2D Convolutions

$$(f * h)(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') h(x - x', y - y') dx' dy'$$

$$(f * h)[r, c] = \sum_{c'=-\infty}^{\infty} \sum_{c'=-\infty}^{\infty} f[r', c'] h[r - r', c - c']$$

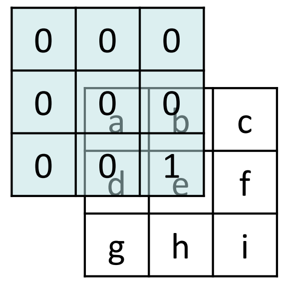
2D Convolution is commutative

I

a	b	c
d	e	f
g	h	i

f

1	0	0
0	0	0
0	0	0



$I \otimes f$

e	f	0
h	i	0
0	0	0

f

1	0	0
0	0	0
0	0	0

I

a	b	c
d	e	f
g	h	i

f$\otimes I$

e	f	0
h	i	0
0	0	0

2D Convolution is associative

- Best use of associativity in separable filters

$$
\begin{aligned}
h(x,y) &= h_1(x)h_2(y) \\
f(x,y) \star h(x,y) &= (f(x,y) \star h_1(x)) \star h_2(y)
\end{aligned}
$$

$$
\begin{bmatrix} +1 & 0 & -1 \\ +2 & 0 & -2 \\ +1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} +1 & 0 & -1 \end{bmatrix}
$$

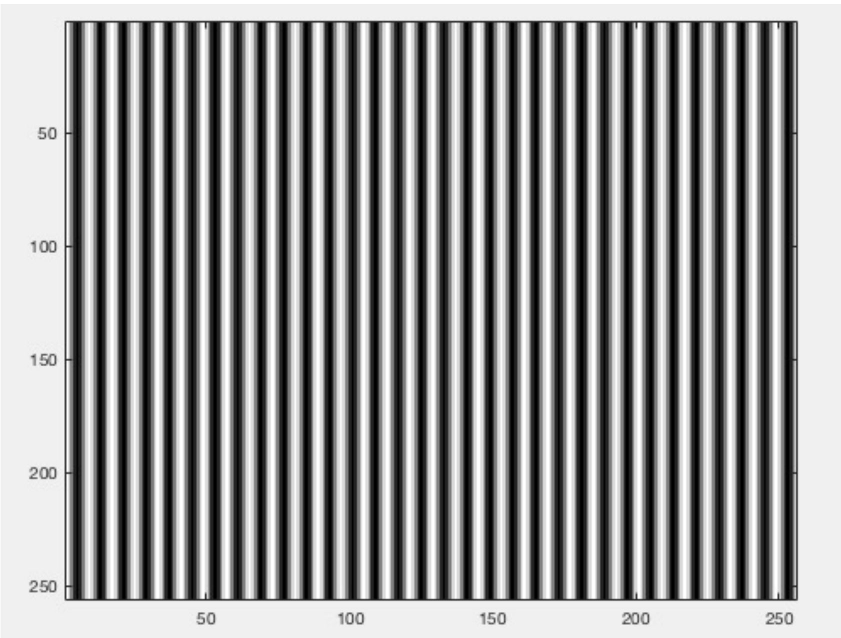
The 2D Fourier Transform

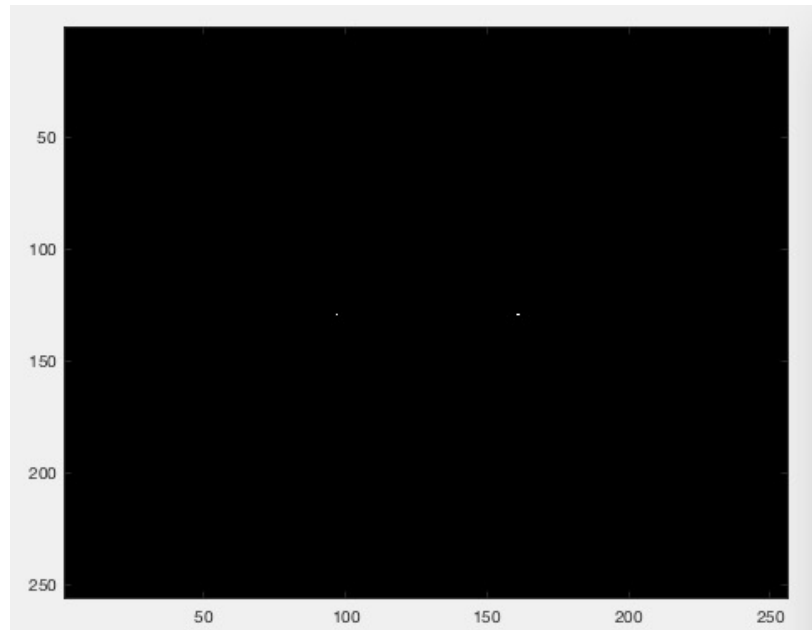
2D Fourier Transform

$$f(x,y) \circ\!\!-\!\!\bullet \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j(\omega_x x+\omega_y y)}dxdy$$

$$F(\omega_x,\omega_y) \bullet\!\!-\!\!\circ \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega_x,\omega_y)e^{j(\omega_x x+\omega_y y)}d\omega_x d\omega_y$$

Example: 1D-cosine as an image





$$f(x,y) = \cos(\omega_0 x) \qquad f(x,y) \circ\!\!-\!\!\bullet \frac{1}{2}(\delta(\omega_x - \omega_0) + \delta(\omega_x + \omega_0)).\delta(\omega_y)$$

Separable functions

$$f(x,y) = f_1(x)f_2(y) \circ\!\!-\!\!\bullet \int_{-\infty}^{\infty} f_1(x)e^{-j\omega_x x}dx \int_{-\infty}^{\infty} f_2(y)e^{-j\omega_y y}dy = F_1(\omega_x)F_2(\omega_y)$$

$$f(x,y) = \cos(\omega_1 x)\cos(\omega_2 y)$$

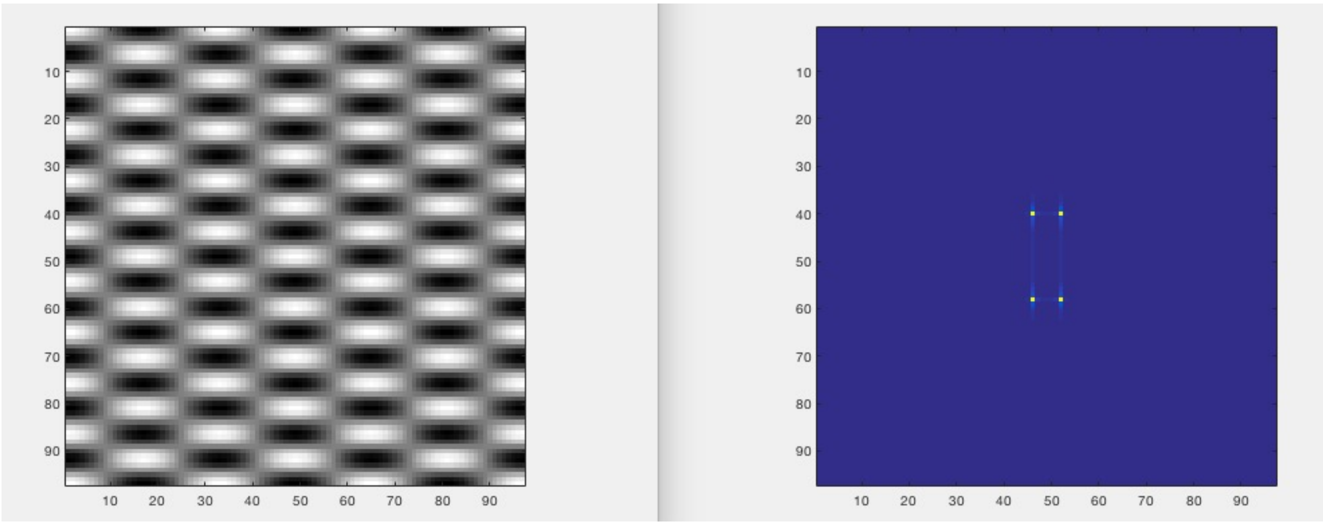
$$\circ\!\!-\!\!\bullet$$

$$\frac{1}{2}(\delta(\omega_x - \omega_1) + \delta(\omega_x + \omega_1))\frac{1}{2}(\delta(\omega_y - \omega_2) + \delta(\omega_y + \omega_2))$$

Separable functions

$$f(x,y) = \cos(\omega_1 x)\cos(\omega_2 y)$$

$$\frac{1}{2}(\delta(\omega_x - \omega_1) + \delta(\omega_x + \omega_1))\frac{1}{2}(\delta(\omega_y - \omega_2) + \delta(\omega_y + \omega_2))$$

