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ESE402 Assignment 3
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Oct. 25, 2021

Problem 1

PROBLEM 1A

Given: $n = 5$, $\sigma^2 = 9$

$H_0 : \mu = 10$, $H_a : \mu \neq 10$

$\Rightarrow$ Test statistic : $\dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}} = \dfrac{12 - 10}{3/\sqrt{5}} = 1.49$

$\Rightarrow$ p-value of z is $2(1 - \Phi(1.49)) = 0.13622$, which is larger than $\alpha = 0.05$

Therefore, we accept H_0 , i.e. $\mu = 10$

PROBLEM 1B

$z_{\alpha/2}$ $\because$ 2 sided test for $\alpha = 0.05 = 1.96$

Thus, 95% confidence Interval $\Rightarrow$

$$\left[\bar{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} , \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}} \right] \Rightarrow \left[12 - 1.96\left(\frac{3}{\sqrt{5}}\right) , 12 + 1.96\left(\frac{3}{\sqrt{5}}\right) \right]$$

$$\Rightarrow [9.37, 14.63]$$

Problem 2

PROBLEM 2A

Given:

$n = 51$, $\hat{p} = \frac{41}{51}$, $\alpha = 0.01$

Hypothesis Test: $H_0 : p > 50\%$; $H_a : p \leq 50\%$

We know 1-sided z-score for $\alpha = 0.01 = 2.326$; rejection region is $z \leq -z_\alpha$

Test Statistic Value $= \dfrac{0.80392 - 0.5}{\sqrt{0.5(1-0.5)/51}} = 4.3408$; p-value $= 1 - \Phi(4.341) \approx 0$

Since ~~β~~ $p \ngtr \alpha = 0.01$, we ~~accept~~ reject H_0

PROBLEM 2B

Confidence Interval for true probability P

$$-2.326 \leq t \leq 2.326$$

$$\Rightarrow -2.326 \leq \frac{0.80392 - p}{\sqrt{0.5(1-0.5)/51}} \quad \Leftarrow \text{~~Lower~~ Upper Bound}$$

$$\Rightarrow p \leq \left(\frac{(-2.326)\,0.5}{\sqrt{51}} - 0.80392\right)(-1)$$

$$\Rightarrow P \leq 0.967$$

Similarly, lower bound:

$$\frac{0.80392 - p}{\sqrt{0.5(1-0.5)/51}} \leq 2.326 \Rightarrow p \geq \left(\frac{2.326(0.5)}{\sqrt{51}} - 0.80392\right)(-1)$$

$$\Rightarrow p \geq 0.641$$

99% 1-sided confidence Interval:

CI: $[0.641, 0.967]$

Problem 3

PROBLEM 3A

$H_0 : p = 0.25$; $H_a : p < 0.25$

$n = 100$, $\hat{p} = 0.2$

i) $\alpha = 0.05$, left tailed z-value: $z_\alpha = 1.65$

Test statistic value $= \dfrac{0.2 - 0.25}{\sqrt{0.25(1-0.25)/100}} \Rightarrow z = -1.155$

Rejection region : $z \leq -z_\alpha$ or $p \leq \alpha$ corresponding p-value $= \Phi(-1.15) = 0.12$

Since $-1.155 \nleq -1.65$, we accept H_0; From p-value perspective $0.12 \geq 0.05$ so we accept H_0

ii) $\alpha = 0.01$, left-tailed z-value : $z_\alpha = 2.33$; we use same z value computed above

Since $-1.155 \nleq -2.33$, we also accept H_0; similarly $\overset{0.12}{p} \nleq \overset{0.01}{\alpha}$, H_0 accepted

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PROBLEM 3B

H_0: $p = 0.25$, $H_a: p \neq 0.25$, $\alpha = 0.05$ & $\alpha = 0.01$

Rejection region: $z \geq z_{\alpha/2}$ OR $z \leq -z_{\alpha/2}$; p-value in this case $= 2\Phi(-1.155)$
$= 0.250$

