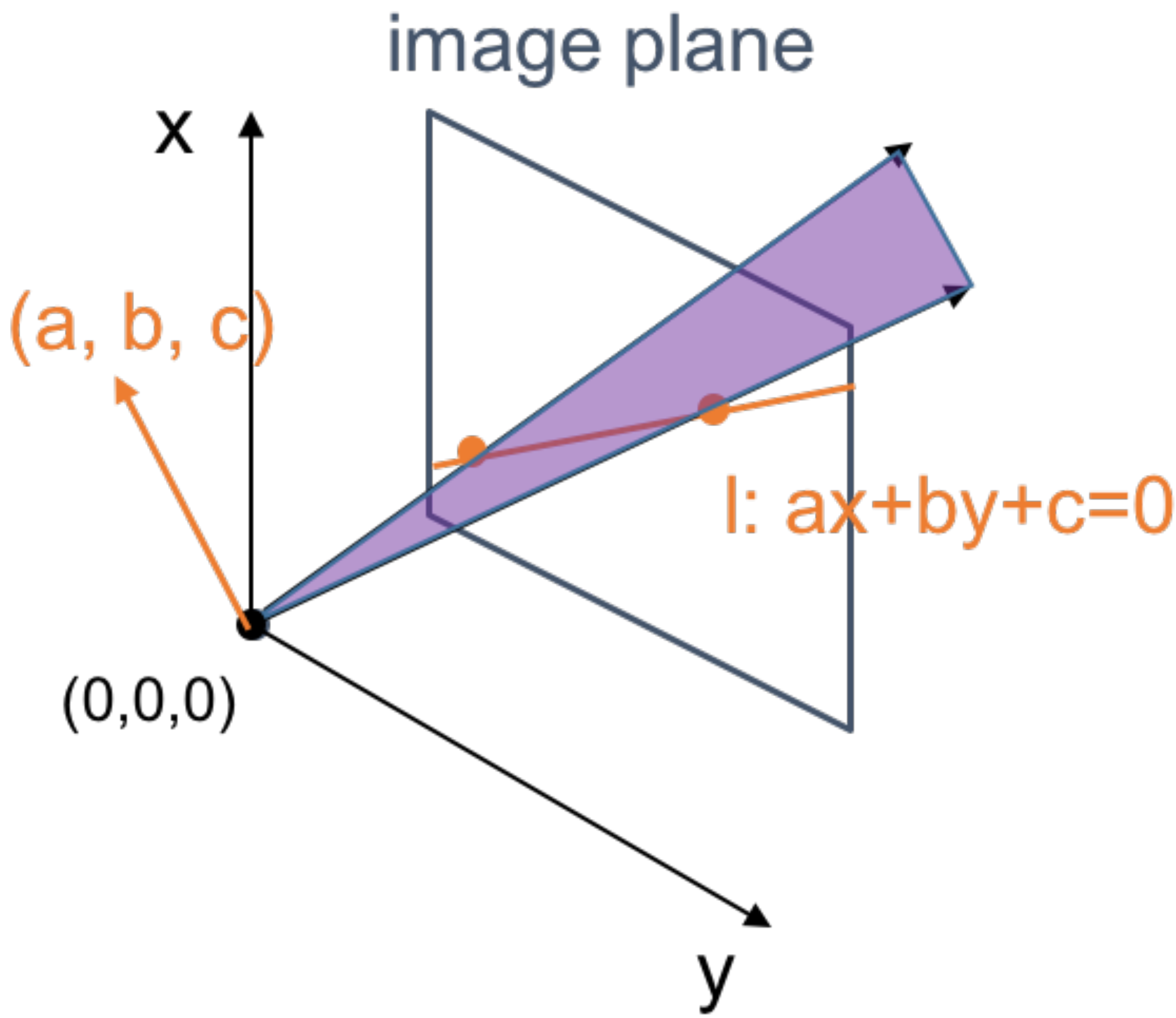


- A line is a *plane* of rays through origin
  - all rays (x,y,z) satisfying: ax + by + cz = 0

$$\text{in vector notation:} \quad 0 = \underbrace{\begin{bmatrix} a & b & c \end{bmatrix}}_{\mathbf{l}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{p}}$$