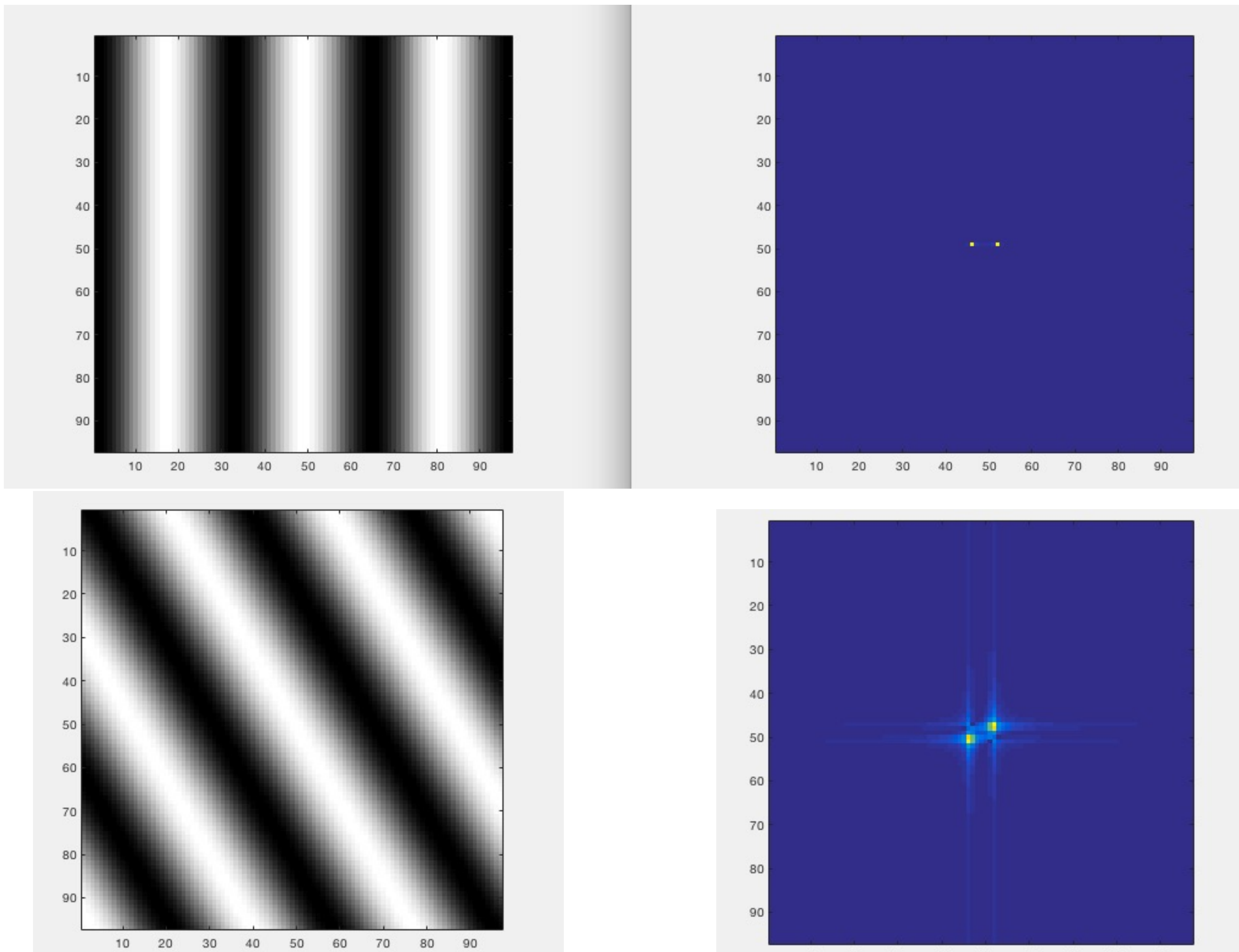


Shift Theorem in 2D

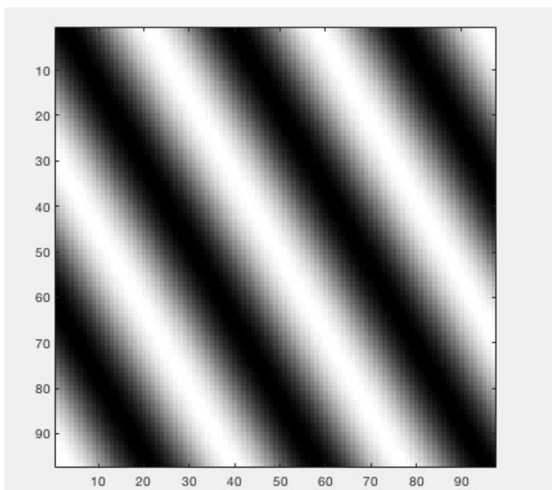
$$f(x - x_0, y - y_0) \circ\!\!-\!\!\bullet\ F(\omega_x, \omega_y) e^{-j(\omega_x x_0 + \omega_y y_0)}$$

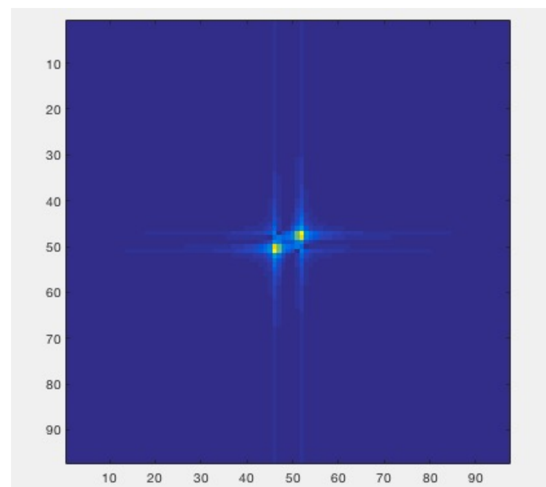
If we know the phases of two 1D signals we can recover their relative displacement?
But can we do that for 2D images?

2D rotation

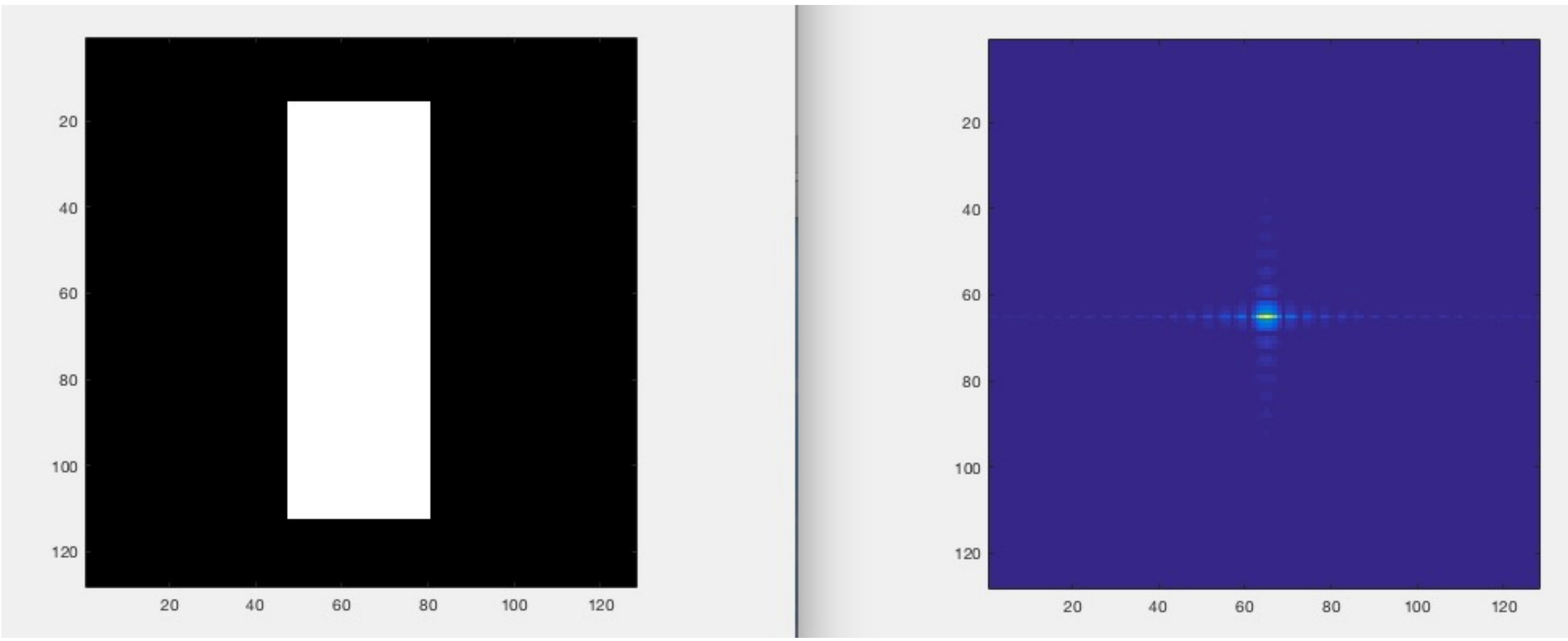

10 20 30 40 50 60 70 80 90
10 20 30 40 50 60 70 80 90
10 20 30 40 50 60 70 80 90
10 20 30 40 50 60 70 80 90
10 20 30 40 50 60 70 80 90
10 20 30 40 50 60 70 80 90
10 20 30 40 50 60 70 80 90
10 20 30 40 50 60 70 80 90

