Hidden Markov Models

Slides mostly from Mitch Marcus and Eric Fosler

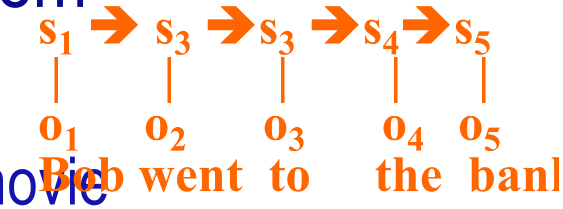
(with lots of modifications).

Learning objectives
Markov model
HMM components

Have you seen
- HMMs?
- dynamic programming?
- Kalman filters or linear dynamical systems?
- LSTMs?

HMMs are dynamic latent variable models

- Given a sequence of *sounds*, find the sequence of *words* most likely to have produced them
- Given a sequence of *images* find the sequence of *locations* most likely to have produced them.
- Given a sequence of *words*, find the sequence of "*meanings*" most likely to have generated them
  - Or *parts of speech*: Noun, verb, adverb, …
  - Or *entity type*: Person, place, company, date, movie
    - E.g. *river bank* vs. *money bank*


s_1 → s_3 → s_3 → s_4 → s_5
o_1 o_2 o_3 o_4 o_5
Bob went to the banl

Conditional Independence

- If we want the joint probability of an entire sequence, the *Markov* assumption lets us treat it as a product of “bigram” conditional probabilities:

$$p(w1,w2,w3,w4) =$$

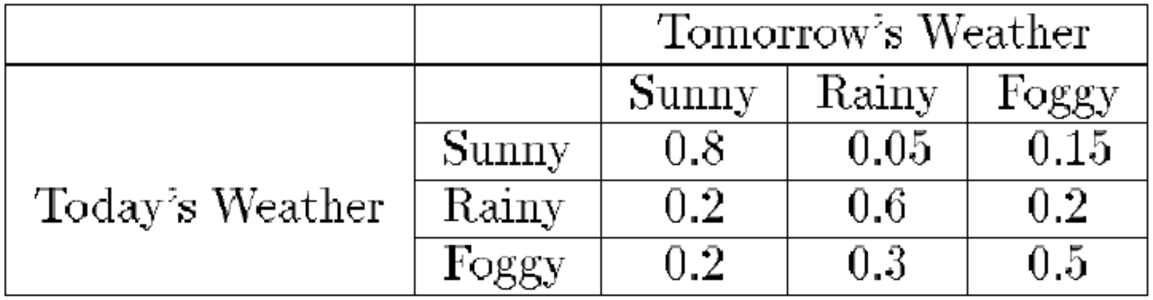
$$p(w1)\ p(w2|w1)\ p(w3|w2,w1)\ p(w4|w3,w2,w1)$$

$$\sim$$

$$p(w1)\ p(w2|w1)\ p(w3|w2) \qquad p(w4|w3)$$

A Markovian weather model:

- Tomorrow is like today, except when it isn’t.