i) $\alpha = 0.05$ 2-tailed z value $= 1.960$

Since $-1.155 \ngeq 1.960$ & $-1.155 \nleq -1.960$, we accept H_0

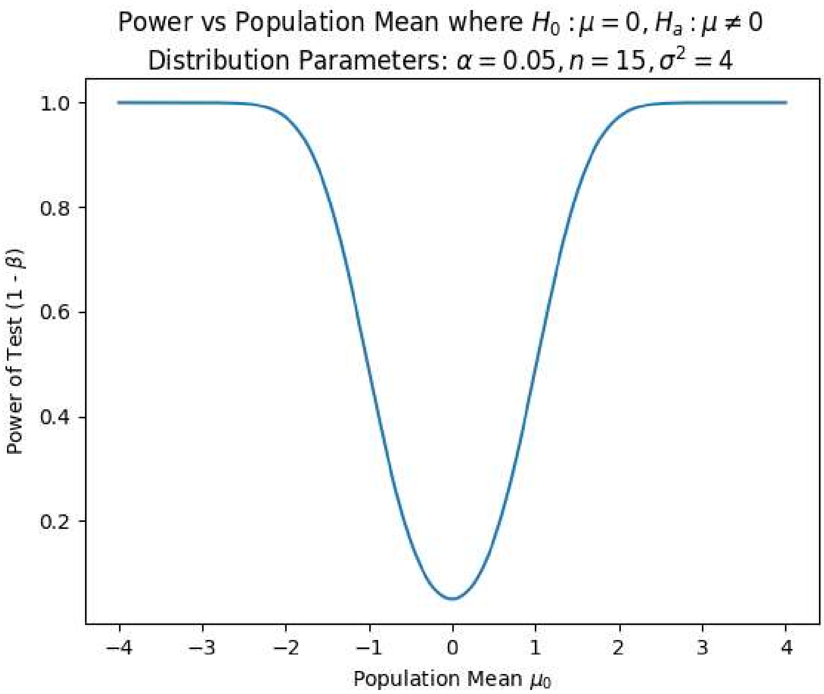
From p-value perspective, accept H_0 $\because 0.250 > 0.05$

ii) $\alpha = 0.01$, $z_{\alpha/2} = 2.576$

Since $-1.155 \ngeq 2.576$ & $-1.155 \nleq -2.576$, we accept H_0

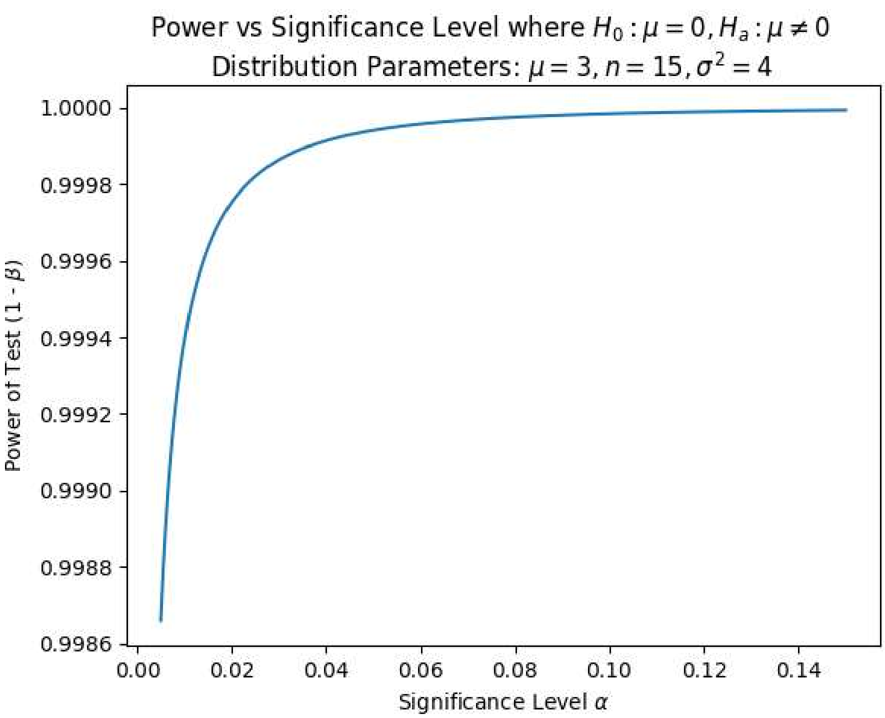
$\because 0.25 > 0.01$, accept H_0 based on p-value test

