Complex Analysis

Definition - A complex no. z is of the form

$z = x + iy$ where $x \in R$, $y \in R$

Complex Plane

$$C = \{ z = x + iy \mid x \in R, y \in R \}$$

$x = Re(z)$, $y = Im(z)$

NOTE :-

i) $Im(z)$ is a real no.

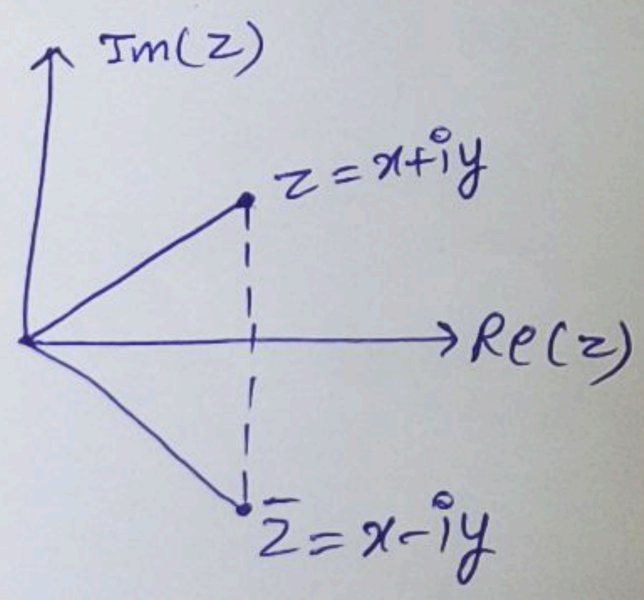
if $y = 0$ then

$z = x + iy = x$

$R \subset C$

ii) we also write $z = x + iy$

conjugate :- $\bar{z} = x - iy$



Let $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

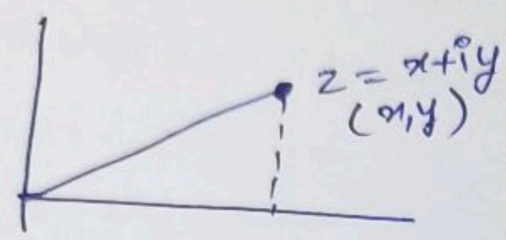
Equality :- $z_1 = z_2$ iff when $x_1 = x_2$ & $y_1 = y_2$

Sum :- $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Product :- $z_1 z_2$

we can also write

$z = x + iy = (x, y)$



<u>Note</u> :- $z = (x, y) = (x, 0) + (0, y)$

Basic Algebraic Properties -

1) Commutative law

i.e. $z_1 + z_2 = z_2 + z_1$

$z_1 z_2 = z_2 z_1$

2) Associative law

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$

3) Distributive law

$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

4) Additive Identity

$0 = (0, 0)$

$z + 0 = z \ \forall \ z \in \mathbb{C}$

5) Multiplicative identity

$1 = (1, 0)$

$z \cdot 1 = z$

6) Additive Inverse

unique ← $-z = -(x, y) = (-x, -y)$

$z + (-z) = 0 = (0, 0)$

7) Multiplicative inverse

$z = (x, y) \neq 0$

$z^{-1} \Rightarrow z z^{-1} = 1 = (1, 0)$ (unique)

$z = (x,y) \neq 0$

Let $z^{-1} = (u, v)$

$(x,y)(u,v) = (1,0)$

$u = \dfrac{x}{x^2+y^2}$, $v = \dfrac{-y}{x^2+y^2}$

$z^{-1} = \left(\dfrac{x}{x^2+y^2}, \dfrac{-y}{x^2+y^2} \right) \quad z \neq 0$

$z = x+iy = (x,y)$

$z^{-1} = u+iv = (u,v)$ $\quad \underline{z \neq 0}$

$zz^{-1} = 1 = (1,0)$

$(x+iy)(u+iv) = (1,0)$

$xu + ixv + iyu - yv = (1,0)$

$xu - yv = 1$ — (i)

$xv + yu = 0$ — (ii)

$-\dfrac{xv}{y} = u$

$-x\left(\dfrac{xv}{y}\right) - yv = 1$

$x^2v + y^2v = -y$

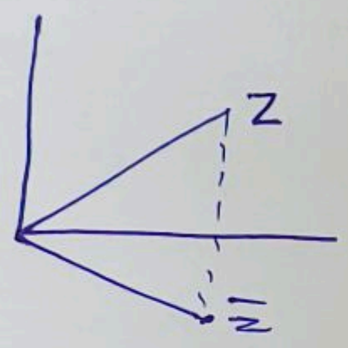
$v = \dfrac{-y}{x^2+y^2}$

$u = x/x^2+y^2$

Absolute value :-

$z = (x,y) = x+iy$

$|z| = \sqrt{x^2+y^2}$



$x = Re(z)$

$y = Im(z)$

$|z|^2 = x^2+y^2$

$Re(z) \leq |Re(z)| \leq |z|$

$Im(z) \leq |Im(z)| \leq |z|$

Let $z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$

$\Rightarrow \quad \overline{(z_1+z_2)} = \bar{z}_1 + \bar{z}_2$

$\Rightarrow \quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2} \qquad z_2 \neq 0$

$\Rightarrow \quad z + \bar{z} = 2Re(z) \qquad z - \bar{z} = 2Im(z)$

$\Rightarrow \quad Re(z) = \dfrac{z+\bar{z}}{2} \qquad Im(z) = \dfrac{z-\bar{z}}{2}$

Δ inequality

$$z\bar{z} = |z|^2$$

$$|z+w|^2 = (z+w)(\overline{z+w}) \leq (|z|+|w|)^2$$

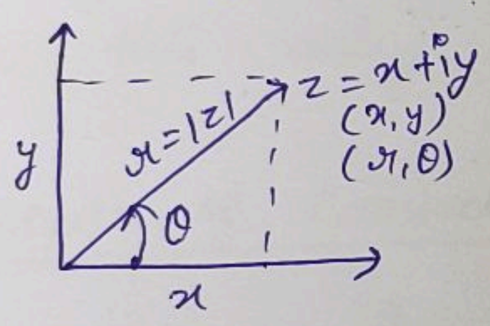
$$|z+w| \leq |z|+|w|$$

Reverse Δ Inequality

$$|z_1+z_2| \geq \big||z_1|-|z_2|\big|$$

Polar form (Exponential form)

Let $z = x+iy$ & $\boxed{z \neq 0}$



Using Polar Coordinates :-

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$z = x+iy$$
$$= r\cos\theta + i r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

Euler's formula :-

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Proof :- $e^x = \sum_0^\infty$

$$ie^{i\theta} = \sum_0^\infty \frac{(i\theta)^n}{n!} = \cos\theta + i\sin\theta$$

Let $f(\theta) = e^{-i\theta}(\cos\theta + i\sin\theta)$

Diff. $f'(\theta) = 0 \cdot (?)$

Integrate

$f(\theta) = \alpha$

$f(0) = 1 \qquad \alpha = 1$

$$\boxed{z = re^{i\theta}}$$

where $r = |z|$ length of the position vector.

θ = angle measured from +ve real axis. to the vector $\vec{z}$. axis is called the general argument.

