CIS 580, Machine Perception, Spring 2021
Homework 2 Solutions

Instructions

- This is an individual homework and worth 100 points
- You must submit your solutions on Gradescope, the entry code is 96JGNN. We recommend that you use LaTeX, but we will accept scanned solutions as well.
- Start early! If you get stuck, please post your questions on Piazza or come to office hours!

Homework

1. (15 pts) For each of the following pairs of points, write down an equation for the line that passes through them (points are in P^2 and $a \neq b \neq c \neq d$):

$$(a)\,[0, a, 0] \textit{ and } [0, 0, a]$$
$$(b)\,[a, a, 1] \textit{ and } [a, a, 2]$$
$$(c)\,[a, b, 0] \textit{ and } [c, d, 0]$$

Answer:

(a) $l = [0, a, 0] \times [0, 0, a] = [a^2, 0, 0]$

(b) $l = [a, a, 1] \times [a, a, 2] = [a, -a, 0]$

(c) $l = [a, b, 0] \times [c, d, 0] = [0, 0, ad - bc]$

2. (15 pts) For each of the following pairs of lines in $\mathbb{P}^2$, write down the point of their intersection.

$$(a)\,x - y + w = 0 \textit{ and } w = 0$$
$$(b)\,3x - w = 0 \textit{ and } 4y - w = 0$$
$$(c)\,x - y + 5w = 0 \textit{ and } x - y + 2w = 0$$

Answer:

(a) $x = [1, -1, 1] \times [0, 0, 1] = [-1, -1, 0]$

(b) $x = [3, 0, -1] \times [0, 4, -1] = [4, 3, 12]$

(c) $x = [1, -1, 5] \times [1, -1, 2] = [3, 3, 0]$

3. (10 pts) Find λ such that the three lines of $\mathbb{P}^2$, $w = 0$, $x + \lambda y + \lambda w = 0$, and $\lambda x + y + \lambda w = 0$ have a common intersection. Which point is the intersection?
Answer: We have the lines $l_1 = [0, 0, 1]$, $l_2 = [1, \lambda, \lambda]$, $l_3 = [\lambda, 1, \lambda]$. If they intersect in the same point we have that

$$l_1 \times l_2 \sim l_1 \times l_3 \implies [-\lambda, 1, 0] \sim [-1, \lambda, 0]$$

The two vectors are equivalent when $\lambda = 1$, $\lambda = -1$. When $\lambda = 1$ then l_1, l_2 are the same lines and the point of intersection of the now 2 lines is $[-1, 1, 0]$. When $\lambda = -1$ then the point of intersection is $[1, 1, 0]$.

4. (20 pts) Find a projective transformation A that preserves the points $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$, and the origin of the coordinate system O and will map the point $p_3 = (1, 1, 1)$ to the points $p'_3 = (3, 2, 1)$? Does the image of line at infinity still lie at infinity? Why?
Answer:
Given that $A = [a_1^T, a_2^T, a_3^T]$ where a_i^T is the i^{th} column of A, we have that:

$$a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a_1^T$$

$$b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = a_2^T$$

$$c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a_1^T$$

So we get that

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Finally for the fourth point we get

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \implies \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

we set $\lambda = 1$ and we get $a = 3$, $b = 2$, $c = 1$ so

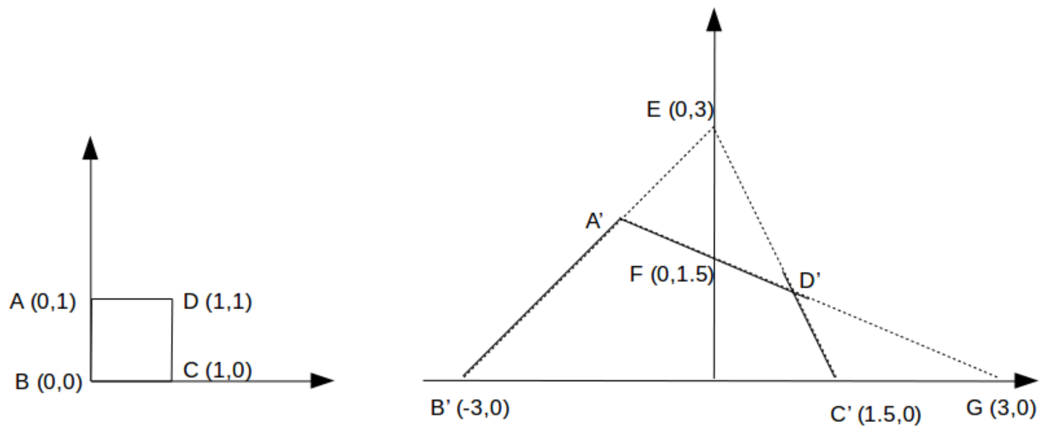
$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The line at infinity becomes:

$$l' = A^{-T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So it stays at infinity

5. (20 pts) Please find a projection transformation P such that $A' \sim PA$, $B' \sim PB$, $C' \sim PC$, $D' \sim PD$ as shown in the following figure. [**Hint**: it's a little tedious to calculate transformation using only A, B, C and D, try to use the intersection of parallel lines.]



Answer: The lines parallel lines BA and CD intersect at $[0, 1, 0]^T$. The parallel lines AD and BC intersect at $[1, 0, 0]^T$. Also point D' can be found by the intersection of the lines $A'G$ and $C'E$ and it is $D' = [1, 1, 1]$. So we have that similar with question 4

$$a\begin{bmatrix}3\\0\\1\end{bmatrix} = P\begin{bmatrix}1\\0\\0\end{bmatrix} = p_1^T$$

$$b\begin{bmatrix}0\\3\\1\end{bmatrix} = P\begin{bmatrix}0\\1\\0\end{bmatrix} = p_2^T$$

$$c\begin{bmatrix}-3\\0\\1\end{bmatrix} = P\begin{bmatrix}0\\0\\1\end{bmatrix} = p_1^T$$

So we have that

$$P = \begin{bmatrix}3a & 0 & -3c\\0 & 3b & 0\\a & b & c\end{bmatrix}$$

And with the final point we get

$$P\begin{bmatrix}1\\1\\1\end{bmatrix} = \lambda\begin{bmatrix}1\\1\\1\end{bmatrix} \implies \begin{bmatrix}3a - 3c\\3b\\a + b + c\end{bmatrix} = \lambda\begin{bmatrix}1\\1\\1\end{bmatrix}$$

by setting $\lambda = 1$ and solving the system of equations we get $a = 1/2$, $b = 1/3$, $c = 1/6$. And the final transformation matrix becomes

$$P = \begin{bmatrix} 3/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 1/3 & 1/6 \end{bmatrix}$$

6. (20 pts) A projective transformation A maps point $(-a, 0, 1)$ to point $(1, 0, 0)$, and maps point $(0, b, 1)$ to point $(0, 1, 0)$. However, it keeps the origin of system $(0, 0, 1)$ and $(1, 1, 1)$ fixed. Please find the transformation A.

   Answer: It is more convenient to first compute the inverse transformation A^{-1}. Similar with the previous questions we get:

$$d\begin{bmatrix} -a \\ 0 \\ 1 \end{bmatrix} = P\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a_1'^T$$

$$e\begin{bmatrix} 0 \\ b \\ 1 \end{bmatrix} = P\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = a_2'^T$$

$$f\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = P\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a_1'^T$$

   which means that

$$A^{-1} = \begin{bmatrix} -da & 0 & 0 \\ 0 & eb & 0 \\ d & e & f \end{bmatrix}$$

   Then we use the fourth point to get

$$A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \implies \begin{bmatrix} -da \\ eb \\ d+e+f \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   by setting $\lambda = 1$ and solving the system of equations we get $d = -1/a$, $e = 1/b$, $f = (ab - a + b)/ab$ So

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{a} & \frac{1}{b} & \frac{ab-a+b}{ab} \end{bmatrix} \implies$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{b}{ab-a+b} & -\frac{a}{ab-a+b} & \frac{ab}{ab-a+b} \end{bmatrix}$$