Lecture 3:

$$X_i \overset{iid}{\sim} \text{dist}(\mu, \sigma)$$
$$\bar{X} = \frac{X_1 + \cdots + X_n}{n}$$

CLT:

$$F_{\bar{X}}(x) = \Pr\{\bar{X} \le x\}$$

$$\Phi_{\mu, \sigma^2/n}(x) : \text{CDF of } N(\mu, \sigma^2/n)$$

Then:

$$\left| F_{\bar{X}}(x) - \Phi_{\mu, \sigma^2/n}(x) \right| \le \frac{c}{n}$$

(for all $x \in \mathbb{R}$)

$\forall\, a, b$

$$\Pr\{\bar{X} \in [a,b]\} = \Pr\{N(\mu, \sigma^2/n) \in [a,b]\} + \frac{c}{n}$$

Gaussian Distribution:

$$X \sim N(\mu, \sigma^2) \longrightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

↳ mean : μ

variance : σ^2

Notation : Note that we'll be using $N(\mu, \sigma^2)$ and σ^2 is the variance

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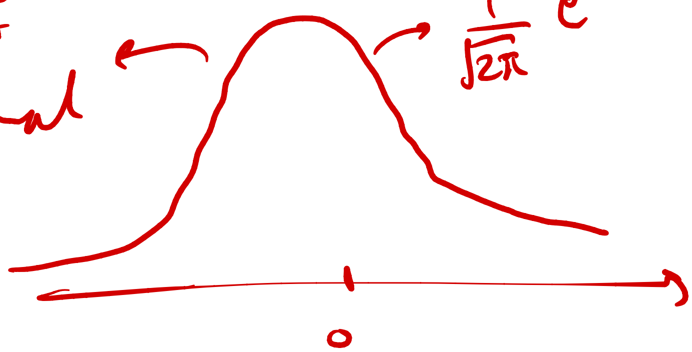
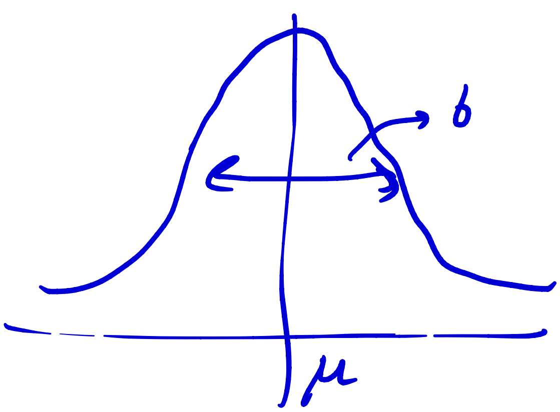
Normal Distribution $\triangleq \underbrace{N(0,1)}_{Z}$

PDF of the Normal Dist.

