Lecture 11:

This Friday (22nd)

Lecture

Next Monday (25th)

Review Session

Quick recap:

$X_1, \dots, X_n \sim \text{dist}(\mu, \sigma^2)$

$$\begin{cases} H_0: & \mu = \mu_0 \\ H_a: & \mu > \mu_0 \end{cases}$$

$$T(X_1, \dots, X_n) = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \underset{H_0}{\overset{H_a}{\gtrless}} z_\alpha$$

α: significance level

P-value:

Definition: The p-value is the probability, calculated assuming the the null hypothesis is true, of obtaining a value of the test statistic at least as contradictory as the value calculated from the available sample.

$$\text{p-value} = \Pr\{T(X_1, \dots, X_n) \geq \overbrace{40}^{\text{obtained from data}} \mid H_0 \text{ is true}\}$$
$$= 0.01$$

The Delivery Example:

A restaurant claims that their delivery time to a certain neighborhood is 30 mins (on average). We would like to verify this claim. So we order food from a bunch of different/random locations at random times and find that the sample-mean of the delivery time is 40 mins.

H_0: claim:

Delivery time $\leq$ 30 mins
(on average)

Restaurant $\to$ Data Acquisition $\to$ $X_1, \dots, X_n$ $\to$ Data Analysis

$$\frac{X_1 + \cdots + X_n}{n} = \underline{\underline{40}}$$

Surprising!

Google $\to$

we've been monitoring data from this restaurant for the past 20 years we can say that, assuming that the average delivery time is less than 30 mins, the probability that you obtain a sample mean of 40 mis or larger is 0.01.

$$\Pr\left\{\frac{X_1+\cdots+X_n}{n} \geq 40 \,\middle|\, \underbrace{E[X_i]=30}_{\text{claim } (H_0) \text{ is true}}\right\} \approx \underbrace{0.01}_{\text{P-value}}$$

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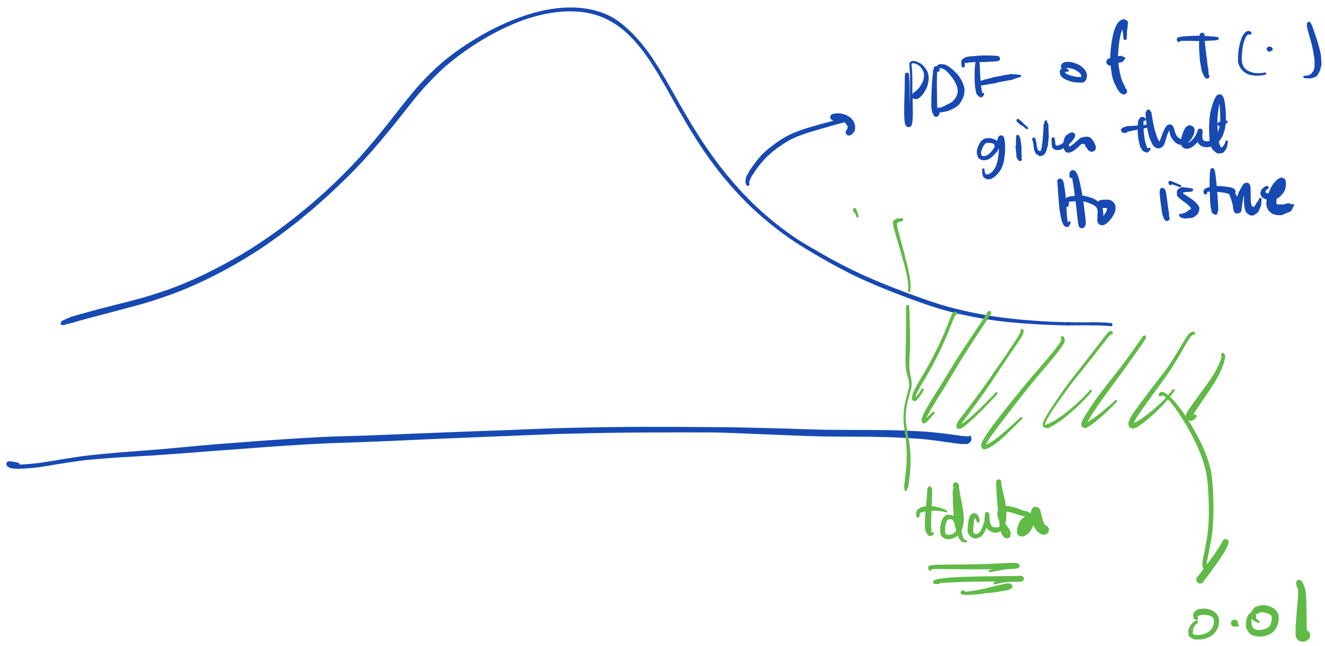
* P-value is computed from data.

