

Problem 1A & 1B

PROBLEM 1 A

$$f(x|\sigma) = \frac{1}{2\sigma} \exp\left\{-\frac{|x|}{\sigma}\right\}$$

Method of Moments to estimate σ

$$\begin{aligned} 1^{\text{st}} \text{ Population moment: } E[X] &= \int_{-\infty}^{\infty} \frac{x}{2\sigma} \exp\left\{-\frac{|x|}{\sigma}\right\} dx = \\ &= \int_{-\infty}^0 \frac{x}{2\sigma} \exp\left\{-\frac{|x|}{\sigma}\right\} dx + \int_0^{\infty} \frac{x}{2\sigma} \exp\left\{-\frac{|x|}{\sigma}\right\} dx \end{aligned}$$

Because integrand is an odd function & symmetric about 0,
the sum of these two pieces will negate each other & equal 0

$$\Rightarrow E[X] = 0$$

Thus, we must go to the second moment.

$$2^{\text{nd}} \text{ Population moment: } E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{2\sigma} \exp\left\{-\frac{|x|}{\sigma}\right\} dx$$

Using the fact that this integrand is an even function, we can
change it into 2 parts & remove absolute value around x

$$E[X^2] = 2 \cdot \int_0^{\infty} \frac{x^2}{2\sigma} \exp\left\{-\frac{x}{\sigma}\right\} dx = \int_0^{\infty} \frac{x^2}{\sigma} \exp\left\{-\frac{x}{\sigma}\right\} dx$$

Now, substitute $t = \frac{x}{\sigma}$ & $dt = \frac{dx}{\sigma}$ to simplify the integral

Limits of integration remain the same

$$\Rightarrow E[X^2] = \frac{1}{\sigma} \int_0^{\infty} (\sigma t)^2 \cdot \exp\{-t\} \cdot \sigma \cdot dt$$

$$= \frac{\sigma}{\sigma} \cdot \sigma^2 \int_0^{\infty} t^2 \cdot \exp\{-t\} dt$$

$$= \sigma^2 \cdot \Gamma(3) = 2! \cdot \sigma^2$$

$$E[X^2] = 2\sigma^2$$

$$\Rightarrow \sigma = \sqrt{\frac{1}{2} E[X^2]}$$

Using sample moment instead of population moment:

$$\sigma = \sqrt{\frac{1}{2n} \sum_{i=1}^n X_i^2}$$

PROBLEM 1B

① Define likelihood function

MLE estimate of $\sigma \Rightarrow$ ② Set 1st derivative of log-likelihood = 0

③ solve for argmax of likelihood

$$\text{lik}(\sigma) = f(x_1, \dots, x_n | \sigma) = f(x_1 | \sigma) \cdot f(x_2 | \sigma) \cdots f(x_n | \sigma)$$

$$\Rightarrow \text{lik}(\sigma) = \prod_{i=1}^n \frac{1}{2\sigma} \exp\left\{-\frac{|x_i|}{\sigma}\right\} = \frac{1}{(2\sigma)^n} \exp\left\{-\frac{(|x_1| + |x_2| + \dots + |x_n|)}{\sigma}\right\}$$

Convert this into log likelihood:

$$\ln(\text{lik}(\sigma)) = -n \cdot \ln(2\sigma) - \frac{|x_1| + \dots + |x_n|}{\sigma}$$

We wish to find σ which maximizes this over $(0, \infty)$

$$1^{\text{st}} \text{ derivative of } \ln(\text{lik}(\sigma)) \Rightarrow \frac{-n}{\sigma} + \frac{|x_1| + \dots + |x_n|}{\sigma^2}$$

$$\text{Setting } 1^{\text{st}} \text{ derivative} = 0, \quad \underbrace{\sigma = 0}_{\text{Trivial Solution}} \quad \text{OR} \quad \sigma = \frac{|x_1| + |x_2| + \dots + |x_n|}{n}$$

$$\Rightarrow \text{MLE estimate of } \sigma = \frac{1}{n} \sum_{i=1}^n |X_i|$$

Problem 1C

PROBLEM 1C

Using asymptotic normality property of the MLE estimator, we know

$$\text{Var}(\hat{\sigma}) = \frac{1}{n I(\hat{\sigma})} \quad \text{where } I(\sigma) = E\left[\left(\frac{\partial}{\partial \sigma} \ln f(X|\sigma)\right)^2\right]$$

