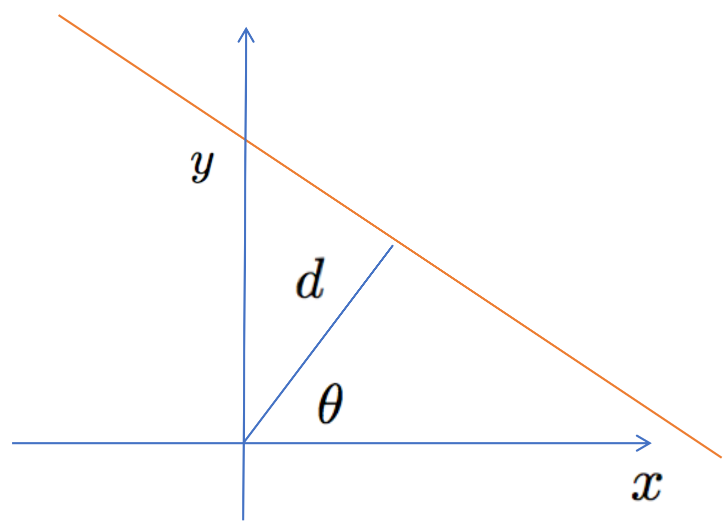
Hough and RANSAC

Kostas Daniilidis

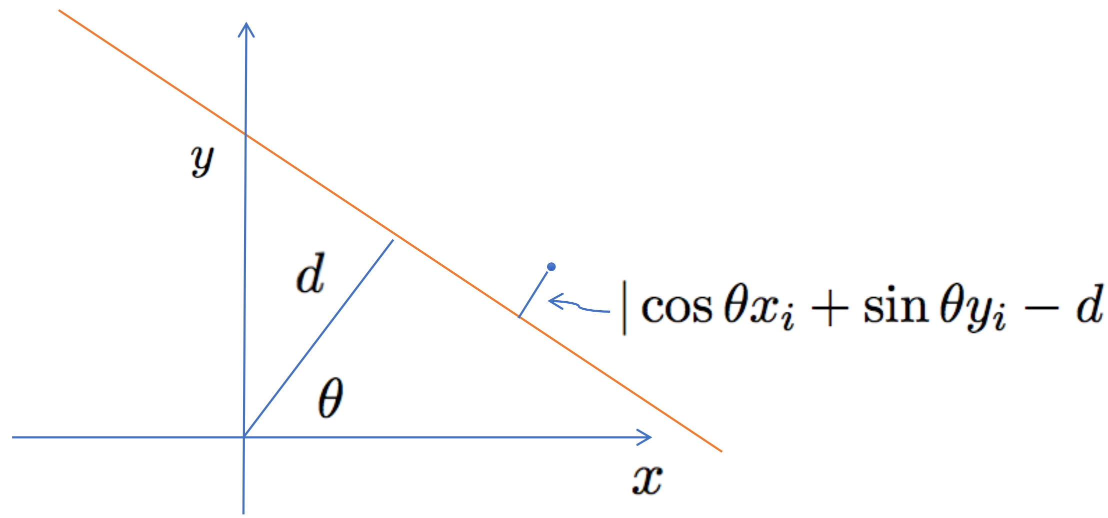
Given data (x_i, y_i) belonging to a line

$$x \cos\theta + y \sin\theta = d$$



we can estimate the best line as

$$\underset{\theta\in[0,2\pi),d\geq 0}{\arg\min} \sum_{i=1} N(x_i \cos\theta + y_i \sin\theta - d)^2$$



Line fitting

We can eliminate d and obtain

$$\arg\min_{\theta\in[0,2\pi)} \sum_{i=1}^{N} ((x_i - \bar{x})\cos\theta + (y_i - \bar{y})\sin\theta)^2$$

which is equivalent to the minimization of the Rayleight quotient

$$\arg\min_{\theta\in[0,2\pi)} \begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix} \underbrace{\sum_{i=1}^{N} \begin{pmatrix}(x_i - \bar{x})^2 & (x_i - \bar{x})(y_i - \bar{y}) \\ (x_i - \bar{x})(y_i - \bar{y}) & (y_i - \bar{y})^2\end{pmatrix}}_{C} \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

Solution is the eigenvector to the minimal eigenvalue of C.

