Bernoulli Distribution

- **Bernoulli distribution** is the **discrete** probability distribution of a random variable which takes the value 1 with probability p and the value 0 with probability $1-p$ that is, the probability distribution of any single experiment that asks a **yes-no question**.

$$P(X = x) = \begin{cases} 1 - p, & x = 0 \\ p, & x = 1 \end{cases}$$

$$E(X) = \mathrm{p}$$
$$Var(X) = \mathrm{p}(1 - \mathrm{p})$$

Binominal Distribution

- **Binominal Distribution** is of the number of successes in a sequence of n independent **Bernoulli experiments**, each asking a yes–no question, and each with its own boolean-valued outcome: a random variable containing a single bit of information: success/yes/true/one (with probability p) or failure/no/false/zero (with probability q = 1 − p).

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

$$E(X) = \mathrm{np}$$
$$Var(X) = \mathrm{np}(1 - \mathrm{p})$$

Poisson Probability Distribution

- The **Poisson probability distribution** describes the number of times some event occurs during a **specified interval**. The interval may be time, distance, area, or volume.

- Assumptions of the Poisson Distribution
  - (1) The probability is proportional to the length of the interval.
  - (2) The intervals are independent.

Poisson Probability Distribution

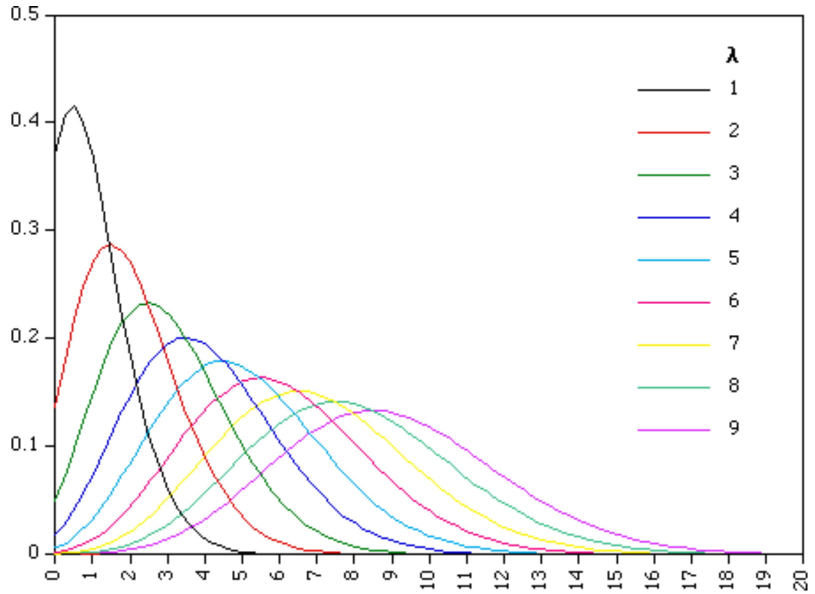
- Examples:
    - ✓ No. of road accidents
    - ✓ No. of death in flood
    - ✓ No. of mistakes per page committed by a typist.
    - ✓ No. of accidents due to falling of trees or roofs.
    - ✓ No. of goals in games of football or hockey etc.

Poisson Probability Distribution

POISSON DISTRIBUTION	$P(x) = \dfrac{\mu^x e^{-\mu}}{x!}$	**[6–7]**

where:

μ (mu) is the mean number of occurrences (successes) in a particular interval.
e is the constant 2.71828 (base of the Napierian logarithmic system).
x is the number of occurrences (successes).
$P(x)$ is the probability for a specified value of x.



Practice 3

- Assume baggage is rarely lost by Northwest Airlines. Suppose a random sample of 1,000 flights shows a total of 300 bags were lost. Thus, the arithmetic mean number of lost bags per flight is 0.3.If the number of lost bags per flight follows a Poisson distribution with $\mu = 0.3$, find the probability of not losing any bags.

$$P(0) = \frac{\mu^x e^{-u}}{x!} = \frac{0.3^0 e^{-.3}}{0!} = .7408$$

Poisson Probability Distribution

- Important Property:

If μ is the average number of successes occurring in a given time interval or region in the Poisson distribution, then the mean and the variance of the Poisson distribution are both equal to μ.

$$E(X) = \mu = np$$

$$Var(X) = \mu = np$$

Exponential Distribution

- **Exponential Distribution** is the probability distribution that describes **the time between events** in a Poisson process, i.e., a process in which events occur **continuously and independently at a constant average rate (denoted as** λ).