2D rotation

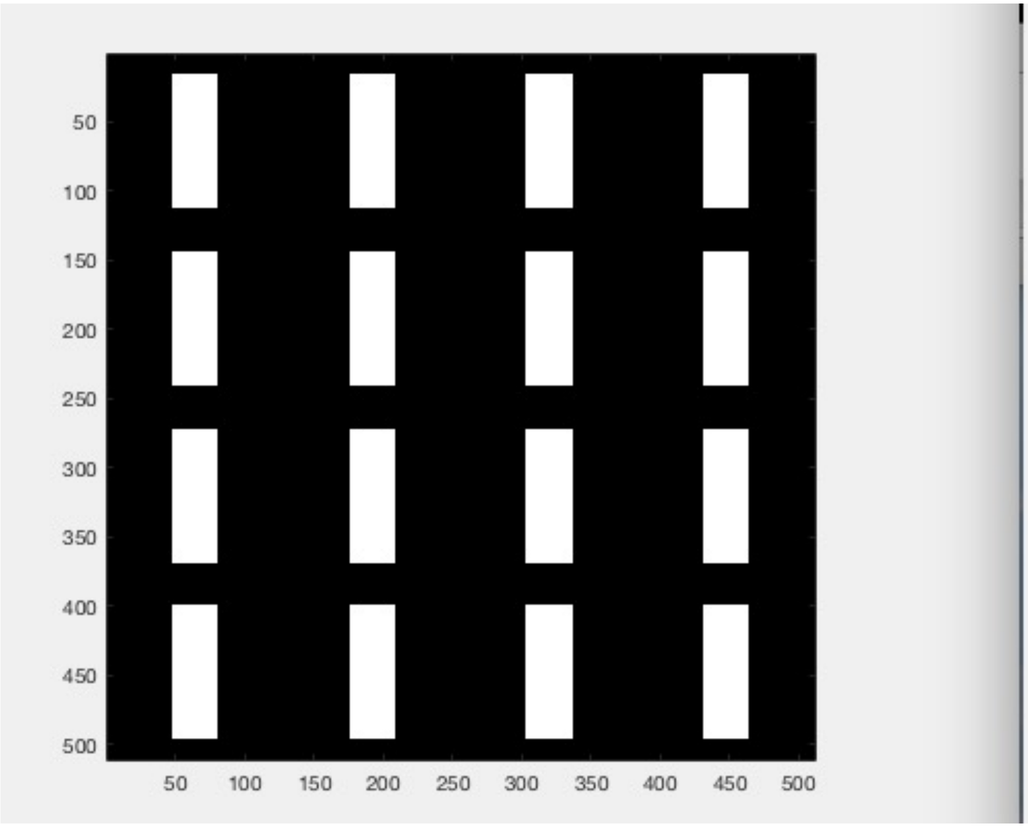

10
20
30
40
50
60
70
80
90
10 20 30 40 50 60 70 80 90

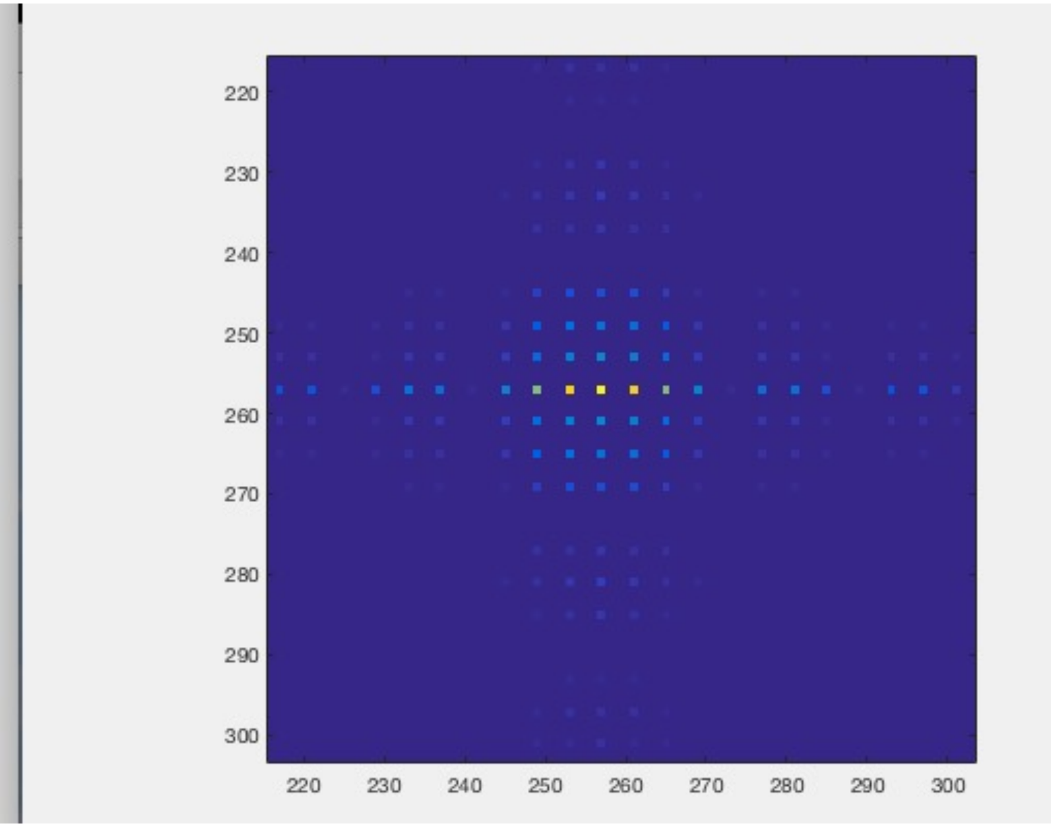

10
20
30
40
50
60
70
80
90
10 20 30 40 50 60 70 80 90

2D Fourier of a box

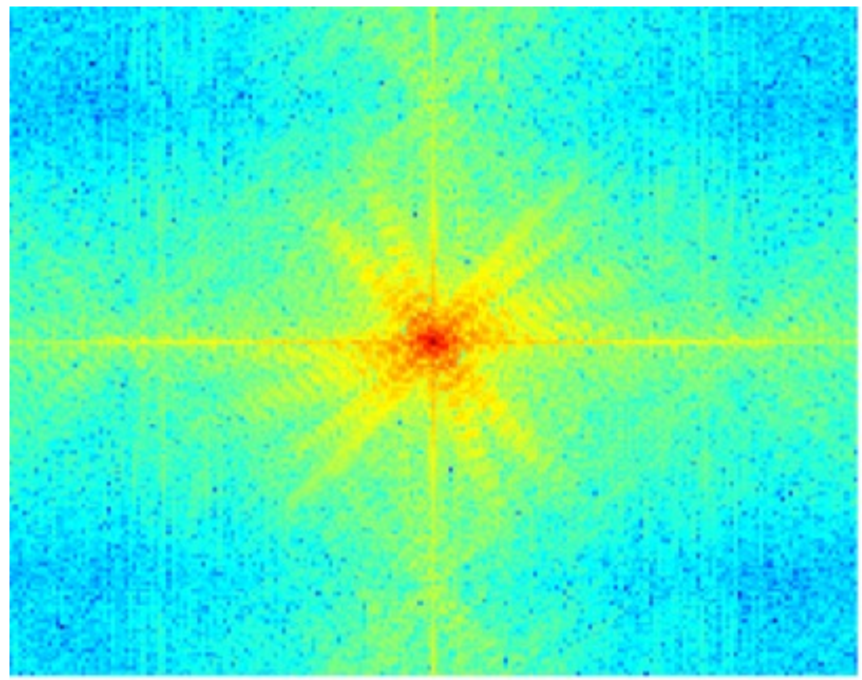


How do we model other periodic patterns?

