Lecture 12:

$$X_1, \ldots, X_n \sim f(x|\theta)$$

Example of a more general hypothesis testing problem:

$$\begin{cases} H_0 : & \theta = \theta_0 \\ H_a : & \theta = \theta_a \end{cases}$$

For such problems, the likelihood function is the main tool to design appropriate tests.

$$lik(\theta) = f(x_1, \ldots, x_n | \theta)$$
$$= f(x_1|\theta) \times \cdots \times f(x_n|\theta)$$

(intuitively, we should have:)

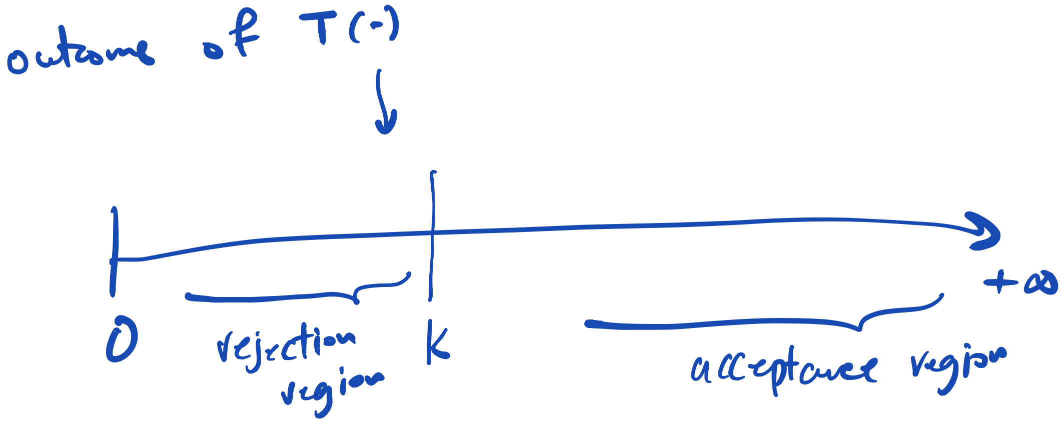
θ_0 $\longrightarrow$ $lik(\theta_0)$ $>$ $lik(\theta_a)$ if H_0 is true

θ_a $\qquad$ $<$ if H_a is true

Test:

$$\boxed{T(x_1, \ldots, x_n) = \frac{lik(\theta_0)}{lik(\theta_a)}}$$

$$lik(\theta) = f(x_1, \ldots, x_n | \theta)$$



$$\frac{lik(\theta_0)}{lik(\theta_a)} \underset{H_a}{\overset{H_0}{\gtrless}} k$$

k is determined based on the significance level α.

Example: Assume $X_1, \dots, X_n \sim f(x|\lambda)$

$$\underbrace{f(x|\lambda) = \lambda e^{-\lambda x} \mathbb{1}\{x \geq 0\}}_{\exp(\lambda) \text{ dist.}}$$

$$\begin{cases} H_0 = \lambda = \lambda_0 \\ H_a = \lambda = \lambda_a \end{cases}$$

(Assume for simplicity $\lambda_a > \lambda_0$)

significance level $\underline{\alpha}$ is also given.

e.g. 0.05

Test: $\dfrac{lik(\lambda_0)}{lik(\lambda_a)} \triangleq T(X_1, \dots, X_n)$

$$T(X_1, \dots, X_n) \underset{H_a}{\overset{H_0}{\gtrless}} k$$

$$\text{lik}(\lambda) = f(x_1, \ldots, x_n \mid \lambda)$$

$$= \prod_{i=1}^{n} f(x_i \mid \lambda)$$

$$= \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$T(x_1, \ldots, x_n) = \frac{\text{lik}(\lambda_0)}{\text{lik}(\lambda_a)} = \frac{\lambda_0^n e^{-\lambda_0 \sum_{i=1}^{n} x_i}}{\lambda_a^n e^{-\lambda_a \sum_{i=1}^{n} x_i}}$$

$$= \left(\frac{\lambda_0}{\lambda_a}\right)^n e^{-(\lambda_0 - \lambda_a) \sum_{i=1}^{n} x_i}$$

$$\rightarrow \quad \left(\frac{\lambda_0}{\lambda_a}\right)^n e^{-(\lambda_0 - \lambda_a) \sum x_i} \underset{H_a}{\overset{H_0}{\gtrless}} K$$

$$-(\lambda_0 - \lambda_a)\sum_{i=1}^{n} X_i \; \underbrace{+ n\log\frac{\lambda_0}{\lambda_a}}\; \underset{H_a}{\overset{H_0}{\gtrless}} \log k$$

$$\Longleftrightarrow \overbrace{(\lambda_a - \lambda_0)}^{\geq 0}\sum X_i \underset{H_a}{\overset{H_0}{\gtrless}} \log k - n\log\frac{\lambda_0}{\lambda_a}$$

$$\Longleftrightarrow \sum_{i=1}^{n} X_i \underset{H_a}{\overset{H_0}{\gtrless}} \overbrace{\left(\frac{\log k - n\log\frac{\lambda_0}{\lambda_a}}{(\lambda_a - \lambda_0)}\right)}^{k'}$$

$$\Longleftrightarrow \underline{\sum_{i=1}^{n} X_i \underset{H_a}{\overset{H_0}{\gtrless}} k'}$$

we have to find k' in terms of α.