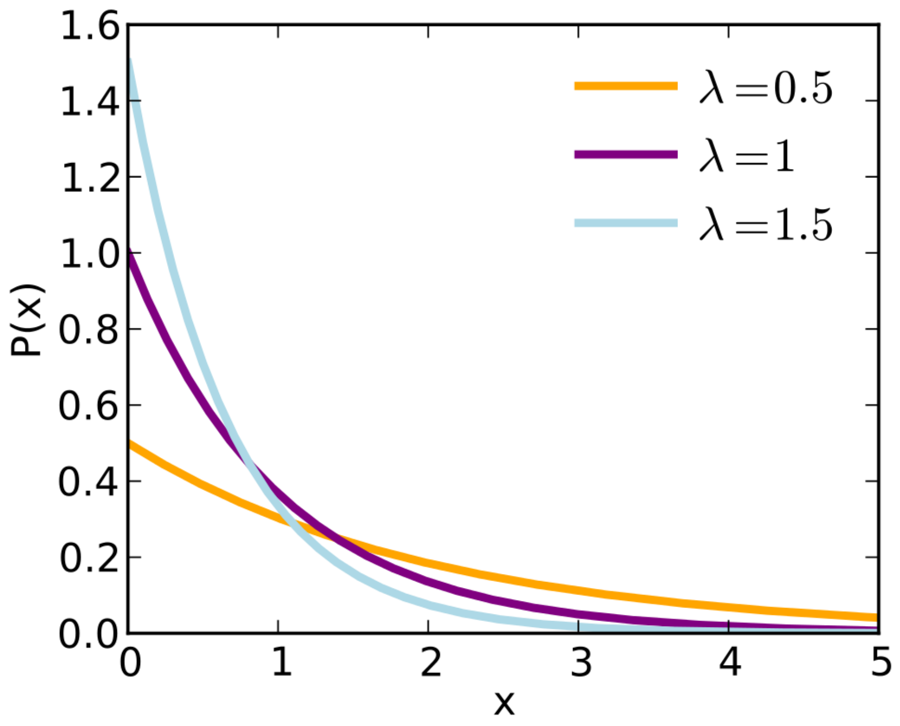
- It is a particular case of the **gamma distribution**.

- Example: queue for coffee

Exponential Distribution

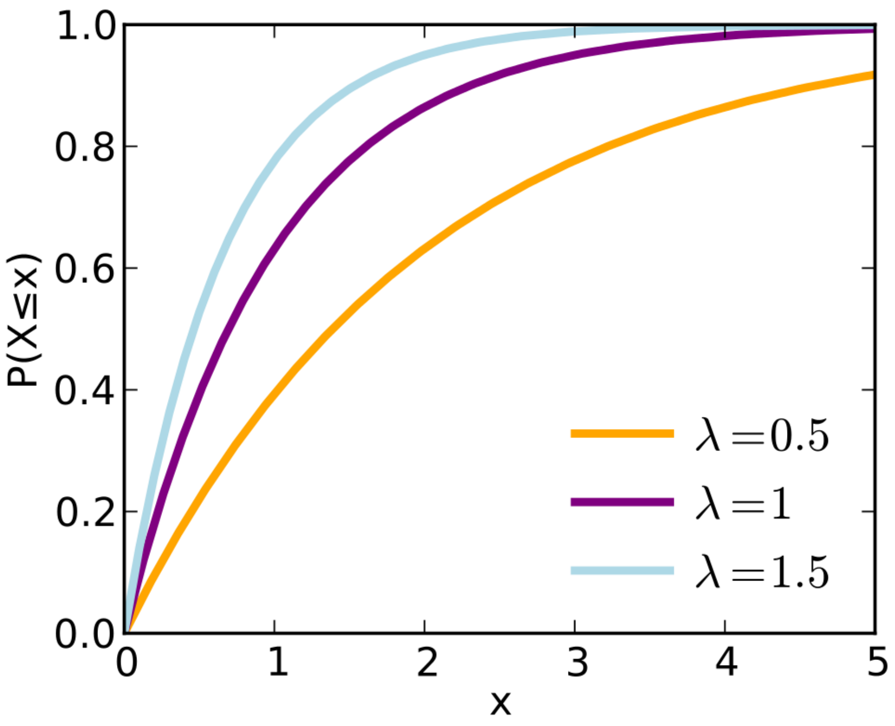
Probability density function

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$



Cumulative distribution function

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$



$$E(X) = \frac{1}{\lambda}$$

$$Var(X) = \frac{1}{\lambda^2}$$

Practice 4

- The time between arrivals of cars at Al's full-service gas pump follows an exponential probability distribution with **a mean time between arrivals 3 minutes**. Al would like to know the probability that the time between two successive arrivals will be **2 minutes or less**.