Integral multiple of 2π.

i.e. $\theta + 2n\pi$, $n = 0, \pm 1, \pm 2$ -----

$$\arg(z) = \theta + 2n\pi \ , \ n \in \mathbb{Z}$$

<u>Principal value of argument</u> :-

denoted by $Arg(z)$ and is a <u>unique value</u> of θ satisfying.

$$\boxed{-\pi < \theta \leq \pi}$$

$$\Rightarrow \arg(z) = Arg(z) + 2n\pi \ , \ n \in \mathbb{Z}$$

<u>Note</u> :- If z is -ve

$Arg\, z = \pi$ and $\boxed{\text{NOT}}$ $-\pi$

<u>Example</u> :- $z = -1 - i$

$$Arg(-1-i) = -3\pi/4 + 2n\pi \quad n \in \mathbb{Z}$$

<u>Example</u> :- $z = -1 - i$

$r = \sqrt{2}$

$$z = \sqrt{2}\, e^{i(-3\pi/4)}$$

If $z = re^{i\theta}$

then $\boxed{z^n = r^n e^{in\theta}}$

Proof - Mathematical induction.

Ex-) Find $(\sqrt{3}+i)$

$\sqrt{3}+i = 2e^{i(\pi/6)}$

De-Movier's Theorem -

$e^{i\theta} = \cos\theta + i\sin\theta$

$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin(n\theta) \quad \forall\ n \in \mathbb{Z}$

Prove this using mathematical induction.

Arguments of Products -

If $z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$\arg(z_1 z_2) = (\theta_1+\theta_2) + 2n\pi \qquad n = 0, \pm 1, \pm 2 \ldots$

Claim I :- $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$

Example- Let $z_1 = -1 \quad z_2 = i$

$\mathrm{Arg}(z_2) = \mathrm{Arg}(-i) = -\pi/2$

But $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) = \pi + \pi/2 = 3\pi/2$

$2\pi + \mathrm{Arg}(z_1 z_2) = -\frac{\pi}{2} + 2\pi = 3\pi/2$

Argument of quotients :-

$$\arg(z_1/z_2) = \arg(z_1 z_2^{-1}) = \arg(z_1) + \arg(z_2^{-1})$$
$$= \arg(z_1) - \arg(z_2)$$

Example :-) Find the principal argument of $z = \frac{-2}{1+i\sqrt{3}}$

Solu.) $\arg(z) = \arg(-2) - \arg(1+i\sqrt{3})$
$= \pi - \pi/3 = 2\pi/3$

one value of $\arg(z) = \frac{2\pi}{3} \in (-\pi, \pi]$

$\text{Arg}(z) = 2\pi/3$

Roots of complex numbers -

Aim :- Non-zero z, find n^{th} root of z.

Need :- Two roots

1.) Equality of complex nos. i.e. if $z_1 = r_1 e^{i\theta_1}$
$z_2 = r_2 e^{i\theta_2}$

$z_1 = z_2$ iff $r_1 = r_2$ & $\theta_1 = \theta_2 + 2k\pi$, $k = 0, \pm 1 \ldots$

2.) If $z = re^{i\theta}$
$z^n = r^n e^{in\theta}$ (By De Movier's Theorem)

Let $z_0 = r_0 e^{i\theta_0}$ be non-zero
and find it's n^{th} root
i.e. find $\sqrt[n]{z_0} = z$

$z^n = z_0$

$$r^n e^{in\theta} = r_0 e^{i\theta_0}$$

$$r^n = r_0 \Rightarrow r = \sqrt[n]{r_0}$$

$$n\theta = \theta_0 + 2k\pi$$

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n} \qquad k = 0, \pm 1 \text{ ----}$$

$$\therefore \quad z = re^{i\theta} = \sqrt[n]{r}\, e^{i(\theta_0/n + 2k\pi/n)}, \quad k = 0, \pm 1 \text{ ----}$$

Ex-) Find all values of

$z_0 = -8i$ and find $z_0^{1/3}$.

Solu.) $-8i = 8e^{(i(-\pi/2 + 2k\pi))} \qquad k = 0, \pm 1 \text{ ----}$

need to find $\underset{re^{i\theta}}{\underset{\Downarrow}{z}}$ s.t. $z^3 = z_0$

$$z_k = 2e^{(i(-\pi/6 + 2k\pi/3))} \qquad k = 0, 1, 2$$

Ex-) Find $\sqrt{z}$, where $z = (\sqrt{3} + i)$?

solu.) $\sqrt{3} + i = 2e^{i(\pi/6 + 2k\pi)} \qquad k = 0, \pm 1, \pm 2 \text{ ----}$

$$z_k = \sqrt{2}\, e^{i(\pi/\sqrt{2} + k\pi)} \qquad k = 0, 1$$

$$\cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2} \qquad \sin^2\frac{\alpha}{2} = \frac{1 - \sin\alpha}{2}$$

$$z_0 = \frac{\pm 1}{\sqrt{2}}\left(\sqrt{2 + \sqrt{3}} + i\left(\sqrt{2 - \sqrt{3}}\right)\right)$$

$$z = -z_0$$

Ex-) $z=1$ find $\sqrt[n]{z}$

Solu.) $1 = \cos 2k\pi + i\sin 2k\pi$ $\quad k = 0, \pm 1 ----$

$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ $\quad k = 0, 1 ----$

Function of a complex variable

Def^n:- Let S be a set of complex nos. A fn f defined on S is a map (rule) that satisfies every value of $z \in S$. a no. $w \in \mathbb{C}$

i.e. $f(z) = w$

$z = x + iy$ $\quad w = u + iv$

$f(z) = u + iv$

$z = re^{i\theta}$

$f(re^{i\theta}) = u(ri\theta) + iv(ri\theta)$

Ex-) Let $f(z) = z^2$

Solu.) $z = x + iy$

$f(z) = f(x+iy) = (x+iy)^2$

$= \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2ixy}_{v(x,y)}$

$f(re^{i\theta}) = r^2 e^{i2\theta}$

$= \underbrace{r^2\cos 2\theta}_{u(r,\theta)} + \underbrace{i^2\sin 2\theta}_{v(r,\theta)}$

$\lim_{x \to x_0} f(x)$ if $R \to R$

Dr. Vikas Goyal.

Limits :-

$$\lim_{z \to z_0} f(z) = f(z_0) = w_0$$

for a given $\varepsilon > 0$, $\exists$ a +ve no. δ such that

$|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

limit if it exists is unique.