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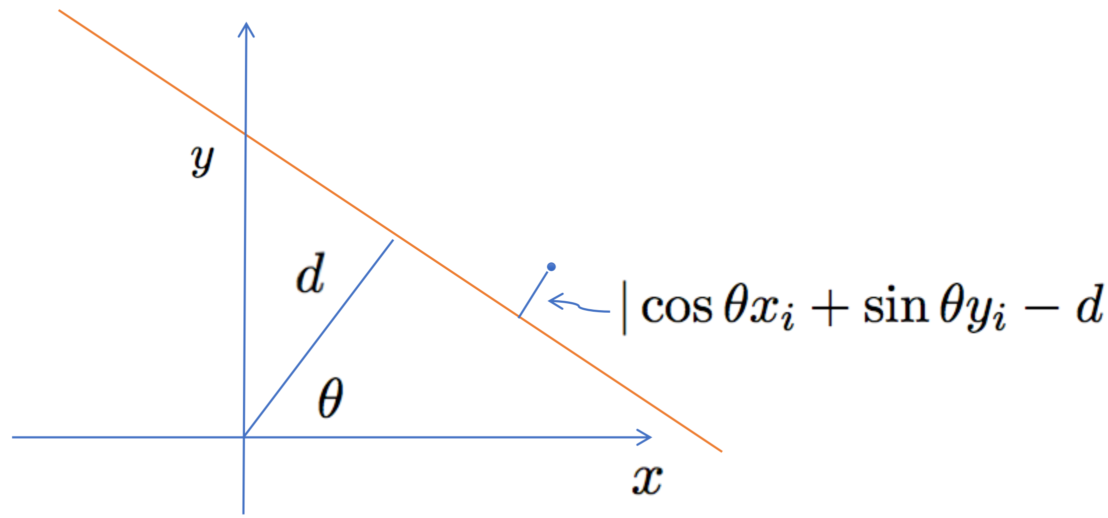
Solution is the eigenvector to the minimal eigenvalue of C.

which corresponds to the maximization of the likelihood function

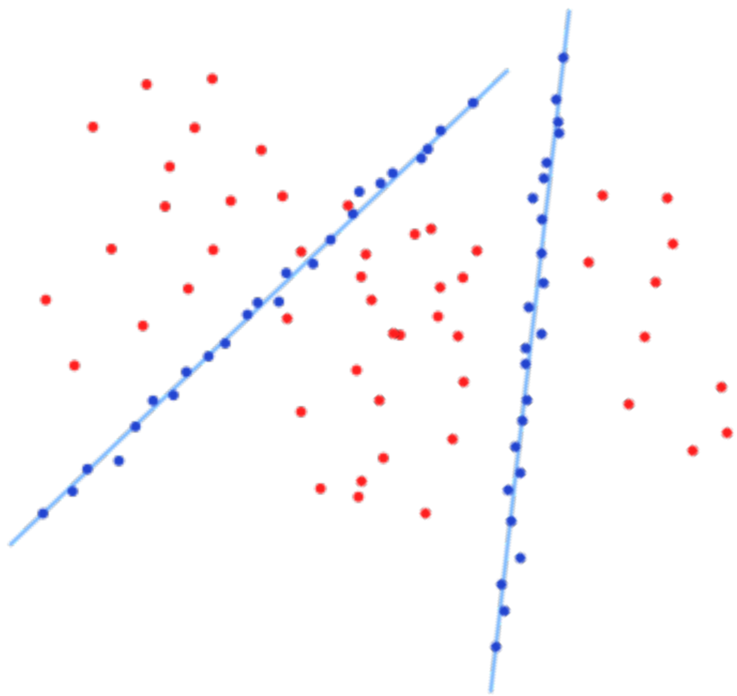
$$e^{-\frac{1}{2\sigma_i^2}(\cos\theta x_i + \sin\theta y_i - d)^2}$$

with $\sigma_i = 1$ being the same for all points.

