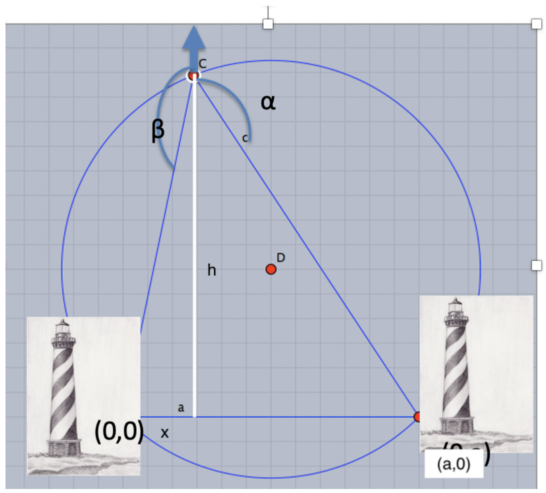
CIS 580: Machine Perception, Spring 2020
Homework 4 Solution

1 Problem 1, 30pts



As shown in the graph, we draw a line from C that is perpendicular to the line connecting $(0, 0)$ and $(a, 0)$. We denote the length of the line as h. Denote the length of the line connecting $(0, 0)$ and the point of intersection as x and the length of the line connecting $(a, 0)$ and the point of intersection as y.
From the triangles, we have

$$\frac{x}{h} = \tan(\pi - \beta)$$
$$\frac{y}{h} = \tan(\pi - \alpha)$$

Moreover, we know that $x + y = a$. Thus,

$$\tan(\pi - \beta)h + \tan(\pi - \alpha)h = a$$

$$-(\tan\alpha + \tan\beta)h = a$$
$$h = -\frac{a}{\tan\alpha + \tan\beta}$$

Moreover,

$$x = h\tan(\pi - \beta) = \frac{a\tan\beta}{\tan\alpha + \tan\beta}$$

Therefore, the coordinate of the ship is $(\frac{a\tan\beta}{\tan\alpha+\tan\beta}, -\frac{a}{\tan\alpha+\tan\beta})$.

2 Problem 2, 30pts

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K[RT] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

where $T = [T_x, T_y, T_z]^T$ is the translation from world coordinate to camera coordinate. Since $K = I$ and Z_c is parallel to the ground, there only exists rotation around the y axis,

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & T_x \\ 0 & 1 & 0 & T_y \\ -\sin\theta & 0 & \cos\theta & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Since the projections of $(0, 0, 0)$ and $(a, 0, 0)$ are (x_1, y_1) and (x_2, y_2),

$$\lambda \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$\lambda \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a\cos\theta + T_x \\ T_y \\ -a\sin\theta + T_z \end{bmatrix}$$

Therefore, $\lambda = -a\sin\theta + T_z$, the projection equations are

$$x_1 = \frac{T_x}{T_z}$$

$$y_1 = \frac{T_y}{T_z}$$

$$x_2 = \frac{a\cos\theta + T_x}{-a\sin\theta + T_z}$$

$$y_2 = \frac{T_y}{-a\sin\theta + T_z}$$

Part b),

$$\frac{x_2}{y_2} = \frac{a\cos\theta + T_x}{T_y} \rightarrow \frac{x_2}{y_2} - \frac{x_1}{y_1} = \frac{a\cos\theta}{T_y}$$

$$T_y = y_1 T_z = y_2(-a\sin\theta + T_z) \rightarrow T_z = \frac{-ay_2\sin\theta}{y_1 - y_2}$$

Solving the above equations gives

$$\tan\theta = \frac{y_2 - y_1}{y_1 x_2 - y_2 x_1}$$

Note that $\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}$,

$$\sin\theta = \sqrt{\frac{(\frac{y_2-y_1}{y_1x_2-y_2x_1})^2}{(\frac{y_2-y_1}{y_1x_2-y_2x_1})^2 + 1}}$$

Substitute into $T_z = \frac{-ay_2 \sin\theta}{y_1 - y_2}$, we have

$$T_z = \frac{|ay_2|}{\sqrt{(y_2 - y_1)^2 + (y_1 x_2 - x_1 y_2)^2}}$$

$$T_x = x_1 T_z = \frac{|ay_2| x_1}{\sqrt{(y_2 - y_1)^2 + (y_1 x_2 - x_1 y_2)^2}}$$

$$T_y = y_1 T_z = \frac{|ay_2| y_1}{\sqrt{(y_2 - y_1)^2 + (y_1 x_2 - x_1 y_2)^2}}$$