$$P(x \leq 2) = 1 - e^{-2/3} = 1 - .5134 = .4866$$

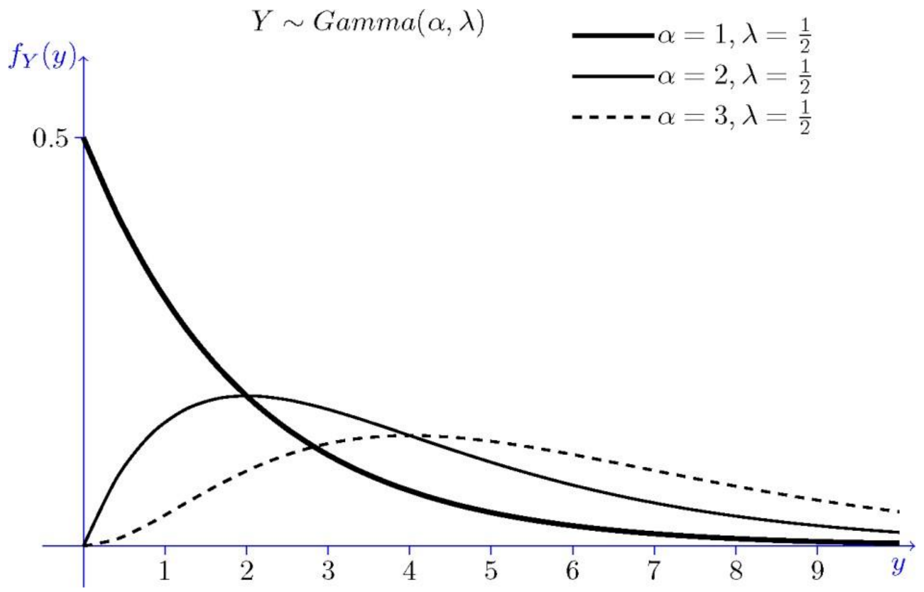
Gamma Distribution

A continuous random variable X is said to have a *gamma* distribution with parameters $\alpha > 0$ and $\lambda > 0$, shown as $X \sim Gamma(\alpha, \lambda)$, if its PDF is given by

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

If we let $\alpha = 1$, we obtain

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$



$$EX = \frac{\alpha}{\lambda}, \qquad Var(X) = \frac{\alpha}{\lambda^2}.$$

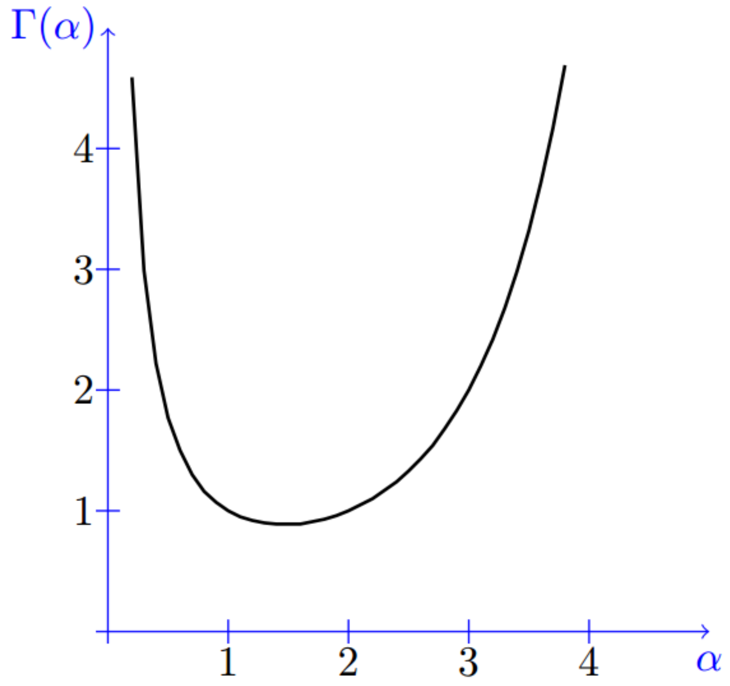
Gamma Distribution

Gamma function: The gamma function [10], shown by $\Gamma(x)$, is an extension of the factorial function to real (and complex) numbers. Specifically, if $n \in \{1, 2, 3, \dots\}$, then

$$\Gamma(n) = (n-1)!$$

More generally, for any positive real number α, $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \mathrm{d}x, \qquad \text{for } \alpha > 0.$$



