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Geometry

SECOND EDITION

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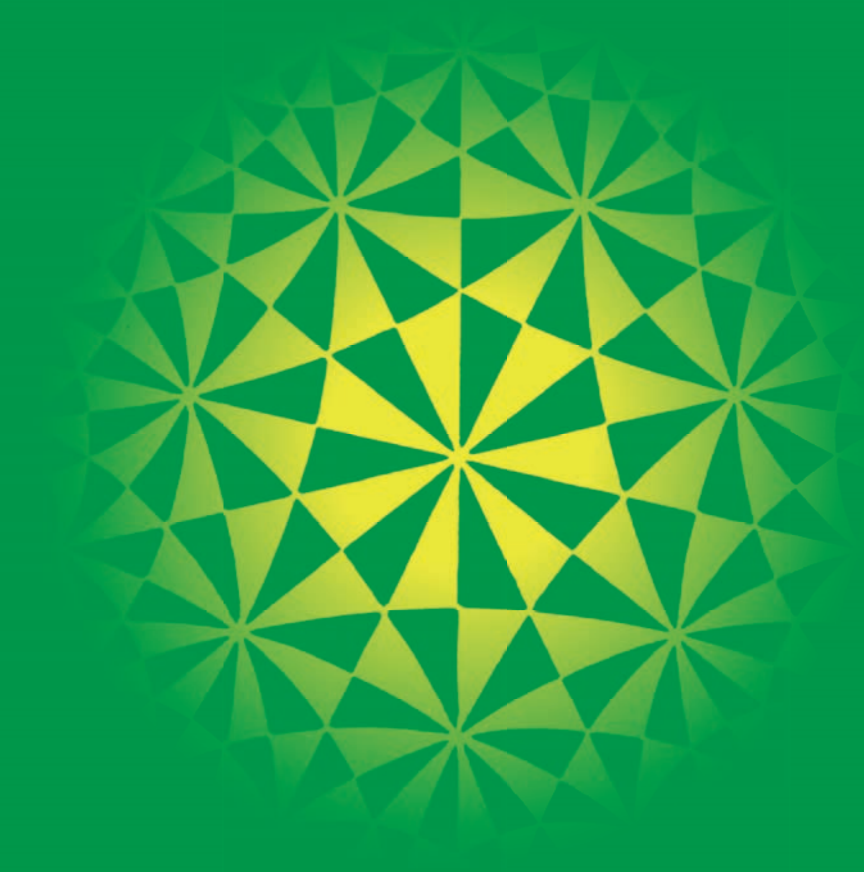

Geometry

SECOND EDITION

DAVID A. BRANNAN
MATTHEW F. ESPLEN
JEREMY J. GRAY

The Open University

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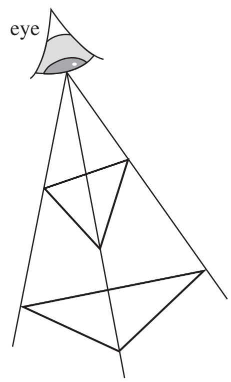
3 Projective Geometry: Lines

Geometry is one branch of mathematics that has an obvious relevance to the ‘real world’. Earlier, we studied some results in Euclidean geometry and we described the group of Euclidean transformations, the isometries. We saw that the Euclidean transformations preserve distances and angles, and have a definite physical significance.

Chapters 1 and 2.

In this chapter we study *projective geometry,* a very different type of geometry, that has important but less obvious applications. It was discovered through artists’ attempts over many centuries to paint realistic-looking pictures of scenes composed of objects situated at differing distances from the eye. How can three-dimensional scenes be represented on a two-dimensional canvas? Projective geometry explains how an eye perceives ‘the real world’, and so explains how artists can achieve realism in their work.

For example, in Computer Graphics and in Art.



In Section 3.1, we look at the development of perspective in Art and explain the concept of a *perspectivity.* We describe Desargues’ Theorem, which concerns a curious property of two triangles whose vertices are in perspective from a single point, and so explain that perspective can play a key role in the statement and the proof of theorems in mathematics.

In Section 3.2, we define the term *projective point* (or *Point*) and call the set of all such Points the *projective plane,* which we denote by $\mathbb{RP}^2$. We also define a *projective line* (or *Line*). To enable us to tackle problems in projective geometry algebraically, we introduce *homogeneous coordinates* to specify the Points in $\mathbb{RP}^2$.

In Section 3.3, we define the projective transformations of $\mathbb{RP}^2$ and use them to define projective geometry. We also prove the *Fundamental Theorem of Projective Geometry,* which states that given two sets of four Points there is a unique projective transformation of $\mathbb{RP}^2$ that maps the Points in one set to the corresponding Points in the other set. This crucial result enables us to apply a preliminary transformation to many geometric problems, thereby simplifying their solution by reducing the arithmetic involved. It turns out that there is a close connection between the idea of perspective in $\mathbb{R}^3$ and projective transformations.

We also require that no three Points in either set lie on a Line.

In Section 3.4, we use the Fundamental Theorem of Projective Geometry to prove several results, including Desargues’ Theorem. We also introduce the concept of *duality,* which involves a remarkable relationship between Points and Lines.

Finally, in Section 3.5, we note that the ideas of distance and ratio of distances along a line have no immediate analogues in $\mathbb{RP}^2$; nevertheless, we are able to define a related quantity called the *cross-ratio* of four collinear Points in $\mathbb{RP}^2$. This quantity is very useful in proving various mathematical results, and it has 'real life' applications – such as in aerial photography.

3.1 Perspective

3.1.1 Perspective in Art

The first 'pictures' were probably Cave Art wall paintings: for example, depictions of animals and hunters. Up to the Middle Ages, most pictures were drawn on walls, floors or ceilings of buildings and were intended to convey messages rather than to be accurate illustrations of what an eye might see. For example, Christian religious art portrayed Christ and the Saints, the Bayeux tapestry outlined events such as the Norman Conquest and the Battle of Hastings, and so on.

(Clockwise from left) Hunters below antelopes. Bambata cave, Zimbabwe © M. Jelliffe; Tomb of Rekhmare, Thebes. 1500 BC © Ronald Sheridan; Bayeux Tapestry: The death of Harold. These prints are reproduced by kind permission of A.A. & A. Ancient Art and Architecture Collection.

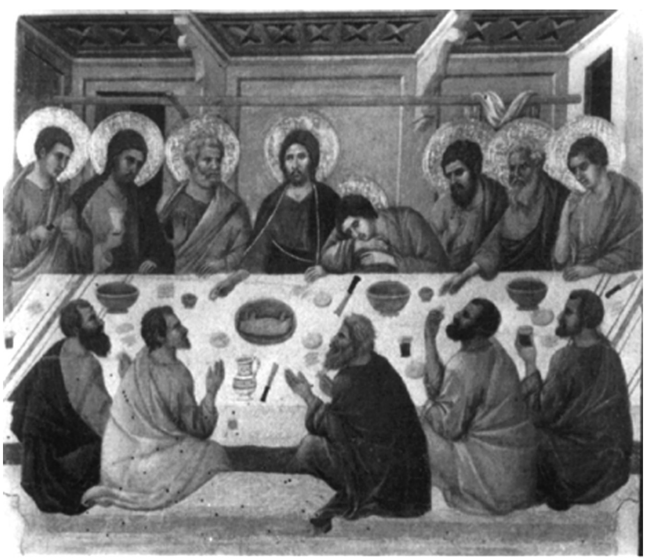
To the modern eye, the people and animals in these pictures appear to be rather stylized, and the whole scene seems very two-dimensional. The events illustrated do not appear to be properly integrated into the background, even if this is included.

Towards the end of the 13th century, early Renaissance artists began to attempt to portray 'real' situations in a realistic way. For example, people at the back of a group would be drawn higher up than those at the front – a technique known as *terraced perspective.*

Simone Martini 'Maestà' Palazzo Pubblico, Sala del Mappamodo, Siena (su concessione del commune di Siena). Foto LENSINI Siena.

As artists struggled to find better techniques to improve the realism of their work, the idea of *vertical perspective* was developed by the Italian school of artists (including Duccio (1255–1318) and Giotto (1266–1337)). To create an impression of depth in a scene, the artist would represent pairs of parallel lines that are symmetrically placed either side of the scene by lines that meet on the centre line of the picture. The method is not totally realistic, since objects do not appear to recede into the distance in the way that might be expected. The problem of depicting 'distant objects looking smaller', with a properly integrated foreground and background, was tackled by many artists, including notably Ambrogio Lorenzetti (c. 1290–1348).

Giotto is sometimes called the 'Father of Modern Painting'.

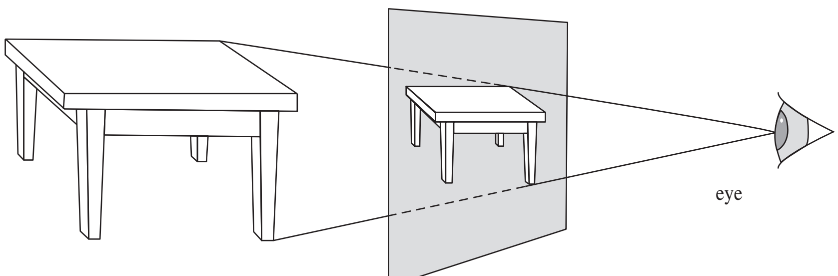


'Last Supper' painted by Duccio; Opera del Duomo, Siena. Foto LENSINI Siena.

The modern system of *focused perspective* was discovered around 1425 by the sculptor and architect Brunelleschi (1377–1446), developed by the painter and architect Leone Battista Alberti (1404–1472), and finally perfected by Leonardo da Vinci (1452–1519).

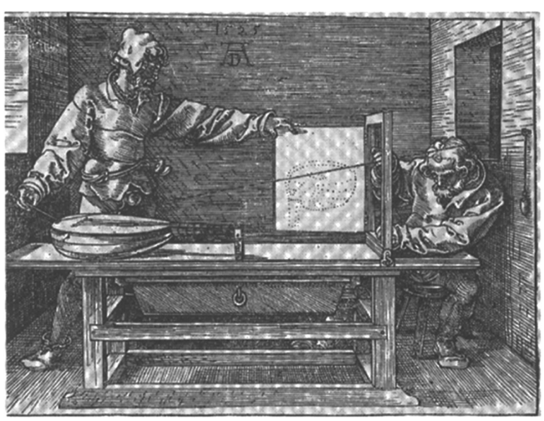
Alberti wrote that the first necessity for a painter is 'to know geometry'.

These artists realized that what the eye actually 'sees' of a scene are the various rays of light travelling from each point in the scene to the eye. An effective way of deciding how to depict a three-dimensional scene on a two-dimensional canvas so as to create a realistic impression is therefore as follows. Imagine a glass screen placed between the eye and the three-dimensional scene. Each line joining the eye to a point of the scene pierces the glass screen at some point. The set of all such points forms an image on the screen known as a *cross-section.* Since the eye cannot distinguish between light rays coming from the points of the actual scene and light rays coming from the corresponding points of the cross-section (since these are in exactly the same direction), the cross-section produces the same impression as the original scene. In other words, the cross-section gives a realistic two-dimensional representation of the three-dimensional scene.



The German artist Albrecht Dürer (1471–1528) introduced the term *perspective* (from the Latin verb meaning 'to see through') to describe this technique, and illustrated it by a series of well-known woodcuts in his book *Underweysung der Messung mit dem Zyrkel und Rychtsscheyed* (1525). The Dürer woodcut below shows an artist peering through a grid on a glass screen to study perspective and the effect of foreshortening.

In English: *Instruction on measuring with compass and straight edge.*
We discuss foreshortening in Subsection 3.1.2.



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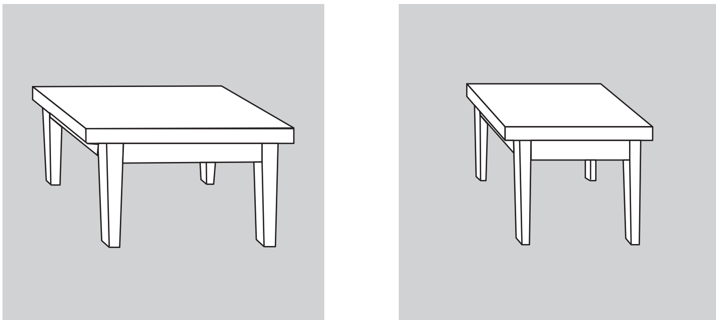
Of course, the picture displayed on the screen is just one representation of the scene. If the screen is placed closer to, or further away from, the eye, the size of the cross-section changes. Also, the screen may be placed at a different angle for a given position of the eye, or the eye itself may be moved to a different position. In each case, a different cross-section is obtained, though they are all related to each other.

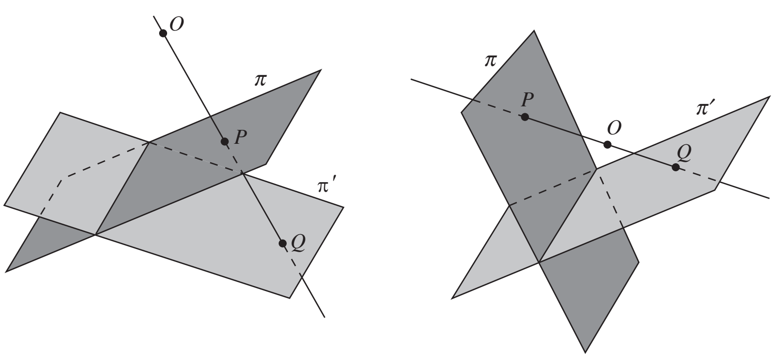


3.1.2 Mathematical Perspective

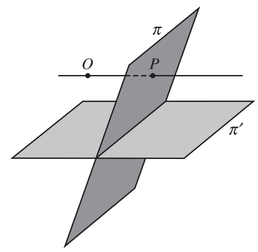
To help us understand the relationship between different representations of a scene, we now look at perspective from a mathematical point of view. In place of an eye and light rays travelling to it, we use the family of all lines in $\mathbb{R}^3$ through a given point. For convenience, this point will often be the origin O. The glass screen is replaced by a plane in $\mathbb{R}^3$ that does not pass through the origin.

In order to compare the cross-sections that appear on different screens, we consider two planes π and π' that do not pass through O. A point P in π and a point Q in π' are said to be *in perspective from O* if there is a straight line through O, P and Q. A *perspectivity from* π *to* π' centred at O is a function that maps a point P of π to a point Q of π' whenever P and Q are in perspective from O. Notice that the planes π and π' may lie on the same side of O as shown on the left below, or they may lie on opposite sides of O as shown on the right.

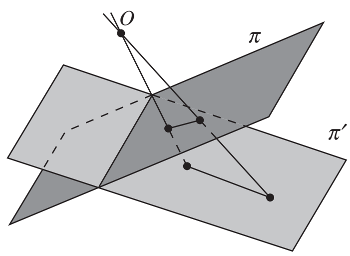
In terms of O representing an eye, the figure on the right corresponds to the observer having the ability to look simultaneously both forwards and backwards!

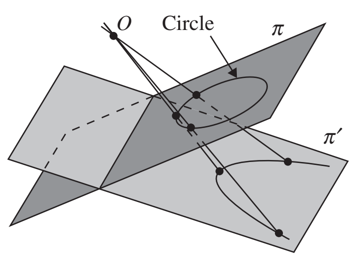


One complication with the above definition of a perspectivity is that the domain of the perspectivity is not necessarily the whole of π. Indeed, if P is any point of π such that OP is parallel to π', as shown in the margin, then P cannot have an image in π', and cannot therefore belong to the domain of the perspectivity. From a mathematical point of view, this need to exclude such exceptional points from the domain of a perspectivity turns out to be rather a nuisance. In Subsection 3.2.3 we shall therefore reformulate the definition of a perspectivity in such a way that these exceptional points can be included in the domain.



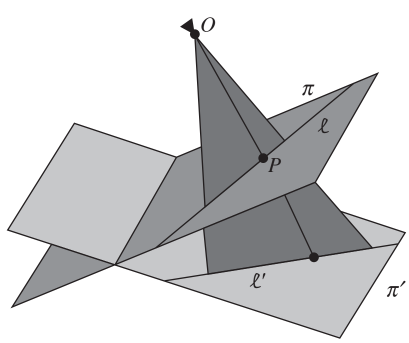
Even with only the preliminary definition of perspectivity given above, it is clear that some features of figures are preserved under a perspectivity, while others are not. For example, the figure on the left below illustrates a particular perspectivity in which a line segment in one plane maps onto a line segment in another plane. This suggests that collinearity is preserved by a perspectivity. On the other hand, the figure on the right illustrates a perspectivity in which a circle in one plane appears to map to a parabolic shape in another plane, which suggests that 'circularity' is not preserved.



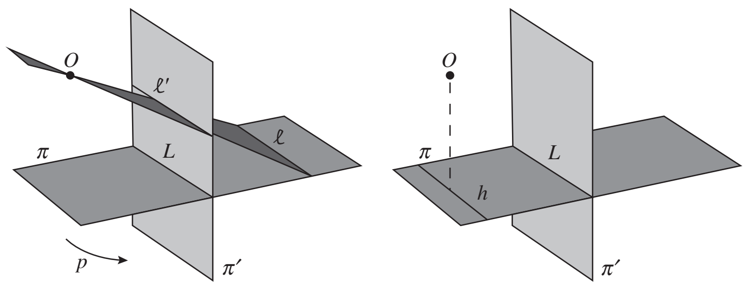


One of our main tasks is to study the images of standard configurations such as lines and conics under perspectivities. This chapter deals with lines; the next chapter deals with conics.

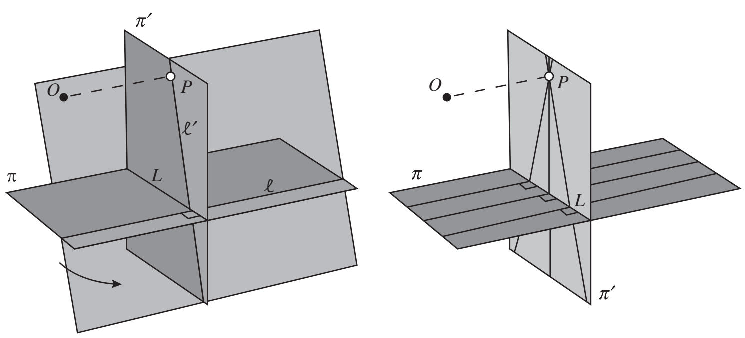
Consider a perspectivity with centre O that maps points in a plane π to points in a plane π'. A convenient way to visualize the image of a line ℓ under the perspectivity is to consider an arbitrary point P on ℓ. As P moves along ℓ, the line OP sweeps out a plane. The line ℓ' where this plane intersects π' is the image of ℓ.



To be specific, consider the perspectivity p with centre O that maps points in a horizontal plane π to points in a vertical plane π', and let L be the line where π and π' intersect. Under p, every line ℓ in π that is parallel to L maps to a horizontal line ℓ' in π'. In particular, L maps to itself. The only exception is the line h that passes through the foot of the perpendicular from O to π. This line does not have an image in π' since the lines joining points of h to O are parallel to π'.



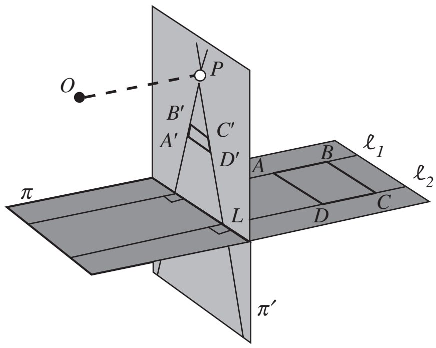
Next, consider the image under the same perspectivity p of a line ℓ in π that is perpendicular to L. To do this, let P denote the foot of the perpendicular from O to the plane π'. Although P is not the image of any point of π, the plane through O and ℓ meets π' in some line ℓ' that passes through P. It follows that the image of ℓ under p is some line ℓ' through P, with the point P itself omitted.



The above argument works for *any* line in π that is perpendicular to L. All such lines are mapped by the perspectivity p to lines in π' that pass through P, and that omit the point P itself.

We may combine our observations concerning lines in π that are parallel to L or perpendicular to L in the following way. Let *ABCD* be a rectangle in π on the opposite side of L from O, with sides *AB* and *CD* that lie on lines ℓ_1 and ℓ_2, perpendicular to L. Then *AD* and *BC* both map onto horizontal lines in π' between L and P. As the side BC recedes from L, its image $B'C'$ under the perspectivity p moves further up π' towards P, becoming shorter as it moves.

Artists describe this shortening of the image on π' of lines of equal length in π as *foreshortening*.

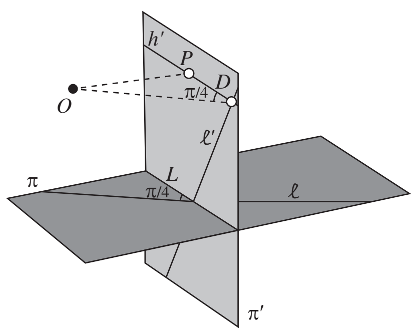


To an observer whose eye is located at O, the lines ℓ_1 and ℓ_2 *appear* to meet 'at infinity', and this corresponds to their images under p *appearing* to meet at P. The point P is called the *principal vanishing point* of the perspectivity p because the images in π' of all lines in π perpendicular to L appear to vanish there.

You can think of ℓ_1 and ℓ_2 as a pair of railroad lines disappearing into the distance.

In fact, a perspectivity has many vanishing points. For instance, let ℓ be any line in π that intersects L at an angle of $\pi/4$. Now let h' be the horizontal line in π' through P, and let D be the point on h' such that OD is parallel to ℓ. Then the plane through O and ℓ meets π' in some line ℓ' that passes through D. It follows that the image of ℓ under p is a line through D, with the point D itself omitted.

Here the symbol π is being used in two different ways: as a label for the embedding plane, and as an angle.



The point D is called a *diagonal vanishing point* of the perspectivity. All lines in the plane π that are parallel to the given line ℓ have images in π' that are lines through D, with the point D itself omitted.

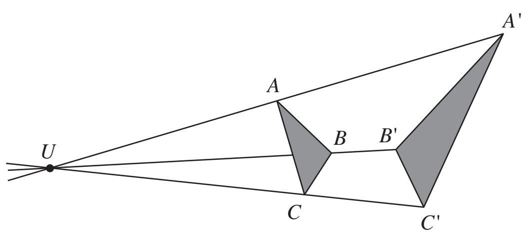
That is, the images appear to *vanish* at D.

In the same way, each point of the horizontal line h' in π' through P is a *vanishing point* for the images of all lines in π in some direction; hence the line h'

is called the *vanishing line.* It corresponds to the 'horizon' in the plane – in other words, to the points 'at infinity' towards which an observer's eye is pointing when looking in a horizontal direction.

3.1.3 Desargues' Theorem

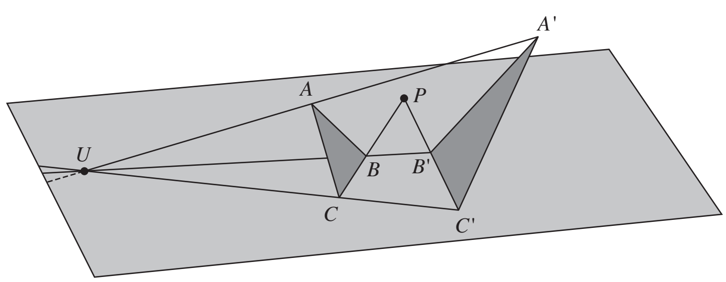
The idea that information in three dimensions can be related to information in two dimensions, and vice versa, plays an important role in mathematics just as it does in Art. For example, consider the following three-dimensional figure that consists of two triangles ΔABC and $\Delta A'B'C'$ which are in perspective from a point U. For the moment we shall assume that no pair of corresponding sides BC and $B'C'$, CA and $C'A'$, and AB and $A'B'$, are parallel.



We shall show that this three-dimensional figure has the property that BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ meet at P, Q, R, respectively, where P, Q and R are collinear. This will enable us to formulate an equivalent two-dimensional result, known as *Desargues' Theorem.*

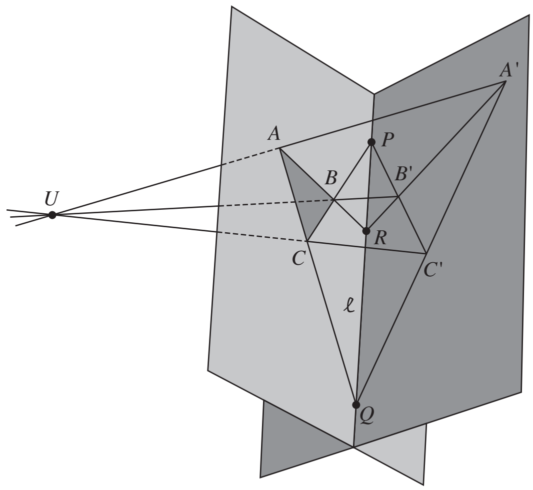
Girard Desargues (1593–1662) was a French engineer and architect.

To prove the three-dimensional result, observe that both BC and $B'C'$ lie in the plane that passes through the points U, B and C. Since BC and $B'C'$ are coplanar but not parallel, they must meet at some point P.



Similarly, the sides CA and $C'A'$ meet at some point Q, and the sides AB and $A'B'$ meet at some point R.