- A line is also represented as a homogeneous 3-vector **l**

Projective Lines



Line Representation

- a line is $\rho = x\cos\theta + y\sin\theta$

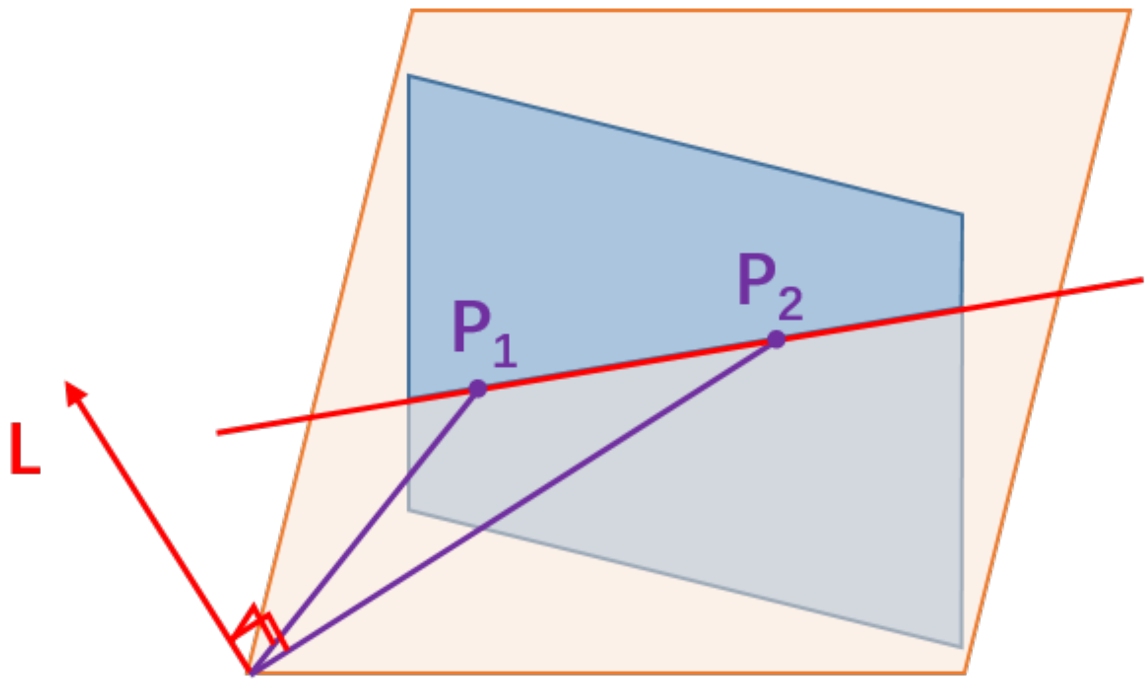
- ρ is the distance from the origin to the line
- θ is the norm direction of the line

- It can also be written as

$$ax + by + c = 0;$$

$$\begin{aligned}
\cos\theta &= \frac{a}{\sqrt{a^2+b^2}} \\
\sin\theta &= \frac{b}{\sqrt{a^2+b^2}} \\
\rho &= -\frac{c}{\sqrt{a^2+b^2}}
\end{aligned}$$

