MEAM 520
Lecture 4: Homogeneous Transformations

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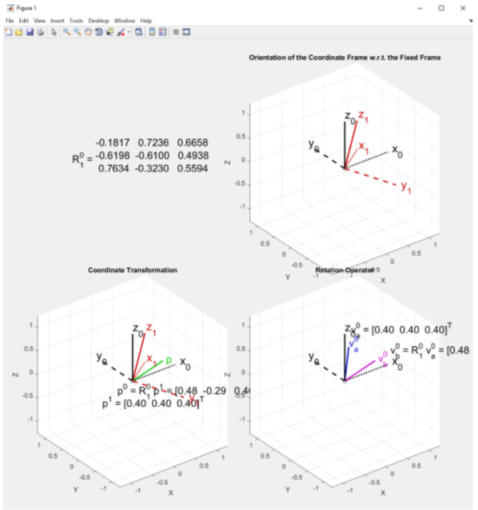
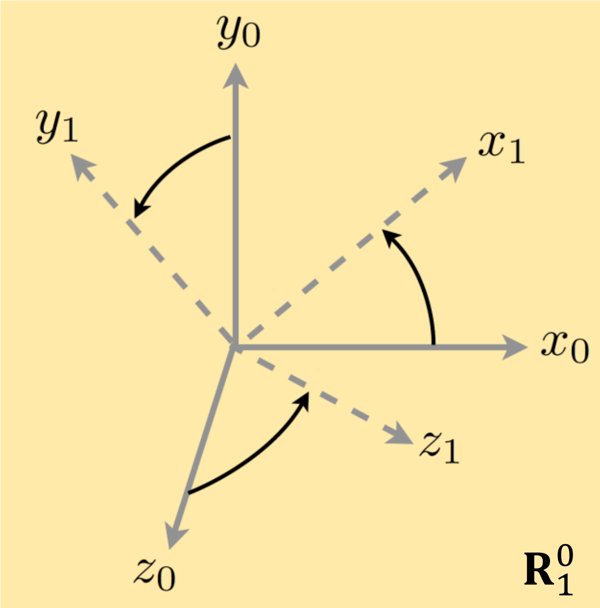
Last Time: Rotation Matrices

$$\mathbf{R}_1^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

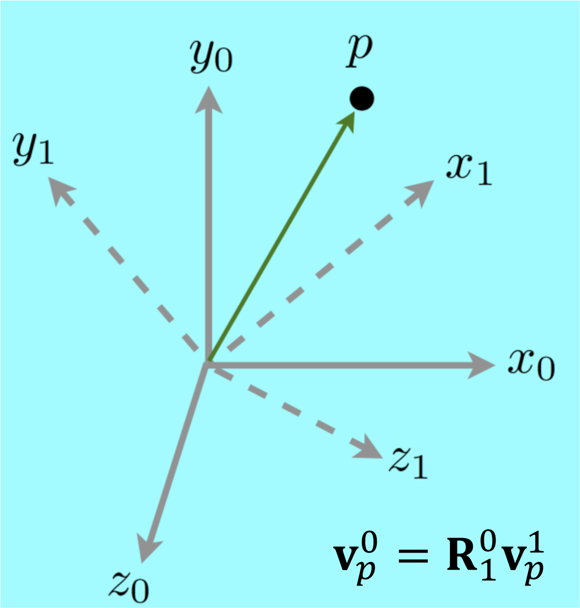
$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \hat{x}_1^0 & \hat{y}_1^0 & \hat{z}_1^0 \end{array}$$



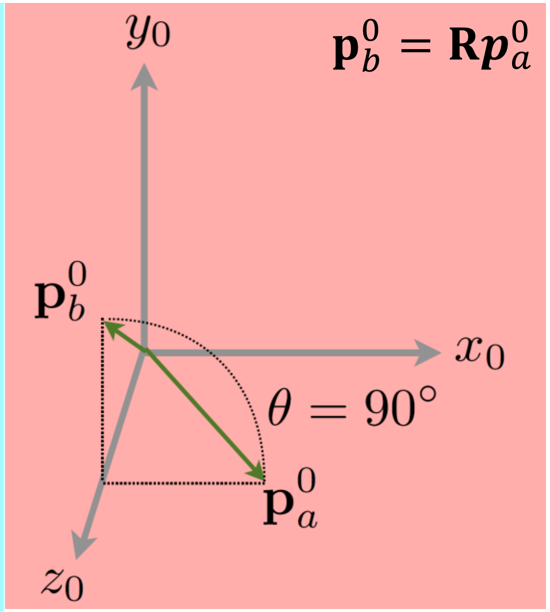
visualizeR.m



Orientation of one coordinate frame with respect to another frame



Coordinate transformation relating the coordinates of a point p in two different frames

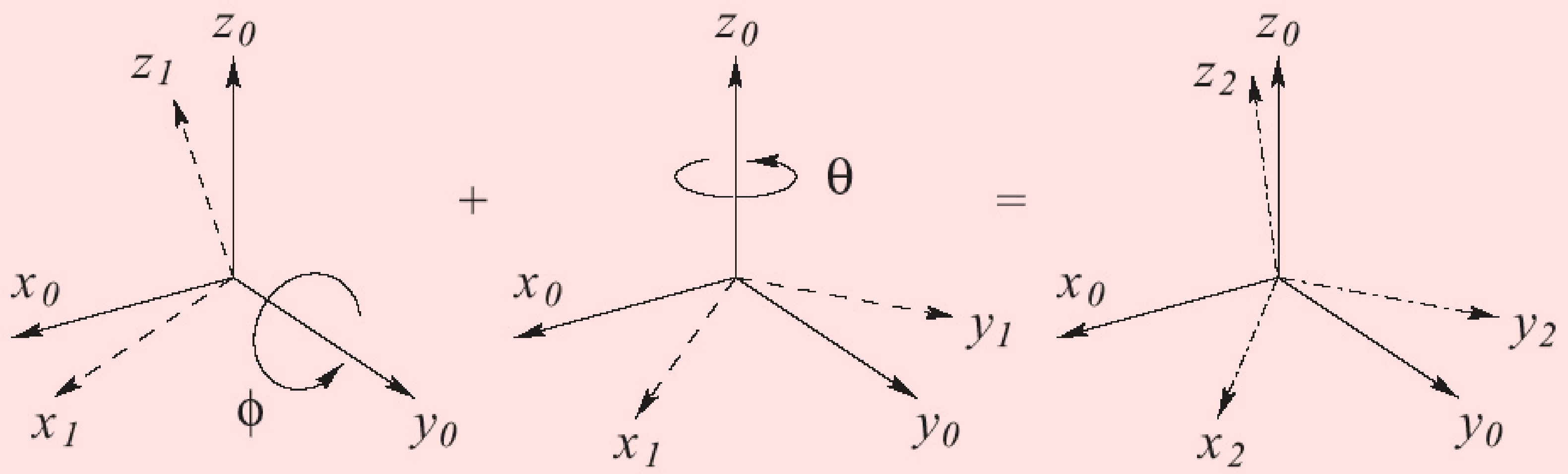


Operator taking a vector and rotating it to yield a new vector in the same coordinate frame

Last Time: Multiplication Order for Rotation Matrices

Rotation about a fixed frame? **pre-multiply**

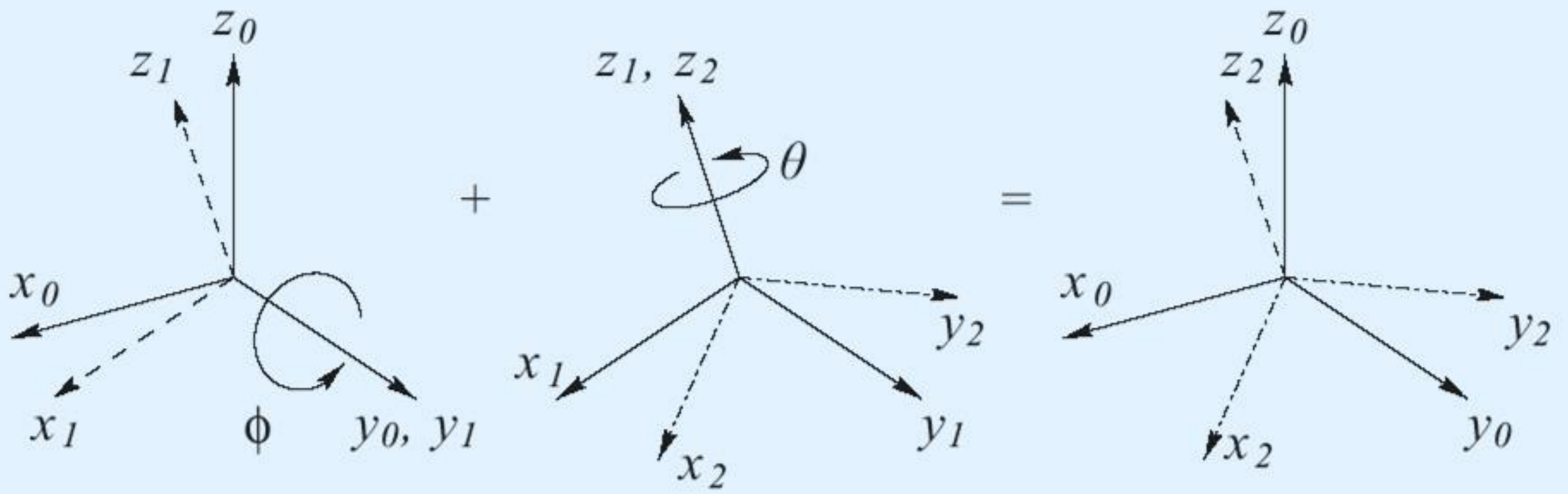
$$\mathbf{R}_2^0 = \mathbf{R}\mathbf{R}_1^0$$



Used in:
Yaw/Pitch/Roll Angles

Rotation about intermediate frame? **post-multiply**

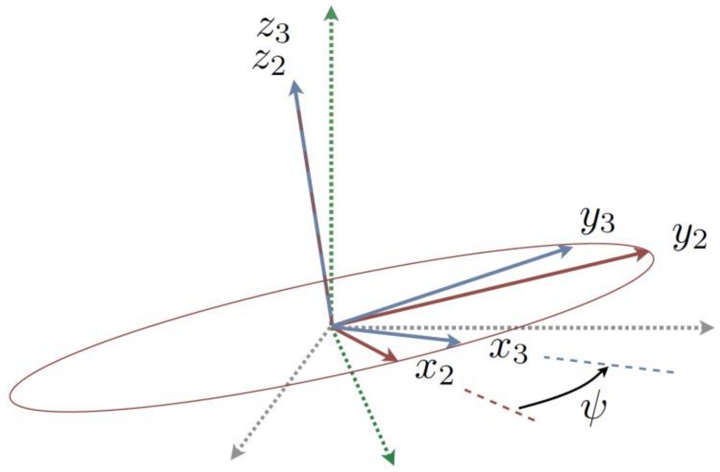
$$\mathbf{R}_2^0 = \mathbf{R}_1^0\mathbf{R}_2^1$$



Used in:
Euler Angles

Euler Angles

Define a set of 3 angles ϕ, θ, ψ to go from 0 → 3 by rotating around the axes of the **current frame**.



