CIS580 Problem Set 3

Sheil Sarda <sheils@seas.upenn.edu>
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Contents

1 Transformation to map facade to rectangle

$$W' \sim P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies \alpha W' = P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = T_1$$

$$X' \sim P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \implies \beta X' = P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = T_2$$

$$Y' \sim P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies \gamma Y' = P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T_3$$

$$Z' \sim P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \implies \delta Z' = P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = T_1 + T_2 + T_3$$

Combining the above equations:

$$\delta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} -b \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ h \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can infer that $\delta = 1$, and simplify the system of equations to:

$$\delta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} -b \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ h \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} = \left(\begin{bmatrix} -b & 0 & 0 \\ 0 & h & 0 \\ 1 & 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving for the inverse of the coefficient matrix, we obtain:

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -1/b \\ 1/h \\ 1/b - 1/h + 1 \end{bmatrix}$$

From this, we obtain the transformation P by multiplying α, β and γ into the above equation:

$$T^{-1} = \alpha \begin{bmatrix} -b \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ h \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{-1}{b} \begin{bmatrix} -b \\ 0 \\ 1 \end{bmatrix} + \frac{1}{h} \begin{bmatrix} 0 \\ h \\ 1 \end{bmatrix} + \left(\frac{1}{b} - \frac{1}{h} + 1 \right) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-1}{b} & \frac{1}{h} & \frac{1}{b} - \frac{1}{h} + 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Taking the inverse of the above transformation matrix, we obtain the matrix T:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{h}{h-b+bh} & \frac{-b}{h-b+bh} & \frac{bh}{h-b+bh} \end{bmatrix}$$

2 Compute distances of patrol car and bridge

2.1 Distance from bridge to junction

Known quantities:

Segment	**Value**
XA	4 cm
AB	4 cm
BY	2 cm
YC	1 cm

Using cross-ratios of the above quantities, we obtain:

$$\begin{aligned}
\frac{AY/AC}{BY/BC} &= \frac{6/7}{2/3} \\
\frac{1-CY}{(0.5-CY)/0.5} &= \frac{6/7}{2/3} && \text{Given junction location and map signs} \\
\frac{1-CY}{(0.5-CY)/0.5} &= \frac{6/7}{2/3} \\
CY &= \frac{9}{7}\cdot(2CY-1)+1 \\
CY &= \frac{18CY}{7} - \frac{2}{7} \\
\implies CY &= \frac{2}{11} = 0.182km
\end{aligned}$$

Using the above information, we can also compute $YB = 0.5 - CY = \frac{7}{22}$.

2.2 Distance from patrol car to junction

Using cross-ratios, we obtain:

$$\begin{aligned}
\frac{XB/XY}{AB/AY} &= \frac{8/10}{4/6} \\
\frac{(0.5+XA)/(BY+0.5+XA)}{0.5/(BY+0.5)} &= \frac{8/10}{4/6} \\
\frac{(18/22)(0.5+XA)}{0.5(18/22+XA)} &= \frac{8/10}{4/6} \\
\frac{(9/22)+(18XA/22)}{(9/22+0.5XA)} &= \frac{8/10}{4/6} \\
(4/6)((9/22)+(18XA/22)) &= (9/22+0.5XA)(8/10) \\
(3/11)+(6XA/11) &= 36/110+(2XA/5) \\
8XA/55 &= 6/110 \\
XA &= (6*55)/(110*8) = 3/8 = 0.375km \\
\implies XC &= 1+\frac{3}{8} = 1.375km
\end{aligned}$$

3 Compute distances from the image

3.1 Distance BC

Using cross-ratios, we obtain:

$$\frac{A_wC_w/A_wD_w}{B_wC_w/B_wD_w} = \frac{8/12}{4/8}$$

We also know,

$$\frac{AC/AD}{BC/BD} = \frac{8/12}{4/8} = \frac{(3+BC)/(5+BC)}{BC/(2+BC)}$$

Simplifying the above expression to solve for BC, we obtain:

$$\frac{8/12}{4/8} = \frac{(3+BC)/(5+BC)}{BC/(2+BC)}$$
$$\frac{4}{3} = \frac{(3+BC)(2+BC)}{BC(5+BC)}$$

Solving the above quadratic equation, we obtain $BC = 2.42$ units.

