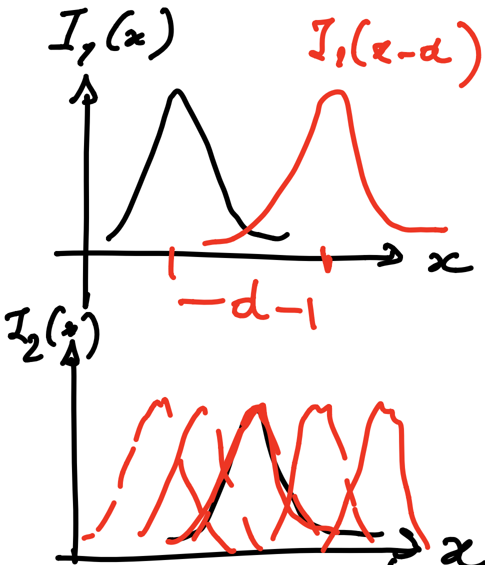
Optical Flow

$$\sum_{\vec{x}\in N}\left(I_1(\vec{x}-\vec{d}) - I_2(\vec{x})\right)^2$$

$$= \sum_{\vec{x}\in N}\left(\underbrace{I_1(\vec{x}) - \vec{d}^T\nabla I_1(\vec{x})}_{\text{Taylor expansion}} - I_2(\vec{x})\right)^2$$

$\vec{d}$ small, $O(\|d\|^2)$ negligible



$$= \sum\left(-\vec{d}^T\nabla I_1(x) - \Delta I(x)\right)^{2}$$

$\Delta I = I_2 - I_1$

linear in $\vec{d}$

$\vec{d}^T A \vec{d} + b^T \vec{d} + c$

quadratic

$$\nabla I_1 = \begin{pmatrix}\partial I_1/\partial x \\ \partial I_1/\partial y\end{pmatrix}$$

To minimize take the derivative !

(minus vanishes)

$$\frac{\partial}{\partial \vec{d}} = 2\sum\left(\vec{d}^{\,T}\nabla I_1 + \Delta I\right)\nabla I_1 = 0$$

$$\left(\sum \nabla I_1 \nabla I_1^T\right)\underbrace{\vec{d}}_{2\times 1} = -\sum \nabla I_1 \Delta I$$

Hessian because it is of the form $\sum a a^T$ it is positive definite hence d will be a minimum.

ΔI at same pixel

I_1

$$\begin{bmatrix} 1 & 3 & 1 \\ 4 & 10 & 8 \\ 1 & 5 & 7 \end{bmatrix}$$

I_2

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 10 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\Delta I = I_2 - I_1 = \begin{bmatrix} 1 & -2 & 2 \\ -3 & -6 & 2 \\ -1 & -4 & [illegible] \end{bmatrix}$$

∇I, at the center

$\overleftrightarrow{x} : [4 \;\; 10 \;\; 8] \cdot [-1 \;\; 0 \;\; 1] = 4$

mask derivative

$\downarrow y : 2$

Do this for the whole neighborhood

$$\left[\sum_{x,y \in N} \begin{pmatrix} \left(\frac{\partial I}{\partial x}\right)^2 & \frac{\partial I}{\partial x}\frac{\partial I}{\partial y} \\ \frac{\partial I}{\partial x}\frac{\partial I}{\partial y} & \left(\frac{\partial I}{\partial y}\right)^2 \end{pmatrix} \right] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sum \Delta I \frac{\partial I}{\partial x} \\ \sum \Delta I \frac{\partial I}{\partial y} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \left[\quad \right]^{-1} \left[\quad \right]$$

$\frac{\partial I}{\partial x} \quad \frac{\partial I}{\partial y} \quad \Delta I$ $\boxed{\text{needed}}$

2 × 2 linear system

Does it always have a solution and if yes is it unique?

$$M\vec{\alpha} = \vec{u}$$

2×2 2×1 2×1

① <u>existence</u> $(M, \vec{u})$

2×3

$\vec{u}$ can be written as a linear combination of the columns of M

$$\begin{cases} 3x + 2y = 1 \\ 3x + 2y = 2 \end{cases}$$

② uniqueness : $\boxed{\mathrm{rank}(M) = 2}$

$\det(M) \neq 0$

Can $\sum_i \nabla I_i \nabla I_i^T$ be singular ?

