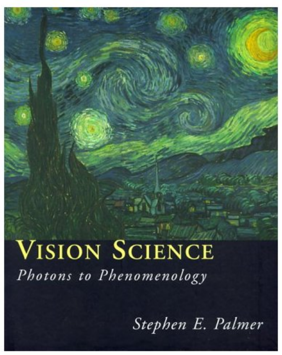
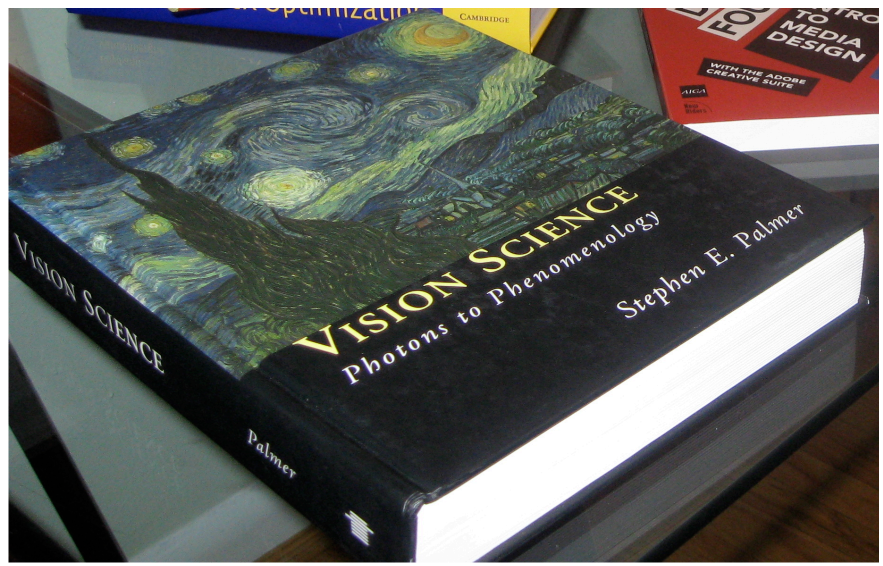
Projective Transformations

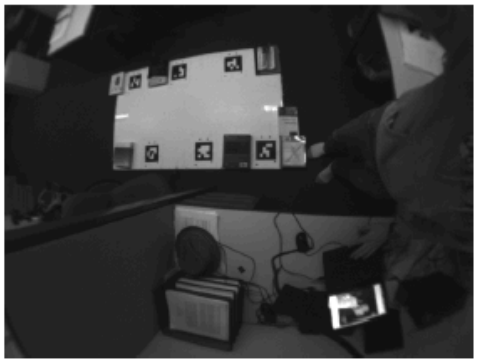
aka Collineations
aka Homographies

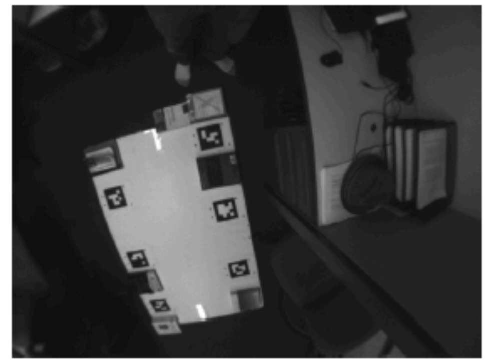
A perspective projection of a plane (like a camera image) is always a projective transformation

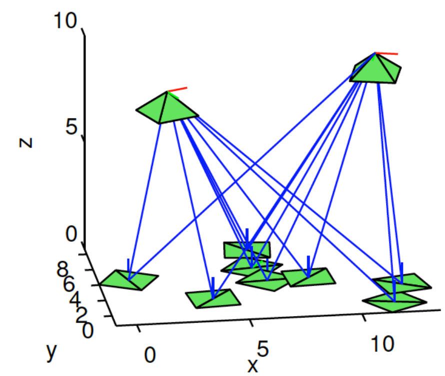




Using the projective transformation the pose of a robot with respect to a planar pattern:







Projective Transformation

Definition

A **projective transformation** is any invertible matrix transformation $\mathbb{P}^2 \to \mathbb{P}^2$.

A projective transformation A maps p to $p' \sim Ap$.

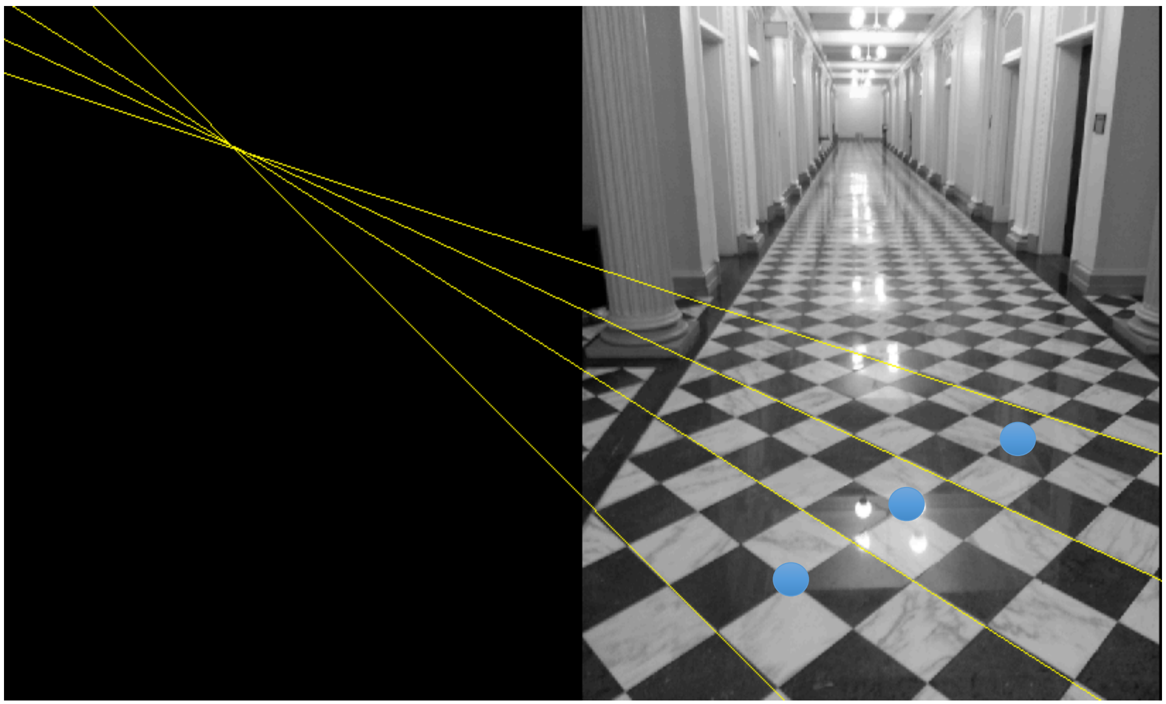
Invertibility means that $\det(A) \neq 0$ and that there exists $\lambda \neq 0$ such that $\lambda p' = Ap$.