Since the points P, Q and R lie both on the plane which contains the triangle ΔABC and on the plane which contains the triangle $\Delta A'B'C'$, they must lie on the line ℓ where the two planes meet. It follows that P, Q and R are collinear.



To obtain the equivalent two-dimensional result, imagine that you are viewing the three-dimensional configuration through a transparent screen. Since this viewing process will not alter the collinearity of points or the coincidence of lines, we may reinterpret the three-dimensional result in terms of the image on the screen to obtain the following theorem.

We give a rigorous proof of Desargues' Theorem in Theorem 1, Subsection 3.4.1.

Theorem 1 Desargues' Theorem
Let ΔABC and $\Delta A'B'C'$ be triangles in $\mathbb{R}^2$ such that the lines AA', BB' and CC' meet at a point U. Let BC and $B'C'$ meet at P, CA and $C'A'$ meet at Q, and AB and $A'B'$ meet at R. Then P, Q and R are collinear.

Strictly speaking, we have not proved this theorem since it is not immediately obvious that ΔABC and $\Delta A'B'C'$ can be obtained as images of triangles in $\mathbb{R}^3$ which have corresponding sides that are not parallel. Nevertheless, the above argument does provide reasonably convincing evidence that the theorem is true.

One remarkable feature of the above argument is the way in which the geometry of the figure on the transparent screen is characterized by the rays of light that enter an eye. Thus a point on the screen corresponds to a single ray of light that enters the eye, a line on the screen corresponds to a plane of rays of light that enter the eye, and so on. The geometry of the figure can be investigated entirely in terms of these rays of light. The screen is needed only to interpret the result in terms of a two-dimensional figure.

In the rest of Chapters 3 and 4, we introduce a geometry known as *projective geometry* that enables us to work with figures on a plane (a screen) as if they correspond to rays of light that enter an eye in the way described above.

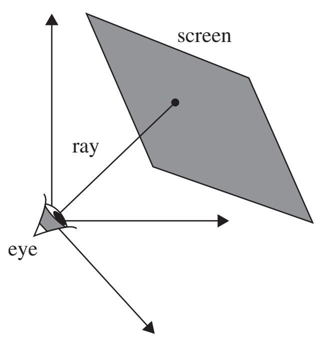
3.2 The Projective Plane $\mathbb{RP}^2$

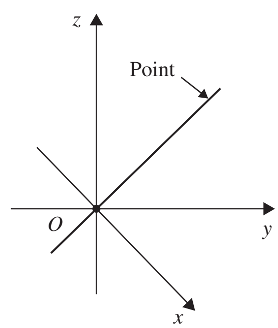
You have already met the Kleinian view that a geometry consists of a group of transformations acting on a space of points. In this section we begin our discussion of projective geometry by investigating its space of points. The group of transformations is discussed in Section 3.3.

Introductary remarks to Chapter 2

3.2.1 Projective Points

Imagine an eye situated at the origin of $\mathbb{R}^3$ looking at a fixed screen. As we mentioned in Subsection 3.1.1, each point of the screen corresponds to the ray of light that enters the eye from the point. This correspondence between points of the screen and rays of light through the origin is the clue that we need to define a space of points for our new geometry.



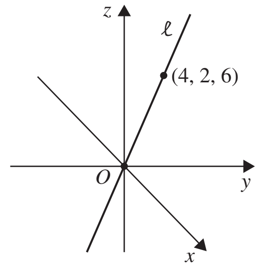


Rather than use the points of the screen directly, we use the rays of light that enable an eye to 'see' the points from the origin. We can express this idea mathematically by defining a *projective point* to be a Euclidean line in $\mathbb{R}^3$ that passes through the origin. In order to avoid confusion with Euclidean points of $\mathbb{R}^3$, we write Point with a capital P whenever we mean a projective point.

It is important that you use the capital letter P in 'Point'.

Definitions **A Point** (or **projective point)** is a line in $\mathbb{R}^3$ that passes through the origin of $\mathbb{R}^3$. The **real projective plane** $\mathbb{RP}^2$ is the set of all such Points.

In order to prove results in projective geometry algebraically, we need to have an algebraic notation that can be used to specify the Points of $\mathbb{RP}^2$. To do this, we use the fact that a line ℓ through the origin O in $\mathbb{R}^3$ is uniquely determined once we have specified a Euclidean point (other than O) that lies on ℓ. For example, there is a unique line ℓ in $\mathbb{R}^3$ through O and the point with Euclidean coordinates (4, 2, 6), so we can use these coordinates to specify a projective point. When doing this we write the coordinates in the form [4, 2, 6], with square brackets to indicate that the coordinates refer to a projective point.

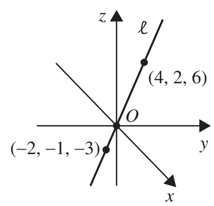


Definition The expression [*a, b, c*], in which the numbers *a, b, c* are not all zero, represents the Point P in $\mathbb{RP}^2$ which consists of the unique line in $\mathbb{R}^3$ that passes through (0, 0, 0) and (*a, b, c*). We refer to [*a, b, c*] as **homogeneous coordinates** of P. If (*a, b, c*) has position vector **v**, then we often denote P by [**v**] and we say that P can be **represented** by **v.**

Note that [0, 0, 0] is *not* defined.

Remark

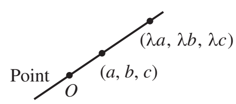
Often we abuse our notation slightly, by talking about 'the Point [*a, b, c*]' when strictly speaking we should say 'the Point with homogeneous coordinates [*a, b, c*]'.



Notice that the homogeneous coordinates of a Point are not unique. For example, the Point with homogeneous coordinates [4, 2, 6] consists of a line that passes through (0, 0, 0) and (4, 2, 6). But this line also passes through (−2, −1, −3), so [4, 2, 6] and [−2, −1, −3] both represent the same Point.

In general, if *(a, b, c)* is any point on a line through the origin, and λ is any real number, then $(\lambda a, \lambda b, \lambda c)$ also lies on the line. Moreover, if (a, b, c) is not at the origin and $\lambda \neq 0$, then $(\lambda a, \lambda b, \lambda c)$ is not at the origin either. It follows that [*a, b, c*] and $[\lambda a, \lambda b, \lambda c]$ both represent the same Point, for any $\lambda \neq 0$. We express this by writing

$$[a, b, c] = [\lambda a, \lambda b, \lambda c], \qquad \text{for any } \lambda \neq 0. \tag{1}$$



Conversely, if there is no non-zero real number λ such that

$$(a', b', c') = (\lambda a, \lambda b, \lambda c),$$

then (a, b, c) and (a', b', c') cannot lie on the same line through the origin, and so the homogeneous coordinates [*a, b, c*] and $[a', b', c']$ must represent different Points in $\mathbb{RP}^2$.

Example 1 Which of the following homogeneous coordinates represent the same Point in $\mathbb{RP}^2$ as [6, 3, 2]?

(a) [18, 9, 6] (b) [12, −6, 4] (c) $\left[1, \frac{1}{2}, \frac{1}{3}\right]$ (d) [1, 2, 3]

Solution

Throughout the solution we use equation (1):

$$[a, b, c] = [\lambda a, \lambda b, \lambda c],$$

for any $\lambda \neq 0$.

(a) This represents the same Point as [6, 3, 2], for if $\lambda = 3$, then

$$[18, 9, 6] = [6\lambda, 3\lambda, 2\lambda] = [6, 3, 2].$$

(b) This represents a Point different from [6, 3, 2], for there is no λ that satisfies the simultaneous equations

$$12 = 6\lambda, -6 = 3\lambda, 4 = 2\lambda.$$

(c) This represents the same Point as [6, 3, 2], for if $\lambda = \frac{1}{6}$, then

$$\left[1, \tfrac{1}{2}, \tfrac{1}{3}\right] = [6\lambda, 3\lambda, 2\lambda] = [6, 3, 2].$$

(d) This represents a Point different from [6, 3, 2], for there is no λ that satisfies the simultaneous equations

$$1 = 6\lambda, \quad 2 = 3\lambda, \quad 3 = 2\lambda. \qquad \square$$

Problem 1 Which of the following homogeneous coordinates represent the same Point in $\mathbb{RP}^2$ as [1, 2, 3]?

(a) [2, 4, 6] (b) $[1, 2, -3]$ (c) $[-1, -2, -3]$ (d) [11, 12, 13]

At first sight it may seem rather unsatisfactory that the coordinates of a Point are not unique. However, this ambiguity can often be turned to our advantage. For example, if a calculation yields a Point of $\mathbb{RP}^2$ with fractional homogeneous coordinates such as $\left[1, \frac{1}{2}, \frac{1}{3}\right]$, then the rest of the calculation may be simpler if we 'clear' the fractions and represent the Point by the integer homogeneous coordinates [6, 3, 2] instead.

Problem 2 For each of the following homogeneous coordinates, find integer homogeneous coordinates which represent the same Point.

(a) $\left[\frac{3}{4}, \frac{1}{2}, -\frac{1}{8}\right]$ (b) $\left[0, 4, \frac{2}{3}\right]$ (c) $\left[\frac{1}{6}, -\frac{1}{3}, -\frac{1}{2}\right]$

Given a collection of homogeneous coordinates, it is not always easy to spot those that represent the same Point. In such cases it is sometimes possible to rewrite the coordinates in a form that makes the comparison easier.

Example 2 Determine homogeneous coordinates of the form [*a, b,* 1] for the Points

$$[2, -1, 4], \qquad [4, 2, 8], \qquad [2\pi, -\pi, 4\pi],$$

$$[200, 100, 400], \qquad \left[-\frac{1}{2}, -\frac{1}{4}, -1\right], \qquad [6, -9, -12].$$

Hence decide which homogeneous coordinates represent the same Points.

Solution According to equation (1), a Point of $\mathbb{RP}^2$ is unchanged if its homogeneous coordinates are multiplied (or divided) by any non-zero real number. Since the third coordinate of each Point is non-zero, we may divide by this third coordinate to obtain homogeneous coordinates of the form $[a, b, 1]$ as follows:

For, dividing by a non-zero number λ is equivalent to multiplying by the non-zero number $1/\lambda$.

$$[2,-1,4]=\left[\tfrac{1}{2},-\tfrac{1}{4},1\right]; \qquad [4,2,8]=\left[\tfrac{1}{2},\tfrac{1}{4},1\right];$$
$$[2\pi,-\pi,4\pi]=\left[\tfrac{1}{2},-\tfrac{1}{4},1\right]; \qquad [200,100,400]=\left[\tfrac{1}{2},\tfrac{1}{4},1\right];$$
$$\left[-\tfrac{1}{2},-\tfrac{1}{4},-1\right]=\left[\tfrac{1}{2},\tfrac{1}{4},1\right]; \qquad [6,-9,-12]=\left[-\tfrac{1}{2},\tfrac{3}{4},1\right].$$

Since $[a,b,1]=[a',b',1]$ if and only if $a=a'$ and $b=b'$, it follows that:

$[2,-1,4]$ and $[2\pi,-\pi,4\pi]$ represent the same Point;

$[4,2,8]$, $[200,100,400]$ and $\left[-\frac{1}{2},-\frac{1}{4},-1\right]$ represent the same Point;

$[6,-9,-12]$ represents none of the other Points. □

Notice that the method used in Example 2 works only if the third coordinates of all the Points are non-zero. If this is not the case, then you may still be able to apply the technique using the first or second coordinates.

Problem 3 Determine homogeneous coordinates of the form $[1, b, c]$ for the Points

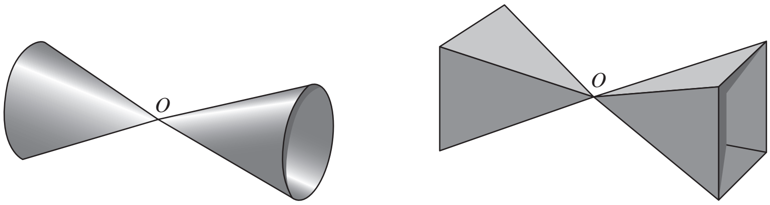
$$[2,3,-5], \qquad [-8,-12,20], \qquad \left[\sqrt{2},\sqrt{3},-\sqrt{5}\right],$$
$$[4,-6,10], \qquad [-20,-30,50], \qquad [74,148,0].$$

Hence decide which homogeneous coordinates represent the same Points.

Having defined projective points, we are now in a position to define a *projective figure.* Just as a figure in Euclidean geometry is defined to be a subset of $\mathbb{R}^2$, so figures in projective geometry are defined to be subsets of $\mathbb{RP}^2$.

Definition **A projective figure** is a subset of $\mathbb{RP}^2$.

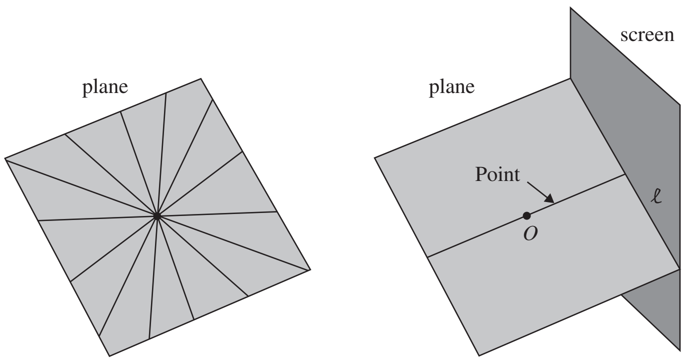
Projective figures are just sets of lines in $\mathbb{R}^3$ that pass through the origin. Thus a double cone with a vertex at O, and a double square pyramid with a vertex at O, are both examples of projective figures, for they can both be formed from sets of lines that pass through the origin of $\mathbb{R}^3$.



3.2.2 Projective Lines

A particularly simple type of projective figure is a plane through the origin. Such a plane is a projective figure because it can be formed from the set of all Points (lines through the origin of $\mathbb{R}^3$) that lie on the plane. Since all but one of these Points can be thought of as rays of light that come from a line on a screen, it seems reasonable to define any plane through the origin to be a *projective line.*

The exception is the ray of light parallel to the screen. We shall discuss the significance of this ray later, in Subsection 3.2.3.



Just as we use 'Point' to refer to a 'projective point', so we use 'Line' to refer to a 'projective line'. The use of a capital L avoids any confusion with lines in $\mathbb{R}^3$.

Definitions **A Line** (or **projective line**) in $\mathbb{RP}^2$ is a plane in $\mathbb{R}^3$ that passes through the origin. Points in $\mathbb{RP}^2$ are **collinear** if they lie on a Line.

Since a Line in $\mathbb{RP}^2$ is simply a plane in $\mathbb{R}^3$ that passes through the origin, it must consist of the set of Euclidean points (x, y, z) that satisfy an equation of the form

$$ax + by + cz = 0,$$

where a, b and c are real and not all zero. We can interpret this fact in terms of $\mathbb{RP}^2$ as follows.

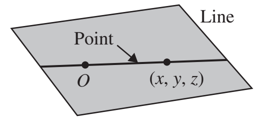
Theorem 1 The general equation of a Line in $\mathbb{RP}^2$ is

$$ax + by + cz = 0, \tag{2}$$

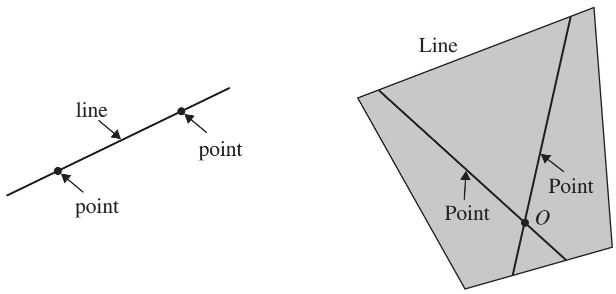
where *a, b, c* are real and not all zero.

Remark

1. The equation of a Line is not unique, for, if $\lambda \neq 0$, then $\lambda ax+\lambda by+\lambda cz = 0$ is also an equation for the Line. We can use this fact to 'clear fractions' from the coefficients just as we did for the homogeneous coordinates of a Point.
2. From the figure in the margin it is clear that a Point lies on a Line, or a Line passes through a Point, if and only if the Point has homogeneous coordinates $[x, y, z]$ which satisfy the equation of the Line. For example, [1, –1, 1] lies on the Line $3x + y - 2z = 0$, but [0, 1, 3] does not.



In Euclidean geometry there is a unique line that passes through any two distinct points, as illustrated on the left of the figure below. Similarly, in projective geometry two distinct Points (lines through the origin) lie on a unique Line (plane through the origin).



We express this observation in the form of a theorem, as follows.

Theorem 2 Collinearity Property of $\mathbb{RP}^2$
Any two distinct Points of $\mathbb{RP}^2$ lie on a unique Line.

It is sometimes possible to find an equation for the Line that passes through two distinct Points of $\mathbb{RP}^2$ simply by spotting an equation of the form (2) that is satisfied by the homogeneous coordinates of both Points.

Example 3 For each of the following pairs of Points, write down an equation for the Line that passes through them.

(a) [3, 2, 0] and [3, 4, 0] (b) [1, 2, 1] and [3, 0, 3]
(c) [1, 0, 0] and [0, 0, 1]

Solution

(a) Both the Points have a z-coordinate equal to 0, so the homogeneous coordinates must satisfy the equation $z = 0$. This equation is of the form (2) with $a = 0, b = 0$ and $c = 1$, so it must be the required equation for the Line.

The equation $x = 3$ is not of the form (2), and so is not the equation of a Line.

(b) The homogeneous coordinates of both Points satisfy $x = z$. This equation is of the form (2) with $a = 1$, $b = 0$ and $c = -1$. It must therefore be the required equation for the Line.

(c) The homogeneous coordinates of both Points satisfy $y = 0$. This equation is of the form (2) with $a = 0$, $b = 1$ and $c = 0$, so it must be the required equation for the Line. □

Problem 4 For each of the following pairs of Points, write down an equation for the Line that passes through them.

(a) [0, 1, 0] and [0, 0, 1] (b) [2, 2, 3] and [3, 3, 7]

But how do we find an equation for a Line through two given Points in cases where it cannot be found by inspection? As an example, consider the Points $[2, -1, 4]$ and $[1, -1, 1]$. We could certainly substitute the values $x = 2$, $y = -1$, $z = 4$ and $x = 1$, $y = -1$, $z = 1$ into equation (2), to obtain the pair of simultaneous equations

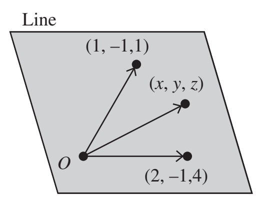
$$2a - b + 4c = 0,$$
$$a - b + c = 0.$$

Then subtracting twice the second equation from the first, we obtain $b = -2c$. So from the second equation it follows that $a = -3c$. If we set $c = -1$, say, then $a = 3$ and $b = 2$, so an equation for the Line is

$$3x + 2y - z = 0.$$

Of course, we could set c to have any non-zero value, but $c = -1$ keeps the calculation simple.

In this case the calculations are fairly straightforward, but there is an alternative method that is often simpler. Notice that the Line in $\mathbb{RP}^2$ through the Points $[2, -1, 4]$ and $[1, -1, 1]$ is the Euclidean plane in $\mathbb{R}^3$ that contains the position vectors of the points $(2, -1, 4)$ and $(1, -1, 1)$ in $\mathbb{R}^3$. A point (x, y, z) lies in this plane if and only if the vector (x, y, z) is a linear combination of the vectors $(2, -1, 4)$ and $(1, -1, 1)$; in other words, if and only if the vectors (x, y, z), $(2, -1, 4)$ and $(1, -1, 1)$ are linearly dependent.



But three vectors in $\mathbb{R}^3$ are linearly dependent if and only if the 3×3 determinant that has these vectors as its rows is zero. It follows that (x, y, z) lies in the plane containing the position vectors $(2, -1, 4)$ and $(1, -1, 1)$ if and only if

$$\begin{vmatrix} x & y & z \\ 2 & -1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 0.$$

Translating this statement back into a statement concerning $\mathbb{RP}^2$, we deduce that the Point $[x, y, z]$ lies on the Line through the Points $[2, -1, 4]$ and $[1, -1, 1]$ if and only if

$$\begin{vmatrix} x & y & z \\ 2 & -1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 0.$$

Expanding this determinant in terms of the entries in its first row, we obtain

$$\begin{aligned}\begin{vmatrix} x & y & z \\ 2 & -1 & 4 \\ 1 & -1 & 1 \end{vmatrix} &= x\begin{vmatrix} -1 & 4 \\ -1 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} + z\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} \\ &= 3x + 2y - z.\end{aligned}$$

Hence an equation for the required Line in $\mathbb{RP}^2$ is

$$3x + 2y - z = 0. \tag{3}$$

Remark

It is always sensible to check your arithmetic by checking that the two given Points actually lie on the Line that you have found. For instance, the answer above is correct, since equation (3) is a homogeneous linear equation in x, y and z, and the equation is satisfied by $x = 2$, $y = -1$, $z = 4$ and by $x = 1$, $y = -1$, $z = 1$.

We may summarize the above method in the form of a strategy, as follows.

Strategy To determine an equation for the Line in $\mathbb{RP}^2$ through the Points $[d, e, f]$ and $[g, h, k]$:

1. write down the equation

$$\begin{vmatrix} x & y & z \\ d & e & f \\ g & h & k \end{vmatrix} = 0;$$

2. expand the determinant in terms of the entries in its first row to obtain the required equation in the form $ax + by + cz = 0$.

Example 4 Find an equation for the Line that passes through the Points $[1, 2, 3]$ and $[2, -1, 4]$.

Solution An equation for the Line is

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{vmatrix} = 0.$$

Now

$$\begin{aligned}\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{vmatrix} &= x\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} - y\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + z\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \\ &= 11x + 2y - 5z.\end{aligned}$$

An equation for the Line is therefore

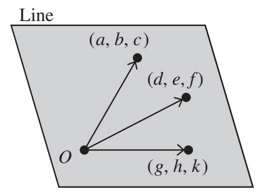
$$11x + 2y - 5z = 0. \qquad \square$$

You can easily check that the Points [1, 2, 3] and [2, −1, 4] lie on this Line.

Problem 5 Determine an equation for each of the following Lines in $\mathbb{RP}^2$:

(a) the Line through the Points [2, 5, 4] and [3, 1, 7];
(b) the Line through the Points [−2, −4, 5] and [3, −2, −4].

A similar technique can be used to check whether three given Points are collinear. Indeed, three Points $[a, b, c]$, $[d, e, f]$, $[g, h, k]$ are collinear if and only if the position vectors of the points (a, b, c), (d, e, f), (g, h, k) are linearly dependent; that is, if and only if



$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = 0.$$

Example 5 Determine whether the Points [2, 1, 3], [1, 2, 1] and [−1, 4, −3] are collinear.

Solution We have

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ -1 & 4 & -3 \end{vmatrix} &= 2\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} \\ &= 2(-6-4) - (-3+1) + 3(4+2) \\ &= -20 + 2 + 18 \\ &= 0. \end{aligned}$$

Since this is zero it follows that [2, 1, 3], [1, 2, 1] and [−1, 4, −3] are collinear. $\square$

We summarize the method of Example 5 in the following strategy.

Strategy To determine whether three Points $[a, b, c]$, $[d, e, f]$, $[g, h, k]$ are collinear:

1. evaluate the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$;
2. the Points $[a, b, c]$, $[d, e, f]$, $[g, h, k]$ are collinear if and only if this determinant is zero.

Problem 6 Determine whether the following sets of Points are collinear.

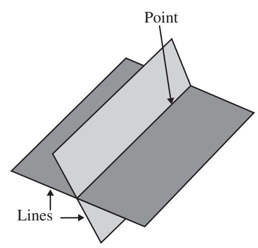
(a) $[1, 2, 3], [1, 1, -2], [2, 1, -9]$ (b) $[1, 2, -1], [2, 1, 0], [0, -1, 3]$

Before rushing to solve a problem using determinants, you should always stop to see if you can solve the problem more easily by inspection. For example, suppose that you are asked to check whether the Points [1, 0, 0], [0, 1, 0], [1, 1, 1] are collinear. Clearly, [1, 0, 0] and [0, 1, 0] lie on the Line $z = 0$, whereas [1, 1, 1] does not, so the Points are not collinear.

Problem 7 Verify that no three of the Points [1, 0, 0], [0, 1, 0], [0, 0, 1] and [1, 1, 1] are collinear.

The Points that you considered in Problem 7 play an important part in our development of the theory of projective geometry, so we give them special names.

Definitions The Points [1, 0, 0], [0, 1, 0], [0, 0, 1] are known as the **triangle of reference.** The Point [1, 1, 1] is called the **unit Point.**



Next, observe that any two distinct Lines necessarily meet at a unique Point. Indeed, a Line in $\mathbb{RP}^2$ is simply a plane in $\mathbb{R}^3$ that passes through the origin, and two distinct planes through the origin of $\mathbb{R}^3$ must intersect in a unique Euclidean line through the origin; that is, in a Point. This is very different to the situation in Euclidean geometry where parallel lines do not meet.

Theorem 3 Incidence Property of $\mathbb{RP}^2$
Any two distinct Lines in $\mathbb{RP}^2$ intersect in a unique Point of $\mathbb{RP}^2$.