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



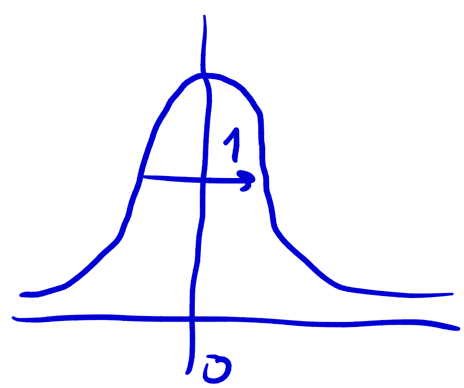
$$N(\mu, \sigma^2) = \mu + \sigma Z$$



first ×σ

then add μ

←

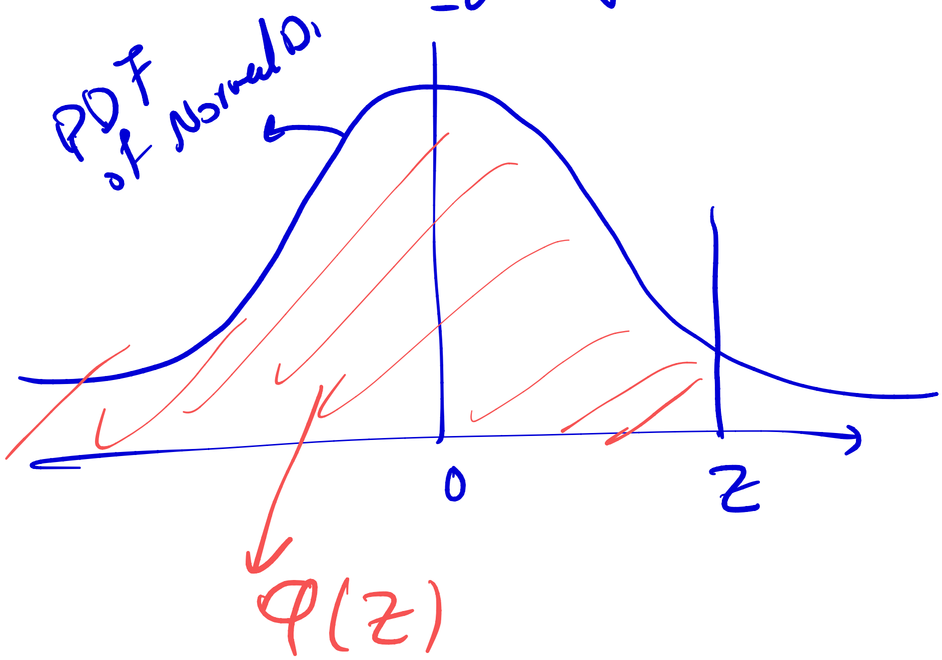


A few definitions:

The Gaussian Cumulative Distribution Function

$$\phi(z) = \Pr\{\underbrace{Z}_{N(0,1)} \le z\}$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2} \, dx$$



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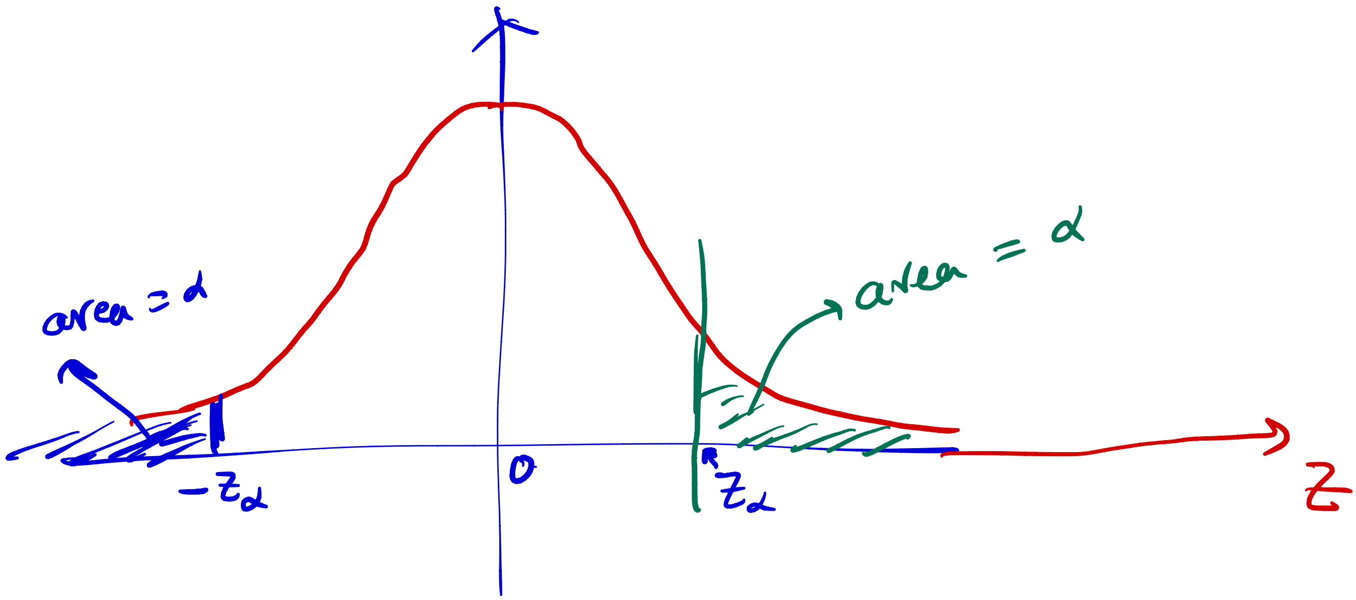
z_α : Given a value $\alpha \in [0,1]$ we define z_α as the point on the z-axis such the area under the normal PDF after z_α is equal to α.