$$= \frac{E\left([l'(\sigma)]^2\right)}{n} = \frac{-E(l''(\sigma))}{n}$$

where $l(\sigma)$ is the log-likelihood function

$$l''(\sigma) = \frac{n}{\sigma^2} - 2 \cdot \frac{|x_1| + \dots + |x_n|}{\sigma^3}$$

$$\Rightarrow E(l''(\sigma)) = \frac{n}{\sigma^2} - \frac{2}{\sigma^3} (E(|x_1| + \dots + |x_n|))$$

$$\text{Solving for } E[|X|] = \int_{-\infty}^{\infty} |x| \cdot f(x) dx = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{2\sigma} \cdot \exp\left\{\frac{-|x|}{\sigma}\right\} dx$$

Using the fact that integrand is even:

$$E[|X|] = \frac{2}{2\sigma} \int_0^{\infty} x \cdot \exp\left\{\frac{-x}{\sigma}\right\} dx \quad \& \text{ substituting again for } t = \frac{x}{\sigma} \quad \& \quad dt = \frac{dx}{\sigma}$$

$$\Rightarrow E[|X|] = \frac{1}{\sigma} \int_0^{\infty} (t\sigma) \cdot \exp\{-t\} \cdot \sigma \cdot dt$$

$$= \sigma \int_0^{\infty} t \cdot \exp\{-t\} dt = \sigma \cdot \Gamma(2) = \sigma$$

$$\Rightarrow E(l''(\sigma)) = \frac{n}{\sigma^2} - \frac{2}{\sigma^3} \cdot n \cdot \sigma = \frac{-n}{\sigma^2}$$

$$\Rightarrow \text{Var}(\hat{\sigma}) = \frac{-1}{E(l''(\sigma))} = \frac{\sigma^2}{n}$$

Problem 2A

PROBLEM 2A

$$I(\theta) = E \left[\left(\frac{\partial}{\partial \theta} \log f(x|\theta) \right)^2 \right]$$

$$= \int [l'(x|\theta)]^2 f(x|\theta) dx \quad \text{where } l' \text{ represents first derivative of log-likelihood function}$$

Assuming we can exchange the order of differentiation & integration

$$\int f'(x|\theta) dx = \frac{\partial}{\partial \theta} \int f(x|\theta) dx = 0$$

($\because \int f(x|\theta) = 1$
& derivative of a constant = 0)

$$\int f''(x|\theta) dx = \frac{\partial^2}{\partial \theta^2} \int f(x|\theta) dx = 0$$

~~Thus~~ Also, $l'(x|\theta) = \frac{\partial}{\partial \theta} (\log f(x|\theta)) = \frac{1}{f(x|\theta)} \cdot f'(x|\theta)$ (chain rule of derivatives)

$$\Rightarrow l''(x|\theta) = \frac{\partial}{\partial \theta} \left[\frac{f'(x|\theta)}{f(x|\theta)} \right] = \frac{f''(x|\theta)f(x|\theta) - [f'(x|\theta)]^2}{[f(x|\theta)]^2} = \frac{f''(x|\theta)}{f(x|\theta)} - \frac{[f'(x|\theta)]^2}{f(x|\theta)}$$

$$\Rightarrow E[l''(x|\theta)] = \int \left[\frac{f''(x|\theta)}{f(x|\theta)} - \frac{[f'(x|\theta)]^2}{f(x|\theta)} \right] f(x|\theta) dx$$

$$= \int f''(x|\theta) dx - E[(l'(x|\theta))^2]$$

$$= -I(\theta)$$

$$\Rightarrow I(\theta) = -E[l''(x|\theta)] = -E \left[\frac{\partial^2}{\partial \theta^2} \log f(x|\theta) \right]$$

Hence proved.

