solutions

1 problem 1

1.1 Find λ such that the three lines of $\mathbb{P}^2$, $w = 0$, $x+\lambda y+\lambda w = 0$, and $\lambda x+y+\lambda w = 0$ have a common intersection. Which point is the intersection?

The three lines can be written as a linear system

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

From the first constraint we know $w = 0$, what remains is to determine λ so some x and y solve

$$\begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

For a non trivial solution the determinant of the matrix must be 0 implying $\lambda = -1, 1$. This results in the point of intersection $(0, 0, 1)X(-1, 1, -1) = (-1, -1, 0) \sim (1, 1, 0)$ when $\lambda = -1$ and $(0, 0, 1)X(1, 1, 1) = (-1, 1, 0) \sim (1, -1, 0)$ when $\lambda = 1$. Note for case $\lambda = -1$, the point found corresponds to the intersection of 3 lines while for $\lambda = 1$, the point found corresponds to the intersection of 2 lines since lines 2 and 3 become the same.

1.2 For each of the following pairs of points in $\mathbb{P}^2$, write down an equation for the line that passes through them:

(a) $(a)\,[2, 3, 4]$ **and** $[3, 5, 1]$

(b) $(b)\,[a, 0, 1]$ **and** $[0, b, 1]$

(a) $l = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -17 \\ 10 \\ 1 \end{pmatrix}$

(b) $l = \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ -a \\ ab \end{pmatrix}$

1.3 For each of the following pairs of lines in $\mathbb{P}^2$, determine the point of intersection:

(a) $x - 2y + w = 0$ **and** $3x + y - 4w = 0$

(b) $x + 5y - 6w = 0$ **and** $7x - 2w = 0$

(a) $p = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix} \cong \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) $p = \begin{pmatrix} 1 \\ 5 \\ -6 \end{pmatrix} \times \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -10 \\ -40 \\ -35 \end{pmatrix} \cong \begin{pmatrix} 2 \\ 8 \\ 7 \end{pmatrix}$

2 problem 2

A projective transformation M preserves the points $p_1 = (1,0,0)$, $p_2 = (0,1,0)$, and the origin of the coordinate system o. However, it maps the point $p_3 = (1,1,1)$ to the points $p_3' = (2,1,1)$, meaning $(2,1,1)^T \sim M(1,1,1)^T$.

2.1 Compute M.

Our projective transformation maps a point p to $\lambda p'$

$$\lambda p' = Mp \tag{3}$$

We are given the mappings for four points described as vectors. We can combine the vectors to form the following matrix

$$\begin{pmatrix} p_1' & p_2' & p_3' & o' \end{pmatrix} = M \begin{pmatrix} p_1 & p_2 & p_3 & o \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} \lambda_1 & 0 & 2\lambda_3 & 0 \\ 0 & \lambda_2 & \lambda_3 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \end{pmatrix} = M \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{5}$$

Dividing through by λ_4 and letting $M = \lambda_4 M'$, $\alpha = \lambda_1/\lambda_4$, $\beta = \lambda_2/\lambda_4$, and $\gamma = \lambda_3/\lambda_4$, we have,

$$\begin{pmatrix} \alpha & 0 & 2\gamma & 0 \\ 0 & \beta & \gamma & 0 \\ 0 & 0 & \gamma & 1 \end{pmatrix} = M \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{6}$$

and

$$\begin{pmatrix} \alpha & 0 & 2\gamma & 0 \\ 0 & \beta & \gamma & 0 \\ 0 & 0 & \gamma & 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{11} + m_{12} + m_{13} & m_{13} \\ m_{21} & m_{22} & m_{21} + m_{22} + m_{23} & m_{23} \\ m_{31} & m_{32} & m_{31} + m_{32} + m_{33} & m_{33} \end{pmatrix} \tag{7}$$

Solving for the entries of M we have

$$M = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

Observing that $\gamma = 1$ allows us to solve for α and β giving

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

2.2 Compute the image of the line at infinity.

The line at infinity is defined $l_\infty = (0, 0, 1)$. Remember that the transformation maps a point p to p',

$$p' = Mp \tag{10}$$

And a point p can be said to be on a line l if

$$l^T M^{-1} p' = 0 \tag{11}$$

Plugging in our mapping we have

$$l^T p = 0 \tag{12}$$
$$l^T M^{-1} p' = 0 \tag{13}$$
$$(l^T M^{-1}) p' = 0 \tag{14}$$

And we see $l^T M^{-1}$ is a line l'; $l' = M^{-T} l$.

Using the above, the result of applying the mapping $M to l_∞ is,

$$l' = M^{-T} l_\infty \tag{15}$$

$$M^T l' = l_\infty \tag{16}$$

$$\begin{pmatrix} 2l'_1 \\ l'_2 \\ l'_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{17}$$

and we see the l_∞ stays at infinity.

3 problem 3

The image of the rectangle-shaped facade of a building has two vanishing points, one at $v_x = (-b, 0)$ corresponding to horizontal lines and one at $v_y = (0, h)$ corresponding to the vertical lines. Find the transformation that will map the facade to a rectangle. Assume that the origin $o = (0, 0)$ and the point $p = (1, 1)$ remain fixed. You are free to use the result of the last question but you can definitely find a solution without it.

