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ESE402 Assignment 2
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Problem 1A & 1B

PROBLEM 1 A

$$f(x \mid \sigma) = \frac{1}{2\sigma} \exp\left\{-\frac{|x|}{\sigma}\right\}$$

Method of Moments to estimate σ

1^{st} Population moment: $$E[X] = \int_{-\infty}^{\infty} \frac{x}{2\sigma} \exp\left\{\frac{-|x|}{\sigma}\right\} dx =$$

$$= \int_{-\infty}^{0} \frac{x}{2\sigma} \exp\left\{-\frac{|x|}{\sigma}\right\} + \int_{0}^{\infty} \frac{x}{2\sigma} \exp\left\{\frac{-|x|}{\sigma}\right\}$$

Because integrand is an odd function & symmetric about 0, the sum of these two pieces will negate each other & equal 0

$\Rightarrow E[X] = 0$ Thus, we must go to the second moment.

2^{nd} Population moment: $$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{2\sigma} \exp\left\{\frac{-|x|}{\sigma}\right\} dx$$

Using the fact that this integrand is an even function, we can change it into 2 parts & remove absolute value around x

$$E[X^2] = 2 \cdot \int_{0}^{\infty} \frac{x^2}{2\sigma} \exp\left\{\frac{-x}{\sigma}\right\} dx = \int_{0}^{\infty} \frac{x^2}{\sigma} \exp\left\{\frac{-x}{\sigma}\right\} dx$$

Now, substitute $t = \frac{x}{\sigma}$ & $dt = \frac{dx}{\sigma}$ to simplify the integral

Limits of integration remain the same

$$\Rightarrow E[X^2] = \frac{1}{\sigma} \int_{0}^{\infty} (\sigma t)^2 \cdot \exp\{-t\} \cdot \sigma \cdot dt$$

$$= \frac{\sigma}{\sigma} \cdot \sigma^2 \int_0^\infty t^2 \cdot \exp\{-t\}\, dt$$

$$= \sigma^2 \cdot \Gamma(3) = 2! \cdot \sigma^2$$

$$E[X^2] = 2\sigma^2$$

$$\Rightarrow \sigma = \sqrt{\frac{1}{2} E[X^2]}$$

Using sample moment instead of population moment:

$$\sigma = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} X_i^2}$$

PROBLEM 1B

MLE estimate of σ $\Rightarrow$
1. Define likelihood function
2. Set 1^{st} derivative of log-likelihood $= 0$
3. solve for argmax of likelihood

$$\text{lik}(\sigma) = f(x_1, \ldots, x_n | \sigma) = f(x_1|\sigma) \cdot f(x_2|\sigma) \cdots f(x_n|\sigma)$$

$$\Rightarrow \text{lik}(\sigma) = \prod_{i=1}^{n} \frac{1}{2\sigma} \exp\left\{\frac{-|x|}{\sigma}\right\} = \frac{1}{(2\sigma)^n} \exp\left\{\frac{-(|x_1| + |x_2| \cdots + |x_n|)}{\sigma}\right\}$$

Convert this into log likelihood:

$$\ln(\text{lik}(\sigma)) = -n \cdot \ln(2\sigma) - \frac{|x_1| \cdots + |x_n|}{\sigma}$$

We wish to find σ which maximizes this over $\langle 0, \infty \rangle$

1^{st} derivative of $\ln(\text{lik}(\sigma)) \Rightarrow \frac{-n}{\sigma} + \frac{|x_1| \cdots + |x_n|}{\sigma^2}$

Setting 1^{st} derivative $= 0$, $\underbrace{\sigma = 0}_{\text{Trivial Solution}}$ OR $\sigma = \frac{|x_1| + |x_2| \cdots + |x_n|}{n}$

$$\Rightarrow \text{MLE estimate of } \sigma = \frac{1}{n} \sum_{i=1}^{n} |X_i|$$

