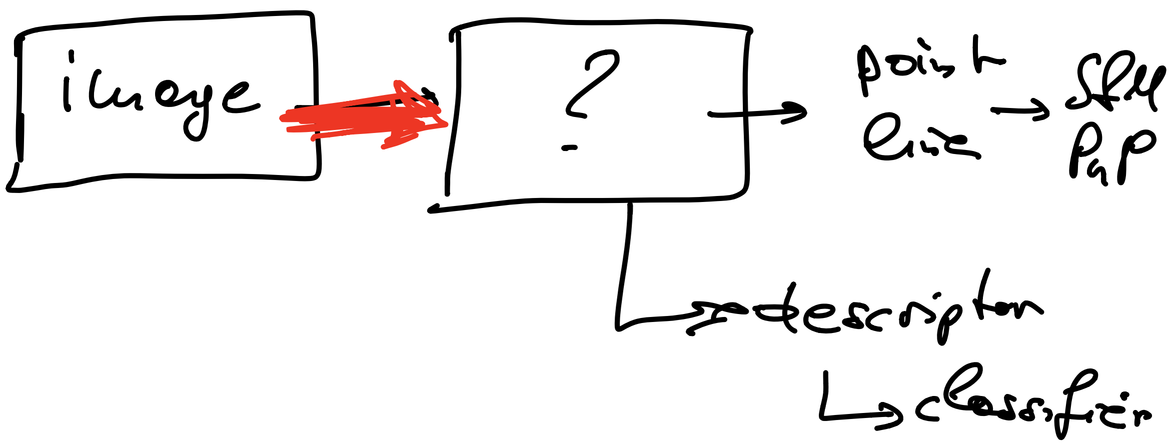


Last time: Any linear (LSI) shift-invariant system can be realized with a convolution.

$$f(t) \longrightarrow \boxed{\text{LSI}} \longrightarrow g(t)$$

$$\exists\, h(t)$$

$$g(t) = \int_{-\infty}^{\infty} f(t')\, h(t-t')\, dt'$$

$$\delta(t) \longrightarrow \boxed{} \longrightarrow h(t)$$

impulse response

because $h(t) = \int_{-\infty}^{\infty} \delta(t')\, h(t-t')\, dt'$

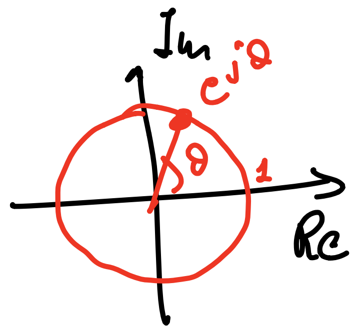
Fourier Transform

Periodic function:

Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} F[n]\, e^{j2\pi \frac{n}{T} t}$$

$$j^2 = -1 = i^2$$



$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{j\left(2\pi\frac{n}{T}\right)t} = e^{j\omega t} = \cos\omega t + j\sin\omega t$$

↳ $\omega = 2\pi \frac{n}{T}$

angular frequence

Hz

period

What is $F[n]$?

↳ coefficient of Fourier series

Fourier Transform

$$F[n] = \int_{-T/2}^{T/2} f(t)\, e^{-j2\pi\frac{n}{T}t}\, dt$$

How is this related to the chord breakdown?

$$\text{chord } f(t) = \sum_{n=-\infty}^{\infty} \underbrace{A_n}_{\text{amplitude } (\in \mathbb{R})} \cos\left(2\pi \frac{n}{T} t + \underbrace{\varphi_n}_{\text{phase}}\right)$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{F[n]}_{\text{complex number}} e^{j2\pi \frac{n}{T} t}$$

it accounts for ← number

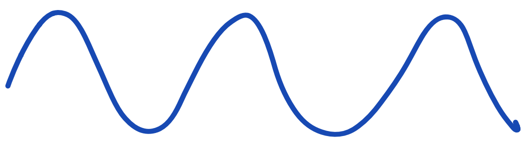
both amplitude A_n and phase φ_n.