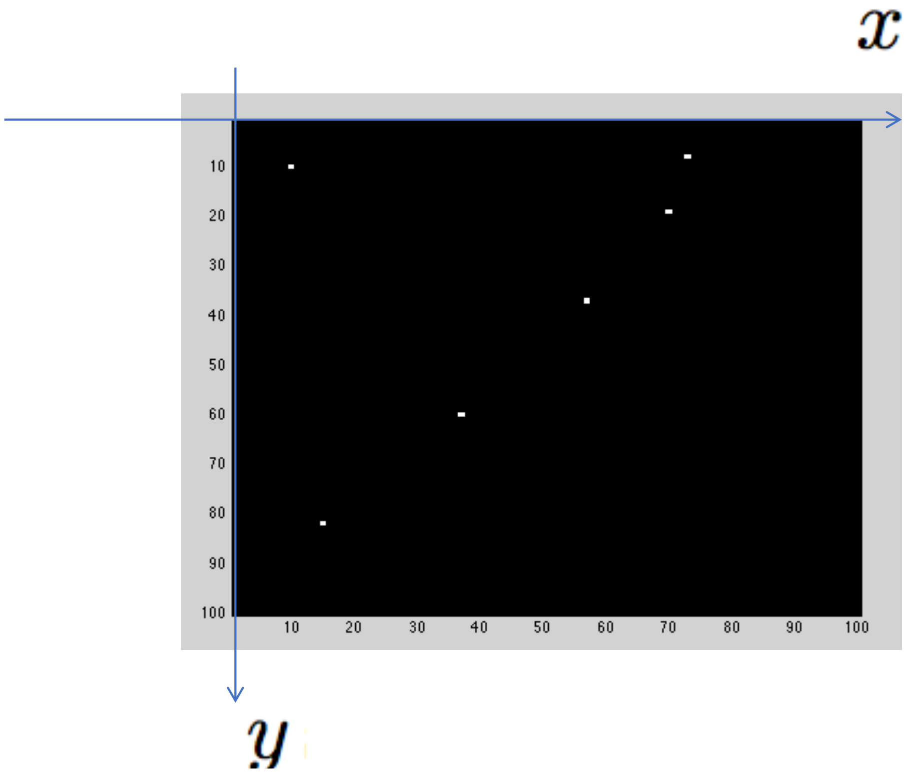


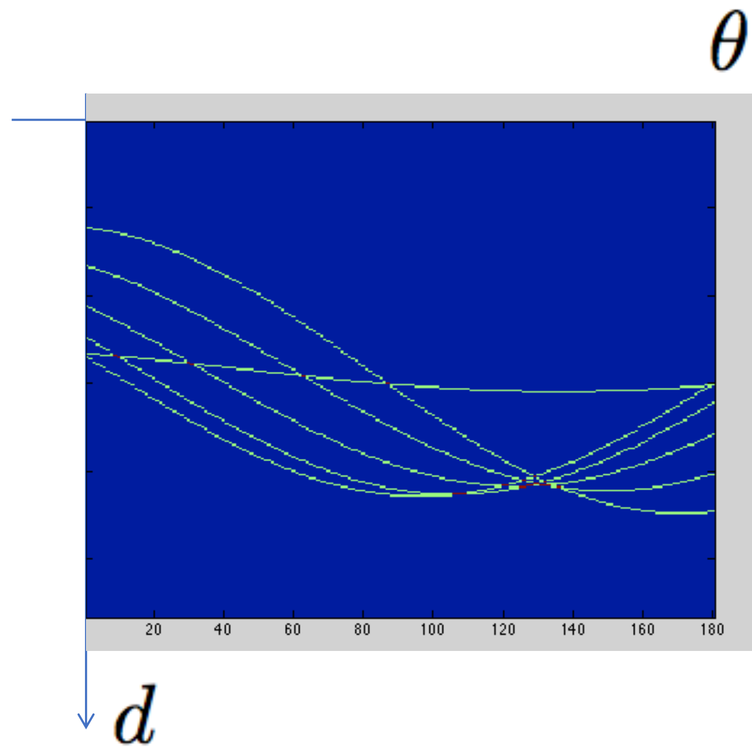
How do we choose which points belong to the line?

What if multiple lines?