Problem 1C

PROBLEM 1C

Using asymptotic normality property of the MLE estimator, we know

$$Var(\hat{\sigma}) = \frac{1}{n I(\hat{\sigma})} \quad \text{where } I(\sigma) = E\left[\left(\frac{\partial}{\partial \sigma} \ln f(X|\sigma)\right)^2\right]$$

$$= \frac{E\left(\left[\ell'(\sigma)\right]^2\right)}{n} = \frac{-E(\ell''(\sigma))}{n}$$

where $\ell(\sigma)$ is the log-likelihood function

$$\ell''(\sigma) = \frac{n}{\sigma^2} - 2 \cdot \frac{|x_1| + \dots |x_n|}{\sigma^3}$$

$$\Rightarrow E(\ell''(\sigma)) = \frac{n}{\sigma^2} - \frac{2}{\sigma^3}\left(E(|X_1| + \dots |X_n|)\right)$$

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solving for $E[|X|] = \int_{-\infty}^{\infty} |x| \cdot f(x)\, dx = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{2\sigma} \cdot \exp\left\{\frac{-|x|}{\sigma}\right\} dx$

· Using the fact that integrand is even:

$$E[|X|] = \frac{2}{2\sigma} \int_0^{\infty} x \cdot \exp\left\{\frac{-x}{\sigma}\right\} dx$$

& substituting again for $t = \frac{x}{\sigma}$ & $dt = \frac{dx}{\sigma}$

$$\Rightarrow E[|X|] = \frac{1}{\sigma} \int_0^{\infty} (t\sigma) \cdot \exp\{-t\} \cdot \sigma \cdot dt$$

$$= \sigma \int_0^{\infty} t \cdot \exp\{-t\}\, dt = \sigma \cdot \Gamma(2) = \sigma$$

$$\Rightarrow E(\ell''(\sigma)) = \frac{n}{\sigma^2} - \frac{2}{\sigma^3} \cdot n \cdot \sigma = \frac{-n}{\sigma^2}$$

$$\Rightarrow Var(\hat{\sigma}) = \frac{-1}{E(\ell''(\sigma))} = \frac{\sigma^2}{n}$$

Problem 2A

PROBLEM 2A

$$I(\theta) = E\left[\left(\frac{\partial}{\partial\theta}\log f(X|\theta)\right)^2\right]$$

$$= \int [l'(x|\theta)]^2 f(x|\theta)\,dx$$ where l' represents first derivative of log-likelihood function

Assuming we can exchange the order of differentiation & integration

$$\int f'(x|\theta)\,dx = \frac{\partial}{\partial\theta}\int f(x|\theta)\,dx = 0$$ ($\because \int f(x|\theta) = 1$ & derivative of a constant $= 0$)

$$\int f''(x|\theta)\,dx = \frac{\partial^2}{\partial\theta^2}\int f(x|\theta)\,dx = 0$$

~~Thus,~~ Also, $l'(x|\theta) = \frac{\partial}{\partial\theta}\left(\log f(X|\theta)\right) = \frac{1}{f(X|\theta)}\cdot f'(X|\theta)$ (chain rule of derivatives)

$$\Rightarrow l''(x|\theta) = \frac{\partial}{\partial\theta}\left[\frac{f'(x|\theta)}{f(x|\theta)}\right] = \frac{f''(x|\theta)f(x|\theta) - [f'(x|\theta)]^2}{[f(x|\theta)]^2} = \frac{f''(x|\theta)}{f(x|\theta)} - [l'(x|\theta)]^2$$

$$\Rightarrow E[l''(x|\theta)] = \int\left[\frac{f''(x|\theta)}{f(x|\theta)} - [l'(x|\theta)]^2\right] f(x|\theta)\,dx$$

$$= \int f''(x|\theta)\,dx - E[(l'(X|\theta))^2]$$

$$= -I(\theta)$$

$$\Rightarrow I(\theta) = -E[l''(x|\theta)] = -E\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right]$$

Hence proved.