Practice 5

Engineers designing the next generation of space shuttles plan to include **two fuel pumps** —one active, the other in reserve. If the primary pump malfunctions, the second is automatically brought on line. Suppose a typical mission is expected to require that fuel be pumped for at most **50 hours**. According to the manufacturer's specifications, **pumps are expected to fail once every 100 hours**.

What are the chances that such a fuel pump system would not remain functioning for the full 50 hours?

Solution

We know that the mean time of failure is 100 hours. So the rate parameter $\lambda = 1/100$. The shape parameter $\alpha = 2$, Since we want to calculate the event that both the pumps fail.

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \frac{1}{100^2\Gamma(2)} e^{-y/100} y^{2-1} = \frac{1}{10000} y e^{-y/100}$$

Therefore, the probability that the system fails to last for 50 hours is:

$$P(Y < 50) = \int_0^{50} \frac{1}{10000} y e^{-y/100} dy$$

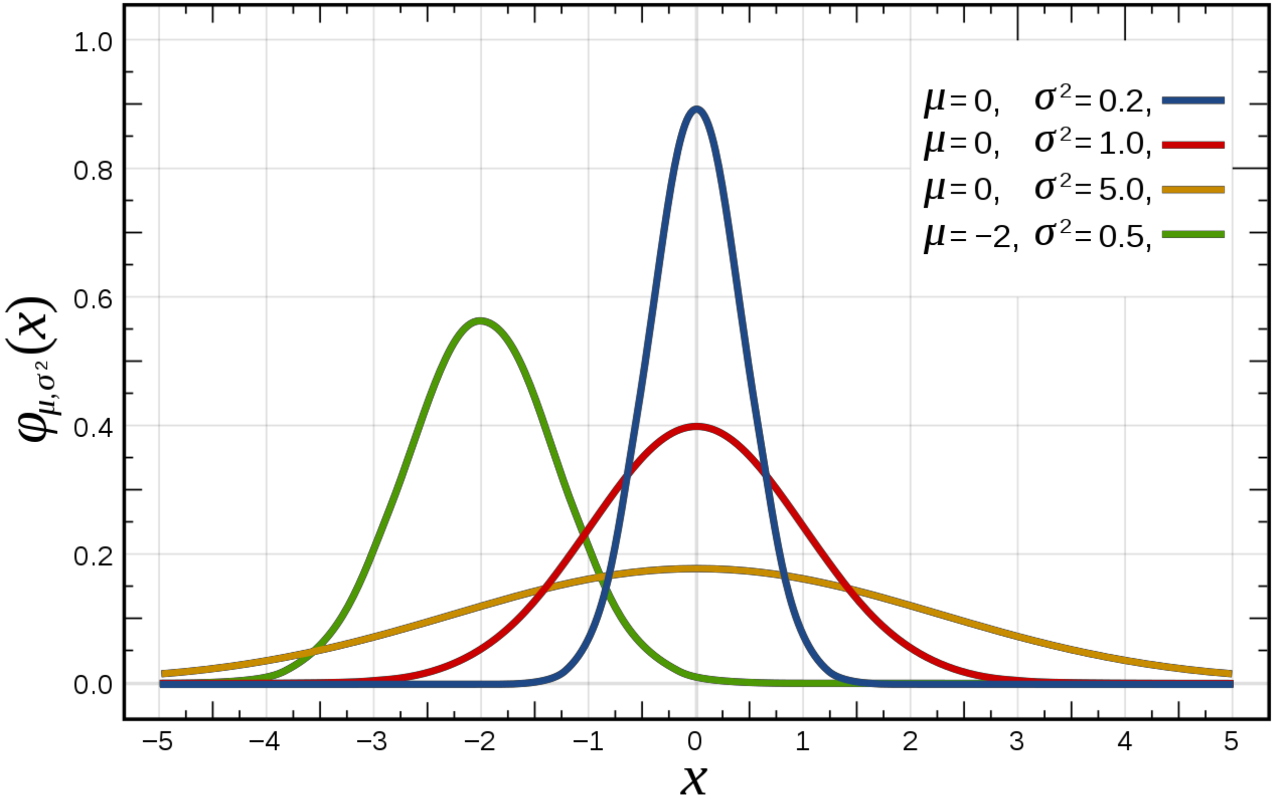
Using Integration by Parts we get: $P(Y < 50) = 1 - \frac{3}{2} \exp^{-1/2}$