Problem 2B & 2C

PROBLEM 2B

Using the standard normal property of $\sqrt{nI(\theta_0)} (\hat{\theta} - \theta_0)$
 we can determine the bounds of the confidence interval as follows:

$$\Pr\left(-Z_{\frac{\alpha}{2}} \leq \sqrt{nI(\theta_0)} (\hat{\theta} - \theta_0) \leq Z_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\Rightarrow \Pr\left(\frac{-Z_{\alpha/2}}{\sqrt{nI(\theta_0)}} \leq \hat{\theta} - \theta_0 \leq \frac{Z_{\alpha/2}}{\sqrt{nI(\theta_0)}}\right) = 1 - \alpha$$

Further, we know that Fisher Information per MLE estimate
 of σ , $I(\sigma) = \frac{n}{\sigma^2}$

Bounds of confidence interval $[a, b]$

$$a = \frac{-Z_{\alpha/2} \sigma}{\sqrt{n \cdot \frac{n}{\sigma^2}}} = \frac{-Z_{\alpha/2} \sigma}{n} + \sigma, \quad \text{Similarly, } b = \frac{Z_{\alpha/2} \sigma}{n} + \sigma$$

$$\Pr\left(\frac{-Z_{\alpha/2} \sigma}{n} + \sigma \leq \hat{\theta} \leq \sigma + \frac{Z_{\alpha/2} \sigma}{n}\right)$$

PROBLEM 2C

$$f(x|a, b) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$\text{lik}(a, b) = \begin{cases} \left(\frac{1}{b-a}\right)^n & \text{if } a \leq x_i \leq b \quad \forall i \in [1, n] \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \log(\text{lik}(a, b)) = \begin{cases} -n \ln(b-a) & \text{if } a \leq x_i \leq b \quad \forall x_i \\ 0 & \text{otherwise} \end{cases}$$

Thus, we wish to maximize $-n \ln(b-a)$

assuming $b-a \leq 1$ (otherwise, log likelihood will become a negative number, so better to choose zero)

In this case,

$$\hat{a}_{MLE} = \min(X_1, \dots, X_n)$$

$$\hat{b}_{MLE} = \max(X_1, \dots, X_n)$$

since this will maximize the # of points that have non-zero probability

Problem 3A

PROBLEM 3A

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & \text{if } x \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

Solving for first moment: $E[X] = \int_{\theta}^{\infty} x \cdot e^{-(x-\theta)} dx$

Substitute $t = x - \theta$, $dt = dx$

limits of integration change: $\theta \rightarrow 0$
 $\infty \rightarrow \infty$

$$\Rightarrow E[X] = \int_0^{\infty} (t+\theta) e^{-t} dt$$

$$= \int_0^{\infty} t e^{-t} dt + \theta \int_0^{\infty} e^{-t} dt$$

PDF integrates to 1

$$E[X] = \Gamma(2) + \theta = 1 + \theta$$

$$\Rightarrow \hat{\theta}_{MOM} = \bar{X} - 1 \quad (\text{substituting sample mean for } E[X])$$

Problem 3B

PROBLEM 3B

$$\text{Lik}(\theta) = \begin{cases} \prod_{i=1}^n e^{-(x_i - \theta)} & \text{if } x_i \geq \theta \quad \forall i \in [1, n] \\ 0 & \text{otherwise} \end{cases}$$

We can re-write the above definition using

$$x_{\min} = \text{Min}(x_1, \dots, x_n)$$

$$\text{Lik}(\theta) = \begin{cases} e^{-n(\bar{x} - \theta)} & \text{provided that } \theta \leq x_{\min} \\ 0 & \text{otherwise} \end{cases}$$

Problem 4A

PROBLEM 4A

$$\theta = \exp\{-\lambda\}$$

We observe that $\Pr(X=0)$ for a Poisson distribution:

$$\Pr(X=0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} = \theta$$

We can design a function $\mathbb{1}(X)$ as follows:

$$\mathbb{1}(X) = \begin{cases} 1 & \text{if } X=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\theta} = \sum_{i=1}^n \frac{\mathbb{1}(x_i)}{n}$$

To prove $\hat{\theta}$ is unbiased, we need to show $E[\hat{\theta}] = \theta$

$$\begin{aligned} E[\hat{\theta}] &= \frac{1}{n} \cdot \sum_{i=1}^n E[\mathbb{1}(x_i)] = E[\mathbb{1}(X)] \quad \text{using linearity of expectation} \\ &= 1 \cdot \Pr(X=0) + 0 \\ &= 1 \cdot e^{-\lambda} \\ &= \theta \end{aligned}$$

Hence proved.