Problem 2B & 2C

PROBLEM 2B

Using the standard normal property of $\sqrt{nI(\theta_0)}\,(\hat{\theta}-\theta_0)$ we can determine the bounds of the confidence interval as follows:

$$Pr\left(-Z_{\frac{\alpha}{2}} \le \sqrt{nI(\theta_0)}\,(\hat{\theta}-\theta_0) \le Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\Rightarrow Pr\left(\frac{-Z_{\alpha/2}}{\sqrt{nI(\theta_0)}} \le \hat{\theta}-\theta_0 \le \frac{Z_{\alpha/2}}{\sqrt{nI(\theta_0)}}\right) = 1-\alpha$$

Further, we know that Fisher Information per MLE estimate of σ, $I(\sigma) = \frac{n}{\sigma^2}$

Bounds of confidence interval $[a,b]$

$$a = \frac{-Z_{\alpha/2}}{\sqrt{n \cdot \frac{n}{\sigma^2}}} + \theta_0 = \frac{-Z_{\alpha/2}\,\sigma}{n} + \sigma,$$

Similarly,

$$b = \frac{Z_{\alpha/2}\,\sigma}{n} + \sigma$$

$$Pr\left(\frac{-Z_{\alpha/2}\,\sigma}{n} + \sigma \le \hat{\theta} \le \sigma + \frac{Z_{\alpha/2}\,\sigma}{n}\right)$$

PROBLEM 2C

$$f(x|a,b) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$lik(a,b) = \begin{cases} \left(\frac{1}{b-a}\right)^n & \text{if } a \le x_i \le b \quad \forall\, i \in [1,n] \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \log(lik(a,b)) = \begin{cases} -n\ln(b-a) & \text{if } a \le x_i \le b \;\forall\, x_i \\ 0 & \text{otherwise} \end{cases}$$

Thus, we wish to maximize $-n \ln(b-a)$
assuming $b-a \leq 1$ (otherwise, log likelihood will become a negative number, so better to choose zero)

In this case,

$$\hat{a}_{MLE} = \min(X_1, \ldots, X_n)$$

$$\hat{b}_{MLE} = \max(X_1, \ldots, X_n)$$

since this will maximize the # of points that have non-zero probability

Problem 3A

PROBLEM 3A

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & \text{if } x \geqslant \theta \\ 0 & \text{otherwise} \end{cases}$$

Solving for first moment: $E[X] = \int_{\theta}^{\infty} x \cdot e^{-(x-\theta)}\, dx$

Substitute $t = x - \theta$, $dt = dx$
limits of integration change: $\theta \to 0$
$\infty \to \infty$

$$\Rightarrow E[X] = \int_0^\infty (t+\theta)\, e^{-t}\, dt$$

$$= \int_0^\infty t e^{-t}\, dt + \theta \underbrace{\int_0^\infty e^{-t}\, dt}_{\text{PDF integrates to 1}}$$

$$E[X] = \Gamma(2) + \theta = 1 + \theta$$

$$\Rightarrow \hat{\theta}_{MOM} = \bar{X} - 1 \quad \text{(substituting sample mean for } E[X])$$

Problem 3B

PROBLEM 3B

$$\text{Lik}(\theta) = \begin{cases} \prod_{i=1}^{n} e^{-(x_i - \theta)} & \text{if } x_i \geq \theta \;\; \forall i \in [1,n] \\ 0 & \text{otherwise} \end{cases}$$

We can re-write the above definition using

$$x_{min} = \text{Min}(X_1, \ldots, X_n)$$

$$\text{Lik}(\theta) = \begin{cases} e^{-n(\bar{x} - \theta)} & \text{provided that } \theta \leq x_{min} \\ 0 & \text{otherwise} \end{cases}$$

Problem 4A

PROBLEM 4A

$\theta = \exp\{-\lambda\}$

We observe that $\Pr(X=0)$ for a Poisson distribution:

$$\Pr(X=0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} = \theta$$

We can design a function $\mathbb{1}(X)$ as follows:

$$\mathbb{1}(X) = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\theta} = \sum_{i=1}^{n} \frac{\mathbb{1}(x_i)}{n}$$

To prove $\hat{\theta}$ is unbiased, we need to show $E[\hat{\theta}] = \theta$

$$E[\hat{\theta}] = \frac{1}{n} \cdot \sum_{i=1}^{n} E[\mathbb{1}(x_i)] = E[\mathbb{1}(X)] \quad \text{using linearity of expectation}$$

$$= 1 \cdot \Pr(X=0) + 0$$

$$= 1 \cdot e^{-\lambda}$$

$$= \theta$$

Hence proved.

