Lecture 10:

$X_1, \ldots, X_n \sim N(\mu, \sigma^2)$

$$\begin{cases} H_0: & \mu = \mu_0 \\ H_a: & \mu > \mu_0 \end{cases}$$

- step 1: (to design T)

$$T(X_1, \ldots, X_n) = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

- Assuming that H_0 is true, the distribution of the outcome of T is:

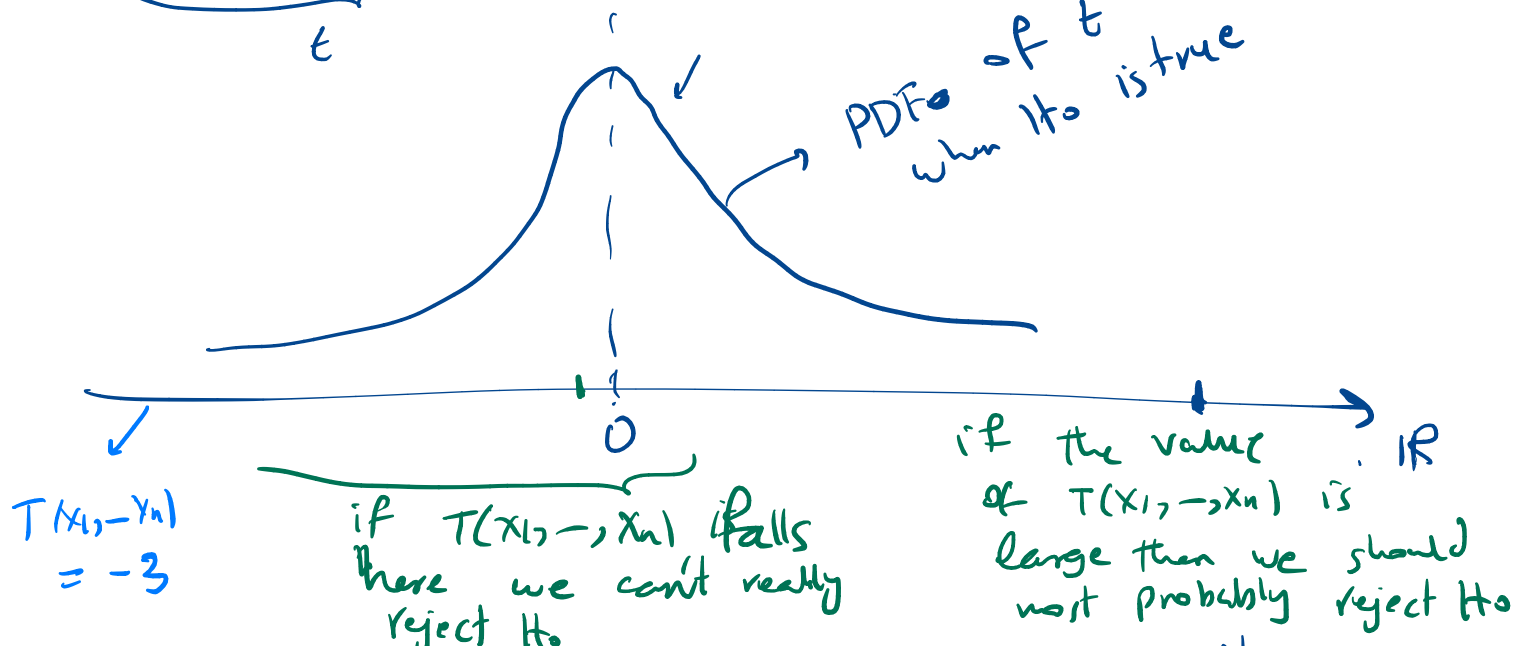
$$T(X_1, \ldots, X_n) = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \sim \underline{N(0,1)}$$

- Step 2: (identify acceptance/reject regions)

Constraint: $\underline{\alpha} = \Pr\{H_0 \text{ is rejected} \mid H_0 \text{ is true}\}$