To find k':

$$\Pr\{\text{type I error}\} = \alpha$$

$\Leftrightarrow Pr\{H_0 \text{ is rejected} \mid \underbrace{H_0 \text{ is true}}\} = \alpha$

$\Leftrightarrow Pr\{\sum_{i=1}^{n} X_i < K' \mid X_i \sim \exp(\lambda_0)\} = \alpha$

$$\boxed{\text{If } X_i \sim \exp(\lambda_0) \Rightarrow \sum_{i=1}^{n} X_i \overset{CLT}{\sim} \frac{n}{\lambda_0} + \frac{\sqrt{n}}{\lambda_0} N(0,1)}$$

$Pr\{\frac{n}{\lambda_0} + \frac{\sqrt{n}}{\lambda_0} N(0,1) < K'\} = \alpha$

CLT:

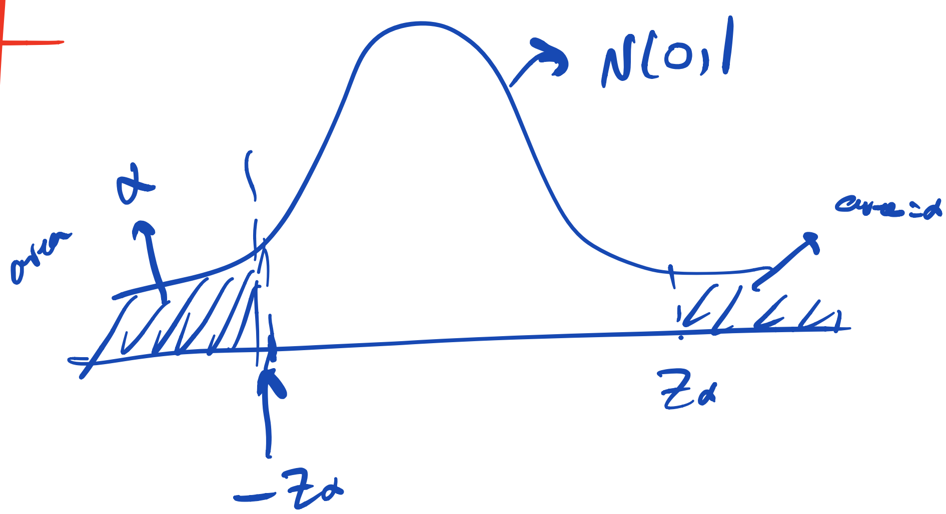
if $X_i \sim dist(\mu, \sigma^2)$

$\sum_{i=1}^{n} X_i = n\mu + \sigma \cdot \sqrt{n} N(0,1)$

$X_i \sim \exp(\lambda_0)$

$\mu = \frac{1}{\lambda_0}, \sigma^2 = \frac{1}{\lambda_0^2}$

$\Leftrightarrow Pr\{N(0,1) < \frac{K' - n/\lambda_0}{\frac{\sqrt{n}}{\lambda_0}}\} = \alpha$



$$\Leftrightarrow \frac{K' - n/\lambda_0}{\sqrt{n}/\lambda_0} = -Z_\alpha$$

$$\Longrightarrow \boxed{k' = -\frac{z_\alpha \cdot \sqrt{n}}{\lambda_0} + \frac{n}{\lambda_0}}$$

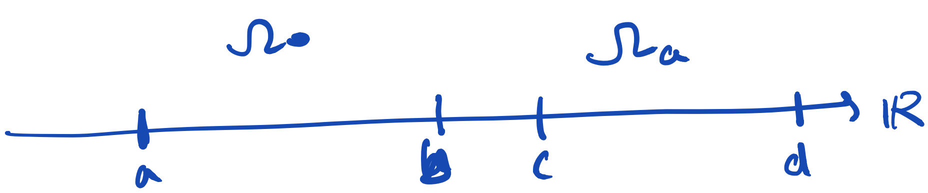
$$\Longrightarrow \boxed{\sum_{i=1}^{n} X_i - \frac{n}{\lambda_0} \underset{H_a}{\overset{H_0}{\gtrless}} -\frac{z_\alpha \sqrt{n}}{\lambda_0}}$$

$X_1, \dots, X_n \sim f(x|\theta)$

$$\begin{cases} H_0: & \theta = \theta_0 \\ H_a: & \theta = \theta_a \end{cases} \xRightarrow{\text{General version}} \begin{cases} H_0: & \theta \in \Omega_0 \\ H_a: & \theta \in \Omega_a \end{cases}$$

e.g. $\Omega_0 = \{\theta_0\}$ and $\Omega_a = \{\theta_a\}$ ← solved using the likelihood test

e.g.



e.g.