Problem 4B

PROBLEM 4B

$$Var(\hat{\theta}) = E[\hat{\theta}^2] - (E[\hat{\theta}])^2$$

Using Linearity of Variance:

$$Var(\hat{\theta}) = Var\left[\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(x_i)\right] = \left(\frac{1}{n}\right)^2 \cdot \left[Var(\mathbb{1}) \cdot n\right] = \frac{1}{n} \cdot Var(\mathbb{1}(x))$$

$$Var(\mathbb{1}(x)) = E[(\mathbb{1}(x))^2] - (E[\mathbb{1}(x)])^2$$

$$= e^{-\lambda} - (e^{-\lambda})^2$$

$$\Rightarrow Var(\hat{\theta}) = \frac{e^{-\lambda}(1 - e^{-\lambda})}{n}$$

Compare with Cramer Rao lower bound

$$Var(\hat{\theta}) = E[\|\hat{\theta} - \theta\|^2] \geq \frac{1}{nI(\theta_0)}$$

We know $I(\lambda) = E\left[\left(\frac{\partial}{\partial\lambda} \log f(x|\lambda)\right)^2\right]$

$$= E\left[\left(\frac{\partial}{\partial\lambda} \log \frac{\lambda^k e^{-\lambda}}{k!}\right)^2\right]$$

$$= E\left[\left(\frac{\partial}{\partial\lambda}\left(-\lambda + \log\lambda^k - \log k!\right)\right)^2\right]$$

$$= E\left[\left(-1 + \frac{k}{\lambda}\right)^2\right] = E\left[1 + \left(\frac{k}{\lambda}\right)^2 - 2\frac{k}{\lambda}\right] \quad (\text{for } x = k)$$

$$= \frac{1}{\lambda^2}E[X^2] - \frac{2}{\lambda}E[x] + 1$$

$$= 1 + \frac{1}{\lambda} - 1 = \frac{1}{\lambda}$$

$$\Rightarrow I(\theta) = E\left[\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2\right]$$

$$= E\left[\left(\frac{\partial}{\partial\theta}\cdot\frac{\partial\theta}{\partial\lambda}\cdot\frac{\partial\lambda}{\partial\theta}\cdot\log f(x)\right)^2\right]$$

$$= \frac{E\left[\left(\frac{\partial}{\partial\lambda}\log f(x)\right)^2\right]}{\left(\frac{\partial\theta}{\partial\lambda}\right)^2} = \frac{I(\lambda)}{\left(\frac{\partial\theta}{d\lambda}\right)^2} = \frac{I(\lambda)}{e^{-2\lambda}} \quad (\text{from def}^n \text{ of } \theta)$$

$$\Rightarrow I(\theta) = \frac{I(\lambda)}{e^{-2\lambda}}$$

$$I(\theta) = \frac{1}{\lambda e^{-2\lambda}}$$

$$Var(\hat{\theta}) \geq \frac{1}{nI(\theta_0)} = \frac{\lambda e^{-2\lambda}}{n}$$

Problem 5A

PROBLEM 5A

$$\hat{\mu} = 1\cdot\frac{2{,}600}{10{,}000} + 2\cdot\frac{5{,}200}{10{,}000} + 3\cdot\frac{2{,}200}{10{,}000} = 1.96$$

$$Var(\hat{\mu}) = (1-1.96)^2\cdot\frac{2{,}600}{10{,}000} + (2-1.96)^2\cdot\frac{5{,}200}{10{,}000} + (3-1.96)^2\cdot\frac{2{,}200}{10{,}000}$$

$$= 0.4784$$

$\Rightarrow$ Std. error = 0.691665 /100

Z-score for 95% confidence Interval = 1.96

$\Rightarrow$ Bounds of CI $[a, b]$

$a = 1.96 - 1.96\,(0.691665$ /100 = 1.94

$b = 1.96 + 1.96\,(0.691665)$ /100 = 1.97

Problem 5B

PROBLEM 5B

$$X = \begin{cases} 1 & p_1 \\ 2 & p_2 \\ 3 & 1-(p_1+p_2) \end{cases}$$

$$\hat{p}_1(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{\mathbb{1}\{x_i = 1\}}{n}$$

Estimating p_1 for the provided sample:

$$\hat{p}_1 = \frac{2{,}600}{10{,}000} + 0 \cdot \frac{7{,}400}{10{,}000} = 0.26$$

Is $\hat{p}_1$ unbiased? Yes, because $E[\hat{p}_1] = p_1$

$$E[\hat{p}_1] = \frac{\left(E[X_1 == 1] + E[X_2 == 1] \cdots\right)}{n} = E[X == 1]$$

$$= 1 \cdot p_1 + 0 \cdot (p_2 + 1 - (p_1 + p_2))$$

$$= p_1$$

Hence Proved.

