CIS580 Problem Set 4

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1 Mapping 2D points

1.1 Use solution for orthogonal Procrustes problem to obtain rotation matrix

1.2 Find solution that solves directly for the rotation angle θ and translation $[T_x, T_Y]$

2 Phone held vertically

2.1 Write projection equations

Given:

$$\lambda \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

Using the above, we can generate a system of 4 equations with 4 unknowns as follows:

$$x_2 = \frac{a \cdot cos(\theta) + T_X}{-a \cdot sin(\theta) + T_Z}$$
$$y_2 = \frac{0 + T_Y}{-a \cdot sin(\theta) + T_Z}$$
$$x_1 = \frac{T_X}{T_Z}$$
$$y_1 = \frac{T_Y}{T_Z}$$

2.2 Solve equations for yaw angle θ and translations $[T_x, T_y, T_z]$

Solving the above system of equations:

$$T_X = x_1 \cdot T_Z$$
$$x_2 = \frac{a \cdot cos(\theta) + x_1 \cdot T_Z}{-a \cdot sin(\theta) + T_Z}$$
$$T_Z \cdot (x_2 - x_1) = a \cdot cos(\theta) + a \cdot sin(\theta) \cdot x_2$$
$$T_Z = \frac{a \cdot cos(\theta) + a \cdot sin(\theta) \cdot x_2}{x_2 - x_1}$$

Another equation in terms of T_Z and θ can be obtained as follows:

$$\begin{aligned} T_Y &= y_1 \cdot T_Z \\ y_2 &= \frac{0 + T_Y}{-a \cdot sin(\theta) + T_Z} \\ T_Z \cdot (y_2 - y_1) &= a \cdot sin(\theta) \cdot y_2 \\ T_Z &= \frac{a \cdot sin(\theta) \cdot y_2}{y_2 - y_1} \end{aligned}$$

Solving for θ:

$$\begin{aligned} \frac{a \cdot sin(\theta) \cdot y_2}{y_2 - y_1} &= \frac{a \cdot cos(\theta) + a \cdot sin(\theta) \cdot x_2}{x_2 - x_1} \\ cos(\theta) &= -x_2 \cdot y_1 + x_1 \cdot y_2 \\ sin(\theta) &= y_1 - y_2 \end{aligned}$$

$$\implies \theta = arctan2\left(\frac{-x_2 \cdot y_1 + x_1 \cdot y_2}{y_1 - y_2}\right)$$

Using this to solve for the other unknowns:

$$\begin{aligned} T_Z &= \frac{a \cdot sin(\theta) \cdot y_2}{y_2 - y_1} & \\ T_X &= x_1 \cdot T_Z & &= \frac{x_1 \cdot a \cdot sin(\theta) \cdot y_2}{y_2 - y_1} \\ T_Y &= y_1 \cdot T_Z & &= \frac{y_1 \cdot a \cdot sin(\theta) \cdot y_2}{y_2 - y_1} \end{aligned}$$

2.3 Conditions on camera position to obtain unique or finite number of solutions

3 Decompose H into rotation R and translation T

Given:

$$g_1 = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \qquad g_2 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$H = \begin{bmatrix} \frac{-4}{8} & 0 & 1 \\ \frac{6}{8} & \frac{\sqrt{3}-12}{8} & \frac{5-4\sqrt{3}}{8} \\ \frac{-2\sqrt{3}}{8} & \frac{7+4\sqrt{3}}{8} & \frac{\sqrt{3}+4}{8} \end{bmatrix}$$

Also, from the provided picture we can obtain the rotation matrices R_1 and R_2:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \qquad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$H' = {R_2}^{-1} \times H \times R_1$$

$$= \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 2 & \frac{-1}{2} \end{bmatrix}$$

We know H' is a composition of a rotation about the Y axis, followed by a translation. Thus, we can also represent it as:

$$H' = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} + \begin{bmatrix} 0 & \frac{T_X}{2} & 0 \\ 0 & \frac{T_Y}{2} & 0 \\ 0 & \frac{T_Z}{2} & 0 \end{bmatrix}$$

Equating the two version of H', we find:

$$\begin{bmatrix} \frac{T_X}{2} & \frac{T_Y}{2} & \frac{T_Z}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}$$

$$\theta = \frac{2 \cdot \pi}{3}$$

We also know that the rotation matrix to go from Camera 1 to Camera 2 in Camera 1 coordinates R_Y is a rotation about the Y axis by π radians. Thus:

$$R_Y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Combining the above information, we can create ${}^1\mathbf{R}_2$:

$${}^1\mathbf{R}_2 = R_1 \times R_Y \times {R_2}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We have verified the above rotation matrix by computing ${}^1\mathbf{R}_2 \times g_1$, and ensuring that it does indeed equal g_2.

Now we use the above information to extract the translation vector from H.