3.2 Distance DV

Using a similar technique as above, we obtain:

$$\frac{B_wD_w/B_wV_w}{C_wD_w/C_wV_w} = \frac{B_wD_w}{C_wD_w} = \frac{8}{4}$$

We also know,

$$\frac{BD/BV}{CD/CV} = 2 = \frac{(2BC)/(2BC+DV)}{2/(2+DV)}$$

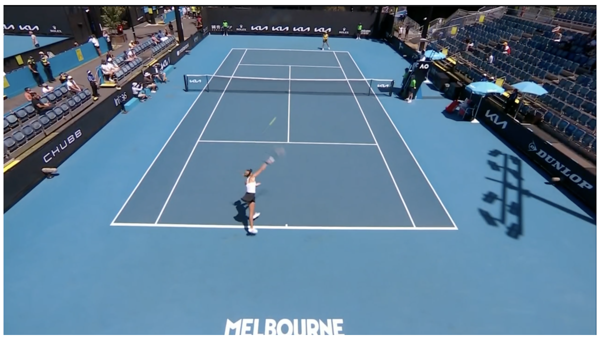
Simplifying the above expression to solve for BC, we obtain:

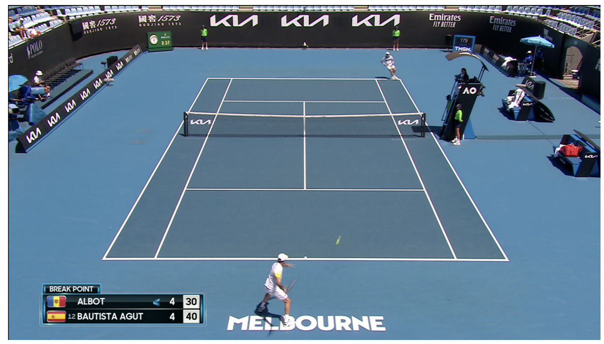
$$2 = \frac{(2BC)/(2BC+DV)}{2/(2+DV)}$$
$$2 = \frac{4.424 \cdot (2+DV)}{2 \cdot (4.424+DV)}$$

Solving the above equation for DV, we obtain $DV = 20.87$ units.

4 Different perspectives in a tennis match

4.1 Why is the perspective different

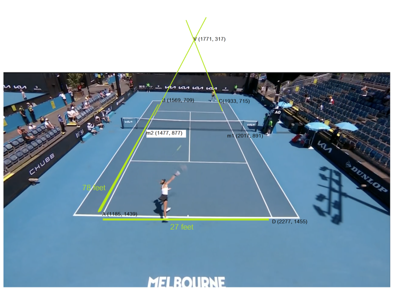


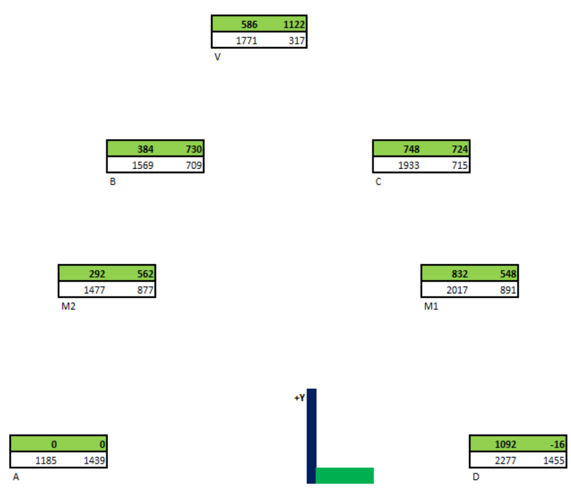


The perspective of the two images above are different because the camera for the left image is located much closer to the court than the right image due to the construction of the stadium.