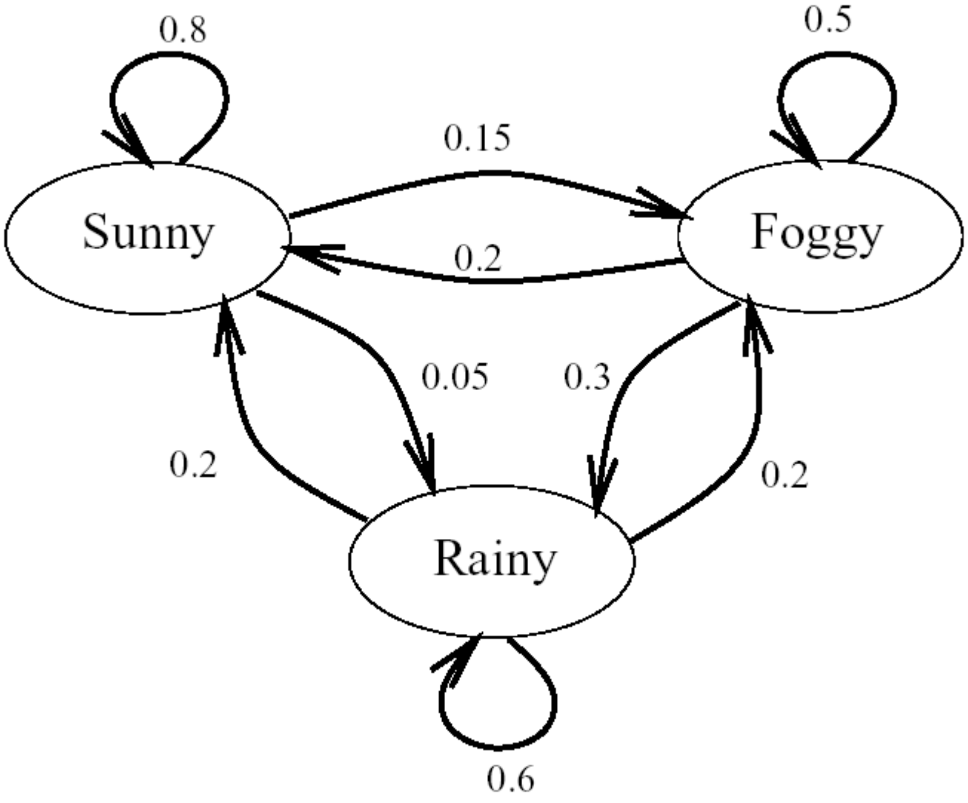
		Tomorrow’s Weather		
Today’s Weather		Sunny	Rainy	Foggy
	Sunny	0.8	0.05	0.15
	Rainy	0.2	0.6	0.2
	Foggy	0.2	0.3	0.5

If you start from any prior distribution over states (weathers) and run long enough you converge to a stationary distribution.

- **Markov matrix gives**

  p(tomorrow’s weather | today’s weather)

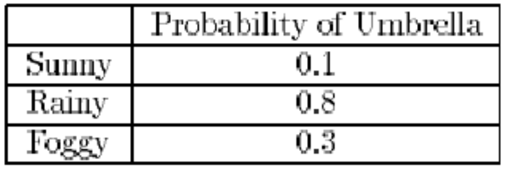
The same model expressed as a graph



Imperfect Knowledge

But sometimes we can only observe a process “as through a glass darkly.” We don’t get to see what is really happening, but only some clues that are more or less strongly indicative of what might be happening.

So you’re bucking for partner in a windowless law office and you don’t see the weather for days at a time … But you do see whether your office mate (who has an actual life) brings in an umbrella or not:



	Probability of Umbrella
Sunny	0.1
Rainy	0.8
Foggy	0.3

How to make predictions?

Now you’re bored with researching briefs,
and you want to guess the weather
from a sequence of umbrella (non)sightings:

$P(w_1, w_2, \ldots w_n \mid u_1, \ldots, u_n)$

You observe u, but not w.

w is the “hidden” part of the
“Hidden Markov Model”

How to do it?

In speech recognition, we will observe the sounds, but not the intended words

Bayes rule rules!

Bayes’ Rule!

$$P(w_1, \ldots, w_n \mid u_1, \ldots, u_n) = \frac{P(u_1, \ldots, u_n | w_1, \ldots, w_n) P(w_1, \ldots, w_n)}{P(u_1, \ldots, u_n)}$$

A Rainy-Day Example

- You go into the office Sunday morning and it’s sunny.
  - w_1 = Sunny
- You work through the night on Sunday, and on Monday morning, your officemate comes in with an umbrella.
  - u_2 = T
- What’s the probability that Monday is rainy?
  - $P(w_2\text{=Rainy} \mid w_1\text{=Sunny}, u_2\text{=T})$ =

    $P(u_2\text{=T}|w_2\text{=Rainy})/P(u_2\text{=T}|\ w_1\text{=Sunny})$ x $P(w_2\text{=Rainy}|\ w_1\text{=Sunny})$