Problem 4A



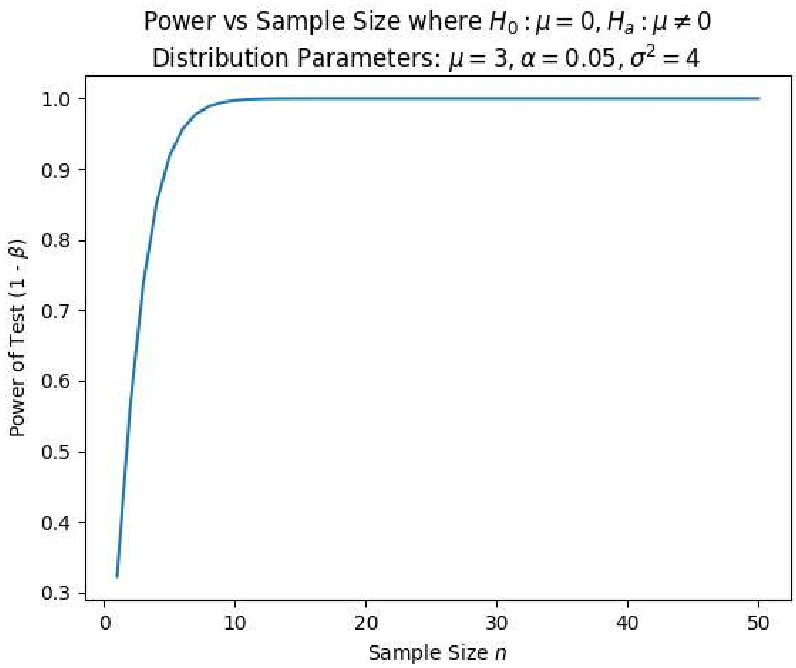
Takeaway: Power of the test increases the further the population mean moves from the hypothesized mean in either direction. This is intuitive because the power measures the inverse probability of type 2 errors, i.e. missing accepting the null hypothesis when it is correct, so as population mean moves in either extreme, the chance of the null hypothesis being true decreases, hence chance of type II errors also decrease.

Problem 4B



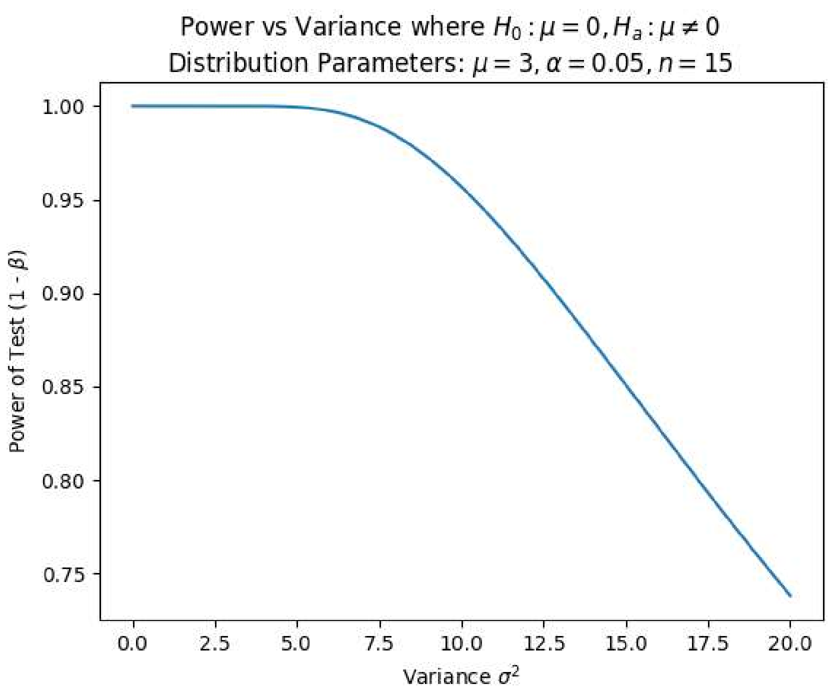
Takeaway: The power of the test increases with alpha, asymptotically reaching 100%. This is intuitive, because Type I and Type II error are at odds with each other, by design. Hence, loosening the tolerated Type I error decreases the probability of committing a Type II error.

Problem 4C



Takeaway: Power increases with sample size, asymptotically reaching 100%. This is intuitive because the greater the n, the more closely the sample mean represents the population mean, resulting in a more powerful conclusion.