Problem 4B

PROBLEM 4B

$$\text{Var}(\hat{\theta}) = E[\hat{\theta}^2] - (E[\hat{\theta}])^2$$

Using Linearity of Variance:

$$\text{Var}(\hat{\theta}) = \text{Var}\left[\frac{1}{n} \sum_{i=1}^n \mathbb{1}(x_i)\right] = \left(\frac{1}{n}\right)^2 \cdot [\text{Var}(\mathbb{1}) \cdot n] = \frac{1}{n} \cdot \text{Var}(\mathbb{1}(x))$$

$$\begin{aligned} \text{Var}(\mathbb{1}(x)) &= E[(\mathbb{1}(x))^2] - (E[\mathbb{1}(x)])^2 \\ &= e^{-\lambda} - (e^{-\lambda})^2 \end{aligned}$$

$$\Rightarrow \text{Var}(\hat{\theta}) = \frac{e^{-\lambda}(1 - e^{-\lambda})}{n}$$

Compare with Cramer Rao lower bound

$$\text{Var}(\hat{\theta}) = E[\|\hat{\theta} - \theta\|^2] \geq \frac{1}{n I(\theta_0)}$$

$$\text{We know } I(\lambda) = E\left[\left(\frac{\partial}{\partial \lambda} \log f(x|\lambda)\right)^2\right]$$

$$= E\left[\left(\frac{\partial}{\partial \lambda} \log \frac{\lambda^k e^{-\lambda}}{k!}\right)^2\right]$$

$$= E\left[\left(\frac{\partial}{\partial \lambda} (-\lambda + \log \lambda^k - \log k!)\right)^2\right]$$

$$= E\left[\left(-1 + \frac{k}{\lambda}\right)^2\right] = E\left[1 + \left(\frac{k}{\lambda}\right)^2 - 2\frac{k}{\lambda}\right] \quad (\text{for } x=k)$$

$$= \frac{1}{\lambda^2} E[X^2] - \frac{2}{\lambda} E[X] + 1$$

$$= 1 + \frac{1}{\lambda} - 1 = \frac{1}{\lambda}$$

$$\begin{aligned}
 \Rightarrow I(\theta) &= E \left[\left(\frac{\partial}{\partial \theta} \log f(x|\theta) \right)^2 \right] \\
 &= E \left[\left(\frac{\partial}{\partial \theta} \cdot \frac{\partial \theta}{\partial \lambda} \cdot \frac{\partial \lambda}{\partial \theta} \cdot \log f(x) \right)^2 \right] \\
 &= E \left[\left(\frac{\partial}{\partial \lambda} \log f(x) \right)^2 \right] = \frac{I(\lambda)}{\left(\frac{\partial \theta}{\partial \lambda} \right)^2} = \frac{I(\lambda)}{e^{-2\lambda}} \quad (\text{from def'n of } \theta)
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow I(\theta) &= \frac{I(\lambda)}{e^{-2\lambda}} \\
 I(\theta) &= \frac{1}{\lambda e^{-2\lambda}}
 \end{aligned}$$

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n I(\theta_0)} = \frac{\lambda e^{-2\lambda}}{n}$$

Problem 5A

PROBLEM 5A

$$\hat{\mu} = 1 \cdot \frac{2,600}{10,000} + 2 \cdot \frac{5,200}{10,000} + 3 \cdot \frac{2,200}{10,000} = 1.96$$

$$\begin{aligned}
 \text{Var}(\hat{\mu}) &= (1-1.96)^2 \cdot \frac{2,600}{10,000} + (2-1.96)^2 \cdot \frac{5,200}{10,000} + (3-1.96)^2 \cdot \frac{2,200}{10,000} \\
 &= 0.4784
 \end{aligned}$$

$$\Rightarrow \text{Std. Error} = 0.691665$$

Z-score per 95% confidence interval = 1.96

\Rightarrow Bounds of CI [a, b]

$$a = 1.96 - 1.96 (0.691665) = 0.60434$$

$$b = 1.96 + 1.96 (0.691665) = 3.315662$$

Problem 5B

PROBLEM 5B

$$X = \begin{cases} 1 & p_1 \\ 2 & p_2 \\ 3 & 1-(p_1+p_2) \end{cases}$$

$$\hat{P}_1(x_1, \dots, x_n) = \sum_{i=1}^n \frac{1\{x_i=1\}}{n}$$

Estimating p_1 for the provided sample:

$$\hat{p}_1 = \frac{2,600}{10,000} + 0 \cdot \frac{7,400}{10,000} = 0.26$$

Is \hat{p}_1 unbiased? Yes, because $E[\hat{p}_1] = p_1$

$$\begin{aligned} E[\hat{p}_1] &= \frac{(E[X_1=1] + E[X_2=1] + \dots)}{n} = E[X=1] \\ &= 1 \cdot p_1 + 0 \cdot (p_2 + 1 - (p_1 + p_2)) \\ &= p_1 \end{aligned}$$

Hence Proved.