Projective lines from two points



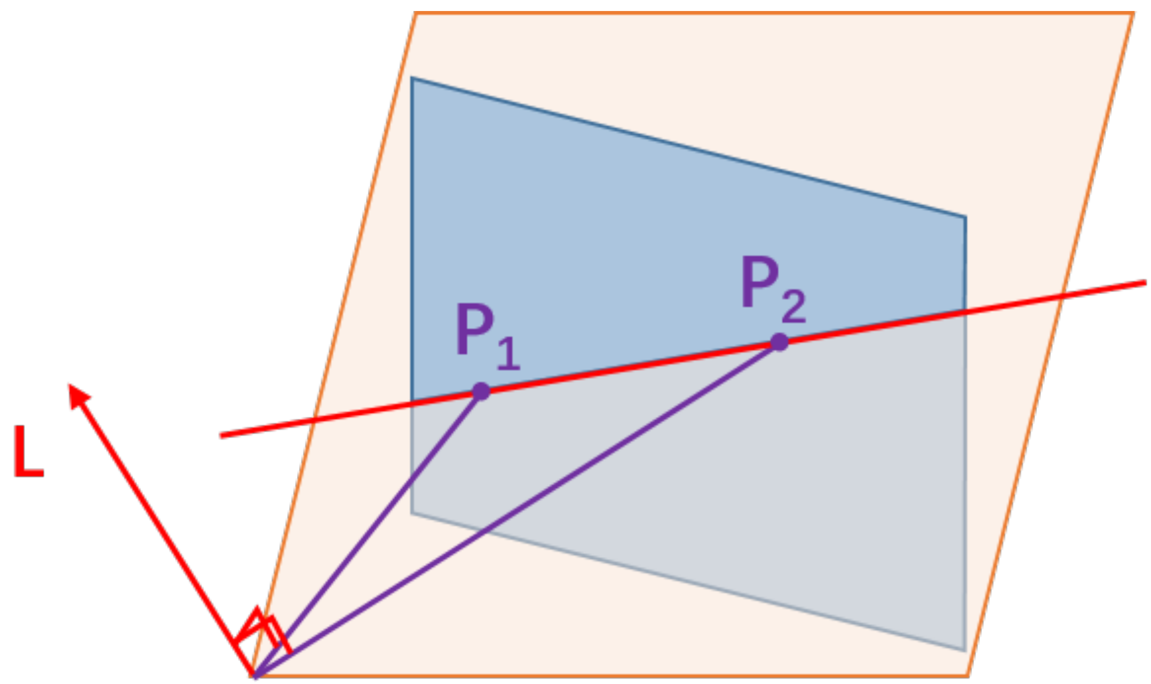
Line passing through two points

Two points: $x \quad x'$

Define a line

l is the line passing two points

$$l = x \times x'$$



Line passing through two points

Two points: x x'

Define a line

$$l = x \times x'$$

l is the line passing two points

Proof:

