# 3D-3D Pose or Procrustes Problem

Given correspondences of points $A_i \in \mathbb{R}^3$ and $B_i \in \mathbb{R}^3$ find the scaling, rotation, and translation transformation, called *similitude* transformation, that satisfies

$$A_i = sRB_i + T$$

for $R \in SO(3)$, $T \in \mathbb{R}$, and $s \in \mathbb{R}^+$.

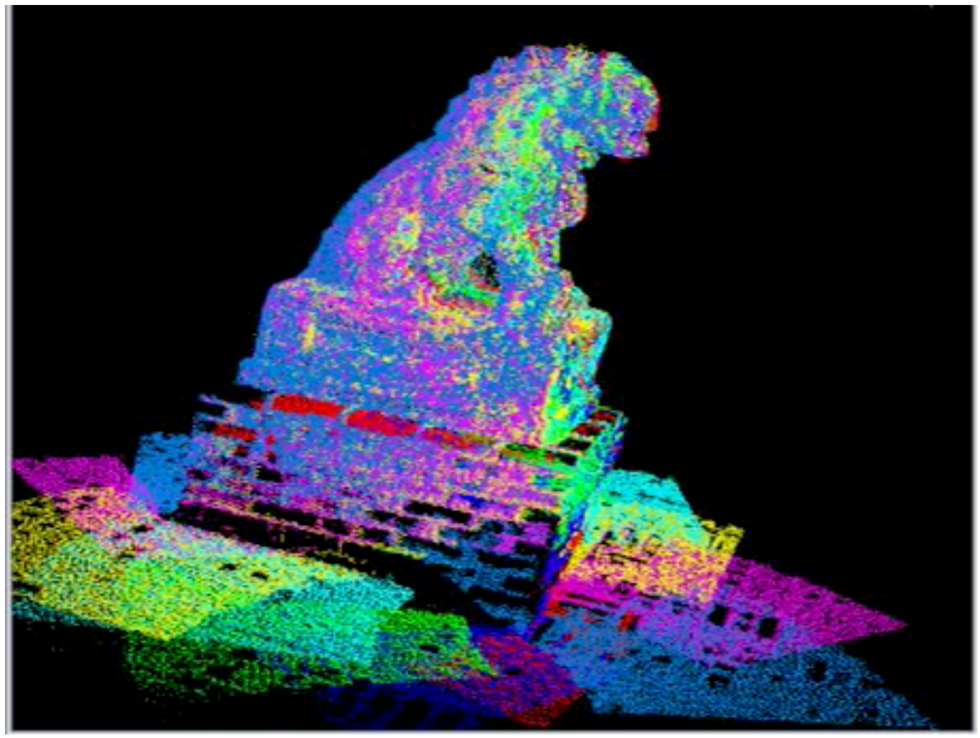
# 3D-3D Pose or Procrustes Problem

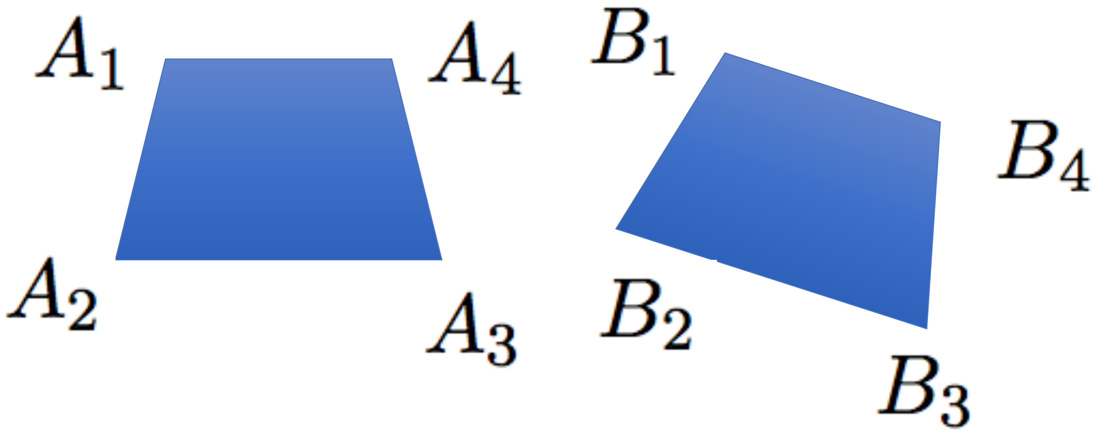
In the camera rigid pose problem scale $s = 1$ is known:

$$Z_i p_i^{cam} = R P_i^{obj} + T$$

This is the last step of the P3P problem or the entire problem of finding rigid pose when we know the depth at every point (e.g., in am RGB-D sensor).