Problem 4D



Takeaway: Power decreases as variance increases. This makes intuitive sense because in a very spread-out distribution, the sample mean might not closely represent the population mean, so it is easy to make incorrect acceptance or rejection decisions.

```python
import numpy as np
import scipy.special as sp
import scipy.stats as st
from matplotlib import pyplot as plt
from time import sleep


def generate_mu():
    # Simulate True Mean of Normal Distribution
    mu_array = np.linspace(-4, 4, 1000)
    return mu_array


def generate_alpha():
    # Simulate Probability of Type 1 Errors
    alpha_error = np.linspace(0.005, 0.15, 1000)
    return alpha_error


def generate_n():
    # Simulate Various Sample Sizes
    max_n = 50
    n_array = np.linspace(1, max_n, max_n)
    return n_array


def generate_var():
    # Simulate Different Variances
    sigma_squared_array = np.linspace(0.001, 20, 1000)
    return sigma_squared_array


def compute_power(mu, z_alpha, n, sigma):
    # Power is probability of rejecting the null hypothesis
    # when it is wrong

    # Find acceptance and rejection regions in terms of sample means
    Xbar_lower_bound = -z_alpha * (sigma / n ** 0.5)
    Xbar_upper_bound = z_alpha * (sigma / n ** 0.5)

    # Find power in terms of Z value
    Z_left_tail = (Xbar_lower_bound - mu) / (sigma / n ** 0.5)
    Z_right_tail = (Xbar_upper_bound - mu) / (sigma / n ** 0.5)

    # Compute and return power
    power = abs(sp.ndtr(Z_left_tail)) + (1 - abs(sp.ndtr(Z_right_tail)))
    return power


def plot_power(scenario):
    if (scenario == 'vary_mu'):
        mu_array = generate_mu()
        z_alpha = 1.96  # two-tailed test value for alpha = 0.05
        n = 15  # number of observations
        sigma = 2  # standard deviation
        power_array = []
        for mu in mu_array:
            power = compute_power(mu, z_alpha, n, sigma)
            power_array.append(power)
```

```
    plt.plot(mu_array, power_array)
    plt.suptitle(
        r'Power vs Population Mean where $H_0 : \mu = 0, H_a : \mu \neq 0$' + '\n' +
        r'Distribution Parameters: $\alpha = 0.05, n = 15, \sigma^2 = 4$')
    plt.xlabel(r'Population Mean $\mu_0$')

elif (scenario == 'vary_alpha'):
    alpha_array = generate_alpha()
    mu = 3.0  # population mean
    n = 15  # number of observations
    sigma = 2  # standard deviation
    power_array = []
    for alpha in alpha_array:
        z_alpha = abs(st.norm.ppf(alpha / 2.0))
        power = compute_power(mu, z_alpha, n, sigma)
        power_array.append(power)

    plt.plot(alpha_array, power_array)
    plt.suptitle(
        r'Power vs Significance Level where $H_0 : \mu = 0, H_a : \mu \neq 0$' + '\n' +
        r'Distribution Parameters: $\mu = 3, n = 15, \sigma^2 = 4$')
    plt.xlabel(r'Significance Level $\alpha$')

elif (scenario == 'vary_n'):
    n_array = generate_n()
    mu = 3.0  # population mean
    z_alpha = 1.96  # two-tailed test value for alpha = 0.05
    sigma = 2  # standard deviation
    power_array = []
    for n in n_array:
        power = compute_power(mu, z_alpha, n, sigma)
        power_array.append(power)

    plt.plot(n_array, power_array)
    plt.suptitle(
        r'Power vs Sample Size where $H_0 : \mu = 0, H_a : \mu \neq 0$' + '\n' +
        r'Distribution Parameters: $\mu = 3, \alpha = 0.05, \sigma^2 = 4$')
    plt.xlabel(r'Sample Size $n$')

elif (scenario == 'vary_sigma_squared'):
    sigma_squared_array = generate_var()
    z_alpha = 1.96  # two-tailed test value for alpha = 0.05
    n = 15  # number of observations
    mu = 3.0  # population mean
    power_array = []
    for sigma_squared in sigma_squared_array:
        sigma = sigma_squared ** 0.5
        power = compute_power(mu, z_alpha, n, sigma)
        power_array.append(power)

    plt.plot(sigma_squared_array, power_array)
    plt.suptitle(
        r'Power vs Variance where $H_0 : \mu = 0, H_a : \mu \neq 0$' + '\n' +
        r'Distribution Parameters: $\mu = 3, \alpha = 0.05, n = 15$')
    plt.xlabel(r'Variance $\sigma^2$')

plt.ylabel(r'Power of Test (1 - $\beta$)')
plt.show()
```

```
def main():
    scenario_array = ['vary_mu',
                      'vary_alpha',
                      'vary_n',
                      'vary_sigma_squared'
                      ]
    for scenario in scenario_array:
        plot_power(scenario)
    return


if __name__ == '__main__':
    main()
```