$$\mathbf{R} = \mathbf{R}_{z,\phi}\mathbf{R}_{y,\theta}\,\mathbf{R}_{z,\psi}$$

$$\mathbf{R} = \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

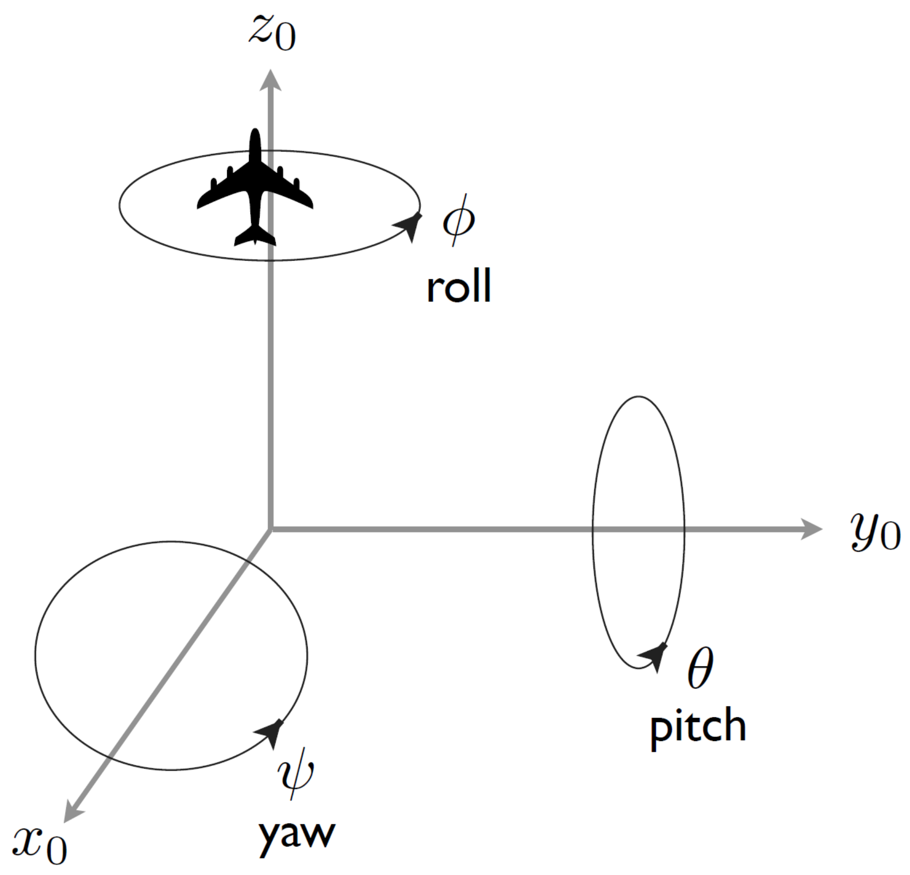
Using Z-Y-Z convention:

1. Rotate by ϕ about z_0
2. Rotate by θ about y_1
3. Rotate by ψ about z_2

$$\mathbf{R} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Yaw, Pitch, Roll Angles

Define a set of 3 angles ϕ, θ, ψ to go from 0 → 3 by rotating around **fixed axes**.



Our book uses X-Y-Z convention.

Think about a plane flying in the z direction. Yaw is left/right, Pitch is up/down, and roll is rotating about z.

Yaw, Pitch, Roll Angles to Rotation Matrices

Pre-multiply using the **basic rotation matrices**

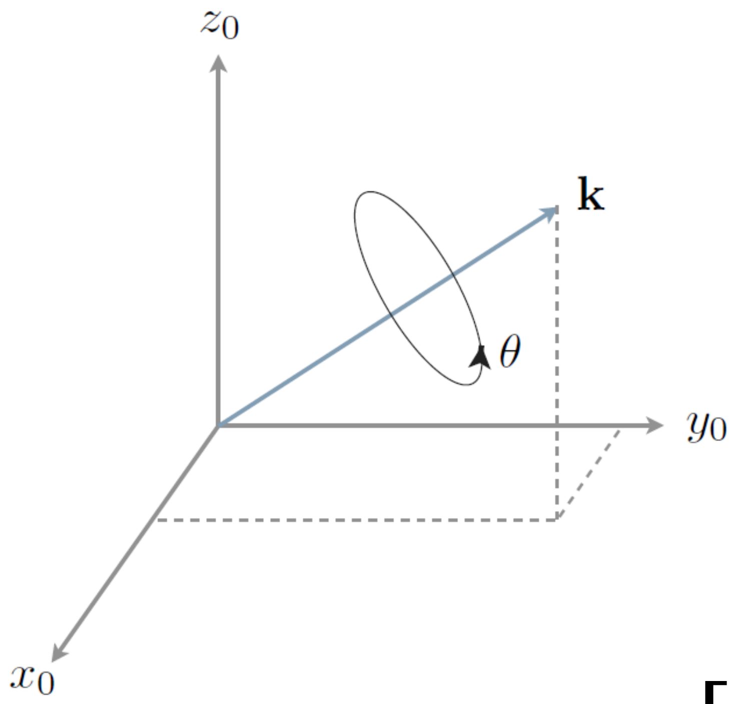
$$\mathbf{R} = \mathbf{R}_{z,\phi}\mathbf{R}_{y,\theta}\,\mathbf{R}_{x,\psi}$$

$$\mathbf{R} = \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} c_\phi c_\theta & c_\phi s_\theta s_\psi - s_\phi c_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & s_\phi s_\theta s_\psi + c_\phi c_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

Angle/Axis Representation

Rotation by an angle about an axis in space



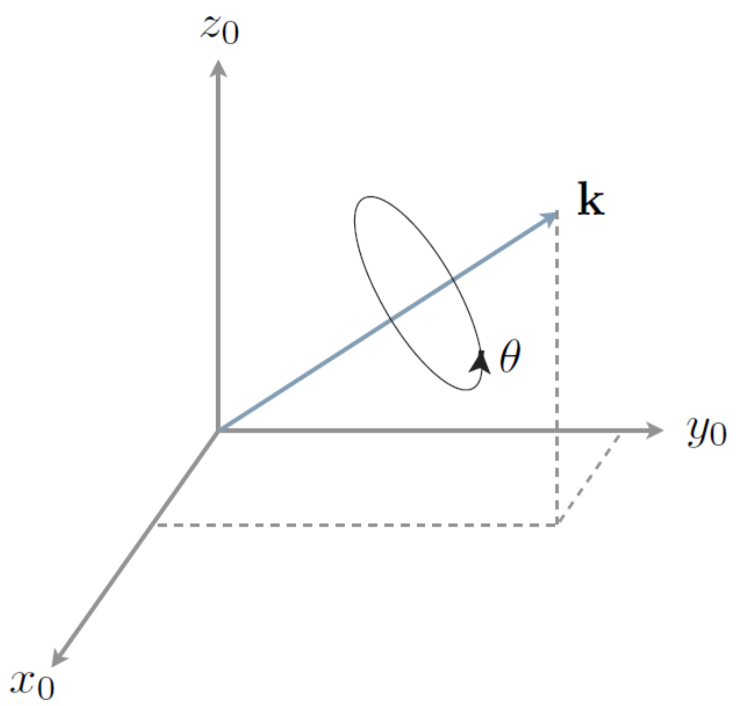
$$\mathbf{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \text{ with } \|\mathbf{k}\| = 1$$

Let $v_\theta = \operatorname{vers}\theta = 1 - c_\theta$

$$\mathbf{R}_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

Angle/Axis Representation

Any rotation matrix can be represented this way!



$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

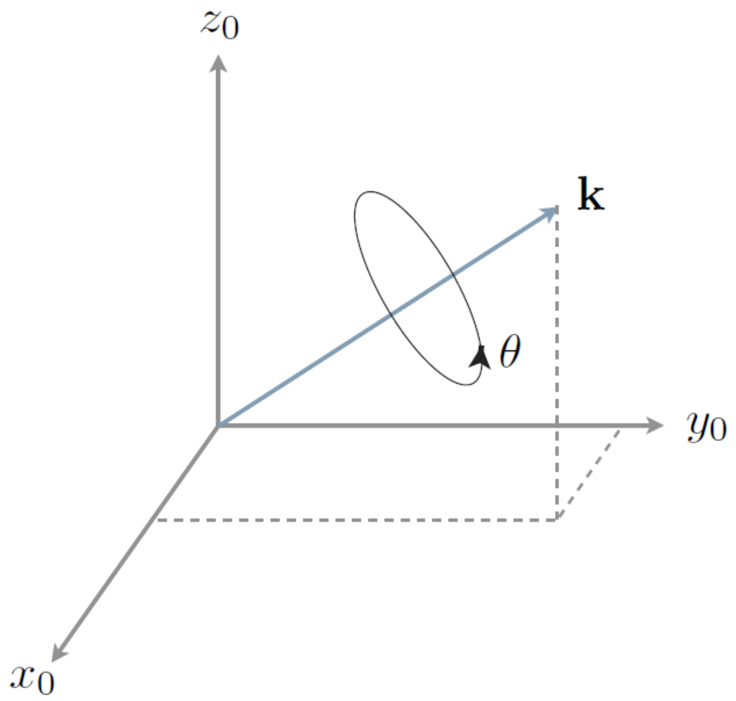
$$\mathbf{k} = \frac{1}{2s_\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

Why?

Angle/Axis Representation

Any rotation matrix can be represented this way!



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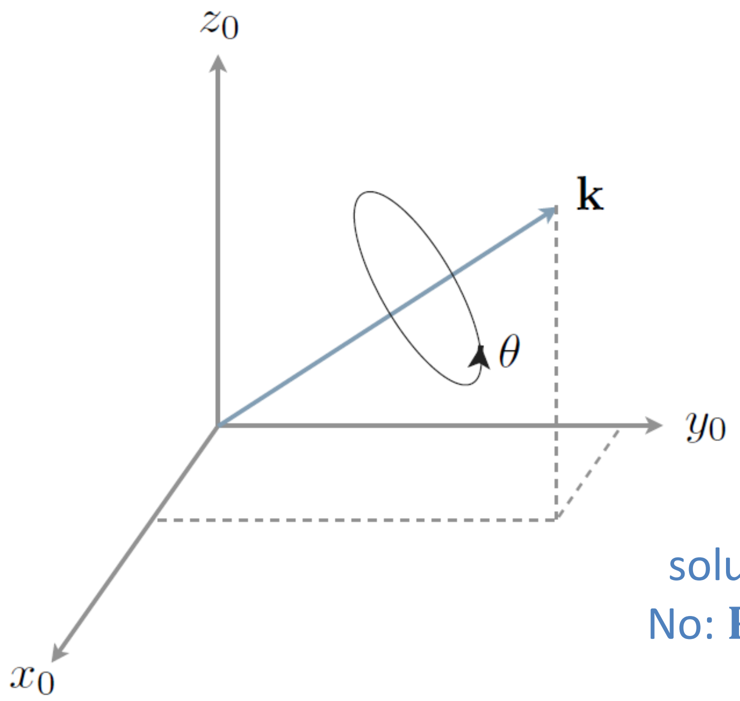
Why? $\mathbf{Rk} = \mathbf{k} \Longrightarrow \mathbf{k}$ is the eigenvector of $\mathbf{R}$ corresponding to eigenvalue $\lambda = 1$

Angle/Axis Representation

Any rotation matrix can be represented this way!



