CIS580 Problem Set 2

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Contents

1 Line that passes through the points

1.1 $[0, a, 0], [0, 0, a]$

$$l = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} a^2 - 0 \\ 0 - 0 \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$$

1.2 $[a, a, 1], [a, a, 2]$

$$l = \begin{bmatrix} a \\ a \\ 1 \end{bmatrix} \times \begin{bmatrix} a \\ a \\ 2 \end{bmatrix} = \begin{bmatrix} 2a - a \\ a - 2a \\ a^2 - a^2 \end{bmatrix} = \begin{bmatrix} a \\ -a \\ 0 \end{bmatrix}$$

1.3 $[a, b, 0], [c, d, 0]$

$$l = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \times \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} = \begin{bmatrix} 0 - 0 \\ 0 - 0 \\ ad - bc \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ad - bc \end{bmatrix}$$

2 Point of Intersection $\in \mathbb{P}^2$

2.1 $x - y + w = 0, w = 0$

$$P = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 - 0 \\ 0 - 1 \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

2.2 $3x - w = 0, 4y - w = 0$

$$P = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 + 4 \\ 0 + 3 \\ 12 - 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix}$$

2.3 $x - y + 5w = 0, x - y + 2w$

$$P = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 + 5 \\ 5 - 2 \\ -1 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

3 Find λ such that three lines intersect

Write the lines in a matrix system of equations:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Given that $w = 0$, we reduce the system of equations to the following:

$$\begin{bmatrix} 1 & \lambda \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Set the determinant equal to zero:

$$\begin{aligned} 1 - \lambda &= 0 \\ (1-\lambda)(1+\lambda) &= 0 \\ \lambda &= -1, 1 \end{aligned}$$

We choose $\lambda = -1$ since in the other case, we do not have three distinct lines. Using this, we compute the point of intersection as follows:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0+1 \\ 1+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

4 Find projective transformation A

We wish to preserve:

$$\begin{aligned} P_1 &= (1,0,0) \\ P_2 &= (0,1,0) \\ O &= (0,0,1) \end{aligned}$$

We wish to map: $P_3 = (1,1,1) \to {P_3}' = (3,2,1)$

$$({P_1}', {P_2}', {P_3}', O) = M(P_1, P_2, P_3, O)$$

$$\begin{bmatrix} \lambda_1 & 0 & 3\lambda_3 & 0 \\ 0 & \lambda_2 & 2\lambda_3 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \end{bmatrix} = M \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Simplify the matrix by dividing by λ_4 on both sides:

$$\begin{bmatrix} a & 0 & 3c & 0 \\ 0 & b & 2c & 0 \\ 0 & 0 & c & 1 \end{bmatrix} = M' \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Compute the matrix product on the RHS of the above equation, and solve for M' as follows:

$$M = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \because c = 1$$

5 Find projective transformation P

$$W' \sim P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies \alpha W' = P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$X' \sim P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \implies \beta X' = P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Y' \sim P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies \gamma Y' = P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Z' \sim P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \implies \delta Z' = P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Combining the above equations:

$$\delta Z' = \alpha W' + \beta X' + \gamma Y'$$

$$\delta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

We can infer that $\delta = 1$, and simplify the system of equations to:

$$\begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} = \left(\begin{bmatrix} 3 & 0 & -3 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving for the inverse of the coefficient matrix, we obtain:

$$\begin{bmatrix} \frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \end{bmatrix} \implies \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$$

From this, we obtain the transformation P by multiplying α, β and γ into the above equation:

$$\begin{bmatrix} \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

6 Determine the projective transformation A

Following the same method as the previous question, we set-up the system of equations as follows:

$$\delta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} -a \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ b \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can infer that $\delta = 1$, and simplify the system of equations to:

$$\begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} = \left(\begin{bmatrix} -a & 0 & 0 \\ 0 & b & 0 \\ 1 & 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving for the inverse of the coefficient matrix, we obtain:

$$\begin{bmatrix} -\frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ \frac{1}{a} & -\frac{1}{b} & 1 \end{bmatrix} \implies \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -\frac{1}{a} \\ \frac{1}{b} \\ \frac{1}{a} - \frac{1}{b} + 1 \end{bmatrix}$$

In the above equation, we know $\gamma = 1$ because the origin is preserved by the transformation. From this result, we obtain the transformation A^{-1} by multiplying α, β and γ into the above equation:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{a} & \frac{1}{b} & 1 \end{bmatrix} \implies A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{a} & -\frac{1}{b} & 1 \end{bmatrix}$$