Problem 5C

PROBLEM 5C

Method of moments to estimate p_1 & p_2

$$\begin{aligned} E[X] &= 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot (1 - (p_1 + p_2)) \\ &= p_1 + 2p_2 + 3 - 3p_1 - 3p_2 \\ &= -2p_1 - p_2 + 3 \end{aligned}$$

$$\begin{aligned} E[X^2] &= 1^2 \cdot p_1 + 2^2 p_2 + 9(1 - (p_1 + p_2)) \\ &= p_1 + 4p_2 + 9 - 9p_1 - 9p_2 \\ &= -8p_1 - 5p_2 + 9 \end{aligned}$$

Substituting to solve for p_1 & p_2

$$E[X^2] - 4E[X] = -5p_2 + 4p_2 + 9 - 12$$

$$\Rightarrow p_2 = -E[X^2] + 4E[X] - 3$$

This implies

$$-2p_1 + E[X^2] - 4E[X] + 3 + 3 = E[X]$$

$$\Rightarrow p_1 = \frac{-E[X^2] - 5E[X] + 3}{2}$$

We can substitute for $E[X]$ & $E[X^2]$ as follows:

$$E[X] = \bar{X} = 1.96$$

$$\begin{aligned} E[X^2] &= \text{Variance} + 1.96^2 \\ &= 0.4784 + (1.96)^2 \end{aligned}$$

$$\Rightarrow p_1 = 0.26$$

$$p_2 = 0.52$$

Problem 5D

PROBLEM 5D

$$X = \begin{cases} 1 & 1/4 \\ 2 & p_2 \\ 3 & 3/4 - p_2 \end{cases}$$

Also:

We know that the joint frequency of the sample is a product of marginal frequencies

$$\text{lik}(p_2) = \prod_{i=1}^3 \cancel{x_i \log p_i} (p_i)^{(\#x_i)} \quad \text{where } i \in [1, 2, 3]$$

$$\Rightarrow \log\text{-likelihood}(p_2) = \sum_{i=1}^3 (\#x_i) \log p_i \quad \text{where } (\#x_i) = \text{count of samples in category } i \text{ (i.e. 1, 2 or 3)}$$

$$\text{lik}(p_2) = \left(\frac{1}{4}\right)^{(\#x_1)} (p_2)^{(\#x_2)} \left(\frac{3}{4} - p_2\right)^{(\#x_3)}$$

$$\Rightarrow \ell(p_2) = x_1 \log\left(\frac{1}{4}\right) + x_2 \log(p_2) + x_3 \log\left(\frac{3}{4} - p_2\right)$$

To find $\arg\max l(p_2)$, take first derivative & set equal to 0

$$\frac{\partial l(p_2)}{\partial p_2} = \frac{x_2}{p_2} - \frac{x_3}{\frac{3}{4} - p_2} = 0$$

$$\Rightarrow \frac{x_2}{p_2} = \frac{x_3}{\frac{3}{4} - p_2} \Rightarrow p_2 (x_3 + x_2) = \frac{3}{4} x_2$$

$$\Rightarrow \hat{p}_2 = \frac{\frac{3}{4} x_2}{(x_3 + x_2)} ; \text{ Numerically, } \hat{p}_2 = \frac{\frac{3}{4} (5200)}{5200 + 2200} = 0.527$$