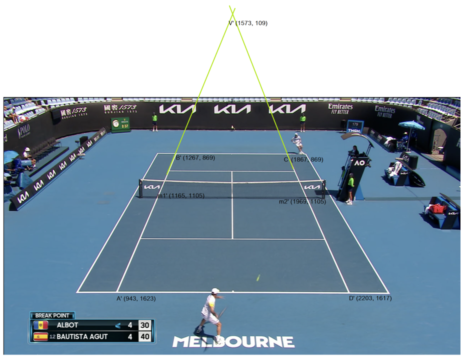
Consequently, the image on the right is more zoomed-in to focus-in on the court whereas the image on the left uses a larger Field of View lens to get the entire court into focus from the shorter distance.

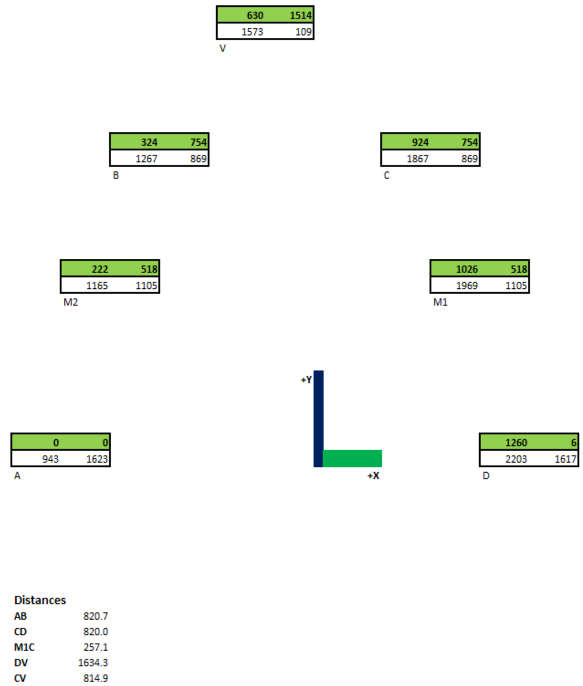
4.2 Find vanishing points using cross-ratios





Distances	
AB	824.8
CD	816.0
M1C	195.0
DV	1245.4
CV	429.7





Known quantities:

Units	**Segment**	**Value**
Image	AD	$\sqrt{1092^2 + 16^2} = 1092$ pixels
Image	BC	$\sqrt{364^2 + 6^2} = 364$ pixels
Image	AB	$\sqrt{730^2 + 384^2} = 825$ pixels
Image	CD	$\sqrt{344^2 + 740^2} = 816$ pixels
Image	M_1M_2	$\sqrt{540^2 + 14^2} = 540$ pixels
Image	$A'D'$	$\sqrt{1260^2 + 6^2} = 1260$ pixels
Image	$B'C'$	$\sqrt{600^2 + 0^2} = 600$ pixels
Image	$A'B'$	$\sqrt{324^2 + 754^2} = 821$ pixels
Image	$C'D'$	$\sqrt{336^2 + 748^2} = 820$ pixels
Image	$M_1'M_2'$	$\sqrt{804^2 + 0^2} = 804$ pixels
World	$AD, A'D'$	27 feet
World	$BC, B'C'$	27 feet
World	$AB, A'B'$	78 feet
World	$CD, C'D'$	78 feet