Problem 5C

PROBLEM 5C

Method of moments to estimate p_1 & p_2

$$E[X] = 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot (1 - (p_1 + p_2))$$

$$= p_1 + 2p_2 + 3 - 3p_1 - 3p_2$$

$$= -2p_1 - p_2 + 3$$

$$E[X^2] = 1^2 \cdot p_1 + 2^2 p_2 + 9(1 - (p_1 + p_2))$$

$$= p_1 + 4p_2 + 9 - 9p_1 - 9p_2$$

$$= -8p_1 - 5p_2 + 9$$

Substituting to solve for P_1 & P_2

$$E[X^2] - 4E[X] = -5P_2 + 4P_2 + 9 - 12$$

$$\Rightarrow P_2 = -E[X^2] + 4E[X] - 3$$

This implies

$$-2P_1 + E[X^2] - 4E[X] + 3 + 3 = E[X]$$

$$\Rightarrow P_1 = \frac{-E[X^2] - 5E[X]}{2} + 3$$

We can substitute for $E[X]$ & $E[X^2]$ as follows:

$$E[X] = \bar{X} = 1.96$$

$$E[X^2] = \text{variance} + 1.96^2$$

$$= 0.4784 + (1.96)^2$$

$$\Rightarrow P_1 = 0.26$$

$$P_2 = 0.52$$

Problem 5D

PROBLEM 5D

$$X = \begin{cases} 1 & 1/4 \\ 2 & P_2 \\ 3 & 3/4 - P_2 \end{cases}$$

Also:
We know that the joint frequency of the sample is a product of marginal frequencies

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$$\text{lik}(P_2) = \prod_{i=1}^{3} \cancel{x_i \log P_i} \, (P_i)^{(\# \text{ of } i)} \quad \text{where } i \in [1, 2, 3]$$

$$\Rightarrow \text{log-likelihood}(P_2) = \sum_{i=1}^{3} (\# x_i) \log P_i$$

where $(\# x_i)$ = count of samples in category i (i.e. 1, 2 or 3)

$$\text{lik}(P_2) = \left(\frac{1}{4}\right)^{(\# x_1)} (P_2)^{(\# x_2)} \left(\frac{3}{4} - P_2\right)^{(\# x_3)}$$

$$\Rightarrow \ell(P_2) = x_1 + \log\left(\frac{1}{4}\right) + x_2 \log(P_2) + x_3 \log\left(\frac{3}{4} - P_2\right)$$

To find argmax $\ell(p_2)$, take first derivative & set equal to 0

$$\frac{\partial \ell(p_2)}{\partial p_2} = \frac{x_2}{p_2} - \frac{x_3}{\frac{3}{4} - p_2} = 0$$

$$\Rightarrow \frac{x_2}{p_2} = \frac{x_3}{\frac{3}{4} - p_2} \Rightarrow p_2(x_3 + x_2) = \frac{3}{4} x_2$$

$$\Rightarrow \hat{p}_2 = \frac{\frac{3}{4} x_2}{(x_3 + x_2)}$$; Numerically, $$\hat{p}_2 = \frac{\frac{3}{4}(5200)}{5200 + 2200} = 0.527$$

95% Confidence Interval using large sample theory

$$\hat{\theta} \pm z\left(\frac{\alpha}{2}\right) \cdot \frac{1}{\sqrt{n I(\hat{\theta})}}$$