$\leq$ significance level ($\underline{\alpha_0}$)
given

$\underbrace{T(X_1, \dots, X_n)}_{t} \in \mathbb{R}$



- Assuming that H_0 is true, the

$$T(X_1, \dots, X_n) \sim \mathcal{N}(0,1)$$

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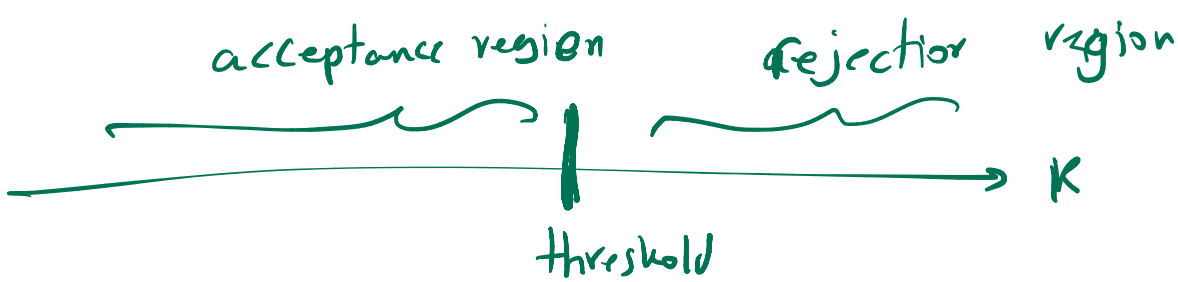


How should we find the threshold?