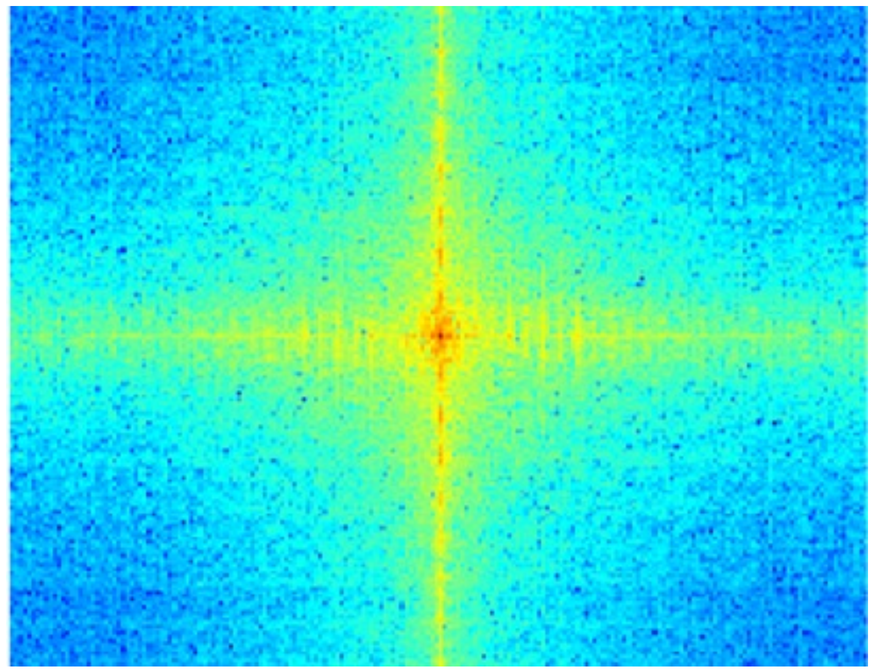




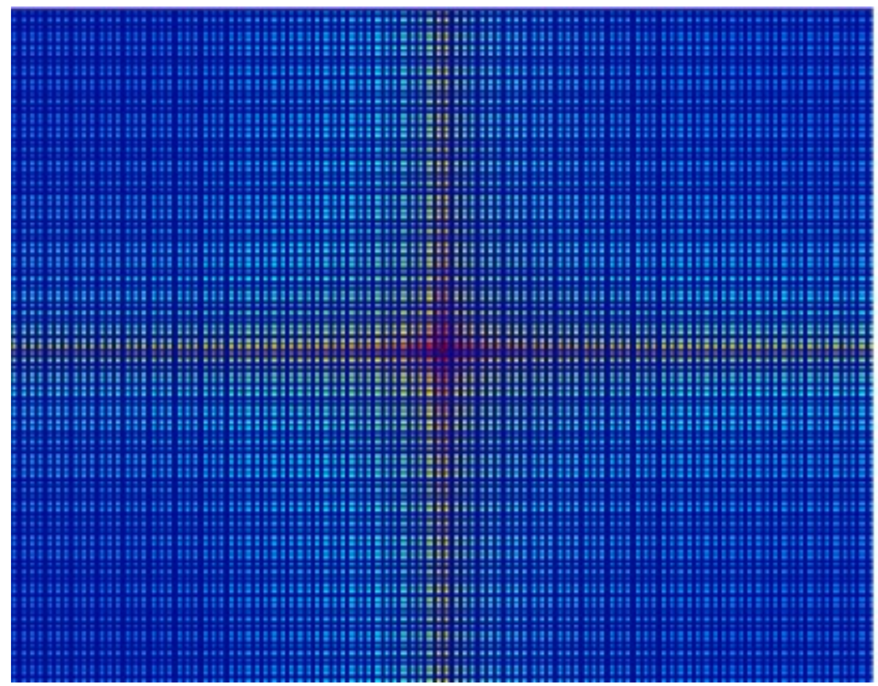
Clue about orientation of edges



Clue about periodicity



Clues about contrast



Fourier and Spatial Frequency FAQ

DTFT (CFT) vs DFT (FFT)

DTFT: $f[n] \circ\!\!-\!\!\bullet F(\omega) = \sum_{n=0}^{L-1} f[n] e^{-i\omega n}$ periodic with period 2π

DFT: $f[n] \circ\!\!-\!\!\bullet F[k] = \sum_{n=0}^{L-1} f[n] e^{-i\frac{2\pi k}{L}n}$ length L because period is L

The same as matlab but $n \to n-1$ and $k \to k-1$.

Matlab help screenshot:

$$Y(k) = \sum_{j=1}^{n} X(j)\, W_n^{(j-1)(k-1)}$$

$$X(j) = \frac{1}{n} \sum_{k=1}^{n} Y(k)\, W_n^{-(j-1)(k-1)},$$

where

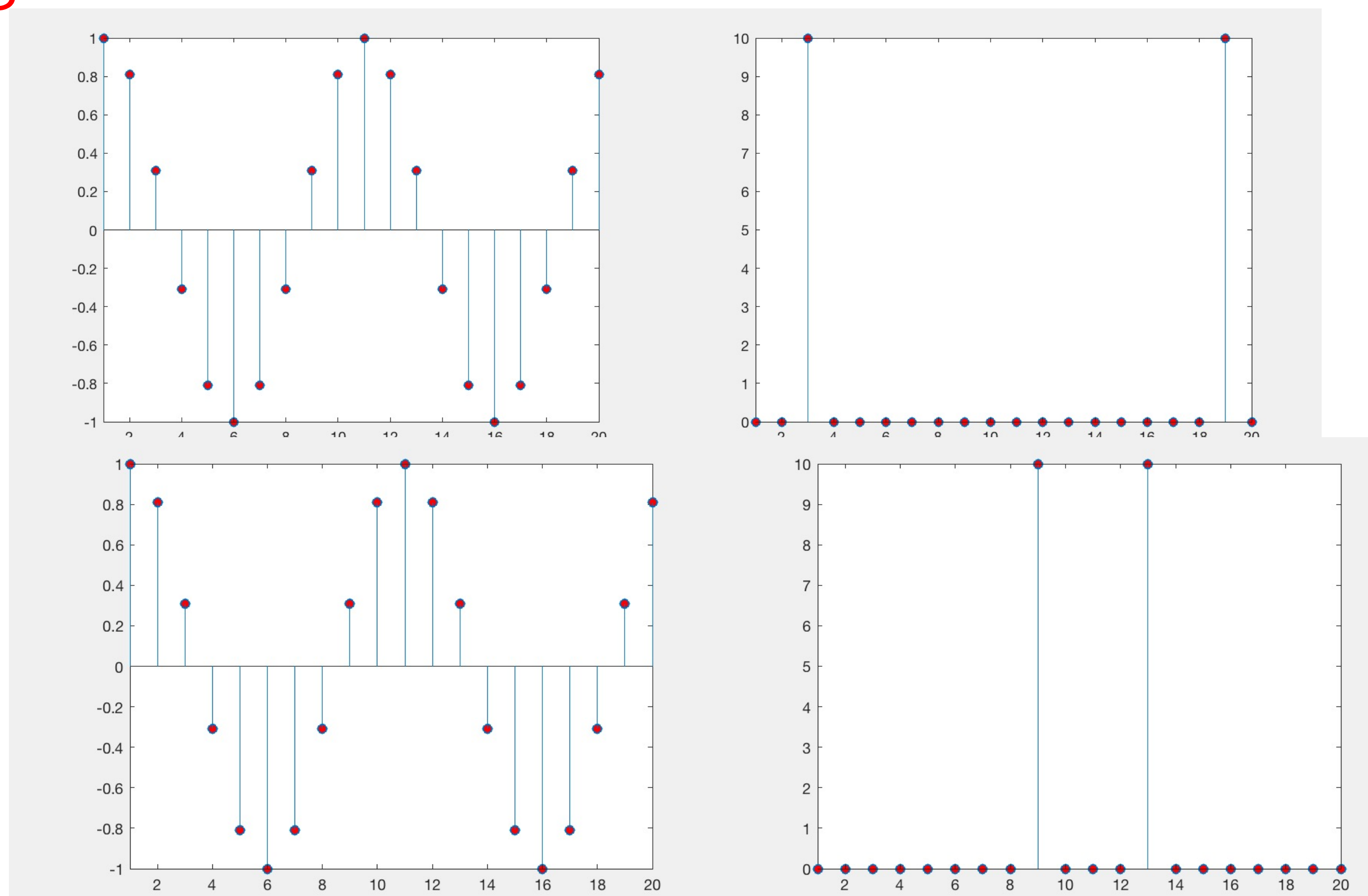
$$W_n = e^{(-2\pi i)/n}$$

Intrepreting the FFT

```
n = [0:19];
fcos = cos(2*pi*n/10);
```

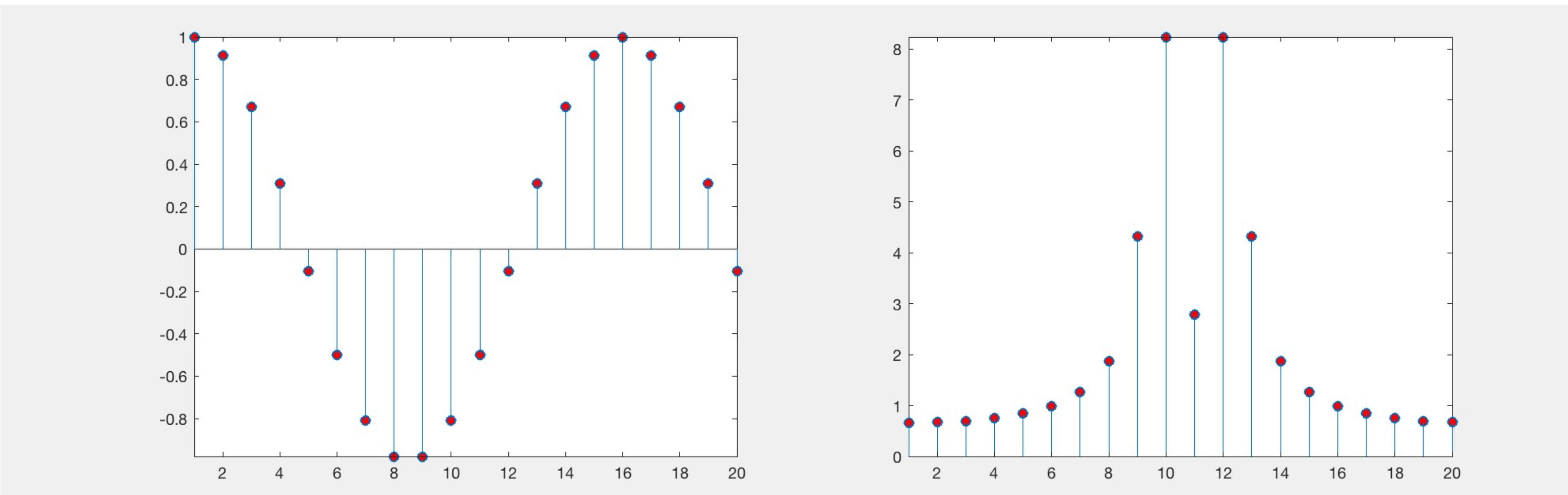
Without fftshift

With fftshift, equivalent to

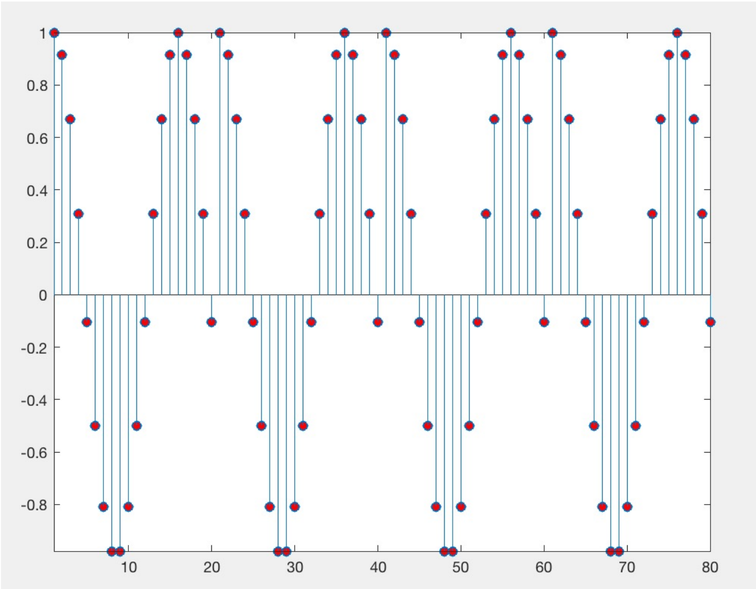


Side effects of DFT (FFT) assumption of periodicity

```
n = [0:19];
fcos = cos(2*pi*n/15);
```