This result neatly complements Theorem 2, the *Collinearity Property* of $\mathbb{RP}^2$.

We can determine the Point of intersection of two Lines simply by solving the equations of the two Lines as a pair of simultaneous equations.

Example 6 Determine the Point of intersection of the Lines in $\mathbb{RP}^2$ with equations $x + 6y - 5z = 0$ and $x - 2y + z = 0$.

We know that there is a unique Point of intersection, by Theorem 3.

Solution At the Point of intersection $[x, y, z]$ of the two Lines, we have

$$x + 6y - 5z = 0,$$
$$x - 2y + z = 0.$$

Subtracting the second equation from the first, we obtain

$$8y - 6z = 0,$$

so that $y = \frac{3}{4} z$. Substituting this into the second equation, we obtain $x = \frac{1}{2} z$.

It follows that the Point of intersection has homogeneous coordinates $\left[\frac{1}{2}z, \frac{3}{4}z, z\right]$ which we can rewrite in the form $\left[\frac{1}{2}, \frac{3}{4}, 1\right]$ or [2, 3, 4]. □

Note that $z \neq 0$, since [0, 0, 0] are not allowed as homogeneous coordinates.

Problem 8 Determine the Point of intersection of each of the following pairs of Lines in $\mathbb{RP}^2$:

(a) the Lines with equations $x - y - z = 0$ and $x + 5y + 2z = 0$;
(b) the Lines with equations $x + 2y - z = 0$ and $2x + y - 4z = 0$.

Problem 9 Determine the Point of $\mathbb{RP}^2$ at which the Line through the Points $[1, 2, -3]$ and $[2, -1, 0]$ meets the Line through the Points $[1, 0, -1]$ and [1, 1, 1].

In some cases we can write down the Point at which two Lines intersect without having to solve any equations at all. For example, the Lines with equations $x = 0$ and $y = 0$ clearly meet at the Point [0, 0, 1].

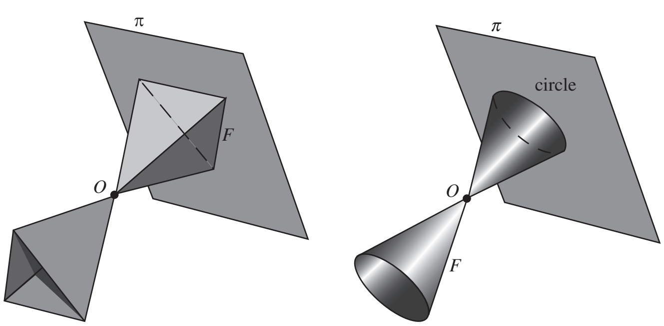
Problem 10 Determine the Point of $\mathbb{RP}^2$ at which the Line through the Points [1, 0, 0] and [0, 1, 0] meets the Line through the Points [0, 0, 1] and [1, 1, 1].

3.2.3 Embedding Planes

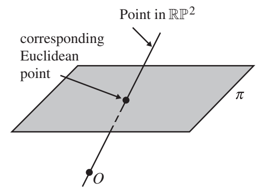
So far we have used three-dimensional space to develop the theory of projective geometry. In practice, however, we want to use projective geometry to study two-dimensional figures in a plane. In order to do this, we now investigate a way of associating figures in a plane with figures in $\mathbb{RP}^2$, and vice versa.

Suppose that a plane π contains a figure F. We can place π into $\mathbb{R}^3$, making sure that it does not pass through the origin, and then construct a corresponding projective figure by drawing in all the Points of $\mathbb{RP}^2$ that pass through the points of F. For example, if F is the triangle shown on the left below, then the corresponding projective figure is a double triangular pyramid. Note that if we change the position of π in $\mathbb{R}^3$, we obtain a different projective figure corresponding to F.

It is a *double* pyramid because the Points which make up the pyramid are lines that emerge from the origin in *both* directions.

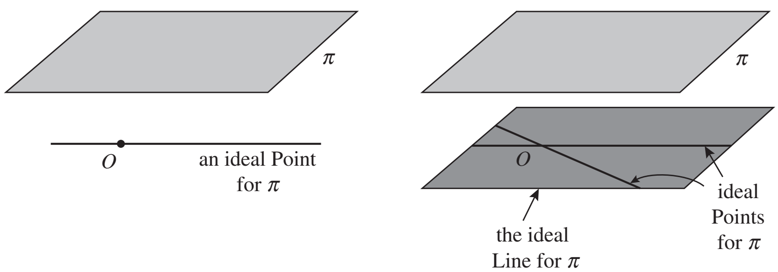


Conversely, suppose that we start with a projective figure F. The corresponding Euclidean figure in π consists of the Euclidean points where the Points of F pierce π. For example, if F is a double cone whose axis is at right angles to the embedding plane, as shown on the right above, then the corresponding Euclidean figure is a circle. Note that if we change the position of π in $\mathbb{R}^3$, we obtain a different plane figure corresponding to F.

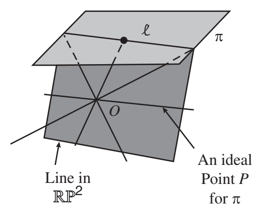


This correspondence between projective figures and Euclidean figures works well provided that each Point of the projective figure pierces the plane π, as shown in the margin. Unfortunately, any Point of $\mathbb{RP}^2$ that consists of a line through the origin *parallel* to π does not pierce π, and so cannot be associated with a point of π. Such a Point is called an **ideal Point** for π.

All the ideal Points for π lie on a plane through O parallel to π. This plane is a projective line known as the **ideal Line** for π.



How can we represent a projective figure on π if the figure includes some of the ideal Points for π? As a simple example, consider the Line illustrated in the margin. This is a projective figure which intersects π in a line ℓ. Every Point of the Line pierces the embedding plane at a point of ℓ except for the ideal Point P which cannot be represented on π. In order to represent the Line completely, we need not only the line ℓ but also the ideal Point P. In other words, the Line is represented by $\ell \cup \{P\}$.



In general, a projective figure can be represented by a figure in π provided that we are prepared to include a subset of Points taken from the ideal Line for π. In order to allow for these additional ideal Points, we introduce the concept of an *embedding plane.*

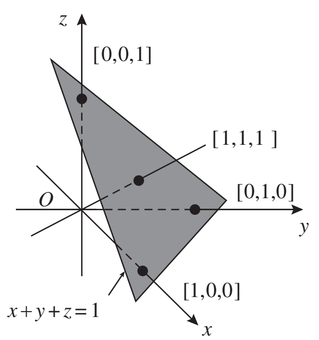
Definitions An **embedding plane** is a plane, π, which does not pass through the origin, together with the set of all ideal Points for π. The plane in $\mathbb{R}^3$ with equation $z = 1$ is called the **standard embedding plane**. The mapping of $\mathbb{RP}^2$ into the standard embedding plane is called the **standard embedding** of $\mathbb{RP}^2$.

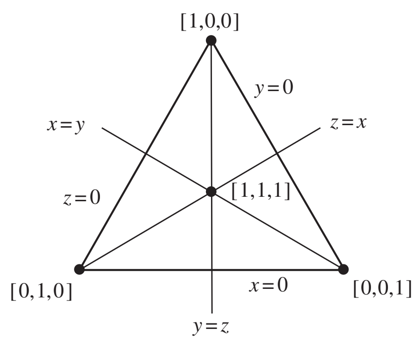
We frequently use π to denote both a plane and an embedding plane, but no confusion should arise.

We may summarize the above discussion by saying that for a given embedding plane, every projective figure in $\mathbb{RP}^2$ corresponds to a figure in the embedding plane, and vice versa. The figure in the embedding plane may include some ideal Points but is otherwise a Euclidean figure.

If two embedding planes are parallel to each other, the same Points of $\mathbb{RP}^2$ correspond to ideal Points of the embeddings; whereas, if the embedding planes are not parallel, different Points of $\mathbb{RP}^2$ correspond to ideal Points of the two embedding planes.

Once we have represented a projective figure in an embedding plane, we can investigate the relationship between its Points and Lines without having to refer to three-dimensional space at all. For example, consider the representation of the triangle of reference and unit Point on the embedding plane $x + y + z = 1$, shown on the left below. If we extract the embedding plane from $\mathbb{R}^3$, as shown on the right, we can use the algebraic theory developed earlier to write down an equation for the Line through any two given Points, without reference to $\mathbb{R}^3$.

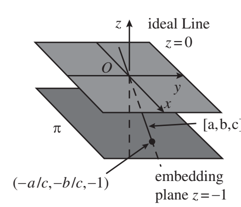




Similarly, we can use the algebraic techniques to calculate the homogeneous coordinates of the Point of intersection of any two given Lines.

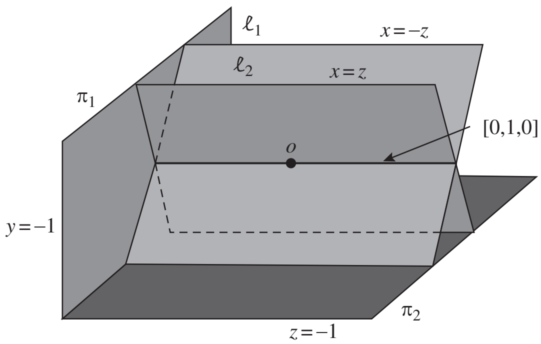
> **Problem 11** On the right-hand diagram above, insert the homogeneous coordinates of the Points where the Lines through [1, 1, 1] meet the sides of the triangle of reference.

Any plane may be used as an embedding plane provided that it does not pass through the origin. For example, if we take π to be the plane $z = -1$, then the ideal Line for π has equation $z = 0$, and the ideal Points are Points of the form $[a, b, 0]$, where a and b are not both zero. Any other Point $[a, b, c]$ has $c \neq 0$ and can therefore be represented in π by the Euclidean point$(-a/c, -b/c, -1)$.



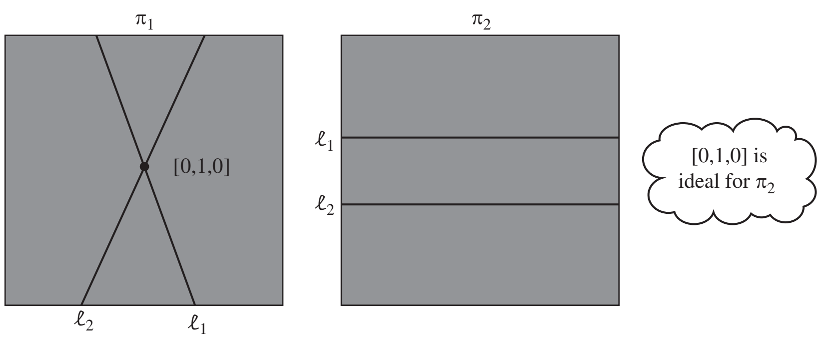
> **Problem 12** Let π be the embedding plane $y = -1$. Describe the ideal Points for π, and specify the Euclidean point of π which represents the Point [2, 4, 6].

Although we can choose any embedding plane to represent figures of $\mathbb{RP}^2$, the representation does depend on the choice. For example, suppose that π_1 is the embedding plane $y = -1$, and that π_2 is the embedding plane $z = -1$. Now consider the projective figure which consists of two Lines ℓ_1 and ℓ_2 with equations $x = -z$ and $x = z$, respectively. These Lines intersect at the Point [0, 1, 0].



On the embedding plane π_1 the Lines ℓ_1 and ℓ_2 are represented by two lines that can be seen to meet at the point corresponding to [0, 1, 0]. However, on the embedding plane π_2 the Point of intersection [0, 1, 0] is an ideal Point and so the Lines ℓ_1 and ℓ_2 are represented by parallel lines that do not appear to meet. The contrast between the two representations of ℓ_1 and ℓ_2 is particularly striking if we extract the two embedding planes from $\mathbb{R}^3$ and lay them side by side, as follows.

This is the mathematical fact which explains why artists sometimes draw parallel lines (such as railroad lines) as intersecting lines.



This example illustrates that Lines which appear to be parallel in one embedding plane may not appear to be parallel in another embedding plane. In the next section you will see that the transformations of projective geometry are chosen so as to ensure that the projective properties of a figure are unaffected by the choice of embedding plane. Since parallelism does depend on the choice of embedding plane, it cannot be a projective property, so the concept of parallel Lines is meaningless in projective geometry.

3.2.4 An equivalent definition of Projective Geometry

In our work on projective geometry, we have used Euclidean points in a plane in $\mathbb{R}^3$ to construct the projective points (Points) of the geometry $\mathbb{RP}^2$, homogeneous coordinates for those Points, and projective lines (Lines).

Equivalently, we could have defined $\mathbb{RP}^2$ as the set of ordered triples $[a, b, c]$, where a, b, c are real and not all zero, with the convention that we regard $[\lambda a, \lambda b, \lambda c]$ and $[a, b, c]$ (where $\lambda \neq 0$) as the same Point in the geometry. We would then have defined projective lines (Lines) as the set of points $[x, y, z]$ in $\mathbb{RP}^2$ that satisfy an equation of the form $ax + by + cz = 0$, where a, b, c are real and not all zero, Then we would continue to develop the theory of projective geometry in the same way as we have done here.

However, we chose to start our work by looking at a model of $\mathbb{RP}^2$ obtained by using an embedding plane π in $\mathbb{R}^3$ that does not pass through the origin. We modeled the projective points $[a, b, c]$ by the Euclidean lines through the origin and the corresponding Euclidean points (a, b, c), plus 'points at infinity' (the ideal Points); and we modeled the projective lines by Euclidean planes through the origin, For convenience, we chose often to use Euclidean points (a, b, c) on a given embedding plane to describe the Euclidean model.

Any plane that does not pass through the origin in $\mathbb{R}^3$ will serve as an embedding plane.

The formal method of defining projective geometry, though, is less intuitive than the description motivated by the $\mathbb{R}^3$ model!

3.3 Projective Transformations

3.3.1 The Group of Projective Transformations

By now you should be familiar with the idea that a geometry consists of a space of points together with a group of transformations which act on that space.

Having introduced the space of projective points $\mathbb{RP}^2$ in Section 3.2, we are now in a position to describe the transformations of $\mathbb{RP}^2$. First we shall define the transformations algebraically, then we give a geometrical interpretation of the transformations using the ideas of perspectivity introduced in Section 3.1, and finally meet the Fundamental Theorem of Projective Geometry.

Subsection 3.2.1

Recall that a point of $\mathbb{R}^3$ (other than the origin) on an embedding plane π (that does not pass through the origin) has coordinates $\mathbf{x} = (x, y, z)$ with respect to the standard basis of $\mathbb{R}^3$, and homogeneous coordinates of the corresponding Point $[\mathbf{x}]$ in $\mathbb{RP}^2$ (which represents the points $\{\lambda\mathbf{x} : \lambda \in \mathbb{R}\}$) are $[\lambda x, \lambda y, \lambda z]$ for some real $\lambda \neq 0$. Since the Points of $\mathbb{RP}^2$ are just lines through the origin of $\mathbb{R}^3$, we need a group of transformations that map the lines through the origin of $\mathbb{R}^3$ onto the lines through the origin of $\mathbb{R}^3$. Suitable transformations of $\mathbb{R}^3$ that do this are the invertible linear transformations.

If $\mathbf{A}$ is the matrix of an invertible linear transformation of $\mathbb{R}^3$ to itself, the transformation maps points $\mathbf{x} = (x, y, z)$ of $\mathbb{R}^3$ to points $\mathbf{Ax}$ of $\mathbb{R}^3$; then the *projective transformation* with matrix $\mathbf{A}$ maps Points $[\mathbf{x}]$ of $\mathbb{RP}^2$ to Points $[\mathbf{Ax}]$ of $\mathbb{RP}^2$. This suggests that we define the transformations of projective geometry as follows.

In fact, any 'continuous' transformation of $\mathbb{RP}^2$ to itself that maps Lines to Lines and that preserves incidences of Lines corresponds to an invertible linear transformation of $\mathbb{R}^3$. We omit a proof of this fact.

Definitions A **projective transformation** of $\mathbb{RP}^2$ is a function $t : \mathbb{RP}^2 \to \mathbb{RP}^2$ of the form

$$t : [\mathbf{x}] \mapsto [\mathbf{Ax}],$$

where $\mathbf{A}$ is an invertible 3×3 matrix. We say that $\mathbf{A}$ is a matrix **associated** with t. The set of all projective transformations of $\mathbb{RP}^2$ is denoted by $P(2)$.

Example 1 Show that the function $t : \mathbb{RP}^2 \to \mathbb{RP}^2$ defined by

$$t : [x, y, z] \mapsto [2x + z, -x + 2y - 3z, x - y + 5z]$$

is a projective transformation, and find the image of [1, 2, 3] under t.

Solution The transformation t has the form $t : [\mathbf{x}] \mapsto [\mathbf{Ax}]$, where $\mathbf{x} = (x, y, z)$ and

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & -3 \\ 1 & -1 & 5 \end{pmatrix}.$$

Now

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} 2 & 0 & 1 \\ -1 & 2 & -3 \\ 1 & -1 & 5 \end{vmatrix} \\ &= 2(10 - 3) - 0 + (1 - 2) \\ &= 13 \neq 0. \end{aligned}$$

So $\mathbf{A}$ is invertible. It follows that t is a projective transformation.

We have

$$t([1, 2, 3]) = [2 + 3, -1 + 4 - 9, 1 - 2 + 15] = [5, -6, 14]. \qquad \square$$

Problem 1 Decide which of the following functions t from $\mathbb{RP}^2$ to itself are projective transformations. For those that are projective transformations, write down a matrix associated with t.

(a) $t : [x, y, z] \mapsto [-2y + 3z, -x + 5y - z, -3x]$
(b) $t : [x, y, z] \mapsto [x - 7y + 4z, -x + 5y - z, x - 9y + 7z]$
(c) $t : [x, y, z] \mapsto [x - 1 + z, 2y - 4z + 5, 2x]$

Problem 2 Let t be the projective transformation associated with the matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & 1 \\ 4 & -3 & 4 \end{pmatrix}.$$

Determine the image under t of each of the following Points.

(a) $[1, 2, -1]$ (b) $[1, 0, 0]$ (c) $[0, 1, 0]$
(d) $[0, 0, 1]$ (e) $[1, 1, 1]$

Since we can multiply the homogeneous coordinates of Points in $\mathbb{RP}^2$ by any non-zero real number λ without altering the Point itself, it follows that if $\mathbf{A}$ is a matrix associated with a particular projective transformation then so is the matrix $\lambda\mathbf{A}$, provided that $\lambda \neq 0$. For example, another matrix associated with the transformation in Example 1 is

$$\mathbf{B} = \begin{pmatrix} -4 & 0 & -2 \\ 2 & -4 & 6 \\ -2 & 2 & -10 \end{pmatrix},$$

for we have $\mathbf{B} = -2\mathbf{A}$.

Problem 3 Write down a matrix with top left-hand entry $\frac{1}{2}$ which is associated with the transformation in Example 1.

Before we can use the projective transformations to define projective geometry, we must first check that they form a group.

Theorem 1 The set of projective transformations $P(2)$ forms a group under the operation of composition of functions.

Recall that a similar result holds for affine transformations.

Proof We check that the four group axioms hold.

G1 CLOSURE Let t_1 and t_2 be projective transformations defined by

$$t_1 : [\mathbf{x}] \mapsto [\mathbf{A_1x}] \quad \text{and} \quad t_2 : [\mathbf{x}] \mapsto [\mathbf{A_2x}],$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are invertible 3×3 matrices. Then

$$\begin{aligned} t_1 \circ t_2([\mathbf{x}]) &= t_1(t_2([\mathbf{x}])) \\ &= t_1([\mathbf{A_2x}]) \\ &= [(\mathbf{A_1A_2})\mathbf{x}]. \end{aligned}$$

Since $\mathbf{A}_1$ and $\mathbf{A}_2$ are invertible, it follows that $\mathbf{A_1A_2}$ is invertible. So by definition $t_1 \circ t_2$ is a projective transformation.

G2 IDENTITY Let $i : \mathbb{RP}^2 \to \mathbb{RP}^2$ be the transformation defined by

$$i : [\mathbf{x}] \mapsto [\mathbf{Ix}],$$

where $\mathbf{I}$ is the 3×3 identity matrix; this is a projective transformation, since $\mathbf{I}$ is invertible.

Let t: $\mathbb{RP}^2 \to \mathbb{RP}^2$ be an arbitrary projective transformation, defined by t :$[\mathbf{x}] \mapsto [\mathbf{Ax}]$, for some invertible 3 $\times$ 3 matrix $\mathbf{A}$. Then for any $[\mathbf{x}] \in \mathbb{RP}^2$,

$$t \circ i([\mathbf{x}]) = [\mathbf{A}(\mathbf{Ix})] = [\mathbf{Ax}]$$

and

$$i \circ t([\mathbf{x}]) = [\mathbf{I}(\mathbf{Ax})] = [\mathbf{Ax}].$$

Thus $t \circ i = i \circ t = t$. Hence i is the identity transformation.

G3 INVERSES Let t : $\mathbb{RP}^2 \to \mathbb{RP}^2$ be an arbitrary projective transformation defined by

$$t\colon [\mathbf{x}] \mapsto [\mathbf{Ax}],$$

for some invertible 3×3 matrix $\mathbf{A}$. Then we can define another projective transformation t' : $\mathbb{RP}^2 \to \mathbb{RP}^2$ by

$$t'\colon [\mathbf{x}] \mapsto [\mathbf{A^{-1}x}].$$

Now, for each $[\mathbf{x}] \in \mathbb{RP}^2$, we have

$$t \circ t'([\mathbf{x}]) = \mathbf{t}([\mathbf{A^{-1}x}]) = [\mathbf{A}(\mathbf{A^{-1}x})] = [\mathbf{x}]$$

and

$$t' \circ t([\mathbf{x}]) = t'([\mathbf{Ax}]) = [\mathbf{A}^{-1}(\mathbf{Ax})] = [\mathbf{x}].$$

Thus t' is an inverse for t.

G4 ASSOCIATIVITY Composition of functions is always associative.

It follows that the set of projective transformations $P(2)$ forms a group. ■

The above proof shows that if t_1 and t_2 are projective transformations with associated matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively, then $t_1 \circ t_2$ is a projective transformation with an associated matrix $\mathbf{A}_1\mathbf{A}_2$. We therefore have the following strategy for composing projective transformations.

Strategy To compose two projective transformations t_1 and t_2:

1. write down matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ associated with t_1 and t_2;
2. calculate $\mathbf{A}_1\mathbf{A}_2$;
3. write down the composite $t_1 \circ t_2$ with which $\mathbf{A}_1\mathbf{A}_2$ is associated.

The proof also shows that if t is a projective transformation with an associated matrix $\mathbf{A}$, then t^{-1} is a projective transformation with associated matrix $\mathbf{A}^{-1}$. We therefore have the following strategy for calculating the inverse of a projective transformation.

See Appendix 2 for one method to calculate $\mathbf{A}^{-1}$.

Strategy To find the inverse of a projective transformation t:

1. write down a matrix $\mathbf{A}$ associated with t;
2. calculate $\mathbf{A}^{-1}$;
3. write down the inverse t^{-1} with which $\mathbf{A}^{-1}$ is associated.

Example 2 Let t_1 and t_2 be projective transformations defined by

$$t_1 : [x, y, z] \mapsto [x + z, x + y + 3z, -2x + z],$$

$$t_2 : [x, y, z] \mapsto [2x, x + y + z, 4x + 2y].$$

Determine the projective transformations $t_2 \circ t_1$ and t_1^{-1}.

Solution The transformations t_1 and t_2 have associated matrices

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ -2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{pmatrix},$$

respectively. It follows that $t_2 \circ t_1$ has an associated matrix

$$\mathbf{A}_2\mathbf{A}_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 5 \\ 6 & 2 & 10 \end{pmatrix},$$

so

$$t_2 \circ t_1 : [x, y, z] \mapsto [2x + 2z, y + 5z, 6x + 2y + 10z].$$

Next, t_1^{-1} has an associated matrix $\mathbf{A}_1^{-1}$ given by

$$\mathbf{A}_1^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{7}{3} & 1 & -\frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix};$$

a simpler matrix associated with t_1^{-1} is then

$$\begin{pmatrix} 1 & 0 & -1 \\ -7 & 3 & -2 \\ 2 & 0 & 1 \end{pmatrix},$$

so

$$t_1^{-1} : [x, y, z] \mapsto [x - z, -7x + 3y - 2z, 2x + z]. \qquad \square$$

Problem 4 Let t_1 and t_2 be projective transformations defined by

$$t_1 : [x, y, z] \mapsto [2x + y, -x + z, y + z],$$

$$t_2 : [x, y, z] \mapsto [5x + 8y, 3x + 5y, 2z].$$

Determine the projective transformations $t_1 \circ t_2$ and t_1^{-1}.

Having shown that the set of projective transformations forms a group under composition of functions, we can now define **projective geometry** to be the study of those properties of figures in $\mathbb{RP}^2$ that are preserved by projective transformations. Those properties that are preserved by projective transformations are known as **projective properties.**

3.3.2 Some Properties of Projective Transformations

We now check two important properties of projective transformations, namely, that they preserve collinearity and incidence.

