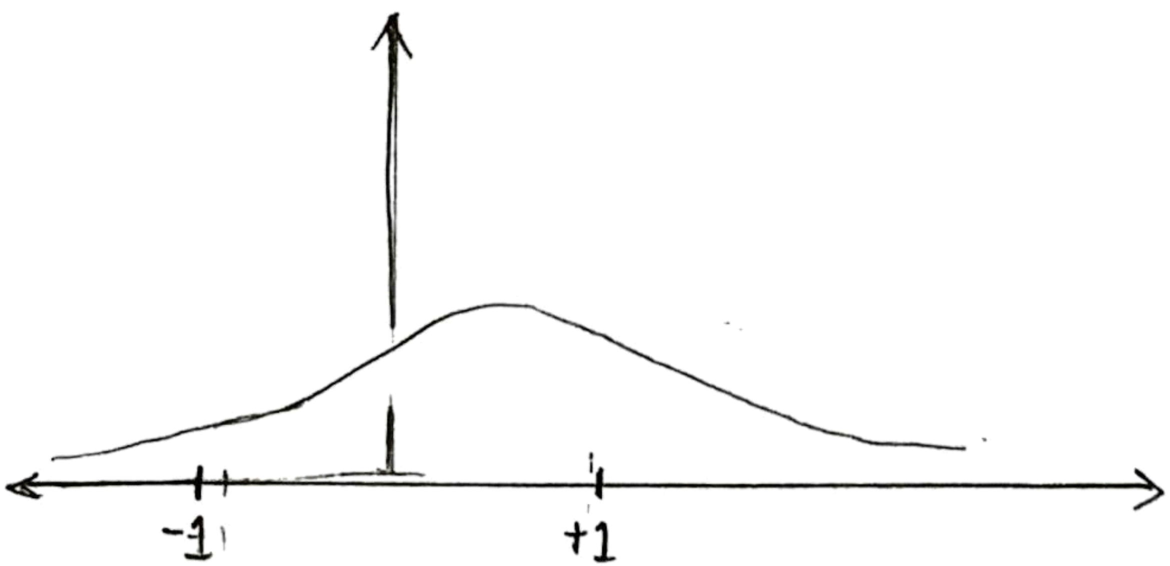
Q1] X_i s are generated IID from $f(x|a,p) = p \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+a)^2}{2}} + (1-p) \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2}}$

Given: $p \in [0,1]$, $a = 1$

a) PDF $f(x|a,p)$ vs x ; $p = 1/4$



In this case, $f(x|a=1, p=1/4)$

$$= \frac{1}{4} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}} + \frac{3}{4} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}$$

(i) Since there's more weight on the normal distribution w/ mean $= 1$, we get a peak closer to $x=1$ than $x=-1$

(ii) Unimodal $\because$ $a \leq 1$

Q1 b] $Var(X_i)$ in terms of p where $X_i \sim f(x|a,p) = \ldots\ldots$ as given

$$Var(X) = E[X^2] - (E[X])^2$$

where

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x|a=1,p) = \int_{-\infty}^{\infty} x^2 \left(\frac{p}{\sqrt{2\pi}} e^{-\frac{(x+a)^2}{2}} + \frac{1-p}{\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2}} \right) dx$$

$$= p\int_{-\infty}^{\infty} \frac{x^2 e^{-\frac{(x+a)^2}{2}}}{\sqrt{2\pi}} dx + 1-p\int_{-\infty}^{\infty} \frac{x^2 e^{-\frac{(x-a)^2}{2}}}{\sqrt{2\pi}} dx$$

where $a=1$ (given) & $\sigma^2 = 1$ (inferred)

$$= \left[(-1)^2 + (1)^2\right] p + \left[(1)^2 + (1)^2\right](1-p)$$

$$= 2p - 2p + 2 = \boxed{2}$$

Also, $E[X] = p\int_{-\infty}^{\infty} \frac{x e^{-\frac{(x+a)^2}{2}}}{\sqrt{2\pi}} dx + (1-p)\int_{-\infty}^{\infty} \frac{x^2 e^{-\frac{(x-a)^2}{2}}}{\sqrt{2\pi}} dx$ where $a=1$ (given)

