CIS580 Problem Set 6

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CIS580 Spring 2021

Contents

1 Mapping 2D points

1.1 Use solution for orthogonal Procrustes problem to obtain rotation matrix

Centroid of first set of four points: $(1.25, 1.50)$ and of the second set of four points: $(1.80, 1.11)$.

$$A = \begin{bmatrix} -1.25 & -0.25 & -0.25 & 1.75 \\ \\ -0.50 & -1.50 & -0.50 & 2.50 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.65 & -1.52 & -0.53 & 2.70 \\ \\ 1.18 & 0.07 & 0.19 & -1.43 \end{bmatrix}$$

Taking the SVD of $A \times B^T$

$$[U, S, V^T] = SVD(A \times B^T)$$

$$\implies R = V \times \begin{bmatrix} 1 & 0 \\ \\ 0 & det(V \times U^T) \end{bmatrix} \times U^T$$

$$= \begin{bmatrix} 0.1234 & 0.9924 \\ \\ -0.9924 & 0.1234 \end{bmatrix}$$

1.2 Find solution that solves directly for the rotation angle θ and translation $[T_x, T_Y]$

$$\sum (x_i{}' - x_i \cdot cos(\theta) + y_i \cdot sin(\theta) - t_x)^2$$
$$+ (y_i{}' - x_i \cdot sin(\theta) - y_i \cdot cos(\theta) - t_y)^2$$

Taking partial derivatives with respect to t_x:

$$\frac{\partial}{\partial t_x} = -2 \cdot (-t_x - x_i \cdot cos(\theta) + y_i \cdot sin(\theta) + x') = 0$$

$$\implies t_x = -\bar{x_i} \cdot cos(\theta) + \bar{y_i} \cdot sin(\theta) + \bar{x'}$$

Taking partial derivatives with respect to t_y:

$$\frac{\partial}{\partial t_y} = -2 \cdot (-t_y - x_i \cdot sin(\theta) + y_i \cdot cos(\theta) + y') = 0$$

$$\implies t_y = -\bar{x_i} \cdot sin(\theta) - \bar{y_i} \cdot cos(\theta) + \bar{y'}$$

Next steps:

1. Combine the equations for t_x and t_y using the original equation
2. Take the derivative with respect to θ and set it equal to 0
3. Solve for t_x and t_y by substituting θ

2 Phone held vertically

2.1 Write projection equations

Given:

$$\lambda \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

Using the above, we can generate a system of 4 equations with 4 unknowns as follows:

$$x_2 = \frac{a \cdot cos(\theta) + T_X}{-a \cdot sin(\theta) + T_Z}$$
$$y_2 = \frac{0 + T_Y}{-a \cdot sin(\theta) + T_Z}$$
$$x_1 = \frac{T_X}{T_Z}$$
$$y_1 = \frac{T_Y}{T_Z}$$

2.2 Solve equations for yaw angle θ and translations $[T_x, T_y, T_z]$

Solving the above system of equations:

$$\begin{aligned}
T_X &= x_1 \cdot T_Z \\
x_2 &= \frac{a \cdot cos(\theta) + x_1 \cdot T_Z}{-a \cdot sin(\theta) + T_Z} \\
T_Z \cdot (x_2 - x_1) &= a \cdot cos(\theta) + a \cdot sin(\theta) \cdot x_2 \\
T_Z &= \frac{a \cdot cos(\theta) + a \cdot sin(\theta) \cdot x_2}{x_2 - x_1}
\end{aligned}$$

Another equation in terms of T_Z and θ can be obtained as follows:

$$\begin{aligned}
T_Y &= y_1 \cdot T_Z \\
y_2 &= \frac{0 + T_Y}{-a \cdot sin(\theta) + T_Z} \\
T_Z \cdot (y_2 - y_1) &= a \cdot sin(\theta) \cdot y_2 \\
T_Z &= \frac{a \cdot sin(\theta) \cdot y_2}{y_2 - y_1}
\end{aligned}$$

Solving for θ:

$$\begin{aligned}
\frac{a \cdot sin(\theta) \cdot y_2}{y_2 - y_1} &= \frac{a \cdot cos(\theta) + a \cdot sin(\theta) \cdot x_2}{x_2 - x_1} \\
cos(\theta) &= -x_2 \cdot y_1 + x_1 \cdot y_2 \\
sin(\theta) &= y_1 - y_2
\end{aligned}$$

$$\implies \theta = arctan2\left(\frac{y_1 - y_2}{-x_2 \cdot y_1 + x_1 \cdot y_2}\right)$$

Using this to solve for the other unknowns:

$$\begin{aligned}
T_Z &= \frac{a \cdot sin(\theta) \cdot y_2}{y_2 - y_1} \\
T_X &= x_1 \cdot T_Z & &= \frac{x_1 \cdot a \cdot sin(\theta) \cdot y_2}{y_2 - y_1} \\
T_Y &= y_1 \cdot T_Z & &= \frac{y_1 \cdot a \cdot sin(\theta) \cdot y_2}{y_2 - y_1}
\end{aligned}$$

2.3 Conditions on camera position to obtain unique or finite number of solutions

$y_1 \neq y_2 \implies$ the length of the rod cannot be zero.

3 Decompose H into rotation R and translation T

Given:

$$g_1 = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \qquad g_2 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$H = \begin{bmatrix} \frac{-4}{8} & 0 & 1 \\ \frac{6}{8} & \frac{\sqrt{3}-12}{8} & \frac{5-4\sqrt{3}}{8} \\ \frac{-2\sqrt{3}}{8} & \frac{7+4\sqrt{3}}{8} & \frac{\sqrt{3}+4}{8} \end{bmatrix}$$

Also, from the provided picture we can obtain the rotation matrices R_1 and R_2:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \qquad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$H' = {R_2}^{-1} \times H \times R_1$$

$$= \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 2 & \frac{-1}{2} \end{bmatrix}$$

We know H' is a composition of a rotation about the Y axis, followed by a translation. Thus, we can also represent it as:

$$H' = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} + \begin{bmatrix} 0 & \frac{T'_X}{2} & 0 \\ 0 & \frac{T'_Y}{2} & 0 \\ 0 & \frac{T'_Z}{2} & 0 \end{bmatrix}$$

Equating the two version of H', we find:

$$\begin{bmatrix} \frac{T'_X}{2} & \frac{T'_Y}{2} & \frac{T'_Z}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}$$

$$\theta = \frac{2 \cdot \pi}{3}$$

We also know that the rotation matrix to go from Camera 1 to Camera 2 in Camera 1 coordinates R_Y is a rotation about the Y axis by θ radians. Thus:

$$R_Y = \begin{bmatrix} \frac{-1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{-\sqrt{3}}{2} & 0 & \frac{-1}{2} \end{bmatrix}$$

Combining the above information, we can create ${}^1\mathbf{R}_2$:

$${}^1\mathbf{R}_2 = R_1 \times R_Y \times {R_2}^{-1} = \begin{bmatrix} -0.500 & -0.750 & 0.433 \\ 0.433 & 0.217 & 0.875 \\ -0.750 & 0.625 & 0.217 \end{bmatrix}$$

Now we use the above information to extract the translation vector from H.

$$T = R_1 \times T'$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1.000 \\ -2.000 \\ 3.464 \end{bmatrix}$$

Thus, H can be decomposed into ${}^1\mathbf{R}_2$ and T.