Suppose

$\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} f(z) = w_1$

for a given $\varepsilon > 0$, $\exists$ δ_1 δ_2 such that

$|f(z) - w_0| < \varepsilon$, whenever $0 < |z - z_0| < \delta_1$

$|f(z) - w_1| < \varepsilon$, whenever $0 < |z - z_0| < \delta_2$

let $\delta \leq (\delta_1, \delta_2)$

$$|w_1 - w_0| = |(f(z) - w_0) - (f(z) - w_1)|$$

$$\leq |f(z) - w_0| + |f(z) - w_1|$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

$\because$ ε is arbitary

$|w_1 - w_0| = 0 \Rightarrow \boxed{w_1 = w_0}$

Ex-) Let $f(z) = \frac{iz}{2}$, $|z|<1$ Show that $\lim_{z\to 1} f(z) = \frac{i}{2}$

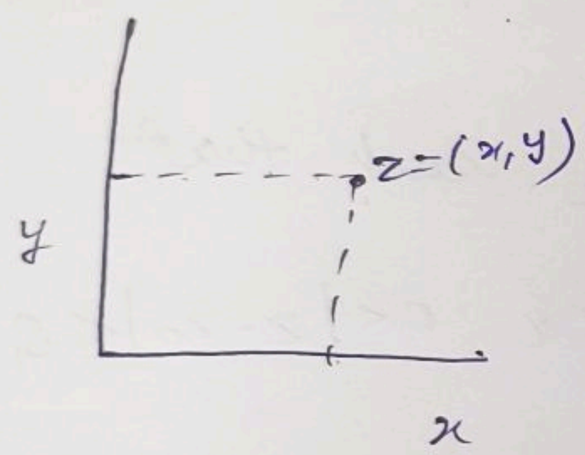
Solu.) $\left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \left| \frac{z-1}{2} \right|$

$\left| f(z) - \frac{i}{2} \right| < \varepsilon$ whenever $0 < |z-1| < 2\varepsilon$

Ex-) Let $f(z) = \frac{z}{\bar{z}}$ show that $\lim_{z\to 0} f(z)$ doesn't exist.

Solu.) $z = (x, y)$

$z \to 0 \Rightarrow (x,y) \to (0,0)$



Let $z = (x, 0)$ approaching $(0,0)$ along x.

i.e. $f(z) = \frac{x+iy}{x-iy} = \frac{x+i0}{x-i0} = 1$

$\lim_{z\to 0} f(z) = 1$

Let $z = (0, y)$ approaching $(0,0)$ along y-axis.

$f(z) = \frac{0+iy}{0-iy} = -1$

$f(z) \to -1$ as $z \to 0$

$\lim_{z\to 0} f(z)$ doesn't exist

Theorems

1) Let f

2) Su

Theorems (Limits) :-

1) Let $f(z) = u(x,y) + i(x,y)$

$z = x + iy$ $\qquad w_0 = u_0 + iv_0$

$z_0 = x_0 + iy_0$

$\lim_{z \to 0} f(z) = w_0$ if and only if

$\lim_{(x,y) \to (0,0)} u(x,y) = u_0$

$\lim_{(x,y) \to (0,0)} v(x,y) = v_0$

2) Suppose $\lim_{z \to z_0} f(z) = w_0$, $\lim_{z \to z_0} g(z) = w_1$

Then

$$\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) = w_0 + w_1$$

$$\lim_{z \to z_0} [f(z)\, g(z)] = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z) = w_0 w_1$$

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} f(z) \Big/ \lim_{z \to z_0} g(z) = w_0 / w_1$$

$w_1 \neq 0$

Continuity :-

Def^n :- A fx^n $f(z)$ is CONTINUOUS at a pnt z_0 if the following is satisfied.

i) $\lim_{z \to z_0} f(z)$ should exist.

ii) $f(z_0)$ is defined

iii) $\lim_{z \to z_0} f(z) = f(z_0)$

iv) containing i) & ii)

v) for a given $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

Derivative :-

The derivative of $f(z)$ at $z = z_0$ is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

let $z = z_0 + \Delta z$ or $\Delta z = z - z_0$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Ex-) Let $f(z)=z^2$. Find $f'(z)$.

Solu-) $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (\Delta z + 2z)$

$= 2z.$

Ex-) Let $f(z) = \bar{z}$. Show that $f'(z)$ doesn't exist for non-zero z.

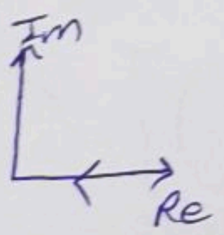
Solu-) $\frac{\Delta w}{\Delta z} = \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$

$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$

$\Delta z = (\Delta x, \Delta y)$

Take the limit $\Delta z \to 0$ along x-axis

i.e. $\Delta z = (\Delta x, 0) = \Delta x + i \cdot 0$ $\quad \overline{\Delta z} = (\Delta x, 0)$



$\Delta x \to 0 \Rightarrow \Delta z \to 0$

$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

Take the limit $\Delta z \to 0$ along y-axis. i.e. $\Delta z = (0, \Delta y)$

$\overline{\Delta z} = (0, -\Delta y)$

$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta y \to 0} \frac{-\Delta y}{\Delta y} = -1$

$\Rightarrow$ limit doesn't exists.

$f(z) = u(x,y) + iv(x,y)$ $\qquad z = x+iy$

Cauchy - Riemann Equations :-

Let $f(z) = u(x,y) + iv(x,y)$ $\qquad f(z) = w$

Let $z_0 = (x_0, y_0)$

i.e. $z_0 = x_0 + iy_0$

$\Delta z = \Delta x + i\Delta y$.

$$\Delta w = f(z_0 + \Delta z) - f(z_0)$$

$$= \left(u(x_0 + \Delta x + y_0 + \Delta y) - u(x_0, y_0)\right) - \left\{v(x_0 + \Delta x, y_0 + \Delta y) + v(x_0, y_0)\right\}$$

Assume that the derivative $f'(z)$ exists.

i.e. $f'(z) = \lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ exists

we have

$$f'(z_0) = \lim_{(\Delta x, \Delta y) \to (0,0)} \left[\left(Re \frac{\Delta w}{\Delta z}\right) + i\left(Im \frac{\Delta w}{\Delta z}\right)\right]$$

(i) Let $\Delta z = (\Delta x, 0)$ i.e. $\Delta x \to 0 \Rightarrow \Delta z \to 0$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x \to 0} \left\{\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\,\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right\}$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(Re \frac{\Delta w}{\Delta z}\right) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

$$= u_x(x_0, y_0)$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\operatorname{Im} \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta y \to 0} \frac{v(x_0 + \Delta x_0) - v(x_0, y_0)}{\Delta x}$$

$$= v_x(x_0, y_0)$$

$$\boxed{f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)} \quad \text{—— (1)}$$

(ii)

Let $\Delta z = (0, \Delta y) \longrightarrow (0 + i\Delta y)$

$\Delta y \to 0 \Rightarrow \Delta z \to 0$

$$\frac{\Delta w}{\Delta z} = \left[\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + \frac{i\, v(x_0, y_0 + \Delta y) - v(x_0 y_0)}{i\Delta y} \right]$$

$$= \left[\frac{v(x_0, y_0 + \Delta y) - v(x_0 y_0)}{\Delta y} - i \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} \right]$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\operatorname{Re} \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0 y_0)}{\Delta y}$$

$$= v_y(x_0, y_0)$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\operatorname{Im} \Delta w / \Delta z \right) = -\lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}$$

$$= u_y(x_0, y_0)$$

$$f'(z_0) = v_y(x_0, y_0) - i u_y(x_0, y_0)$$

$$\boxed{f'(z_0) = -i[u_y(x_0, y_0) + i v_y(x_0, y_0)]} \quad \text{--- (2)}$$

from (1) & (2) (since $f'(z_0)$ is assumed to exist) then :-

$$\boxed{\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ u_y(x_0, y_0) &= -v_x(x_0, y_0) \end{aligned}}$$

These are called Cauchy Riemann Equations.