$$T(X_1, \ldots, X_n) \rightarrow T(\text{ data gathered})$$

$$= t_{data}$$

$$\text{P-value}(\underline{\underline{\text{data}}}) = \Pr\left\{T(X_1, \ldots, X_n) \geq t_{data} \,\middle|\, H_0 \text{ is true}\right\}$$

$$= \underbrace{0.01}$$



$$\begin{cases} H_0 : & E[X_i] \le 30 \\ H_a : & E[X_i] > 30 \end{cases}$$

$$\text{Restaurant Delivery} \longrightarrow \boxed{\text{Data Analy.}} \longrightarrow \underbrace{X_1, \ldots, X_n}$$

$$\downarrow$$

$$\underline{T(X_1, \ldots, X_n)}$$

$$\text{Data} \longrightarrow T(\text{Data}) \longrightarrow t_{data} = 40$$

P_value $\{$

$= \Pr\{ T(X_1, \ldots, X_n) \geq \underline{40} \mid H_0 \text{ is true} \}$

$= 0.01$

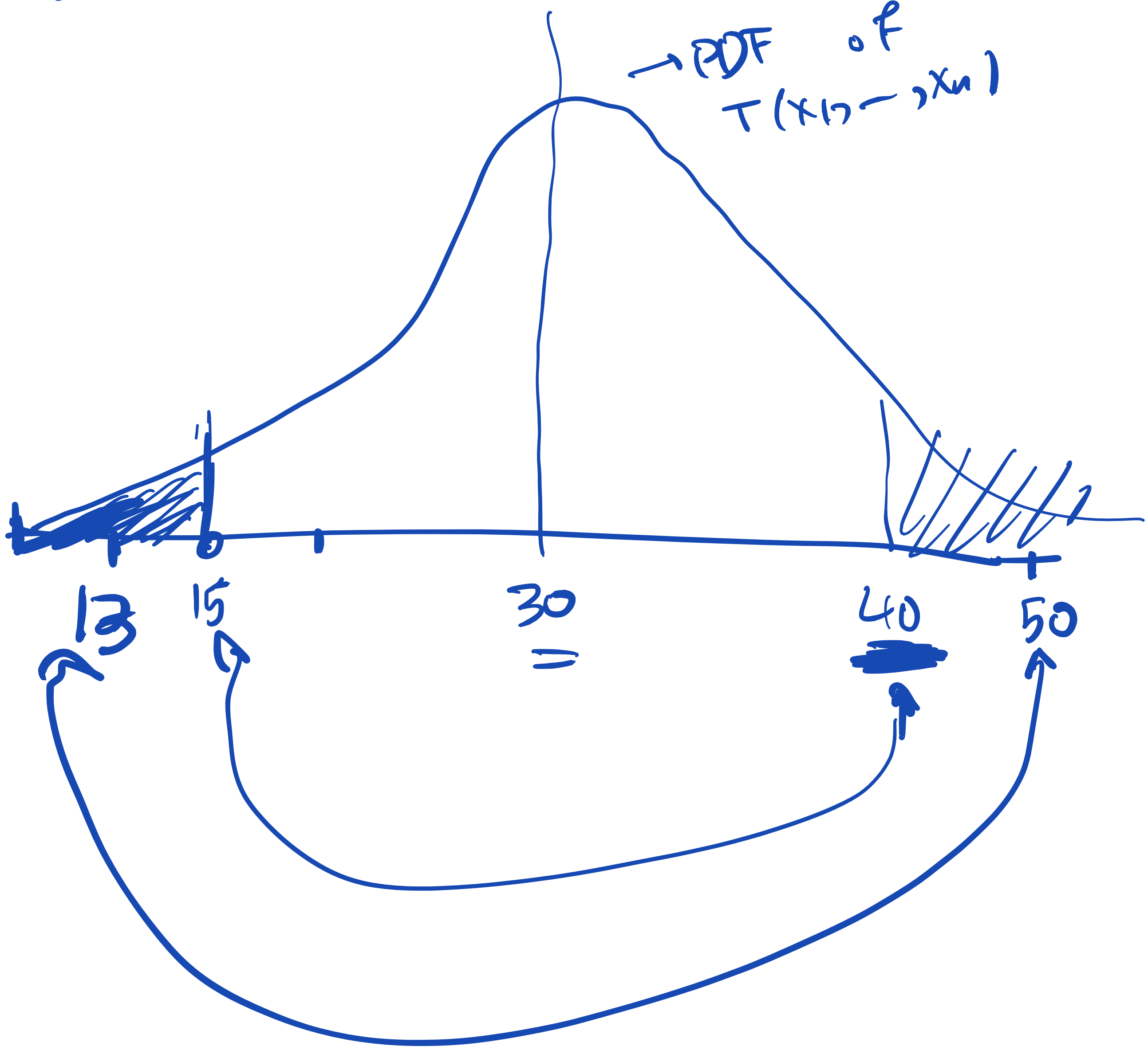
$\begin{cases} H_0 : & \text{average} = 30 \\ H_a : & \text{average} \neq 30 \end{cases}$