Is axis/angle solution unique?
No: $\mathbf{R}_{k,\theta} = \mathbf{R}_{-k,-\theta}$

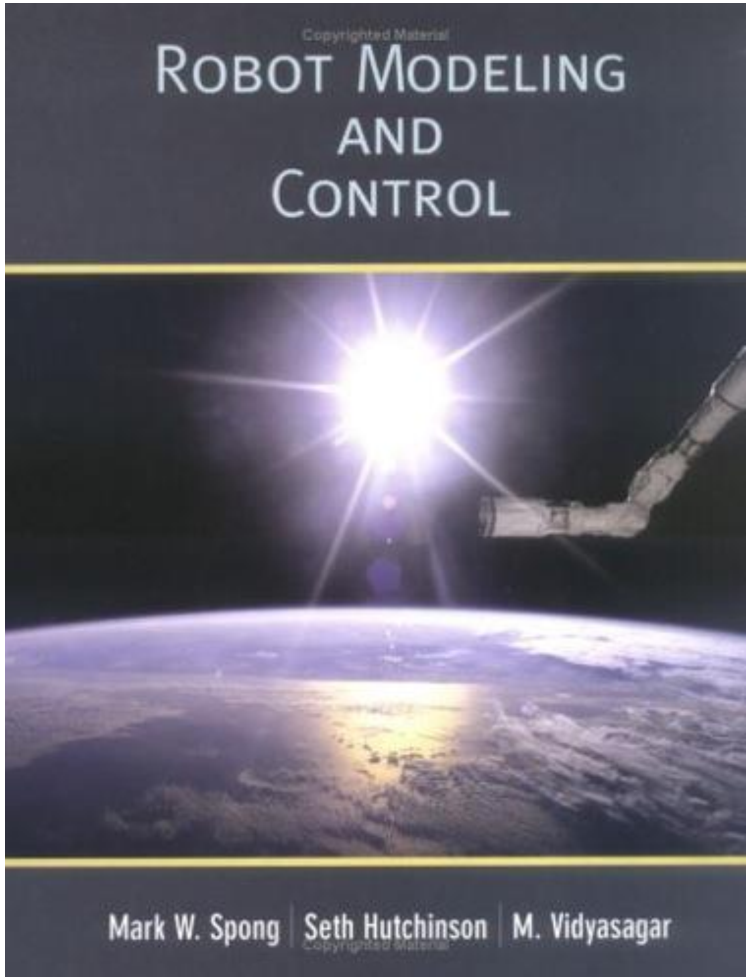
$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\mathbf{k} = \frac{1}{2s_\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

Why? $\mathbf{Rk} = \mathbf{k} \Longrightarrow \mathbf{k}$ is the eigenvector of $\mathbf{R}$ corresponding to eigenvalue $\lambda = 1$

Today: Homogeneous Transformations



Chapter 2: Rigid Motions

- Read Sec. 2.6 - 2.8

Lab 1 is posted (pre-lab due 9/16, lab due 9/23)

Lab 1: Kinematic Characterization of the Lynx

MEAM 520, University of Pennsylvania

September 9, 2020

This lab consists of two portions, with a pre-lab due on **Wednesday, September 16, by midnight (11:59 p.m.)** and a lab (code+report) due on **Wednesday, September 23, by midnight (11:59 p.m.)**. Late submissions will be accepted until midnight on Saturday following the deadline, but they will be penalized by 25% for each partial or full day late. After the late deadline, no further assignments may be submitted; post a private message on Piazza to request an extension if you need one due to a special situation. This assignment is worth 50 points.

You may talk with other students about this assignment, ask the teaching team questions, use a calculator and other tools, and consult outside sources such as the Internet. To help you actually learn the material, what you submit must be your own work, not copied from any other individual or team. Any submissions suspected of violating Penn's Code of Academic Integrity will be reported to the Office of Student Conduct. When you get stuck, post a question on Piazza or go to office hours!

Individual vs. Pair Programming

Work closely with your partner throughout the lab, following these guidelines, which were adapted from "All I really needed to know about pair programming I learned in kindergarten," by Williams and Kessler, *Communications of the ACM*, May 2000. This article is available on Canvas under Files / Resources.

- Start with a good attitude, setting aside any skepticism, and expect to jell with your partner.
- Don't start alone. Arrange a meeting with your partner as soon as you can.
- Use just one setup, and sit side by side. For a programming component, a desktop computer with a large monitor is better than a laptop. Make sure both partners can see the screen.
- At each instant, one partner should be driving (writing, using the mouse/keyboard, moving the robot) while the other is continuously reviewing the work (thinking and making suggestions).
- Change driving/reviewing roles at least every 30 minutes, *even if one partner is much more experienced than the other.* You may want to set a timer to help you remember to switch.
- If you notice an error in the equation or code that your partner is writing, wait until they finish the line to correct them.
- Stay focused and on-task the whole time you are working together.
- Take a break periodically to refresh your perspective.
- Share responsibility for your project; avoid blaming either partner for challenges you run into.
- Recognize that working in pairs usually takes more time than working alone, but it produces better work, deeper learning, and a more positive experience for the participants.

1

Forward Kinematics for Lynx robot

Pre-lab:

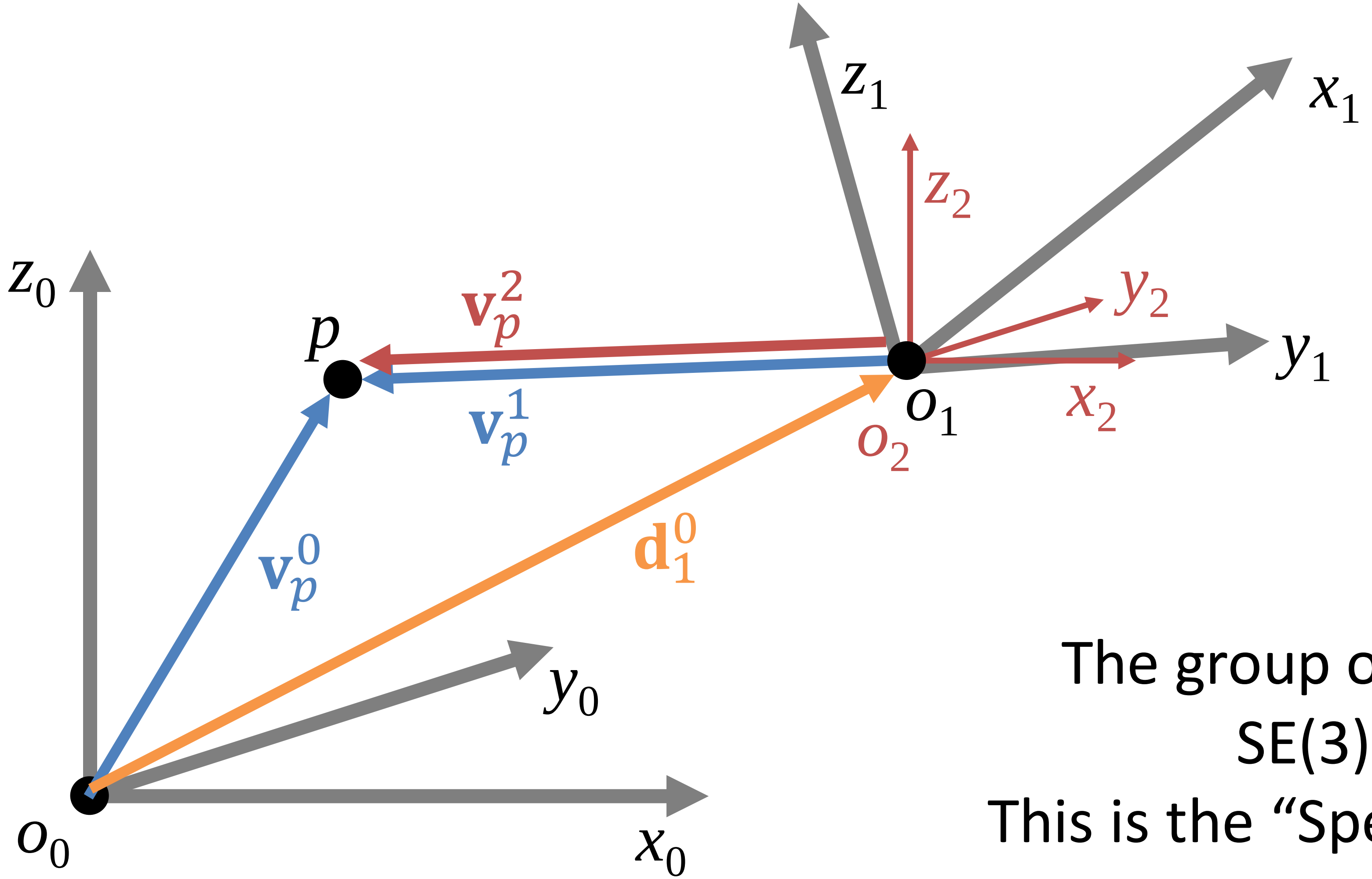
- Must be your own individual work
- Covers up to today's content

Lab:

- Do this in pairs
- Covers up to next Tuesday's content

Rigid Motions

combine pure **translation** and pure **rotation**



$$\mathbf{v}_p^0 = \mathbf{v}_p^2 + \mathbf{v}_{o_2}^0$$

$$\mathbf{v}_p^0 = \mathbf{R}_1^2 \mathbf{v}_p^1 + \mathbf{d}_1^0$$

$$\mathbf{v}_p^0 = \mathbf{R}_1^0 \mathbf{v}_p^1 + \mathbf{d}_1^0$$

The group of all rigid motions is
$SE(3) = \mathbb{R}^3 \times SO(3)$.
This is the "Special Euclidean Group."