A Line in $\mathbb{RP}^2$ is a plane in $\mathbb{R}^3$ that passes through the origin. It therefore consists of the set of points (x, y, z) of $\mathbb{R}^3$ that satisfy an equation of the form

$$ax + by + cz = 0,$$

where a, b and c are not all zero. We can write this condition equivalently in the matrix form $\mathbf{Lx} = 0$, where $\mathbf{L}$ is the non-zero row matrix $(a\ b\ c)$ and $\mathbf{x} = (x\ y\ z)^T$.

Now let t be a projective transformation defined by $t : [\mathbf{x}] \mapsto [\mathbf{Ax}]$, where $\mathbf{A}$ is an invertible 3×3 matrix, and let $[\mathbf{x}]$ be an arbitrary Point on the Line $\mathbf{Lx} = 0$. Then the image of $[\mathbf{x}]$ under t is a Point $[\mathbf{x}']$ where $\mathbf{x}' = \mathbf{Ax}$. Since $\mathbf{x}$ satisfies the equation $\mathbf{Lx} = 0$, it follows that $\mathbf{x}'$ satisfies $\mathbf{L}(\mathbf{A}^{-1}\ \mathbf{x}') = 0$, or $(\mathbf{LA}^{-1})\ \mathbf{x}' = 0$. Dropping the dash, we conclude that the image of the Line $\mathbf{Lx} = 0$ under t is the Line with equation

$$\left(\mathbf{LA}^{-1}\right)\mathbf{x} = 0.$$

Here, $\mathbf{LA}^{-1}$ is non-zero, for if

$$\mathbf{LA}^{-1} = \mathbf{0},$$

then

$$\mathbf{0} = \left(\mathbf{LA}^{-1}\right)\mathbf{A} = \mathbf{L}\left(\mathbf{A}^{-1}\mathbf{A}\right) = \mathbf{L},$$

which is not the case.

Since the image of a Line in $\mathbb{RP}^2$ is a Line, it follows that collinearity is preserved under a projective transformation.

Notice that if $\mathbf{B}$ is any matrix associated with t^{-1}, then $\mathbf{B} = \lambda\mathbf{A}^{-1}$ for some non-zero real number λ, and so $\left(\mathbf{LA}^{-1}\right)\mathbf{x} = 0$ if and only if $(\mathbf{LB})\mathbf{x} = 0$. It follows that the image of the Line can equally well be written as $(\mathbf{LB})\mathbf{x} = 0$. (For instance, since $\mathbf{A}^{-1} = \operatorname{adj}(\mathbf{A})/\det(\mathbf{A})$ so that t^{-1} also has $\operatorname{adj}(\mathbf{A})$ as an associated matrix, we can express the image of the Line as $(\mathbf{L}\operatorname{adj}(\mathbf{A}))\mathbf{x} = 0$.)

We therefore summarize the above discussion in the form of a strategy, as follows.

Strategy To find the image of a Line

$$ax + by + cz = 0$$

under a projective transformation $t : [\mathbf{x}] \mapsto [\mathbf{AX}]$:

1. write the equation of the Line in the form $\mathbf{Lx} = 0$, where $\mathbf{L}$ is the matrix $(a\ b\ c)$;
2. find a matrix $\mathbf{B}$ associated with t^{-1};
3. write down the equation of the image as $(\mathbf{LB})\mathbf{x} = 0$.

Example 3 Find the image of the Line $2x + y - 3z = 0$ under the projective transformation t_1 defined by

$$t_1 : [x, y, z] \mapsto [x + z, x + y + 3z, -2x + z].$$

Solution The equation of the Line can be written in the form $\mathbf{Lx} = 0$, where

$$\mathbf{L} = (2 \quad 1 \quad -3).$$

In Example 2 we showed that $t_1{}^{-1}$ has an associated matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & -1 \\ -7 & 3 & -2 \\ 2 & 0 & 1 \end{pmatrix}.$$

So

$$\mathbf{LB} = (2 \quad 1 - 3) \begin{pmatrix} 1 & 0 & -1 \\ -7 & 3 & -2 \\ 2 & 0 & 1 \end{pmatrix} = (-11 \quad 3 - 7).$$

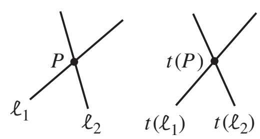
It follows that the required image has equation

$$-11x + 3y - 7z = 0. \qquad \square$$

Problem 5 Find the image of the Line $x + 2y - z = 0$ under the projective transformation t_1 defined by

$$t_1 : [x, y, z] \mapsto [2x + y, -x + z, y + z].$$

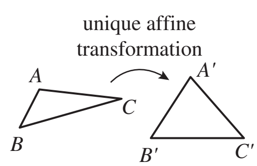
Next, we consider the incidence property. If two Lines intersect at the Point P, then P lies on both Lines. So if t is a projective transformation, then $t(P)$ lies on the images of both Lines. It follows that the image under t of the Point of intersection of the two Lines is the Point of intersection of the images of the two Lines. In other words, incidence is also preserved under a projective transformation.



Theorem 2 Collinearity and incidence are both projective properties.

3.3.3 Fundamental Theorem of Projective Geometry

In Chapter 2 we discussed the Fundamental Theorem of Affine Geometry which states that given any two sets of three non-collinear points of $\mathbb{R}^2$ there is a unique affine transformation which maps the points in one set to the corresponding points in the other set. So an affine transformation is uniquely determined by its effect on any given triangle.



In this subsection we explore an analogous result for projective geometry known as the *Fundamental Theorem of Projective Geometry.* We begin by asking you to tackle the following problem.

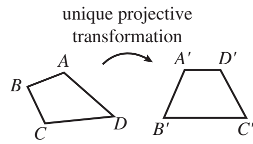
Problem 6 Let t_1 and t_2 be the projective transformations with associated matrices

$$\mathbf{A}_1 = \begin{pmatrix} -4 & -1 & 1 \\ -3 & -2 & 1 \\ 4 & 2 & -1 \end{pmatrix} \text{ and } \mathbf{A}_2 = \begin{pmatrix} -8 & -6 & -2 \\ -3 & 4 & 7 \\ 6 & 0 & -4 \end{pmatrix},$$

respectively. Find the images of the Points $[1, -1, 1]$, $[1, -2, 2]$ and $[-1, 2, -1]$ under t_1 and t_2.

You should have found that both of the projective transformations t_1 and t_2 map the Points $[1, -1, 1]$, $[1, -2, 2]$ and $[-1, 2, -1]$ to the Points $[-2, 0, 1]$, $[0, 3, -2]$ and $[1, -2, 1]$, respectively. Notice, however, that t_1 and t_2 are not the same projective transformation, since their matrices are not multiples of each other. It follows that, unlike affine transformations, projective transformations are not uniquely determined by their effect on three (non-collinear) Points.

This raises the question as to whether it is possible to specify how many Points *are* required to determine a projective transformation. According to the Fundamental Theorem of Projective Geometry, the answer is four. In fact the theorem states that given any two sets of four Points, no three of which are collinear, there is a *unique* projective transformation that maps the Points in one set to the corresponding Points in the second set. Thus, in projective geometry a transformation is uniquely determined by its effect on a quadrilateral.



To understand why a triangle is insufficient to determine a projective transformation uniquely, consider what happens when we look for a projective transformation that maps the triangle of reference to three given non-collinear Points.

Recall that the Points [1, 0, 0], [0, 1, 0] and [0, 0, 1] are known as the *triangle of reference.*

Example 4 Find a projective transformation t that maps the Points $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$ to the non-collinear Points $[1, -1, 1], [1, -2, 2]$ and $[-1, 2, -1]$, respectively.

Solution Let $\mathbf{A}$ be a matrix associated with t, and let the first column of $\mathbf{A}$ be $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Then since

$$\left[\begin{pmatrix} a & * & * \\ b & * & * \\ c & * & * \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right],$$

Here the asterisks * denote unspecified numbers.

it follows that we may take $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ as the first column of $\mathbf{A}$.

Similarly, since

$$\left[\begin{pmatrix} * & d & * \\ * & e & * \\ * & f & * \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} d \\ e \\ f \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\right]$$

and

$$\left[\begin{pmatrix} * & * & g \\ * & * & h \\ * & * & k \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} g \\ h \\ k \end{pmatrix}\right] = \left[\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}\right],$$

it follows that a suitable transformation is given by $t : [\mathbf{x}] \mapsto [\mathbf{A}x]$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & 2 \\ 1 & 2 & -1 \end{pmatrix}.$$

□

Notice that because the Points $[1, -1, 1]$, $[1, -2, 2]$ and $[-1, 2, -1]$ are not collinear it follows that the columns of $\mathbf{A}$ are linearly independent, so that $\mathbf{A}$ is invertible.

This example illustrates the fact that we can always find a projective transformation $t : [\mathbf{x}] \mapsto [\mathbf{Ax}]$ which maps the triangle of reference to three non-collinear Points simply by writing the homogeneous coordinates of the Points as the columns of $\mathbf{A}$. Notice, however, that the transformation we obtain is not unique. Indeed, if the Points $[1, -1, 1]$, $[1, -2, 2]$ and $[-1, 2, -1]$ in Example 4 are rewritten in the form $[u, -u, u]$, $[\upsilon, -2\upsilon, 2\upsilon]$ and $[-w, 2w, -w]$, for some non-zero real numbers u, υ, w, then the matrix becomes

$$\mathbf{A} = \begin{pmatrix} u & \upsilon & -w \\ -u & -2\upsilon & 2w \\ u & 2\upsilon & -w \end{pmatrix}.$$

The corresponding transformation $t : [\mathbf{x}] \mapsto [\mathbf{Ax}]$ still maps the triangle of reference to the Points $[1, -1, 1]$, $[1, -2, 2]$ and $[-1, 2, -1]$, as required, but the effect that t has on the other Points of $\mathbb{RP}^2$ depends on the numbers u, v and w.

So if we wish to specify t uniquely we need to assign particular values to u, v and w. We can do this by specifying the effect that t has on a fourth Point [1, 1, 1].

Recall that the Point [1, 1, 1] is known as the *unit Point.*

Example 5 Find the projective transformation t which maps the Points [1, 0, 0], [0, 1, 0], [0, 0, 1] and [1, 1, 1] to the Points $[1, -1, 1]$, $[1, -2, 2]$, $[-1, 2, -1]$ and [0, 1, 2], respectively.

Solution If $\mathbf{A}$ is the matrix associated with t, then its columns must be multiples of the homogeneous coordinates $[1, -1, 1]$, $[1, -2, 2]$, $[-1, 2, -1]$; that is,

$$\mathbf{A} = \begin{pmatrix} u & \upsilon & -w \\ -u & -2\upsilon & 2w \\ u & 2\upsilon & -w \end{pmatrix}.$$

Also, to ensure that t maps [1, 1, 1] to [0, 1, 2] we must choose u, v and w so that

$$\left[\begin{pmatrix} u & v & -w \\ -u & -2v & 2w \\ u & 2v & -w \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right].$$

We can do this by solving the equations

$$\begin{aligned} u + v - w &= 0, \\ -u - 2v + 2w &= 1, \\ u + 2v - w &= 2. \end{aligned}$$

Adding the second and third equations we obtain $w = 3$. If we then subtract the first equation from the third we obtain $v = 2$. Finally, if we substitute v and w into the first equation we obtain $u = 1$. The required projective transformation is therefore given by $t : [\mathbf{x}] \mapsto [\mathbf{A}\mathbf{x}]$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & -4 & 6 \\ 1 & 4 & -3 \end{pmatrix}.$$

□

The columns of **A** are still linearly independent because they are non-zero multiples of the linearly independent vectors $(1, -1, 1)$, $(1, -2, 2)$ and $(-1, 2, -1)$.

It is natural to ask whether the method used in this example can be adapted to find a projective transformation which maps the triangle of reference and unit Point to *any* four given Points. The answer is usually yes, but since collinearity is a projective property, and since no three of the Points [1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1] are collinear, the method must fail if three of the four given Points lie on a Line. Provided we exclude this possibility, the answer is yes!

Strategy To find the projective transformation which maps

$$\begin{aligned} &[1, 0, 0] \quad \text{to} \quad [a_1, a_2, a_3], \\ &[0, 1, 0] \quad \text{to} \quad [b_1, b_2, b_3], \\ &[0, 0, 1] \quad \text{to} \quad [c_1, c_2, c_3], \\ &[1, 1, 1] \quad \text{to} \quad [d_1, d_2, d_3], \end{aligned}$$

where no three of $[a_1, a_2, a_3], [b_1, b_2, b_3], [c_1, c_2, c_3], [d_1, d_2, d_3]$ are collinear:

1. find u, v, w such that

$$\begin{pmatrix} a_1u & b_1v & c_1w \\ a_2u & b_2v & c_2w \\ a_3u & b_3v & c_3w \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix};$$

We explain why the method works in the Remark that follows the strategy.

2. write down the required projective transformation in the form $t : [\mathbf{x}] \mapsto [\mathbf{Ax}]$, where $\mathbf{A}$ is any non-zero real multiple of the matrix

$$\begin{pmatrix} a_1u & b_1v & c_1w \\ a_2u & b_2v & c_2w \\ a_3u & b_3v & c_3w \end{pmatrix}.$$

The non-zero multiple can be used to clear fractions from the entries of the matrix $\mathbf{A}$.

Remark

To see why this strategy always works, notice that we can rewrite the equation from Step 1 in the form

$$u\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + v\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + w\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

From this we can make the following observations.

(a) The equation in Step 1 must have a unique solution for u, v, w because the required values of u, v and w are simply the coordinates of (d_1, d_2, d_3) with respect to the basis of $\mathbb{R}^3$ formed from the three linearly independent vectors $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$.

(b) The values *of* u, v and w must all be non-zero, because otherwise three of the vectors $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3)$ would be linearly dependent.

(c) Since the columns of $\mathbf{A}$ are non-zero, multiples of the linearly independent vectors $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$ it follows that $\mathbf{A}$ is invertible, and hence that t is a projective transformation.

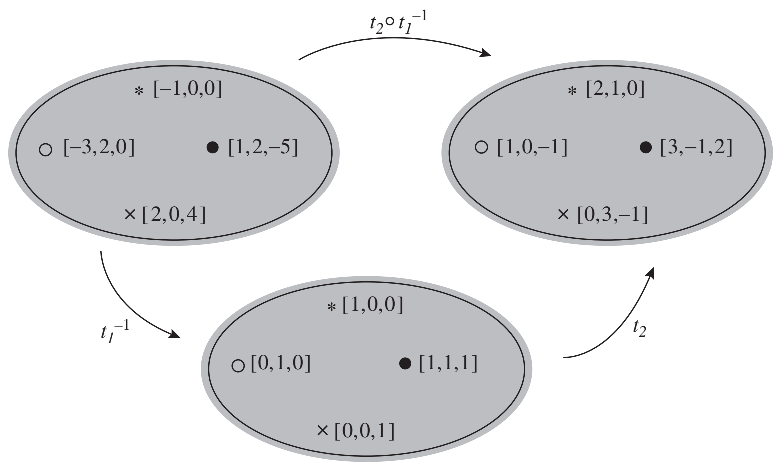
Since no three of the Points $[a_1, a_2, a_3]$, $[b_1, b_2, b_3]$, $[c_1, c_2, c_3]$, $[d_1, d_2, d_3]$ are collinear, it follows that *any three* of the vectors (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3) must be linearly independent.

There is no need to check whether any three of the four given Points are collinear because any failure of this condition will emerge in the process of applying the strategy. Indeed, if the equation in Step 1 fails to yield unique non-zero values for u, v and w, then it must be because three of the Points $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3)$ lie on a Line.

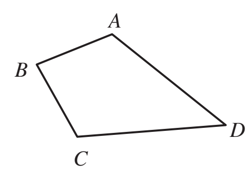
Problem 7 Use the above strategy to find the projective transformation which maps the Points [1, 0, 0], [0, 1, 0], [0, 0, 1] and [1, 1, 1] to the Points:

(a) $[-1, 0, 0], [-3, 2, 0], [2, 0, 4]$ and $[1, 2, -5]$, respectively;
(b) [1, 0, 0], [0, 0, 1], [0, 1, 0] and [3, 4, 5], respectively;
(c) [2, 1, 0], $[1, 0, -1], [0, 3, -1]$ and $[3, -1, 2]$, respectively.

Now consider the transformation t_1 in Problem 7(a). The inverse of this, t_1^{-1}, is a projective transformation which maps the Points $[-1, 0, 0], [-3, 2, 0]$, [2, 0, 4] and $[1, 2, -5]$ back to the triangle of reference and unit Point. So if, after applying this inverse, we apply the projective transformation t_2 in Problem 7(c), then the overall effect of the composite $t_2 \circ t_1^{-1}$ is that of a projective transformation which sends the Points $[-1, 0, 0], [-3, 2, 0]$, [2, 0, 4] and $[1, 2, -5]$ directly to the Points [2, 1, 0], $[1, 0, -1], [0, 3, -1]$ and $[3, -1, 2]$, respectively.



In a similar way we can find a projective transformation which maps any set of four Points to any other set of four Points. The only constraint is that no three of the Points in either set can be collinear. In the following statement of the Fundamental Theorem we express this constraint by requiring that each of the four sets of Points lie at the vertices of some quadrilateral, where a *quadrilateral* is defined as follows. A *quadrilateral* is a set of four Points A, B, C and D (no three of which are collinear), together with the Lines AB, BC, CD and DA.

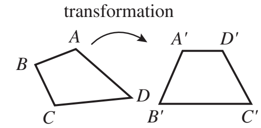


Theorem 3 The Fundamental Theorem of Projective Geometry
Let $ABCD$ and $A'B'C'D'$ be two quadrilaterals in $\mathbb{RP}^2$. Then:

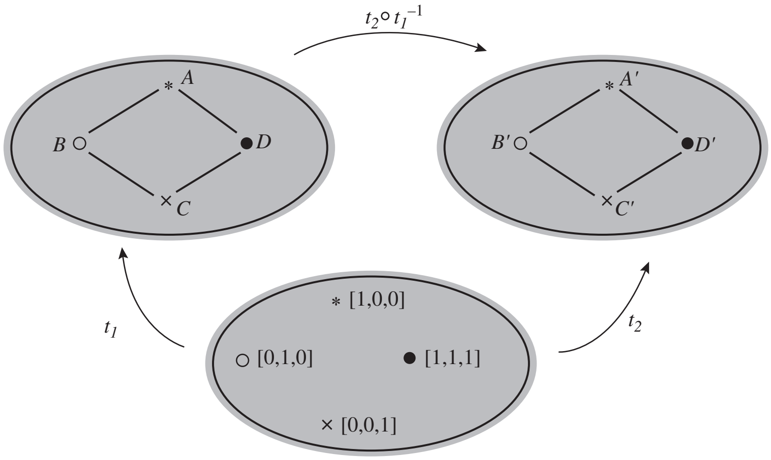
(a) there is a projective transformation t which maps

$$A \text{ to } A', B \text{ to } B', C \text{ to } C', D \text{ to } D';$$

(b) the projective transformation t is unique.



Proof According to the strategy above, there is a projective transformation t_1 which maps the Points [1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1] to the Points A, B, C, D, respectively. Similarly, there is a projective transformation t_2 which maps the Points [1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1] to the Points A', B', C', D', respectively.



(a) The composite $t = t_2 \circ t_1^{-1}$ is then a projective transformation which maps A to A', B to B', C to C', D to D'.

(b) To check uniqueness of t, we first check that the identity transformation is the only projective transformation which maps each of the Points [1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1] to themselves. In fact any projective transformation with this property must have an associated matrix which is some non-zero multiple of the matrix

This follows from the discussion leading to the strategy above.

$$\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix}, \quad \text{where} \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Such a matrix must be (a non-zero multiple of) the identity matrix, and so the transformation must indeed be the identity.

Next suppose that t and t' are two projective transformations which satisfy the conditions of the theorem. Then the composites $t_2^{-1} \circ t \circ t_1$ and $t_2^{-1} \circ t' \circ t_1$ must both be projective transformations which map each of the Points [1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1] to themselves. Since this implies that both composites are equal to the identity, we deduce that

$$t_2^{-1} \circ t \circ t_1 = t_2^{-1} \circ t' \circ t_1.$$

If we now compose both sides of this equation with t_2 on the left and with t_1^{-1} on the right, then we obtain $t = t'$, as required. ■

The Fundamental Theorem tells us that there is a projective transformation which maps any given quadrilateral onto any other given quadrilateral. So we have the following corollary.

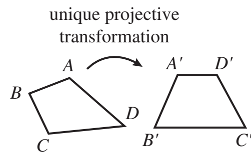
Corollary All quadrilaterals are projective-congruent.

By *projective-congruent* we mean that there is a projective transformation that maps any quadrilateral onto any other quadrilateral.

If we actually need to find the projective transformation which maps one given quadrilateral onto another given quadrilateral, we simply follow the strategy used to prove part (a) of the Fundamental Theorem.

Strategy To determine the projective transformation t which maps the vertices of the quadrilateral $ABCD$ to the corresponding vertices of the quadrilateral $A'B'C'D'$:

1. find the projective transformation t_1 which maps the triangle of reference and unit Point to the Points A, B, C, D, respectively;
2. find the projective transformation t_2 which maps the triangle of reference and unit Point to the Points A', B', C', D', respectively;
3. calculate $t = t_2 \circ t_1^{-1}$.



Example 6 Find the projective transformation t which maps the Points [1, −1, 2], [1, −2, 1], [5, −1, 2], [1, 0, 1] to the Points [−1, 3, −2], [−3, 7, −5], [2, −5, 4], [−3, 8, −5], respectively.

Solution We follow the steps in the above strategy.

(a) Any matrix associated with the projective transformation t_1 which maps the Points [1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1] to the Points $[1, -1, 2]$, $[1, -2, 1], [5, -1, 2]$, [1, 0, 1], respectively, must be a multiple of the matrix

$$\begin{pmatrix} u & v & 5w \\ -u & -2v & -w \\ 2u & v & 2w \end{pmatrix}, \quad \text{where} \begin{pmatrix} u & v & 5w \\ -u & -2v & -w \\ 2u & v & 2w \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Solving the equations

$$u + v + 5w = 1,$$
$$-u - 2v - w = 0,$$
$$2u + v + 2w = 1,$$

we obtain $u = \frac{1}{2}, v = -\frac{1}{3}, w = \frac{1}{6}$. So a suitable choice of matrix for t_1 is

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{3} & \frac{5}{6} \\ -\frac{1}{2} & \frac{2}{3} & -\frac{1}{6} \\ 1 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad \text{or more simply } \mathbf{A}_1 = \begin{pmatrix} 3 & -2 & 5 \\ -3 & 4 & -1 \\ 6 & -2 & 2 \end{pmatrix}.$$

It is simpler to multiply the first matrix by 6 to obtain integer entries. This does not alter the projective transformation t_1 with which the matrix is associated.

(b) Any matrix associated with the projective transformation t_2 which maps the Points [1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1] to the Points $[-1, 3, -2], [-3, 7, -5], [2, -5, 4], [-3, 8, -5]$, respectively, must be a multiple of the matrix

$$\begin{pmatrix} -u & -3v & 2w \\ 3u & 7v & -5w \\ -2u & -5v & 4w \end{pmatrix}, \text{ where} \begin{pmatrix} -u & -3v & 2w \\ 3u & 7v & -5w \\ -2u & -5v & 4w \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ -5 \end{pmatrix}.$$

Solving the equations

$$-u - 3v + 2w = -3,$$
$$3u + 7v - 5w = 8,$$
$$-2u - 5v + 4w = -5,$$

we obtain $u = 2, v = 1, w = 1$. So a suitable choice of matrix for t_2 is

$$\mathbf{A}_2 = \begin{pmatrix} -2 & -3 & 2 \\ 6 & 7 & -5 \\ -4 & -5 & 4 \end{pmatrix}.$$

(c) A matrix associated with the inverse, t_1^{-1}, of t_1 is $\mathbf{A}_1^{-1}$, which we can calculate to be

$$\mathbf{A}_1{}^{-1} = \begin{pmatrix} -\frac{1}{12} & \frac{1}{12} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{12} & -\frac{1}{12} \end{pmatrix};$$

then a simpler matrix associated with t_1^{-1} is

$$\mathbf{B} = \begin{pmatrix} 1 & -1 & -3 \\ 0 & -4 & -2 \\ -3 & -1 & 1 \end{pmatrix}.$$

The required projective transformation is therefore $t : [\mathbf{x}] \mapsto [\mathbf{A}x]$, where

$$\mathbf{A} = \mathbf{A}_2\mathbf{B} = \begin{pmatrix} -2 & -3 & 2 \\ 6 & 7 & -5 \\ -4 & -5 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & -3 \\ 0 & -4 & -2 \\ -3 & -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -8 & 12 & 14 \\ 21 & -29 & -37 \\ -16 & 20 & 26 \end{pmatrix}.$$

□

Problem 8 Find the projective transformation t that maps the Points $[-1, 0, 0], [-3, 2, 0], [2, 0, 4], [1, 2, -5]$ to the Points $[2, 1, 0], [1, 0, -1], [0, 3, -1], [3, -1, 2]$, respectively.

Problem 9 Find the projective transformation t that maps the Points $[1, 0, -3], [1, 1, -2], [3, 3, -5], [6, 4, -13]$ to the Points $[3, -5, 3], [\frac{1}{2}, -1, 0], [3, -5, 6], [8, -13, 12]$, respectively.