Important Fact:
Sum of independet Gaussians is a Gaussian.

$$\begin{cases} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{cases} \qquad X_1 \perp\!\!\!\perp X_2$$

Then
$$\begin{cases} X_1 + X_2 = N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \\ Var(X_1 + X_2) = Var(X_1) + Var(X_2) \\ \qquad\qquad\qquad = \sigma_1^2 + \sigma_2^2 \end{cases}$$

Two r.v.'s are called independent if

$$Pr\{X_1 \in A, X_2 \in B\} = Pr\{X_1 \in A\} \cdot Pr\{X_2 \in B\}$$

$\forall A, B$

Q: If $X_1, -, X_n$ are iid and $N(\mu, \sigma)$ then what is the distribution of

$$\frac{X_1 + \cdots + X_n}{n} ?$$

A: $X_i \sim N(\mu, \sigma^2)$ then:

$$\begin{cases} X_1 + \cdots + X_n = N(\mu \cdot n, \quad n\sigma^2) \\ Var(X_1 + \cdots + X_n) = Var(X_1) + \cdots + Var(X_n) \end{cases}$$

$$\sigma^2 + \cdots + \sigma^2$$

$$= n\sigma^2$$

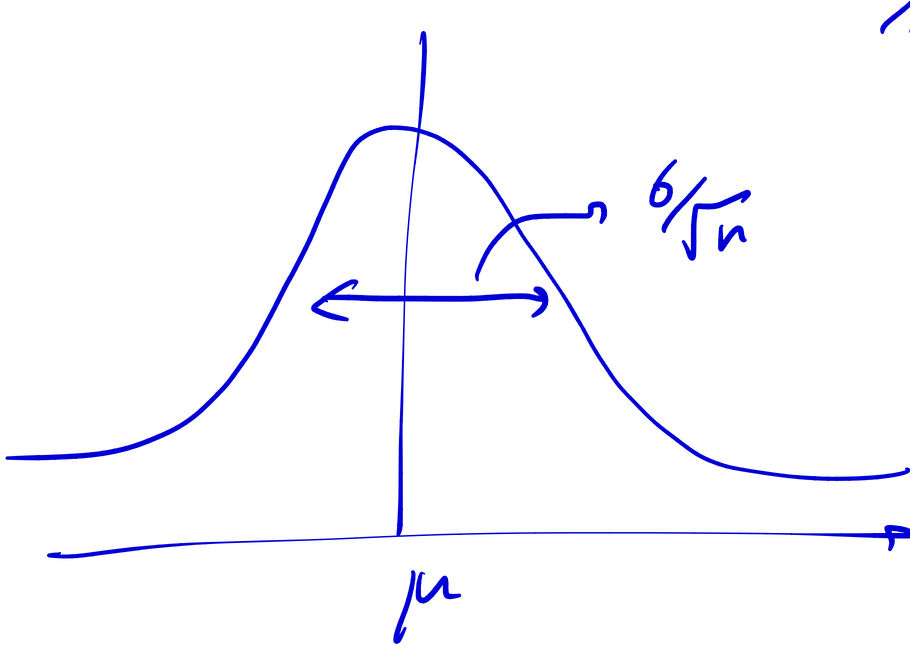
$$\bar{X} = \frac{1}{n}\left(X_1 + \cdots + X_n \right)$$

$$\rightarrow \begin{cases} E[\bar{X}] = \frac{1}{n} E[X_1 + \cdots + X_n] = \underline{\underline{\mu}} \\[2ex] Var(\bar{X}) = \frac{1}{n^2} Var(X_1 + \cdots + X_n) \\[1ex] \qquad\qquad = \frac{\sigma^2}{n} \end{cases}$$