Problem 5

PROBLEM 5

$H_0: \lambda = \lambda_0$; $H_a: \lambda = \lambda_1$ where $\lambda_1 > \lambda_0$; Given α_0, determine rejection region

Likelihood Ratio Test:

a) Likelihood of λ_0 : $\prod_{i=1}^{n} \left(\frac{e^{-\lambda_0} \lambda_0^{x_i}}{x_i!} \right)$

b) Likelihood of λ_1 : $\prod_{i=1}^{n} \left(\frac{e^{-\lambda_1} \lambda_1^{x_i}}{x_i!} \right)$

Likelihood ratio test: $\frac{(a)}{(b)} = \frac{e^{-n\lambda_0} (\lambda_0)^{\sum_{i=1}^{n} x_i}}{e^{-n\lambda_1} (\lambda_1)^{\sum_{i=1}^{n} x_i}} < c$

$$= \left(\frac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^{n} x_i} < c \cdot e^{n(\lambda_0 - \lambda_1)}$$

Taking log of both sides $\Rightarrow$

$$= \sum_{i=1}^{n} x_i \cdot \log\left(\frac{\lambda_0}{\lambda_1}\right) < n(\lambda_0 - \lambda_1) + \log c$$

We flip inequality because $\log(\lambda_0/\lambda_1)$ is negative $\because \lambda_1 > \lambda_0$ $\Rightarrow$

$$\underline{\underline{\sum_{i=1}^{n} x_i}} > \underbrace{\frac{n(\lambda_0 - \lambda_1) + \log c}{\log(\lambda_0/\lambda_1)}}_{\text{constant} = c_1}$$

$\Downarrow$

We know sum of independent Poisson RVs w/ param. λ_0 is another Poisson$(n\lambda_0)$

$$\Rightarrow \quad P_{\lambda = \lambda_0}\left(\sum_{i=1}^{n} x_i > c_1\right) = P(Y > c_1) \quad \text{where } Y \sim \text{Poisson}(n\lambda_0)$$

$\Rightarrow$ Rejection region should be inverse CDF of the Poisson$(n\lambda_0)$ distribution for α_0

Problem 6

PROBLEM 6

$$L(\theta) = \prod_{i=1}^{n} \theta \exp\{-\theta x_i\}$$

$$\Rightarrow \ell(\theta) = n \log \theta - \theta \sum_{i=1}^{n} x_i$$

$$\Rightarrow \ell'(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{\bar{X}}$$

Likelihood Ratio test: $H_0: \theta = \theta_0, H_a: \theta \neq \theta_0$

$$\frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\theta_0^n \exp\{-\theta_0 \sum_{i=1}^{n} x_i\}}{\hat{\theta}^n \exp\{-\hat{\theta} \sum_{i=1}^{n} x_i\}}$$

$$= \left(\frac{\theta_0^n}{\hat{\theta}^n}\right) \exp\left\{-n\bar{X}(\theta_0 - \hat{\theta})\right\}$$

$$= \left(\theta_0 \bar{X} \exp\{-\theta_0 \bar{X} + 1\}\right)^n$$

Now, we wish to determine rejection region:

$$\left(\theta_0 e^{-1} \bar{X} \exp\{-\theta_0 \bar{X}\}\right)^n \leq c_1$$

$$\theta_0 e^{-1} \bar{X} \exp\{-\theta_0 \bar{X}\} \leq c_1^{1/n}$$

$$\bar{X} \exp\{-\theta_0 \bar{X}\} \leq \underbrace{e\theta_0^{-1} c_1^{1/n}}_{=c}$$

$\Rightarrow$ rejection region corresponds to $\bar{X} \exp\{-\theta_0 \bar{X}\} \leq c$ where c is defined above