Observe that we will write either $p' \sim Ap$ or $\lambda p' = Ap$.

A projective transformation is also known as **collineation or homography.**

A projective transformation preserves incidence:

- Three collinear points are mapped to three collinear points.
- and three concurrent lines are mapped to three concurrent lines.



Projective transformation of lines

If A maps a point to Ap, then where does a line l map to?

Line equation in original plane

$$l^T p = 0$$

Line equation in image plane $p' \sim Ap$

$$l^T A^{-1} p' = 0$$

implies that $l' = A^{-T} l$.

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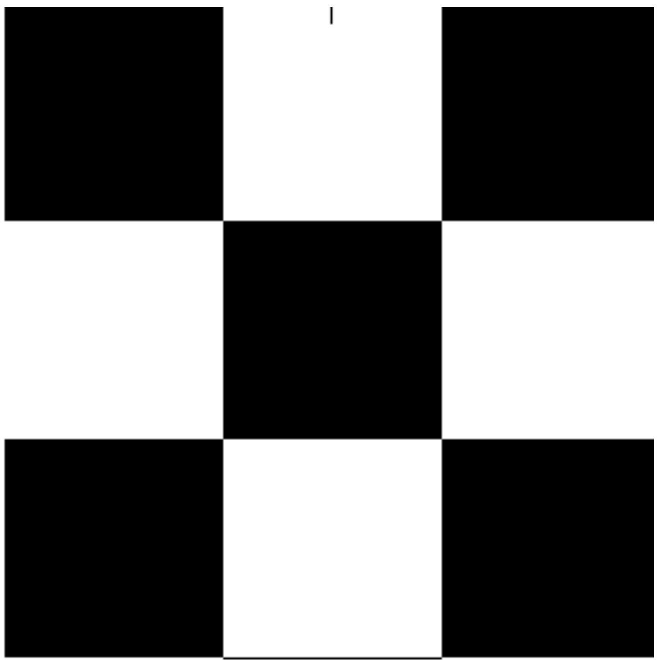
Compute Projective Transformations Using 4 Points

Kostas Daniilidis

How can we compute the projective transformation between

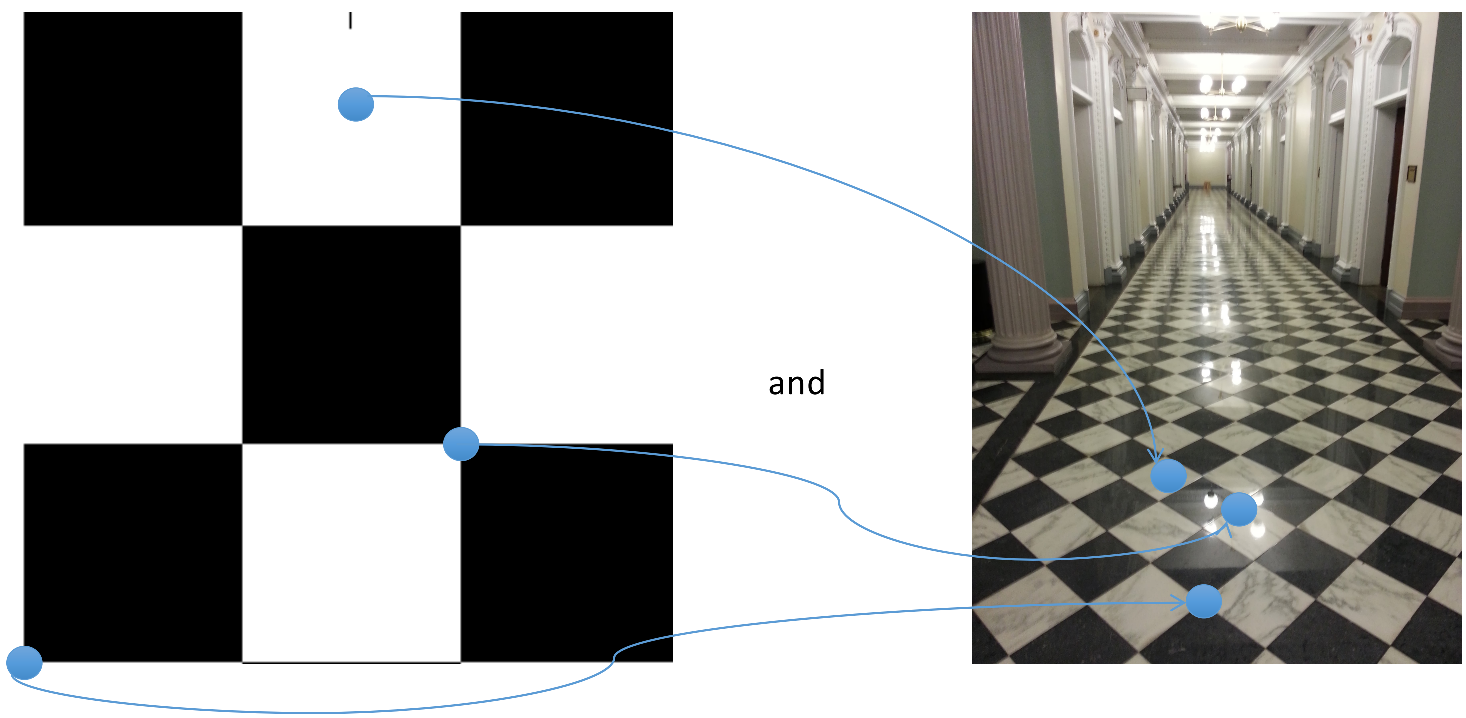


Floor tiles measured in [m]

and

Points in pixel coordinates

The result of such a transformation would map any point in one plane to the corresponding point in the other



Floor tiles measured in [m]