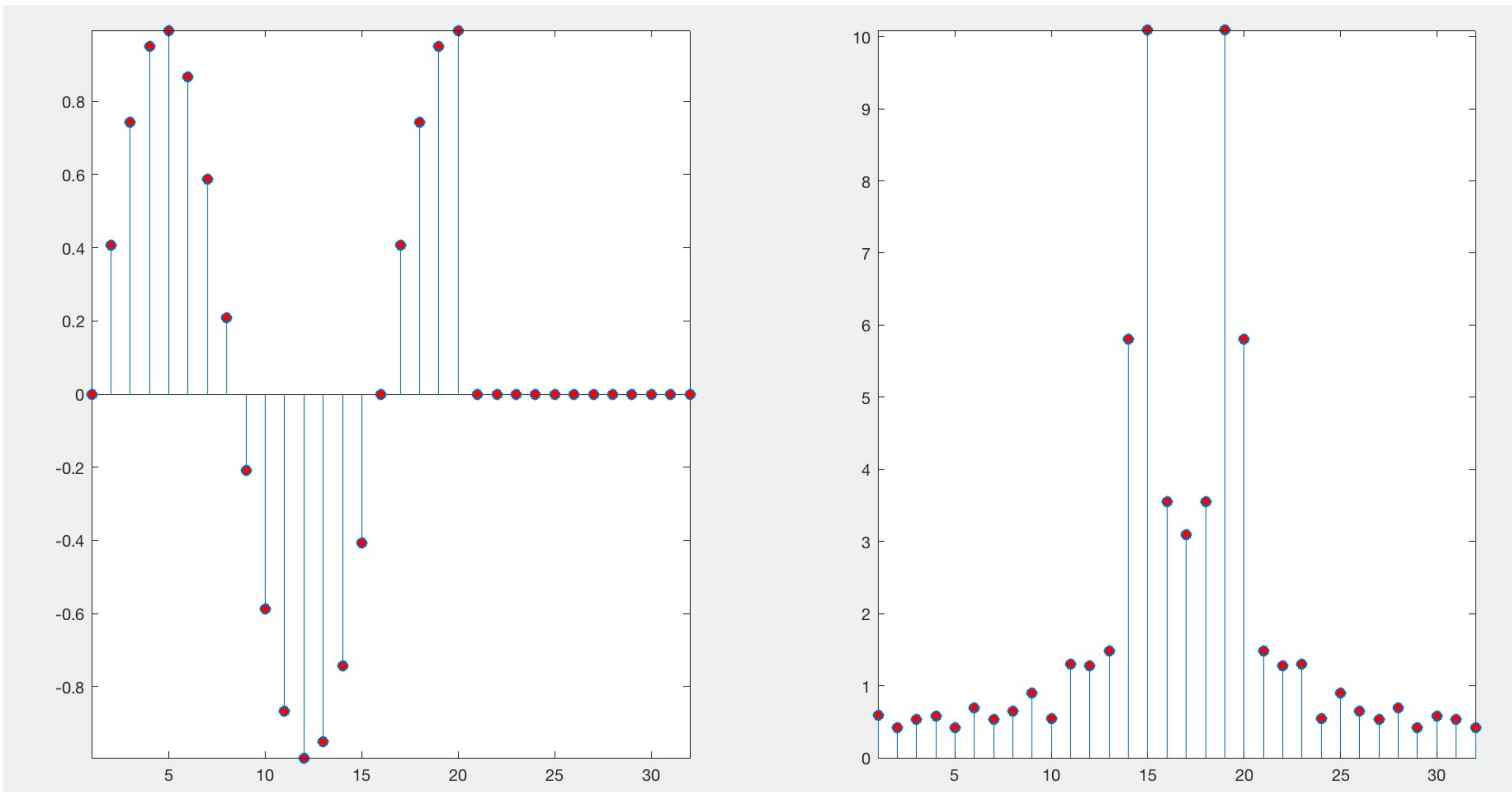


Because DFT assumes periodicity

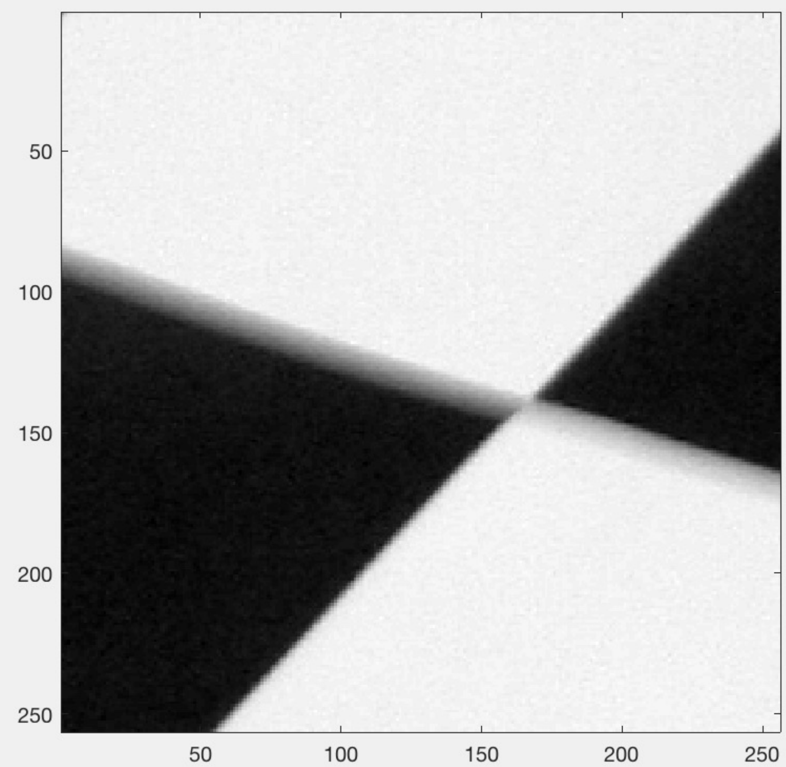


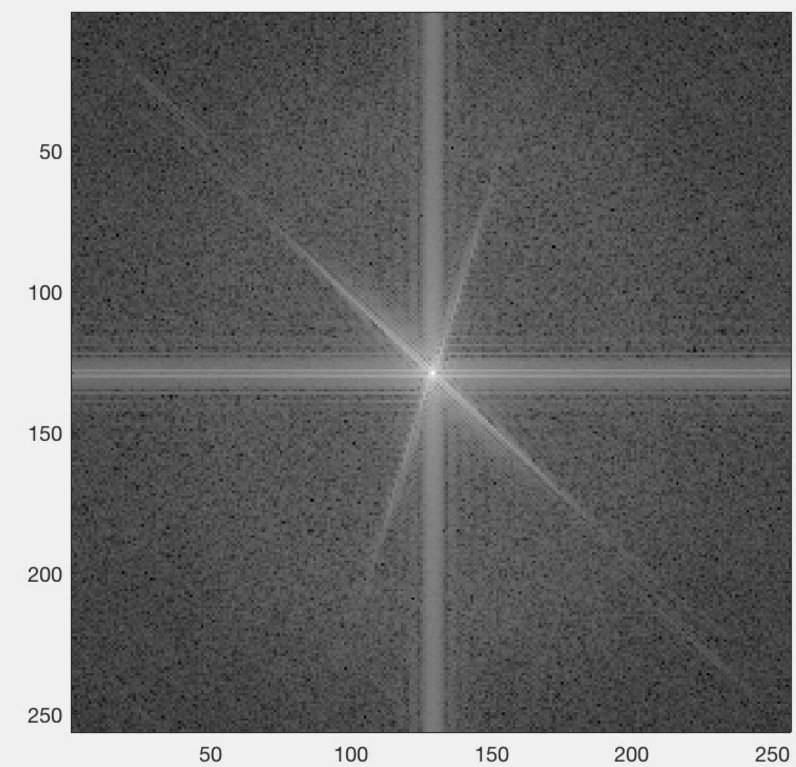
This is not anymore a pure cosine

Padding fft(f,32)

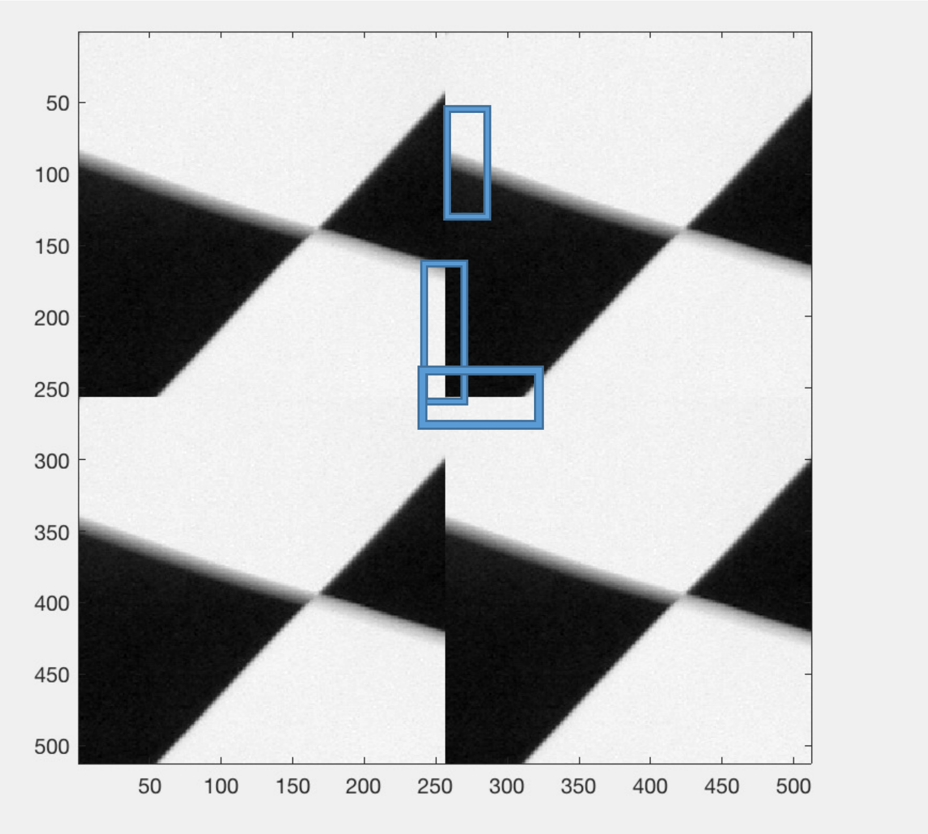


Why do we get here vertical and horizontal frequencies?

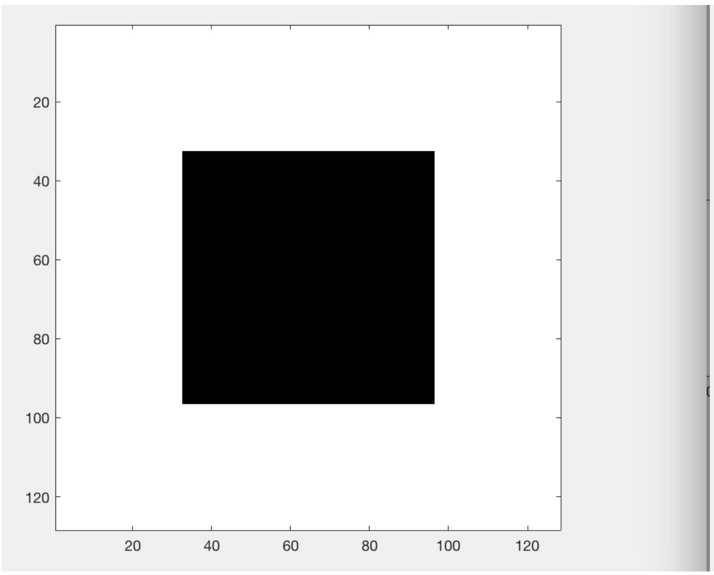


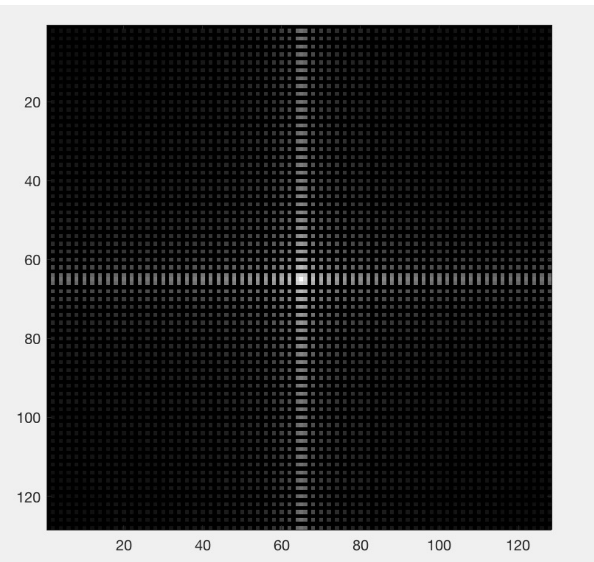


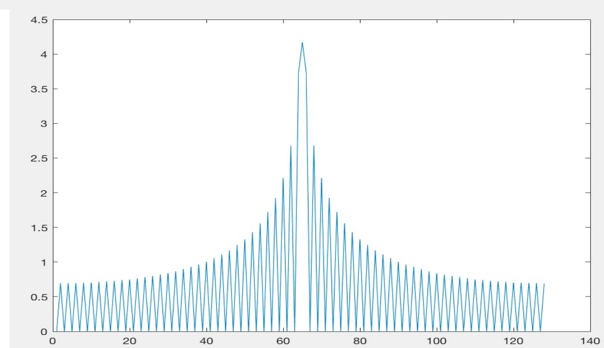
Because of the replication!



