ESE 402/542 Recitation 3: Basic Tools of Estimation

Last Time...

- MGF and how to use it to derive moments of a random variable

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- Some basics on CLT–visually and intro to usage

Today

- CLT examples and practice

CLT 1: Applying transforms before CLT

- **Scenario:** we have an object with initial mass m_0. It is then periodically subjected to impacts that leave a proportion of the mass left. Let's say there were a total of n impacts, each leaving $X_i \in (0.1, 1)$ proportion of the mass, where X_i are iid. We want to understand the distribution of the remaining mass after n impacts, Z_n.

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- How can we transform the problem such that we can apply CLT? (Hint: We apply CLT to sums of things...how can we get sums from this?)

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  1. $\log(X_i)$ are iid. Yep.
  2. $\operatorname{var}(\log(X_i)) < \infty$. Lower bound on $\log(X_i)$ is $\log(0.1) = -1$, upper bound is $\log(1) = 0$. Bounded rvs have finite variance (intuitive, but check Popoviciu's inequality if interested).
- Good to go!

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- Problem source: Rice Chapter 5 Example G

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  - Games are iid...probably not exactly, but let's say it's close enough.
  - Finite variance. Each game is either won (1) or lost (0) – Bernoulli, which has finite variance

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- Crunch the numbers! $\hat{p} - z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \approx 0.5215 > 0.5$. Reject the null hypothesis. There is statistical significant evidence that home-field advantage exists (under our assumptions).

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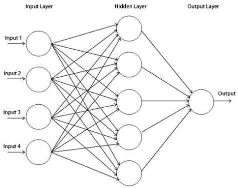
- Source: Anderson, 2014

CLT 3: An Application to Deep Learning



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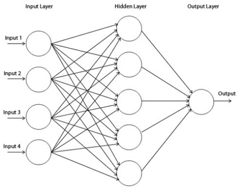

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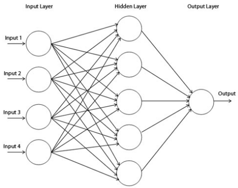

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- Neural net is randomly initialized with $a_i \sim \mathcal{N}(0, 1)$, $w_{ij} \sim \mathcal{N}(0, 1/n)$, all independent.
- Want to know distribution of $f(x)$ as number of hidden neurons $n \to \infty$, for FIXED x.

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- What is behavior of $\sum_{i=1}^{k} w_{ij} x_j$? Mean? Variance? Say $\|x\|_2^2 = \sum_{i=1}^{k} x_k^2 = 1$ for simplicity. Remember w_{ij} are independent and mean 0, $w_{ij} \sim \mathcal{N}(0, 1/n)$.

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- Source: Prof. Hassani's topics class last semester.