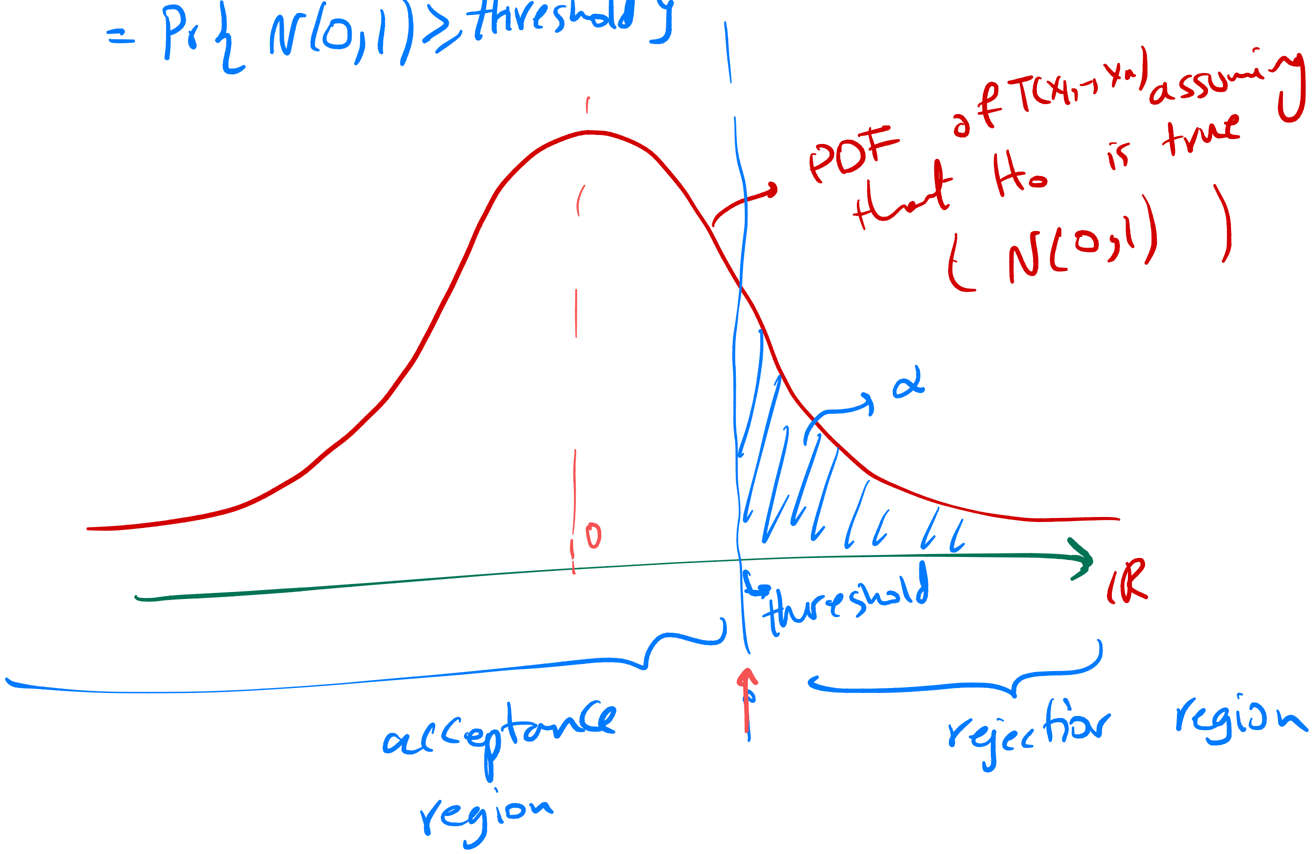
→ we still need to make sure that

$\alpha \le \alpha_0$ (e.g. 0.01).

$\alpha = \Pr\{ \underbrace{H_0 \text{ is rejected}} \mid \underbrace{H_0 \text{ is true}} \}$

$= \Pr\{ T(X_1, \ldots, X_n) \in \text{rejection region} \mid T(X_1, \ldots, X_n) \sim N(0,1) \}$

$= \Pr\{ N(0,1) \ge \text{threshold} \}$

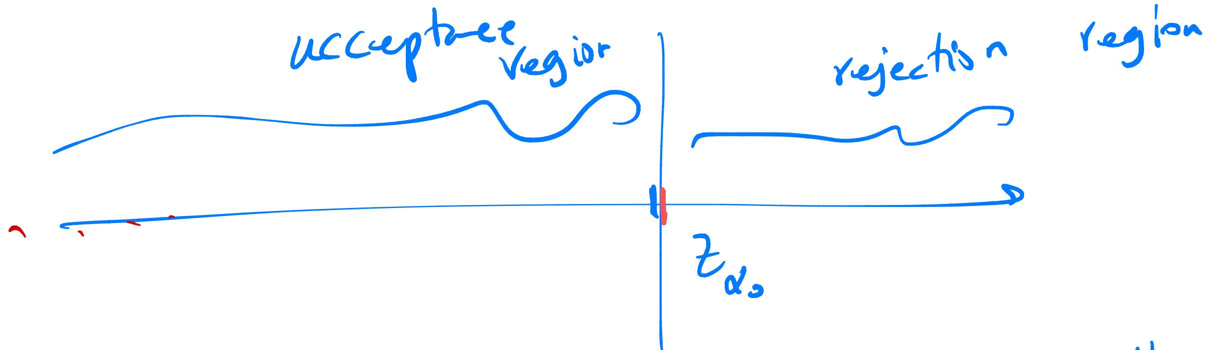


we want α to be less that α_0.

$$\alpha = \alpha_0 \iff \boxed{\text{threshold} = z_{\alpha_0}}$$

Hypothesis testing problem

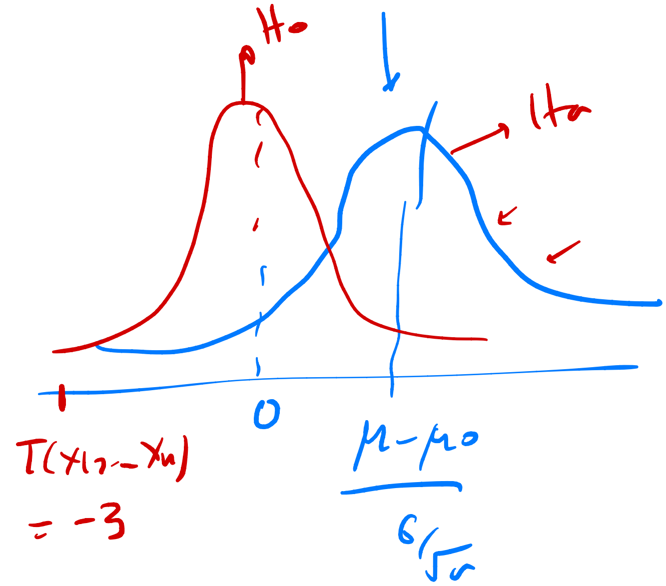
$$\begin{cases} H_0: \mu = \mu_0 \\ H_a: \mu > \mu_0 \end{cases} \longrightarrow T(X_1, \dots, X_n) = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$