Theorem :- suppose that

$f(z) = u(x, y) + i v(x, y)$

$f'(z)$ exists at a pnt $z_0 = (x_0, y_0)$

Then the 1st order derivative of u & v exists at (x_0, y_0) and also must satisfy the C-R eqn

$u_x = v_y$ & $u_y = -v_x$

Theorem :- Suppose that

$$f(z) = u(x,y) + iv(x,y)$$

and $f'(z)$ exists at a pnt $z = z_0 = x_0 + iy_0$ then the 1st partial derivative of $u(x,y)$ and $v(x,y)$ must exist. at (x_0, y_0) and they must satisfy the Cauchy - Riemann Eqn given by $v_x = v_y$, $u_y = -v_x$ at (x_0, y_0)

also $f'(z_0) = u_x + iv_x$

when the partial derivatives are evaluated at (x_0, y_0).

Ex-) Let $f(z) = z^2 = x^2 - y^2 + 2ixy$

$$f'(z) = 2z$$

To verify that C-R = are satisfied, we have

$$u(x,y) = x^2 - y^2 , \quad v(x,y) = 2xy$$

$$\left.\begin{aligned} u_x &= 2x \\ u_y &= -2y \\ v_x &= 2y \\ v_y &= 2x \end{aligned}\right\} \Rightarrow \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Also $f'(z) = u_x + iv_x = 2x + 2iy = 2z$

Note:- C-R= are necessary for the existence of the derivative of f at $z = z_0$ this can be used to locate the pnt where $f'(z)$ doesn't exist.

Ex-) Let $f(z) = |z|^2$

$u(x, y) = x^2 + y^2$ $\quad u_x = 2x \quad u_y = 2y$

$v(x, y) = 0$ $\quad v_x = 0 \quad v_y = 0$

$f'(z)$ doesn't exist

exist at any $z \neq 0$.

Sufficient condition for differentiability

Theorem:- Let the fx^n

$$f(z) = u(x, y) + i v(x, y)$$

be defined in some ε-neighbourhood of $z = z_0 = (x_0, y_0)$ and suppose

i) the 1st partial derivative of u & v wrt x & y do exist everywhere in the neighbourhood of $z = z_0$.

ii) These partial derivatives are continuous at (x_0, y_0) & satisfy the C-R eqns.

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right] \text{ at } (x_0, y_0)$$

Then $f'(z)$ exists at $z = z_0$ and

$$f'(z_0) = u_x + i v_x \quad \text{at } (x_0, y_0)$$

C-R eqns (Polar Coordinates)

$$f(z), \quad z = re^{i\theta} \qquad z \neq 0$$

$$f(z) = u(r,\theta) + i v(r,\theta)$$

$$\frac{\partial w}{\partial r} = u_r = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial r} \longrightarrow u_r = u_x \cos\theta + u_y \sin\theta$$

$$\frac{\partial w}{\partial \theta} = u_\theta = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial \theta} \longrightarrow \underbrace{u_\theta = -u_x r\sin\theta + u_y r\cos\theta}_{①}$$

$$\left.\begin{aligned} v_r &= v_x \cos\theta + v_y \sin\theta \\ v_\theta &= -v_x r\sin\theta + v_y r\sin\theta \end{aligned}\right] - ②$$

Now, C-R eqn since

$$u_x = v_y, \quad u_y = -v_x$$

This gives

$$\left.\begin{aligned} v_r &= -u_y \cos\theta + u_x \sin\theta \\ v_\theta &= u_y r\sin\theta + u_r r\cos\theta \end{aligned}\right] - ③$$

from ①, ②, ③ we get $\boxed{\begin{aligned} r u_r &= v_\theta \\ u_\theta &= -r v_r \end{aligned}}$ at $z \neq 0$

Theorem (Polar form) :-

Let $f(z) = u(r,\theta) + i v(r,\theta)$ be defined in some ε-nbd s.t. $z_0 \neq 0 = r_0 e^{i\theta_0}$ and suppose that -

a) 1st partial derivatives of u & v wrt r & θ exist everywhere in the ε nbd of $z = z_0$.

b) These partial derivative must be cont. at (r_0, θ_0) and satisfy C-R eqn

$$\boxed{r u_r = v_\theta, \quad u_\theta = -r v_r}$$

Then $f'(z_0)$ exists

$$\boxed{f'(z_0) = e^{i\theta}(u_r + i v_r)}$$

Ex-) Let us consider

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

i.e. $u(r,\theta) = \dfrac{\cos\theta}{r}$ $\quad v(r,\theta) = -\dfrac{\sin\theta}{r}$

we have, $r u_r = -\dfrac{\cos\theta}{r} = v_\theta$

$u_\theta = -\dfrac{\sin\theta}{r} = -r v_r$

$\Rightarrow$ C-R eqn satisfied.

$$f'(z) = e^{i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right)$$

$$= e^{-i\theta} \cdot \frac{e^{-i\theta}}{r^2} = -\frac{1}{r^2(e^{i\theta})^2} = \frac{-1}{z^2}$$

Analytic functions -

A fx^n $f(z)$ is analytic at $z = z_0$ if it has a derivative at every pnt in some nbd at $z = z_0$.

$f(z)$ is said to be analytic in some domain if it is analytic at every pnt of that domain.

Ex-) Let $f(z) = \frac{1}{z}$ is analytic at each $z \neq 0$ but $f(z) = |z|^2$ is not analytic at any pnt since its derivative doesn't exist except at $z = 0$

Harmonic Functions -

Def^n :- A real valued fx^n $u(x,y)$ is said to be harmonic in some domain D in the x-y planes of throughout the given domain, the 1st and 2nd partial derivatives are continuous and satisfies the relation.

$$u_{xx} + u_{yy} = 0$$

called the Laplace eq^n.

Example :- Consider the function

$$u(x,y) = z^4 \sin x$$

~~Then in any~~ This is harmonic in any domain of the x-y plane, since it satisfies

$$u_{xx} + u_{yy} = 0$$

Theorem :- If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then the component fx^{n}s $u(x,y)$ & $v(x,y)$ are harmonic in D.

Proof :- Since $f(z)$ is analytic in D, C-R eqn must be satisfied in D.

i.e. $u_x = v_y$ & $u_y = -v_x$

Differentiate x & y

$u_{xx} = v_{yx}$ $\qquad$ $u_{yx} = -v_{xx}$ —— 1

$u_{xy} = v_{yy}$ $\qquad$ $u_{yy} = -v_{xy}$ —— 2

Since 2nd order partial derivatives are conti.

i.e. $u_{xy} = u_{yx}$

$v_{xy} = v_{yx}$

$(1) - (2) \Rightarrow \quad u_{xx} + u_{yy} = 0$

$v_{xx} + v_{yy} = 0$

$\Rightarrow \quad u(x,y) + iv(x,y) \longrightarrow$ harmonic

<u>Example</u> :- Consider the function $f(z) = \frac{i}{z^2}$

<u>Solu</u>:) It is analytic at all $z \neq 0$.

$$f(z) = i/z^2 = \frac{i}{z^2} \times \frac{\bar{z}^2}{\bar{z}^2} = \frac{i\bar{z}^2}{|z|^4} = \frac{2xy + i(x^2-y^2)}{(x^2+y^2)^2}$$

$$\left.\begin{aligned} u(x,y) &= \frac{2xy}{(x^2+y^2)^2} \\ v(x,y) &= \frac{x^2-y^2}{(x^2+y^2)^2} \end{aligned}\right\}$$ are harmonic $\forall (x,y) \neq (0,0)$ i.e. $u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$

<u>Harmonic Conjugate</u>

<u>Def^n</u> :- If $u(x,y)$, $iv(x,y)$ are harmonic in a domain D. and their 1st order partial derivatives satisfy the C-R = in D then $v(x,y)$ is said to be harmonic conj. of $u(x,y)$.