95% Confidence Interval using large sample theory

$$\hat{\theta} \pm Z\left(\frac{\alpha}{2}\right) \cdot \frac{1}{\sqrt{n I(\hat{\theta})}}$$

$$Z_{\alpha/2} = 1.96 \text{ for 95\% CI}$$

$$\text{We know } \frac{\partial}{\partial p_2} l(p_2) = \frac{x_2}{p_2} - \frac{x_3}{\frac{3}{4} - p_2}$$

$$E\left[\left(\frac{\partial}{\partial p_2} l(p_2)\right)^2\right] = I(\hat{\theta}) = \left(\frac{x_2}{p_2} - \frac{4x_3}{3-4p_2}\right)^2 \cdot p_1 + \left(\frac{x_2}{p_2} - \frac{4x_3}{3-4p_2}\right)^2 p_2 + \left(\frac{x_2}{p_2} - \frac{4x_3}{3-4p_2}\right)^2 p_3$$

$$\begin{aligned}
 I(\hat{\theta}) &= \left(\frac{5200}{0.527} - \frac{4 \cdot 2200}{3 - 4(0.527)} \right)^2 \left(\frac{1}{4} \right) \\
 &+ \left(\frac{5200}{0.527} - \frac{4 \cdot 2200}{3 - 4(0.527)} \right)^2 (0.527) \\
 &+ \left(\frac{5200}{0.527} - \frac{4 \cdot 2200}{3 - 4(0.527)} \right)^2 \left(\frac{3}{4} - 0.527 \right) \\
 &= 2.896
 \end{aligned}$$

$$\Rightarrow 0.527 \pm \frac{1.96}{\sqrt{10,000 \times 2.896}}$$

$$\begin{aligned}
 \Rightarrow [a, b] \quad a &= 0.527 - 0.0115 = 0.5155 \\
 b &= 0.527 + 0.0115 = 0.5385
 \end{aligned}$$

Problem 6A & 6B

```

C:\Users\sheil\AppData\Local\Programs\Python\Python39\python.exe "C:/Users/sheil/OneDr
Mean computed without inbuilt functions      : 4.2457769104068035
Mean using inbuilt numpy function            : 4.245776910406805
Deviation between both methods                : 1.7763568394002505e-15
=====
Variance computed without inbuilt functions: 18.22406983005141
Variance using inbuilt numpy function        : 18.224069830051395
Deviation between both methods                : 1.4210854715202004e-14
=====
90% Confidence Interval Bounds               : (3.833021657112814, 4.658532163700796)

```



```

import csv
import numpy as np
import scipy.stats as st

fname = "data_HW2.csv"

def computeMeanWithoutInbuilt(data_array):
    sum = 0.0
    for d in data_array:
        sum += d
    sample_mean = sum / len(data_array)
    return sample_mean

def computeVarianceWithoutInbuilt(data_array):
    sample_mean = computeMeanWithoutInbuilt(data_array)
    total_squared_deviation = 0.0
    for d in data_array:
        total_squared_deviation += (d - sample_mean) ** 2
    sample_variance = total_squared_deviation / len(data_array)
    return sample_variance

with open(fname) as f:
    reader = csv.reader(f)
    next(reader) # skip header
    string_data = [r[0] for r in reader]
    data_array = np.asarray(string_data, dtype=np.float64, order="C")

    sample_mean_without_inbuilt = computeMeanWithoutInbuilt(data_array)
    sample_mean_with_inbuilt = np.mean(data_array)
    sample_mean_deviation = np.abs(sample_mean_without_inbuilt -
    sample_mean_with_inbuilt)

    sample_variance_without_inbuilt = computeVarianceWithoutInbuilt(data_array)
    sample_variance_with_inbuilt = np.nanvar(data_array)
    sample_variance_deviation = np.abs(sample_variance_without_inbuilt -
    sample_variance_with_inbuilt)

    ninety_percent_confidence_interval = st.t.interval(alpha=0.90, df=len(data_array) -
1, loc=np.mean(data_array),
                                                    scale=0.25)

    print("Mean computed without inbuilt functions      : " +
str(sample_mean_without_inbuilt))
    print("Mean using inbuilt numpy function           : " +
str(sample_mean_with_inbuilt))
    print("Deviation between both methods              : " + str(sample_mean_deviation))
    print("=====")
    print("Variance computed without inbuilt functions: " +
str(sample_variance_without_inbuilt))
    print("Variance using inbuilt numpy function         : " +
str(sample_variance_with_inbuilt))
    print("Deviation between both methods              : " +
str(sample_variance_deviation))
    print("=====")
    print("90% Confidence Interval Bounds                  : " +
str(ninety_percent_confidence_interval))

```