$$T(X_1, \dots, X_n) \begin{cases} > z_{\alpha_0} & \to \text{reject } H_0 \\ < z_{\alpha_0} & \to \text{do not reject } H_0 \end{cases}$$

if H_a was true $\to$ $\underline{\mu > \mu_0}$

$$T(X_1, \dots, X_n) = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \overbrace{\frac{\mu - \mu_0}{\sigma/\sqrt{n}}}^{>0} + N(0,1)$$



Let's compute β for the example above.

$\beta = \Pr\{\text{type II error}\}$

$= \Pr\{H_0 \text{ is accepted} \mid H_0 \text{ is } \underline{\text{false}}\}$

$$\begin{cases} H_0: & \mu = \mu_0 \\ H_a: & \mu > \mu_0 \end{cases} \qquad \begin{cases} T(X_1,\dots,X_n) = \dfrac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \\ T(X_1,\dots,X_n) \underset{\text{accept}}{\overset{\text{reject}}{\gtrless}} Z_{\alpha_0} \end{cases}$$

Assuming that H_a is true, $X_1,\dots,X_n \sim N(\underset{\mu>\mu_0}{\mu},\sigma^2)$

Assuming $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ $\longrightarrow$ $\bar{X} = \frac{X_1 + \dots + X_n}{n} \sim N(\mu, \frac{\sigma^2}{n})$

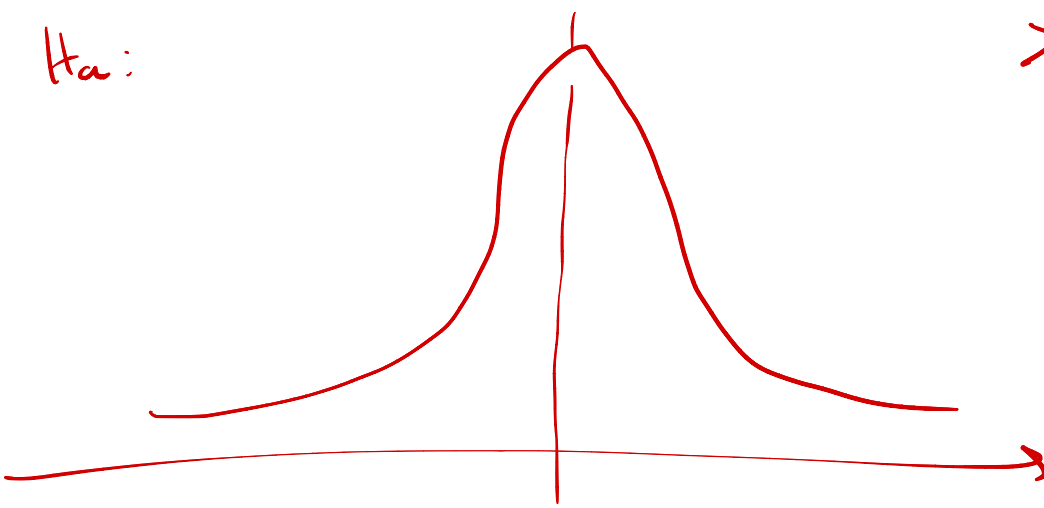
$\mu > \mu_0$

$$\Rightarrow \bar{X} = \mu + \frac{\sigma}{\sqrt{n}} N(0,1)$$

recall: $T(X_1, \dots, X_n) = \dfrac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$

$$\Rightarrow T(X_1, \dots, X_n) \sim \underbrace{\frac{\mu - \mu_0}{\frac{\sigma}{\sqrt{n}}}}_{>0} + N(0,1)$$