Theorem – A f^n $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if and only if $v(x,y)$ is harmonic conjugate of $u(x,y)$.

Note :- If $v(x,y)$ is a harmonic conjugate of $u(x,y)$ in same domain, then $u(x,y)$ in general need not be harmonic conjugate of $v(x,y)$

Ex-) $\left.\begin{array}{l} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{array}\right] \Rightarrow f(z) = z^2$ (analytic $\forall z$)

Also, $u(x,y)$ & $v(x,y)$ are harmonic

$v(x,y)$ is harmonic conj. of $u(x,y)$

But, $u(x,y)$ is not harmonic conj. of $v(x,y)$.

bcoz $2xy + i(x^2 - y^2) = f(z)$ is not analytic.

If $u(x,y)$ is harmonic in simply connected domain D

Harmonic Conjugate (contd.)

Note :- If $u(x,y)$ is harmonic in some simply connected domain D then its harmonic Conjugate $v(x,y)$ exists in the same domain D and is unique (subject to an additive constant).

Definition:- domain D is said to be simply connected if every closed contour in D contains pnts only of D.

Method:- of finding harmonic conjugate.

Ex-) Let $u(x,y) = y^3 - 3x^2y$. find it's harmonic conj.

Sol(n.) $u(x,y) = y^3 - 3x^2y$ is harmonic everywhere in the xy-plane

$$\because \quad u_{xx} + u_{yy} = 0$$

To find harmonic conjugate $v(x,y)$ they must satisfy CR=

i.e. $u_x = v_y$

$u_y = -v_x$

$u_x = -6xy = v_y$

Integrate wrt y keeping x = fixed.

$$v(x,y) = -3xy^2 + \varphi(x)$$

$\varphi(x) \rightarrow$ arbitrary integration Const.

Now, $v_x = -3y^2 + \varphi'(x) = -u_y$

$$-u_y = -3y^2 + 3x^2$$

$$\Rightarrow \varphi'(x) = 3x^2$$

$$\Rightarrow \varphi(x) = x^3 + c$$

$c \rightarrow$ arbitrary Const.

i.e. $v(x,y) = -3xy^2 + x^3 + c$

$f(z) = u(x,y) + iv(x,y)$

$= (y^3 - 3x^2y) + i(-3xy^2 + x^3 + c)$

$= i(z^3 + c)$ $\qquad z = x + iy$

Elementary functions :-

The exponential fn's -

$z = x + iy$

$e^z = e^{x+iy} = e^x . e^{iy}$

$e^{iy} = \cos y + i \sin y$

(i) $e^{x_1} . e^{x_2} = e^{x_1 + x_2}$ $\qquad x_1, x_2 \in R$ (Additive prop.)

$z_1, z_2 \in C$

$e^{z_1} . e^{z_2} = e^{z_1 + z_2}$

$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$ $\qquad$ we have $e^0 = 1$

$\frac{1}{e^{z_2}} = e^{-z_2}$

$\frac{d}{dz} e^z = e^z \ \forall z \Rightarrow f(z) = e^z$ is an entire fn.

(ii) $e^z \neq 0 \quad z \in C$

$e^z = e^{x+iy} = e^x . e^{iy} = \rho e^{i\theta}$

$\rho = e^x, \ \theta = y$

$|e^z| = e^x$ $\qquad \rho$ and $(e^z) = y + 2n\pi$

$n = 0, \pm 1, \pm 2 \ldots$

$|e^z| > 0$

we also have

$$e^{z+i2\pi} = e^z \cdot e^{i2\pi}$$
$$= e^z \qquad (e^{i2\pi} = 1)$$

e^z is periodic f^n with period $2\pi i$.

$e^x > 0 \qquad x \in R$

$e^z \qquad z \in C$ can be $-ve$

Eg-) $e^{ix} = -1$

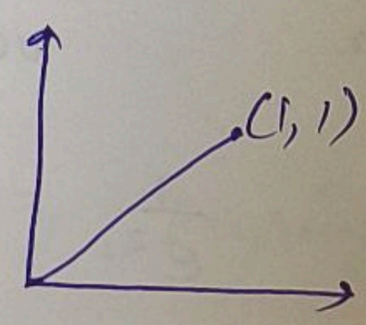
or $e^{i(2n+1)\pi} = -1 \qquad \forall \; n = 0, \pm 1, \pm 2 \ldots$

The logarithmic functions

$$e^w = z$$

ex-) find z

$e^z = 1 + i$



Solu.)

$$z = x + iy$$
$$e^z = e^{x+iy} = 1 + i$$
$$= e^x \cdot e^{iy} = \sqrt{2}\, e^{i\pi/4}$$

$\Rightarrow e^x = \sqrt{2}$ & $y = \pi/4 + 2n\pi \qquad n = 0, \pm 1, \pm 2 \ldots$

$x = \ln\sqrt{2}, \quad y = \pi/4 + 2n\pi$

$z = x + iy = \frac{1}{2}\ln 2 + \left(2n + \frac{1}{4}\right)\pi i. \qquad n = 0, \pm 1, \ldots$

(i) $-e^{w} = z$ where $z \neq 0$ with $z = x + iy = re^{i\theta}$ $(-\pi < \theta \leq \pi)$

$\text{Q}\ w = u + iv$

i.e. $e^{u+iv} = re^{i\theta}$

$e^{u} \cdot e^{iv} = re^{i\theta}$

$e^{u} = r$, $v = \theta + 2n\pi$ $\quad n = 0, \pm 1, \pm 2 \ldots$

or $u = \ln r$, $v = \theta + 2n\pi$

Eqn (i) is satisfied iff

(2) — $w = \ln r + i(\theta + 2n\pi)$, $n = 0, \pm 1, \pm 2 \ldots$

multivalued fxn.

If $\boxed{\log z = \ln r + i(\theta + 2n\pi)}$ — (3)

we have

$e^{\log z} = z$ satisfied for $z \neq 0$

Ex-) Let $z = -1 - i\sqrt{3}$ Find $\ln z$

Solu.) $\log z = \ln(-1 - i\sqrt{3}) = \ln 2 + i(-2\pi/3 + 2n\pi)$

$n = 0, \pm 1, \pm 2 \ldots$

Remark :- show that $\log(e^{z}) \neq z$

Proof :- $\log z = \ln|z| + i(\theta + 2n\pi)$

$= \ln|z| + i \arg z$

we have

$|e^{z}| = |e^{x+iy}| = e^{x}$

$\arg(e^{z}) = y + 2n\pi$ $\quad n = 0, \pm 1, \pm 2 \ldots$

$$\log(e^z) = \ln|e^z| + i\arg(e^z)$$
$$= \ln e^x + i(y + 2n\pi)$$
$$= (x + iy) + i \cdot 2n\pi$$
$$= z + i \cdot 2n\pi \quad , \quad n = 0, \pm 1, \pm 2 \text{ ----}$$

<u>Principle value of $\log z$:-</u>

Denoted by $\operatorname{Log} z$ and defined by taking $n = 0$ in eq^n (3). i.e. $\operatorname{Log} z = \ln r + i\theta$.