3.3.4 Geometrical Interpretation of Projective Transformations

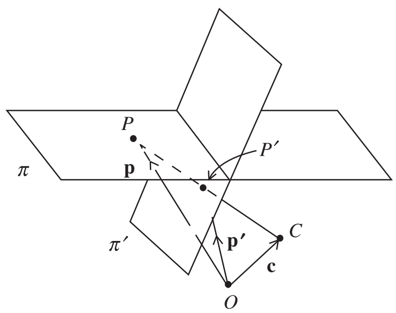
You may omit this subsection at a first reading, as it is quite hard going.

In this subsection we discuss the relationship between projective transformations and the perspectivities introduced in Section 3.1.

Starting from the geometric definition of perspective in Subsection 3.1.1, we will define the term *perspective transformation*, show how a perspective transformation may be interpreted as a projective transformation, and finally prove that any projective transformation can be expressed as the composite of at most three perspective transformations.

Recall that a *projective transformation* of $\mathbb{RP}^2$ is a map $[\mathbf{x}] \to [\mathbf{Ax}]$, where $\mathbf{A}$ is an invertible 3×3 matrix.

So, let π and π' be two embedding planes in $\mathbb{R}^3$ that do not pass through the origin O in $\mathbb{R}^3$, and let C ($\neq O$) be another point in $\mathbb{R}^3$ such that OC is not parallel to either π or π'. Let C have position vector $\mathbf{c}$ (based at O). Also, let σ denote an arbitrary perspectivity from the point C that maps the plane π to the plane π'.



Note that O, C, P and P' are coplanar.

Now, the perspectivity σ will map any point P (with position vector $\mathbf{p}$) in π onto some point P' (with position vector $\mathbf{p}'$) in π', so long as the vector $\mathbf{p} - \mathbf{c}$ is not parallel to the plane π'. We then define the perspective transformation associated with σ to be the mapping of $\mathbb{R}^3$ to itself that maps the line $[\mathbf{p} - \mathbf{c}]$ onto the line $[\mathbf{p}' - \mathbf{c}]$. But since C, P and P' are collinear, it follows that the vectors $\mathbf{p} - \mathbf{c}$ and $\mathbf{p}' - \mathbf{c}$ must be parallel (equivalently, that $[\mathbf{p} - \mathbf{c}] = [\mathbf{p}' - \mathbf{c}]$); hence there is some real number t such that

Recall that a line through the origin in $\mathbb{R}^2$ is a 1-dimensional vector space.

The value of t will depend on the particular point P under discussion.

$$\mathbf{p}' - \mathbf{c} = t(\mathbf{p} - \mathbf{c}).$$

We can then rewrite this formula in the form

$$\mathbf{p}' = t(\mathbf{p} - \mathbf{c}) + \mathbf{c},$$

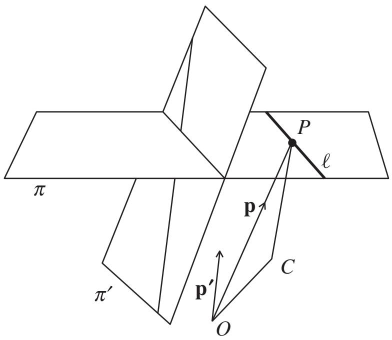
or

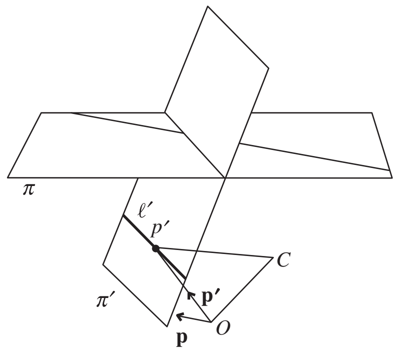
$$\mathbf{p}' = t\mathbf{p} + (1 - t)\mathbf{c},$$

so that

$$[\mathbf{p}] \mapsto [t\mathbf{p} + (1 - t)\mathbf{c}].$$

In this way, the perspectivity σ gives a one-one mapping from π onto π', except that it is not defined on the line ℓ in π where π cuts the plane through C parallel to π'; also, there are no points of π that map onto points of π' on the line ℓ' where π' cuts the plane through C parallel to π. We use the fact that the ideal Points for a plane correspond to the directions of lines in the plane, rather than actual points in the plane, to extend our definition of the map.





So, first, let P be a point of π that lies on the line ℓ; then the points O, C, P are not collinear, since OC is not parallel to the plane π'. Denote the position vector $\overrightarrow{OP}$ by $\mathbf{p}$. Let the plane through O, C and P meet the plane π' in a line, and let $\mathbf{p}'$ be a position vector based at O that is parallel to this line. Then we specify that our (extended) map σ maps the line $[\mathbf{p}]$ through O onto the line $[\mathbf{p}']$ through O.

Similarly, let P' be a point of π' that lies on the line ℓ'; then the points O, C, P' are not collinear, since OC is not parallel to the plane π. Denote the position vector $\overrightarrow{OP'}$ by $\mathbf{p}'$. Let the plane through O, C and P' meet the plane π in a line, and let $\mathbf{p}$ be a position vector based at O that is parallel to this line. Then we specify that our (further extended) map σ maps the line $[\mathbf{p}]$ through O onto the line $[\mathbf{p}']$ through O.

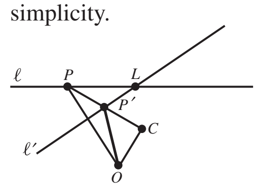
Finally, we specify that the extended map σ maps the line through O that is parallel to the line of intersection of π and π' onto itself.

In this way, we have constructed a transformation that is a one-one mapping of $\pi \cup \{\text{the ideal Points for } \pi\}$ onto $\pi' \cup \left\{\text{the ideal Points for } \pi'\right\}$ associated in a natural way with the given perspectivity σ, and we call it the associated **perspective transformation**. This maps the family of Euclidean lines through O onto itself, in other words $\mathbb{RP}^2$ onto itself.

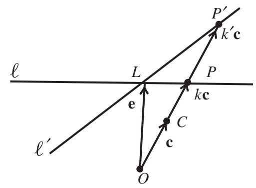
We now explain why we can think of this perspective transformation as a projective transformation.

First, we consider a perspectivity in $\mathbb{R}^2$ as this will prove useful later in our discussion. So, let ℓ and ℓ' be two lines in $\mathbb{R}^2$ that do not pass through the origin O in $\mathbb{R}^2$, let L be the common point of the two lines, and let $C (\neq O)$ be another point in $\mathbb{R}^2$ such that OC is not parallel to either ℓ or ℓ'. Consider any perspectivity σ, with centre some point C, say, that maps ℓ 'onto' ℓ'.

In our discussion we shall omit discussion of 'the exceptional points', for simplicity.



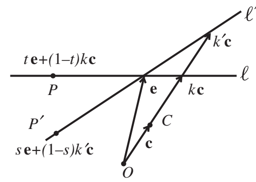
We are interested in the map τ that sends lines through O to lines through O that is obtained from σ as follows. Let σ map the point P on ℓ onto the point P' on ℓ'. Then we define $\tau(OP) = OP'$.



It is clear that $\tau(OL) = OL$, because L is fixed by σ.

It is also clear that $\tau(OC) = OC$, because if the line OC meets ℓ at P and the line ℓ' at P' then O, C, P and P' are collinear and so $\tau(OC) = OC$.

We therefore choose to take as basis vectors in $\mathbb{R}^2$ the vector $\mathbf{e} = \overrightarrow{OL}$ and the vector $\mathbf{c} = \overrightarrow{OC}$.



We now find the effect of the map τ on a line through the origin O. We shall suppose that the line OC meets the line ℓ at the point with position vector $k\mathbf{c}$ and the line ℓ' at the point with position vector $k'\mathbf{c}$. Any point P on the line ℓ then has position vector $t\mathbf{e} + (1-t)k\mathbf{c}$, for some real number t, and any point P' on the line ℓ' has position vector $s\mathbf{e} + (1-s)k'\mathbf{c}$, for some real number s.

Next, the line OP consists of the points with coordinates $u(\mathbf{e} + m\mathbf{c}) = u\mathbf{e} + mu\mathbf{c}$ for a fixed value of m and varying values of u. This gives two (equivalent) expressions for the position of the point P, namely $t\mathbf{e}+(1-t)k\mathbf{c}$ and $u\mathbf{e}+mu\mathbf{c}$. It follows that we must have

$$u = t \quad \text{and} \quad mu = (1-t)k.$$

Dividing the second equation by the first, we get

$$m = \frac{1-t}{t}k$$

so that

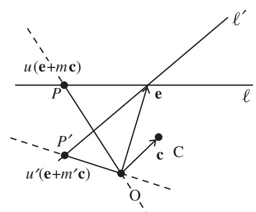
$$tm = k - kt,$$

which yields the formula

$$t = \frac{k}{m+k}.$$

Similarly, the line OP' consisting of points $u'(\mathbf{e} + m'\mathbf{c}) = u'\mathbf{e} + m'u'\mathbf{c}$, for a fixed value of m' and varying values of u', meets the line ℓ' at the point P' where

$$s = \frac{k'}{m'+k'}.$$



Now, we have $\tau(OP) = OP'$ if and only if the points C, P and P' are collinear; that is, if and only if there is a real number r such that

$$r(t\mathbf{e} + (1-t)k\mathbf{c} - \mathbf{c}) = s\mathbf{e} + (1-s)k'\mathbf{c} - \mathbf{c}.$$

This is the case if and only if

$$rt = s \quad \text{and} \quad r((1-t)k - 1) = (1-s)k' - 1.$$

We can eliminate r by dividing the second equation by the first equation, so obtaining

$$\frac{(1-t)k-1}{t} = \frac{(1-s)k'-1}{s}.$$

Then, if we substitute $t = \frac{k}{m+k}$ and $s = \frac{k'}{m'+k'}$ into this equation, after some manipulation we obtain the remarkable result that

$$m' = m\frac{(k-1)k'}{(k'-1)k}.$$

It follows that, in terms of the basis elements $\mathbf{e}$ and $\mathbf{c}$, the map τ of the family of lines through O to itself given by

$$\left[\begin{pmatrix}1\\m\end{pmatrix}\right] \mapsto \left[\begin{pmatrix}1\\m'\end{pmatrix}\right]$$

can be represented by the matrix

$$\begin{pmatrix}1 & 0\\ 0 & \frac{(k-1)k'}{(k'-1)k}\end{pmatrix}.$$

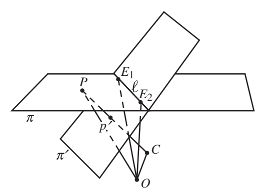
For convenience, we write this matrix in the form

$$\begin{pmatrix}1 & 0\\ 0 & r\end{pmatrix},$$

where r is a fixed number that depends only on the geometry of the two lines ℓ and ℓ' and on our choice of the point C. Furthermore, $r \neq 0$ since $k \neq 1$ (for $k = 1$ implies that C lies on π) and $k' \neq 0$ (for $k' = 0$ implies that O lies on π').

Now we consider the situation in $\mathbb{R}^3$, in relation to $\mathbb{RP}^2$. Consider an arbitrary perspectivity σ, with centre some point C, say, that maps a plane π 'onto' a plane π', with neither O nor C lying on π or π'. As before, we shall suppose that the line OC meets the plane π at a point with position vector $k\mathbf{c}$ and the plane π' at a point with position vector $k'\mathbf{c}$.

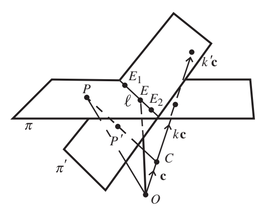
Once again, we are interested in the map τ that sends lines (in $\mathbb{R}^3$) through O to lines through O that is obtained from σ as follows. Let σ map a point P in π to a point P' in π'. Then we define $\tau(OP) = OP'$.



Let E_1 and E_2 be any two points on the common line ℓ of the two planes. Then, clearly, $\tau(OE_1) = OE_1$ and $\tau(OE_2) = OE_2$ because every point of ℓ is fixed by σ.

It is also clear that $\tau(OC) = OC$, because if the line OC meets the plane π at Q and the plane π' at Q', then O, C, Q and Q' are collinear, and so $\tau(OC) = OC$.

We therefore now take as basis vectors in $\mathbb{R}^3$ the vectors $\mathbf{e}_1 = \overrightarrow{OE_1}$, $\mathbf{e}_2 = \overrightarrow{OE_2}$, and $\mathbf{c} = \overrightarrow{OC}$.



We now find the effect of τ on a line in $\mathbb{R}^3$ through the origin O. We can simplify our task by observing that the lines OC, OP and OP' all lie in a plane, π'' say; let this plane meet the line common to the given planes π and π' at a point E. We then define $\mathbf{e} = \overrightarrow{OE}$.

We can now apply our earlier discussion of the planar case to the restriction of the mapping τ to the plane π''. If we denote the lines OP and OP' by the parametrizations

$$OP = u(\mathbf{e} + m\mathbf{c}) = u\mathbf{e} + um\mathbf{c}, \quad \text{where } m \text{ is fixed and } u \text{ varies,}$$

$$OP' = u'(\mathbf{e} + m'\mathbf{c}) = u'\mathbf{e} + u'm'\mathbf{c}, \quad \text{where } m' \text{ is fixed and } u' \text{ varies,}$$

it follows from our earlier discussion that m' can be expressed in the form rm, where r is a fixed number that depends only on the geometry of the two planes π and π' and on our choice of the point C. Hence we can rewrite the parametrization of OP' as

$$OP' = u'(\mathbf{e} + rm\mathbf{c}) = u'\mathbf{e} + u'rm\mathbf{c}, \quad \text{where } r,\ m \text{ are fixed and } u' \text{ varies.}$$

Next, we can express the position vector $\mathbf{e}$ of the point E in the form

$$\mathbf{e} = t\mathbf{e}_1 + (1-t)\mathbf{e}_2, \quad \text{for some number } t.$$

Then we have that the mapping τ maps the line

$$OP = u(t\mathbf{e}_1 + (1-t)\mathbf{e}_2 + m\mathbf{c}) = ut\mathbf{e}_1 + u(1-t)\mathbf{e}_2 + um\mathbf{c}$$

onto the line

$$OP' = u'(t\mathbf{e}_1 + (1-t)\mathbf{e}_2 + rm\mathbf{c}) = u't\mathbf{e}_1 + u'(1-t)\mathbf{e}_2 + u'rm\mathbf{c}.$$

It follows analogously to the two-dimensional situation that, in terms of the basis elements $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{c}$, the mapping τ of the family of lines in $\mathbb{R}^3$ to itself can be represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{pmatrix},$$

Notice that this maps the line represented by the points $[t, 1-t, 0]$ to itself pointwise, as it should.

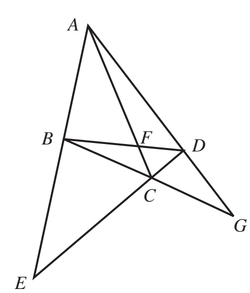
where the (non-zero) constant r depends only on the position of the planes π and π'. Since this is an invertible 3×3 matrix, this is our required description of the perspective transformation as a projective transformation.

We now go the other way round, and obtain a projective transformation as a sequence of three perspective transformations. We have a lot of freedom, because we can choose the centres of perspectivities and the planes.

Suppose we are given a projective transformation τ and two embedding planes π and π'. Let $[\mathbf{a}],\ [\mathbf{b}],\ [\mathbf{c}],\ [\mathbf{d}]$ be any four non-collinear Points, and $[\mathbf{a}'],\ [\mathbf{b}'],\ [\mathbf{c}'],\ [\mathbf{d}']$ be any other four non-collinear Points; we can represent these Points by (Euclidean) points A, B, C, D in π and $A',\ B',\ C',\ D'$ in π', respectively.

We will use in our discussion the existence and uniqueness parts of the Fundamental Theorem of Projective Geometry: namely, that there is one and only one projective transformation mapping four non-collinear Points to any other four non-collinear Points. This means that if we can find a composite of three perspective transformations that maps $[\mathbf{a}],\ [\mathbf{b}],\ [\mathbf{c}],\ [\mathbf{d}]$ to $[\mathbf{a}'],\ [\mathbf{b}'],\ [\mathbf{c}'],\ [\mathbf{d}']$, respectively, then this composite must be a projective transformation (by the existence part) and it must equal the given projective transformation τ (by the uniqueness part).

Subsection 3.3.3, Theorem 3



We now exhibit a sequence of three perspectivities the composite of which maps A, B, C, D in π to $A',\ B',\ C',\ D'$ in π'. In our discussion it is convenient to let AB and CD meet at E, AC and BD meet at F, and AD and BC meet at G, with analogous definitions of $E',\ F'$ and G'. (You may find the figures below helpful to follow through the argument; though, for simplicity, we have omitted the initial plane π and the points F', F'' and F'''.)

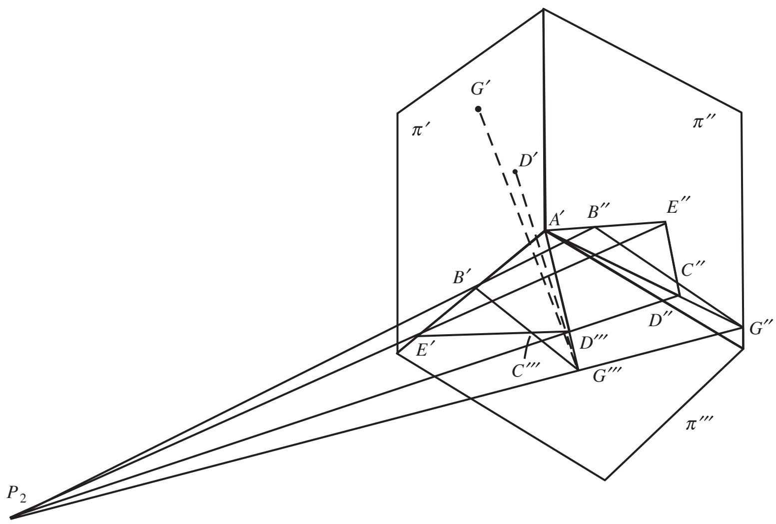
1. The first perspectivity is from the plane π to a plane π'' that passes through A'. The centre of this perspectivity is an arbitrary point P_1 on the line AA' (if $A = A'$, then P_1 can be chosen to lie anywhere not on π or π'). This perspectivity maps A to A' and B, C, D, E, F, G to B'', C'', D'', E'', F'', G'', say, respectively. We can assume, by suitably varying π'', that $B'B''$ and $E'E''$ are not parallel.

$A \mapsto A'$

$B \mapsto B''$, $C \mapsto C''$,

$D \mapsto D''$, $E \mapsto E''$,

$F \mapsto F''$, $G \mapsto G''$



2. Now, by the definition of E' the points A', B', E' lie on a line through A', and since the points A, B, E are collinear the points A', B'', E'' lie on a line through A'; so these five points lie in a plane, and because the lines $B'B''$ and $E'E''$ are not parallel they meet in a point, P_2 say. We then pick a plane π''' through the line $A'B'E'$ and map the plane π'' onto π''' by the perspectivity with centre P_2. This sends A', B'', E'' to A', B', E', respectively, and the points C'', D'', F'', G'' to, say, C''', D''', F''', G''', respectively. As before, we can assume that $D'D'''$ and $G'G'''$ are not parallel.

A' fixed;
$B'' \mapsto B'$, $E'' \mapsto E'$

$C'' \mapsto C'''$, $D'' \mapsto D'''$,

$F'' \mapsto F'''$, $G'' \mapsto G'''$

3. Now, since A, D, G are collinear, the points A', D', G' lie on a line through A' and the points A', D''', G''' lie on a line through A', so these five points lie in a plane; and because the lines $D'D'''$ and $G'G'''$ are not parallel they meet in a point, P_3 say. We then map the plane π''' onto the plane π' by the perspectivity with centre P_3. This map sends A', B', D''', E''', G''' to A', B', D', E', G', respectively.

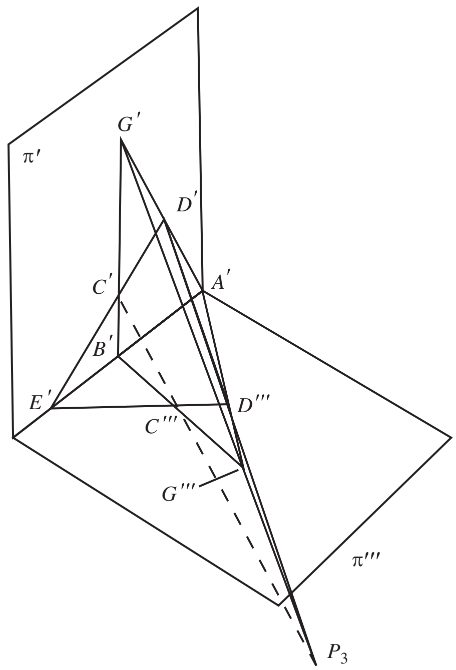
   This third perspectivity sends the line $B'G'''$ to the line $B'G'$ and the line $D'''E'$ to the line $D'E'$, so it maps the point C''' to the point C'.

Refer here to the figure below.

A', B', E' fixed

$D''' \mapsto D'$, $G''' \mapsto G'$

Thus the composite of these three perspectivities maps the points A, B, C, D in π to A', B', C', D' in π', as required.



Theorem 4 Perspectivity Theorem
Every projective transformation can be expressed as the composite of three perspective transformations.

3.4 Using the Fundamental Theorem of Projective Geometry

In Section 3.2 we described how an embedding plane π can be used to represent projective space $\mathbb{RP}^2$. The Points of $\mathbb{RP}^2$ are represented by Euclidean points in π and the Lines of $\mathbb{RP}^2$ are represented by Euclidean lines in π.

In general, any Euclidean figure in an embedding plane corresponds to a projective figure in $\mathbb{RP}^2$, and visa versa. This correspondence enables us to compare Euclidean theorems about a figure in an embedding plane with projective theorems about the corresponding projective figure. Provided that the theorems are concerned exclusively with projective properties, such as collinearity and incidence, then a Euclidean theorem will hold if and only if the corresponding projective theorem holds.

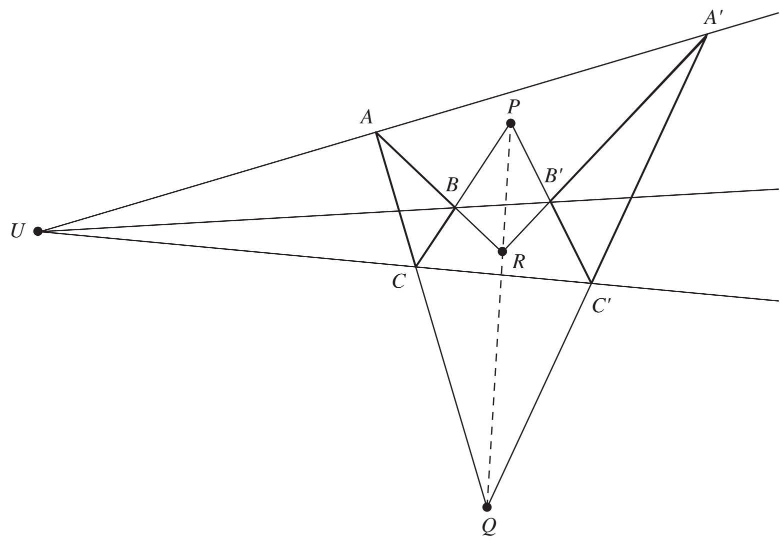
The Euclidean figure may have some ideal Points attached to it.

3.4.1 Desargues' Theorem and Pappus' Theorem

The advantage of interpreting a Euclidean theorem as a projective theorem in this way is that we can often obtain a much simpler proof of the theorem than would be possible using Euclidean geometry directly. We illustrate this by using projective geometry to prove the theorem of Desargues.

We introduced Desargues' Theorem in Subsection 3.1.3.

Theorem 1 Desargues' Theorem
Let ΔABC and $\Delta A'B'C'$ be triangles in $\mathbb{R}^2$ such that the lines AA', BB' and CC' meet at a point U. Let BC and $B'C'$ meet at P, CA and $C'A'$ meet at Q, and AB and $A'B'$ meet at R. Then P, Q and R are collinear.



Proof Because this theorem is concerned exclusively with the projective properties of collinearity and incidence we can interpret it as a projective theorem in $\mathbb{RP}^2$. Moreover, by the Fundamental Theorem of Projective Geometry we know that any configuration of the theorem is projective-congruent to a configuration of the theorem in which $A = [1, 0, 0]$, $B = [0, 1, 0]$, $C = [0, 0, 1]$ and $U = [1, 1, 1]$. If we can prove the theorem in this special case then we can use the fact that projective-congruence preserves projective properties to deduce that the theorem holds in general.

To prove the special case we use the algebraic techniques described in Section 3.2. First observe that the Line AU passes through the Points [1, 0, 0] and [1, 1, 1], and therefore has equation $y = z$. Since A' is a Point on AU, it must have homogeneous coordinates of the form $[a, b, b]$, for some real numbers a and b. Now, $b \neq 0$, since $A \neq A'$; so we may write the homogeneous coordinates of A' in the form $[p, 1, 1]$ (where $p = a/b$).

For $[a, 0, 0] = [1, 0, 0]$.