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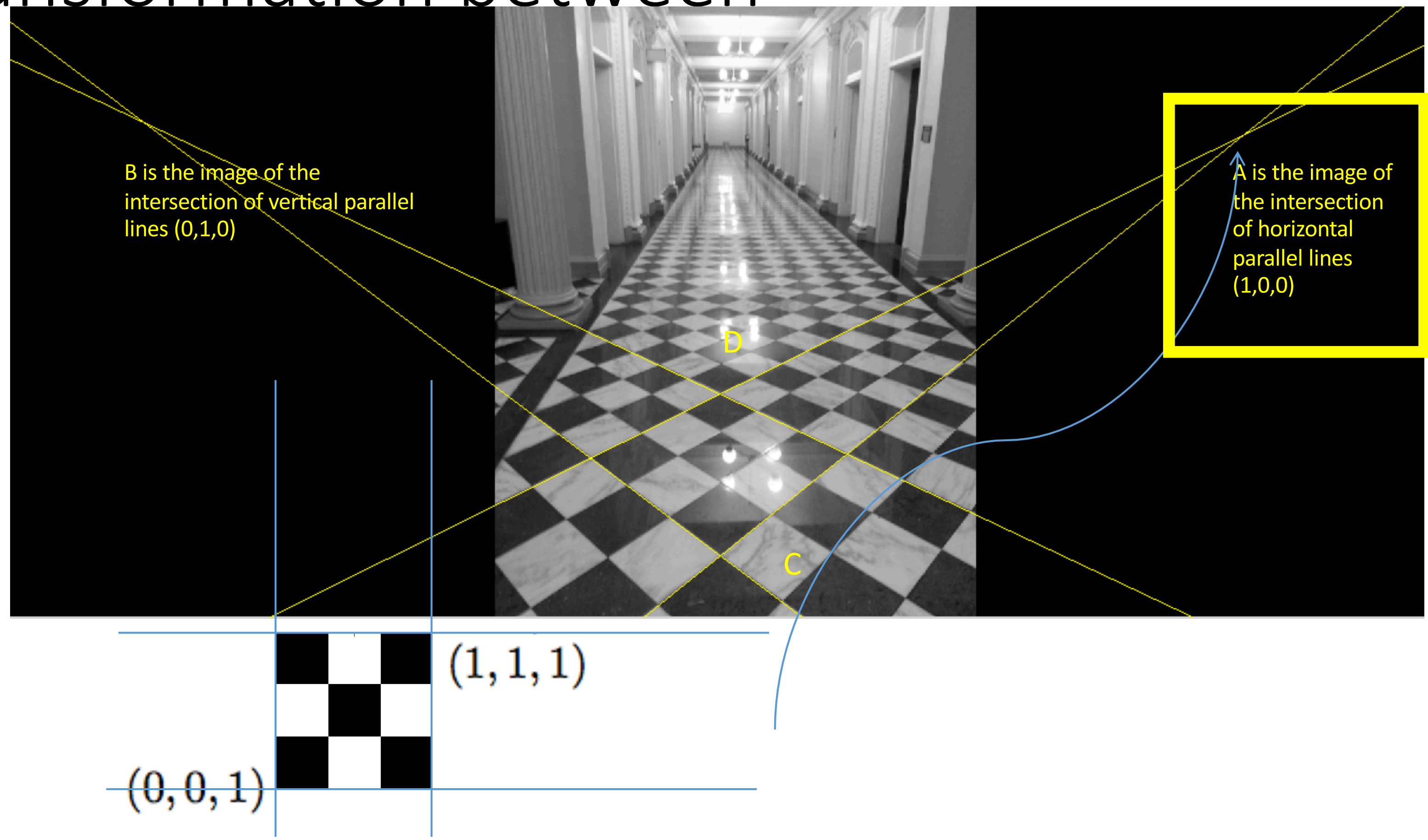

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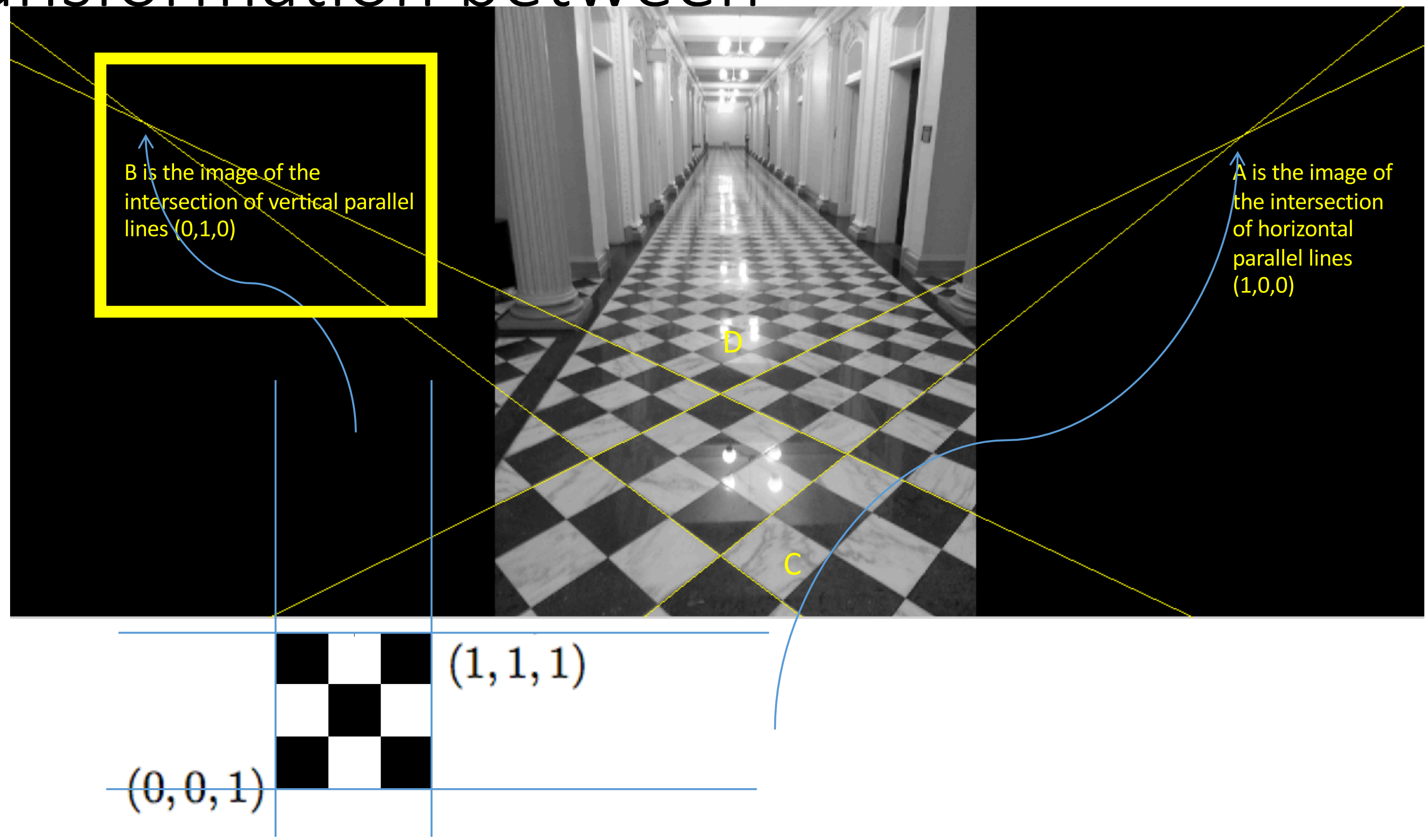