This is a single valued fn.

<u>Remark</u> :- $\boxed{\log z = \operatorname{Log} z + i \cdot 2n\pi}$ $\quad n = 0, \pm 1, \pm 2$ ----

<u>Example</u> :-) Let $z = 1$ find $\log(1)$ and $\operatorname{Log}(1)$.

<u>Solu</u>.) $\log(1) = \ln 1 + i(0 + 2n\pi)$

$$\log(1) = 2n\pi \cdot i \qquad n = 0, \pm 1 \text{ -----}$$

$$\operatorname{Log}(1) = \ln 1 + i \cdot 0 = \ln 1 = 0$$

<u>Example</u>:-) Let $z_1 = -1$ & $z_2 = -1$ find $\log(z_1)$ or $\log(z_2)$.

<u>Solu</u>.) $\log(-1) = \ln 1 + i(\pi + 2n\pi)$

$$= (2n+1)\pi i \qquad n = 0, \pm 1, \pm 2 \text{ ---}$$

$$\operatorname{Log}(-1) = \pi i$$

Some identities (involving inequalities)

If $z_1 \neq 0$, $z_2 \neq 0$

then $\log(z_1 z_2) = \log z_1 + \log z_2$ —— (*)

follows in the lines of

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Proof – $|z_1 z_2| = |z_1|\,|z_2|$

$$\ln|z_1 z_2| = \ln|z_1| + \ln|z_2|$$

Now,

$$\log(z_1 z_2) = \ln|z_1 z_2| + i\arg(z_1 z_2)$$
$$= \ln|z_1| + \ln|z_2| + i(\arg(z_1) + \arg(z_2))$$
$$= \ln|z_1| + i\arg(z_1) + \ln|z_2| + i\arg(z_2)$$
$$= \log z_1 + \log z_2$$

Example:-) Let $z_1 = -1$, $z_2 = -1$

$n = 0, \pm 1, \pm 2 \ldots$

Solu.) $\log(1) = \log z_1 z_2 = 2n\pi i$

$$\log z_1 = \log z_2 = \log(-1) = (2n+1)\pi i$$

$$\mathrm{Log}(1) = 0, \quad \mathrm{Log}(-1) = \pi i$$

$\Rightarrow$ (*) is not true in general for principal log fn (Log).

$\log(z_1/z_2) = \log z_1 - \log z_2$

Real exponents –

$z \neq 0$

find z^n we have

$$z^n = e^{n \log z} \qquad n = 0, \pm 1, \pm 2 \ldots$$

we also have

$$z^{1/n} = \exp\left(\frac{1}{n} \log z\right) \qquad n = 0, \pm 1, \pm 2 \ldots$$

RHS :-

$\exp\left(\frac{1}{n} \log z\right)$ lies in distinct values (which are the nth roots of z).

Proof –

$$\exp\left(\frac{1}{n} \log z\right) = \exp\left(\frac{1}{n} \ln r + i\left(\frac{\theta + 2k\pi}{n}\right)\right\}$$

$k = 0, 1 \ldots$

$$= \sqrt[n]{r} \exp\left\{ i\left(\frac{\theta + 2k\pi}{n}\right)\right\}$$

$k = 0, \pm 1, \pm 2 \ldots$

Prove that –

$\exp\left[i - \frac{2k\pi}{n}\right]$ lies in distinct values for

$k = 0, 1, 2 \ldots n-1$

Complex Exponents -

$z \neq 0$ & exponent $c \in \mathbb{C}$

then $z^c = e^{c \log z}$

Example) Find z^{-2i}

Solu.) $z^{-2i} = \exp(-2i \log i)$

Now $\log i = \ln 1 + i(\pi/2 + 2n\pi)$

$= (2n + \frac{1}{2})\pi i \quad n = 0, \pm 1, \pm 2 \ldots$

$z^{-2i} = \exp[(2n + \frac{1}{2})\pi] \quad n = 0, \pm 1, \pm 2 \ldots$

Note :-

1) $i^{-2i} = \exp[(4n + \frac{1}{2})\pi] \quad n = 0, \pm 1, \pm 2 \ldots$

are all real numbers.

Trignometic fx^ns -

⇒ **Euler's formula :-**

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x \quad x \in \mathbb{R}$

$e^{ix} - e^{-ix} = 2i \sin x$

$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$\dfrac{e^{ix} + e^{-ix}}{2} = \cos x$

we can write for $z \in \mathbb{C}$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{--- (i)} \qquad \cos z = e^{iz} + e^{-iz}/2$$

$\because$ e^{iz} & e^{-iz} are entire (i.e. analytic in the finite complex plane).

we have, $\sin z$ & $\cos z$ are also entire (their linear combination is entire fn).

$$\because \frac{d}{dz}(e^{iz}) = ie^{iz} \quad \text{or} \quad \frac{d}{dz}(e^{-iz}) = -ie^{-iz}$$

$$\Rightarrow \frac{d}{dz}(\sin z) = \cos z \quad (\text{from eqn (i)})$$

$$\Rightarrow \frac{d}{dz}(\cos z) = -\sin z$$

$$\Rightarrow \sin(-z) = -\sin z$$
$$\cos(-z) = \cos z$$

Also, $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$$\Rightarrow \sin 2z = 2 \sin z \cos z$$
$$\cos 2z = 1 - 2\sin^2 z$$
$$= 2\cos^2 z - 1$$
$$= \cos^2 z - \sin^2 z$$

$$\Rightarrow \sin(z + \pi/2) = \cos z$$
$$\sin(z - \pi/2) = -\cos z$$

$\Rightarrow \sin(z+2n\pi) = \sin z \qquad \cos(z+2n\pi) = \cos z$

Hyperbolic functions -

$\sinh y = e^{y} - e^{-y}/2 \qquad \cosh y = e^{y} + e^{-y}/2 \qquad y \in R$

$\Rightarrow \sin(iy) = i \sinh y$ — (1) $\qquad \cos(iy) = \cosh y$ — (2)

$z = x + iy$

$\sin(z) = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$\cos(z) = \cos(x+iy) = \cancel{\cos x \cos y}\ \cos x \cosh y - i \sin x \sinh y$

$|\sin(z)|^2 = \sin^2 x + \sinh^2 y$

$|\cos(z)|^2 = \cos^2 x + \sinh^2 y$

$\Rightarrow \sinh y \to \infty$ as $y \to \infty$

$\therefore$ $\sin z$ & $\cos z$ are not bounded fn's.