Fourier of the box is Fourier of rect(x)rect(y)

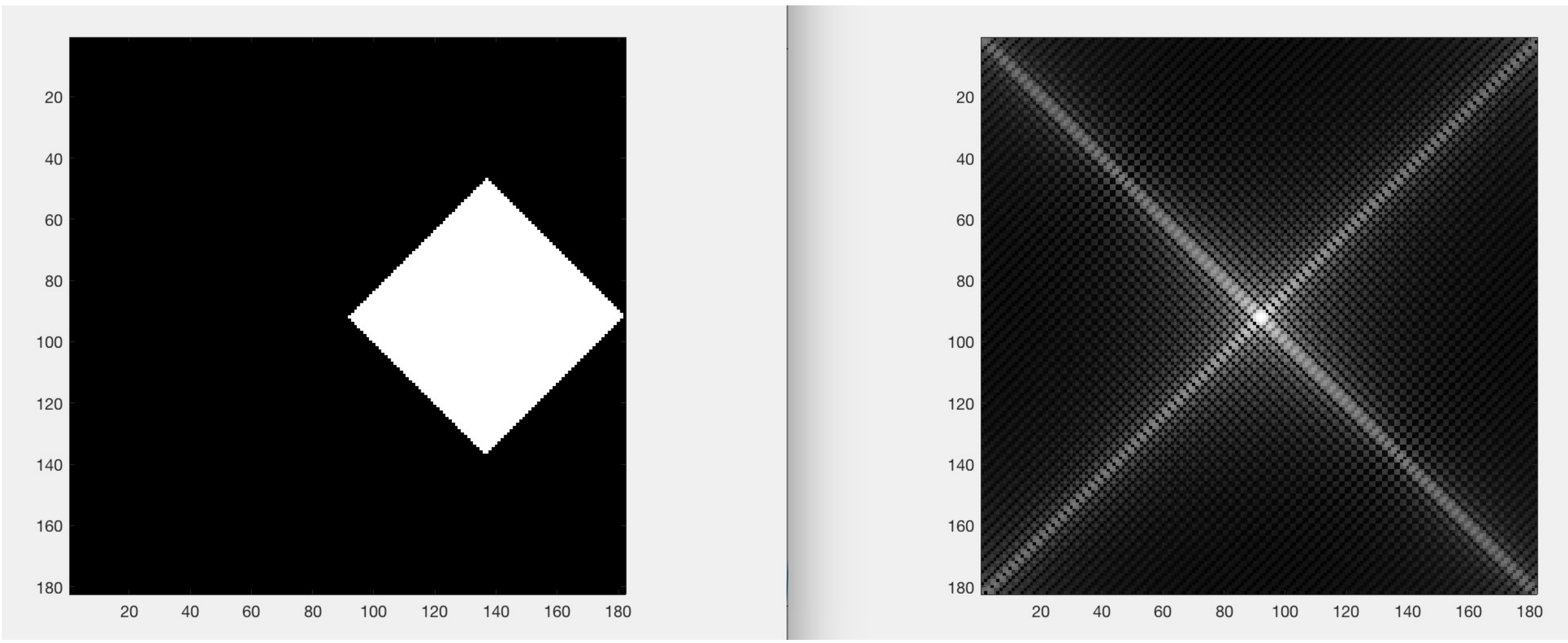


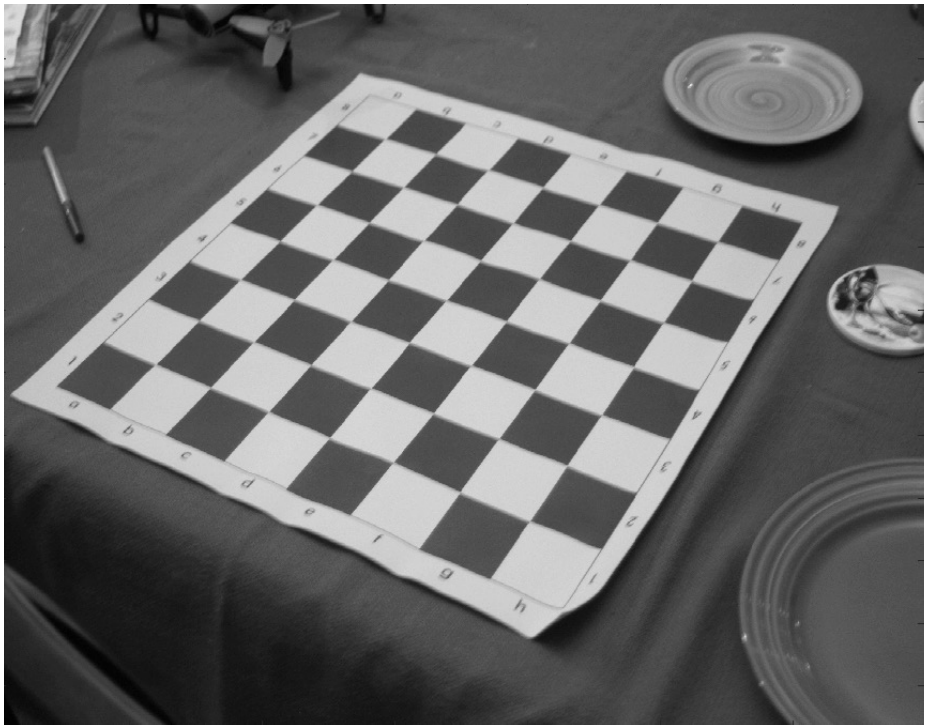


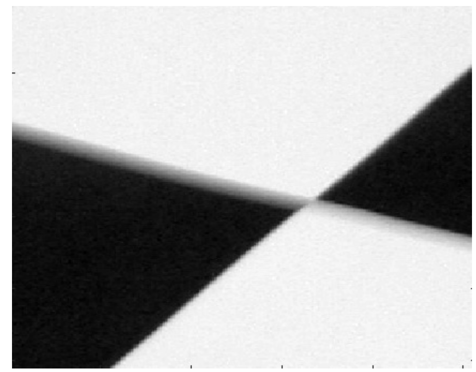


Middle row

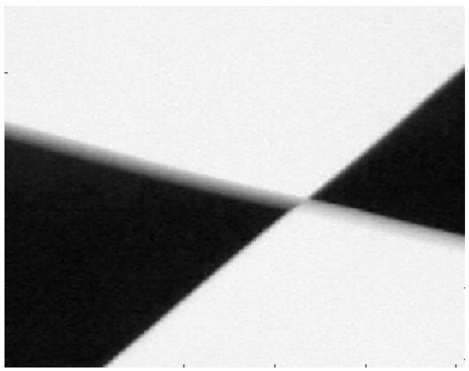
But here no horizontal or vertical components