Gaussian/Normal Distribution

- The normal distribution is useful because of the **central limit theorem**
- The pdf of Gaussian is:

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

μ is the mean or expectation of the distribution (and also its median and mode),
σ is the standard deviation, and
σ^2 is the variance



Gaussian/Normal Distribution

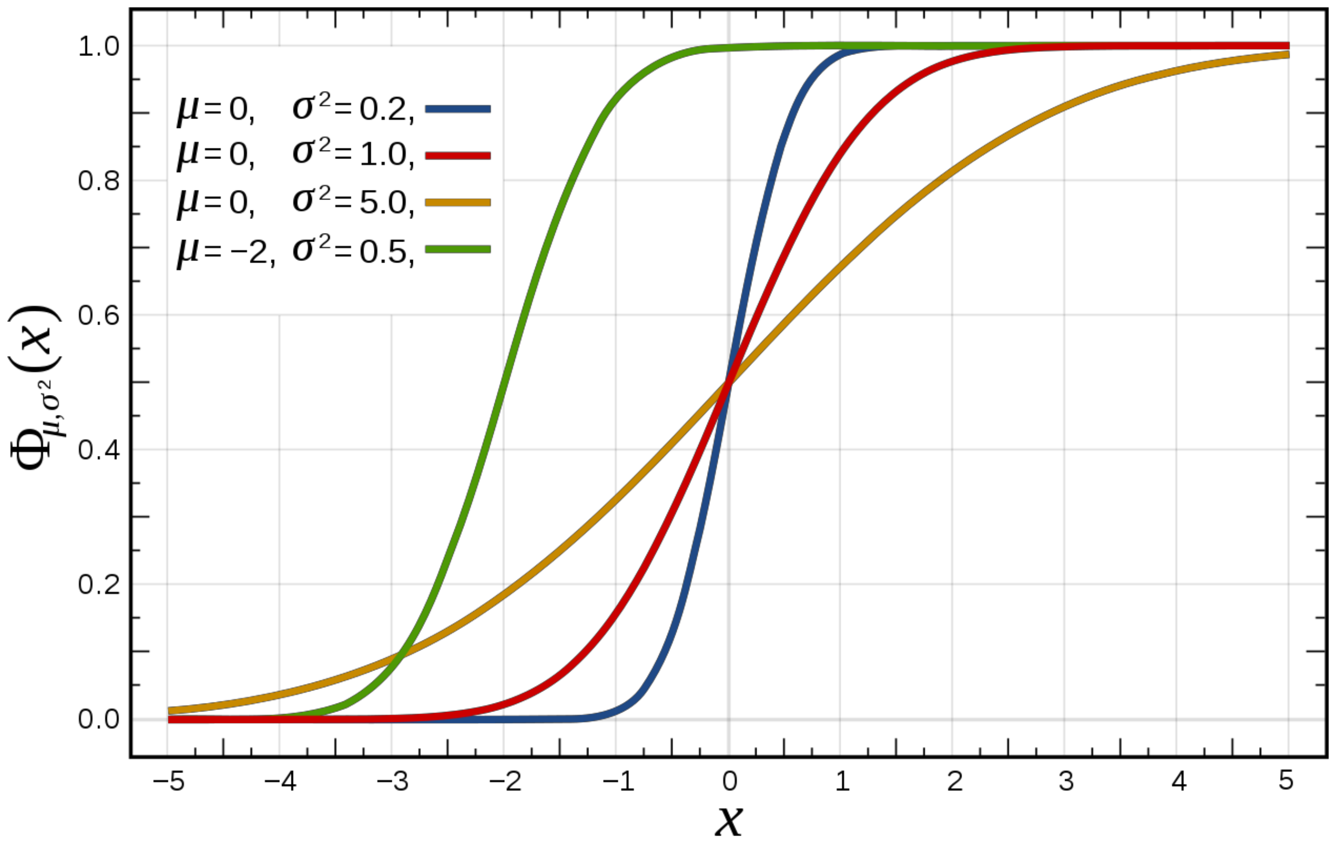
- The cdf of **standard Gaussian** is:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