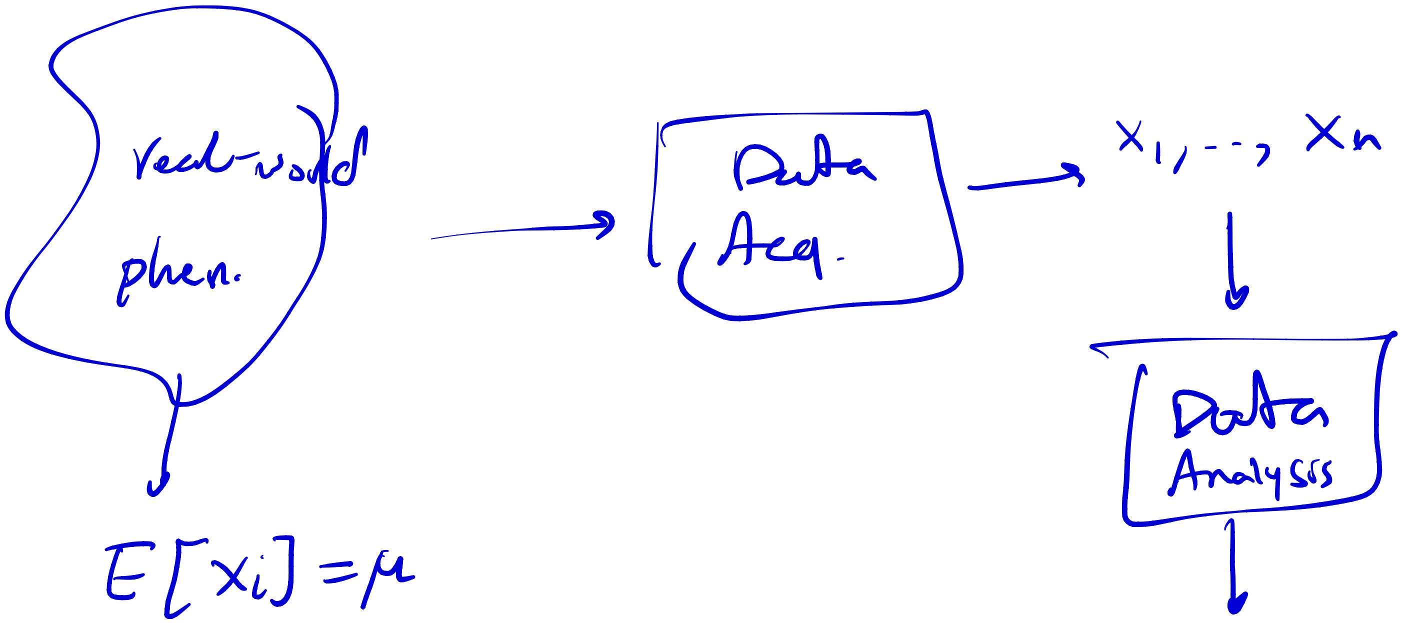
$$\Rightarrow \boxed{\bar{X} = N\left(\mu, \frac{\sigma^2}{n}\right)}$$

Confidence intervals:

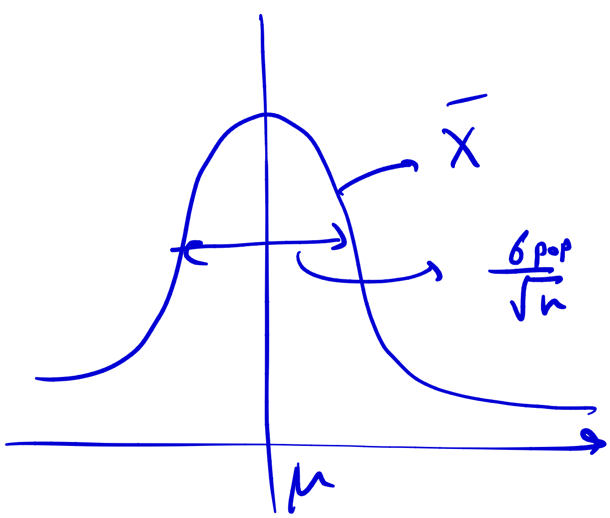
$$\frac{X_1+\dots+X_n}{n} \to \mu, \quad \sigma^2/n$$



So far we've seen that the variance of $\bar{X}$ decays like σ^2/n. The question that we'd like to answer now is "How can we obtain guarantees on how good the estimator $\bar{X}$ is ?