$$x \cdot (x \times x') = 0$$
$$x' \cdot (x \times x') = 0$$

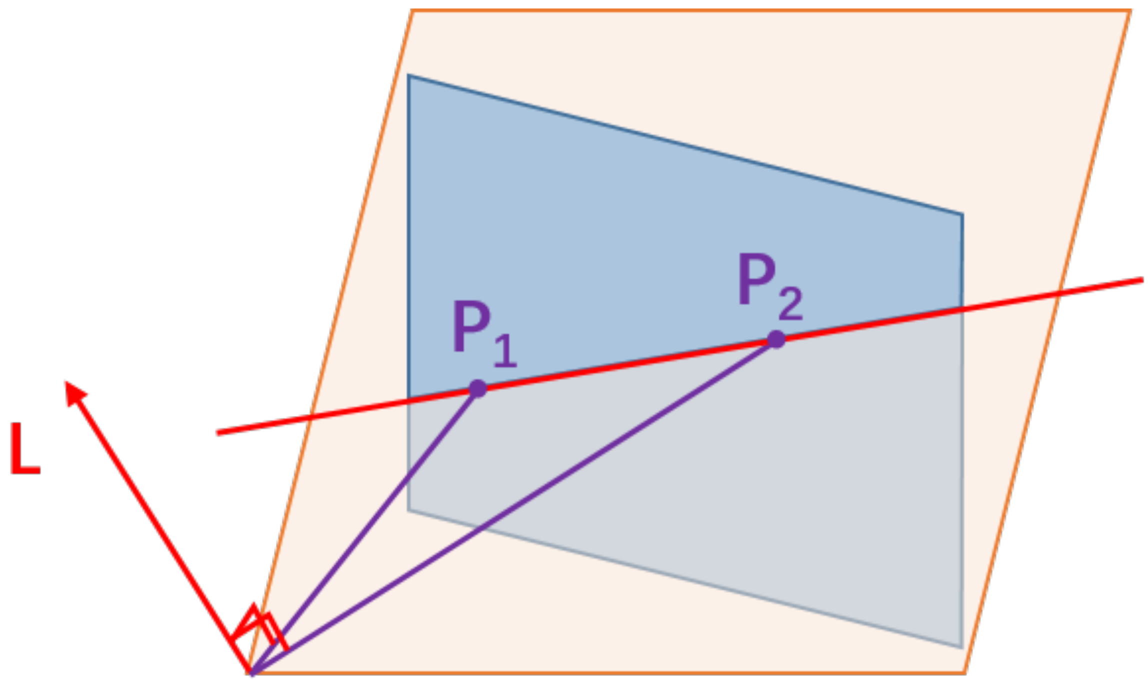
$$x \cdot l = 0$$
$$x' \cdot l = 0$$

Line passing through two points

- M

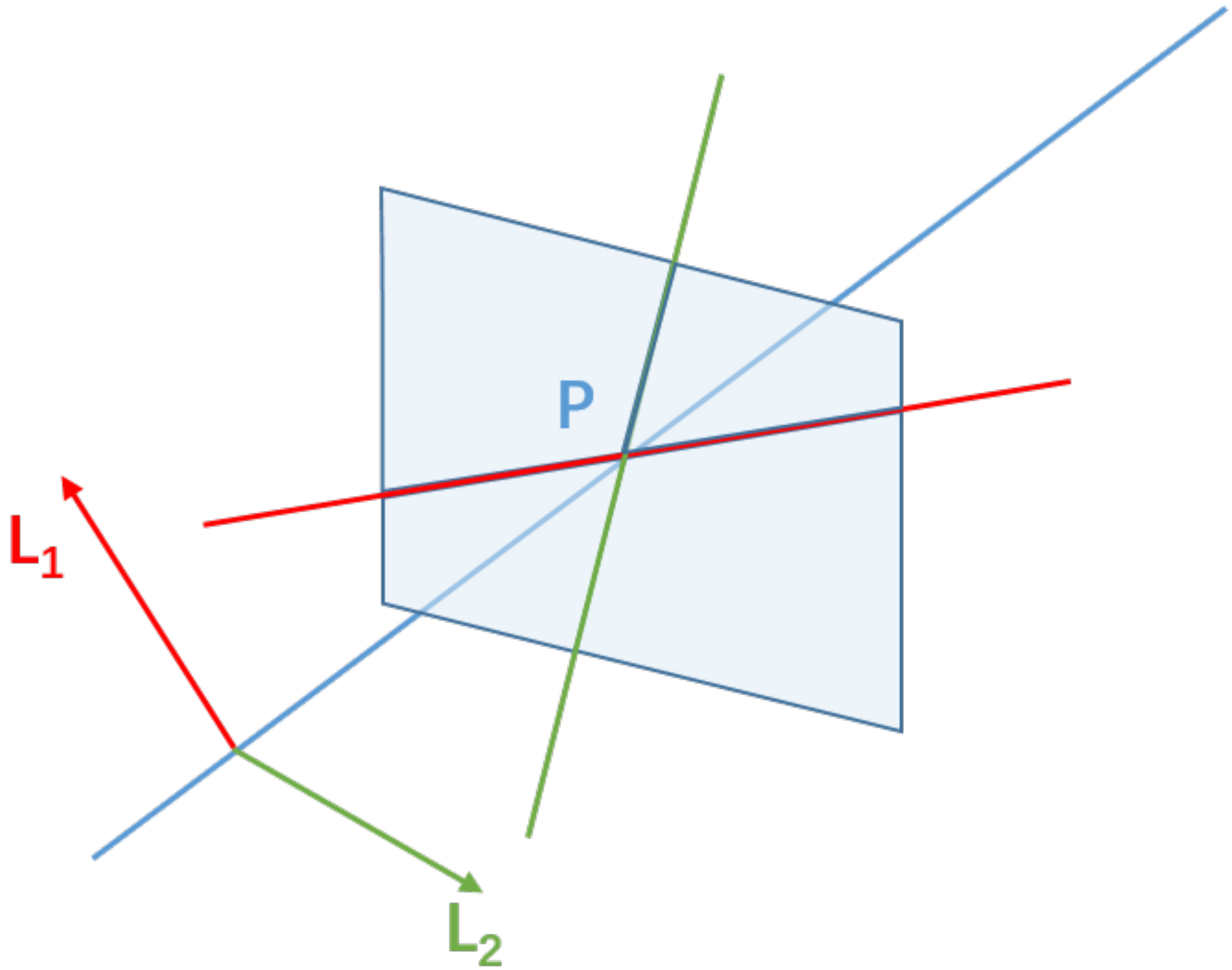
$$
\begin{aligned}
l &= x \times x' \\
&= \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ x'_1 & x'_2 & x'_3 \end{vmatrix} \\
&= (x_2 x'_3 - x_3 x'_2) i \\
&\quad + (x_3 x'_1 - x_1 x'_3) j \\
&\quad + (x_1 x'_2 - x_2 x'_1) k \\
&= (x_2 x'_3 - x_3 x'_2, x_3 x'_1 - x_1 x'_3, x_1 x'_2 - x_2 x'_1)^T
\end{aligned}
$$