Similarly, the homogeneous coordinates of the Points B' and C' may be written in the form $[1, q, 1]$ and $[1, 1, r]$, respectively, for some real numbers q and r.

We omit the details of the calculations.

We now find the Point P where BC and $B'C'$ intersect. The Line BC has equation $x = 0$. Since the Line $B'C'$ passes through the Points $B' = [1, q, 1]$ and $C' = [1, 1, r]$, it must have equation

$$\begin{vmatrix} x & y & z \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0,$$

which we may rewrite in the form

$$(qr-1)x-(r-1)y+(1-q)z=0.$$

It follows that at the Point P of intersection of the Lines BC and $B'C'$ we must have $x=0$ and $(r-1)y=(1-q)z$, so that P has homogeneous coordinates $[0,1-q,r-1]$.

Similarly, the Points Q and R have homogeneous coordinates $[1-p, 0, r-1]$ and $[1-p, q-1, 0]$, respectively.

We omit the details of the calculations.

Now, the Points P, Q and R are collinear if

$$\begin{vmatrix} 0 & 1-q & r-1 \\ 1-p & 0 & r-1 \\ 1-p & q-1 & 0 \end{vmatrix} = 0.$$

But

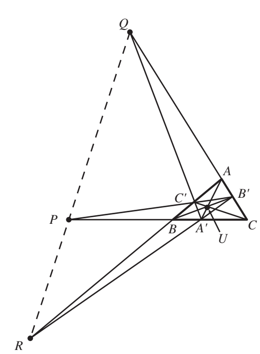
$$\begin{aligned}
&\begin{vmatrix} 0 & 1-q & r-1 \\ 1-p & 0 & r-1 \\ 1-p & q-1 & 0 \end{vmatrix} \\
&\quad = -(1-q)\begin{vmatrix} 1-p & r-1 \\ 1-p & 0 \end{vmatrix} + (r-1)\begin{vmatrix} 1-p & 0 \\ 1-p & q-1 \end{vmatrix} \\
&\quad = -(1-q)(1-p)(1-r)+(r-1)(1-p)(q-1) \\
&\quad = 0.
\end{aligned}$$

It follows that P, Q and R are collinear, as asserted. The general result now holds, by projective-congruence. ■

When using the Fundamental Theorem to simplify proofs of results in projective geometry, we do not usually refer to projective-congruence. Instead, *so long as the properties involved are projective properties*, we content ourselves with an initial remark of the type: 'By the Fundamental Theorem of Projective Geometry, we may choose the four Points..., no three of which are collinear, to be the triangle of reference and the unit Point; that is, to have homogeneous coordinates [1, 0, 0], [0, 1, 0], [0, 0, 1] and [1, 1, 1], respectively'.

Problem 1 Let ΔABC be a triangle in $\mathbb{R}^2$, and let U be any point of $\mathbb{R}^2$ that is not collinear with any two of the points A, B and C. Let the lines AU, BU and CU meet the lines BC, CA and AB at the points A', B' and C', respectively. Next, let the lines BC and $B'C'$ meet at P, AC and $A'C'$ meet at Q, and AB and $A'B'$ meet at R. Prove that P, Q and R are collinear.

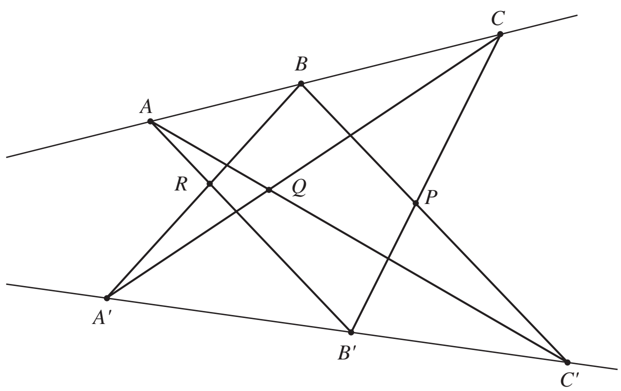
Hint: Let A, B, C be the vertices of the triangle of reference, and let U be the unit Point. Then determine the homogeneous coordinates of the Points A', B' and C'.



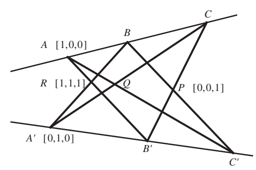
Next we use the Fundamental Theorem of Projective Geometry to prove Pappus' Theorem.

Theorem 2 Pappus' Theorem
Let A, B and C be three points on a line in $\mathbb{R}^2$, and let A', B' and C' be three points on another line. Let BC' and $B'C$ meet at P, CA' and $C'A$ meet at Q, and AB' and $A'B$ meet at R. Then P, Q, R are collinear.

This theorem is named after Pappus, a Greek mathematician who discovered it in the 3rd century AD.



Proof We interpret the theorem as a projective theorem, so: by the Fundamental Theorem of Projective Geometry we may choose the four Points A, A', P, R, no three of which are collinear, to be the triangle of reference and the unit Point; that is, to have homogeneous coordinates [1, 0, 0], [0, 1, 0], [0, 0, 1] and [1, 1, 1], respectively.



First observe that the Line AR passes through the Points [1, 0, 0] and [1, 1, 1], and must therefore have equation $y = z$. Since B' is a Point on AR, it must have homogeneous coordinates of the form $[a, b, b]$ for some real numbers a and b. Now, $b \neq 0$ since $A \neq B'$, so we may write the homogeneous coordinates of B' in the form $[r, 1, 1]$ (where $r = a/b$).

Similarly, the Point B lies on the Line $x = z$ through the Points $A' = [0, 1, 0]$ and $R = [1, 1, 1]$, so it must have homogeneous coordinates of the form $[1, s, 1]$.

Next we find the Point C where the Line AB intersects the Line $B'P$. Since the Line AB passes through the Points $A = [1, 0, 0]$ and $B = [1, s, 1]$, it must have equation $y = sz$. Also since the Line $B'P$ passes through the Points $B' = [r, 1, 1]$ and $P = [0, 0, 1]$ it must have equation $x = ry$. At the Point C where AB meets $B'P$ we have $y = sz$ and $x = ry$, so $C = [rs, s, 1]$.

Similarly, C' is the point where the Line BP intersects the Line $A'B'$. Since $B = [1, s, 1]$ and $P = [0, 0, 1]$, BP has equation $y = sx$; and, since $A' = [0, 1, 0]$ and $B' = [r, 1, 1]$, $A'B'$ has equation $x = rz$. It follows that $C' = [r, rs, 1]$.

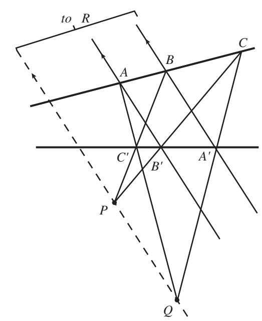
Finally we find the point Q where AC' intersects $A'C$. Since the Line AC' passes through the Points $A = [1, 0, 0]$ and $C' = [r, rs, 1]$ it must have equation $y = rsz$. Also the Line $A'C$ passes through the Points $A' = [0, 1, 0]$ and $C = [rs, s, 1]$ so it must have equation $x = rsz$. At the Point Q where AC' intersects $A'C$ we have $y = rsz$ and $x = rsz$, so $Q = [rs, rs, 1]$.

To complete the proof we simply observe that the Points $R = [1, 1, 1]$, $Q = [rs, rs, 1]$ and $P = [0, 0, 1]$ all lie on the Line $x = y$. It follows that P, Q and R are collinear. ■

A fortunate choice of Points for the triangle of reference and unit Point meant that we did not have to use the determinant criterion for collinearity at the final stage of the argument.

Although we can sometimes simplify the proof of a Euclidean theorem by using projective geometry, there is another more subtle reason for interpreting a Euclidean theorem as a projective theorem. By doing so we can often avoid having to make special provision for exceptional cases, such as when two lines are parallel. In projective geometry, Lines which correspond to a pair of parallel lines in an embedding plane actually meet and are therefore no different to any other Lines.

As an example, consider the diagram in the margin. This illustrates the situation that occurs in Pappus' Theorem when the Point of intersection R of $A'B$ and AB' is an ideal Point for the embedding plane. The above proof of Pappus' Theorem is able to cope with this situation because it uses arguments from $\mathbb{RP}^2$! Our interpretation of the theorem on an embedding plane in this situation is that the Points P and Q must be collinear with the ideal Point R at which $A'B$ and AB' meet. That is, PQ must be parallel to both $A'B$ and AB'.

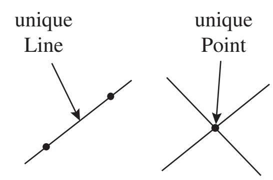


Problem 2 Give a Euclidean interpretation of Desargues' Theorem on an embedding plane π in the case where Q is an ideal Point for π.

3.4.2 Duality

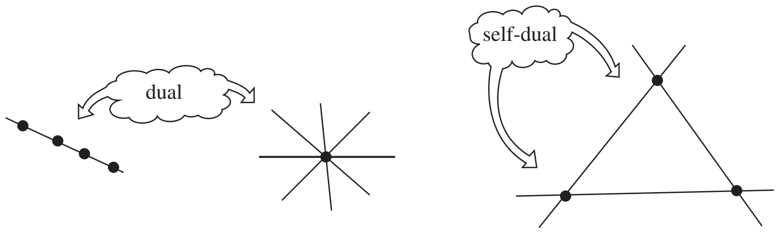
Recall that two key projective properties that we have met so far have a certain symmetry between them.

Collinearity Property	Incidence Property
Any two distinct *Points* lie on a unique *Line.*	Any two distinct *Lines* meet in a unique *Point.*

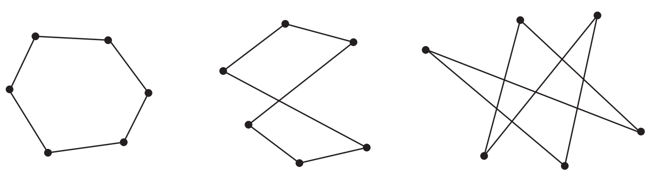


We can obtain one property from the other simply by interchanging the words 'Point' and 'Line', and making whatever other changes are needed to ensure that the sentence makes sense. We say that this interchanging process *dualizes* one statement into the other, and that each statement is the *dual* of the other.

For example, 'a family of Points on a Line' becomes 'a family of Lines through a Point' under dualization. Similarly, 'a triangle' or 'a family of three non-collinear Points and the three Lines joining them' dualizes to 'a family of three non-concurrent Lines and the three Points where they meet', which is again a triangle. Since a triangle is thus dual to a triangle, we say that triangles are *self-dual* figures.



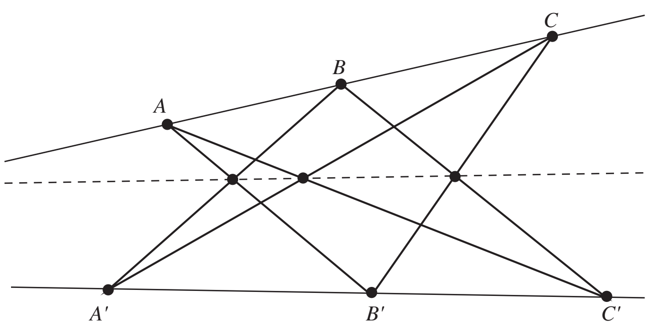
The dualization process is particularly interesting when applied to theorems. We shall illustrate this in the context of Pappus' Theorem. In order to do this, it is helpful to rephrase Pappus' Theorem using the term *hexagon.* As you would expect, a **hexagon** in $\mathbb{RP}^2$ consists of six Points joined by six Lines. The figure below illustrates (Euclidean) hexagons in an embedding plane; the corresponding hexagons in $\mathbb{RP}^2$ are the corresponding six Points and six Lines — that we model as six lines and six planes in $\mathbb{R}^3$.



We can now rephrase Pappus' Theorem in the following form.

Theorem 3 Pappus' Theorem (rephrased)
Let the vertices A, B', C, A', B and C' of a hexagon lie alternately on two different Lines. Then the Points of intersection of opposite sides B'C and BC', CA' and $C'A$, AB' and $A'B$, are collinear.

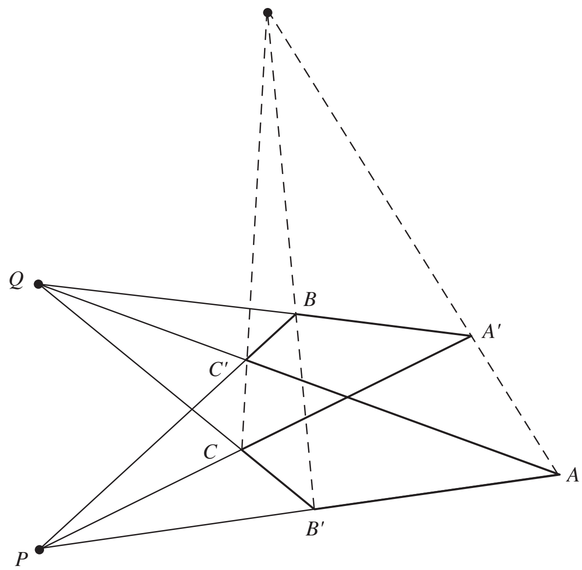
You should compare this formulation with that in Subsection 3.4.1.



If we dualize this theorem, we obtain the following theorem.

Theorem 4 Brianchon's Theorem
Let the sides $AB', B'C, CA', A'B, BC', C'A$ of a hexagon pass alternately through two (different) Points P and Q in $\mathbb{RP}^2$. Then the Lines joining opposite vertices A and A$'$, B and B', C and C', are concurrent.

Charles J. Brianchon (1785–1864) was one of many distinguished French geometers who studied under Gaspard Monge (1746–1818).



Problem 3 Prove Brianchon's Theorem.
Hint: Let P, C, Q, C' be the Points [1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1], respectively.

It turns out that the dual of any true statement concerning Points, Lines and their projective properties remains true after dualization; that is, if we dualize any theorem in projective geometry, then the statement that we obtain is itself a theorem.

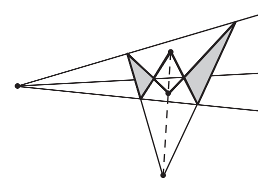
We do not prove this assertion, as it would take us beyond the scope of this book.

Problem 4 Earlier you saw that 'three Points $[a, b, c]$, $[d, e, f]$, $[g, h, k]$ are collinear if and only if $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = 0'$. Write down the dual result of this statement.

Subsection 3.2.2, Strategy

We end this subsection by forming the dual of Desargues' Theorem, as follows.

Desargues' Theorem	Dual Theorem
Let two *triangles* be such that the *Lines* joining corresponding *vertices* meet at a *Point*. Then the *Points* of intersection of the corresponding *sides* of the two *triangles* are *collinear*.	Let two *triangles* be such that the *Points* through which corresponding *sides* pass are *collinear*. Then the *Lines* through the corresponding *vertices* of the two *triangles* are *concurrent*.

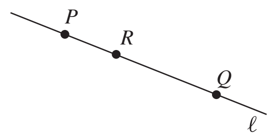


Note that the dual theorem is simply the converse result for Desargues' Theorem! Thus the Principle of Duality enables us to deduce that the converse of Desargues' Theorem holds.

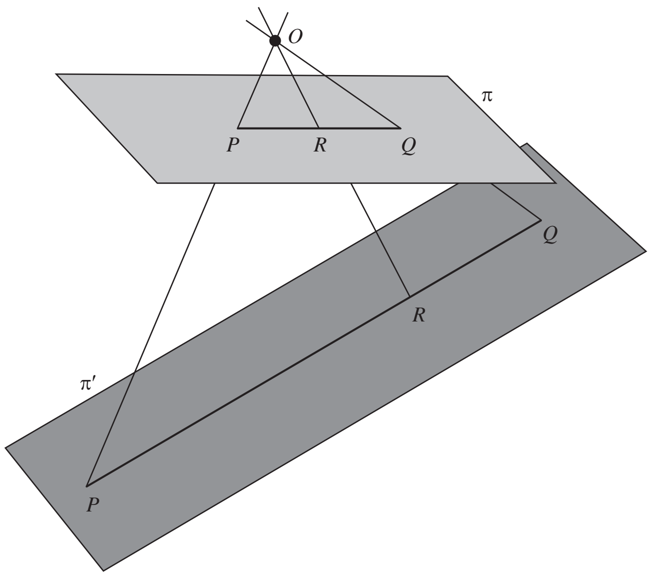
3.5 Cross-Ratio

3.5.1 Another Projective Property

Earlier, in Subsection 2.2.1, we noted that *ratio of lengths along a line* is an affine property. Thus, in affine geometry, if we are given two points P and Q on a line ℓ, then we can locate the position of a third point R along ℓ by specifying the ratio $PR : RQ$. In particular, it is possible to talk about the point midway between P and Q.

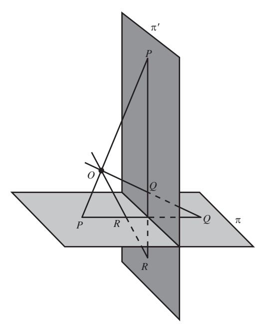


In projective geometry it is meaningless to talk about the Point midway between two other Points. In one embedding plane π a Point R may appear to be midway between the Points P and Q, whereas in another embedding plane π' the ratio $PR : RQ$ may be very different.



This ambiguity arises from the fact that perspectivities do not preserve the ratio of lengths along a line, so: *ratio of lengths along a line* is not a projective property.

In some embedding planes, such as the plane π' illustrated in the margin, the Point R does not even appear to lie between P and Q, so *betweenness* is not a projective property either!



Fortunately, there is a quantity, known as *cross-ratio,* that is preserved under all projective transformations. To see how this is defined, consider four collinear Points $A = [\mathbf{a}]$, $B = [\mathbf{b}]$, $C = [\mathbf{c}]$, $D = [\mathbf{d}]$ in $\mathbb{RP}^2$. We can express the fact that A, B, C, D are collinear by writing $\mathbf{c}$ and $\mathbf{d}$ as linear combinations of $\mathbf{a}$ and $\mathbf{b}$. Thus we can write

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{and} \quad \mathbf{d} = \gamma\mathbf{a} + \delta\mathbf{b},$$

for suitable real numbers $\alpha, \beta, \gamma, \delta$.

The cross-ratio is then defined to be the ratio of the ratios $\frac{\beta}{\alpha}$ and $\frac{\delta}{\gamma}$.

Definition Let A, B, C, D be four collinear Points in $\mathbb{RP}^2$ represented by position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and let

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{and} \quad \mathbf{d} = \gamma\mathbf{a} + \delta\mathbf{b}.$$

Then the **cross-ratio** of A, B, C, D is

$$(ABCD) = \frac{\beta}{\alpha} \Big/ \frac{\delta}{\gamma}.$$

Equivalently, we can write $(ABCD) = \frac{\beta\gamma}{\alpha\delta}$.

Of course, before we can be sure that this definition makes sense, we must ensure that it does not depend on the particular choice of position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ that are used to represent the Points A, B, C, D. We shall check this shortly, but first we illustrate how cross-ratios are calculated.

Example 1 Let $A = [1, 2, 3]$, $B = [1, 1, 2]$, $C = [3, 5, 8]$, $D = [1, -1, 0]$ be Points of $\mathbb{RP}^2$. Calculate the cross-ratio *(ABCD)*.

Solution First, we have to find real numbers α and β such that the following vector equation holds:

$$(3, 5, 8) = \alpha(1, 2, 3) + \beta(1, 1, 2).$$

Note that we have not verified that A, B, C, D are collinear; but if they were not, the equations for $\alpha, \beta, \gamma, \delta$ could not be solved.

Comparing corresponding coordinates on both sides of this vector equation, we deduce that

$$3 = \alpha + \beta, \quad 5 = 2\alpha + \beta \quad \text{and} \quad 8 = 3\alpha + 2\beta.$$

Solving these equations gives $\alpha = 2$, $\beta = 1$.

Next, we find real numbers γ and δ such that the vector equation

$$(1, -1, 0) = \gamma(1, 2, 3) + \delta(1, 1, 2)$$

holds. Comparing corresponding coordinates on both sides of this vector equation, we deduce that

$$1 = \gamma + \delta, \quad -1 = 2\gamma + \delta \quad \text{and} \quad 0 = 3\gamma + 2\delta.$$

Solving these equations gives $\gamma = -2$, $\delta = 3$.

It follows from the definition of cross-ratio that

$$(ABCD) = \frac{\beta}{\alpha} \Big/ \frac{\delta}{\gamma} = \frac{1}{2} \Big/ \frac{3}{-2} = -\frac{1}{3}.$$

□

Problem 1 Calculate the cross-ratio *(ABCD)* for each of the following sets of collinear Points in $\mathbb{RP}^2$.

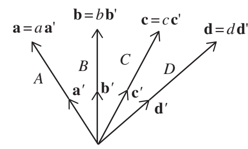
(a) $A = [1, -1, -1]$, $B = [1, 3, -2]$, $C = [3, 5, -5]$, $D = [1, -5, 0]$
(b) $A = [1, 2, 3]$, $B = [2, 2, 4]$, $C = [-3, -5, -8]$, $D = [3, -3, 0]$

You may have noticed that the Points A, B, C, D in Problem 1 (b) are the same as those which appear in Example 1. The only difference is that different homogeneous coordinates are used to represent the Points in each case. As we mentioned after the definition of cross-ratio, the value of the cross-ratio *(ABCD)* does not depend on the homogeneous coordinates that are used to represent A, B, C, D, so it is not surprising that the cross-ratio turned out to have the value $-\frac{1}{3}$ in both cases.

Theorem 1 The cross-ratio *(ABCD)* is independent of the homogeneous coordinates that are used to represent the collinear Points A, B, C, D.

Proof Suppose that $A = [\mathbf{a}]$, $B = [\mathbf{b}]$, $C = [\mathbf{c}]$, $D = [\mathbf{d}]$, and let

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{and} \quad \mathbf{d} = \gamma\mathbf{a} + \delta\mathbf{b}. \tag{1}$$



Now suppose that $A = [\mathbf{a}']$, $B = [\mathbf{b}']$, $C = [\mathbf{c}']$, $D = [\mathbf{d}']$. Then

$$\mathbf{a} = a\mathbf{a}', \quad \mathbf{b} = b\mathbf{b}', \quad \mathbf{c} = c\mathbf{c}', \quad \mathbf{d} = d\mathbf{d}',$$

where a, b, c, d are some non-zero real numbers.

By substituting these expressions into the equations (1), we obtain

$$c\mathbf{c}' = \alpha a\mathbf{a}' + \beta b\mathbf{b}' \quad \text{and} \quad d\mathbf{d}' = \gamma a\mathbf{a}' + \delta b\mathbf{b}',$$

which we can rewrite in the form

$$\mathbf{c}' = \alpha'\mathbf{a}' + \beta'\mathbf{b}' \quad \text{and} \quad \mathbf{d}' = \gamma'\mathbf{a}' + \delta'\mathbf{b}', \tag{2}$$

where $\alpha' = \alpha a/c, \beta' = \beta b/c, \gamma' = \gamma a/d, \delta' = \delta b/d$.

We can now check that equations (1) and (2) yield the same value for the cross-ratio:

$$\begin{aligned}\frac{\beta'}{\alpha'}\bigg/\frac{\delta'}{\gamma'} &= \frac{\beta b/c}{\alpha a/c}\bigg/\frac{\delta b/d}{\gamma a/d}\\ &= \frac{\beta b}{\alpha a}\bigg/\frac{\delta b}{\gamma a}\\ &= \frac{\beta}{\alpha}\bigg/\frac{\delta}{\gamma}.\end{aligned}$$

So, as expected, the cross-ratio is independent of the choice of homogeneous coordinates. ■

The next problem illustrates that although the value of the cross-ratio *(ABCD)* is independent of the choice of homogeneous coordinates that are used to represent A, B, C, D, the value of the cross-ratio does depend on the order in which the Points A, B, C, D appear.

Problem 2 Calculate the cross-ratios *(BACD)* and *(ACBD)* for the four Points used in Problem 1(a).

When answering Problem 2 you may have noticed that *(BACD)* is the reciprocal of the value which we obtained for *(ABCD)* in Problem l(a). Also, *(ACBD)* is equal to $1 -$ *(ABCD)*. The next result shows that this is not simply chance!

Note the way in which the Points are changed from their original ordering in the various cross-ratios. We take the reciprocal when swapping the first or last pair of Points, and we subtract from 1 when swapping the inner or outer pair of Points.