Look at the line equation in a different way



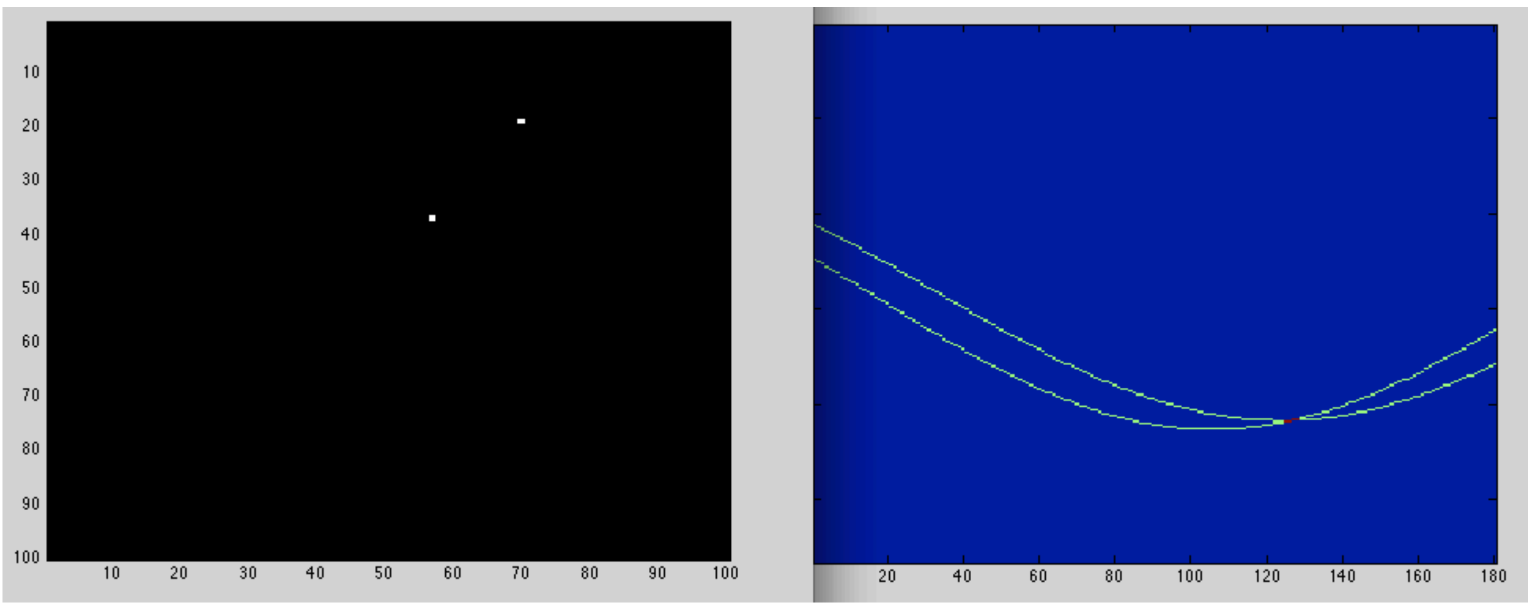


$$d = \cos\theta x_i + \sin\theta y_i$$

For fixed (x_i, y_i) this is the equation of a curve in the (θ, d) coordinate system.

Line through two points (x_1, y_1) and (x_2, y_2) can be found as the intersection (θ, d) of the two curves:

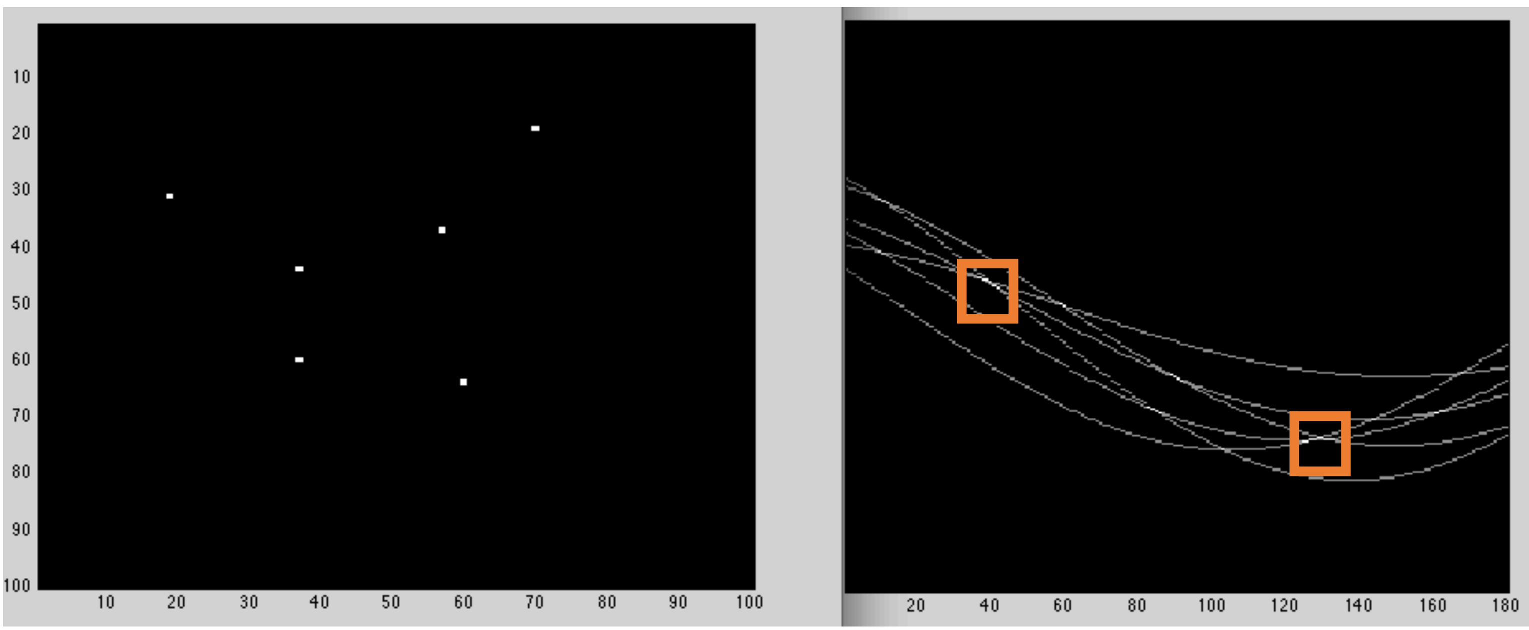
$$\begin{aligned} d &= \cos\theta x_1 + \sin\theta y_1 \\ d &= \cos\theta x_2 + \sin\theta y_2 \end{aligned}$$



If we have multiple lines (and/or) outliers we can estimate the voting function

$$V(\theta, d) = \int \delta(d - x\cos\theta + y\sin\theta)\, dx\, dy$$

where δ the Dirac function.



$$V(\theta, d) = \int \delta(d - x\cos\theta + y\sin\theta)\, dx\, dy$$

is a special case of the Radon transform used in tomography.

Johann Radon 1887-1956

Hough transform: Discretization of the parameter space

$$V[\theta_j, d_j] = \sum_i \delta'(d_j - x_i \cos\theta_j + y_i \sin\theta_j)$$

where δ' some threshold function voting for θ_j, d_j if $|d_j - x_i \cos\theta_j + y_i \sin\theta_j|$ is small.
Finding lines means finding the local maxima of $V[\theta_j, d_j]$ which have a sufficient number of votes.

United States Patent Office 3,069,654

Patented Dec. 18, 1962

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3,069,654

METHOD AND MEANS FOR RECOGNIZING COMPLEX PATTERNS

Paul V. C. Hough, Ann Arbor, Mich., assignor to the United States of America as represented by the United States Atomic Energy Commission

