Homework 3

ESE 402/542

October 30, 2020

Type or scan your answers as a single PDF file and submit on Canvas. Include your code in your PDF submission if you are attempting Problem 4.
Problems 2, 3, and 5 are bonus problems for extra credit.
Problems 1, 4, and 6 are ungraded.

Problem 1. (Ungraded) Suppose that $X_1, \ldots, X_n$ are an i.i.d. random sample of size n with sample mean $\bar{X} = 12$ and sample variance $s^2 = 5$.

(a) Let $n = 5$ and suppose the samples are drawn from a Normal distribution with unknown mean μ and known variance $\sigma^2 = 9$. Let the null hypothesis be $H_0 : \mu = 10$ and the alternative hypothesis be $H_a : \mu \neq 10$. Calculate the relevant test statistic value and p-value. Determine the decision rule for $\alpha = 0.05$.

Solution: Since we know that the population distribution is Normal with a known variance, the relevant test statistic is:

$$\begin{aligned} z &= \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \\ &= \frac{12 - 10}{\frac{3}{\sqrt{5}}} \\ &\approx 1.491 \end{aligned}$$

The p-value of this statistic is $2\Phi(-1.491) = 0.135$. Thus, we fail to reject H_0 at $\alpha = 0.05$.

(b) Using the acceptance region of this test, construct a 95% confidence interval. (Hint: Think about how we can reverse our hypothesis test to construct the equivalent 95% confidence interval)

Solution: Reverse the hypothesis test to construct the 95% confidence interval:

$$\begin{aligned} CI_{95\%} &= 12 \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \\ &= 12 \pm 1.96 \frac{3}{\sqrt{5}} \\ &= [9.37, 14.63] \end{aligned}$$

Problem 2. (Bonus) Federal investigators identified a strong association between chemicals in dry-wall and electrical problems, and there is also strong evidence of respiratory difficulties due to the emission of hydrogen sulfide gas. An extensive examination of 51 homes found that 41 had such problems. Suppose these 51 were randomly sampled from the population of all homes having drywall.

(a) Does the data provide strong evidence for concluding that more than 50 % of all homes with drywall have electrical/environmental problems? Carry out a test of hypotheses using $\alpha = .01$.

Solution: Let $X = 41$ represent the number of homes with chemicals in the drywall and electrical problems. Let $n = 51$ represent the number of homes randomly selected from the population of all homes having drywall. We can obtain the sample proportion as follows:

$$\begin{aligned} \hat{p} &= \frac{X}{n} \\ &= \frac{41}{51} \\ &= 0.8039 \end{aligned}$$

$H_0 : p = 0.5$

$H_a : p > 0.5$

We assume conditions for a normal approximation and find the value of the test statistic:

$$\begin{aligned} z &= \frac{\hat{p} - p}{\sqrt{p(1-p)/n}} \\ &= \frac{0.8039 - 0.5}{\sqrt{0.5(1-0.5)/51}} \\ &= 4.34 \end{aligned}$$

From the standard normal distribution table, the critical value at $\alpha = 0.01$ for a right tailed test is 2.33.

Since the test statistic 4.34 is greater than critical value 2.33, we can reject the null hypothesis. This indicates that there is strong evidence to conclude that the true proportion is greater than 50%.

(b) Calculate a lower bound of a 99 % confidence interval for the percentage of all such homes that have electrical/environmental problems.

Solution: A lower confidence bound using a confidence interval of 99% for the percentage of all such homes that have electrical/environmental problems is calculated as follows:

$$= \hat{p} - Z^* \sqrt{p(1-p)/n}$$

$$= 0.8039 - 2.33 \frac{0.5(1-0.5)}{51}$$
$$0.8039 - 0.1631$$
$$= 0.6408$$

Problem 3. (Bonus) Consider a random sample of size $n = 100$ with sample proportion $\hat{p} = 0.2$ from a population with a true unknown proportion p.