    (likelihood of umbrella)/normalization x prior

Bayes rule for speech

- **To find the most likely word**
  - Start with a prior of how likely each word is
  - And the likelihood of each set of sounds given the word
- **The most likely word is the one most likely to have generated the sounds heard**

The “fundamental equation of speech recognition”:

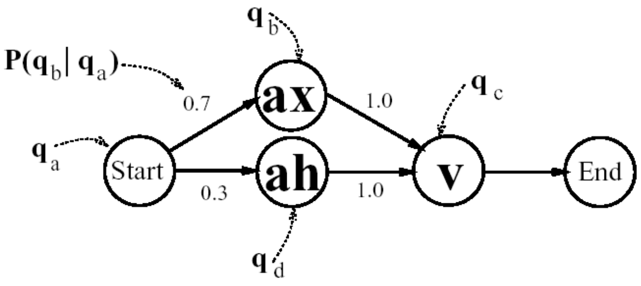
$$\mathit{argmax}_w\, P(w|u) = \mathit{argmax}_w\, P(u|w)\, P(w) \,/\, P(u)$$

Speech Recognition

- **Markov model for words in a sentence**

  P(I like cabbages) = P(I|START)P(like|I)P(cabbages|like)

- **Markov model for sounds in a word**
  - Model the relation of words to sounds by breaking words down into pieces



HMMs can also extract meaning

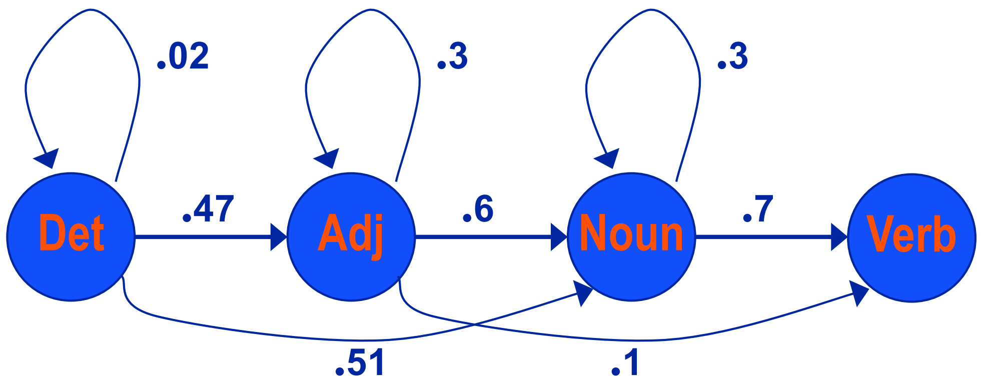
- Natural language is ambiguous
  - “Banks banks in banks on banks.”
- Sequence of hidden states are the “meanings” (what the word refers to) and words are the percepts
- The (Markov) Language Model says how likely each meaning is to follow other meanings
  - All meanings of “banks” may produce the same percept

HMMs: Midpoint Summary

- **Language can be modeled by HMMs**
  - Predict words from sounds
  - Captures priors on words
  - Hierarchical
    - Phonemes to morphemes to words to phrases to sentences
  - Was used in all commercial speech recognition software
    - Now replaced by deep networks
- **Markov assumption, HMM definition**
  - “Fundamental equation of speech recognition”

Hidden Markov Models (HMMs)

Part of Speech tagging is often done using HMMs



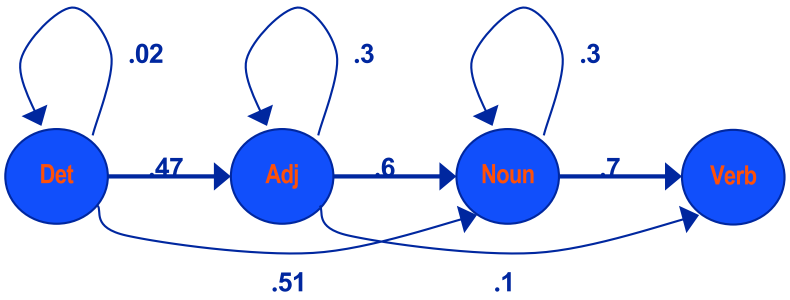
P(w|Det)	
a	.4
the	.4

P(w|Adj)	
good	.02
low	.04

P(w|Noun)	
price	.001
deal	.0001

HMM: The Model

- **Starts in some initial state s_i**
- **Moves to a new state s_j with probability $p(s_j\,|s_i) = a_{ij}$**
  - The matrix *A* with elements a_{ij} is the *Markov transition matrix*
- **Emits an observation o_v (e.g. a word, w) with probability $p(o_v\,|s_i) = b_{iv}$**