Complex Integration :-

$w : [a,b] \to \mathbb{C}$ satisfy

$w(t) = u(t) + i v(t)$

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$$

Example- Find $\int_0^1 (1+it)^2\,dt$

$= \cancel{\int_0^1 1^2\,dt + \int}\ \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt$

$= \frac{2}{3} + i$

Fundamental theorem of Calculus -

Let $w(t) = u(t) + i v(t)$] continuous $\forall\ t \in [a,b]$

$W(t) = U(t) + i V(t)$]

Let $W'(t) = w(t) \quad \forall\ t \in [a,b]$

$U'(t) = u(t)$

$V'(t) = v(t)$

$$\int_a^b w(t)\,dt = U(t)\Big|_a^b + i v(t)\Big|_a^b$$

$$= \big(U(b) + iV(b)\big) - \big(U(a) + iV(a)\big)$$

$$= W(b) - W(a) = W(t)\Big|_a^b$$

Example - $\frac{d}{dt}\left(e^{it}/i\right) = e^{it}$

solu.) $\int_0^{\pi/4} e^{it}\,dt = \frac{e^{it}}{i}\Big|_0^{\pi/4} = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)$

Definition (curve) :-

A curve in the complex plane by a fx^n $\gamma(t)$

$\gamma(t) = x(t) + i y(t) \quad t \in R = [a,b]$ i.e.

$\gamma : [a,b] \longrightarrow \mathbb{C}$

Smooth Curve -

A curve is smooth if

$\gamma'(t) = x'(t) + iy'(t)$ is continuous & non-zero $\forall\ t \in [a,b]$.

⇒ **Contour** :- It is a smooth curve obtained by joining finitely many smooth curves end to end.

⇒ **Contour integral** -

Let $C = \gamma(t)$ $t \in [a,b]$ be a Contour &

let

$f(z) : \mathbb{C} \longrightarrow \mathbb{C}$ be continuous, then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

Note :- Contour integral is independent of the parametrization of the curve.

Ex-) Find $\int_C (z-a)^n dz$, where C: circle of radius r & center at a.

Soln.) Let $\gamma(t) = a + re^{it}$ $t \in [0, 2\pi]$

$$\int_C (z-a)^n dz = \int_0^{2\pi} (re^{it})^n \cdot ire^{it}\,dt$$

$$= i \cdot r^{n+1} \int_0^{2\pi} e^{i(n+1)t}\,dt$$

$$= ir^{n+1}\left[\int_0^{2\pi} \cos(n+1)t\,dt + i\int_0^{2\pi} \sin(n+1)t\,dt\right]$$

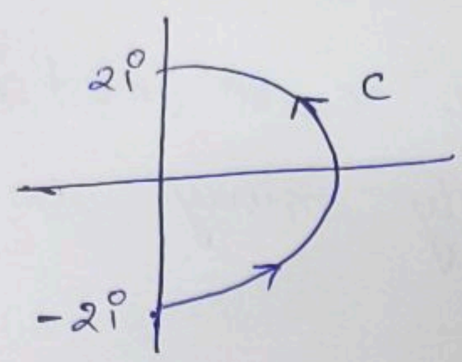
$$= 2\pi i, \quad n = -1$$
$$= 0, \quad n \neq -1$$

Ex-) Find the value of the integral

$$I = \int_C \bar{z}\, dz$$ where C: Right hand half of the circle

$z = 2e^{i\theta}$

$-\frac{\pi}{2} \le \theta \le \pi/2$



$f(z) = \bar{z}$

solu.)

$$I = \int_{-\pi/2}^{\pi/2} \overline{(2e^{i\theta})}\,(2e^{i\theta})'\, d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} e^{-i\theta}\, i \cdot e^{i\theta}\, d\theta$$

$$= 4i\int_{-\pi/2}^{\pi/2} d\theta \Rightarrow 4\pi i$$

Upper Bounds -

Lemma :- If $w(t)$ is a piece wise continuous fx^n defines on $t \in [a,b]$ then

$$\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt \quad \text{—— (1)}$$

Proof :- If the value of the integral on LHS is zero then (1) is True.

Let us assume that

$$\int_a^b w(t)dt = r_0 e^{i\theta_0} \neq 0$$

$$r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt$$

real number

i.e. $r_0 = Re\int_a^b e^{-i\theta_0} w(t)dt = \int_a^b Re(e^{-i\theta_0} w(t)dt)$

$$\leq \int_a^b |e^{-i\theta_0} w(t)|\,dt$$

$$r_0 \leq \int_a^b |w(t)|\,dt$$

ML Inequality :-

Theorem :- Let C be a contour of length L and let $f(z)$ be continuous on C and let

$$|f(z)| \leq M \text{ on } C, \text{ then}$$

$$\left|\int_C f(z)dz\right| \leq ML$$

Proof :- Let $z = z(t)$ $t \in [a,b]$ be a parametric representation of C, then

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))\, z'(t)\,dt\right|$$

$$\leq \int_a^b |f(z(t))\, z'(t)|\,dt$$

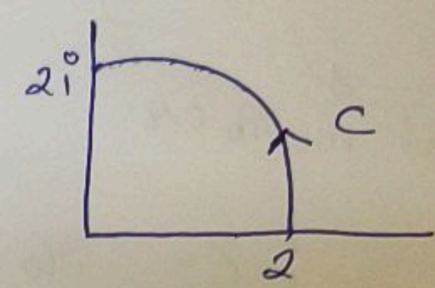
$$= \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

$$\leq M\int_a^b |z'(t)|\,dt \leq ML$$

Illustration of ML inequality :-

Ex-) Let C be the one of the circle $|z| = 2$ for $z = i$ for $z = 2i$ then show that

$$\left| \int_C \frac{z+4}{z^3-1}\, dz \right| \leq \frac{6\pi}{7}$$



Solu.) $f(z) = \dfrac{z+4}{z^3-1}$

$$|z+4| \leq |z| + 4 = 6$$

& $|z^3-1| \geq \big||z|^3 - 1\big| = 7 \Rightarrow \dfrac{1}{|z^3-1|} \leq \dfrac{1}{7}$

$$\left|\frac{z+4}{z^3-1}\right| = \frac{|z+4|}{|z^3-1|} \leq \frac{6}{7} \qquad \text{i.e. } |f(z)| \leq M = \frac{6}{7}$$

Also, $L = \pi$

$$\int_C \left|\frac{z+1}{z^3-1}\, dz\right| \leq \frac{6\pi}{7}$$

i) <u>ε- neighbourhood</u> :-

Let $z_0 \in \mathbb{C}$. An ε- nbd of $z = z_0$ is the set of values of z that satisfy $|z - z_0| < \varepsilon$.

ii) <u>Deleted neighbourhood</u> :-

$z = z_0$ contour of pnts z in the ε- nbd of z_0 except the pnt z_0 itself i.e. $0 < |z - z_0| < \varepsilon$.

iii) <u>Interior pnt</u> :-

A pnt $z = z_0$ is said to be an interior pnt of a set S if a nbd of z_0 contains points that belong to S. only.

iv) <u>open set</u> :-

A set S is called an open set if it contains none of it's boundary pnts.

or If set S is open iff each of it's pnts is an interior pnts.

v) <u>closet set</u> :- A set S is closed if it contains all of it's boundary pnts.

vi) Some sets are neither open nor closed. i.e. S is not open means that $\exists$ a boundary pnt. that belongs to S. Also S is not closed.

means that $\exists$ at least one pnt of the boundary that doesn't belong to S.

$$0 < |z - z_0| \leq r$$

set of complex nos. $\mathbb{C}$ (both open & closed bcoz it has no boundary).

vii) <u>Connected set</u> :- A set S is connected if any 2 pnts in S can be joined by finitely many polygonal line segments that entirely fall in S.