$$= -p + 1 - p = -2p + 1 \qquad \Rightarrow (E[X])^2 = 4p^2 + 1 - 4p$$

$$\Rightarrow Var(X) = 2 - 1 + 4p - 4p^2$$

$$= \boxed{-4p^2 + 4p + 1}$$

Q1 c] $E[X] = -2p+1$ (from (b))

$$\Rightarrow \frac{\sum_{i=1}^{n} X_i}{n} = -2p+1$$

$$\Rightarrow \hat{p} = \frac{n - \sum_{i=1}^{n} X_i}{2n}$$

Q1 d] $E[\hat{p}] \stackrel{?}{=} p$

$$= \frac{1}{2n} E\left[n - \sum_{i=1}^{n} X_i\right] = \frac{1}{2n}\left(n - \sum_{i=1}^{n} E[X_i]\right)$$

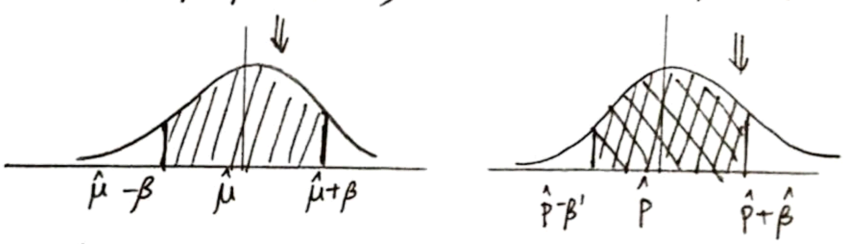
$$= \frac{1}{2\cancel{n}}\left(\cancel{n} - \cancel{n}(-2p+1)\right)$$

$$= \frac{1}{2} + \frac{2p}{2} - \frac{1}{2} = \boxed{p}$$

Hence, $\hat{p}$ is unbiased

Q1 e] Given;
$\mu = E[X_1]$
$\hat{\mu} = \bar{X}$ (sample mean)

TODO: Find β' : $Pr(\mu \in [\hat{\mu} - \beta, \hat{\mu} + \beta]) = Pr(p \in [\hat{p} - \beta', \hat{p} + \beta'])$
$\forall \beta > 0$



We know $\beta = z\left(\frac{\alpha}{2}\right) \cdot \frac{\hat{\sigma}}{\sqrt{n}}$, $\hat{p} = \frac{n - \sum_{i=1}^{n} X_i}{2n}$

$$\Rightarrow z\left(\frac{\alpha}{2}\right) = \frac{\sqrt{n}\,\beta}{\hat{\sigma}}$$

$$\beta' = \left(\frac{\cancel{\sqrt{n}}\,\beta}{\hat{\sigma}}\right)\left(\frac{\sqrt{Var(\hat{p})}}{\cancel{\sqrt{n}}}\right)$$

$$= \frac{\beta}{\hat{\sigma}} \cdot \sqrt{Var\left(\frac{n}{2n} - \sum_{i=1}^{n} \frac{X_i}{2n}\right)} = \frac{\beta}{\hat{\sigma}} \sqrt{\frac{1}{(2n)^2} \sum Var(X_i)}$$

$$= \frac{\beta}{\hat{\sigma}(2n)} \sqrt{n\hat{\sigma}^2}$$

Therefore, $\beta' = \frac{\beta}{\hat{\sigma}(2n)} \cdot \sqrt{n\hat{\sigma}^2} = \frac{\beta}{2}\sqrt{\frac{n\cancel{\hat{\sigma}^2}}{n^2\cancel{\hat{\sigma}^2}}} = \frac{\beta\sqrt{n}}{2}$

$$\boxed{\beta' = \frac{\beta\sqrt{n}}{2}}$$

Q1 f] Bounds of Confidence Interval for p w/ a given α :

$$\hat{p} \pm z\left(\frac{\alpha}{2}\right) \cdot \frac{\sqrt{\frac{1}{(2n)^2} \cdot n\hat{\sigma}^2}}{\sqrt{n}}$$