(a) For the test $H_0 : p = 0.25$ versus $H_a : p < 0.25$, calculate the relevant test statistic value and p-value. Determine the decision rule for $\alpha = 0.05$ and $\alpha = 0.01$.

Solution: Since $np_0 = 25 > 10$ and $n(1 - p_0) = 75 > 10$ we can conduct a z test. We calculate the test statistic as follows:

$$z = \frac{\hat{p} - p}{\sqrt{p(1-p)/n}}$$
$$= \frac{0.2 - 0.25}{\sqrt{0.25(1-0.25)/100}}$$
$$= -1.155$$

The p-value is then $\Phi(-1.155) = 0.124$. Thus, we fail to reject the null hypothesis at both $\alpha = 0.05$ and $\alpha = 0.01$.

(b) For the test $H_0 : p = 0.25$ versus $H_a : p \neq 0.25$, calculate the relevant test statistic value and p-value. Determine the decision rule for $\alpha = 0.05$ and $\alpha = 0.01$.

Solution: The statistic is still $z = -1.155$. The p-value of this test is $2\Phi(-1.155) = 0.248$. Thus, we again fail to reject the null hypothesis at $\alpha_0 = 0.05$ and $\alpha_0 = 0.01$.

Problem 4. (Ungraded) Recall that α represents the probability of a type I error. On the other hand, β represents the probability of a type II error. The power of a test is the probability that the null hypothesis is rejected when it is false, and is therefore defined as $1 - \beta$. For this problem, you will explore how power depends on multiple factors.

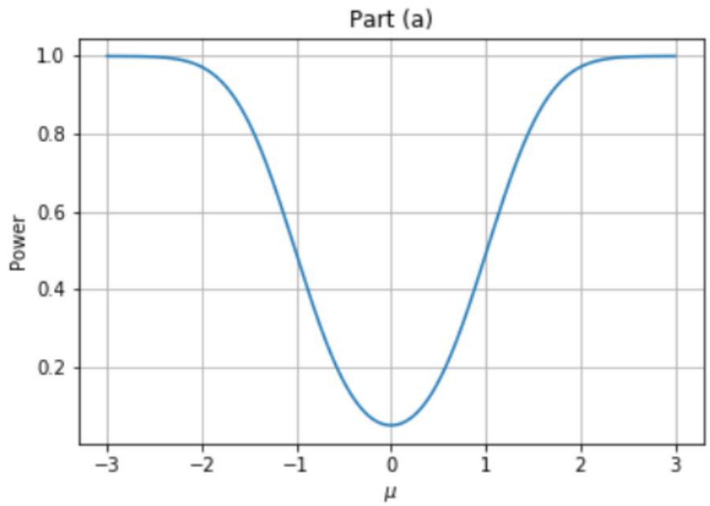
(a) Suppose that $X_1, \ldots, X_n$ are an i.i.d. random sample of size $n = 15$ drawn from a Normal distribution with unknown mean μ and known variance $\sigma^2 = 4$. Using Python, plot the power of the test $H_0 : \mu = 0$ versus $H_a : \mu \neq 0$ as the true mean μ varies at level $\alpha = 0.05$. Make sure to display your graph clearly.

Solution: First, recall that for a two-sided z test, the type II error probability is:

$$\beta = \Phi(z_{\frac{\alpha}{2}} + \frac{0-\mu}{\frac{\sigma}{\sqrt{n}}}) - \Phi(-z_{\frac{\alpha}{2}} + \frac{0-\mu}{\frac{\sigma}{\sqrt{n}}})$$

Since power $= 1 - \beta$, we can calculate power and obtain plots as follows:

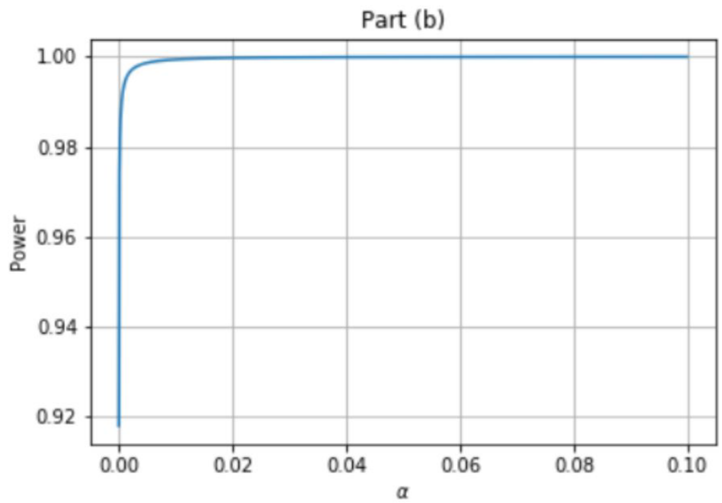
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.stats import norm

alpha = 0.05
n = 15
var = 4
mu = 3

# (a)

mu_list = np.arange(-3, 3.01, 0.01)
beta_a = norm.cdf(norm.ppf(1-alpha/2)+(0-mu_list)/np.sqrt(var/n))
        norm.cdf(-norm.ppf(1-alpha/2)+(0-mu_list)/np.sqrt(var/n))
power_a = 1 - beta_a

plt.subplot(1, 1, 1)
plt.plot(mu_list, power_a)
plt.xlabel(r'$\mu$')
plt.ylabel('Power')
plt.title('Part (a)')
plt.grid()
plt.savefig('a.png')
```



(b) Now using the same information from part (a) except for fixing μ at $\mu = 3$, plot the power of the test as α varies.

Solution:

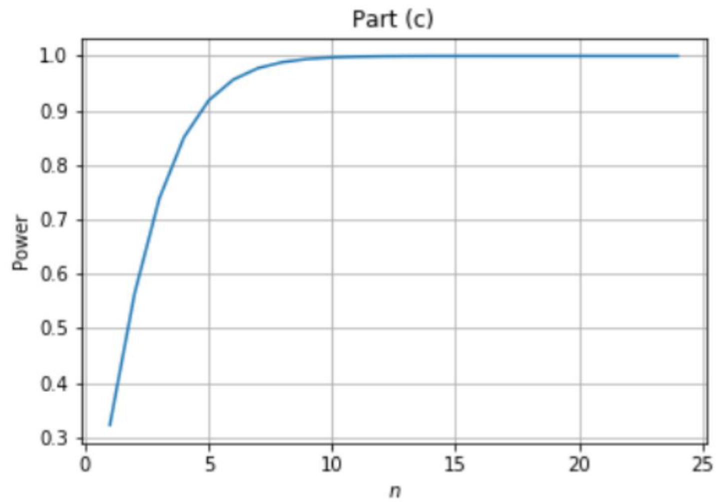
```
# (b)

alpha_list = np.arange(0.00001, 0.1, 0.0001)
beta_b = norm.cdf(norm.ppf(1-alpha_list/2)+(0-mu)/np.sqrt(var/n)) -
        norm.cdf(-norm.ppf(1-alpha_list/2)+(0-mu)/np.sqrt(var/n))
power_b = 1 - beta_b

plt.subplot(1, 1, 1)