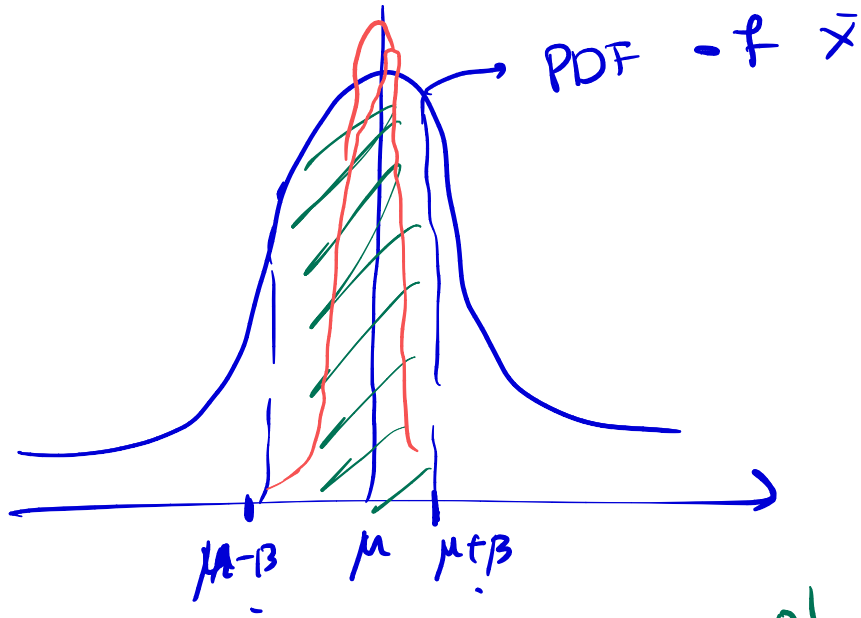
$$\bar{X} = \frac{x_1 + \dots + x_n}{n}$$



(1) idea: $\Pr\{\bar{X} = \mu\} = ?$

think of β to be a small number

(2) right question $\Pr\{\mu \in [\bar{X} - \beta, \bar{X} + \beta]\} = ?$



$$\Pr\left\{ \mu \in \left[\bar{X} - \overset{0.01}{\beta},\ \bar{X} + \overset{0.01}{\beta} \right] \right\} = ?$$

Equivalently, we'd like to know for what value of β we have:

$$\Pr\left\{ \mu \in \underbrace{\left[\bar{X} - \beta,\ \bar{X} + \beta \right]} \right\} = \underbrace{\overbrace{1-\alpha}^{\text{e.g. } 0.95}}_{\text{given}}$$

As the first step, assume that

$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$.

$$\Pr\{\bar{X} \in [\mu-\beta, \mu+\beta]\}$$

$$= \Pr\{\mu \in [\bar{X}-\beta, \bar{X}+\beta]\}$$

$$\bar{X} \in [\mu-\beta, \mu+\beta]$$

$$\begin{cases} \bar{X} \le \mu+\beta \iff \bar{X}-\beta \le \mu \\ \bar{X} \ge \mu-\beta \iff \mu \le \bar{X}+\beta \end{cases}$$

$$\bar{X} \in [\mu-\beta, \mu+\beta] \iff \mu \in [\bar{X}-\beta, \bar{X}+\beta]$$