$$f(t) \circ\!\!-\!\!\bullet F(\omega)$$

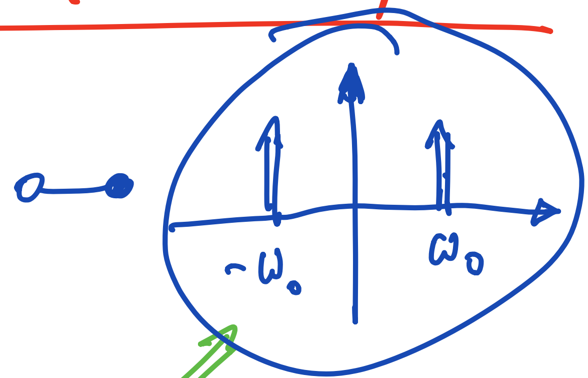
Non-periodic functions have a $\boxed{\text{continuous}}$ Fourier transform

$F(\omega)$ as opposed to $F[n]$

Example of Fourier Transform



$1 \cdot \cos(\omega_0 t)$

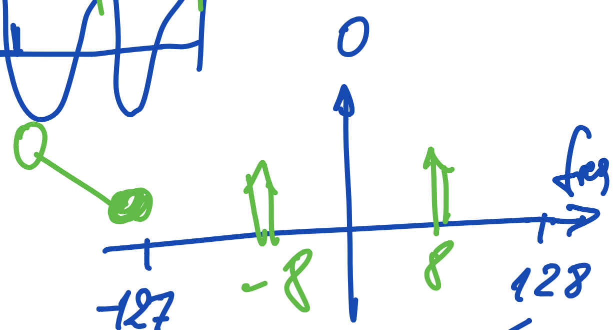


what the heck is a negative frequency!

$$\frac{1}{2}\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)$$

8 cycles

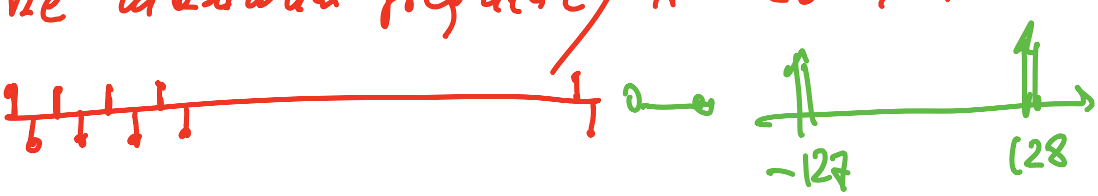
256 pixels



discrete-time signal

or image in pixels

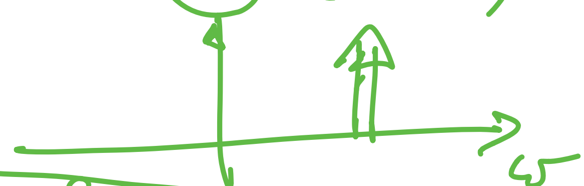
In a discrete signal of length 256
the maximum frequency is 128 cycles



$\boxed{\text{Minimum period is 2 pixels}}$

Maximum frequency is length of signal L divided by 2. $= \frac{L}{2}$

$$e^{j\omega_0 t} \circ\!\!-\!\!\bullet\ 2\pi\,\delta(\omega-\omega_0)$$



no negative frequence

Phase / Modulation Theorem

Shift Theorem: $f(t-t_0) \circ\!\!-\!\!\bullet\ F(\omega)e^{-j\omega t_0}$

Modulation Theorem: $f(t)e^{j\omega_0 t} \circ\!\!-\!\!\bullet\ F(\omega-\omega_0)$