Q1 g] $E[X] = p(-a) + (1-p)(a) = -2pa + a = a(-2p+1)$

$E[X^2] = p\left[(-a)^2 + 1\right] + (1-p)\left[a^2+1\right]$

$= a^2 + 1$

$\Rightarrow a = \sqrt{E[X^2] - 1} \quad \Rightarrow \left(\frac{E[X]}{\sqrt{E[X^2]-1}} - 1\right) \cdot \frac{-1}{2} = p$

$\Rightarrow p = \frac{-E[X]}{2\sqrt{E[X^2]-1}} + \frac{1}{2}$

Therefore, $\boxed{a = \sqrt{\frac{\sum_{i=1}^{n} X_i^2}{n} - 1}}$

$$\boxed{p = -\frac{\frac{\sum_{i=1}^{n} X_i}{n}}{2\sqrt{\frac{\sum_{i=1}^{n} X_i^2}{n} - 1}} + \frac{1}{2}}$$

Q2 a] $\because$ all X_i s are IID, the joint density function is product of individual PDF functions

Q2 b] $= \frac{1}{n} \log f(X_1, \ldots, X_n | \sigma_0) = \frac{1}{n} \sum_{i=1}^{n} \log(X_i | \sigma_0)$

$$= \frac{1}{n} \sum_{i=1}^{n} \log\left(\frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}} \right)$$

$$= \ln\left(\frac{1}{2\sigma}\right) + \frac{1}{n}\sum_{i=1}^{n}\left(-\frac{|X_i|}{\sigma}\right) \quad \text{where } \sigma = \sigma_0$$

$$\ln\left(\frac{1}{2\sigma}\right) + \frac{-1}{\sigma} E\underbrace{[\cancel{|X_i|}]}_{\hat{\sigma}} = \ln\left(\frac{1}{2\sigma}\right) - \frac{\hat{\sigma}}{\sigma}$$

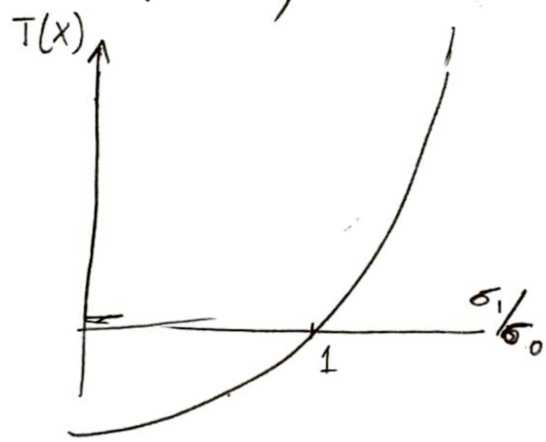
Q2 c] $T(X_1, \ldots X_n) \sim ?$ when H_0 is true ;

Given: $H_0 : \sigma = \sigma_0$; $\sigma_0 < \sigma_1$

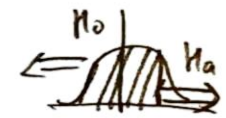
$$T(X_1, \ldots X_n) = \underbrace{\ln\left(\frac{1}{2\sigma_0}\right) - \frac{\hat{\sigma}}{\sigma}}, - \underbrace{\left(\ln\left(\frac{1}{2\sigma_1}\right) - \frac{\hat{\sigma}}{\sigma}\right)}$$

$$= \ln\left(\frac{\sigma_1}{\sigma_0}\right)$$

When $\sigma = \sigma_0$



Q2 d] Find Z-statistic for α, this will be a 1-sided test



Design threshold that will designate acceptance & rejection regions for the T(X) computed from data