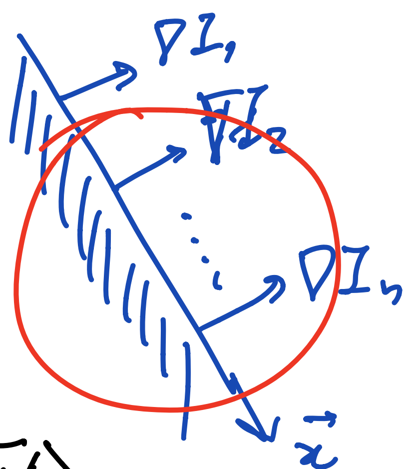
$(\mathrm{rank} = 1)$

$\Longleftrightarrow$ $\exists$ vector x :

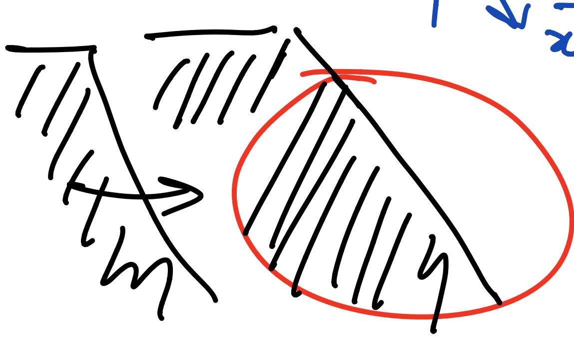
$$\left[\sum \nabla I_i \nabla I_i^T\right] x = 0$$

$\Longleftrightarrow$ all ∇I_i are parallel

$\rightarrow$ rank = 1



Aperture problem

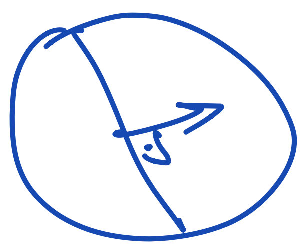


we cannot compute flow!

Aperture problem:

All gradients are parallel!

In this case we can compute only the projection of flow $\vec{d}$ $\perp$ edge:



normal flow!

When is $\sum_i \nabla I_i \nabla I_i^T$ zero rank? $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

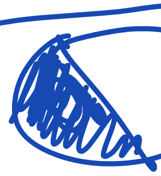
$\Leftrightarrow \nabla I_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

constant color!

? static picture $\Delta I = 0$

$$(M)\vec{d} = 0$$

$\vec{d} = 0$ if $\text{rank}(M)=2$ ok ✓



? singular ⇒ no solution

$\text{rank}(M)=1$ $\quad \Delta I = 0$

$$\nabla I_i \vec{d} = 0$$

∇I_i are parallel

What do we do in practice?

Reject flow when

$\det(M) < \varepsilon$ threshold !

Potential Problems

① motion is too large !

(violation of Taylor expansion)

→ solution → multiple iteration

→ coarse to fine approach *

* I_1, I_2 are 512×512 $\|\vec{d}\| = 10$ pix

build 256×256 $\|\vec{d}\| = 5$ pix

warp $I_1(x - 2d_{64}) \Rightarrow \|\vec{d}_{128}\| < 1$ 128×128 $\|\vec{d}\| = 2.5$ pix

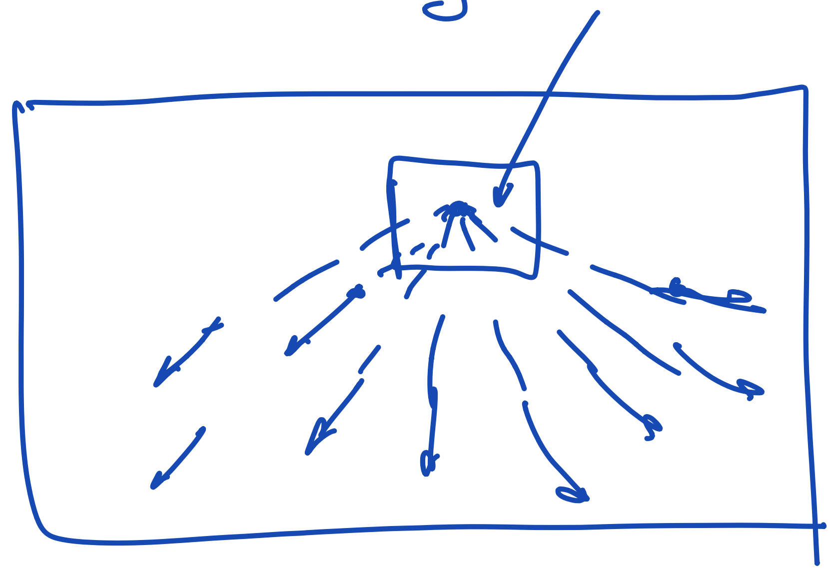
$\Uparrow$

first $\vec{d}_{64} \Longleftarrow$ 64×64 $\|\vec{d}\| = 1.25$ pix

② occlusion

or no color constancy

3. flow is not local in this neighborhood



The described method produced the KLT-tracker!

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illusion