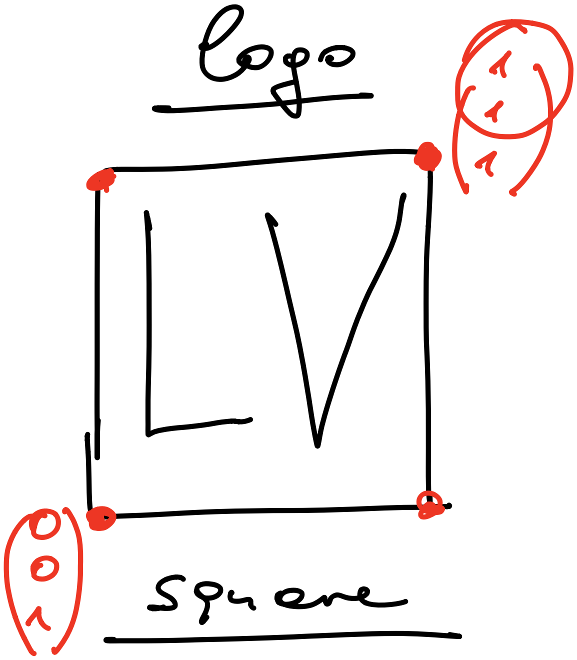
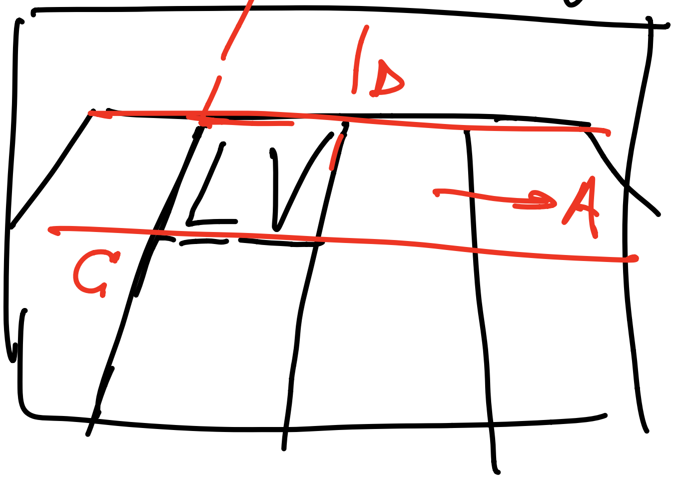
# Brief recap on projective transformations

logo




square

virtual billboarding




Special 4-point algorithm

Let the transformation be $H$

$$A \sim H\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad C \sim H\begin{pmatrix}0\\0\\1\end{pmatrix} \leftarrow \text{origin}$$

$$B \sim H\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad D \sim H\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$H = \begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix}$$

$H = ?$

$$H\begin{pmatrix}1\\0\\0\end{pmatrix} = h_1$$

'=' mean multiple !

$$\underline{A} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \underline{B} = \begin{pmatrix}500\\-15000\\1\end{pmatrix} \quad \underline{C} = \begin{pmatrix}100\\200\\1\end{pmatrix} \quad \underline{D} = \begin{pmatrix}200\\100\\1\end{pmatrix}$$

just is the picture above

$$\boxed{\begin{gathered}\alpha A = h_1 \quad \beta B = h_2 \quad \gamma C = h_3 \\ \delta D = h_1 + h_2 + h_3\end{gathered}}$$

$$\Downarrow$$

$$\boxed{\delta D = \alpha A + \beta B + \gamma C} \quad 3\text{eq}$$

Set $\delta = 1$

$$D = \alpha A + \beta B + \gamma C$$

$$\underset{3\times 1}{D} = \underset{3\times 3}{\begin{pmatrix}A & B & C\end{pmatrix}} \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$$

$$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \begin{pmatrix}A & B & C\end{pmatrix}^{-1} D \Rightarrow H = \underset{3\times 3}{\begin{pmatrix}\alpha A & \beta B & \gamma C\end{pmatrix}}$$

① When is $(A\ B\ C)$ invertible?

collinearity $\underbrace{\det\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}}_{\text{triangle area}} = 0$

② Why did I set $\delta = 1$ without loss of generality (WLOG)

If $p' \sim Hp$ then any other $\lambda H$ is also a proj. transformation that maps $p$ to $p'$.