$z_{\alpha/2} = 1.96$ for 95% CI

We know $$\frac{\partial}{\partial p_2} \ell(p_2) = \frac{x_2}{p_2} - \frac{x_3}{\frac{3}{4} - p_2}$$

$$E\left[\left(\frac{\partial}{\partial p_2} \ell(p_2)\right)^2\right] = I(\hat{\theta}) = \left(\frac{\cancel{x_2}\,0}{p_2} - \frac{4\cancel{x_3}\,0}{3-4p_2}\right)^2 \cdot p_1 + \left(\frac{x_2}{p_2} - \frac{4x_3}{3-4p_2}\right)^2 p_2$$

$$+ \left(\frac{x_2}{p_2} - \frac{4x_3}{3-4p_2}\right) p_3$$

$$I(\hat{\theta}) = \left(\frac{5200}{0.527} - \frac{4 \cdot 2200}{3-4(0.527)}\right)^2 \left(\frac{1}{4}\right)$$
$$+ \left(\frac{5200}{0.527} - \frac{4 \cdot 2200}{3-4(0.527)}\right)^2 (0.527)$$
$$+ \left(\frac{5200}{0.527} - \frac{4 \cdot 2200}{3-4(0.527)}\right)^2 \left(\frac{3}{4} - 0.527\right)$$
$$= 2.896$$

$$\Rightarrow 0.527 \pm \frac{1.96}{\sqrt{10{,}000 \times 2.896}}$$

$$\Rightarrow [a, b] \qquad a = 0.527 - 0.0115 = 0.5155$$
$$b = 0.527 + 0.0115 = 0.5385$$

Problem 6A & 6B

```
C:\Users\sheil\AppData\Local\Programs\Python\Python39\python.exe "C:/Users/sheil/OneDr
Mean computed without inbuilt functions    : 4.2457769104068035
Mean using inbuilt numpy function          : 4.245776910406805
Deviation between both methods             : 1.7763568394002505e-15
==========
Variance computed without inbuilt functions: 18.22406983005141
Variance using inbuilt numpy function      : 18.224069830051395
Deviation between both methods             : 1.4210854715202004e-14
==========
90% Confidence Interval Bounds             : (3.833021657112814, 4.658532163700796)
```

```
import csv
import numpy as np
import scipy.stats as st

fname = "data_HW2.csv"


def computeMeanWithoutInbuilt(data_array):
    sum = 0.0
    for d in data_array:
        sum += d
    sample_mean = sum / len(data_array)
    return sample_mean


def computeVarianceWithoutInbuilt(data_array):
    sample_mean = computeMeanWithoutInbuilt(data_array)
    total_squared_deviation = 0.0
    for d in data_array:
        total_squared_deviation += (d - sample_mean) ** 2
    sample_variance = total_squared_deviation / len(data_array)
    return sample_variance


with open(fname) as f:
    reader = csv.reader(f)
    next(reader)  # skip header
    string_data = [r[0] for r in reader]
    data_array = np.asarray(string_data, dtype=np.float64, order="C")

    sample_mean_without_inbuilt = computeMeanWithoutInbuilt(data_array)
    sample_mean_with_inbuilt = np.mean(data_array)
    sample_mean_deviation = np.abs(sample_mean_without_inbuilt -
sample_mean_with_inbuilt)

    sample_variance_without_inbuilt = computeVarianceWithoutInbuilt(data_array)
    sample_variance_with_inbuilt = np.nanvar(data_array)
    sample_variance_deviation = np.abs(sample_variance_without_inbuilt -
sample_variance_with_inbuilt)

    ninety_percent_confidence_interval = st.t.interval(alpha=0.90, df=len(data_array) -
1, loc=np.mean(data_array),
                                                       scale=0.25)

    print("Mean computed without inbuilt functions    : " +
str(sample_mean_without_inbuilt))
    print("Mean using inbuilt numpy function          : " +
str(sample_mean_with_inbuilt))
    print("Deviation between both methods             : " + str(sample_mean_deviation))
    print("==========")
    print("Variance computed without inbuilt functions: " +
str(sample_variance_without_inbuilt))
    print("Variance using inbuilt numpy function      : " +
str(sample_variance_with_inbuilt))
    print("Deviation between both methods             : " +
str(sample_variance_deviation))
    print("==========")
    print("90% Confidence Interval Bounds             : " +
str(ninety_percent_confidence_interval))
```