Problem 1

1. $\mathcal{L}(\theta|\underset{\sim}{x}) = \prod_{i=1}^{n} (\theta+1) x_i^{\theta}$

$$\ell(\theta|\underset{\sim}{x}) = \sum_{i=1}^{n} \log(\theta+1) + \sum_{i=1}^{n} \theta \log x_i$$

$$= n \log(\theta+1) + \theta \sum_{i=1}^{n} \log x_i$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta+1} + \sum_{i=1}^{n} \log x_i$$

$$\frac{\partial \ell}{\partial \theta} = 0 \Rightarrow \frac{n}{\theta+1} = -\sum_{i=1}^{n} \log x_i$$

$$\frac{n}{\sum_{i=1}^{n} \log x_i} = -\theta - 1$$

$$\boxed{\hat{\theta}_{mle} = -1 - \frac{n}{\sum_{i=1}^{n} \log x_i}}$$

2. $\mathcal{L}(\theta|x) = (\theta+1) x_i^{\theta}$

$$\ell(\theta|x) = \log(\theta+1) + \theta \log x_i$$

$$\frac{d\ell}{d\theta} = \frac{1}{\theta+1} + \log x_i \qquad \frac{d^2\ell}{d\theta} = -\frac{1}{(\theta+1)^2}$$

$$I(\theta) = -E\left[-\frac{1}{(\theta+1)^2} \Big| \theta\right]$$

$$= \int_0^1 \frac{1}{(\theta+1)^2} f(x|\theta)\, dx$$

$$= \frac{1}{(\theta+1)^2} \int_0^1 (\theta+1) x^{\theta}\, dx$$

$$= \frac{1}{(\theta+1)^2} \left(x^{\theta+1}\right)\Big|_0^1$$

$$= \frac{1}{(\theta+1)^2}$$

Asymptotic Variance of $\hat{\theta}_{mle}$: $\frac{1}{n I(\theta)} = \frac{(\theta+1)^2}{n}$

3. $\mathcal{L}(\theta|\underset{\sim}{x}) = \prod_{i=1}^{n} (\theta+1) x_i^{\theta}$

$= (\theta+1)^n \prod_{i=1}^{n} x_i^{\theta}$

$= (\theta+1)^n \left(\prod_{i=1}^{n} x_i\right)^{\theta}$

By the factorization theorem, $\boxed{\prod_{i=1}^{n} x_i \text{ is sufficient}}$

Problem 2

1. $H_0 : \theta = \theta_0$

$H_a : \theta = \theta_1$

2. For simple vs. simple hypothesis, a LRT is optimal (most powerful)

$\mathcal{L}(\theta|\underset{\sim}{x}) = \prod_{i=1}^{n} \left(\frac{1}{\theta} \mathbb{1}_{0 \le x_i \le \theta}\right)$

$= \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}_{0 \le x_i \le \theta}$

$$\Lambda(\underset{\sim}{x}) = \frac{\mathcal{L}(\theta_0|\underset{\sim}{x})}{\mathcal{L}(\theta_1|\underset{\sim}{x})} = \begin{cases} \left(\frac{\theta_1}{\theta_0}\right)^n & \text{if } x_{(n)} \le \theta_1 \\ \infty & \text{o. w.} \end{cases}$$

Want to reject H_0 when $\Lambda(\underset{\sim}{x}) < k$ for some constant k

$P(\text{Type I error} \mid H_0 \text{ is true}) = P(\Lambda(\underset{\sim}{x}) < k \mid \theta = \theta_0)$

$= P\left(\left(\frac{\theta_1}{\theta_0}\right)^n < k \mid x_{(n)} \le \theta_1, \theta = \theta_0\right) P(x_{(n)} \le \theta_1 \mid \theta = \theta_0)$

$+ \underbrace{P(\infty < k \mid x_{(n)} > \theta_1, \theta = \theta_0)}_{= 0} P(x_{(n)} > \theta_1 \mid \theta = \theta_0)$

$= \mathbb{1}\left(\left(\frac{\theta_1}{\theta_0}\right)^n < k\right)\left(\frac{\theta_1}{\theta_0}\right)^n$

Solve for $\mathbb{1}\left(\left(\frac{\theta_1}{\theta_0}\right)^n < k\right)\left(\frac{\theta_1}{\theta_0}\right)^n = \left(\frac{\theta_1}{\theta_0}\right)^n \implies k = 1$

$\boxed{\text{So, we reject } H_0 \text{ if } x_{(n)} \le \theta_1}$

3. $\beta = P(\text{Type II error})$

$= P(\text{Fail to Reject} \mid H_0 \text{ is False})$

$= P(\Lambda(\underset{\sim}{x}) \geq 1 \mid \theta = \theta_1)$

$$= P\left(\left(\frac{\theta_1}{\theta_0}\right)^n \geq 1 \,\Big|\, x_{(n)} < \theta_1, \theta = \theta_1\right) \overbrace{P(x_{(n)} \leq \theta_1 \mid \theta = \theta_1)}^{= 1}$$

$$+ P(\infty > k \mid x_{(n)} < \theta_1, \theta = \theta_1) \underbrace{P(x_{(n)} > \theta_1 \mid \theta = \theta_1)}_{= 0}$$

$$= P\left(\left(\frac{\theta_1}{\theta_0}\right)^n \geq 1 \,\Big|\, x_{(n)} < \theta_1, \theta = \theta_1\right)$$

$= 0$

Power $= 1 - \beta = 1 - 0$

$\boxed{\text{Power} = 1}$

4. $P(X_{(n)} < k \mid \theta = \theta_0) = \alpha$

$\left(\frac{k}{\theta_0}\right)^n = \alpha$

$$\boxed{k = \theta_0 (\alpha)^{1/n}, \quad \alpha < \left(\frac{\theta_1}{\theta_0}\right)^n}$$