Filed Mar. 25, 1960, Ser. No. 17,715

6 Claims. (Cl. 340—146.3)

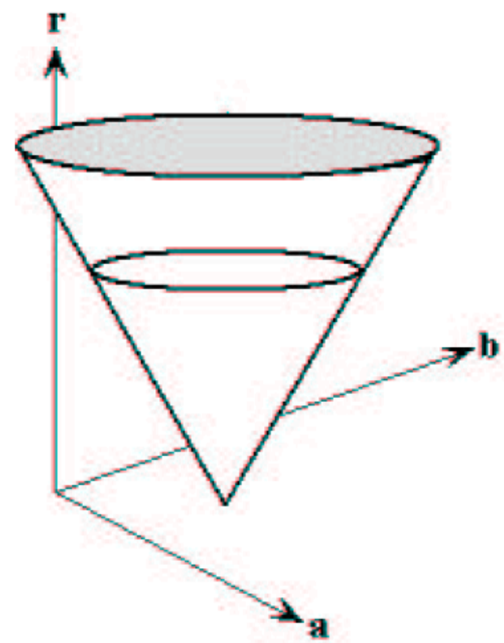
2

of the point on the line segment from the horizontal midline 109 of the framelet 108.

(3) Each line in the transformed plane is made to have an intercept with the horizontal midline 101 of the picture 100 equal to the horizontal coordinate of its respective point on the line segment in framelet 108.

Thus, for a given reference point 110 on line segment 102 a line 110A is drawn in the plane transform 102A. The reference point 110 is approximately midway between

Hough transform can be used for circle detection



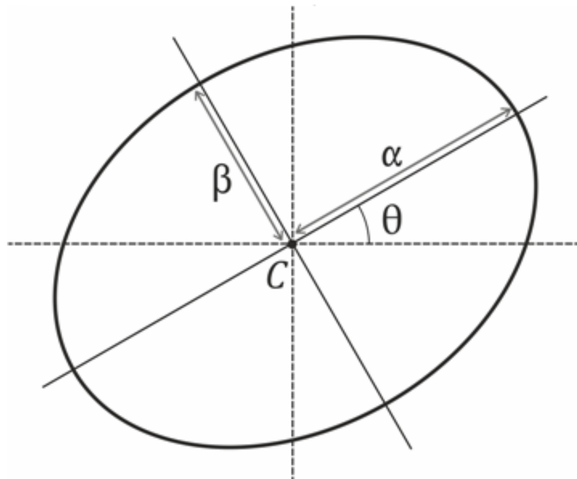
If we fix the (x, y) in the circle equation

$$(x-a)^2 + (y-b)^2 = r$$

we obtain the equation of a cone in (a, b, r) Hough space. Observe that we used r and not r^2.

Let us look at the ellipse equation

$$\begin{pmatrix} x - c_x & y - c_y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha^2} & 0 \\ 0 & \frac{1}{\beta^2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x - c_x \\ y - c_y \end{pmatrix} = 1$$



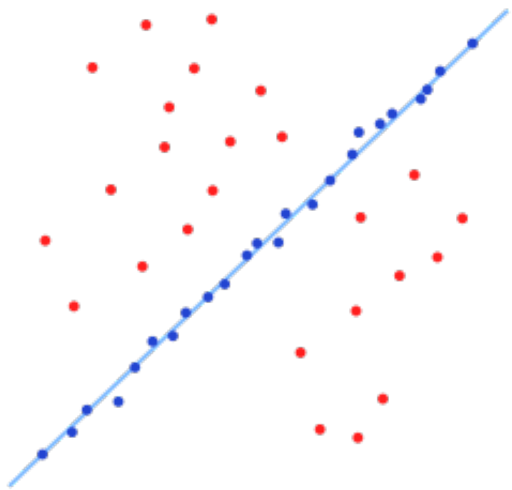
Hough transform needs the computation of $V(c_x, c_y, \alpha, \beta, \theta)$ and search for maxima in 5-dimensional space. Hough becomes intractable beyond lines and circles.

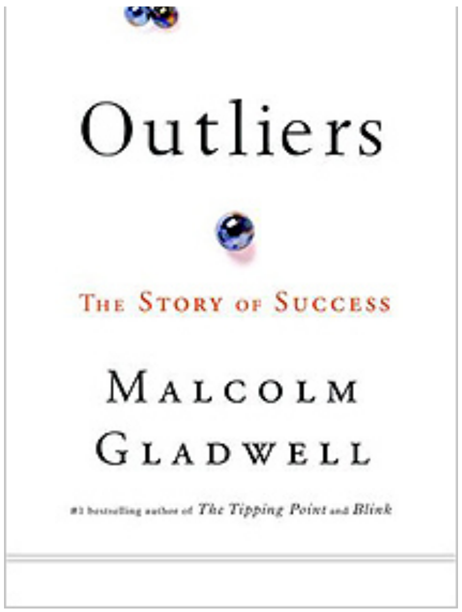
RANSAC

Let us suppose that our only problem is to detect outliers

A point is an *outlier* if it does not fit the underlying probability likelihood model.