Theorem 2 Let A, B, C, D be four distinct collinear Points in $\mathbb{RP}^2$, and let $(ABCD) = k$. Then

$$(BACD) = (ABDC) = 1/k,$$
$$(ACBD) = (DBCA) = 1 - k.$$

Proof Let **a**, **b**, **c**, **d** be any position vectors in $\mathbb{R}^3$ in the directions of the Points A, B, C, D, respectively, of $\mathbb{RP}^2$, and let $\alpha, \beta, \gamma, \delta$ be real numbers such that

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{and} \quad \mathbf{d} = \gamma\mathbf{a} + \delta\mathbf{b}.$$

Then, by definition of cross-ratio, the cross-ratio $(ABCD)$ of the four Points A, B, C, D is the quantity

$$(ABCD) = \frac{\beta}{\alpha} \Big/ \frac{\delta}{\gamma} = \frac{\beta\gamma}{\alpha\delta} = k, \text{say}.$$

To determine (*BACD*), we interchange the roles of A and B in the evaluation of *ABCD* above; it follows that, since

$$\mathbf{c} = \beta\mathbf{b} + \alpha\mathbf{a} \quad \text{and} \quad \mathbf{d} = \delta\mathbf{d} + \gamma\mathbf{a},$$

the cross-ratio (*BACD*) is the quantity

$$(BACD) = \frac{\alpha}{\beta} \Big/ \frac{\gamma}{\delta} = \frac{\alpha\delta}{\beta\gamma} = \frac{1}{k}.$$

To determine (*ABDC*), we interchange the roles of C and D in the evaluation of (*ABCD*) above; it follows that, since

$$\mathbf{d} = \gamma\mathbf{a} + \delta\,\mathbf{b} \quad \text{and} \quad \mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b},$$

the cross-ratio (*ABDC*) is the quantity

$$(ABDC) = \frac{\delta}{\gamma} \Big/ \frac{\beta}{\alpha} = \frac{\alpha\delta}{\beta\gamma} = \frac{1}{k}.$$

This completes the first part of the proof.

To evaluate (*ACBD*), we use the equations

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{and} \quad \mathbf{d} = \gamma\mathbf{a} + \delta\mathbf{b} \tag{3}$$

to express **b** and **d** in terms of **a** and **c**, as follows.

From the first equation in (3) we have

$$\begin{aligned}\mathbf{b} &= (\mathbf{c} - \alpha\mathbf{a})/\beta \\ &= (-\alpha/\beta)\mathbf{a} + (1/\beta)\mathbf{c}. \qquad (4)\end{aligned}$$

β cannot be zero, for if it were then we would have $\mathbf{c} = \alpha\mathbf{a}$; this cannot happen since A and C are distinct Points.

If we then substitute this expression for $\mathbf{b}$ into the second equation in (3), we obtain

$$\begin{aligned}\mathbf{d} &= \gamma\mathbf{a} + \delta((-\alpha/\beta)\mathbf{a} + (1/\beta)\mathbf{c}) \\ &= ((\beta\gamma - \alpha\delta)/\beta)\mathbf{a} + (\delta/\beta)\mathbf{c}. \qquad (5)\end{aligned}$$

It follows from the coefficients of $\mathbf{a}$ and $\mathbf{c}$ in equations (4) and (5) that

$$\begin{aligned}(ACBD) &= \frac{1/\beta}{-\alpha/\beta}\Bigg/\frac{\delta/\beta}{(\beta\gamma - \alpha\delta)/\beta} \\ &= -\left(\frac{\beta\gamma - \alpha\delta}{\alpha\delta}\right) \\ &= 1 - \frac{\beta\gamma}{\alpha\delta} \\ &= 1 - k.\end{aligned}$$

Finally, we can use the previous parts of the proof to evaluate $(DBCA)$, as follows:

This avoids the algebra involved in expressing $\mathbf{c}$ and $\mathbf{a}$ in terms of $\mathbf{d}$ and $\mathbf{b}$.

$$\begin{aligned}(DBCA) &= 1/(BDCA) && \text{(swap first two Points)} \\ &= (BDAC) && \text{(swap last two Points)} \\ &= 1 - (BADC) && \text{(swap middle two Points)} \\ &= 1 - 1/(ABDC) && \text{(swap first two Points)} \\ &= 1 - (ABCD) && \text{(swap last two Points)} \\ &= 1 - k.\end{aligned}$$

■

Example 1

Earlier, we showed that the cross-ratio $(ABCD)$ of the four collinear Points $A = [1, 2, 3]$, $B = [1, 1, 2]$, $C = [3, 5, 8]$, $D = [1, -1, 0]$ in $\mathbb{RP}^2$ is $-\frac{1}{3}$. Theorem 2 enables us to deduce that

$$\begin{aligned}(BACD) &= -3, \qquad & (ABDC) &= -3, \\ (ACBD) &= \tfrac{4}{3}, \qquad & (DBCA) &= \tfrac{4}{3}.\end{aligned}$$

Problem 3 Let the Points $A = [1, -1, -1]$, $B = [1, 3, -2]$, $C = [3, 5, -5]$, $D = [1, -5, 0]$ be collinear Points of $\mathbb{RP}^2$. By applying Theorem 2 to the solution of Problem l(a), determine the values of the cross-ratios $(ABDC)$, $(DBCA)$ and $(ACBD)$.

The next theorem confirms that cross-ratio is preserved by projective transformations.

Theorem 3 Let t be a projective transformation, and let A, B, C, D be any four collinear Points in $\mathbb{RP}^2$. If $A' = t(A), B' = t(B), C' = t(C), D' = t(D)$, then

$$(ABCD) = (A'B'C'D').$$

Proof Let t be the projective transformation $t : [\mathbf{x}] \mapsto [\mathbf{Ax}]$, where $\mathbf{A}$ is an invertible 3×3 matrix. If $A = [\mathbf{a}], B = [\mathbf{b}], C = [\mathbf{c}], D = [\mathbf{d}]$, and

$$\mathbf{a}' = \mathbf{Aa},\ \mathbf{b}' = \mathbf{Ab},\ \mathbf{c}' = \mathbf{Ac},\ \mathbf{d}' = \mathbf{Ad},$$

then $A' = [\mathbf{a}'],\ B' = [\mathbf{b}'],\ C' = [\mathbf{c}'],\ D' = [\mathbf{d}']$.

Since A, B, C, D are collinear, we can write

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{and} \quad \mathbf{d} = \gamma\mathbf{a} + \delta\mathbf{b}, \tag{6}$$

so

$$(ABCD) = \frac{\beta}{\alpha}\bigg/\frac{\delta}{\gamma}.$$

Multiplying each equation in (6) through by $\mathbf{A}$, we obtain

$$\mathbf{c}' = \alpha\mathbf{a}' + \beta\mathbf{b}' \quad \text{and} \quad \mathbf{d}' = \gamma\mathbf{a}' + \delta\mathbf{b}',$$

so that

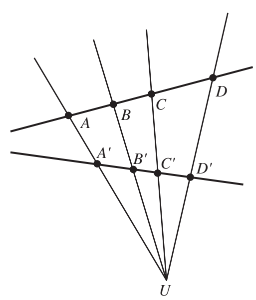
$$(A'B'C'D') = \frac{\beta}{\alpha}\bigg/\frac{\delta}{\gamma}.$$

It follows that

$$(A'B'C'D') = (ABCD).$$

■

We now use Theorem 3 to prove that if four distinct Points on a Line are in perspective with four distinct Points on another Line, then the cross-ratios of the four Points on each Line are equal.

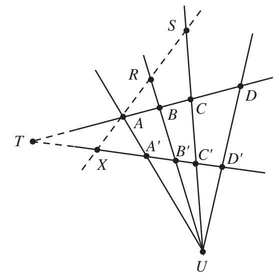


Theorem 4 Let A, B, C, D be four distinct Points on a Line, and let A', B', C', D' be four distinct Points on another Line such that AA', BB', CC', DD' all meet at a Point U. Then

$$(ABCD) = (A'B'C'D').$$

Proof By the Fundamental Theorem of Projective Geometry, there is a unique projective transformation t which maps B to B', C to C', B' to B, and C' to C. We shall show that $t(A) = A'$ and $t(D) = D'$, and hence by Theorem 3 it follows that $(ABCD) = (A'B'C'D')$.

First observe that the composite $t \circ t$ fixes the Points B, C, B' and C'. By the Fundamental Theorem of Projective Geometry, the only projective transformation which does this is the identity transformation, so $t \circ t = i$ and $t = t^{-1}$.



Next observe that t maps the Line BC onto the Line $B'C'$, and vice versa; so the Point T at which BC and $B'C'$ intersect must be fixed by t. Also, t maps the Lines BB' and CC' onto themselves, so their Point of intersection U must be fixed by t.

Now let X be the image of A under t. Then X lies on $B'C'$. We want to show that $X = A'$.

Suppose that $X \neq A'$; then AX cannot pass through U so it must intersect BB' at R and CC' at S, where R, S and U are distinct Points.

Since t is self-inverse, it maps X back to A and therefore maps AX onto itself. But this implies that t fixes the four Points R, S, T, U; so by the Fundamental Theorem of Projective Geometry t must be the identity transformation. This is a contradiction with the hypothesis that the Lines $ABCD$ and $A'B'C'D'$ are different. It follows that we must conclude that $X = A'$, that is, $t(A) = A'$. A similar argument shows that $t(D) = D'$.

Finally, it follows by Theorem 3 that $(ABCD) = (A'B'C'D')$, as required. ■

In affine geometry, if we are given two points A and B, then the ratio AC/CB uniquely determines a third point C on the line AB. We now explore the analogous result for projective geometry, namely that if we are given any three collinear Points A, B, C in $\mathbb{RP}^2$, then the value of the cross-ratio $(ABCD)$ uniquely determines a fourth Point D.

Theorem 5 Unique Fourth Point Theorem
Let A, B, C, X, Y be collinear Points in $\mathbb{RP}^2$ such that

$$(ABCX) = (ABCY).$$

Then $X = Y$.

Proof Let $A = [\mathbf{a}]$, $B = [\mathbf{b}]$, $C = [\mathbf{c}]$, $X = [\mathbf{x}]$, $Y = [\mathbf{y}]$. Since A, B, C, X, Y are collinear, it follows that there are real numbers α, β, γ, δ, λ, μ such that

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}, \quad \mathbf{x} = \gamma\mathbf{a} + \delta\mathbf{b} \quad \text{and} \quad \mathbf{y} = \lambda\mathbf{a} + \mu\mathbf{b}. \tag{7}$$

Then

$$(ABCX) = \frac{\beta\gamma}{\alpha\delta} \quad \text{and} \quad (ABCY) = \frac{\beta\lambda}{\alpha\mu}.$$

Since $(ABCX) = (ABCY)$, it follows that

This is the hypothesis of the theorem.

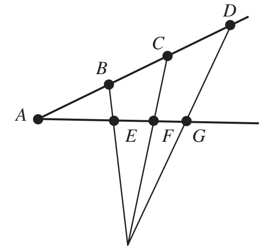
$$\frac{\gamma}{\delta} = \frac{\lambda}{\mu},$$

so $\lambda = \gamma\mu/\delta$. If we substitute this value of λ into the expression for $\mathbf{y}$ in equation (7), we obtain

$$\mathbf{y} = (\gamma\mu/\delta)\mathbf{a} + \mu\mathbf{b} = (\mu/\delta)(\gamma\mathbf{a} + \delta\mathbf{b}) = (\mu/\delta)\mathbf{x}.$$

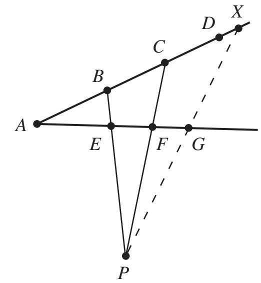
Since $\mathbf{y}$ is a scalar multiple of $\mathbf{x}$, it follows that $X = Y$, as required. ■

In Theorem 4 we showed that the cross-ratios $(ABCD)$ and $(A'B'C'D')$ are equal if the Points A', B', C', D' are in perspective with the Points A, B, C, D. Our next result is a partial converse of this result.



Theorem 6 Let A, B, C, D and A, E, F, G be two sets of collinear Points (on different Lines in $\mathbb{RP}^2$) such that the cross-ratios $(ABCD)$ and $(AEFG)$ are equal. Then the Lines BE, CF and DG are concurrent.

Proof Let P be the Point at which the Lines BE and CF meet, and let X be the Point at which the Line PG meets the Line $ABCD$. Then the Points A, B, C and X are in perspective from P with the Points A, E, F and G, so that



$$(ABCX) = (AEFG).$$

Since we know that $(AEFG) = (ABCD)$, it follows that

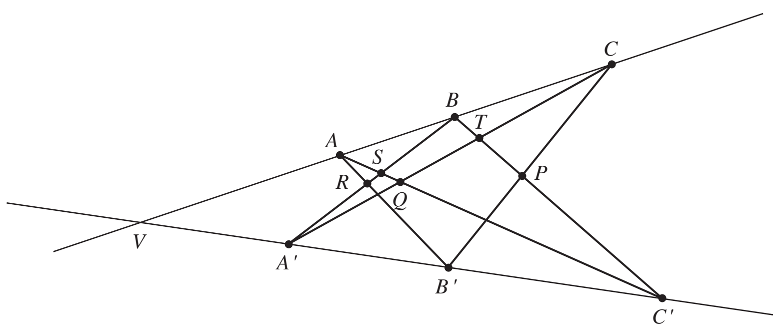
$$(ABCX) = (ABCD).$$

By Theorem 5, we must therefore have $X = D$. Hence the Points A, B, C, D and the Points A, E, F, G are in perspective from P. ■

We can now use Theorem 6 together with the other properties of cross-ratio to give a second proof of Pappus' Theorem.

Theorem 7 Pappus' Theorem
Let A, B and C be three Points on a Line in $\mathbb{RP}^2$, and let A', B' and C' be three Points on another Line. Let BC' and $B'C$ meet at P, CA' and $C'A$ meet at Q, and AB' and $A'B$ meet at R. Then P, Q and R are collinear.

You met this Theorem earlier, in Subsection 3.4.1.



Proof Let V be the Point of intersection of the two given Lines. Also let the Lines BA' and AC' meet at the Point S, and the Lines BC' and CA' meet at the Point T.

Now, the Points V, A', B', C' are in perspective from A with the Points B, A', R, S, so that

$$(VA'B'C') = (BA'RS). \tag{8}$$

By Theorem 4

Similarly, the Points V, A', B', C' are in perspective from C with the Points B, T, P, C', so that

$$(VA'B'C') = (BTPC'). \tag{9}$$

It follows from equations (8) and (9) that

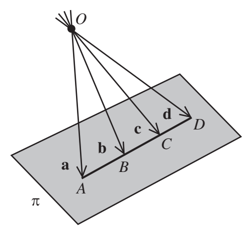
$$(BA'RS) = (BTPC'),$$

so that by Theorem 6 the Lines $A'T, RP, SC'$ are concurrent.

We may rephrase this statement as follows: the Line RP passes through the Point where $A'T$ meets SC'; that is, the Line RP passes through Q. In other words, P, Q and R are collinear. ■

3.5.2 Cross-Ratio on Embedding Planes

So far, we have calculated a given cross-ratio ($ABCD$) by applying the definition of cross-ratio directly to the Points A, B, C, D. However, it is sometimes convenient to evaluate the cross-ratio by examining the representation of the Points on some embedding plane.



See Appendix 2.

Suppose that four collinear Points of $\mathbb{RP}^2$ pierce an embedding plane π at the points A, B, C, D with position vectors **a**, **b**, **c**, **d**, respectively.

According to the Section Formula, we can write **c** and **d** in the form

$$\mathbf{c} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b} \quad \text{and} \quad \mathbf{d} = \mu\mathbf{a} + (1-\mu)\mathbf{b},$$

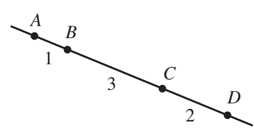
where $(1-\lambda) : \lambda$ is the ratio $AC : CB$, and $(1-\mu) : \mu$ is the ratio $AD : DB$. Then from the definition of cross-ratio

$$(ABCD) = \frac{1-\lambda}{\lambda} \Big/ \frac{1-\mu}{\mu},$$

so

$$(ABCD) = \frac{AC}{CB} \Big/ \frac{AD}{DB}. \tag{10}$$

Example 2 In an embedding plane, the points A, B, C, D lie in order along a line with the distances AB, BC, CD being 1 unit, 3 units and 2 units, respectively. Determine the cross-ratios ($ABCD$), ($BACD$) and ($ACBD$).



Solution Using equation (10) and the sign convention for ratios, we have

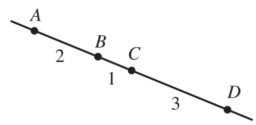
$$(ABCD) = \frac{AC}{CB} \Big/ \frac{AD}{DB} = \left(-\frac{4}{3}\right) \Big/ \left(-\frac{6}{5}\right) = \frac{10}{9},$$

$$(BACD) = \frac{BC}{CA} \Big/ \frac{BD}{DA} = \left(-\frac{3}{4}\right) \Big/ \left(-\frac{5}{6}\right) = \frac{9}{10}$$

and

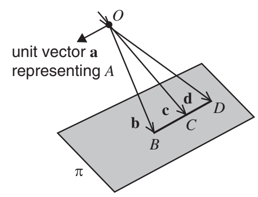
$$(ACBD) = \frac{AB}{BC} \bigg/ \frac{AD}{DC} = \left(\frac{1}{3}\right) \bigg/ \left(-\frac{6}{2}\right) = -\frac{1}{9}. \qquad \square$$

Problem 4 The points A, B, C, D lie in order along a line with the distances AB, BC, CD being 2 units, 1 unit and 3 units, respectively. Determine the cross-ratios $(ABCD)$ and $(DBCA)$.



Sometimes one of the Points whose cross-ratio we are trying to find turns out to be an ideal Point for the embedding plane. In such cases, formula (10) cannot be used since some of the distances in the formula will not be defined.

To be specific, suppose that the Points A, B, C, D are collinear, but that A is an ideal Point for the embedding plane π, as shown in the margin. As before, we can let $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be the position vectors of the points B, C, D on π, but we take $\mathbf{a}$ to be a unit vector along A. Then



$$\mathbf{c} = -(CB)\mathbf{a} + \mathbf{b} \quad \text{and} \quad \mathbf{d} = -(DB)\mathbf{a} + \mathbf{b}.$$

From the definition of cross-ratio, it follows that

$$(ABCD) = \frac{1}{-CB} \bigg/ \frac{1}{-DB} = \frac{DB}{CB}. \tag{11}$$

We can now obtain the corresponding formulas for the cases where B, C or D is an ideal Point, by applying Theorem 2. For example, if B is an ideal Point, then

$$\begin{aligned}(ABCD) &= \frac{1}{(BACD)} && \text{(swap first two terms)}\\ &= (BADC) && \text{(swap last two terms)}\\ &= \frac{CA}{DA} && \text{by equation (11).}\end{aligned}$$

Problem 5 Use Theorem 2 To prove that:

(a) $(ABCD) = \frac{AC}{BC}$ if D is an ideal Point;

(b) $(ABCD) = \frac{BD}{AD}$ if C is an ideal Point.

We now summarize the various formulas for cross-ratio in the form of a strategy, as follows.

Strategy To use an embedding plane to calculate the cross-ratio of four collinear Points:

1. if the four Points pierce the embedding plane at A, B, C, D, then

$$(ABCD) = \frac{AC}{CB} \bigg/ \frac{AD}{DB};$$

2. if one of the Points is an ideal Point for the embedding plane, then

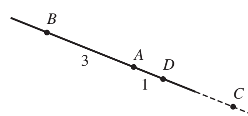
$$(ABCD) = \frac{DB}{CB} \text{ if } A \text{ is ideal,}$$

$$(ABCD) = \frac{CA}{DA} \text{ if } B \text{ is ideal,}$$

$$(ABCD) = \frac{BD}{AD} \text{ if } C \text{ is ideal,}$$

$$(ABCD) = \frac{AC}{BC} \text{ if } D \text{ is ideal.}$$

Example 3 Determine $(ABCD)$ for the collinear points A, B, C, D illustrated in the margin, where C is an ideal Point.

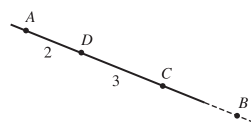


Solution Since C is an ideal Point, we have

$$(ABCD) = \frac{BD}{AD} = \frac{4}{1} = 4.$$

□

Problem 6 Determine $(ABCD)$ for the collinear points A, B, C, D illustrated in the margin, where B is an ideal Point.



3.5.3 An Application of Cross-Ratio

Earlier, we described how projective geometry can be used to obtain two-dimensional representations of three-dimensional scenes. We now describe how cross-ratios can be used to obtain information about a three-dimensional scene from a two-dimensional representation of the scene. We do this in the context of aerial photography.

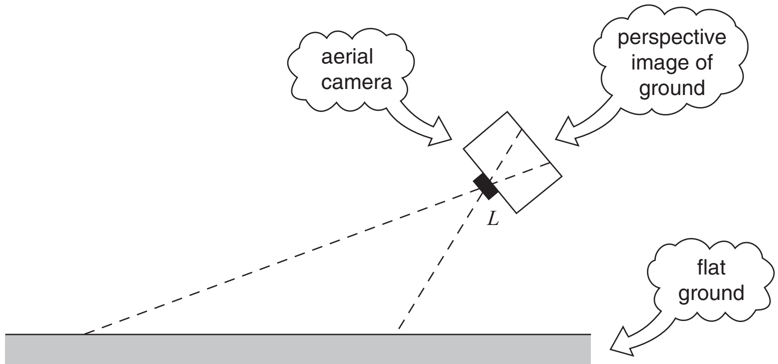
Subsection 3.1.1

For simplicity, consider an aerial camera that takes pictures on a flat film behind its lens, L, of features on a flat piece of land in front of L. Since a point on the ground lies on the same line through L as its image on the film, we can regard the process of taking a photograph as a perspectivity centred at L.

Since collinearity is invariant under a perspectivity, the image of any line ℓ on the ground is a line on the film. Moreover, the cross-ratio of any four collinear points is invariant under a perspectivity, so the cross-ratio of any four points on ℓ must be equal to the cross-ratio of their images on the film.

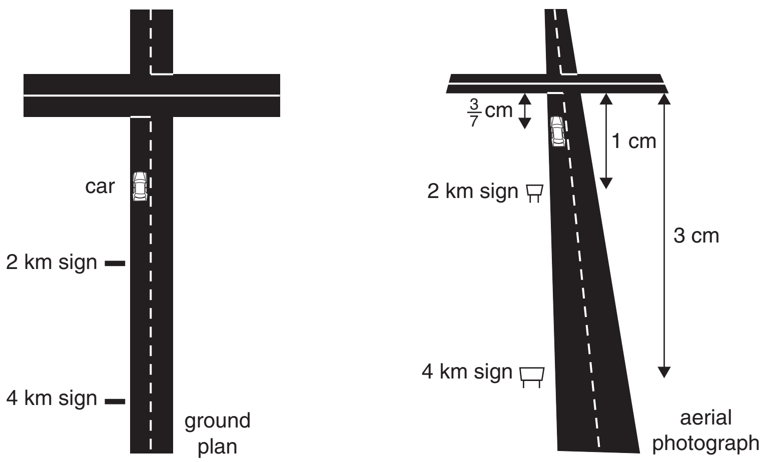
Section 3.3, Theorem 2

Theorem 3



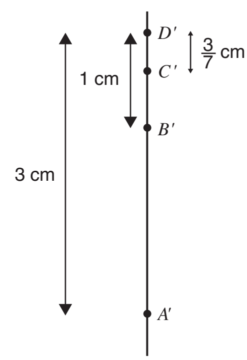
Example 4 An aerial camera photographs a car travelling along a straight road on flat ground towards a junction. Before the junction there are two warning signs at distances of 4 km and 2 km from the junction. On the film the signs are 1 cm and 3 cm from the junction, and the car is $\frac{3}{7}$ cm from the junction. How far is the car from the junction on the ground?

Strictly speaking, the car is not in line with the two signposts. Consequently, the distances marked on the photograph are approximations measured along the line of the left-hand kerb of the road.

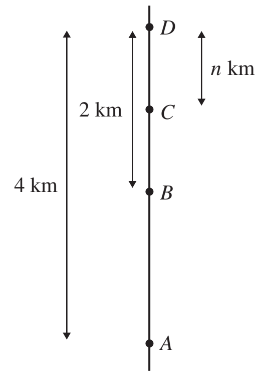


Solution Let A and B denote the signs, C denote the car, and D denote the junction, and let A', B', C', D' be their images on the film. Then

$$\begin{aligned}(A'B'C'D') &= \frac{A'C'}{C'B'} \Big/ \frac{A'D'}{D'B'} \\ &= \left(-\frac{18/7}{4/7}\right) \Big/ \left(-\frac{3}{1}\right) \\ &= \frac{3}{2}.\end{aligned}$$



Now let the car be n km from the junction. Then



$$\begin{aligned}(ABCD) &= \frac{AC}{CB}\Big/\frac{AD}{DB}\\ &= \left(-\frac{4-n}{2-n}\right)\Big/\left(-\frac{4}{2}\right)\\ &= \frac{4-n}{2(2-n)}.\end{aligned}$$

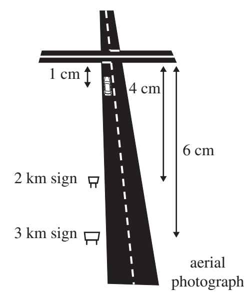
Since $(ABCD)$ and $(A'B'C'D')$ must be equal, it follows that

$$\frac{4-n}{2(2-n)} = \frac{3}{2}.$$

Hence

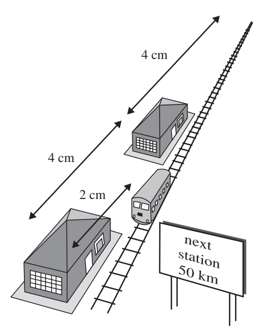
$$4-n = 3(2-n).$$

and so $n = 1$. That is, the car is 1 km from the junction. □



Problem 7 An aerial camera photographs a car travelling along a straight road on flat ground towards a junction. Before the junction there are two warning signs, at distances of 2 km and 3 km from the junction. On the film the signs are 4 cm and 6 cm from the junction, and the car is 1 cm from the junction. How far is the car from the junction on the ground?