Homogeneous Transforms

$$\mathbf{v}_p^0 = \mathbf{R}_1^0 \mathbf{v}_p^1 + \mathbf{d}_1^0$$

$$\begin{bmatrix} \mathbf{v}_p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{d}_1^0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_p^1 \\ 1 \end{bmatrix}$$

homogeneous representation of a vector

homogeneous transformation matrix

Homogeneous Transformation Matrix

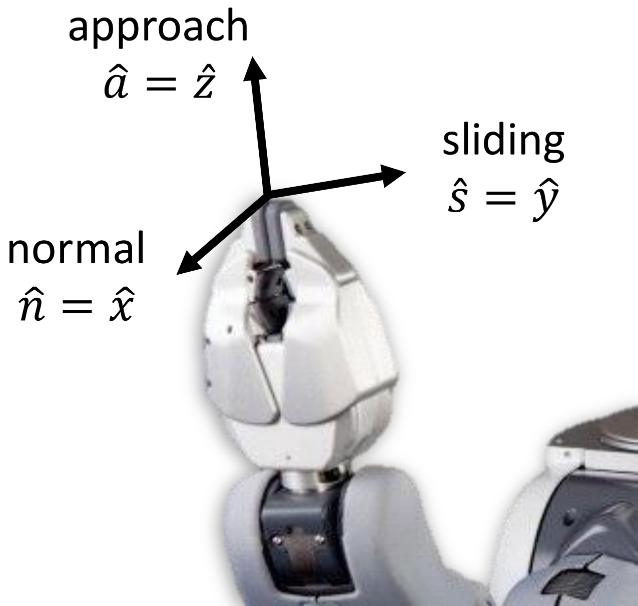
A **homogeneous transformation** is a matrix representation of rigid motion, defined as

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix}$$

where $\mathbf{R}$ is the 3x3 rotation matrix and $\mathbf{d}$ is the 3x1 translation vector

$$\mathbf{H} = \begin{array}{c} \begin{array}{cccc} & \mathbf{R} & & \mathbf{d} \end{array} \\ \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

three zeros one one



Homogeneous Representation

A **homogeneous representation of a vector** is formed by concatenating the original vector with a unit scalar

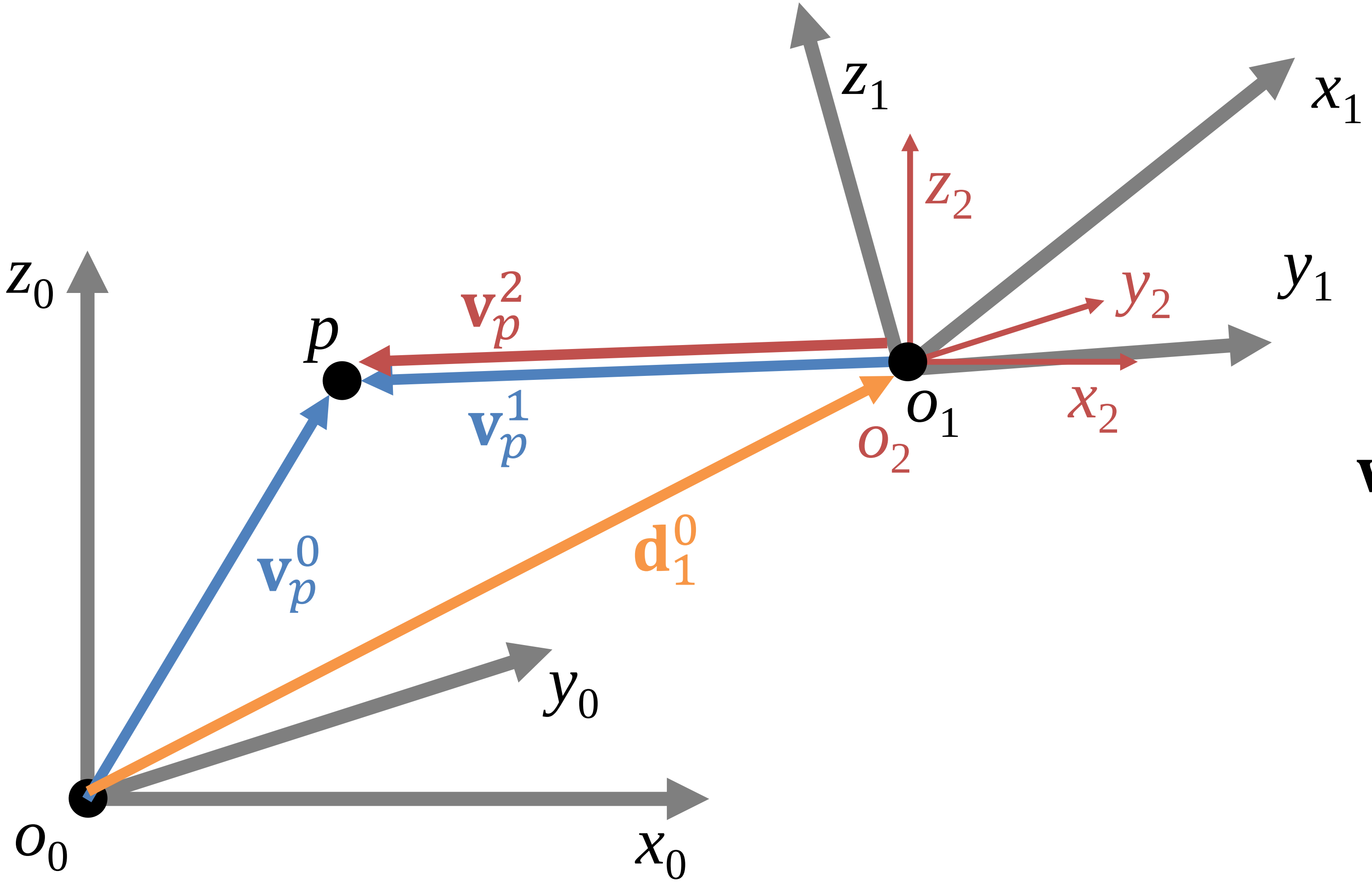
$$\mathbf{P} = \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}$$

where $\mathbf{p}$ is the 3x1 vector

$$\mathbf{P} = \begin{array}{c} \mathbf{v}_p^1 \\ \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \end{array}$$

Rigid Motions

combine pure **translation** and pure **rotation**



$$\mathbf{v}_p^0 = \mathbf{R}_1^0 \mathbf{v}_p^1 + \mathbf{d}_\mathbf{1}^0$$

$$\mathbf{v}_p^0 = \begin{array}{cc} \begin{array}{cccc} & \mathbf{R}_1^0 & & \mathbf{d}_1^0 \end{array} & \mathbf{v}_p^1 \\ \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \end{array}$$

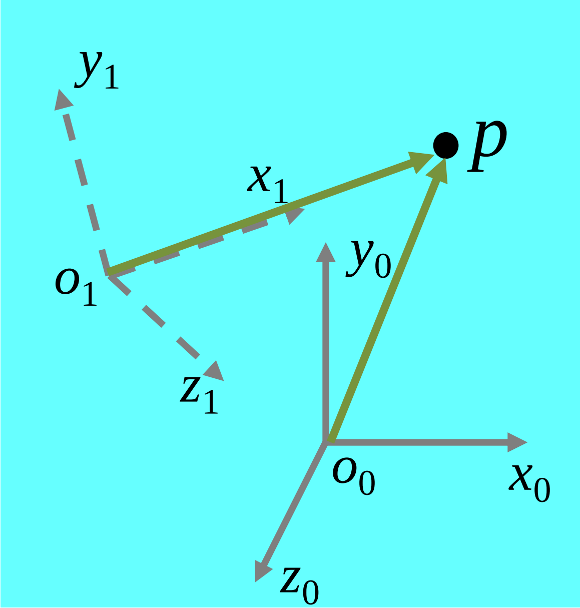
$$\mathbf{P}^0 = \mathbf{H}_1^0 \mathbf{P}^1$$

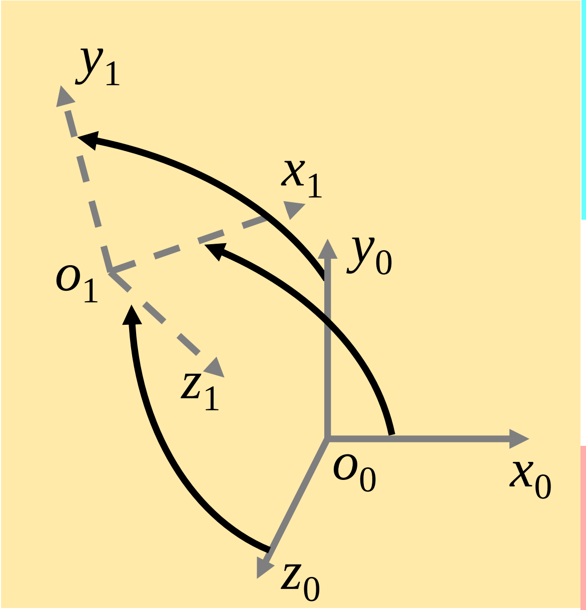
Transformation Matrices

Serve 3 purposes:

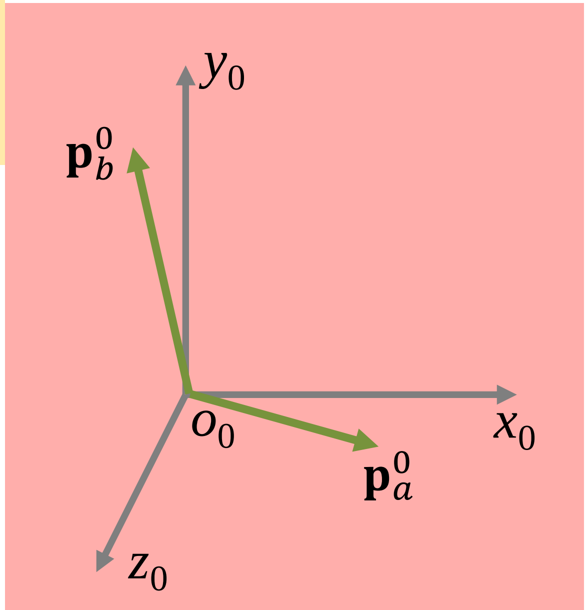
1. Coordinate transformations relating coordinates of a point *p* in two different frames
2. Orientation of a transformed coordinate frame with respect to a fixed frame
3. Operator taking a vector and transforming it to yield a new vector in the same coordinate frame

$\mathbf{P}^0 = \mathbf{H}_1^0\mathbf{P}^1$





$\mathbf{H}_1^0$

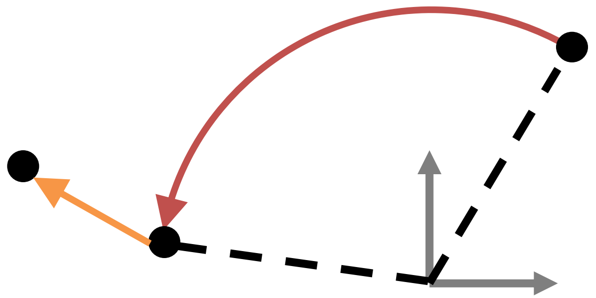


$\mathbf{B} = \mathbf{HA}$

Rigid Motions (Operator Interpretation)

Homogeneous transformations perform **rotation then translation**

$$\mathbf{v}_p^0 = \mathbf{R}\mathbf{v}_p^1 + \mathbf{d}$$

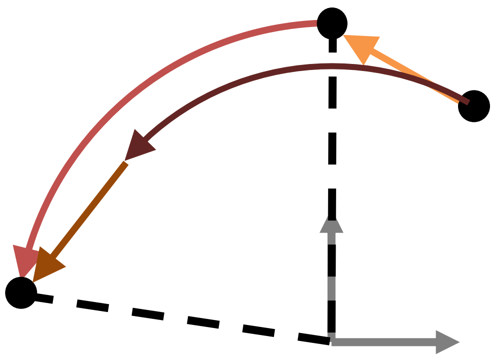


What would **translation then rotation** look like?

$$\mathbf{v}_p^0 = \mathbf{R}\left(\mathbf{v}_p^1 + \mathbf{d}\right)$$

$$\mathbf{v}_p^0 = \mathbf{R}\mathbf{v}_p^1 + \mathbf{R}\mathbf{d}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{R}\mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix}$$