3D-3D Registration enables the creation of 3D models from multiple point clouds:

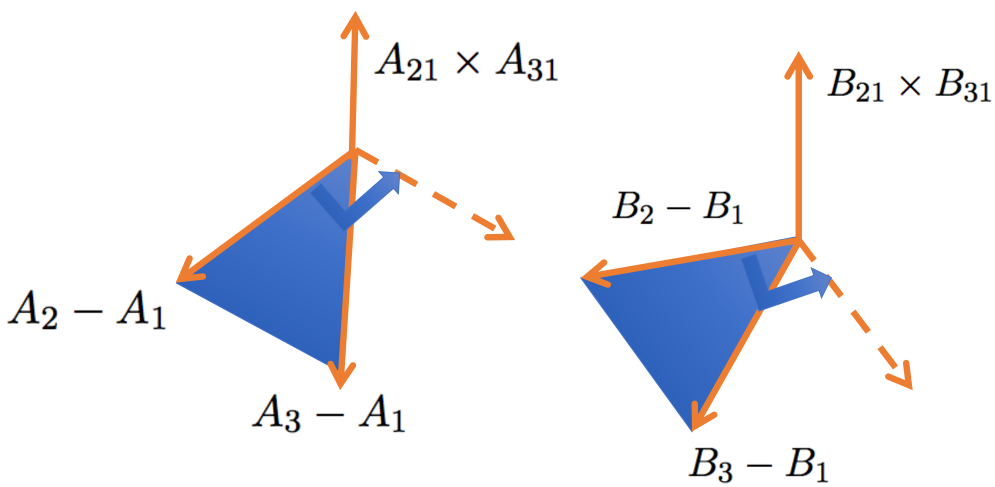


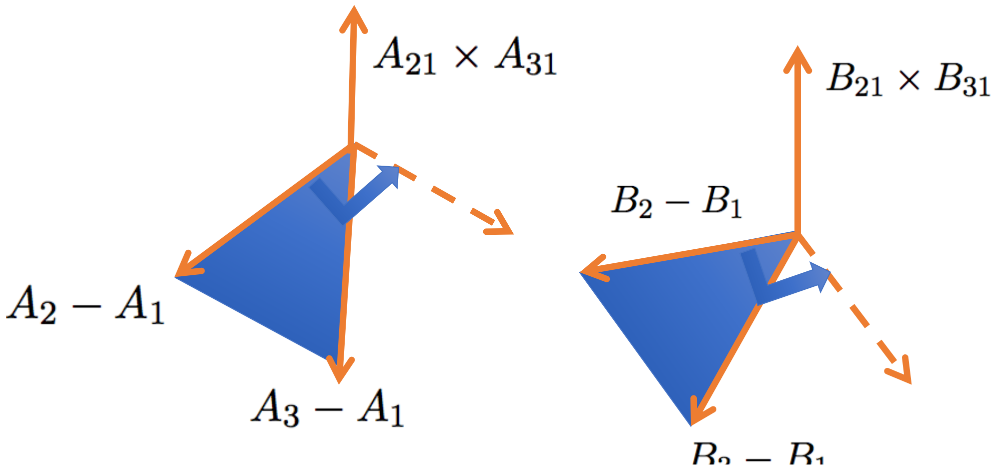



How do we solve for $R, T$ from $n$ point correspondences?

$$A_i = RB_i + T$$

What is the minimal number of points needed?






Three non-collinear points suffice: each triangle $A_{i=1...3}$ and $B_{i=1...3}$ make an orthogonal basis

$$\begin{pmatrix} A_{21} & (A_{21} \times A_{31}) \times A_{21} & A_{21} \times A_{31} \end{pmatrix}$$

and

$$\begin{pmatrix} B_{21} & (B_{21} \times B_{31}) \times B_{21} & B_{21} \times B_{31} \end{pmatrix}$$

Rotation between two orthogonal bases is unique.