$$p' \sim Hp \Leftrightarrow \rho p' = Hp$$

$$\Downarrow$$

$$p' \sim \lambda H p \Leftrightarrow \lambda \rho p' = \lambda H p$$

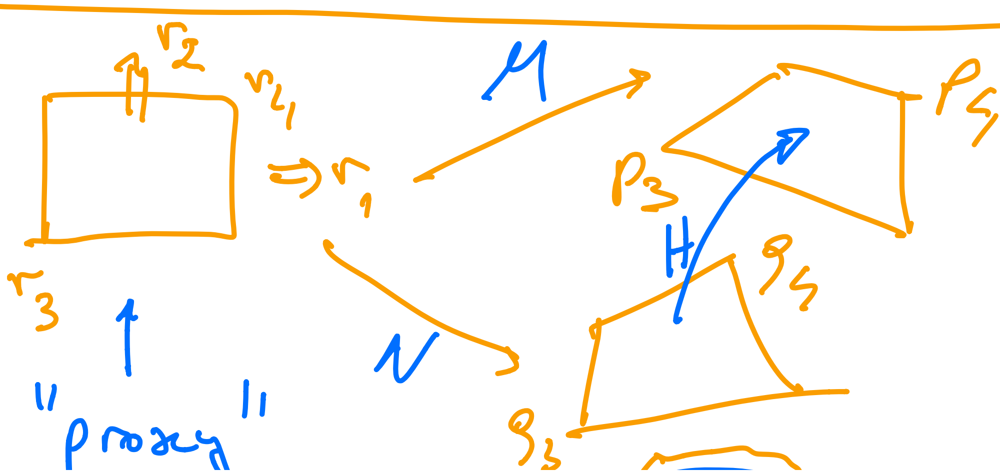
We can find $H$ upto a scalar! If you find

$H = \begin{pmatrix} 5 & 6 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and I find

$$H = \begin{pmatrix} 10 & 12 & 14 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ we both

are correct.

$\boxed{\text{Given \underline{4 point correspondences} with at least 3 being non-collinear we can recover a unique (upto a scale factor) projective transformation.}}$



$$p_i \sim M r_i \sim M N^{-1} q_i \sim H q_i$$

$$q_i \sim N r_i \Rightarrow r_i \sim N^{-1} q_i$$

What happens if we have $N > 4$ point correspondences?

$$p' \sim Hp \qquad \lambda \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} h_{11} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

divide by 3rd equation

$$x' = \frac{\mu h_{11} x + \mu h_{12} y + \mu h_{13}}{\mu h_{31} x + \mu h_{32} y + \mu h_{33}}$$

$$y' = \frac{h_{21} x + h_{22} y + h_{23}}{\quad}$$

If $H$ is a solution then $\mu H$ is a solution too!

$$
\underbrace{\begin{pmatrix} -x & -y & -1 & 0 & 0 & 0 & x'x & x'y & x' \\ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \\ & & & & \vdots & & & & \end{pmatrix}}_{2 \times 9}
\underbrace{\begin{pmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{pmatrix}}_{9 \times 1}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

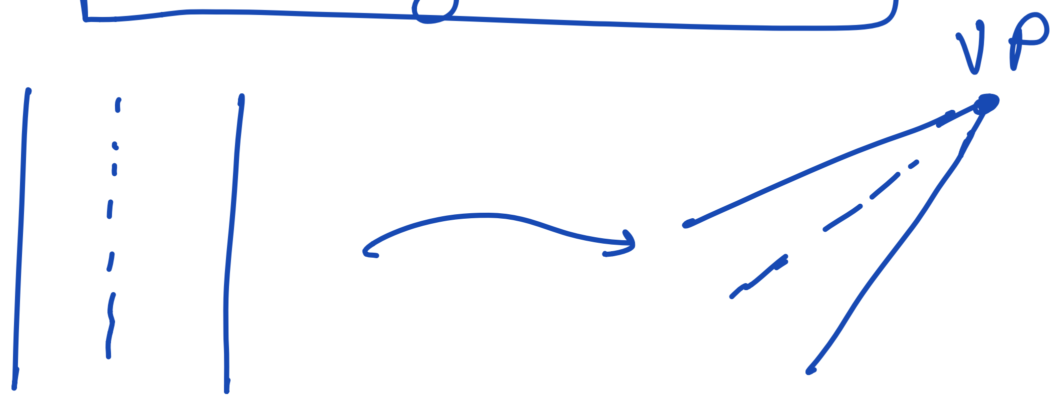
4 points $\quad 8 \times 9 \qquad 9 \times 1 = 8 \times 1$

N points $\quad 2N \times 9$

$$
A = \underset{2N \times 9}{U}\ \underset{9 \times 9}{S}\ \underset{9 \times 9}{V^T}
$$