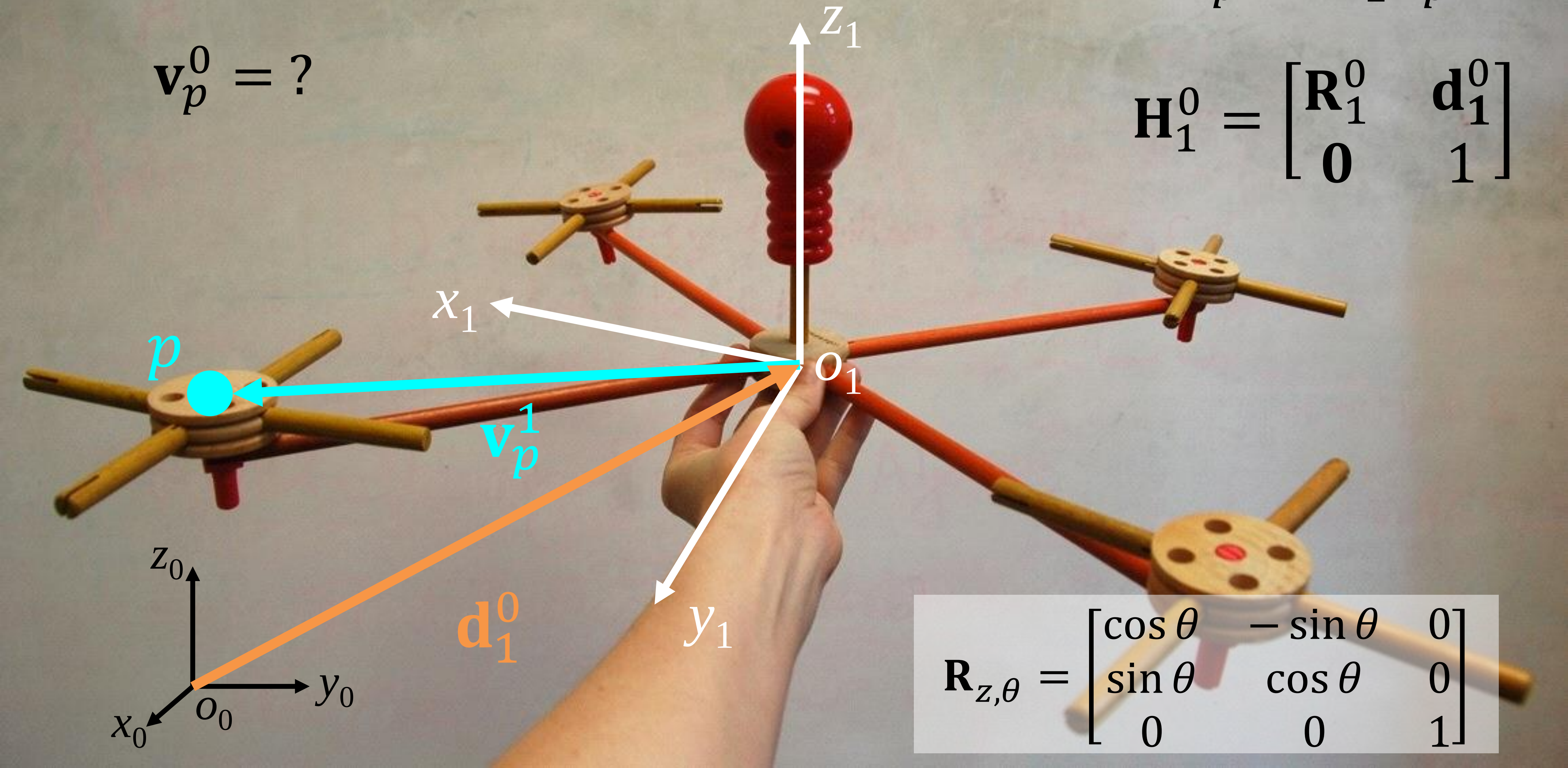


Example with quadrotor model

$$\mathbf{v}_p^0 = \mathbf{H}_1^0 \mathbf{v}_p^1$$

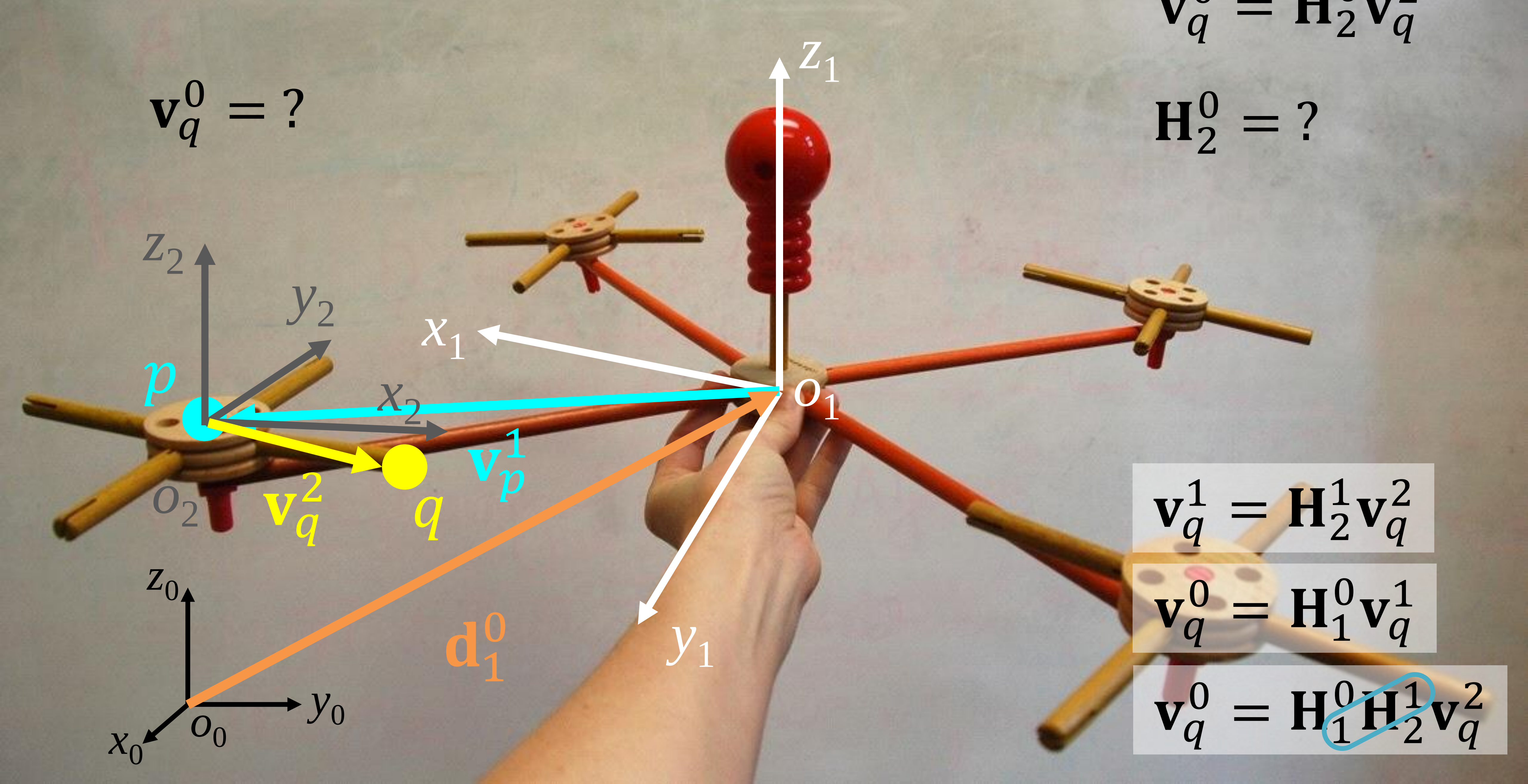
$$\mathbf{v}_p^0 = ?$$

$$\mathbf{H}_1^0 = \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{d}_1^0 \\ \mathbf{0} & 1 \end{bmatrix}$$



$$\mathbf{R}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example with quadrotor model



$$\mathbf{v}_q^0 = ?$$

$$\mathbf{v}_q^0 = \mathbf{H}_2^0 \mathbf{v}_q^2$$

$$\mathbf{H}_2^0 = ?$$

$$\mathbf{v}_q^1 = \mathbf{H}_2^1 \mathbf{v}_q^2$$

$$\mathbf{v}_q^0 = \mathbf{H}_1^0 \mathbf{v}_q^1$$

$$\mathbf{v}_q^0 = \mathbf{H}_1^0 \mathbf{H}_2^1 \mathbf{v}_q^2$$

Compositions of Homogeneous Transformations

Composition of multiple transforms is the same as for rotation matrices

post-multiply when successive transformations
are relative to the intermediate frame

$$\mathbf{H}_2^0 = \mathbf{H}_1^0 \mathbf{H}_2^1$$

pre-multiply when successive transformations
are relative to the fixed world frame

$$\mathbf{H}_2^0 = \mathbf{H}\,\mathbf{H}_1^0$$

Composition via an Intermediate Frame

$$\mathbf{H}_2^0 = \mathbf{H}_1^0 \mathbf{H}_2^1 = \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{d}_1^0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_2^1 & \mathbf{d}_2^1 \\ \mathbf{0} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_2^0 & \mathbf{R}_1^0 \mathbf{d}_2^1 + \mathbf{d}_1^0 \\ \mathbf{0} & 1 \end{bmatrix}$$

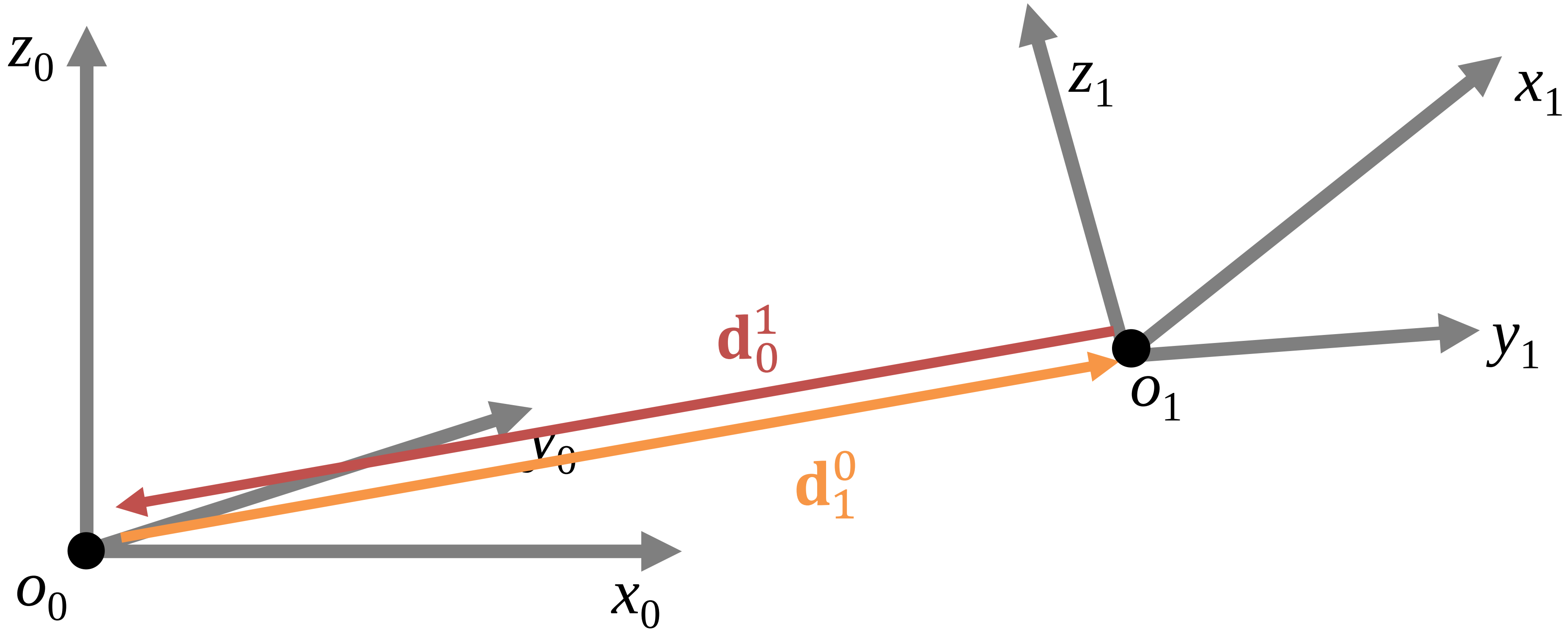
Q: What is the inverse of a homogeneous transformation?