$$T(X_1, \ldots, X_n) = \frac{X_1 + \cdots + X_n}{n}$$

$$t_{data} = T(\text{Data}) = \underline{\underline{40}}$$

P_value $= \Pr\{ T(X_1, \ldots, X_n)$ is "more contradictory" than $t_{data} \mid H_0$ is true $\}$

$$\begin{cases} \underline{T}(x_1, \dots, x_n) \geq 40 \\ T(x_1, \dots, x_n) \leq 30 - f(t_{data}) \end{cases}$$



* The lower the p-value $\Longleftrightarrow$ the more surprising the evedience (data) is $\Longleftrightarrow$ the more unrealistic our null hypothesis looks.

* (more formal) The lowe the P-value is, the more evidence there is in the sample data against the null hypothesis.

* P_value is computed as a function of the sample data.

Decision rule based on the P-value:

$$\underbrace{T \qquad \text{data} : x_1, \dots, x_n}$$

↳ Compute the P-value.

↳ $$\text{P\_value} \underset{\text{accept } H_0}{\overset{\text{reject } H_0}{\lessgtr}} \underbrace{\text{threshold}}_{=\alpha}$$

Significance level $= \alpha$

$$\boxed{\text{P\_value} \underset{\text{accept } H_0}{\overset{\text{reject } H_0}{\lessgtr}} \alpha}$$

the p-value rule

* The following two procedures are equivalent:

Designing acceptance/rejection regions

$$\Updownarrow$$

the P-value rule

Example:

Our simplest hypothesis testing problem:

$$X_1, \dots, X_n \sim N(\mu, \sigma^2)$$

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu > \mu_0 \end{cases}$$

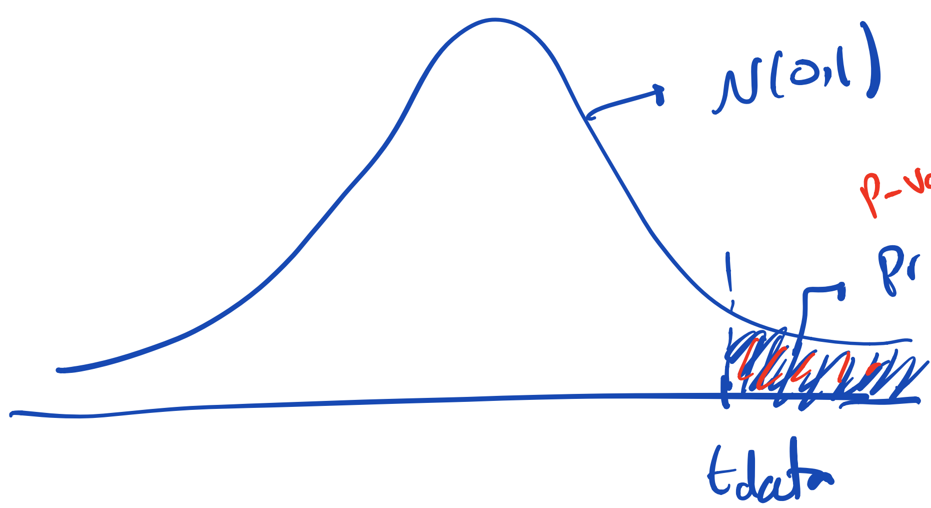
Test: $\underbrace{\dfrac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}}_{T(X_1, \dots, X_n)} \begin{array}{c} \text{reject } H_0 \\ \uparrow \\ > \\ < \\ \downarrow \\ \text{accept } H_0 \end{array} z_\alpha$

P_value:

Data: $x_1, \dots, x_n \longrightarrow T(x_1, \dots, x_n) \triangleq t_{data}$

$$\Pr\{ T(X_1, \dots, X_n) \geq t_{data} \mid \underbrace{H_0 \text{ is true}}_{T(X_1, \dots, X_n) \sim N(0,1)} \}$$

$$= \Pr\{ N(0,1) \geq t_{data} \}$$



p-value (data)