- Calculate Gaussian with mean μ and sd of σ:

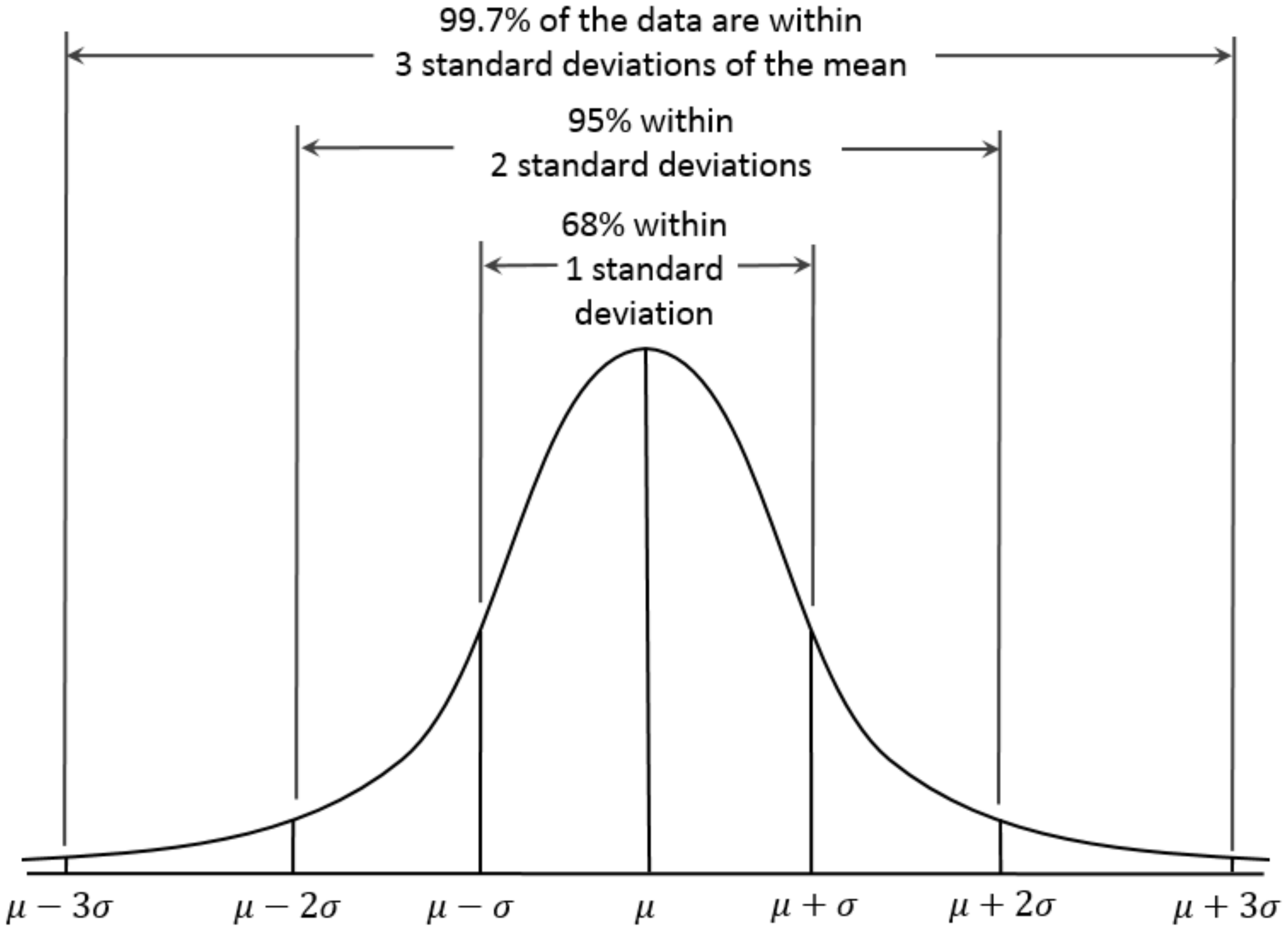
$$F(x) = \varphi\left(\frac{x-\mu}{\sigma}\right)$$

Refer to the Table



Gaussian/Normal Distribution

About 68% of values drawn from a normal distribution are within one standard deviation σ away from the mean; about 95% of the values lie within two standard deviations; and about 99.7% are within three standard deviations. This fact is known as the3-sigma rule.