Inverse Transform

$$\mathbf{H}_0^1 = \begin{bmatrix} \mathbf{R}_0^1 & \mathbf{d}_\mathbf{0}^1 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} & \\ \mathbf{0} & 1 \end{bmatrix}$$

Inverse Transform

$$\mathbf{H}_0^1 = \begin{bmatrix} \mathbf{R}_0^1 & \mathbf{d}_0^1 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \overbrace{(\mathbf{R}_1^0)^\top}^{(\mathbf{R}_1^0)^{-1}} & \overbrace{-(\mathbf{R}_1^0)^\top \mathbf{d}_1^0}^{?} \\ \mathbf{0} & 1 \end{bmatrix}$$



Other Properties of Homogeneous Transforms

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix}$$

- $\|\mathbf{HP} - \mathbf{HQ}\| = \|\mathbf{P} - \mathbf{Q}\|$ Why? $\|\mathbf{HP} - \mathbf{HQ}\| = \left\|\left(\mathbf{Rv}_p + \mathbf{d}\right) - \left(\mathbf{Rv}_q + \mathbf{d}\right)\right\| = \left\|\mathbf{R}\left(\mathbf{v}_p - \mathbf{v}_q\right)\right\|$

  Geometric interpretation: $\mathbf{H}$ is distance-preserving

- $\langle \mathbf{HP} - \mathbf{HQ}, \mathbf{HS} - \mathbf{HQ} \rangle = \langle \mathbf{P} - \mathbf{Q}, \mathbf{S} - \mathbf{Q} \rangle$

  Geometric interpretation: $\mathbf{H}$ is angle-preserving

Basic Homogeneous Transformations

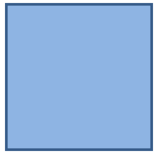
$$\mathrm{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

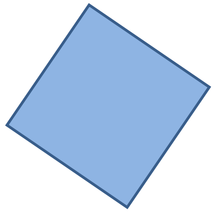
$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices (in 2D)

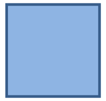
Homogeneous (3DOF)



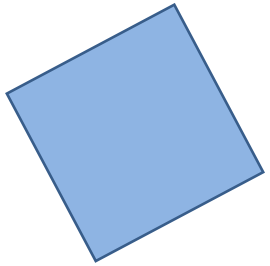
$$\rightarrow \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$



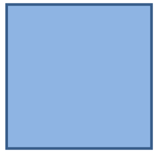
Similarity (4DOF)



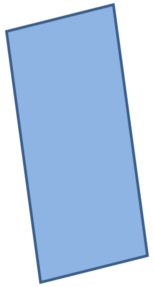
$$\rightarrow \begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$



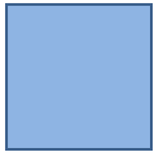
Affine (6DOF)



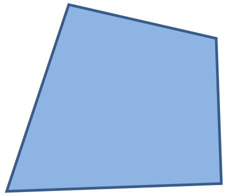
$$\rightarrow \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$



Projective (8DOF)

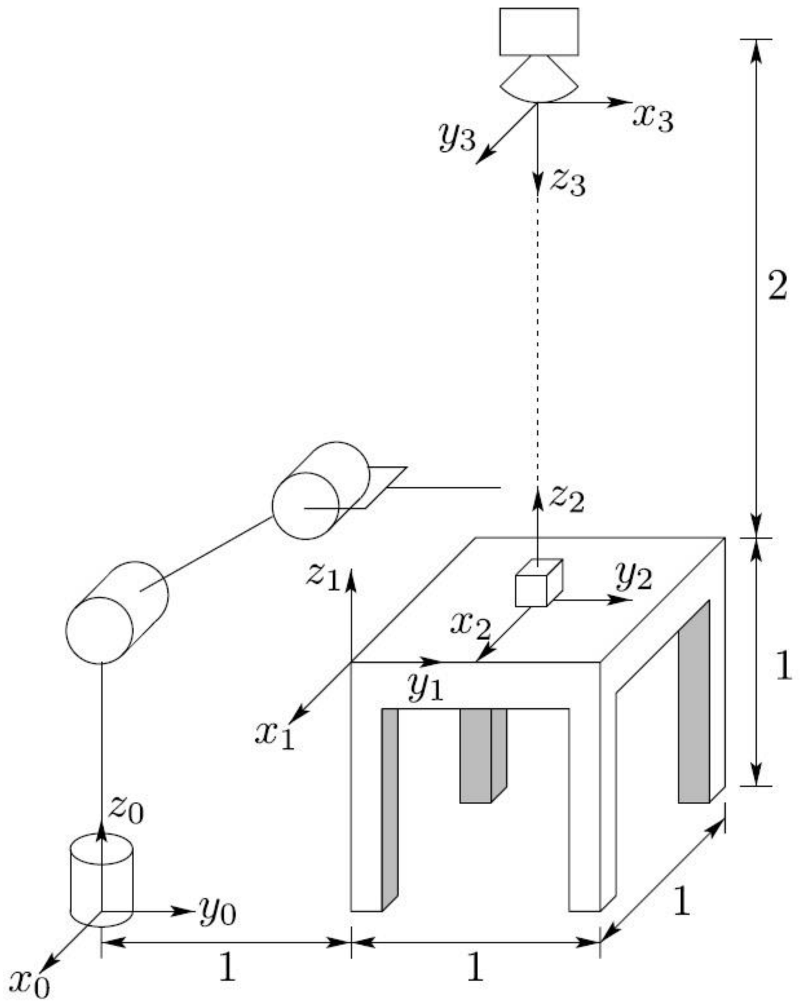


$$\rightarrow \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \rightarrow$$



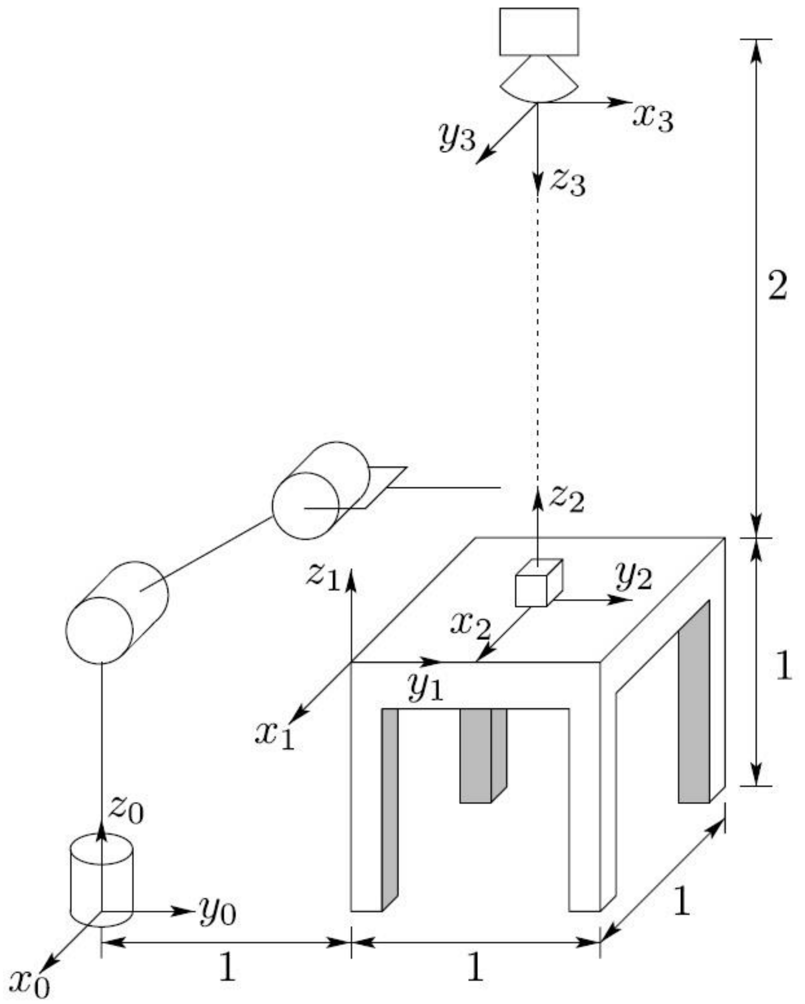
Practice:

1. Write the homogeneous transformations relating frame 0, frame 1, and frame 2 to the camera frame 3.
2. Find the homogeneous transformation relating cube frame 2 to base frame 0.
3. Write an expression for the homogeneous transformation relating cube frame 2 to robot frame 0 as a function of the others.
4. Check this.



Practice:

Write the homogeneous transformations relating frame 0, frame 1, and frame 2 to the camera frame 3.