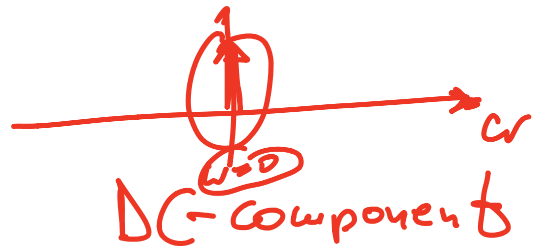
AM

carrier frequency

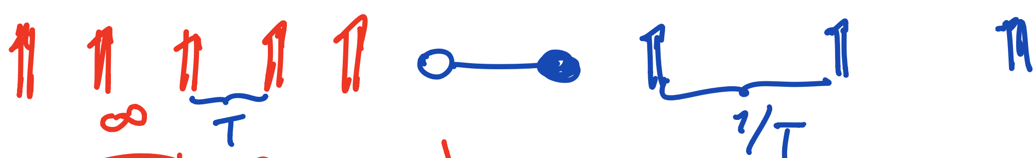
Impulse Theorem

$$\delta(t) \circ\!\!-\!\!\bullet\ 1$$

$$1 \circ\!\!-\!\!\bullet\ 2\pi\delta(\omega)$$

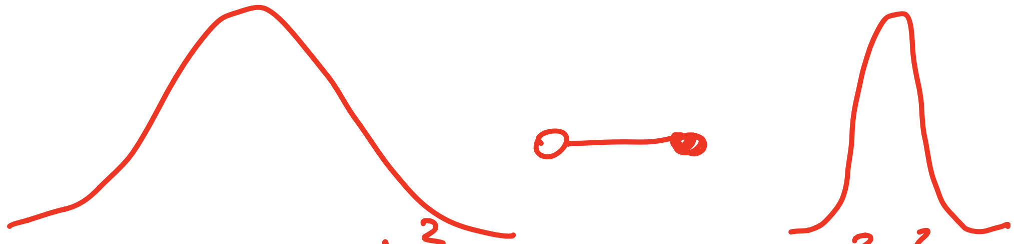


Comb Theorem



$$\sum_{n=-\infty}^{\infty}\delta(t-nT) \circ\!\!-\!\!\bullet\ \frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi n}{T}\right)$$

Fourier of Gaussian



$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{t^2}{2\sigma^2}} \;\circ\!\!-\!\!\bullet\; e^{-\frac{\omega^2\sigma^2}{2}}$$

Hz

$$\omega = 2\pi s$$

angular

$$= 2\pi \frac{n}{T}$$

cycles

$$e^{-\frac{\omega^2\sigma^2}{2}} = e^{-\frac{4\pi^2 s^2\sigma^2}{2}} = e^{-2\pi^2 s^2\sigma^2}$$

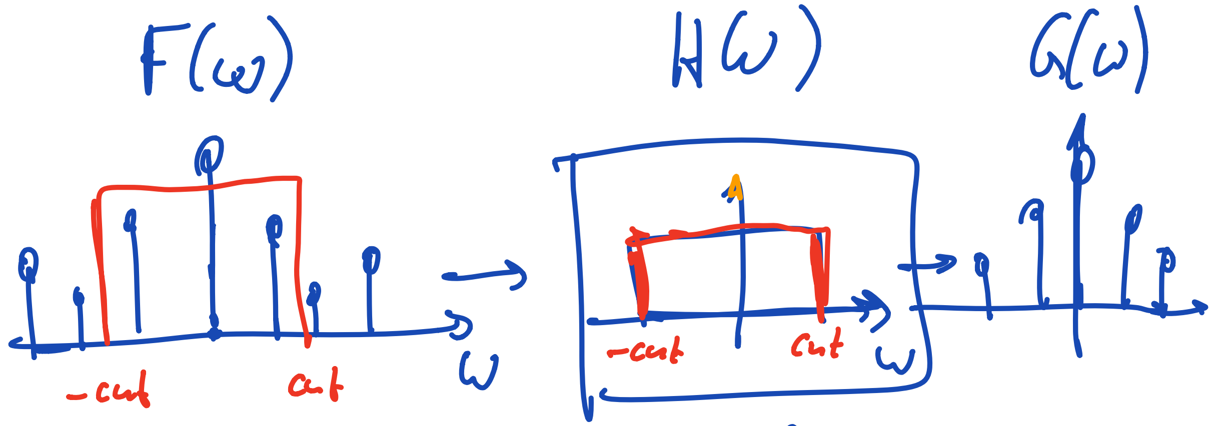
slide

Convolution Theorem

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow g(t) = \int f(t')\,h(t-t')\,dt'$$