Projective lines from two points



When does the line has the form (a, b, 0)?
When does the line has the form (0, 0, 1)?

Points from two lines

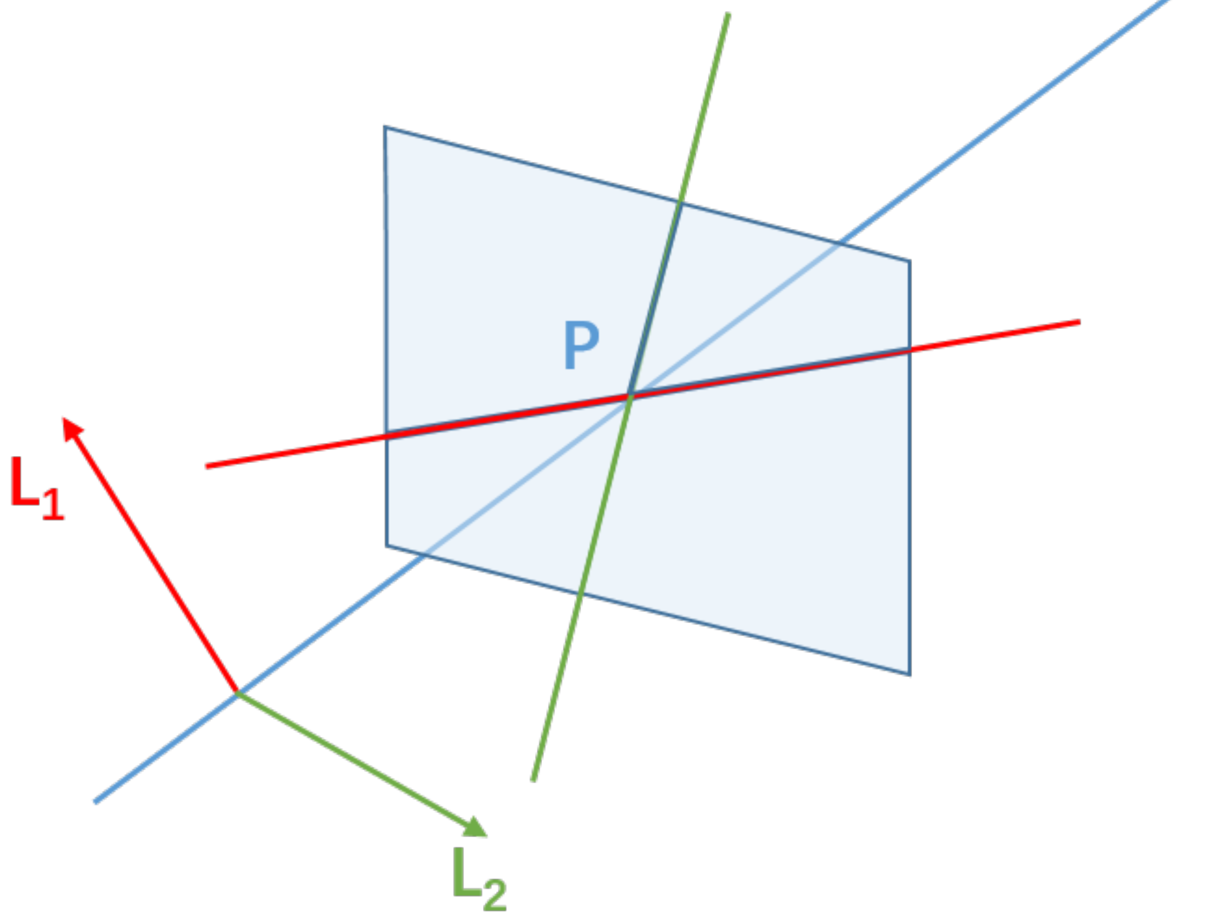


Intersection of lines