Assuming H_a:



$\frac{\mu - \mu_0}{\frac{\sigma}{\sqrt{n}}} > 0$

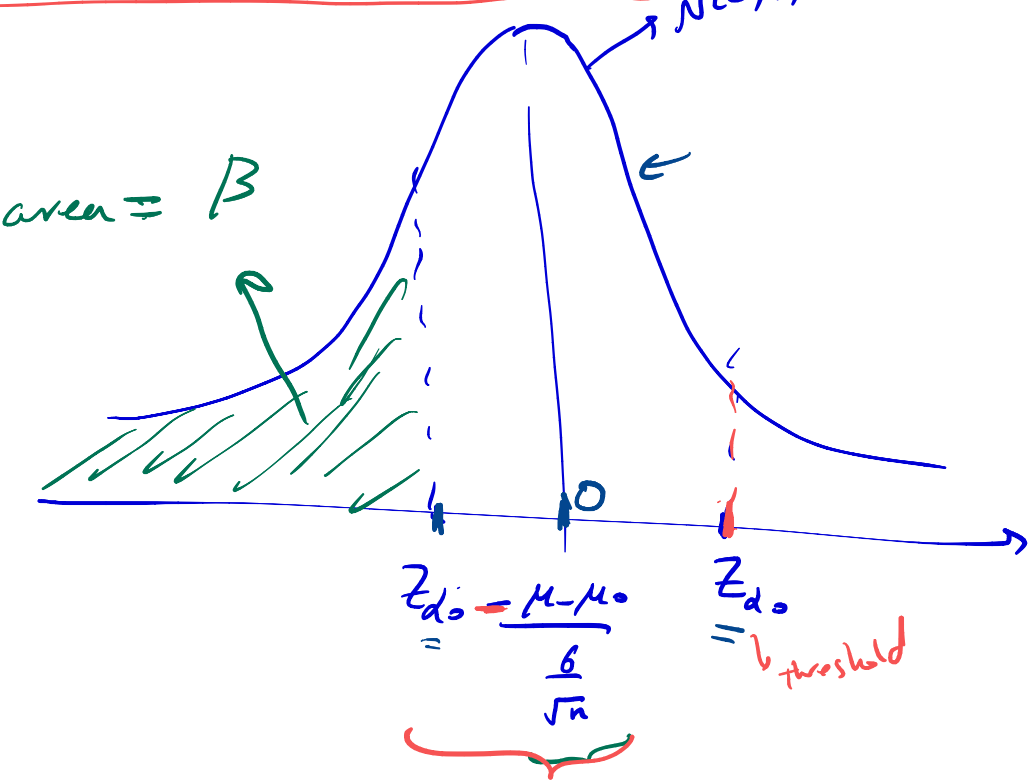
$$\beta = \Pr\left\{ H_0 \text{ is accepted} \;\middle|\; \begin{array}{l} H_a \text{ is true} \\ \& \; H_0 \text{ is false} \end{array} \right\}$$

$$= \Pr\left\{ T(X_1, \dots, X_n) \leq Z_{\alpha_0} \;\middle|\; T(X_1, \dots, X_n) \sim \frac{\mu - \mu_0}{\sigma/\sqrt{n}} + N(0,1) \right\}$$

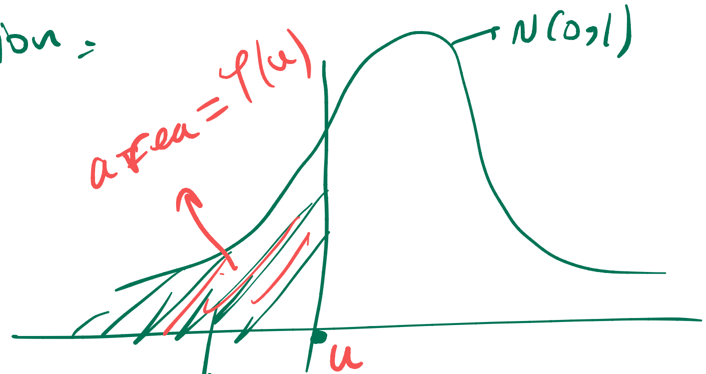
$$= \Pr\left\{ \frac{\mu-\mu_0}{\sigma/\sqrt{n}} + N(0,1) \le z_{\alpha_0} \right\}$$