plt.plot(alpha_list, power_b)
plt.xlabel(r'$\alpha$')
plt.ylabel('Power')
plt.title('Part (b)')
plt.grid()
plt.savefig('b.png')
plt.show()
```



(c) Fix α back to $\alpha = 0.05$ and plot the power of the test as n varies.

Solution:

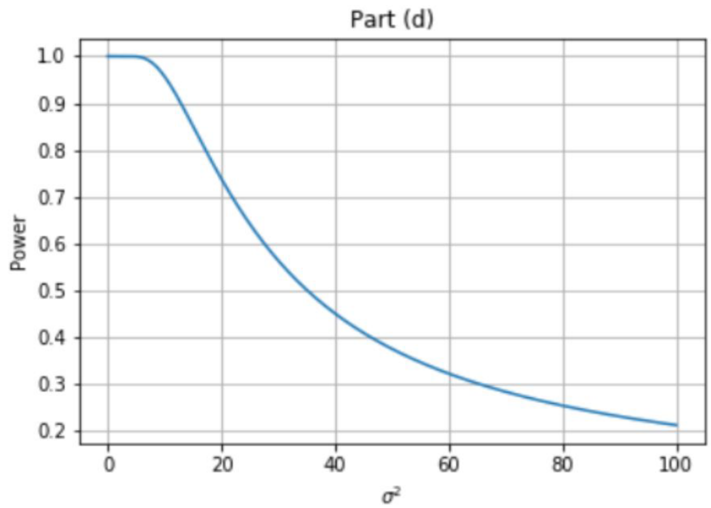
```
# (c)

n_list = np.arange(1, 25)
beta_c = norm.cdf(norm.ppf(1-alpha/2)+(0-mu)/np.sqrt(var/n_list))
        norm.cdf(-norm.ppf(1-alpha/2)+(0-mu)/np.sqrt(var/n_list))
power_c = 1 - beta_c

plt.subplot(1, 1, 1)
plt.plot(n_list, power_c)
plt.xlabel(r'$n$')
plt.ylabel('Power')
plt.title('Part (c)')
plt.grid()
plt.savefig('c.png')
plt.show()
```



(d) Fix n back to $n = 15$ and plot the power of the test as σ^2 varies.

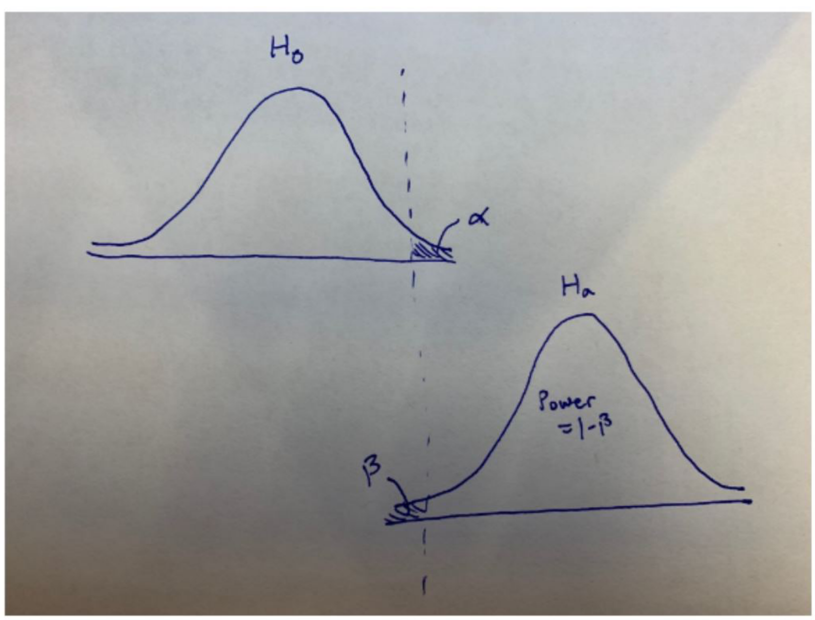
Solution:

```
# (d)

var_list = np.arange(0.00001, 100, 0.01)
beta_d = norm.cdf(norm.ppf(1-alpha/2)+(0-mu)/np.sqrt(var_list/n)) -
        norm.cdf(-norm.ppf(1-alpha/2)+(0-mu)/np.sqrt(var_list/n))
power_d = 1 - beta_d

plt.subplot(1, 1, 1)
plt.plot(var_list, power_d)
plt.xlabel(r'$\sigma^2$')
plt.ylabel('Power')
plt.title('Part (d)')
plt.grid()
plt.savefig('d.png')
plt.show()
```



(e) Compare and interpret your results.

Solution: The results make intuitive sense. Power is the probability of correctly rejecting the null hypothesis. So when the true mean is near/far from the null hypothesis, if the null hypothesis is false, it is unlikely/likely to reject the null hypothesis. When the probability of a type I error is low/high, it becomes harder/easier to reject the null hypothesis in general so power is lower/higher. When the sample size is small/large, in general we need a larger/smaller effect to reject the null so power is smaller/larger. And when the population variance is small/large, in general we need a smaller/larger effect to reject the null so power is larger/smaller. Visualizing how the two distributions change with each parameter in the graphic below helps to understand these relationships:



Problem 5. (Bonus) Let $X_1, ..., X_n$ be a sample from a Poisson distribution. Find the likelihood ratio for testing $H_0 : \lambda = \lambda_0$ versus $H_a : \lambda = \lambda_1$, where $\lambda_1 > \lambda_0$. Use the fact that the sum of independent Poisson random variables follows a Poisson distribution to explain how to determine a rejection region for a test at significance level α_0.

Solution: The likelihood ratio is found as follows:

$$\begin{aligned}\Lambda &= \frac{lik(\lambda_0)}{lik(\lambda_1)} \\ &= \frac{f(X_1, ..., X_n|\lambda_0)}{f(X_1, ..., X_n|\lambda_1)} \\ &= \frac{\prod_{i=1}^n e^{-\lambda_0 \frac{\lambda_0^{X_i}}{X_i!}}}{\prod_{i=1}^n e^{-\lambda_1 \frac{\lambda_1^{X_i}}{X_i!}}} \\ &= e^{n(\lambda_1 - \lambda_0)} (\frac{\lambda_0}{\lambda_1}) \sum_{i=1}^n X_i\end{aligned}$$

Since $\frac{\lambda_0}{\lambda_1} < 1$, a large value of $\sum_{i=1}^n X_i$ corresponds to a small value of λ, which favors the alternate hypothesis. Thus, we can reject the null hypothesis with $\sum_{i=1}^n X_i > c$, where c is determined by the significance level α.

Since X_i is of $Poisson(\lambda)$, $\sum_{i=1}^n X_i$ is of $Poisson(n\lambda)$. Thus:

$$\alpha = P(rejectH_0|H_0 = P(\sum_{i=1}^n X_i > c|\lambda = \lambda_0) = 1 - F(y)$$

where $F(y)$ is the cdf of the Poisson distribution with parameter $n\lambda_0$.

Problem 6. (Ungraded) Let $X_1, ..., X_n$ be a random sample from an exponential distribution with the density function $f(x|\theta) = \theta exp(-\theta x)$. Derive a likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$, and show that the rejection region is of the form $\bar{X} exp(\theta_0 \bar{X}) \leq c$.

Solution:

$$\Lambda = \frac{lik(\theta_0)}{max(lik(\theta))}$$
$$= \frac{\theta_0^n exp(-\sum_{i=1}^n \theta_0 X_i)}{\hat{\theta}^n exp(-\sum_{i=1}^n \hat{\theta} X_i)}$$

where $\hat{\theta} = \frac{1}{x}$ is the MLE. Plugging in the MLE, we get the expression of Λ as:

$$\Lambda = \frac{\theta_0^n exp(-n\bar{X}\theta_0)}{(\frac{1}{x})^n exp(-n)} = \theta_0^n exp(n)[\bar{X} exp(-\theta_0 \bar{X})]^n$$

A small value of $\bar{X}$ can give us a small Λ. Thus, we can construct a rejection region of:

$$S = \{X : \bar{X} exp(-\theta_0 \bar{X}) \leq c\}$$

such that:

$$P(\bar{X} exp(-\theta_0 \bar{X}) \leq c) = \alpha$$