$\theta \in \mathbb{R}^2$

Generalized Likelihood Test:

$X_1, \dots, X_n \sim f(x|\theta)$

$$\begin{cases} H_0 : \theta \in \Omega_0 \\ H_a : \theta \in \Omega_a \end{cases}$$

Test: $T(X_1, \dots, X_n) \overset{\text{idea 1}}{=} \dfrac{\int_{\theta\in\Omega_0} lik(\theta)\, d\theta}{\int_{\theta\in\Omega_a} lik(\theta_a)\, d\theta}$

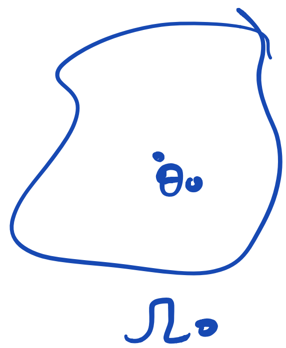
$\overset{\text{idea 2}}{=} \boxed{\dfrac{\max_{\theta\in\Omega_0} lik(\theta)}{\max_{\theta\in\Omega_a} lik(\theta)}}$

intuition:

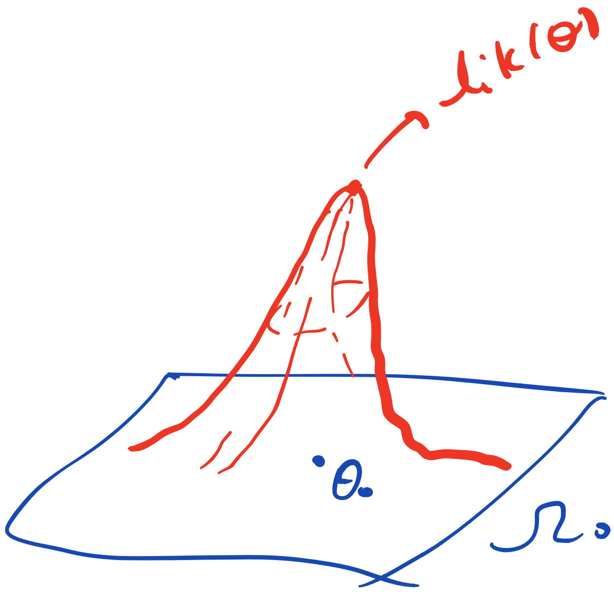
Let's assume that H_0 is true.

$X_1, \ldots, X_n \sim f(x|\theta_0)$

$\theta_0 \in \Omega_0$



$\Rightarrow \quad lik(\theta_0) = \max_{\theta \in \Omega_0} lik(\theta)$



if $\theta_0 \in \Omega_0$ is the true parameter the

$\max_{\theta \in \Omega_0} lik(\theta) = lik(\theta_0)$

- Generalized Likelihood Test:

$$T(X_1, \ldots, X_n) \triangleq \frac{\max_{\theta \in \Omega_0} lik(\theta)}{\max_{\theta \in \Omega} lik(\theta)}$$

- Acceptance/Rejection regions:

$$T(X_1, \dots, X_n) \underset{H_a}{\overset{H_0}{\gtrless}} K$$

is determined in terms of α.

Hypothesis testing problem → design T
↳ acc/rej regions
↳ Type I error → α
Type II error → β

- Remember that our goal is to design a test + acc/rej regions such that
(1) α is less than a given significance level
(2) β is as small as possible

- **Definition** Power of a test $\triangleq 1-\beta$.

- Among all the tests + acc/reject regions with α less than a given level, our ~~to~~ goal should to find the most powerful one (i.e. smallest β).

Neyman-Pearson Lemma:

Consider the following hypothesis testing problem:

$$\boxed{\begin{array}{l} X_1, \dots, X_n \sim f(x|\theta) \\ \begin{cases} H_0: & \theta=\theta_0 \\ H_a: & \theta=\theta_a \end{cases} \\ \text{significance level} = \alpha \end{array}}$$

Among all the tests that we can design for the hypothesic problem above, the likelihood test is the most powerful one.

$51 \rightarrow 41$

H_0 : ~~5%~~

H_a : $> 50\%$

$\alpha \downarrow \quad \beta \uparrow$

$$\min \beta(S)$$

over all the solutions

$$\underline{\alpha(S)} \le \alpha_0 = 0.01$$

$\theta \in \Omega_0 \rightarrow \theta = \theta_0$

$\theta \in \Omega_a \rightarrow \theta \neq \theta_0$