$$e^{-\frac{1}{2\sigma_i^2}(\cos\theta x_i + \sin\theta y_i - d)^2}$$



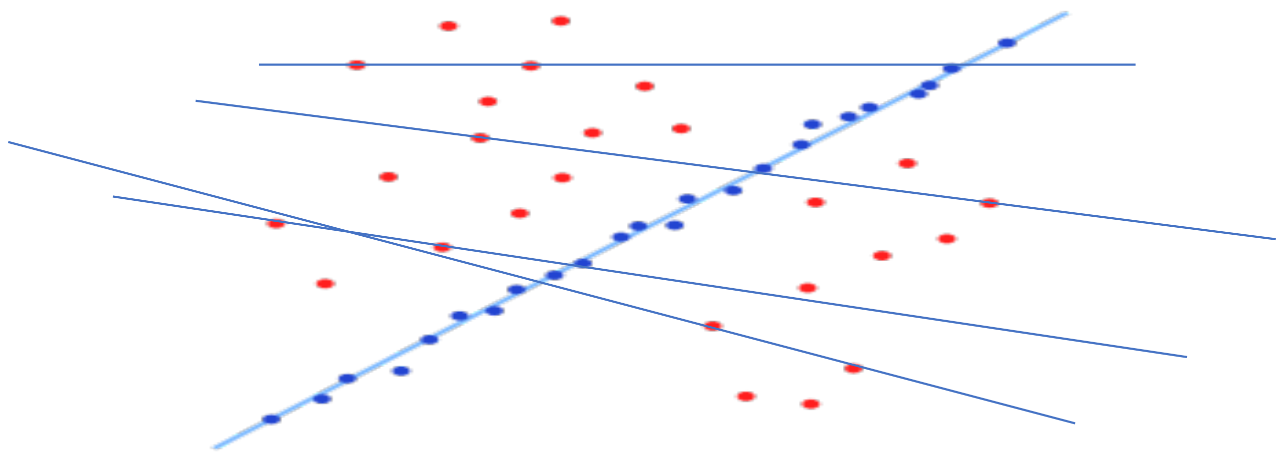


Sample Consensus

1. **Hypothesis:** Each minimal sample set (two points) defines a line.
2. **Test:** Which points of the dataset satisfy the hypothesis (no. of inliers)

Exhaustive Search:

1. Choose all $\binom{n}{2}$ pairs.
2. Keep the one with the maximum number of inliers (above a threshold).



RANdom SAmple Consensus or RANSAC

Graphics and Image Processing

J. D. Foley
Editor

Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography

Martin A. Fischler and Robert C. Bolles
SRI International

and analysis conditions. Implementation details and computational examples are also presented.

Key Words and Phrases: model fitting, scene analysis, camera calibration, image matching, location determination, automated cartography.

CR Categories: 3.60, 3.61, 3.71, 5.0, 8.1, 8.2

I. Introduction

We introduce a new paradigm, Random Sample Consensus (RANSAC), for fitting a model to experimental data; and illustrate its use in scene analysis and automated cartography. The application discussed, the location determination problem (LDP), is treated at a level beyond that of a mere example of the use of the RANSAC paradigm; new basic findings concerning the conditions under which the LDP can be solved are presented and a comprehensive approach to the solution of this problem that we anticipate will have near-term practical applications is described.

To a large extent, scene analysis (and, in fact, science in general) is concerned with the interpretation of sensed data in terms of a set of predefined models. Conceptually, interpretation involves two distinct activities: First, there is the problem of finding the best match between the data and one of the available models (the classification

RANdom SAmple Consensus or RANSAC

Sample minimal sample sets instead of exhaustively traverse them over k iterations.