$$= \Pr\left\{ N(0,1) \le z_{\alpha_0} - \frac{\mu-\mu_0}{\sigma/\sqrt{n}} \right\}$$

$$= \varphi\left(z_{\alpha_0} - \frac{\mu-\mu_0}{\sigma/\sqrt{n}} \right)$$



Notation:



$$\int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \varphi(u)$$

gaussian error function

CDF of normal dist.

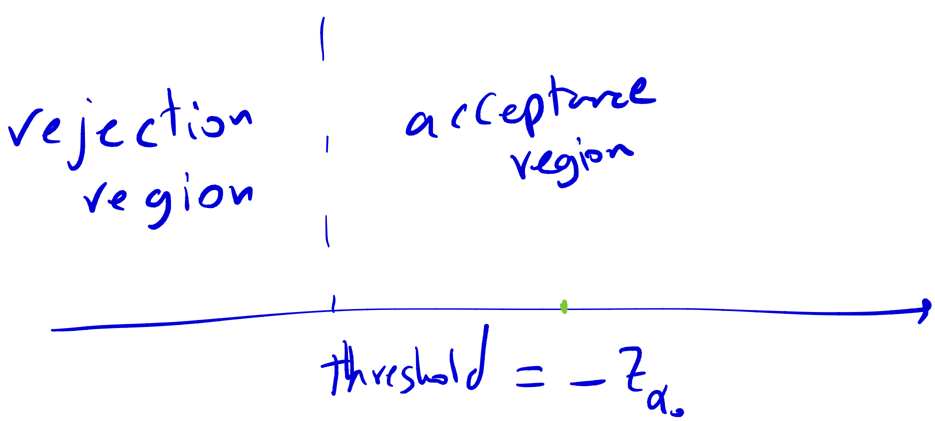
Example 2:

$X_1, \dots, X_n \sim N(\mu, \sigma^2)$

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu < \mu_0 \end{cases}$$

Similar to the previous example

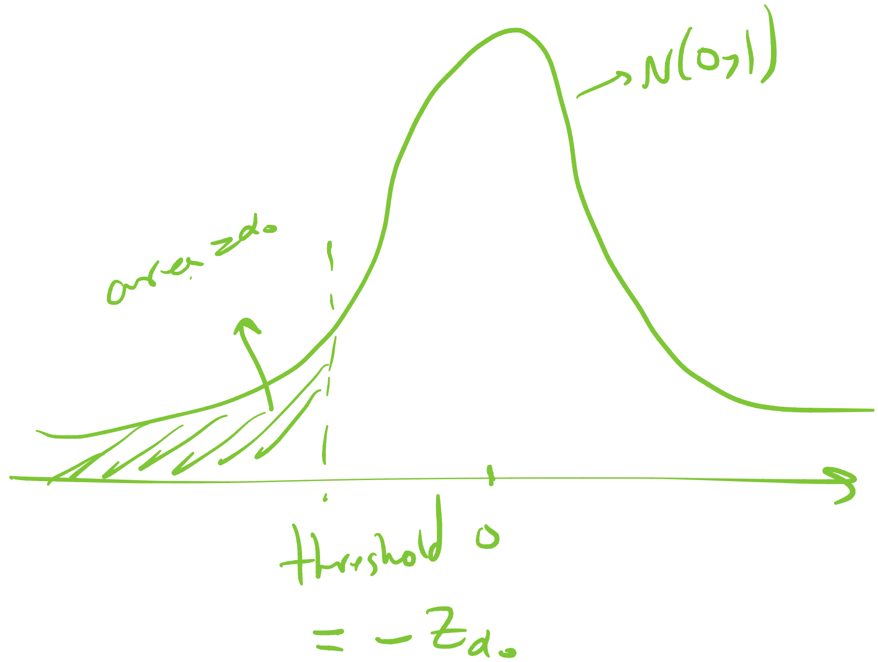
we can take
$$T(X_1, \dots, X_n) = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$