So we'd like to compute

$$\Pr\left\{ \bar{X} \in [\mu-\beta, \mu+\beta] \right\} \leftarrow$$

$$X_i \sim N(\mu, \sigma) \Rightarrow \bar{X} = N\left(\mu, \sigma^2/n\right)$$

$$\Pr\left\{ \mu-\beta \le \bar{X} \le \mu+\beta \right\}$$

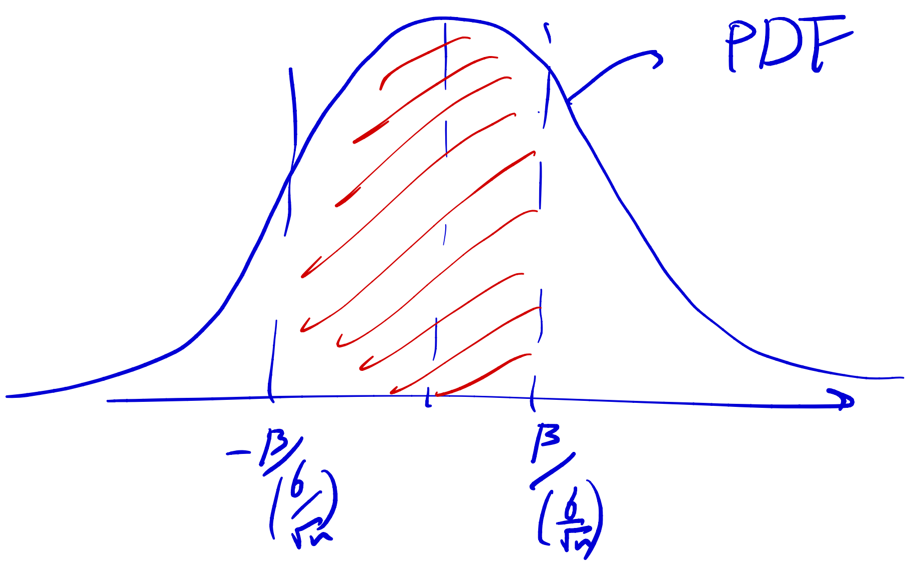
$$= \Pr\left\{ -\beta \le \bar{X}-\mu \le \beta \right\}$$

$$= \Pr\left\{ \frac{-\beta}{\frac{\sigma}{\sqrt{n}}} \le \underbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}_{Z} \le \beta/\left(\frac{\sigma}{\sqrt{n}}\right) \right\}$$

$$\bar{X} = N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) = \mu + \frac{\sigma}{\sqrt{n}} Z^{\rightarrow N(0,1)}$$

$$\Rightarrow \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = Z^{\rightarrow N(0,1)}$$

$$= \Pr\left\{ -\beta \Big/ \frac{\sigma}{\sqrt{n}} \le Z \le \beta \Big/ \left(\frac{\sigma}{\sqrt{n}}\right) \right\}$$