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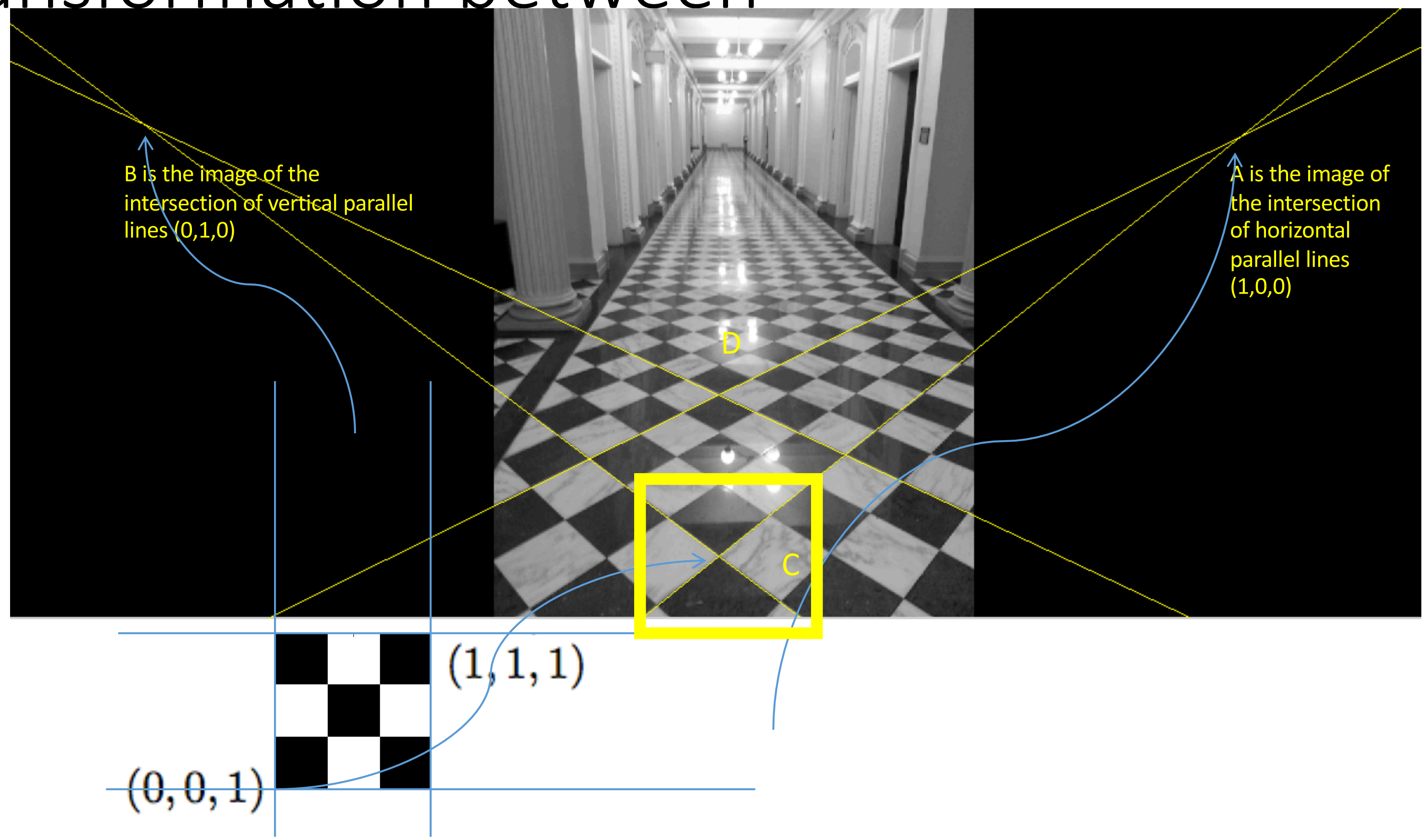