Part c), when $x_1 = x_2, y_1 = y_2$, then

$$\frac{T_y}{T_z} = \frac{T_y}{-a\sin\theta + T_z}$$

Thus, $\sin\theta = 0$. Moreover, since

$$\frac{T_x}{T_z} = \frac{a\cos\theta + T_z}{-a\sin\theta + T_z}$$

, we have $\cos\theta = 0$, which contradicts with the fact that $\sin\theta = 0$. Therefore, we must have $T_y = 0$ and $\frac{T_x}{T_z} = -\cot(\theta)$. There exist infinite solutions because we only know T_x, T_z to a scale.

3 Problem 3, 40pts +Extra Credit

We obtain the rotation R by rotating around x axis by $\pi/2$ and rotating around z axis by θ.

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/2) & -\sin(\pi/2) \\ 0 & \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \\ \sin\theta & \cos\theta & 0 \end{bmatrix}$$

Since the circle is on $Z = 0$ plane,

$$H = \begin{bmatrix} r_1 & r_2 & t \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & T_x \\ 0 & 0 & T_y \\ \sin\theta & \cos\theta & T_z \end{bmatrix}$$

The circle on the ground is given by

$$\begin{bmatrix} X & Y & 1 \end{bmatrix} C \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = 0$$

The transformed circle is given by

$$\begin{bmatrix} X' & Y' & 1 \end{bmatrix} C' \begin{bmatrix} X' \\ Y' \\ 1 \end{bmatrix} = 0$$

where

$$\begin{bmatrix} X' \\ Y' \\ 1 \end{bmatrix} = H \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}, \begin{bmatrix} X' & Y' & 1 \end{bmatrix} = H^T \begin{bmatrix} X & Y & 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} X & Y & 1 \end{bmatrix} (H^{-1})^T C H^{-1} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = 0$$

Thus,

$$C' = (H^{-1})^T C H^{-1}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ -\frac{T_x\cos\theta + T_z\sin\theta}{T_y} & \frac{T_x\sin\theta - T_z\cos\theta}{T_y} & \frac{1}{T_y} \\ \sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\rho^2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\frac{T_x\cos\theta + T_z\sin\theta}{T_y} & \sin\theta \\ -\sin\theta & \frac{T_x\sin\theta - T_z\cos\theta}{T_y} & \cos\theta \\ 0 & \frac{1}{T_y} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{T_x}{T_y} & 0 \\ -\frac{T_x}{T_y} & \frac{T_x^2 + T_z^2 - \rho^2}{T_y^2} & \frac{T_z}{T_y} \\ 0 & \frac{T_z}{T_y} & 1 \end{bmatrix}$$

Part b), Denote C as

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Therefore,

$$-\frac{T_x}{T_y} = C_{12}$$

$$\frac{T_x^2 + T_z^2 - \rho^2}{T_y^2} = C_{22}$$

$$\frac{T_z}{T_y} = C_{32}$$

Solving this yields

$$T_y = \sqrt{\frac{\rho^2}{C_{12}^2 + C_{32}^2 - C_{22}}}$$

$$||T|| = \sqrt{T_x^2 + T_y^2 + T_z^2} = \sqrt{T_y^2(C_{12}^2 + 1 + C_{32}^2)}$$

$$= \rho\sqrt{\frac{C_{12}^2 + C_{32}^2 + 1}{C_{12}^2 + C_{32}^2 - C_{22}}}$$

Part c), We cannot recover the yaw angle because the equation of the mapped circle is independent of the rotation.

Part d), General formula for conic section can be written as

$$\begin{bmatrix} X & Y & 1 \end{bmatrix} \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = 0$$

Set discriminant $B^2 - 4AC > 0$ for hyperbola,

$$B^2 - 4AC = \frac{4T_x^2}{T_y^2} - 4\frac{T_x^2 + T_z^2 - \rho^2}{T_y^2} > 0$$

$$\frac{\rho^2 - T_z^2}{T_y^2} > 1$$

The cross section yields a hyperbola when $\rho^2 > T_z^2$.