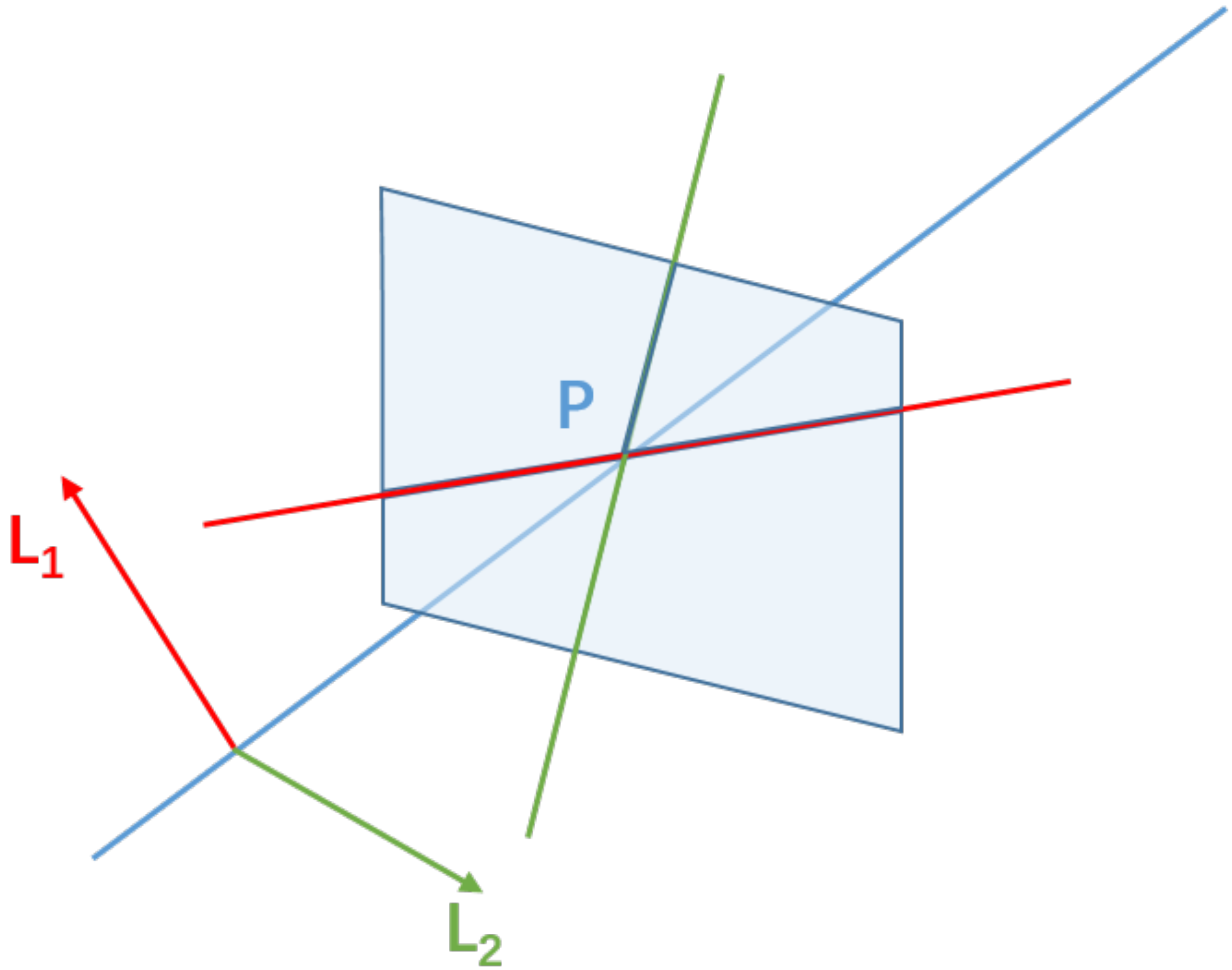
Given two lines: l , l'

Define a point $x = l \times l'$

X is the intersection of the two lines



Points from two lines



When P has the form (x,y,0)?