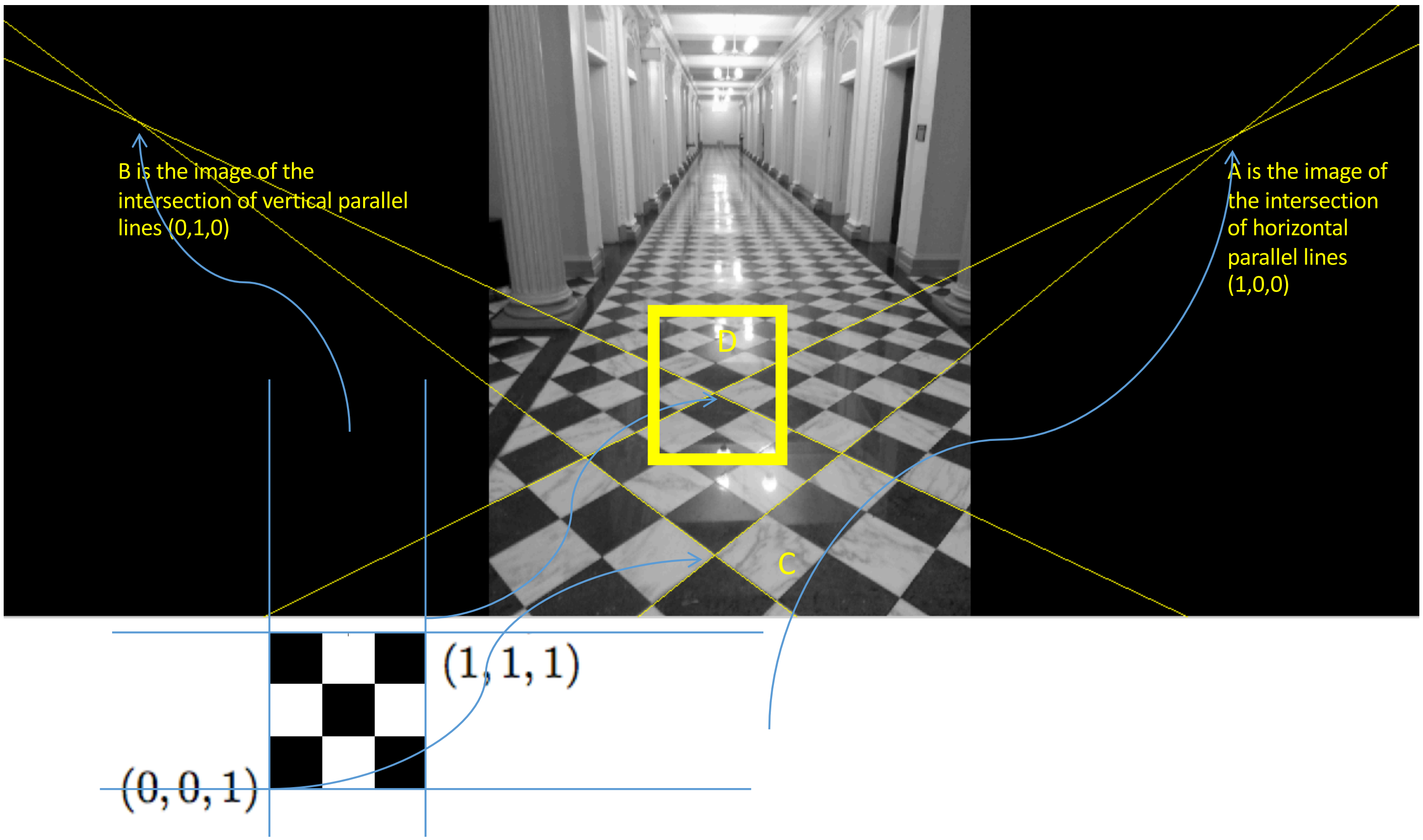
Assume that a mapping A maps the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ to the non-collinear points A,B,C

with coordinate vectors a, b and $c \in \mathbb{P}^2$. Then the following is a possible projective transformation:

$$\begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with 3 degrees of freedoms α, β and γ. This means 3 points do not suffice to compute a projective transformation.

Let us introduce a 4th point D



Let us assume that the same A maps $(1, 1, 1)$ to the point d. Then, the following should hold:

$$\lambda d = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

hence

$$\lambda d = \alpha a + \beta b + \gamma c.$$

There always exist such $\lambda, \alpha, \beta, \gamma$ because four elements of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ are always linearly dependent.

Because a, b, c are not collinear, there exist unique $\alpha/\lambda, \beta/\lambda, \gamma/\lambda$ for writing this linear combination.

Since A is the same as A/λ we solve for α, β, γ such that $d = \alpha a + \beta b + \gamma c$, which can be written as a linear system

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = d.$$

Since a, b, c are not collinear we can always find a unique triple α, β, γ. The resulting projective transformation is $A = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix}$.

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Four points not three of them collinear suffice to recover unambiguously a projective transformation.

Knowledge of this projective transformation makes Virtual Billboards possible!

Computing proj. transformations with more points

$$\mathbf{x}' \sim H\mathbf{x}$$

$$\lambda \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{aligned} \lambda x' &= h_{11}x + h_{12}y + h_{13} \\ \lambda y' &= h_{21}x + h_{22}y + h_{23} \\ \lambda &= h_{31}x + h_{32}y + h_{33} \end{aligned}$$

Computing proj. transformations with more points

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$-h_{11}x - h_{12}y - h_{13} + h_{31}xx' + h_{32}yx' + h_{33}x' = 0$$

$$-h_{21}x - h_{22}y - h_{23} + h_{31}xy' + h_{32}yy' + h_{33}y' = 0$$

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} h = 0$$

$$a_x = \begin{pmatrix} -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \end{pmatrix}$$

$$a_y = \begin{pmatrix} 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \end{pmatrix}$$

$$h = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{21} & h_{22} & h_{23} & h_{31} & h_{32} & h_{33} \end{pmatrix}^T$$

Computing proj. transformations with more points

Our matrix H has 8 degrees of freedom, and so, as each point gives 2 sets of equations, we will need 4 points to solve for h uniquely. So, given four points (such as the corners provided for this assignment), we can generate vectors a_x and a_y for each, and concatenate them together:

$$A = \begin{pmatrix} a_{x,1} \\ a_{y_1} \\ \vdots \\ a_{x,n} \\ a_{y,n} \end{pmatrix}$$

$$Ah = 0$$

As A is a 8x9 matrix, there is a unique null space. Normally, we can use MATLAB's **null** function, however, due to noise in our measurements, there may not be an h such that Ah is exactly 0. Instead, we have, for some small $\vec{\epsilon}$:

$$Ah = \vec{\epsilon} \tag{18}$$

To resolve this issue, we can find the vector h that minimizes the norm of this $\vec{\epsilon}$. To do this, we must use the SVD, which we will cover in week 3. For this project, all you need to know is that you need to run the command:

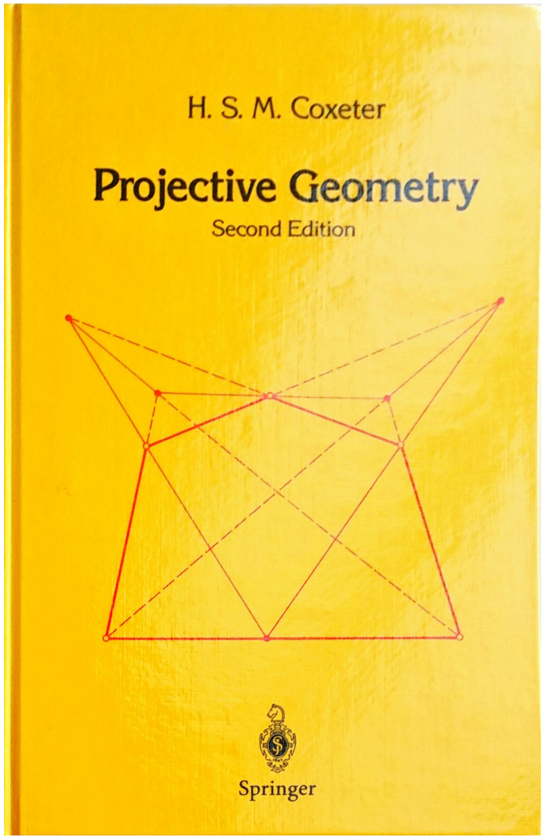
```
[U, S, V] = svd(A);
```

The vector h will then be the last column of V, and you can then construct the 3x3 homography matrix by reshaping the 9x1 h vector.