Vanishing Point for Image 1

$$
\begin{aligned}
\frac{DC/DV}{M_1C/M_1V} &= \frac{C_wD_w/D_wV_w}{M_{1w}C_w/M_{1w}V_w} \\
\frac{816/DV}{195/M_1V} &= 2 \cdot M_{1w}V_w/D_wV_w \\
\frac{816}{195} \cdot \frac{M_1V}{DV} &= 2 \cdot M_{1w}V_w/D_wV_w \\
\frac{195+CV}{816+CV} &= \frac{195}{816} \cdot 2 \cdot 1 \\
(195+CV) \cdot 816 &= 390 \cdot (816+CV) \\
CV \cdot (816-390) &= 816 \cdot (390-195) \\
CV &= \frac{816 \cdot 195}{426} \\
CV &= 374
\end{aligned}
$$

Given the slope of the line and the distance CV, we can compute V as follows:

$$
m = \frac{548-724}{832-748} = \frac{724-V_y}{748-V_x} \tag{1}
$$

$$
374 = \sqrt{(724-V_y)^2 + (748-V_x)^2} \tag{2}
$$

Solving the above system of equations using Mathematica, we obtain the possible vanishing points as:

$$
\begin{aligned}
V_x &\to 909.093, V_y \to 386.472 \\
V_x &\to 586.907, V_y \to 1061.53
\end{aligned}
$$

Double-checking the answer using cross-products:
Image 1 Vertical Vanishing Point

$$
L_1 = \begin{bmatrix} 748 \\ 724 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1092 \\ -16 \\ 1 \end{bmatrix} = \begin{bmatrix} 724+16 \\ 1092-748 \\ -16*748-724*1092 \end{bmatrix} = \begin{bmatrix} 740 \\ 344 \\ -802576 \end{bmatrix}
$$

$$
L_2 = \begin{bmatrix} 384 \\ 730 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 730 \\ -384 \\ 0 \end{bmatrix}
$$

Now, computing $L_1 \times L_2$ to obtain the vanishing point:

$$\begin{bmatrix} 740 \\ 344 \\ -802576 \end{bmatrix} \times \begin{bmatrix} 730 \\ -384 \\ 0 \end{bmatrix} = \begin{bmatrix} 576 \\ 1095 \\ 1 \end{bmatrix}$$

Vanishing Point for Image 2

$$\begin{aligned} \frac{D'C'/D'V'}{M_1{}'C'/M_1{}'V'} &= \frac{C_w{}'D_w{}'/D_w{}'V_w{}'}{M_{1w}{}'C_w{}'/M_{1w}{}'V_w{}'} \\ \frac{820/DV}{257/M_1V} &= 2 \cdot M_{1w}V_w/D_wV_w \\ \frac{820}{257} \cdot \frac{M_1V}{DV} &= 2 \cdot M_{1w}V_w/D_wV_w \\ \frac{257+CV}{820+CV} &= \frac{257}{820} \cdot 2 \cdot 1 \\ (257+CV) \cdot 820 &= 514 \cdot (820+CV) \\ CV \cdot (820-514) &= 820 \cdot (514-257) \\ CV &= \frac{820 \cdot 257}{306} \\ CV &= 689 \end{aligned}$$

Given the slope of the line and the distance CV, we can compute V as follows:

$$m = \frac{518-754}{1026-924} = \frac{754-V_y}{924-V_x} \tag{3}$$

$$689 = \sqrt{(754-V_y)^2 + (924-V_x)^2} \tag{4}$$

Solving the above system of equations using Mathematica, we obtain the possible vanishing points as:

$$\begin{aligned} V_x \to 1197.35, V_y \to 121.544 \\ V_x \to 650.65, V_y \to 1386.46 \end{aligned}$$

4.3 Find vanishing points for court baselines

Image 1 Horizontal Vanishing Point

$$L_1 = \begin{bmatrix} 384 \\ 730 \\ 1 \end{bmatrix} \times \begin{bmatrix} 748 \\ 724 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 364 \\ -268024 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1092 \\ -16 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 1092 \\ 0 \end{bmatrix}$$