The idea here is to map the vanishing points to ideal points (points at infinity) Assume the inverse mapping $H = P^{-1}$. Since the horizontal point at infinity is mapped to vanishing point , vertical point at infinity to v_y and the origin is fixed.

$$\begin{pmatrix} -\alpha b & 0 & 0 \\ 0 & \beta h & 0 \\ \alpha & \beta & \gamma \end{pmatrix} = H \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

$$\tag{19}$$

For normalization assume $\gamma = 1$.Thus,

$$\begin{pmatrix} -\alpha b & 0 & 0 \\ 0 & \beta h & 0 \\ \alpha & \beta & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{20}$$

$$\tag{21}$$

$$\begin{aligned} \alpha &= -\lambda/b \\ \beta &= \lambda/h \\ -\lambda/b + \lambda/h + 1 &= \lambda \end{aligned}$$

Thus $\lambda = \frac{bh}{bh-b+h}, \alpha = \frac{-h}{bh-b+h}, \beta = \frac{b}{bh-b+h}$

$$H = \begin{pmatrix} \frac{bh}{bh-b+h} & 0 & 0 \\ 0 & \frac{bh}{bh-b+h} & 0 \\ \frac{-h}{bh-b+h} & \frac{b}{bh-b+h} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-1}{b} & \frac{1}{h} & \frac{bh-b+h}{bh} \end{pmatrix} \tag{22}$$

$$P = H^{-1} = \begin{pmatrix} -1/\alpha b & 0 & 0 \\ 0 & 1/\beta h & 0 \\ 1/b & -1/h & 1 \end{pmatrix} = \begin{pmatrix} \frac{bh-b+h}{bh} & 0 & 0 \\ 0 & \frac{bh-b+h}{bh} & 0 \\ 1/b & -1/h & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{h}{bh-b+h} & \frac{-b}{bh-b+h} & \frac{bh}{bh-b+h} \end{pmatrix} \tag{23}$$

4 problem 4

4.1 Find the projective transformation A which will keep the points $p_1 = (0,0,1)$ and $p_2 = (1,1,1)$ fixed and will map point $p_3 = (1,0,1)$ to $p_3' = (1,0,0)$ and point $p_4 = (0,1,1)$ to $p_4' = (0,1,0)$?

$$\begin{pmatrix} 0 & \beta & \gamma & 0 \\ 0 & \beta & 0 & 1 \\ \alpha & \beta & 0 & 0 \end{pmatrix} = A \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{24}$$

The first point tells us the last row of A,

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & \alpha \end{pmatrix} \tag{25}$$

The last point tells us about the middle column

$$\begin{pmatrix} * & 0 & 0 \\ * & 1 & 0 \\ * & -\alpha & \alpha \end{pmatrix} \tag{26}$$

The third point tells us about the first column

$$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha & -\alpha & \alpha \end{pmatrix} \tag{27}$$

The second point says "all rows have the same sum" so it must be true that $\gamma = -\alpha = 1$.

4.2 Find the projective transformation A which satisfies $K' \sim AK$, $L' \sim AL$, $M' \sim AM$, $N \sim AN$.

H=

$$\begin{pmatrix} -4\alpha & 0 & -4\gamma \\ 0 & 4\beta & 0 \\ \alpha & \beta & \gamma \end{pmatrix} \tag{28}$$

$$H \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{29}$$

therefore

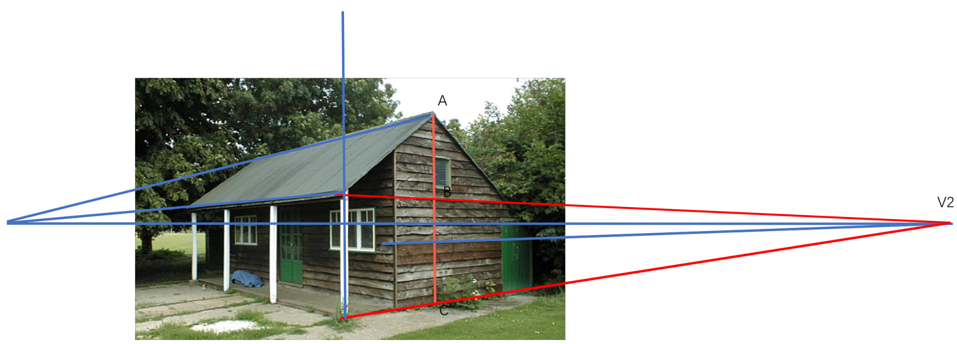
$$H \sim \begin{pmatrix} 12 & 0 & -4 \\ 0 & 16 & 0 \\ 4 & 4 & 1 \end{pmatrix} \tag{30}$$

Although the question does not ask us to solve for n'_x and n'_y we can at this point.

5 problem 5

For this problem of Distance Transfer, we first connect two parallel lines in the world, on the right facade of the building, thus finding the first horizontal vanishing point V_2. Next, we do the same for the left side and we find the second horizontal vanishing point V_1. We connect the two lines and find the horizon. The vertical vanishing point lies at infinity. Since the vertical lines are vertical in the image and the vanishing point lies at infinity, the horizon line should be horizontal in the image, which could double check V_1 and V_2.

Next, we draw a line from the bottom of the pillar to the bottom of the house to the horizon (which meet as expected in V_1, since that line is also parallel with the ones used to find V_2. Next, we draw a line from V_1 to the uppermost part of the pillar and name B the point of intersection with the red line. We put an A on top and a C on bottom. We calculate the distances: $(BC)^{image} = 13/16(in), (AC)^{image} = 3/2(in), (BC)^{world} = 3(m)$. Since the vertical vanishing point is at infinity, the cross ratios become: $\frac{(AC)^{image}}{(BC)^{image}} = \frac{(AC)^{world}}{(BC)^{world}} \implies (AC)^{world} = 5.54m$


A
B
C
V2