Point at infinity

Example: Consider two *parallel* horizontal lines:

$$x = 1; x = 2;$$

Intersection =

$$det[(i, j, k); (-1, 0, 1); (-1, 0, 2)]$$

$$= (0, 1, 0)$$

Point at infinity in the direction of y

Point at infinity, Ideal points

$$l = (a, b, c) \qquad\qquad l' = (a, b, c')$$

Intersection:

$$\begin{aligned}
l \times l' &= l \times l' \\
&= \begin{vmatrix} i & j & k \\ a & b & c \\ a & b & c' \end{vmatrix} \\
&= (bc' - bc, ca - c'a, ab - ab)^T \\
&= (c' - c)(b, -a, 0)^T
\end{aligned}$$

Any point $(x_1, x_2, 0)$ is intersection of lines at infinity

Points at infinity

- Under projective transformation,
  - All parallel lines intersects at the point at infinity

$$\text{line } l = (a, b, c)^T \text{ intersects at } (b, -a, 0)^T$$

  - One point at infinity ⇔ one parallel line direction

- Where are the points at infinity in the image plane?

Line at infinity

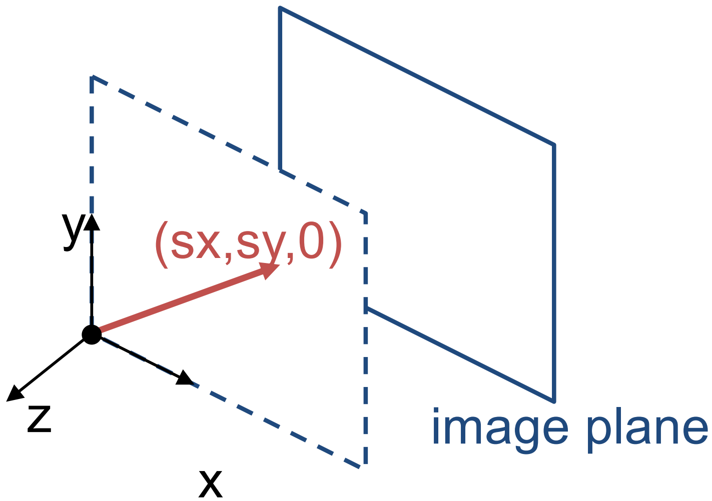
- A line passing all points at infinity:

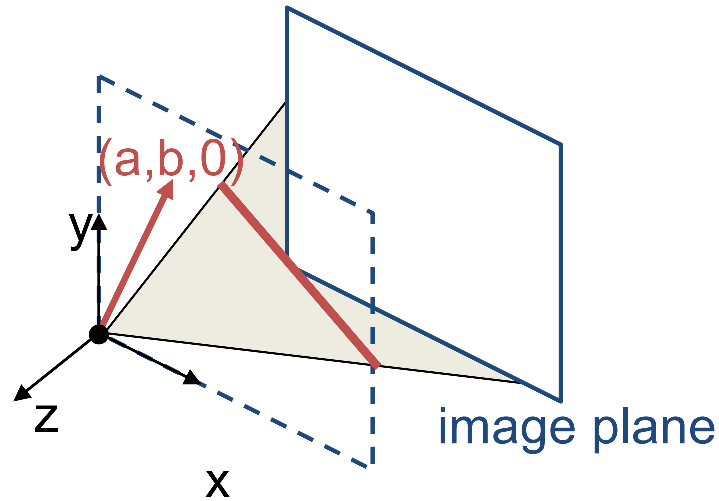
$$l_\infty = (0, 0, 1)^T$$

  - Because :

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = 0$$

"Ideal" points and lines





- Ideal point ("point at infinity")
  - p ≅ (x, y, 0) – parallel to image plane
  - It has infinite image coordinates

Ideal line

- l ≅ (a, b, 0) – parallel to image plane
- Corresponds to a line in the image (finite coordinates)