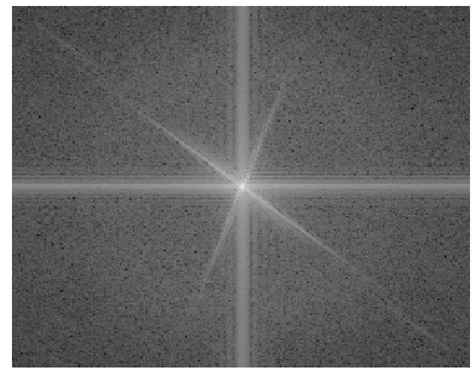






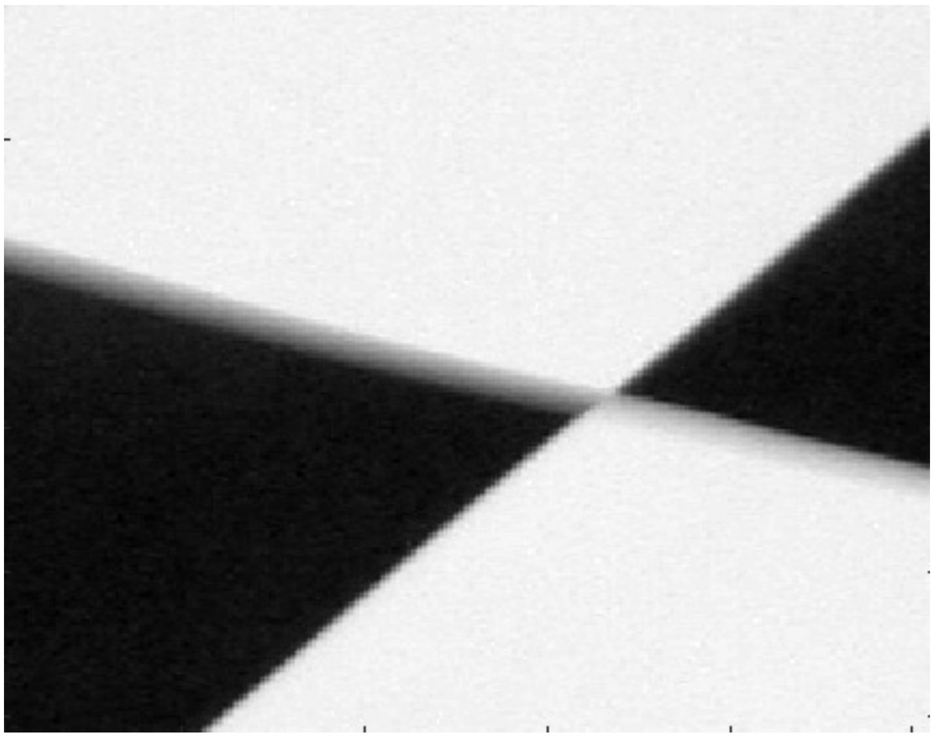
Fourier Transform

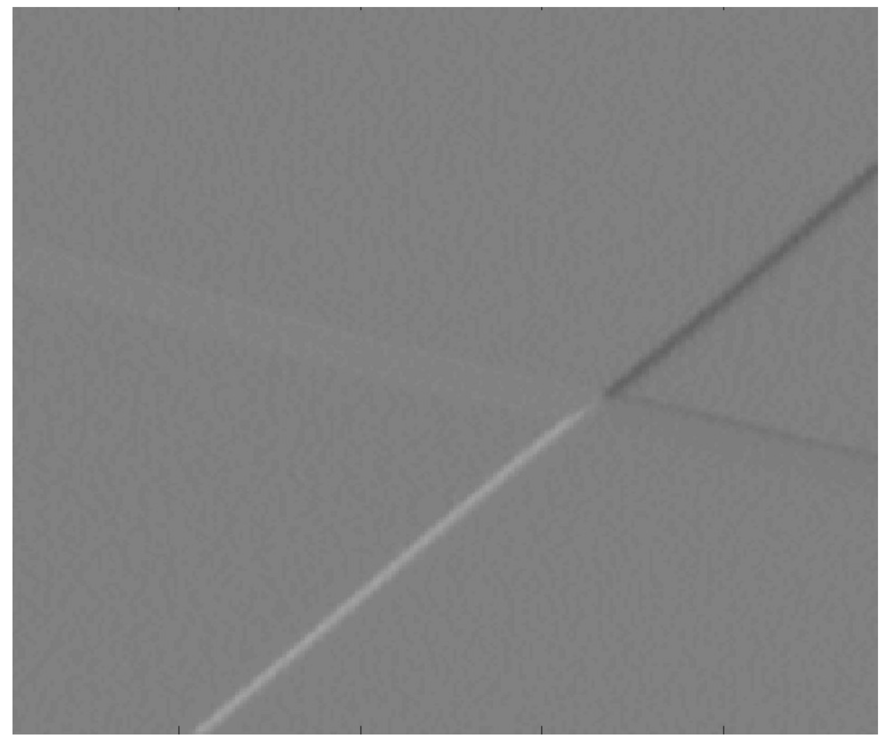


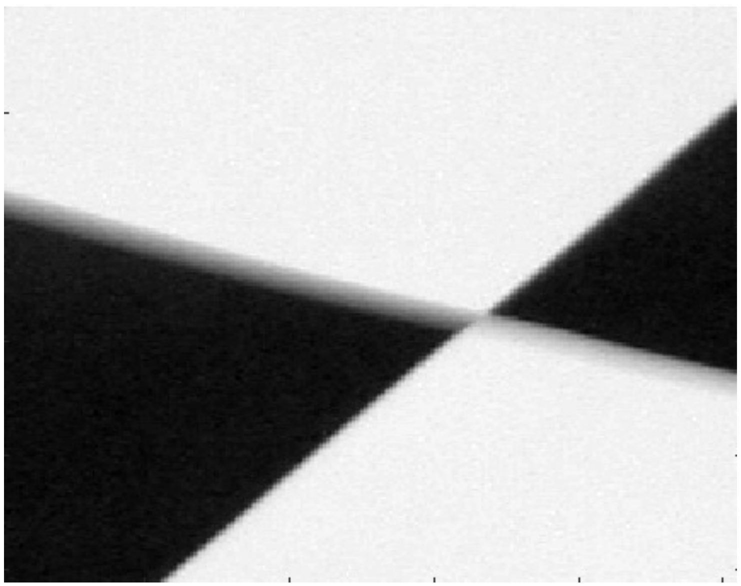
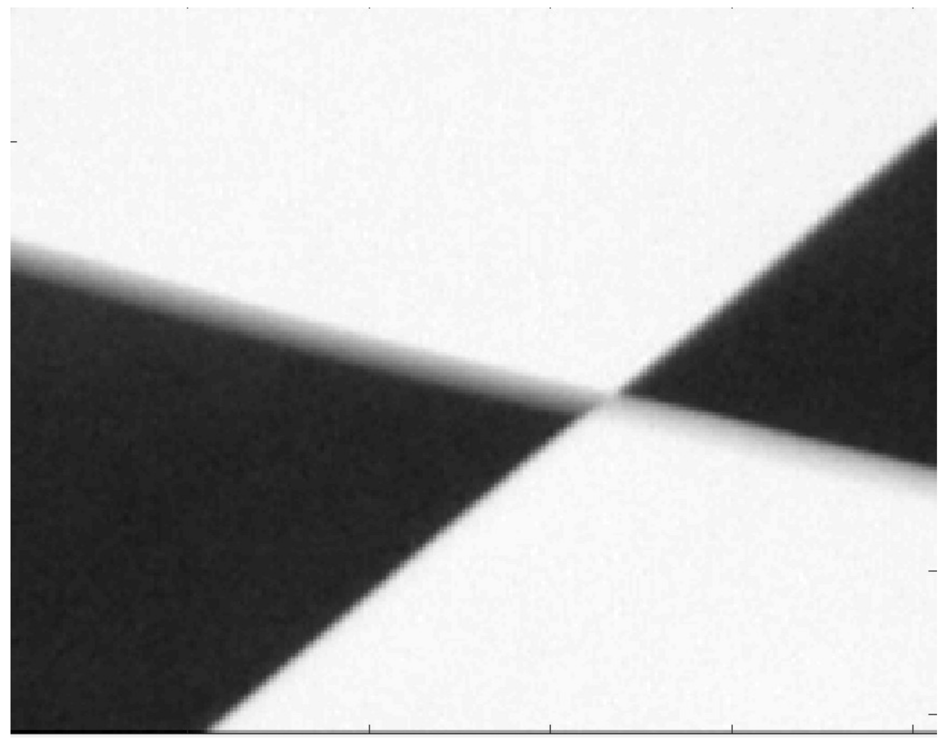


```
ftim = fft2(im);
imagesc(log(abs(fftshift(ftim))));
```

Sobel in x-direction

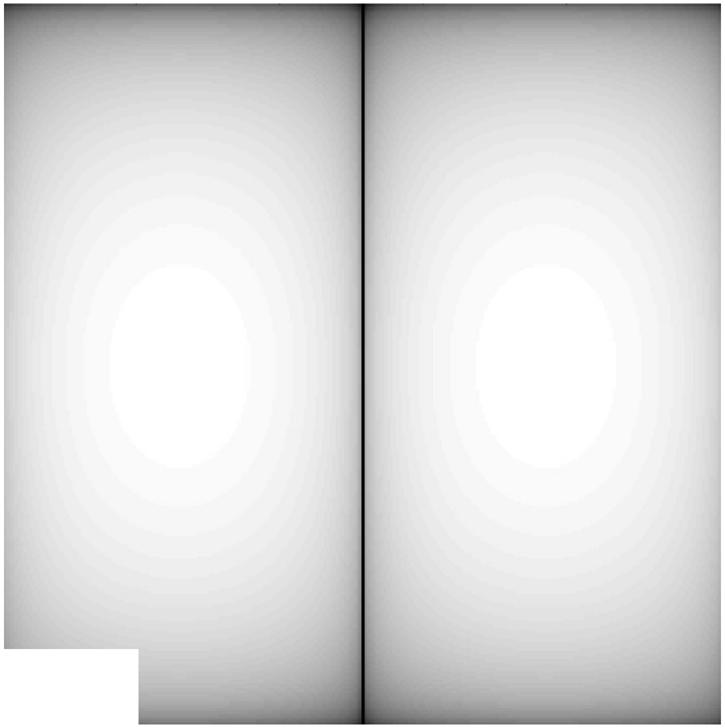




Fourier of the Sobel filter in x-direction

$$\begin{bmatrix} +1 & 0 & -1 \\ +2 & 0 & -2 \\ +1 & 0 & -1 \end{bmatrix}$$



```
ftsobelx = fft2(sobelx,256,256);
figure(21);
imagesc(log(abs(fftshift(ftsobelx))));colormap(gray);axis image;
```

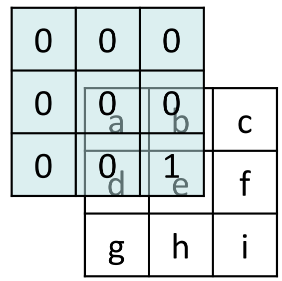
2D Convolutions

$$(f * h)(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y')h(x - x', y - y')dx'dy'$$

$$(f * h)[r, c] = \sum_{c'=-\infty}^{\infty} \sum_{c'=-\infty}^{\infty} f[r', c']h[r - r', c - c']$$

2D Convolution is commutative

$$I = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline\end{array} \qquad f = \begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array}$$



$$I \otimes f = \begin{array}{|c|c|c|}\hline e & f & 0 \\ \hline h & i & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array}$$

$$f = \begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \qquad I = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline\end{array}$$

$$\mathrm{f} \otimes I = \begin{array}{|c|c|c|}\hline e & f & 0 \\ \hline h & i & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array}$$

2D Convolution is associative

- Best use of associativity in separable filters

$$
\begin{aligned}
h(x, y) &= h_1(x) h_2(y) \\
f(x, y) \star h(x, y) &= (f(x, y) \star h_1(x)) \star h_2(y)
\end{aligned}
$$

$$
\begin{bmatrix} +1 & 0 & -1 \\ +2 & 0 & -2 \\ +1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} +1 & 0 & -1 \end{bmatrix}
$$