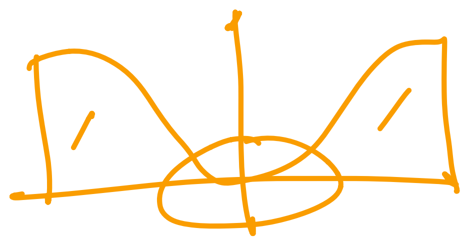
LSI

$$F(\omega) \qquad H(\omega) \qquad \boxed{G(\omega) = F(\omega)\cdot H(\omega)}$$



Low Pass Filter

high-pass



Fourier

cos, sin, $e^{j\omega t}$, $e^{\frac{t^2}{2\sigma^2}}$

Theorem: shift, modulation, convolution

2D Convolution
2D Fourier

reflection

$$h[-x,-y]$$

$$\begin{array}{c|ccc} & -1 & 0 & 1 \\ \hline -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} & -1 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}$$

$$h[x,y]$$

$$h[-1,-1]=1$$

offset

$$\begin{array}{ccc|c} 0 & 0 & 0 & \\ 0 & a & b & c \\ 0 & d & e & f \\ \hline g & h & i & \end{array}$$

centered convolution

$$\begin{array}{|c|c|c|} \hline e & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

Separable 2D filters

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

$$h(x,y) = h_1(x)\, h_2(y)$$

$$\overset{N \times N}{f(x,y)} *_{2D} \overset{M \times M}{h(x,y)}$$

$N^2 \cdot M^2$ operations

$$= f(x,y) *_{2D} \left(h_1(x) *_{2D} h_2(y)\right)$$

$$= \left(f(x,y) *_{2D} h_1(x)\right) *_{2D} h_2(y)$$

$N^2 \cdot M$ [ ] $+$ $N^2 \cdot M$ [ ]

$N^2 M^2$ vs. $N^2 \cdot 2M$

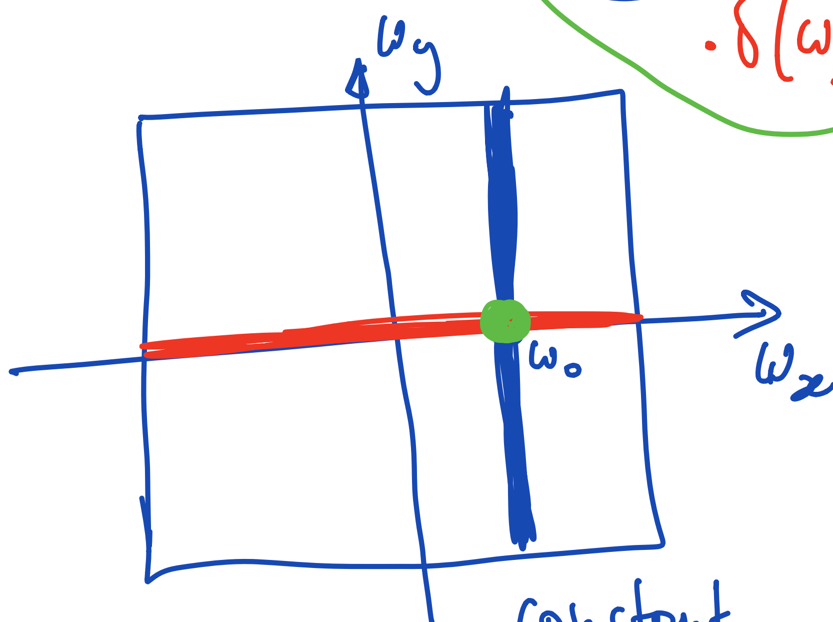
non-separable vs. separable

$M=3$: $9N^2$ vs $6N^2$

$M=7$: $49N^2$ vs. $15N^2$

1D Fourier $\quad e^{j\omega_0 t} \;\circ\!\!-\!\!\bullet\; \delta(\omega-\omega_0)$

2D Fourier $\quad e^{j\omega_0 x} \;\circ\!\!-\!\!\bullet\; \delta(\omega_x-\omega_0)\cdot\delta(\omega_y)$



constant along ω_y

Signal and System

Willsky

Online class Shree Nayar