Now, computing $L_1 \times L_2$ to obtain the vanishing point:

$$\begin{bmatrix} 6 \\ 364 \\ -268024 \end{bmatrix} \times \begin{bmatrix} 16 \\ 1092 \\ 0 \end{bmatrix} = \begin{bmatrix} 402036 \\ -5891 \\ 1 \end{bmatrix}$$

Image 2 Horizontal Vanishing Point

$$L_1 = \begin{bmatrix} 324 \\ 754 \\ 1 \end{bmatrix} \times \begin{bmatrix} 924 \\ 754 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 600 \\ -452400 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1260 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1260 \\ 0 \end{bmatrix}$$

Now, computing $L_1 \times L_2$ to obtain the vanishing point:

$$\begin{bmatrix} 0 \\ 600 \\ -452400 \end{bmatrix} \times \begin{bmatrix} -6 \\ 1260 \\ 0 \end{bmatrix} = \begin{bmatrix} 158340 \\ 754 \\ 1 \end{bmatrix}$$

Finding the homography that maps tennis court to image plane

Image 1

$$W' \sim P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies \alpha W' = P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = P_3$$

$$X' \sim P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \implies \beta X' = P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = P_2$$

$$Y' \sim P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies \gamma Y' = P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = P_1$$

$$Z' \sim P \begin{bmatrix} 27 \\ 78 \\ 1 \end{bmatrix} \implies \delta Z' = P \begin{bmatrix} 27 \\ 78 \\ 1 \end{bmatrix} = 27P_1 + 78P_2 + P_3$$

Combining the above equations:

$$\delta \begin{bmatrix} 748 \\ 724 \\ 1 \end{bmatrix} = 27\gamma \begin{bmatrix} 402036 \\ -5891 \\ 1 \end{bmatrix} + 78\beta \begin{bmatrix} 576 \\ 1095 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can infer that $\alpha = 1$, and simplify the system of equations to:

$$\alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \delta \begin{bmatrix} 748 \\ 724 \\ 1 \end{bmatrix} - 78\beta \begin{bmatrix} 576 \\ 1095 \\ 1 \end{bmatrix} - 27\gamma \begin{bmatrix} 402036 \\ -5891 \\ 1 \end{bmatrix}$$

$$\alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \delta \begin{bmatrix} 748 \\ 724 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -44928 \\ -85410 \\ -78 \end{bmatrix} + \gamma \begin{bmatrix} -10854972 \\ 159057 \\ -27 \end{bmatrix}$$

$$\begin{bmatrix} \delta & \beta & \gamma \end{bmatrix} = \left(\begin{bmatrix} 748 & -44928 & -10854972 \\ 724 & -85410 & 159057 \\ 1 & -78 & -27 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving for the inverse of the coefficient matrix, we obtain:

$$\begin{bmatrix} \delta \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 3.0027 \\ 0.0256 \\ 0.0001 \end{bmatrix}$$

From this, we obtain the transformation P by multiplying α, β and γ into the above equation:

$$\begin{bmatrix} 748 \\ 724 \\ 1 \end{bmatrix} = \gamma \begin{bmatrix} 402036 \\ -5891 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 576 \\ 1095 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 27 \\ 78 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13.4944 & 4.9186 & 0 \\ -0.1977 & 9.3505 & 0 \\ 0 & 0.0085 & 0.3300 \end{bmatrix} \times \begin{bmatrix} 27 \\ 78 \\ 1 \end{bmatrix}$$

Image 2

$$W' \sim P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies \alpha W' = P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = P_3$$
$$X' \sim P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \implies \beta X' = P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = P_2$$
$$Y' \sim P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies \gamma Y' = P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = P_1$$
$$Z' \sim P \begin{bmatrix} 27 \\ 78 \\ 1 \end{bmatrix} \implies \delta Z' = P \begin{bmatrix} 27 \\ 78 \\ 1 \end{bmatrix} = 27P_1 + 78P_2 + P_3$$