P(w|Det)

a	.4
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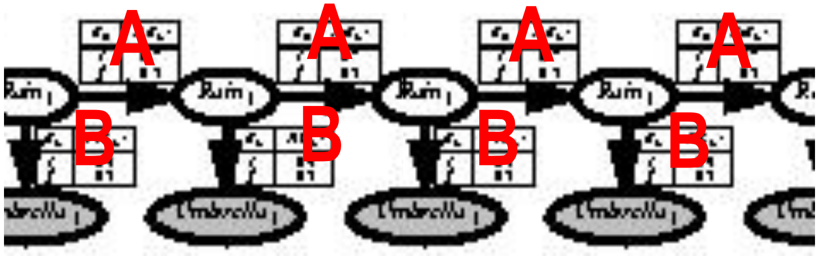
A HMM is a dynamic Bayesian Network

There is a node for the hidden state and for the emission (observed state) at each time step, but the probability tables are the same at all times.



A: Markov transition matrix
B: Emission probabilities

Recognition using an HMM

To find the tag sequence T which maximizes $P(T|W)$, we note that:

Note that the ‘tags’ *t* are the hidden states (*s*) and the words *w* are the emissions *(o)*

$$P(T|W) \propto \pi(t_1) * \prod_{i=1}^{n-1} a(t_i, t_{i+1}) * \prod_{i=1}^{n} b(t_i, w_i)$$

Transitions Emissions

So we need to find

$$\hat{t}_{1,n} = \arg\max_{t_{1,n}} \pi(t_1) * \prod_{i=1}^{n-1} a(t_i, t_{i+1}) * \prod_{i=1}^{n} b(t_i, w_i)$$

$$\begin{aligned} P(T) * P(W|T) \quad = & \\ & P(t_1) * P(t_2|t_1) * P(t_3|t_2) * \ldots * P(t_n|t_{n-1}) * \\ & P(w_1|t_1) * P(w_2|t_2) * \ldots * P(w_n|t_n) \end{aligned}$$

Parameters of an HMM

- ***States***: A set of states $S = s_1,\ldots,s_k$
- ***Markov transition probabilities***: $A = a_{1,1},a_{1,2},\ldots,a_{k,k}$ Each $a_{i,j} = p(s_j \mid s_i)$ represents the probability of transitioning from state s_i to s_j.
- ***Emission probabilities***: A set B of functions of the form $b_i(o_t) = p(o|s_i)$ giving the π_i probability of observation o_t being emitted by s_i
- ***Initial state distribution***: the probability that s_i is a start state

The Three Basic HMM Problems

- ***Problem 1 (Evaluation):*** Given the observation sequence $O=o_1,\dots,o_T$ and an HMM model $\lambda = (A,B,\pi)$, compute the probability of O given the model.
- ***Problem 2 (Decoding):*** Given the observation sequence $O=o_1,\dots,o_T$ and an HMM model $\lambda = (A,B,\pi)$
  find the state sequence that best explains the observations

Problem 3 (Learning): Pick the model parameters $\lambda = (A,B,\pi)$ to maximize $P(O \mid \lambda)$

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What you should know

- **HMMs**
  - Markov assumption
  - Markov transition matrix, Emission probabilities
- **Many other models generalize HMM**
  - Emission can be a real valued (e.g. Gaussian) function of the hidden state
  - The hidden state can be a real valued vector instead of a "one hot" discrete state
    - Instead of moving with a Markov Transition Matrix between states, one moves with Gaussian noise between real states
  - Nonlinear versions give dynamical neural nets