$\boxed{\text{solution is last column of } V}$

# Vanishing Points




VP is the projection of a point at infinity (itself it might be at infinity or not)

## Case 1: Picture of a plane

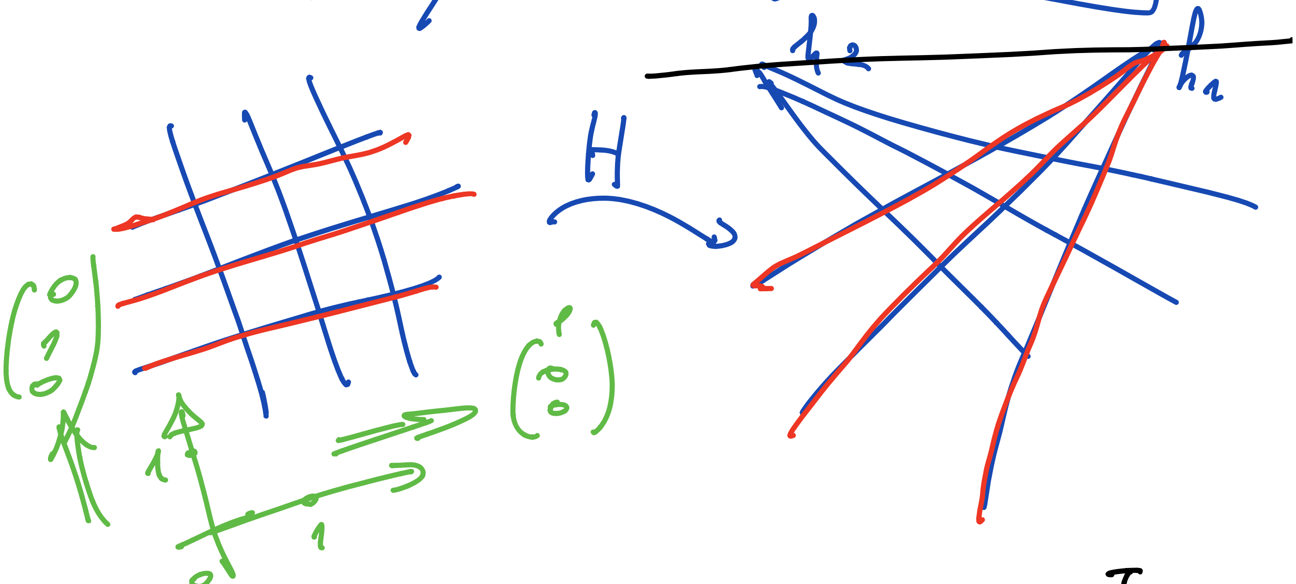
can be modelled with a projective transformation

$$p' \sim H p$$

$$H\begin{pmatrix}1\\0\\0\end{pmatrix} \sim h_1$$

$$H\begin{pmatrix}0\\1\\0\end{pmatrix} \sim h_2$$

are orthogonal vanishing points




horizon equation $(h_1 \times h_2)^T \begin{pmatrix}x\\y\\1\end{pmatrix} = 0$

↓

is the projection of the line at infinity

Assume all points at infinity are on ~~a~~ line (invisible).

$\rightarrow w = 0$ lie on $(0\ 0\ 1)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0$

$H$

$(h_1 \times h_2)^T \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = 0$

if $\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} \sim H \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

| | |
|---|---|
| point at infinity $w=0$ | VP |
| line at infinity | horizon |

line at infinity $\boxed{(0\ 0\ 1)}\, p = 0$ ($l_\infty$)