Chi-Square Distribution

- **Chi-squared distribution** (also chi-square or χ2-distribution) with **k degrees of freedom** is the distribution of a sum of the squares of k independent standard normal random variables.

If Z1, ..., Zk are independent, standard normal random variables, then the sum of their squares,

$$Q = \sum_{i=1}^{k} Z_i^2,$$

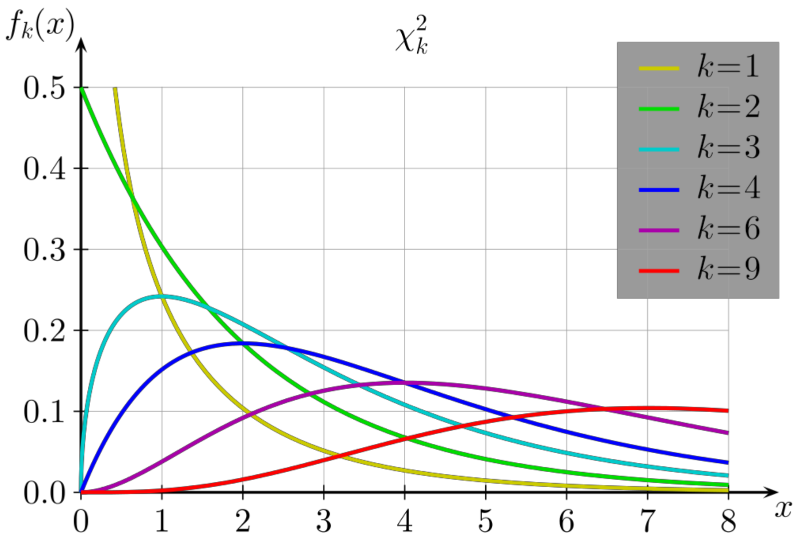
Chi-Square Distribution

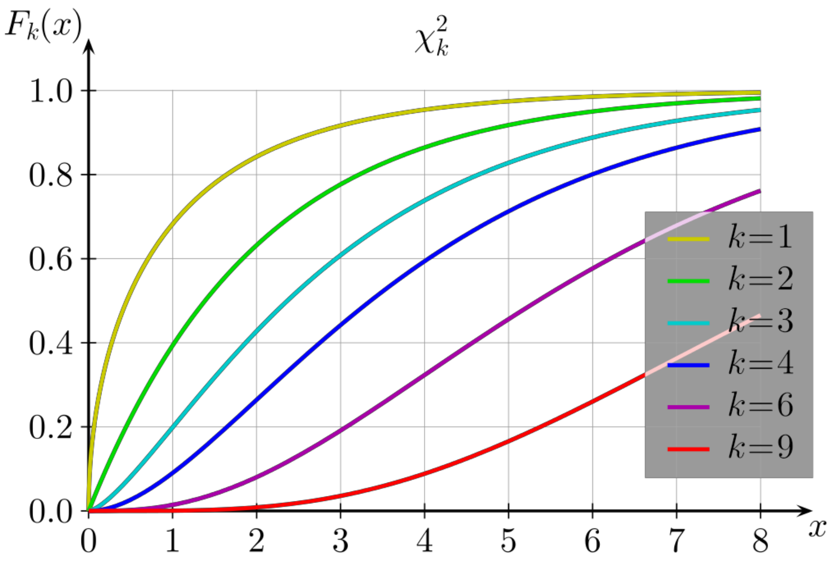
Probability density function

$$f(x;\,k) = \begin{cases} \dfrac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}}\,\Gamma\left(\frac{k}{2}\right)}, & x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Cumulative distribution function

$$F(x;\,k) = \frac{\gamma(\frac{k}{2},\,\frac{x}{2})}{\Gamma(\frac{k}{2})} = P\left(\frac{k}{2},\,\frac{x}{2}\right)$$





$$E(X) = k$$

$$Var(X) = 2k$$

Chi-Square Distribution - Extension

- The Chi-square is used most commonly to compare the incidence (or proportion) of a characteristic in one group to the incidence (or proportion) of a characteristic in other group(s).