Microsoft Office Lens App

Office Lens


H. S. M. Coxeter
Projective Geometry
Springer


H. S. M. Coxeter
Projective Geometry
Second Edition
Springer

What happens when the original set of points is not a square?

Find projective transformation mapping $(a, b, c, d) \rightarrow (a', b', c', d')$:

To determine this mapping we go through the four canonical points.

We find the mapping from $(1, 0, 0), etc$ to (a, b, c, d) and we call it T:

$$a \sim T(1, 0, 0)^T, etc$$

We find the mapping from $(1, 0, 0), etc$ to (a', b', c', d') and we call it T':

$$a' \sim T'(1, 0, 0)^T, etc$$

Then, back-substituing $(1, 0, 0)^T \sim T^{-1}a, etc$ we obtain that

$$a' = T'T^{-1}a, etc$$

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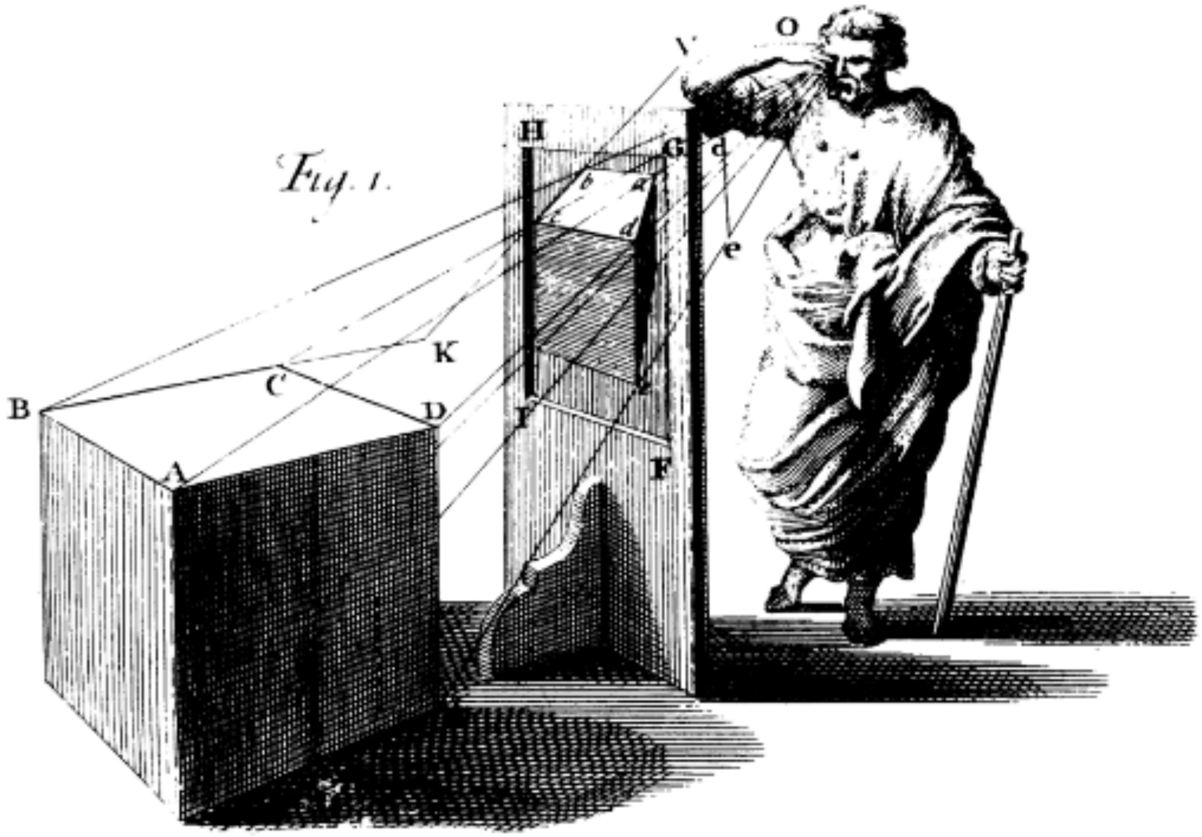