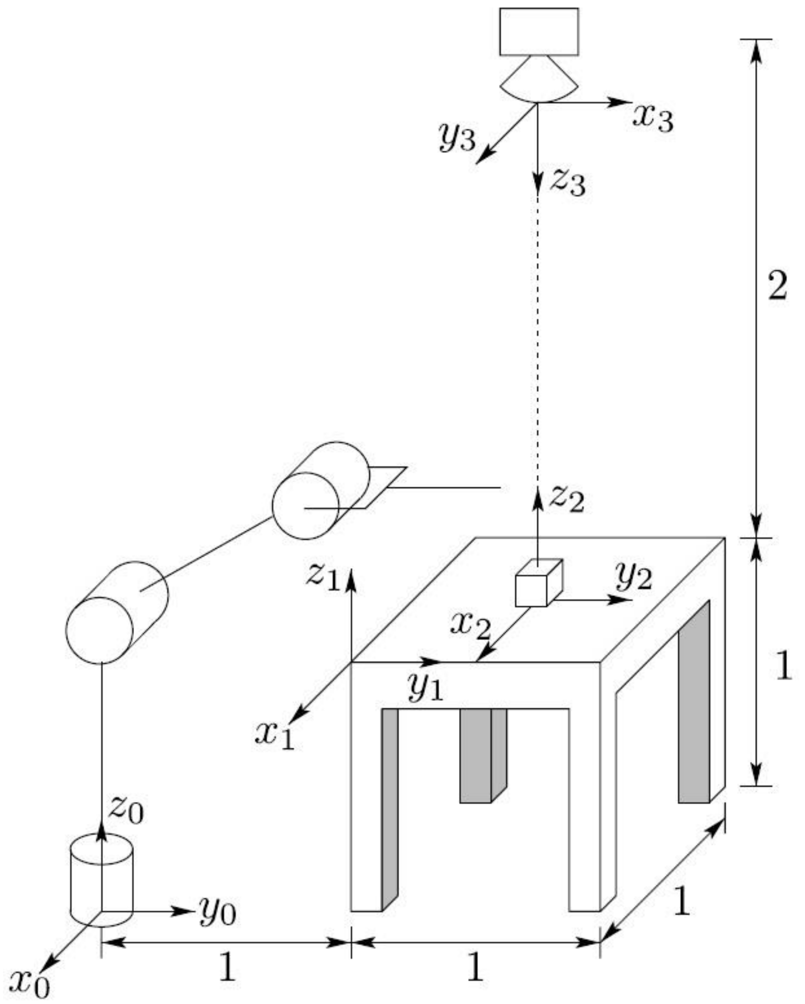
Practice:

Write the homogeneous transformations relating frame 0, frame 1, and frame 2 to the camera frame 3.

$$\mathbf{H}_0^3 = \begin{bmatrix} 0 & 1 & 0 & -1.5\text{m} \\ 1 & 0 & 0 & 0.5\text{m} \\ 0 & 0 & -1 & 3\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

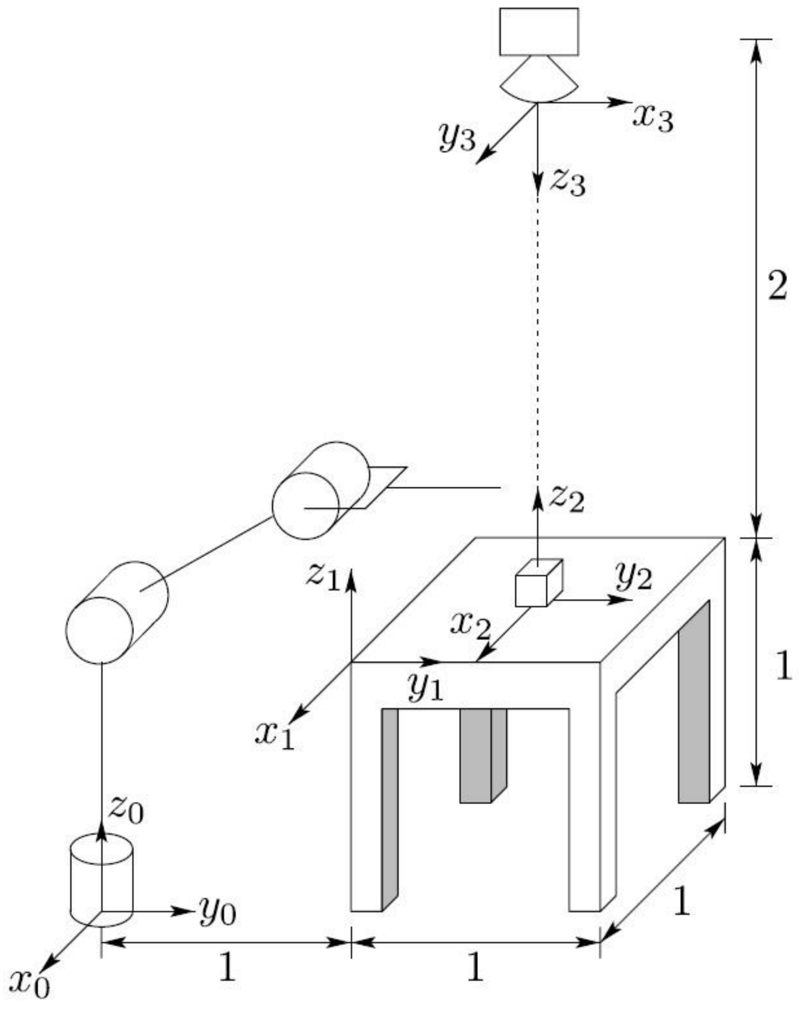
$$\mathbf{H}_1^3 = \begin{bmatrix} 0 & 1 & 0 & -0.5\text{m} \\ 1 & 0 & 0 & 0.5\text{m} \\ 0 & 0 & -1 & 2\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_2^3 = \begin{bmatrix} 0 & 1 & 0 & 0\text{m} \\ 1 & 0 & 0 & 0\text{m} \\ 0 & 0 & -1 & 2\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Practice:

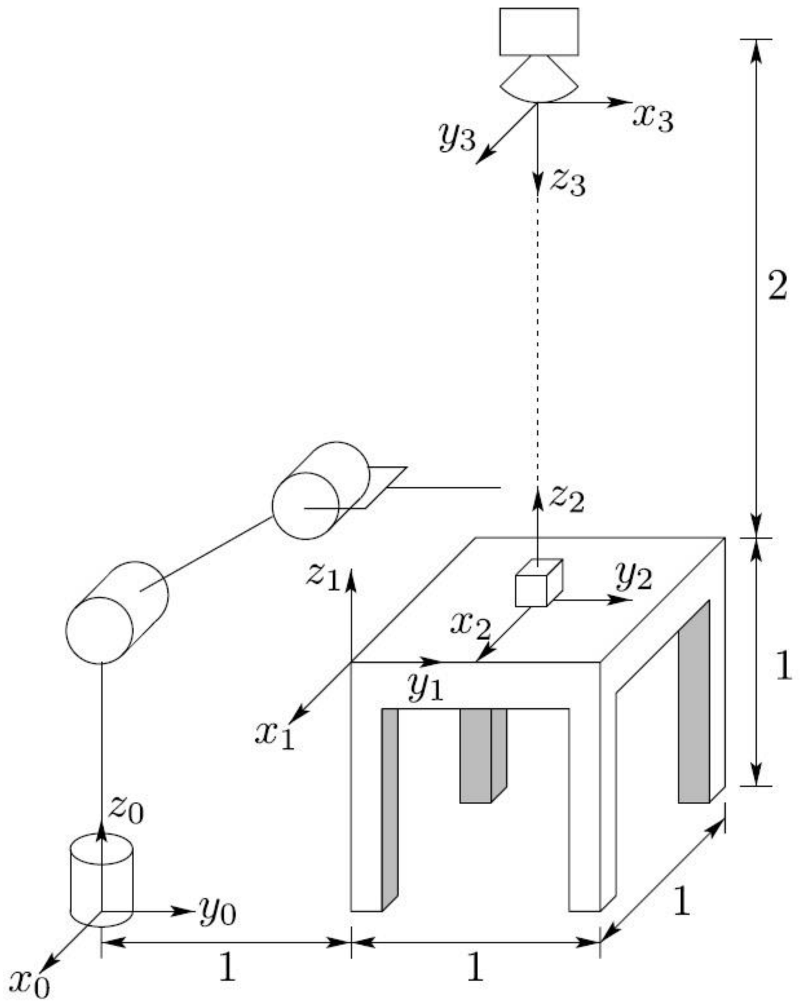
Find the homogeneous transformation relating cube frame 2 to base frame 0.



Practice:

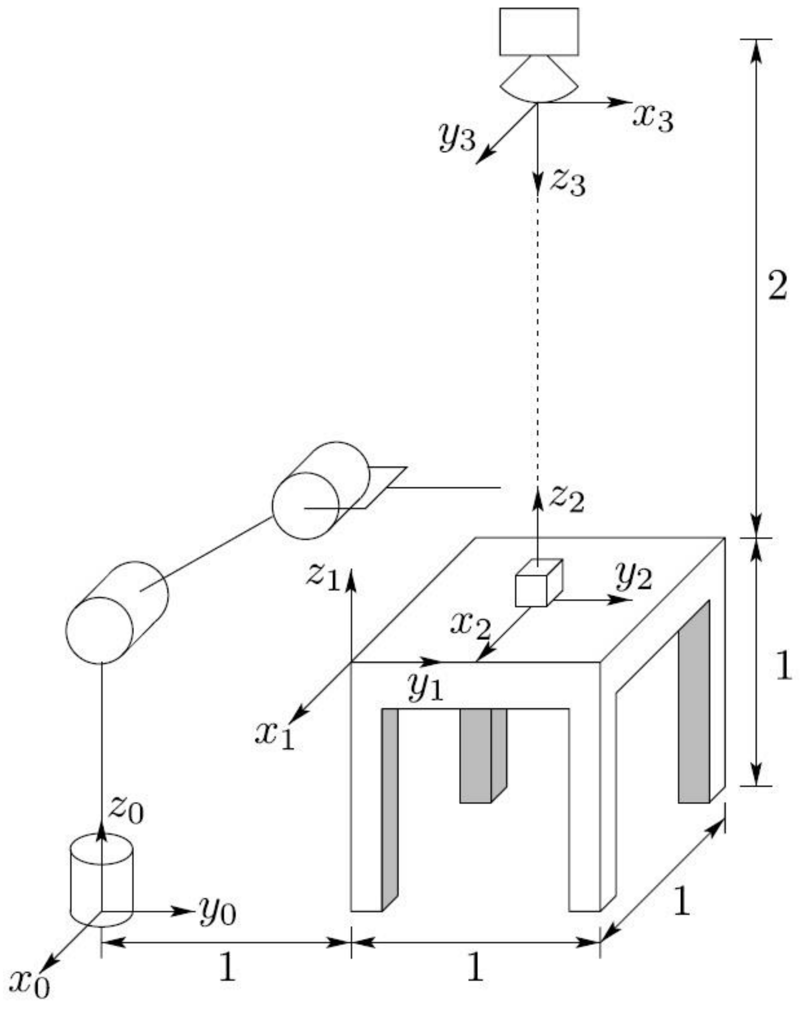
Find the homogeneous transformation relating cube frame 2 to base frame 0.

$$\mathbf{H}_2^0 = \begin{bmatrix} 1 & 0 & 0 & -0.5\text{m} \\ 0 & 1 & 0 & 1.5\text{m} \\ 0 & 0 & 1 & 1\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Practice:

Write an expression for the homogeneous transformation relating cube frame 2 to base frame 0 as a function of the others.