$\Pr\{ N(0,1) \geq t_{data} \}$

$= 1 - \underbrace{\Phi}_{\text{CDF of the normal distribution}}(t_{data})$

Recall that the p_value rule was:

$$\text{p\_value}(\text{data}) \underset{\text{accept}}{\overset{\text{reject } H_0}{\lessgtr}} \alpha$$

$$\Updownarrow$$

$$\Pr\{N(0,1) \geq t_{data}\} \underset{\text{reject } H_0}{\overset{\text{accept } H_0}{\gtrless}} \alpha$$

$$\Updownarrow$$

claim:

$$t_{data} \underset{\text{accept } H_0}{\overset{\text{reject } H_0}{\gtrless}} z_\alpha$$

$$\underbrace{\text{p\_value}(\text{data})} = 1 - \Phi(t_{data})$$

$$1 - \Phi(t_{data}) \underset{\text{accept } H_0}{\overset{\text{reject } H_0}{\lessgtr}} \alpha \qquad (\text{p\_value rule})$$

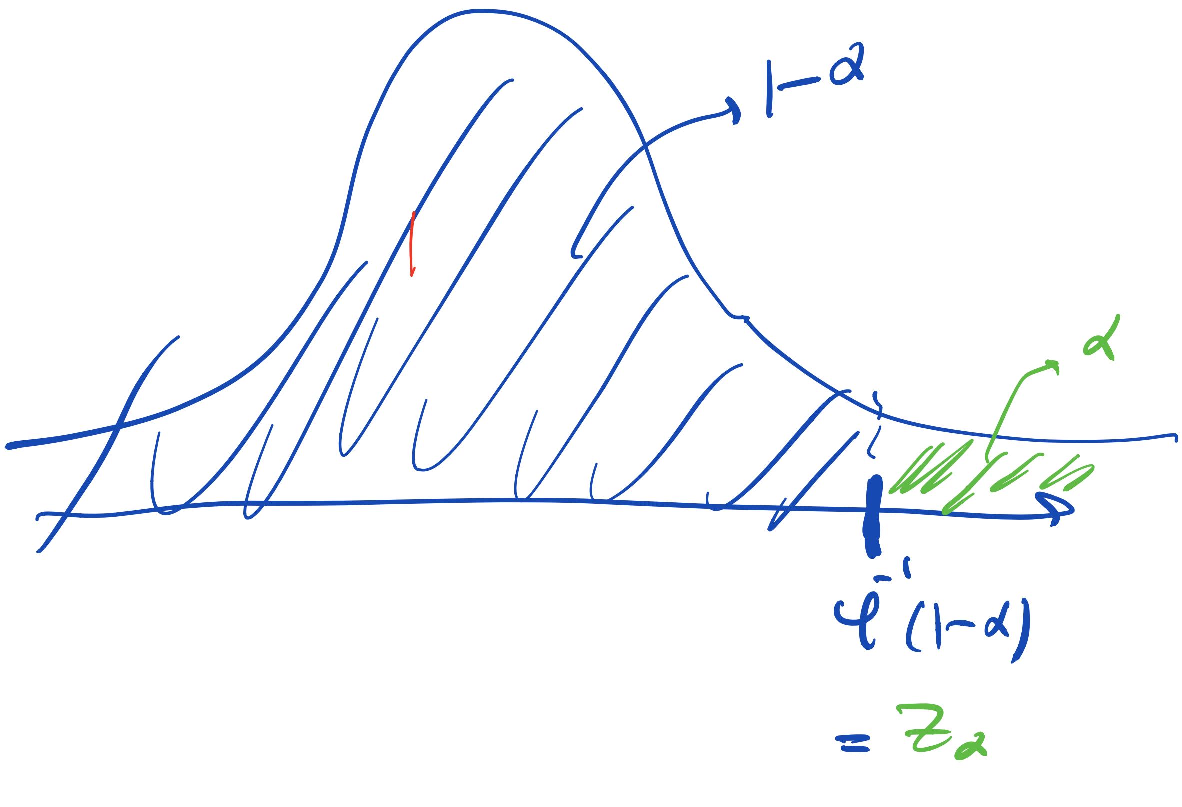
$$\Uparrow$$

$$\Phi(t_{data}) \underset{\text{accept } H_0}{\overset{\text{reject } H_0}{\gtrless}} 1-\alpha$$

$$\Updownarrow$$

$$\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} = t_{data} \underset{\text{accept } H_0}{\overset{\text{reject } H_0}{\gtrless}} \Phi^{-1}(1-\alpha) = z_\alpha$$

acceptance/rejection region



- So far we've only discussed hypothesis testing problems that involve the mean of the data.

- Consider the following (more general) hypothesis testing problem.

Data: $X_1, \ldots, X_n \sim f(x|\theta)$

$$\begin{cases} H_0: & \theta = \theta_0 \\ H_a: & \theta = \theta_a \end{cases}$$