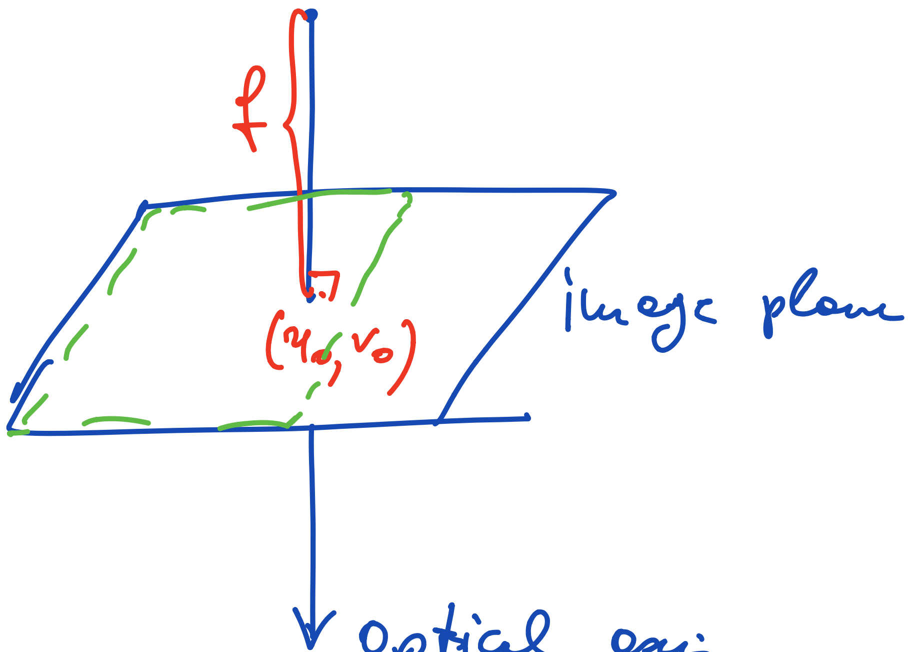
horizon $\boxed{(h_1 \times h_2)^T}\, p' = 0$ ($l_h$)

$\boxed{p' \sim Hp}$

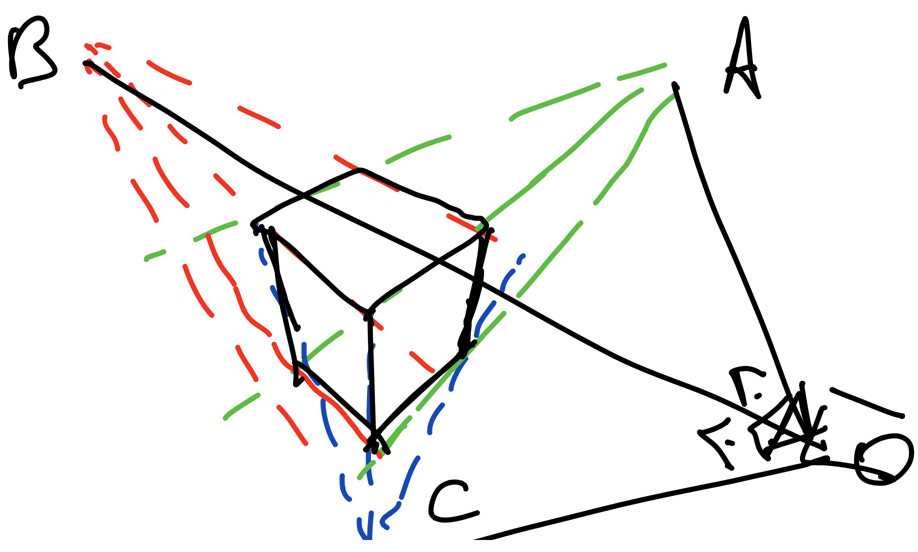
$l \not\sim Hl$

$\boxed{l' \sim H^{-T} l}$

intrinsic $K = \begin{pmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{pmatrix}$



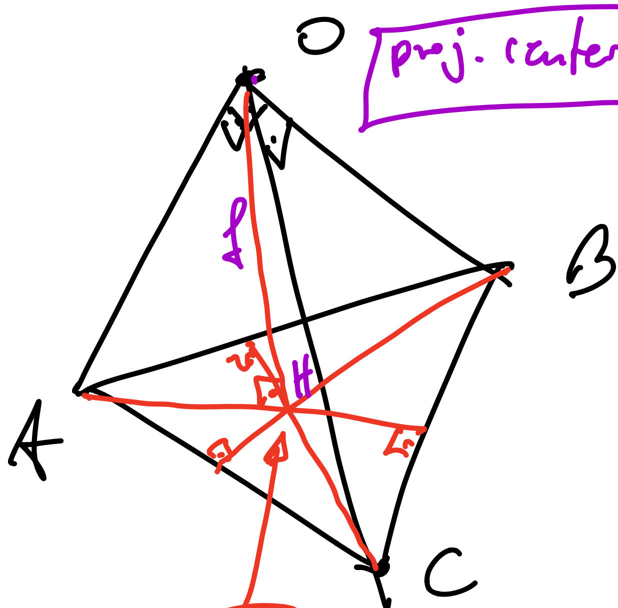

$\boxed{\text{How to compute } f, u_0, v_0 \text{ from 3 orthogonal vanishing points}}$



$OA \perp OB$

$OB \perp OC$

$OC \perp OA$




proj. center

A, B, C image plane (pixels)

1. orthocenter is image center

2. $f = OH$