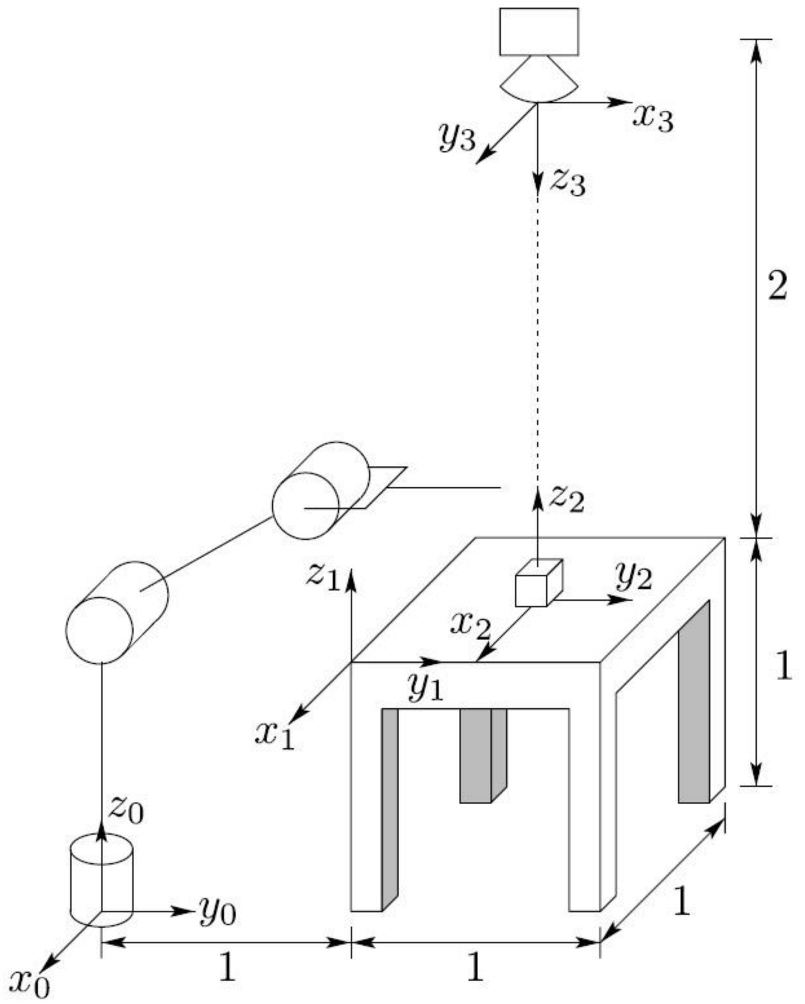
Practice:

Write an expression for the homogeneous transformation relating cube frame 2 to base frame 0 as a function of the others.

$$\mathbf{H}_0^3 = \begin{bmatrix} 0 & 1 & 0 & -1.5\text{m} \\ 1 & 0 & 0 & 0.5\text{m} \\ 0 & 0 & -1 & 3\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{H}_1^3 = \begin{bmatrix} 0 & 1 & 0 & -0.5\text{m} \\ 1 & 0 & 0 & 0.5\text{m} \\ 0 & 0 & -1 & 2\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_2^3 = \begin{bmatrix} 0 & 1 & 0 & 0\text{m} \\ 1 & 0 & 0 & 0\text{m} \\ 0 & 0 & -1 & 2\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{H}_2^0 = \begin{bmatrix} 1 & 0 & 0 & -0.5\text{m} \\ 0 & 1 & 0 & 1.5\text{m} \\ 0 & 0 & 1 & 1\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_2^0 = (\mathbf{H}_0^3)^{-1}\mathbf{H}_2^3$$



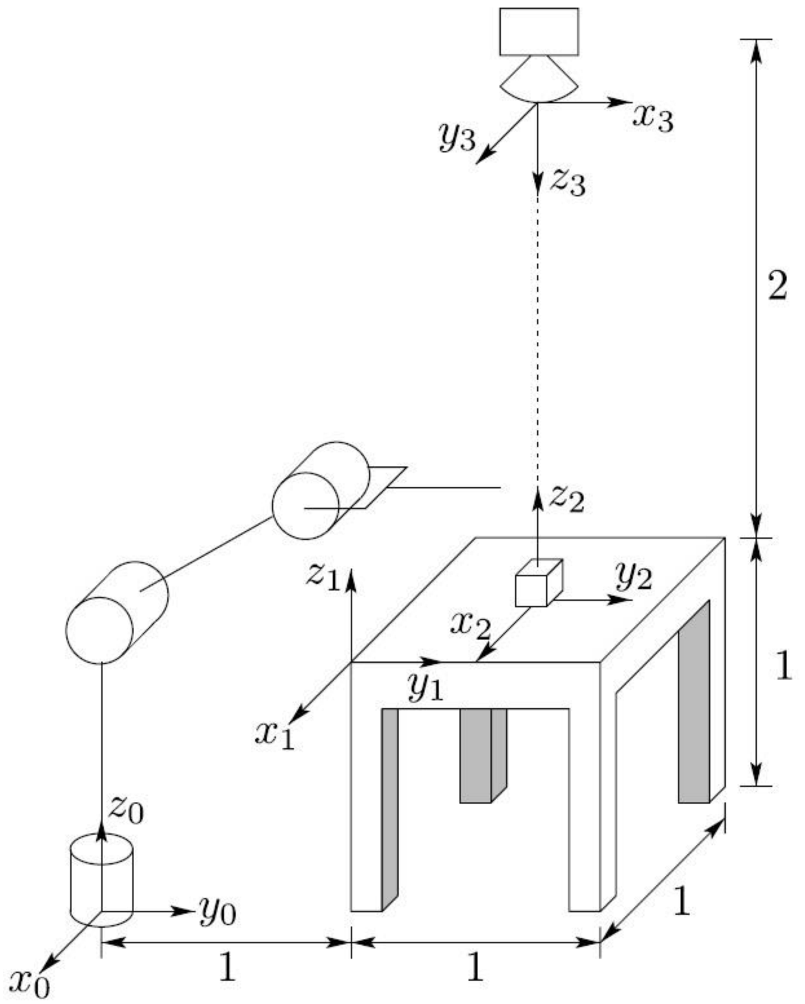
Practice:

Check

$$\mathbf{H}_0^3 = \begin{bmatrix} 0 & 1 & 0 & -1.5\text{m} \\ 1 & 0 & 0 & 0.5\text{m} \\ 0 & 0 & -1 & 3\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{H}_1^3 = \begin{bmatrix} 0 & 1 & 0 & -0.5\text{m} \\ 1 & 0 & 0 & 0.5\text{m} \\ 0 & 0 & -1 & 2\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_2^3 = \begin{bmatrix} 0 & 1 & 0 & 0\text{m} \\ 1 & 0 & 0 & 0\text{m} \\ 0 & 0 & -1 & 2\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{H}_2^0 = \begin{bmatrix} 1 & 0 & 0 & -0.5\text{m} \\ 0 & 1 & 0 & 1.5\text{m} \\ 0 & 0 & 1 & 1\text{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_2^0 = (\mathbf{H}_0^3)^{-1}\mathbf{H}_2^3$$



Computational Considerations

What is the difference between:

$$\mathbf{P}^0 = \mathbf{H}_1^0 \mathbf{P}^1 \qquad \text{and} \qquad \mathbf{p}_v^0 = \mathbf{R}_1^0 \mathbf{p}_v^1 + \mathbf{d}_1^0$$

Computational Considerations

What is the difference between:

practical systems use this

$$\mathbf{P}^0 = \mathbf{H}_1^0 \mathbf{P}^1$$

and

$$\boxed{\mathbf{p}_v^0 = \mathbf{R}_1^0 \mathbf{p}_v^1 + \mathbf{d}_1^0}$$

4 elements x (4 mults + 3 adds)
28 operations

3 elements x (3 mults + 3 adds)
18 operations

multiplying by 0s and 1s is wasteful

Computational Considerations

What is the difference between:

$$\mathbf{p}_v^0 = \left(\mathbf{R}_1^0(\mathbf{R}_2^1\mathbf{R}_3^2)\right)\mathbf{p}_v^3 \qquad \text{and} \qquad \mathbf{p}_v^0 = \mathbf{R}_1^0\left(\mathbf{R}_2^1(\mathbf{R}_3^2\mathbf{p}_v^3)\right)$$

Computational Considerations

What is the difference between:

Use this if $\mathbf{R}$ matrices constant

$$\boxed{\mathbf{p}_v^0 = \left(\mathbf{R}_1^0(\mathbf{R}_2^1\mathbf{R}_3^2)\right)\mathbf{p}_v^3}$$

and

Use this if $\mathbf{R}$ matrices change

$$\boxed{\mathbf{p}_v^0 = \mathbf{R}_1^0\left(\mathbf{R}_2^1(\mathbf{R}_3^2\mathbf{p}_v^3)\right)}$$

multiplying 3x3 matrices takes
27 mults and 18 adds

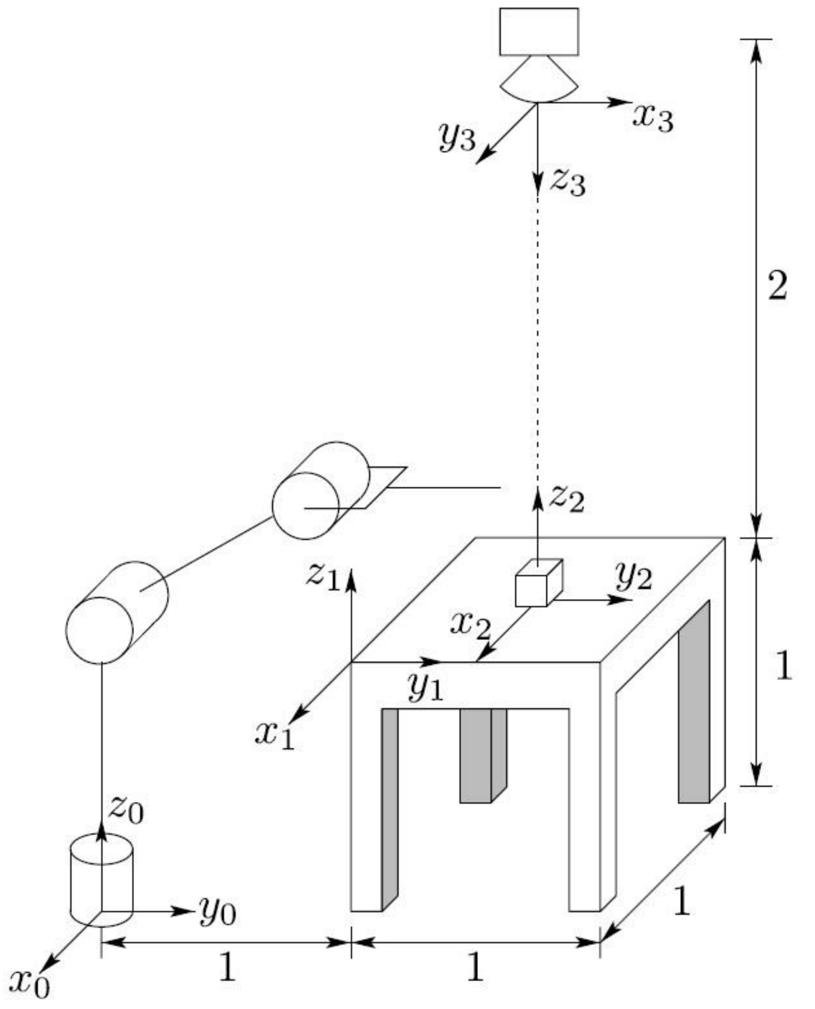
105 operations

multiplying a 3x3 matrix by a 3x1 vector takes
9 mults and 6 adds

45 operations

Example

What is the best way to organize the computation to minimize the calculation effort?



We'll never worry about speed in lab, so it's fine to use $\mathbf{P}^0 = \left(\mathbf{H}_1^0(\mathbf{H}_2^1\mathbf{H}_3^2)\right)\mathbf{P}^3$

Next time: Forward Kinematics for Serial Manipulators

Chapter 3: Forward and Inverse Kinematics

- Read Sec. 3.intro - 3.2