- Chi-Square Test

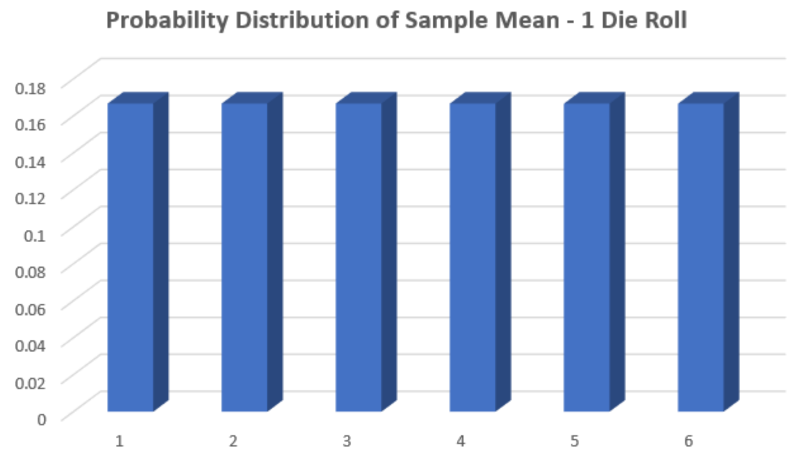
$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

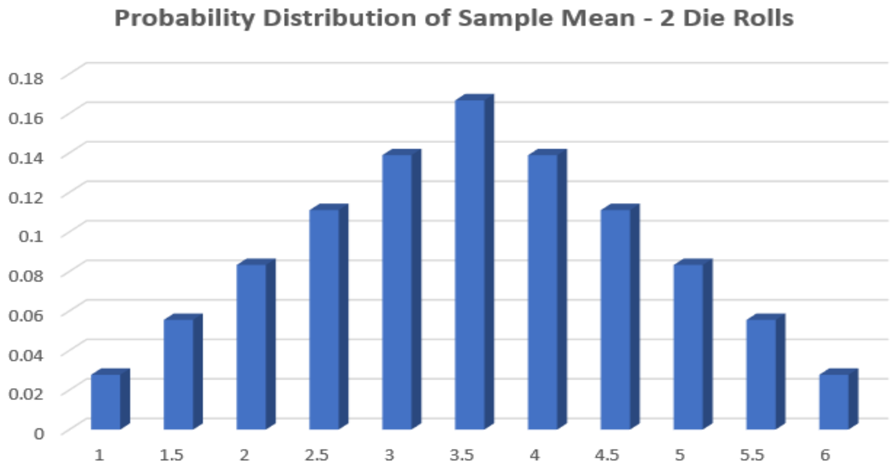
O stands for the observed frequency,

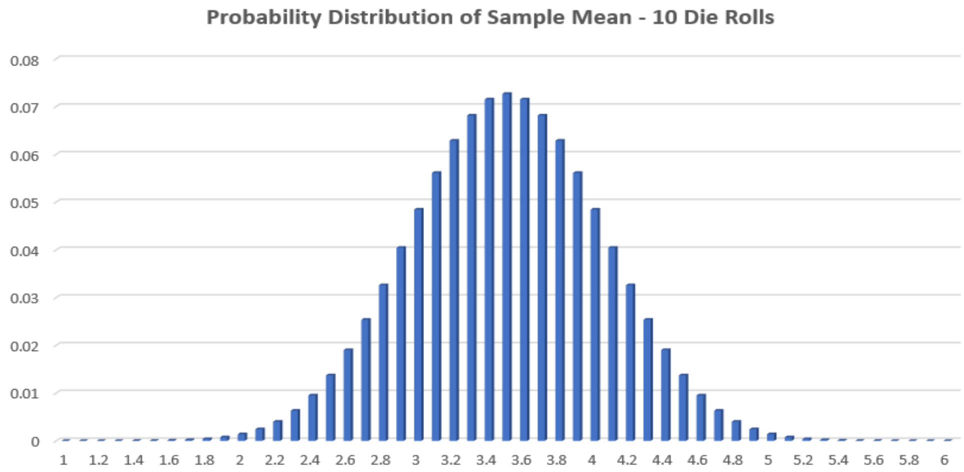
E stands for the expected frequency.

Central Limit Theorem – Roll a dice

- Suppose you roll a single die for 1, 2, 10 times







Central Limit Theorem

- Imagine there is some population with a mean μ and standard deviation σWe can collect samples of size n where the value of n is “large enough”
  - We can then calculate the **mean** of each sample
  - If we create a histogram of those means, then the resulting histogram look much like **a normal distribution**
  - It does not matter what the distribution of the original population is.

Central Limit Theorem