If two lines that are known to be parallel on the ground appear to meet on the film, then the point of intersection on the film corresponds to the ideal Point where the 'parallel lines meet'. We can therefore use the above technique even when one of the Points is ideal, for we can use the second part of the strategy in Subsection 3.5.2 to calculate the cross-ratio whenever one of the Points is ideal.



Problem 8 An aerial camera photographs a train travelling between two stations along a straight track on flat ground. The stations are 50 km apart. When the film is inspected, the stations are 4 cm apart, the train is midway between the stations, and the rails appear to meet (or vanish) 4 cm beyond the station towards which the train is travelling. How far has the train to travel to the next station?

3.6 Exercises

Section 3.2

1. (a) Write down numbers a, b, c and d such that

$$[1, a, b] = \left[-\tfrac{1}{2}, 3, 4\right] \quad \text{and} \quad [c, d, 2] = [3, 0, 1].$$

(b) Which of the following homogeneous coordinates represent the same Point of $\mathbb{RP}^2$ as $[4, -8, 2]$?

(i) $[1, 4, -2]$ (ii) $\left[\frac{1}{4}, -\frac{1}{2}, \frac{1}{8}\right]$ (iii) $\left[-\frac{1}{2}, -2, 1\right]$

(iv) $[-2, 4, -1]$ (v) $\left[-\frac{1}{8}, -\frac{1}{2}, \frac{1}{4}\right]$

2. Determine an equation for each of the following Lines in $\mathbb{RP}^2$:
   (a) the Line through the Points $[1, 2, 3]$ and $[3, 0, -2]$;
   (b) the Line through the Points $[1, -1, -1]$ and $[2, 1, -3]$.
3. Determine whether each of the following sets of Points are collinear:
   (a) $[1, -1, 0]$, $[1, 0, -1]$ and $[2, -1, -1]$;
   (b) $[1, 0, 1]$, $[0, 1, 2]$ and $[1, 2, 3]$.
4. Determine the Point of intersection of each of the following pairs of Lines in $\mathbb{RP}^2$:
   (a) the Lines with equations $x - 2y + z = 0$ and $x - y - z = 0$;
   (b) the Lines with equations $x + 2y + 5z = 0$ and $3x - y + z = 0$.
5. Determine the Point of $\mathbb{RP}^2$ at which the Line through the Points $[8, -1, 2]$ and $[1, -2, -1]$ meets the Line through the Points $[0, 1, -1]$ and $[2, 3, 1]$.
6. Determine the Point of $\mathbb{RP}^2$ at which the Line through the Points $[1, 2, 2]$ and $[2, 3, 3]$ meets the Line through the Points $[0, 1, 2]$ and $[0, 1, 3]$.

Section 3.3

In these exercises, you may find the following list of matrices and their inverses useful.

$$\mathbf{A}: \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & -1 \\ 2 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 4 \\ -1 & 3 & 2 \\ 3 & -3 & 3 \end{pmatrix}$$

$$\mathbf{A}^{-1}: \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ -\frac{2}{3} & -1 & -\frac{4}{3} \\ -1 & -2 & -2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -7 & -2 \\ 3 & -4 & -\frac{4}{3} \\ -2 & 3 & 1 \end{pmatrix}$$

1. Determine which of the following transformations t of $\mathbb{RP}^2$ are projective transformations. For those that are projective transformations, write down a matrix associated with t.

(a) $t : [x, y, z] \mapsto [2x, y + 3z, 1]$
(b) $t : [x, y, z] \mapsto [x, x - y + 3z, x + y]$
(c) $t : [x, y, z] \mapsto [2y, y - 4z, x]$
(d) $t : [x, y, z] \mapsto [x + y - z, y + 3z, x + 2y + 2z]$

2. Determine the images of the Points [1, 2, 3], [0, 1, 0] and [1, −1, 1] under the projective transformation t associated with the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

3. Let

$$t_1 : [x, y, z] \mapsto [2x + y, -x + z, y + z],$$
$$t_2 : [x, y, z] \mapsto [x + y, 3x - z, 4y - 2z]$$

be projective transformations from $\mathbb{RP}^2$ to $\mathbb{RP}^2$.
(a) Write down matrices associated with each of t_1 and t_2.
(b) Determine formulas for $t_2 \circ t_1$ and $t_2 \circ t_1^{-1}$.

4. Find the image of the Line $x + 2y + 3z = 0$ under the projective transformation t_1 defined in Exercise 3.

5. Determine matrices for the projective transformations which map the Points [1, 0, 0], [0, 1, 0], [0, 0, 1] and [1, 1, 1] onto the following Points:
(a) [−2, 0, 1], [0, 1, −1], [−1, 2, −1] and [−1, 1, −1];
(b) [0, 1, 0], [1, 0, 0], [−1, −1, 1] and [2, 1, 1];
(c) [0, 1, −3], [1, 1, −1], [4, 2, 3] and [7, 4, 3].

6. Use the results of Exercise 5 to determine the projective transformations that map:
(a) the Points

$$[-2, 0, 1],\ [0, 1, -1],\ [-1, 2, -1],\ [-1, 1, -1]$$

to the Points

$$[0, 1, 0],\ [1, 0, 0],\ [-1, -1, 1],\ [2, 1, 1],$$

respectively;
(b) the Points

$$[0, 1, 0],\ [1, 0, 0],\ [-1, -1, 1],\ [2, 1, 1]$$

to the Points

$$[0, 1, -3],\ [1, 1, -1],\ [4, 2, 3],\ [7, 4, 3],$$

respectively;
(c) the Points

$$[0, 1, -3],\ [1, 1, -1],\ [4, 2, 3],\ [7, 4, 3]$$

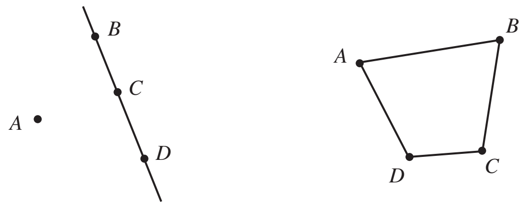
to the Points

$$[-2, 0, 1],\ [0, 1, -1],\ [-1, 2, -1],\ [-1, 1, -1],$$

respectively.

Section 3.4

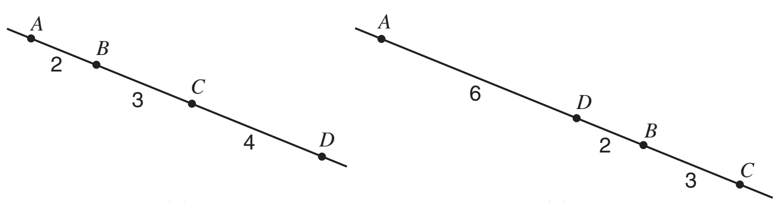
1. For which of the following configurations of Points A, B, C and D in $\mathbb{RP}^2$ is there a projective transformation sending A, B, C to the triangle of reference and D to the unit Point?



2. Let ΔABC be a triangle in $\mathbb{R}^2$, and let U be any point of $\mathbb{R}^2$ that is not collinear with any two of the points *A, B, C*. Let the Lines *BC* and *AU* meet at *P, CA* and *BU* meet at *Q*, and *AB* and *CU* meet at *R*. Prove that *P, Q, R* cannot be collinear.

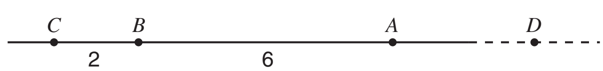
Section 3.5

1. For each of the following sets of Points A, B, C, D, calculate the cross-ratio $(ABCD)$.
   (a) $A = [2, 1, 3],\ B = [1, 2, 3],\ C = [8, 1, 9],\ D = [4, -1, 3]$
   (b) $A = [2, 1, 1],\ B = [-1, 1, -1],\ C = [1, 2, 0],\ D = [-1, 4, -2]$
   (c) $A = [-1, 1, 1],\ B = [0, 0, 2],\ C = [5, -5, 3],\ D = [-3, 3, 7]$
2. For the Points A, B, C, D in Exercise 1(a), determine the cross-ratios $(BACD)$, $(BDCA)$ and $(ADBC)$.
3. For each set of collinear points A, B, C, D illustrated below, calculate the cross-ratio $(ABCD)$.

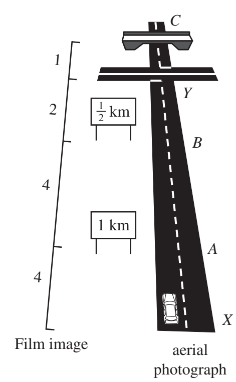


(a) (b)

4. Calculate the cross-ratio $(ABCD)$ for the collinear points A, B, C, D illustrated below, where D is an ideal Point.



5. The diagram in the margin represents an aerial photograph of a straight road on flat ground. At A there is a sign 'Junction 1 km', at B a sign 'Junction



$\frac{1}{2}$ km', and C is the road junction. Also, a police patrol car is at X, and a bridge is at Y. The distances marked on the left of the diagram are measured in cm from the photograph.

Calculate the actual distances (in km) of the patrol car and the bridge from the junction.

Summary of Chapter 3

Section 3.1: Perspective

1. Renaissance artists used **terraced perspective** and later **vertical perspective** in an attempt to portray 'real' scenes in a realistic way. The modern system of **focused perspective** was discovered by Brunelleschi and finally perfected by Leonardo da Vinci; it is well illustrated by the woodcuts of Albrecht Dürer.

   The family of lines joining an eye to each point of a scene meets a screen in front of the eye, and the image on the screen is called a **cross-section** (or **section**). The cross-section gives a realistic two-dimensional representation of the three-dimensional scene.
2. For two planes π and π' that do not pass through the origin O in $\mathbb{R}^3$, points P in π and Q in π' are **in perspective from *O*** if there is a straight line through O, P and Q.

   A **perspectivity from** π **to** π' centred at O is a function that maps a point P of π to a point Q of π' whenever P and Q are in perspective from O. (The planes π and π' may lie on the same or on opposite sides of O.)
3. The domain of a perspectivity may not be the whole of π; for, if P is any point of π such that OP is parallel to to π', then P cannot have an image in π'.

   The image of a line ℓ under a perspectivity is another line, possibly minus one point.

   Foreshortening is the effect under a perspectivity of lines of equal lengths at different distances from a screen corresponding to different lengths on the screen.
4. Two parallel lines in a horizontal plane π appear to an observer to meet at a **vanishing point** on a vertical screen π'; this is the **principal vanishing point** if the lines are perpendicular to the line of intersection of π and π', and a **diagonal vanishing point** otherwise.

   The family of vanishing points is a line, the **vanishing line**, that corresponds to the horizon line in a picture.
5. **Desargues' Theorem** Let ΔABC and $\Delta A'B'C'$ be triangles in $\mathbb{R}^2$ such that the lines AA', BB' and CC' meet at a point U. Let BC and $B'C'$meet at P, CA and $C'A'$meet at Q, and AB and $A'B'$meet at R. Then P, Q and R are collinear.

Section 3.2: The Projective Plane $\mathbb{RP}^2$

1. A **Point** (or **projective point**) is a line in $\mathbb{R}^3$ that passes through the origin of $\mathbb{R}^3$.

   The **real projective plane** $\mathbb{RP}^2$ is the set of all such Points.
2. The expression $[a, b, c]$, in which the numbers a, b and c are not all zero, represents the Point P of $\mathbb{RP}^2$ which consists of the unique line in $\mathbb{R}^3$ that passes through (0,0,0) and (a, b, c). We refer to $[a, b, c]$ as **homogeneous coordinates** of P.

   If (a, b, c) has position vector $\mathbf{v}$, then we often denote P by $[\mathbf{v}]$ and we say that P can be **represented** by $\mathbf{v}$.

   It makes NO sense to write the expression [0,0,0], since not all of a, b and c can be zero.
3. The homogeneous coordinates $[a, b, c]$ and $[\lambda a, \lambda b, \lambda c]$ (where $\lambda \neq 0$) represent the same Point of $\mathbb{RP}^2$; that is, $[a, b, c] = [\lambda a, \lambda b, \lambda c]$, for any $\lambda \neq 0$.

   If there is no non-zero real number λ such that $[a, b, c] = \left[\lambda a', \lambda b', \lambda c'\right]$, then the homogeneous coordinates $[a, b, c]$ and $\left[a', b', c'\right]$ represent different Points of $\mathbb{RP}^2$.

   Further, $\left[a', b', 1\right] = \left[a'', b'', 1\right]$ if and only if $a' = a''$ and $b' = b''$.
4. A **projective figure** is a subset of $\mathbb{RP}^2$.
5. A **Line** (or **projective line**) in $\mathbb{RP}^2$ is a plane in $\mathbb{R}^3$ that passes through the origin. Points of $\mathbb{RP}^2$ are **collinear** if they lie on a Line.
6. The general equation of a Line in $\mathbb{RP}^2$ is $ax + by + cz = 0$, where a, b, c are real and not all zero.
7. **Collinearity Property of** $\mathbb{RP}^2$ Any two distinct Points of $\mathbb{RP}^2$ lie on a unique Line.

   Strategy To determine an equation for the Line in $\mathbb{RP}^2$ through the Points $\left[d, e, f\right]$ and $\left[g, h, k\right]$:

   1. write down the equation $\begin{vmatrix} x & y & z \\ d & e & f \\ g & h & k \end{vmatrix} = 0$;
   2. expand the determinant in terms of the entries in its first row to obtain the required equation in the form $ax + by + cz = 0$.

   Sometimes it is possible to 'spot' the equation of a Line through two Points without using the determinant.
8. **Strategy** To determine whether three Points $[a, b, c]$, $[d, e, f]$ and $[g, h, k]$ are collinear:

   1. evaluate the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$;
   2. the Points $[a, b, c]$, $[d, e, f]$ and $[g, h, k]$ are collinear if and only if this determinant is zero.
9. The Points [1,0,0], [0,1,0], [0,0,1] are known as the **triangle of reference**. The Point [1,1,1] is called the **unit Point**.

10. **Incidence Property of** $\mathbb{RP}^2$ Any two distinct Lines in $\mathbb{RP}^2$ intersect in a unique Point of $\mathbb{RP}^2$.
11. Let π be any plane in $\mathbb{R}^3$ that does not pass through the origin O. Then there is a one-one correspondence between the points of π and those Points of $\mathbb{RP}^2$ which pierce π. Those Points of $\mathbb{RP}^2$ which do not pierce π are called **ideal Points** for π.

    The set of ideal Points for π is a plane through O parallel to π, called the **ideal Line** for π.
12. An **embedding plane** is a plane, π, which does not pass through the origin, together with the set of all ideal Points for π.

    The plane in $\mathbb{R}^3$ with equation $z = 1$ is called the **standard embedding plane**. The mapping of $\mathbb{RP}^2$ into the standard embedding plane is called the **standard embedding** of $\mathbb{RP}^2$.
13. Parallelism is not a projective property.

Section 3.3: Projective Transformations

1. A **projective transformation** of $\mathbb{RP}^2$ is a function $t\colon \mathbb{RP}^2 \to \mathbb{RP}^2$ of the form $t : [\mathbf{x}] \mapsto [\mathbf{Ax}]$, where $\mathbf{A}$ is an invertible 3×3 matrix. We say that $\mathbf{A}$ is a matrix **associated** with t. The set of all projective transformations is denoted by $P(2)$.

    If $\mathbf{A}$ is a matrix **associated** with t, then so is $\lambda\mathbf{A}$ for any non-zero number λ.
2. The set of projective transformations $P(2)$ forms a group under the operation of composition of functions. In particular, if t_1 and t_2 are projective transformations with associated matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively, then $t_1 \circ t_2$ and t_1^{-1} are projective transformations with associated matrices $\mathbf{A}_1\mathbf{A}_2$ and $\mathbf{A}_1^{-1}$.

    Strategy To compose two projective transformations t_1 and t_2:
    1. write down matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ associated with t_1 and t_2;
    2. calculate $\mathbf{A}_1\mathbf{A}_2$;
    3. write down the composite $t_1 \circ t_2$ with which $\mathbf{A}_1\mathbf{A}_2$ is associated.

    Strategy To find the inverse of a projective transformation t:
    1. write down a matrix $\mathbf{A}$ associated with t;
    2. calculate $\mathbf{A}^{-1}$;
    3. write down the inverse t^{-1} with which $\mathbf{A}^{-1}$ is associated.
3. **Strategy** To find the image of a Line $ax + by + cz = 0$ under a projective transformation $t : [\mathbf{x}] \mapsto [\mathbf{Ax}]$:
    1. write the equation of the Line in the form $\mathbf{Lx} = 0$, where $\mathbf{L}$ is the matrix $(a\,b\,c)$;
    2. find a matrix $\mathbf{B}$ associated with t^{-1};
    3. write down the equation of the image as $(\mathbf{LB})\,\mathbf{x} = 0$.
4. **Projective geometry** is the study of those properties of figures in $\mathbb{RP}^2$ that are preserved by projective transformations. Collinearity and incidence are both projective properties.

5. A **quadrilateral** is a set of four Points A, B, C and D (no three of which are collinear), together with the Lines AB, BC, CD and DA.
   All quadrilaterals are projective-congruent.
6. **Strategy** To find the projective transformation which maps [1,0,0] to $[a_1, a_2, a_3]$, [0,1,0] to $[b_1, b_2, b_3]$, [0,0,1] to $[c_1, c_2, c_3]$, [1,1,1] to $[d_1, d_2, d_3]$, where no three of $[a_1, a_2, a_3]$, $[b_1, b_2, b_3]$, $[c_1, c_2, c_3]$ and $[d_1, d_2, d_3]$ are collinear:
   1. find u, v, w for which $\begin{pmatrix} a_1u & b_1v & c_1w \\ a_2u & b_2v & c_2w \\ a_3u & b_3v & c_3w \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$;
   2. write down the required projective transformation in the form t: $[\mathbf{x}] \mapsto [\mathbf{Ax}]$, where $\mathbf{A}$ is any non-zero real multiple of the matrix $\begin{pmatrix} a_1u & b_1v & c_1w \\ a_2u & b_2v & c_2w \\ a_3u & b_3v & c_3w \end{pmatrix}$.
7. **Fundamental Theorem of Projective Geometry** Let $ABCD$ and $A'B'C'D'$ be two quadrilaterals in $\mathbb{RP}^2$. Then:
   (a) there is a projective transformation t which maps A to A', B to B', C to C', D to D';
   (a) the projective transformation t is unique.
8. **Strategy** To determine the projective transformation t which maps the vertices of the quadrilateral $ABCD$ to the corresponding vertices of the quadrilateral $A'B'C'D'$:
   1. find the projective transformation t_1 which maps the triangle of reference and unit Point to the Points A, B, C, D, respectively;
   2. find the projective transformation t_2 which maps the triangle of reference and unit Point to the Points A', B', C', D', respectively;
   3. calculate $t = t_2 \circ t_1^{-1}$.
9. With any given perspectivity σ we can construct an associated **perspective transformation** that is a one-one mapping of $\pi \cup \{\text{the ideal Points for } \pi\}$ onto $\pi' \cup \left\{\text{the ideal Points for } \pi'\right\}$. This maps $\mathbb{RP}^2$ onto itself.
   Every projective transformation can be expressed as the composite of three perspective transformations.

Section 3.4: Using the Fundamental Theorem of Projective Geometry

1. **Desargues' Theorem** Let ΔABC and $\Delta A'B'C'$ be triangles in $\mathbb{R}^2$ such that the lines AA', BB' and CC' meet at a point U. Let BC and $B'C'$ meet at P, CA and $C'A'$ meet at Q, and AB and $A'B'$ meet at R. Then P, Q and R are collinear.
2. The Fundamental Theorem is often used to simplify proofs of results in projective geometry, where the properties involved are projective properties. Generally, we do not explicitly refer to the corresponding auxiliary projective transformation t concerned, but simply comment that "By the

Fundamental Theorem of Projective Geometry, we may choose the four Points ... (no three of which are collinear) to be the triangle of reference and the unit Point; that is, to have homogeneous coordinates [1,0,0], [0,1,0], [0,0,1] and [1,1,1], respectively."

3. **Pappus' Theorem** Let A, B and C be three points on a line in $\mathbb{R}^2$, and let A', B' and C' be three points on another line. Let BC' and $B'C$ meet at P, CA' and $C'A$ meet at Q, and AB' and $A'B$ meet at R. Then P, Q and R are collinear.
4. Any Euclidean figure in an embedding plane corresponds to a projective figure in $\mathbb{RP}^2$. It follows that a Euclidean theorem concerned with projective properties (such as collinearity and coincidence) holds if and only if the corresponding projective theorem holds.
5. The **dual** of a statement about the projective properties of some figure in $\mathbb{RP}^2$ is the corresponding statement about $\mathbb{RP}^2$ in which the terms 'Point' and 'Line' are interchanged, and such other changes are made that ensure that the sentence makes sense.

   A triangle (three non-collinear Points and the Lines joining them) is self-dual.
6. A **hexagon** in $\mathbb{RP}^2$ consists of six Points joined by the six Lines joining them in turn.

   Pappus' Theorem (rephrased) Let the vertices A, B', C, A', B and C' of a hexagon lie alternately on two different Lines. Then the Points of intersection of opposite sides $B'C$ and BC', CA' and $C'A$, and AB' and $A'B$, are collinear.
7. **Brianchon's Theorem** (the dual of Pappus' Theorem) Let the sides $AB', B'C, CA', A'B, BC'$, $C'A$ of a hexagon pass alternately through two (different) Points P and Q in $\mathbb{RP}^2$. Then the Lines joining opposite vertices A and A', B and B', C and C', are concurrent.
8. **Converse of Desargues' Theorem** (also its dual) Let two triangles be such that the Points through which corresponding sides pass are collinear. Then the Lines through the corresponding vertices of the two triangles are concurrent.

Section 3.5: Cross-Ratio

1. Let A, B, C, D be four collinear Points in $\mathbb{RP}^2$ represented by the position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and let $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$ and $\mathbf{d} = \gamma\mathbf{a} + \delta\mathbf{b}$. Then the **cross-ratio** of A, B, C, D is $(ABCD) = \frac{\beta}{\alpha} \Big/ \frac{\delta}{\gamma}$.

   The cross-ratio $(ABCD)$ is independent of the homogeneous coordinates that are used to represent the collinear Points A, B, C, D.
2. Let A, B, C, D be four distinct collinear Points in $\mathbb{RP}^2$, and let $(ABCD) = k$. Then $(BACD) = (ABDC) = 1\big/k$ and $(ACBD) = (DBCA) = 1 - k$.
3. Let t be a projective transformation, and let A, B, C, D be any four collinear Points in $\mathbb{RP}^2$. If $A' = t(A)$, $B' = t(B)$, $C' = t(C)$, $D' = t(D)$, then $(ABCD) = \left(A'B'C'D'\right)$.

4. Let A, B, C, D be four distinct Points on a Line, and let A', B', C', D' be four distinct Points on another Line such that AA', BB', CC', DD' all meet at a Point U. Then $(ABCD) = \left(A'B'C'D'\right)$.
5. **Unique Fourth Point Theorem** Let A, B, C, X, Y be collinear Points in $\mathbb{RP}^2$ such that $(ABCX) = (ABCY)$. Then $X = Y$.
6. Let A, B, C, D and A, E, F, G be two sets of collinear Points (on different Lines in $\mathbb{RP}^2$) such that the cross-ratios $(ABCD)$ and $(AEFG)$ are equal. Then the Lines BE, CF and DG are concurrent.
7. Let four collinear Points of $\mathbb{RP}^2$ pierce an embedding plane at the points A, B, C, D with position vectors **a**, **b**, **c**, **d**, respectively. Then, if we can write **c** and **d** in the form $\mathbf{c} = \lambda\mathbf{a} + (1-\lambda)\,\mathbf{b}$ and $\mathbf{d} = \mu\mathbf{a} + (1-\mu)\,\mathbf{b}$, we have
$$(ABCD) = \frac{1-\lambda}{\lambda}\bigg/\frac{1-\mu}{\mu} = \frac{AC}{CB}\bigg/\frac{AD}{DB}.$$
8. **Strategy** To use an embedding plane to calculate the cross-ratio of four collinear Points:
   1. if the four Points pierce the embedding plane at A, B, C, D, then $(ABCD) = \frac{AC}{CB}\big/\frac{AD}{DB}$;
   2. if one of the Points is an ideal Point for the embedding plane, then
$$\begin{aligned}
(ABCD) &= \frac{DB}{CB} && \text{if } A \text{ is ideal,}\\
(ABCD) &= \frac{CA}{DA} && \text{if } B \text{ is ideal,}\\
(ABCD) &= \frac{BD}{AD} && \text{if } C \text{ is ideal,}\\
(ABCD) &= \frac{AC}{BC} && \text{if } D \text{ is ideal.}
\end{aligned}$$
9. Cross-ratios can be used to measure distances on the ground from aerial photographs, since the cross-ratio of any four points on a line on the ground equals the cross-ratio of their images on the film.