$$\Pr\{\text{type I error} \mid \underbrace{H_0 \text{ is true}}\}$$

$$= \Pr\{T(X_1, \dots, X_n) \leq \text{threshold} \mid H_0 \text{ is true}\}$$

$$= \Pr\{N(0,1) \leq \text{threshold}\} = \alpha_0$$



$$T(X_1, \ldots, X_n) = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \;\underset{\text{reject } H_0}{\overset{\text{accept } H_0}{\lesseqgtr}}\; -z_{\alpha_0}$$

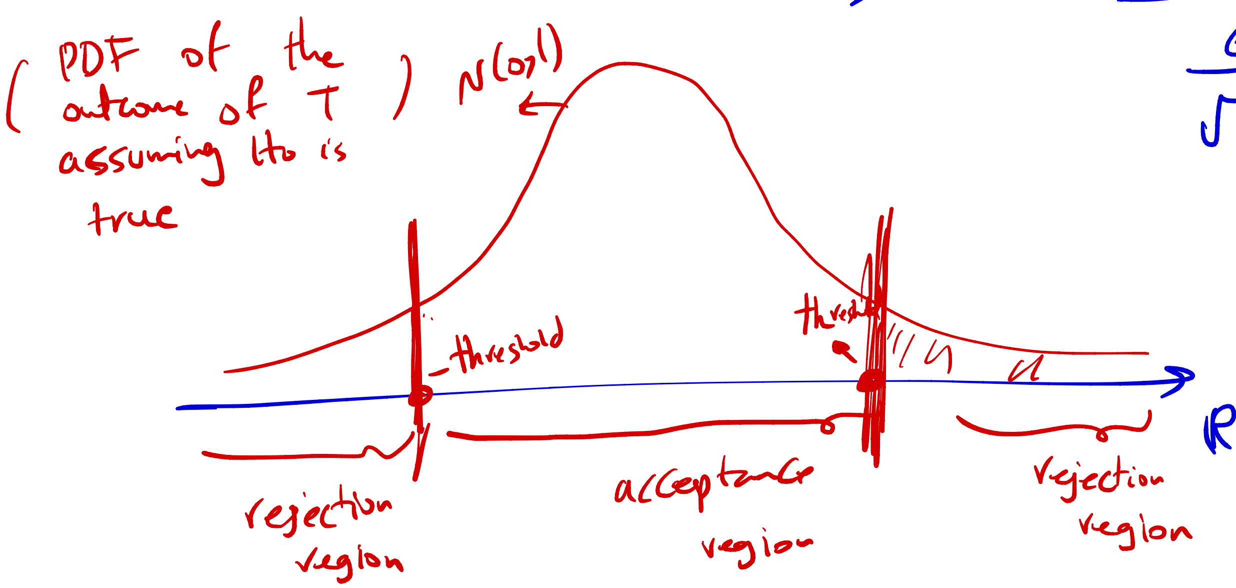
Example 3:

$$X_1, \dots, X_n \sim N(\mu, \sigma^2)$$

$$\begin{cases} H_0: \mu = \mu_0 \\ H_a: \mu \neq \mu_0 \end{cases}$$

Similar to the previous examples, since we're testing the mean, a good statistic can be:

$$T(X_1, \dots, X_n) = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

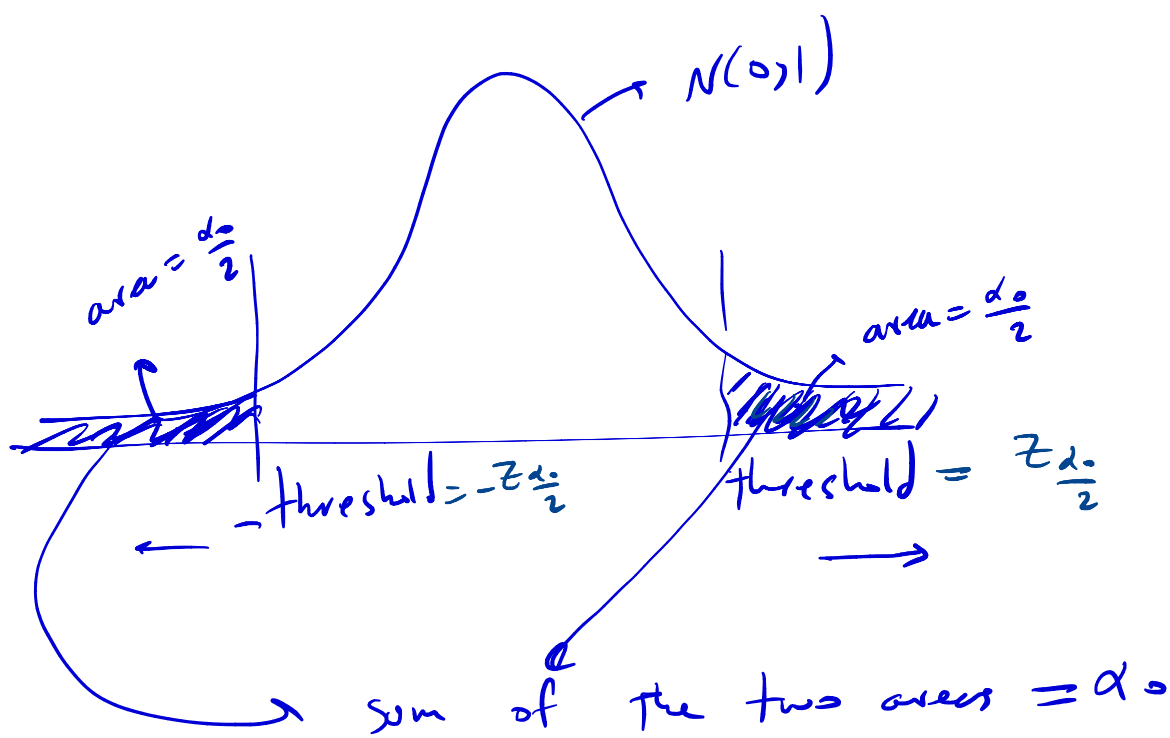


In order to find the threshold, we need use the constraint and make sure it's satisfied:

$$\Pr\{\text{type I error}\} = \alpha_0$$

$$\Pr\left\{ \begin{array}{l} T(x_1, \dots, x_n) \geq \text{threshold} \\ \text{or } T(x_1, \dots, x_n) \leq -\text{threshold} \end{array} \middle| \begin{array}{l} H_0 \text{ is} \\ \text{true} \end{array} \right\} = \underline{\alpha_0}$$

$$= \Pr\left\{ \begin{array}{c} N(0,1) \geq \text{threshold} \\ \text{or} \\ N(0,1) \leq -\text{threshold} \end{array} \right\}$$



$$\Rightarrow \quad \left| T(X_1, \ldots, X_n) \right| \begin{array}{c} \text{reject } H_0 \\ > \\ < \\ \text{accept } H_0 \end{array} z_{\frac{\alpha_0}{2}}$$

A more general setting (large n, known):

$$X_1, \ldots, X_n \sim \text{dist}(\underline{\mu}, \sigma^2)$$

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu > \mu_0 \end{cases}$$

(remember when n is large, by CLT, we have $\bar{X} \sim N(\mu, \sigma^2/n)$)

$$T(X_1, \ldots, X_n) = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

since $\bar{X}$ is (approximately, for large n) distributed according to $N(\mu, \sigma^2/n)$, all the derivations follow exactly as the previous examples.

$$T(X_1, \ldots, X_n) = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \underset{\text{accep}}{\overset{\text{reject}}{\gtrless}} Z_{\alpha_0}$$

Similar claims follow for the other examples of Hypothesis testing that we considered above.

Finally, so far we have assumed that σ^2 is known.

$$T(X_1, \ldots, X_n) = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

what should we do if σ is not known? use $\hat{\sigma}$ instead.

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\left(\sum_{i=1}^{n} (X_i - \bar{X})^2\right)}$$