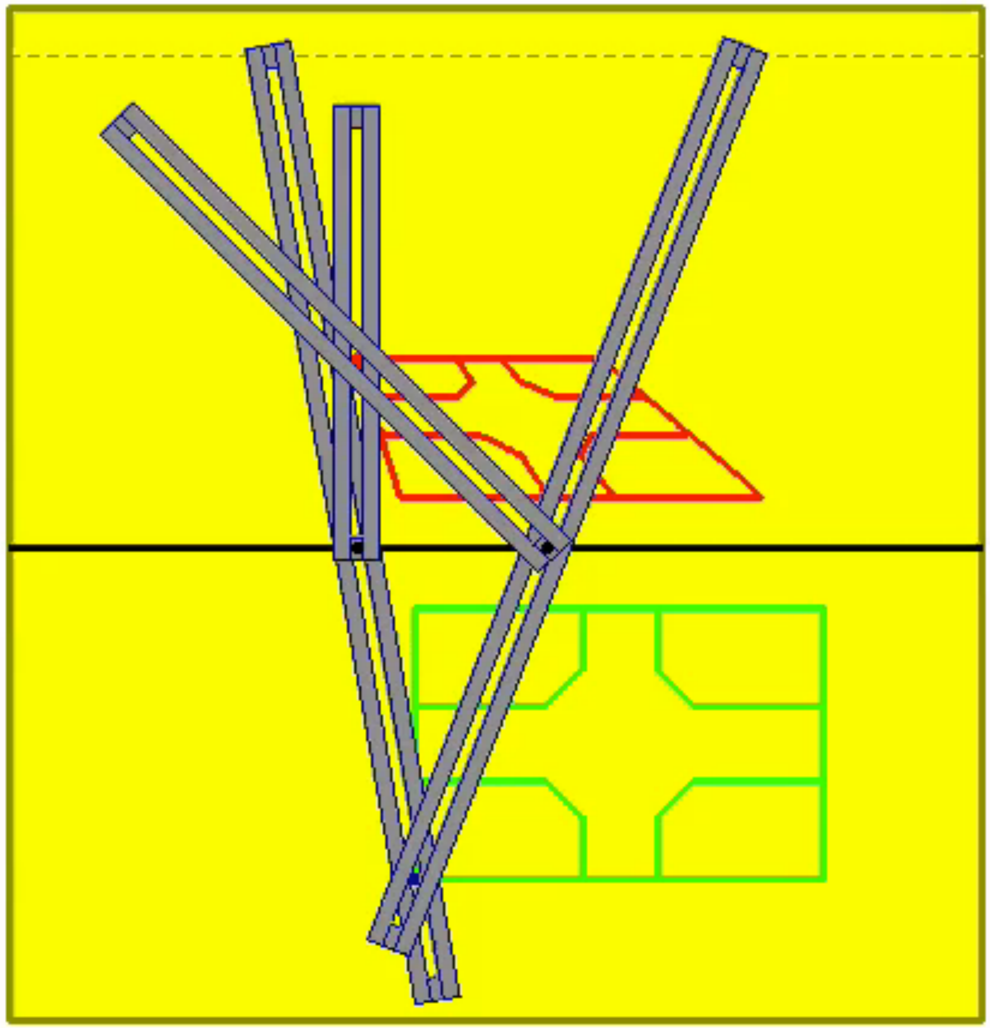
Fig. 1.
V
O
H
G
c
b
a
d
e
K
C
B
D
A
I
F

Bi-perspectograph 1752

LAMBERT'S TWO-DIMENSIONAL PERSPECTOGRAPH (1).

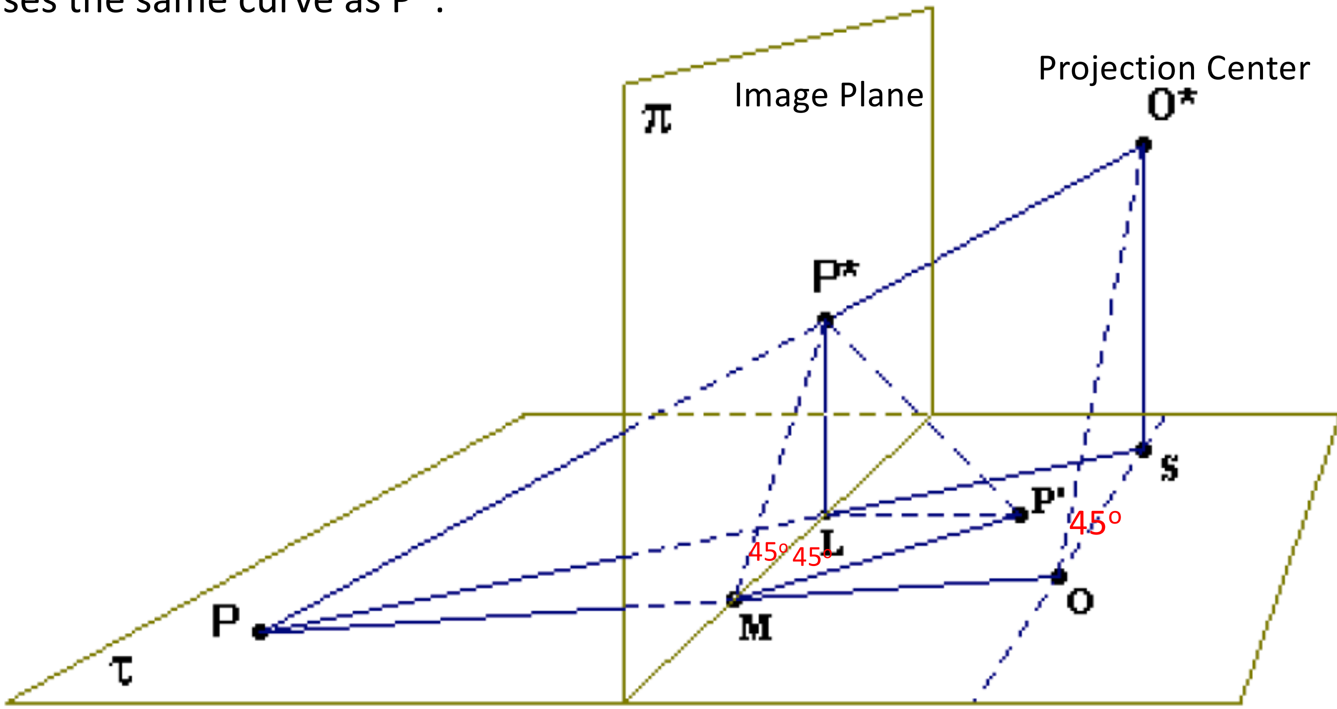
(From: J. H. Lambert, "*Anlage zur Perspektive*", manuscript, August 1752; "*Essai sur la Perspective*", edizione Peiffer – Laurent, 1981)

Mechanical realization



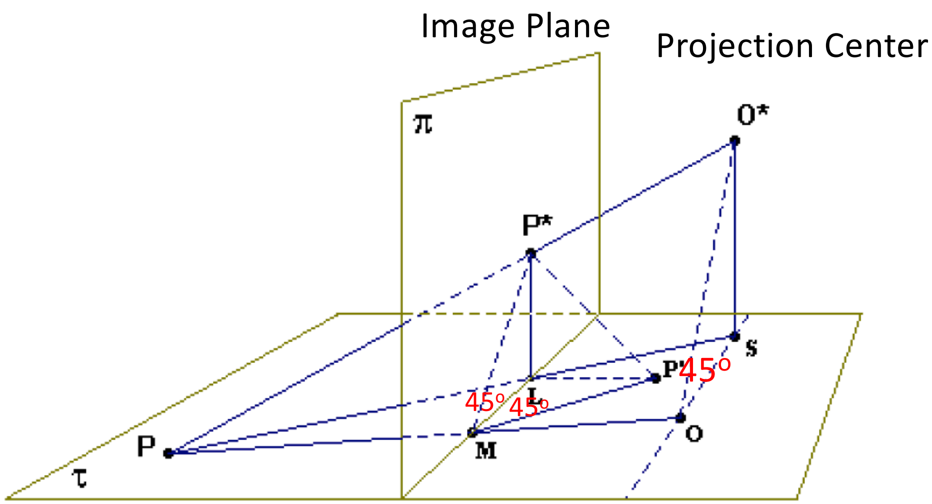
Geometric explanation of bi-perspectograph:

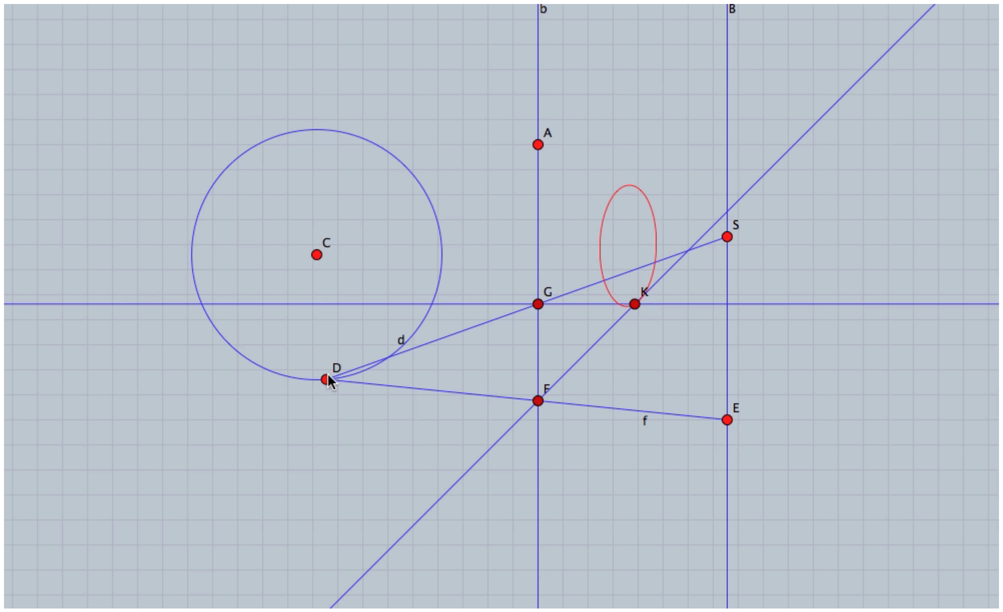
How can we draw a congruent copy of the image plane on the ground plane?
Select O such that $OS=SO^*$. This means angle $SOO^*=45^O$ and hence angle LMP^* is 45^O as well.
Draw line at L perpendicular to LM. Draw line at M with fixed angle 45^O. Call their intersection P'. Then triangle P^*LM is congruent to P'LM.
Hence, P' traverses the same curve as P*.



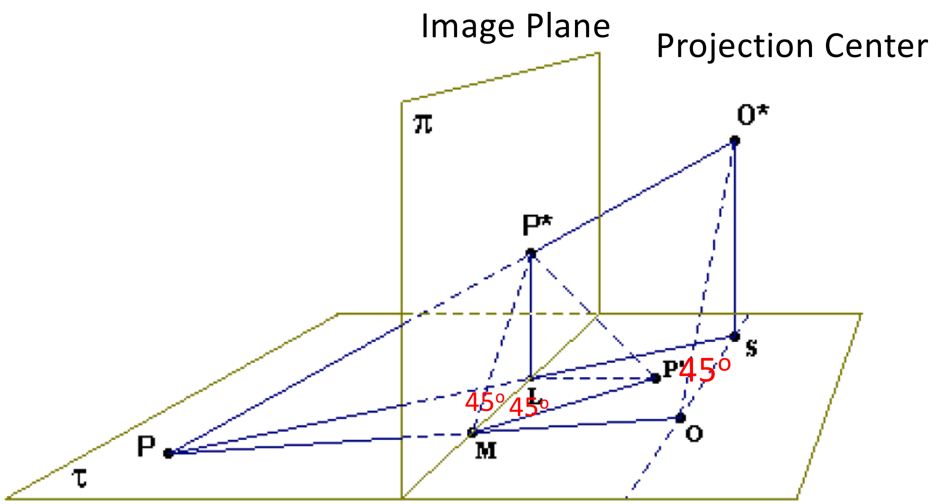
J. H. Lambert, Anlage zur Perspektive, edizione Peiffer - Laurent, 1981

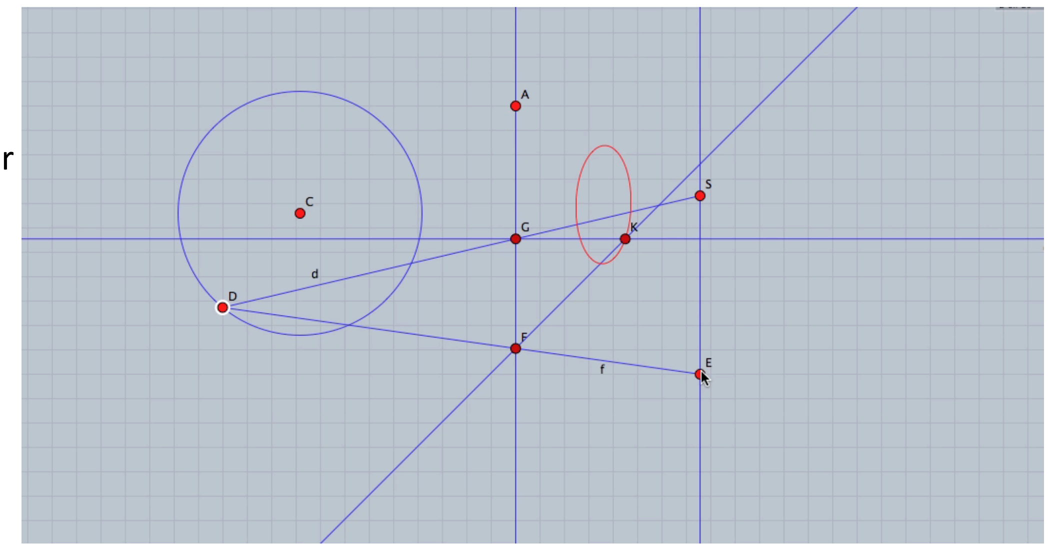
A circle is projected into an ellipse





Effect of height of camera on ellipse shape: the lower the camera, the more squeezed is the ellipse.





Effect of distance of projection center from image plane: Only the size not the shape of the ellipse changes!