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Repeat for k iterations
1.  Choose a minimal sample set
2.  Count the inliers for this set
3.  Keep maximum, if it exceeds a desired number of inliers
stop.
```

Assume that the minimal sample set has M points ($M = 2$ for the case of line fitting).

What is the probability that your minimal sample set is a set of inliers ?

If the probability of a point to be an inlier is ϵ then the probability of choosing an inlier pair is ϵ^M.

In k iterations the probability of NON hitting a single inlier pair is

$$(1 - \epsilon^M)^k$$

Thus the probability of choosing at least one inlier pair in k iterations is

$$p = 1 - (1 - \epsilon^M)^k.$$

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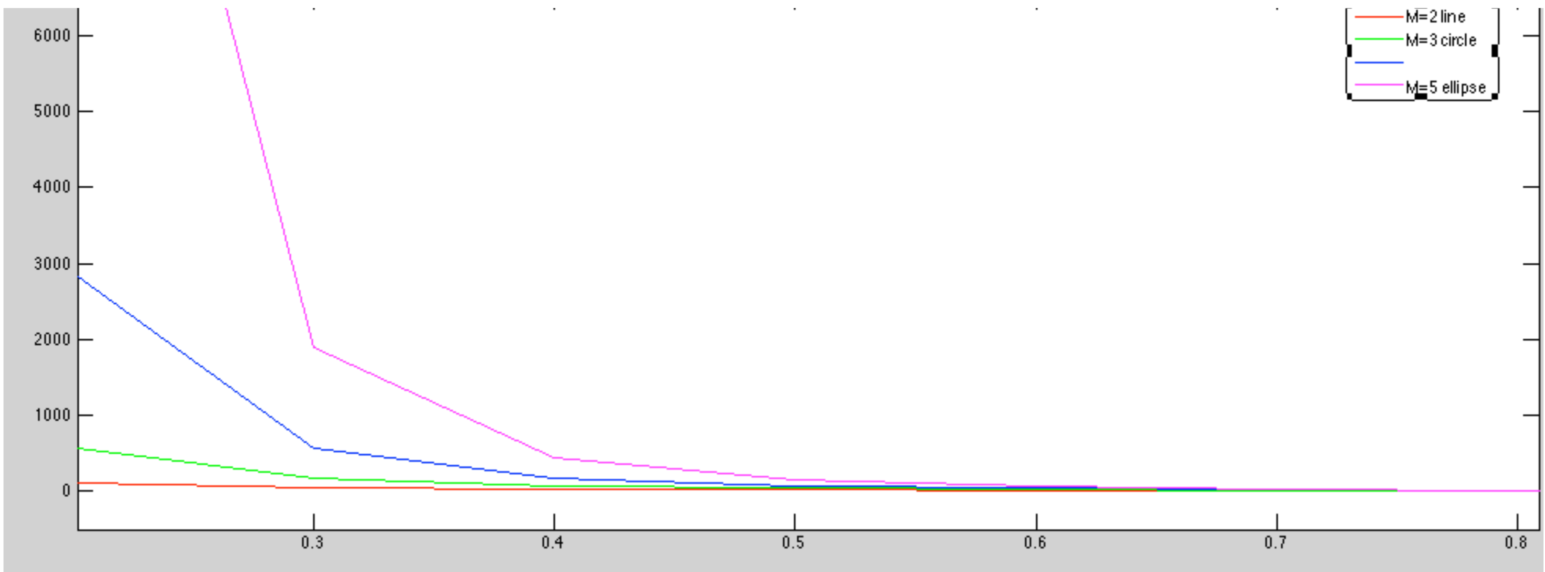
How tractable is RANSAC ?

$$k = \frac{\log(1-p)}{\log(1-\epsilon^M)}$$

Number of iterations needed to hit an inlier with probability 0.99 for the cases of line, circle, homograph, and ellipse fitting



Fraction of inliers

RANSAC vs Hough

- RANSAC can deal only with one model (inliers vs outliers) while Hough detects multiple models
- RANSAC is more efficient when fraction of outliers is low
- RANSAC requires the solution of a minimal set problem,
  - For example, solve of a system of 5 polynomial equations for 5 unknowns
- Hough needs a bounded parameter space
- Hough is intractable for large number of unknowns