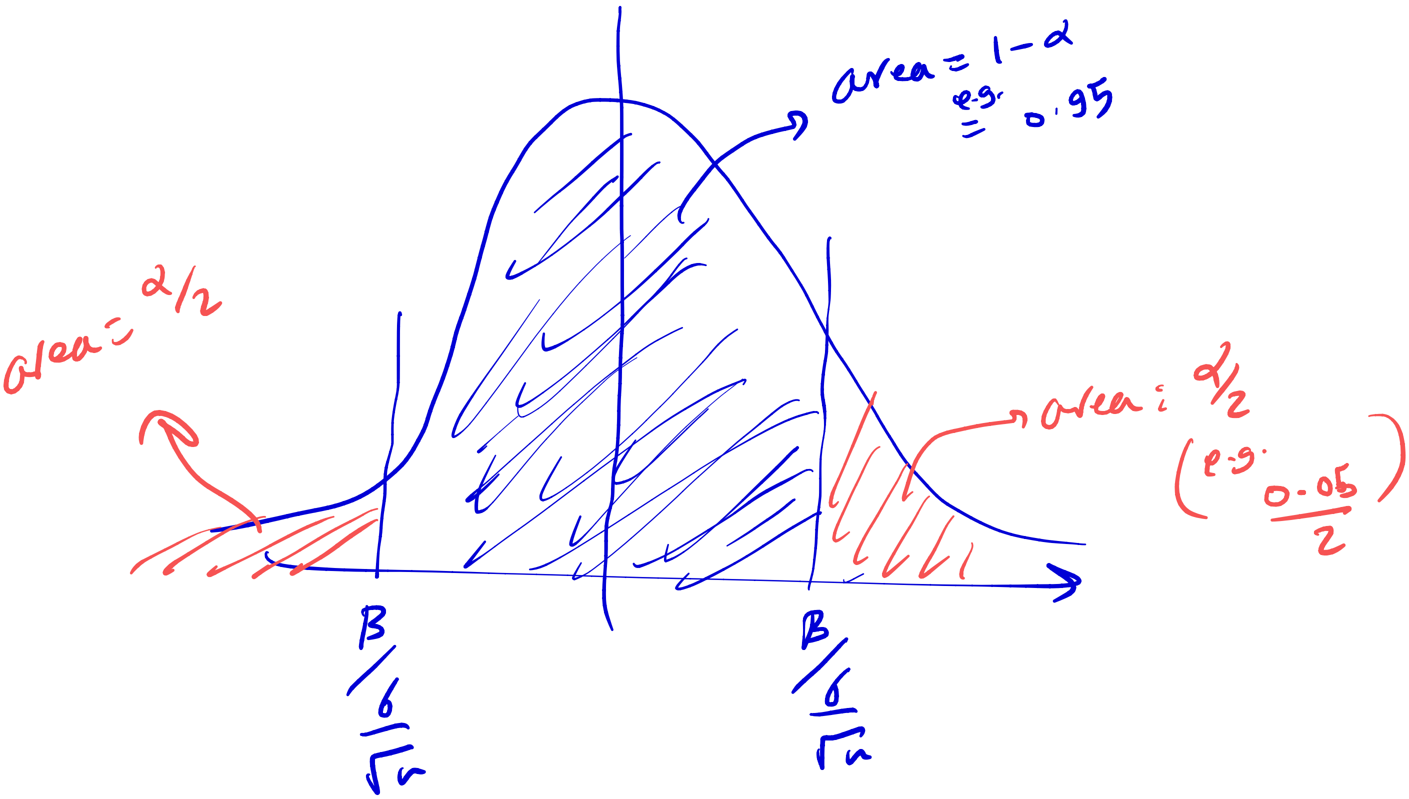


Recall that we wanted to

Solve:

$$\rightarrow \Pr\left\{\mu \in [\bar{X} - \beta, \bar{X} + \beta]\right\} = \overbrace{1-\alpha}^{0.95}$$

$$= \Pr\left\{ -\frac{\beta}{\sigma/\sqrt{n}} \le Z \le \frac{\beta}{\left(\frac{\sigma}{\sqrt{n}}\right)} \right\} = 1-\alpha$$



$$\frac{\beta}{\sigma/\sqrt{n}} = Z_{\alpha/2} \quad \left(\begin{array}{l} \text{e.g.} \\ Z_{0.025} \\ = 1.96 \end{array} \right)$$

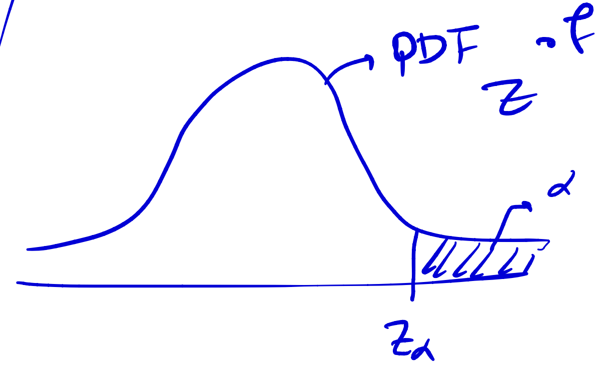
$$\beta = \frac{\sigma}{\sqrt{n}} \cdot z_{\alpha/2}$$

Equivalently,

$$\Pr\left\{ \mu \in \left[\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2},\ \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right] \right\}$$

$$= 1 - \alpha$$

Recall



- So far, all we've obtained ~~was~~ is for the case where $X_i \sim \mathcal{N}(\mu, \sigma^2)$.

- The more general setting is when

$$X_i \overset{iid}{\sim} \text{dist}(\mu, \sigma^2)$$

In this case, we ask:

for what value of β the following holds:

$$\Pr\left\{ \mu \in [\bar{X} - \beta, \bar{X} + \beta] \right\} = 1 - \alpha$$

In this case, from the CLT, we know that:

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) + \cancel{\text{negligible error for } n \text{ large}}$$

And in the exact same way as the Gaussian case derived above, we obtain that

$$\Pr\left\{ \mu \in \left[\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right] \right\}$$

$$= 1 - \alpha .$$

(There is always a CLT error of $\frac{C}{n}$ which we neglect.)