We solve a minimization problem for $N > 3$ point correspondences:

$$\min_{R,T} \sum_{i}^{N} \|A_i - RB_i + T\|^2$$

After differentiating with respect to $T$ we observe that the translation is the difference between the centroids:

$$T = \frac{1}{N}\sum_{i}^{N} A_i - R\frac{1}{N}\sum_{i}^{N} B_i = \bar{A} - R\bar{B}$$

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We subtract the centroids $\bar{A}$ and $\bar{B}$ and rewrite the objective function as

$$\min_R \|A - RB\|_F^2$$

where

$$A = \begin{pmatrix} A_1 - \bar{A} & \dots & A_N - \bar{A} \end{pmatrix}$$

and

$$B = \begin{pmatrix} B_1 - \bar{B} & \dots & B_N - \bar{B} \end{pmatrix}$$

We rewrite the Frobenius norm using the trace of the matrix

$$\|A - RB\|_F^2 = tr(A^T A) + tr(B^T B) - tr(A^T RB) - tr(B^T R^T A)$$

and observe that only the two last terms depend on the unknown $R$ yielding a maximization problem.

Even without using the properties of the trace we can see that both last terms are equal to

$$\sum_i^N R(B_i - \bar{B})(A_i - \bar{A})^T = tr(RBA^T)$$

The 3D-3D pose problem reduced to

$$\max_R tr(RBA^T)$$


If the SVD of $BA^T$ is $USV^T$ and $Z = V^T RU$

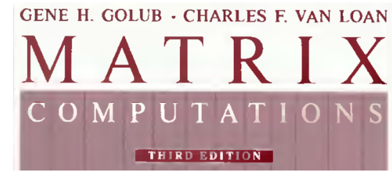
$$tr(RBA^T) = tr(RUSV^T) = tr(ZS) = \sum_1^3 z_{ii}\sigma_i \leq \sum_1^3 \sigma_i$$

and, hence, the upper bound is obtained by setting

$$Z = I \qquad V^T RU = I \qquad R = VU^T$$

We guarantee that $\det(R) = 1$ by inserting

$$R = V \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(VU^T) \end{pmatrix} U^T$$




### 6.4.1 Rotation of Subspaces

Suppose $A \in \mathbb{R}^{m \times p}$ is a data matrix obtained by performing a certain set of experiments. If the same set of experiments is performed again, then a different data matrix, $B \in \mathbb{R}^{m \times p}$, is obtained. In the *orthogonal Procrustes problem* the possibility that $B$ can be rotated into $A$ is explored by solving the following problem:

$$\text{minimize } \| A - BQ \|_F , \qquad \text{subject to } Q^T Q = I_p . \tag{6.4.1}$$

We show that optimizing $Q$ can be specified in terms of the SVD of $B^T A$. The *matrix trace* is critical to the derivation. The trace of a matrix is the sum of its diagonal entries:

$$\mathrm{tr}(C) \;=\; \sum_{i=1}^{n} c_{ii} , \qquad C \in \mathbb{R}^{n \times n}.$$

It is easy to show that if $C_1$ and $C_2$ have the same row and column dimension, then

$$\mathrm{tr}(C_1^T C_2) \;=\; \mathrm{tr}(C_2^T C_1) . \tag{6.4.2}$$

Returning to the Procrustes problem (6.4.1), if $Q \in \mathbb{R}^{p\times p}$ is orthogonal, then

$$\begin{aligned}
\| A - BQ \|_F^2 &= \sum_{k=1}^{p} \| A(:,k) - B\cdot Q(:,k) \|_2^2 \\
&= \sum_{k=1}^{p} \| A(:,k) \|_2^2 + \| BQ(:,k) \|_2^2 \; - \; 2Q(:,k)^T B^T A(:,k) \\
&= \| A \|_F^2 \; + \; \| BQ \|_F^2 \; - \; 2\sum_{k=1}^{p} \left[Q^T(B^T A)\right]_{kk} \\
&= \| A \|_F^2 \; + \; \| B \|_F^2 \; - \; 2\mathrm{tr}(Q^T(B^T A)).
\end{aligned}$$

Thus, (6.4.1) is equivalent to the problem

$$\max_{Q^T Q = I_p} \; \mathrm{tr}(Q^T B^T A)\,.$$

If $U^T(B^T A)V \;=\; \Sigma \;=\; \mathrm{diag}(\sigma_1,\ldots,\sigma_p)$ is the SVD of $B^T A$ and we define the orthogonal matrix $Z$ by $Z \;=\; V^T Q^T U$, then by using (6.4.2) we have

$$\mathrm{tr}(Q^T B^T A) \;=\; \mathrm{tr}(Q^T U \Sigma V^T) \;=\; \mathrm{tr}(Z\Sigma) \;=\; \sum_{i=1}^{p} z_{ii}\sigma_i \;\le\; \sum_{i=1}^{p} \sigma_i\,.$$

The upper bound is clearly attained by setting $Z = I_p$, i.e., $Q = UV^T$.

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