E.g. $|z| < 1$ is connected

$1 < |z| < 2$ is connected

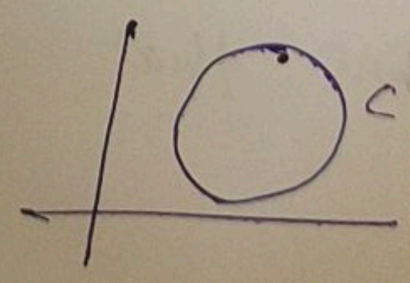
viii) <u>Domain</u> - connectes open set.

ix) <u>Region</u> - Domain that may contain some, none or all boundary pnts.

<u>Cauchy - Theorem</u> :- (CG theorem)

<u>Theorem</u> - If a f^n $f(z)$ is analytic at all pnts interior and on the boundary of a simple closed contour C, then

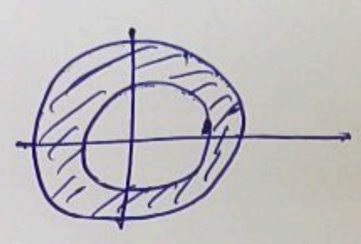
$$\int f(z)\,dz = 0$$



Simply Connected Domain -

Def^n :- A set s is said to be simply connected if many simple closed contour within it contains only pnts of s.

Annular region $1 < |z - z_0| < 2$ is not simply connected



Note :- The closed contour in the CG theorem need not be simple.

CG theorem :- If a fn $f(z)$ is analytic throughout a simply connected domain D then,

$$\int_C f(z)\,dz = 0,$$ where c is any/every closed contour lying in D.

ex-) show that

$$\int \frac{ze^z}{(z+9)^5}\,dz = 0 \quad \text{where } c : |z| < 2$$

soln.) as $c = |z| < 2$ is is simply connected

$f(z) = \dfrac{ze^z}{(z+9)^5}$ is analytic then $|z| < 2$

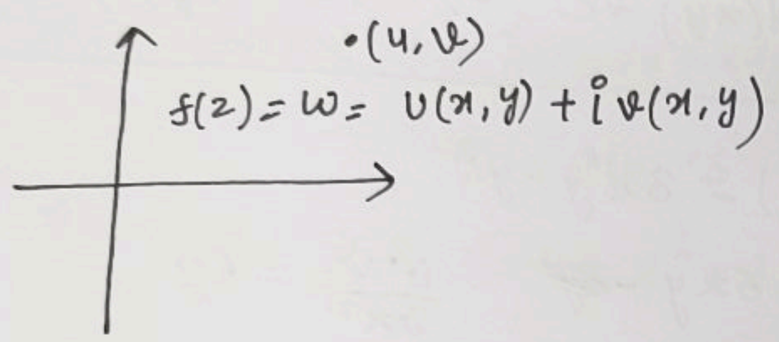
$z = \pm i$ (singularity lie outside c).

Analytic functions :-

$w = f(z)$ is said to be analytic at pnt $z = z_0$ if

$$\frac{dw}{dz} = f'(z) = \lim_{dz \to 0} \frac{f(z+dz) - f(z)}{dz}$$

exists & is unique at z_0.



w plane
$w = f(z)$

C-R eqn :-

$w = u + iv$

and we have to determine whether this w is analytic or differentiable then the condition to be satisfied is :-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Eg:- $f(z) = z^2 \longrightarrow$ Hence analytic.

Solu:) $f(z) = (x+iy)^2$

$f(z) = x^2 + 2ixy - y^2$

$u = x^2 - y^2$

$v = 2xy$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$

$\frac{\partial v}{\partial y} = 2x \qquad \frac{\partial v}{\partial x} = 2y$

Ex-) $f(z) = e^{-x}(\cos y + i \sin y)$

$= e^{-x}\cos y + i(e^{-x}\sin y)$

$u = e^{-x}\cos y \qquad v = e^{-x}\sin y$

$\frac{\partial u}{\partial x} = -e^{-x}\cos y \qquad \frac{\partial v}{\partial x} = -e^{-x}\sin y$

$\frac{\partial u}{\partial y} = -e^{-x}\sin y \qquad \frac{\partial v}{\partial y} = -e^{-x}\cos y$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &\neq \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} &\neq \frac{\partial u}{\partial y} \end{aligned}\right] \text{ f(z) is not analytic.}$$

Ex-) $f(z) = \sin z = \sin(x+iy)$

$f(z) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \sinh y \cos x$

$$\boxed{\begin{aligned} \cos iy &= \cosh y \\ \sin iy &= i \sinh y \end{aligned}} \quad ★★$$

$u = \sin x \cosh y$

$v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y \qquad \frac{\partial v}{\partial y} = \cos x \cosh y$

$\frac{\partial u}{\partial y} = \sin x \sinh y \qquad \frac{\partial v}{\partial x} = -\sin x \cosh y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ Analytic $\forall x^n$.

$f(z) = u + iv \qquad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \cos x \cosh y + i(-\sin x \cosh y)$

$f'(z) = \cos x \cosh y - \sin x \cosh y$

$f'(z) = \cos(x+iy) = \cos z$

Harmonic functions :-

$f(z) = u + iv$

if $u = x^2 - y^2$

then find v

v is called Harmonic conjugate of u.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

diff. wrt x

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad -(a)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

diff. wrt y

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \quad -(b)$$

(a) + (b)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Eg ⇒ $u = x^2 - y^2$. Find v

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence it is a harmonic fx^n and u is a part of analytic fx^n.

If v is given then

$$\boxed{\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0}$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$dv = 2y\,dx + 2x\,dy$$

$$dv = 2(y\,dx + x\,dy)$$

$$\int dv = 2\int d(xy) + c \Rightarrow v = 2xy + c.$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Ex-) $v(x,y) = 3x^2y - y^3$

Soln.) $\frac{\partial v}{\partial x} = 6xy \qquad \frac{\partial^2 v}{\partial x^2} = 6y$

$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 \qquad \frac{\partial^2 v}{\partial y^2} = -6y$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow$ ∵ Harmonic fx^n ✓

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

$$du = (3x^2 - 3y^2)dx - 6xy\,dy$$

$$du = 3x^2dx - 3(y^2dx + 2xy\,dy)$$

$$= 3x^2dx - 3\int d(xy^2) + c$$

$$u = x^3 - 3xy^2 + c$$

Ex-) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2x + 1$

Soln.) $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x + 2$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6$$

$$\frac{\partial u}{\partial y} = -6xy - 6y$$

$$\frac{\partial^2 u}{\partial y^2} = -6x - 6$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

∵ Harmonic fx^n ✓

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = \frac{-\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$dv = (6xy + 6y)dx + (3x^2 - 3y^2 + 6x + 2)dy$$

$$dv = (6xy\,dx + 3x^2 dy)$$

$$\int dv = \int d(3x^2y) + \int 6(y\,dx + x\,dy)$$

$$\int -3y^2 dy + \int 2\,dy + c$$

$$dv = 3x^2y + 6xy - y^3 + 2y + c$$