Combining the above equations:

$$\delta \begin{bmatrix} 924 \\ 754 \\ 1 \end{bmatrix} = 27\gamma \begin{bmatrix} 158340 \\ 754 \\ 1 \end{bmatrix} + 78\beta \begin{bmatrix} 651 \\ 1386 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can infer that $\alpha = 1$, and simplify the system of equations to:

$$\alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \delta \begin{bmatrix} 924 \\ 754 \\ 1 \end{bmatrix} - 78\beta \begin{bmatrix} 651 \\ 1386 \\ 1 \end{bmatrix} - 27\gamma \begin{bmatrix} 158340 \\ 754 \\ 1 \end{bmatrix}$$

$$\alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \delta \begin{bmatrix} 924 \\ 754 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -50778 \\ -108108 \\ -78 \end{bmatrix} + \gamma \begin{bmatrix} -4275180 \\ -20358 \\ -27 \end{bmatrix}$$

$$\begin{bmatrix} \delta & \beta & \gamma \end{bmatrix} = \left(\begin{bmatrix} 924 & -50778 & -4275180 \\ 754 & -108108 & -20358 \\ 1 & -78 & -27 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving for the inverse of the coefficient matrix, we obtain:

$$\begin{bmatrix} \delta \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 2.2010 \\ 0.0153 \\ 0.0003 \end{bmatrix}$$

From this, we obtain the transformation P by multiplying α, β and γ into the above equation:

$$\begin{bmatrix} 924 \\ 754 \\ 1 \end{bmatrix} = \gamma \begin{bmatrix} 158340 \\ 754 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 651 \\ 1386 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 27 \\ 78 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21.1528 & 4.5240 & 0 \\ 0.1007 & 9.6318 & 0 \\ 0.0001 & 0.0069 & 0.4543 \end{bmatrix} \times \begin{bmatrix} 27 \\ 78 \\ 1 \end{bmatrix}$$

4.4 Compute the focal length of each image

Horizon of Image 1

We can compute the horizon by taking the cross-product of the horizontal and Vertical vanishing points of this image, as computed in the previous parts.

$$h = \begin{bmatrix} 402036 \\ -5891 \\ 1 \end{bmatrix} \times \begin{bmatrix} 576 \\ 1095 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6986 \\ -401460 \\ 443622636 \end{bmatrix}$$

$$\implies V_A = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -6986 \\ -401460 \\ 443622636 \end{bmatrix}$$

$$= \begin{bmatrix} 401460 \\ -6986 \\ 0 \end{bmatrix}$$

From the previous part, we also know that the homography H is as follows:

$$\begin{bmatrix} 13.4944 & 4.9186 & 0 \\ -0.1977 & 9.3505 & 0 \\ 0 & 0.0085 & 0.3300 \end{bmatrix}$$

Horizon of Image 2

We can compute the horizon by taking the cross-product of the horizontal and Vertical vanishing points of this image, as computed in the previous parts.

$$h = \begin{bmatrix} 158340 \\ 754 \\ 1 \end{bmatrix} \times \begin{bmatrix} 651 \\ 1386 \\ 1 \end{bmatrix} = \begin{bmatrix} -632 \\ -157689 \\ 218968386 \end{bmatrix}$$

$$\implies V_A = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -632 \\ -157689 \\ 218968386 \end{bmatrix} = \begin{bmatrix} 157689 \\ -632 \\ 0 \end{bmatrix}$$

From the previous part, we also know that the homography H is as follows:

$$\begin{bmatrix} 21.1528 & 4.5240 & 0 \\ 0.1007 & 9.6318 & 0 \\ 0.0001 & 0.0069 & 0.4543 \end{bmatrix}$$

4.5 Compute the vanishing points using intersection of parallel lines

5 Correct perspective of reflection painting

I believe this painting is not perspectively correct, since the lines connecting the points in the image, such as the stool or the girls lower body does not converge to the same vanishing point.

Hence, according to the linked article, this implies the image is not correct.